delft hydraulics laboratory

analytical approaches to non-steady bedload transport

report on basic research

S 78 part IV

May 1973

analytical approaches to non-steady bedload transport

C.B. Vreugdenhil and M. de Vries

report on basic research

S 78 part IV

May 1973

CONTENTS

FIGURES

ANALYTICAL APPROACHES TO NON-STEADY BEDLOAD TRANSPORT

1 Introduction

Time dependent variations of a riverbed due to natural causes and human interference can be estimated for a number of cases by means of a mathematical model [4, 5] . This model, however, is a numerical one and it is therefore not very suitable for rough guesses and first approximations on the influence of various parameters to the time scale of problems in river morphology.

In this report the basic equations of these morphological processes are considered again with the aim to obtain analytical solutions. Naturally this requires assumptions and comparison with the numerical solution of the basic equations is necessary.

The problem is described by the equations of motion and continuity of water and sand. The depth (h), flow velocity (v), bedlevel (z), and sediment transport (s) are the dependent variables, Only one space dimension (x) is used. As has been shown earlier [5] the flow can be considered quasi-steady because the adjustment of the bedlevel is much slower than the one of the waterlevel. Therefore the equation of motion of the fluid becomes the differential equation for backwater curves. This leads to a hyperbolic differential equation for z. This hyperbolic model is discussed in chapter 2.

A stronger schematization is obtained if it is assumed that the water movement is uniform during transient stages. This results in a parabolic model similar to the one used by Ashida and Michiue [1] and treated in chapter 3.

Both models are used to describe the problem of degradation of a riverbed due to the drop of the waterlevel over a certain distance. The hyperbolic and parabolic models are compared in chapter 4 whereas chapter 5 deals with the comparison with the numerical solution obtained without linearization of the basic equations. The conclusions of this study are formulated in chapter 6.

2 Hyperbolic model

The basic equations for the unit width read:

$$v \frac{\partial v}{\partial x} + g \frac{\partial h}{\partial x} + g \frac{\partial z}{\partial z} = - \frac{g \, v|v|}{C^2 h} \tag{2-1}$$

$$v \frac{\partial h}{\partial x} + h \frac{\partial v}{\partial x} = 0 \tag{2-2}$$

$$\frac{\partial z}{\partial t} + \frac{\partial s}{\partial x} = 0 \tag{2-3}$$

$$s = f(v) \tag{2-4}$$

As indicated in chapter 1 these equations hold for a quasi steady water-movement. The sediment transport s (x, t) is a function of the flow velocity v and other parameters which are assumed to be constant.

It requires some algebra to combine these flow equations into one (hyperbolic) differential equation for the bedlevel z.

Therefore first of all s can be eliminated from Eqs. (2-3) and (2-4) which gives

$$\frac{\partial z}{\partial t} + f'(v) \frac{\partial v}{\partial x} = 0 \tag{2-5}$$

with $f' = ds/dv$

Secondly h and $\partial h/\partial x$ can be eliminated from Eqs. (2-1) and (2-2); this gives

$$(v - \frac{gq}{v^2}) \frac{\partial v}{\partial x} + g \frac{\partial z}{\partial x} = - g \frac{v^2|v|}{C^2 q} \tag{2-6}$$

where q is the discharge per unit width.

The derivative $\partial v/\partial x$ can now be eliminated from Eqs. (2-5) and (2-6), thus:

$$- \left(v - \frac{gq}{v^2}\right) \frac{\partial z}{\partial t} + g\, f' \frac{\partial z}{\partial x} + g\, f' \frac{v^2 |v|}{C^2 q} = 0 \qquad (2\text{-}7)$$

This can be considered as

$$v = F\,(\alpha,\beta) \qquad (2\text{-}8)$$

with

$$\alpha = \partial z/\partial t \text{ and } \beta = \partial z/\partial x \qquad (2\text{-}9)$$

Thus $$- \left(v - \frac{gq}{v^2}\right) \alpha + g\, f' \beta + g\, f' \frac{v^3}{C^2 q} = 0 \qquad (2\text{-}10)$$

where it is assumed that the flow is in the positive x-direction. Differentiation with respect to α gives

$$- \left(v - \frac{gq}{v^2}\right) - \alpha \left(1 + \frac{2\,gq}{v^3}\right) \frac{\partial v}{\partial \alpha} + g\, f'' \beta \frac{\partial v}{\partial \alpha} =$$

$$= - \left(g\, f'' \frac{v^3}{C^2 q} + 3\, g\, f'' \frac{v^2}{C^2 q}\right) \frac{\partial v}{\partial \alpha} \qquad (2\text{-}11)$$

or

$$\frac{\partial v}{\partial \alpha} = \frac{\frac{gq}{v^2} - v}{\alpha \left(1 + \frac{2\,gq}{v^3}\right) - g\, f'' \beta - g\, f'' \frac{v^3}{C^2 q} - 3\, g\, f' \frac{v^2}{C^2 q}} \qquad (2\text{-}12)$$

Differentiation of Eq. (2-10) with respect to β yields

$$- \left(1 + \frac{2\,gq}{v^3}\right) \alpha \frac{\partial v}{\partial \beta} + g\,f' + g\,\beta\,f'' \frac{\partial v}{\partial \beta} =$$

$$= - \left(g\,f'' \frac{v^3}{C^2 q} + 3\,g\,f' \frac{v^2}{C^2 q}\right) \frac{\partial v}{\partial \beta} \tag{2-13}$$

or

$$\frac{\partial v}{\partial \beta} = \frac{g\,f'}{\alpha \left(1 + \frac{2\,gq}{v^3}\right) - g\,f''\,\beta - g\,f'' \frac{v^3}{C^2 q} - 3\,g\,f' \frac{v^2}{C^2 q}} \tag{2-14}$$

Apparently

$$\frac{g\,f'}{\frac{gq}{v^2} - v} \frac{\partial v}{\partial \alpha} - \frac{\partial v}{\partial \beta} = 0 \tag{2-15}$$

The term

$$c = \frac{g\,f'}{\frac{gq}{v^2} - v} \tag{2-16}$$

is just equal to the celerity of a small disturbance at the bed for Froude numbers not too close to unity [5].

Hence

$$c \frac{\partial v}{\partial \alpha} - \frac{\partial v}{\partial \beta} = 0 \tag{2-17}$$

Turning back to Eq. (2-8) it follows

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial v}{\partial \beta} \cdot \frac{\partial \beta}{\partial x} \tag{2-18}$$

or with Eqs. (2-10), (2-12) and (2-17)

$$\frac{\partial z}{\partial t} + f' \frac{\partial v}{\partial \alpha} \left(\frac{\partial^2 z}{\partial x \partial t} + c \frac{\partial^2 z}{\partial x^2} \right) = 0 \tag{2-19}$$

or introducing $D = - f'.\ \partial v/\partial \alpha.\ c$ (2-20)

$$\frac{\partial z}{\partial t} - D \frac{\partial^2 z}{\partial x^2} - \frac{D}{c} \frac{\partial^2 z}{\partial x \partial t} = 0 \tag{2-21}$$

The parameter D will now be studied which means that $\partial v/\partial \alpha$ has to be considered.

Eq. (2-10) yields

$$\alpha = \frac{g\ f'}{v - \frac{gq}{v^2}} \left(\beta + \frac{v^3}{C^2 q}\right) \tag{2-22}$$

Using Eqs. (2-12), (2-16) and (2-22) the expression (2-20) for D becomes

$$D = \frac{f'}{\dfrac{\left(\beta + \frac{v^3}{C^2 q}\right)\left(1 + \frac{2\,gq}{v^3}\right)}{\frac{gq}{v^2} - v} + \dfrac{f''}{f'}\left(\beta + \dfrac{v^3}{C^2 q}\right) + 3\,\dfrac{v^2}{C^2 q}} \tag{2-23}$$

If the initial bottom slope is denoted by i_o there results

$$\frac{v^3}{C^2 q} = \frac{v^3 h_o^3 i_o}{q^3} = \left(\frac{v}{v_o}\right)^3 i_o$$

Introducing further the Froude number

$$F = v/\sqrt{gh}$$

Eq. (2-23) becomes

$$D = \frac{1}{3}\frac{f'v}{i_o} \frac{1}{\frac{1}{3}\left\{\frac{\beta}{i_o} + \left(\frac{v}{v_o}\right)^3\right\}\left\{\frac{F^2+2}{1-F^2} + \frac{f''v}{f'}\right\} + \left(\frac{v}{v_o}\right)^3} \qquad (2\text{-}24)$$

As an example the simple power-law transport relation

$$f = av^b \qquad (2\text{-}25)$$

can be introduced. This yields

$$D = \frac{1}{3}\frac{bs}{i_o} \frac{1}{\frac{1}{3}\left\{\frac{\beta}{i_o} + \left(\frac{v}{v_o}\right)^3\right\}\left\{\frac{2+F^2}{1-F^2} + b - 1\right\} + \left(\frac{v}{v_o}\right)^3} \qquad (2\text{-}26)$$

In general for nearly uniform flow $-\beta/i_o$ and v/v_o approach unity and

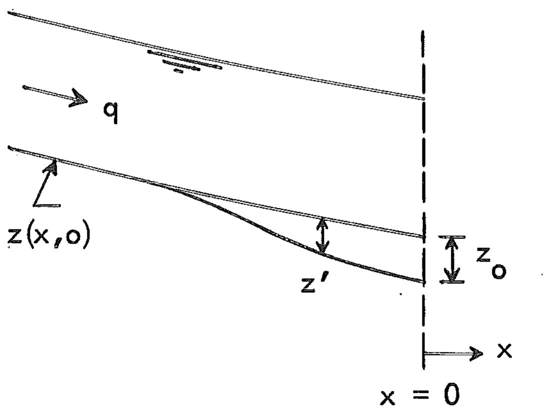
$$D = \frac{1}{3}\frac{f'v}{i_o} \qquad (2\text{-}27)$$

which specializes to

$$D = \frac{1}{3} b \frac{s}{i_o} \qquad (2\text{-}28)$$

if the power-law relation (2-25) is used.

Thus the process is described by the hyperbolic differential equation (2-21), the celerity c and the coefficient D being given by Eqs (2-16) and (2-24). Analytical solutions of this equation generally are not possible as c and D depend on the velocity. However, by linearizing Eq (2-21) an equation with constant coefficients is obtained. This amounts to evaluating c and D at the initial uniform-flow situation and treating them as constants.



As an example the degradation problem for a river will be taken. At x = 0 the bed level is lowered by an amount z_o. A new dependent variable z' will be defined as the change in the bed level:

$$z' = z(x,o) - z$$

The differential equation (2-21) also applies to z'. Boundary and initial conditions are:

$$z'(x,o) = o \tag{2-29}$$
$$z'(o,t) = z_o \tag{2-30}$$

Hereafter the primes are omitted as long as no confusion is possible. The solution of the problem is derived in the Appendix.

3 Parabolic model

A parabolic model is reached by assuming from the very beginning that the water movements remains uniform during transient stages.

The equation of motion now reads

$$-g \frac{v|v|}{C^2h} = g \frac{\partial z}{\partial x} \tag{3-1}$$

Thus

$$\frac{\partial z}{\partial x} = - \frac{v^3}{C^2q} \tag{3-2}$$

or

$$\frac{\partial^2 z}{\partial x^2} = - 3 \frac{v^2}{C^2q} \frac{\partial v}{\partial x} \tag{3-3}$$

From this equation and Eq. (2-5) the term $\partial v/\partial x$ can be eliminated leading to

$$\frac{\partial z}{\partial t} - \frac{C^2q\ f'}{3v^2} \frac{\partial^2 z}{\partial x^2} = 0 \tag{3-4}$$

or

$$\frac{\partial z}{\partial t} - D \frac{\partial^2 z}{\partial x^2} = 0 \tag{3-5}$$

with

$$D = \frac{C^2q\ f'}{3v^2} = \frac{1}{3} \frac{f'v}{i_o} \left(\frac{v_o}{v}\right)^3 \tag{3-6}$$

Evaluating this at the initial situation yields the same result for D as Eq. (2-27). However, the assumption of uniform flow throughout the process leads to a different equation (3-5), which is of the parabolic type. This is attractive as analytical solutions can easily be obtained but the validity of the parabolic model requires further attention. Eq. (3-5) is equivalent to the diffusion equation derived by Ashida and Michiue [1]. The equation has earlier been derived by Culling [3] using qualitative arguments and not specifying the parameter D.

Also the parabolic model can be used to solve the problem of the degradating river. Taking again z as the variation of the bedlevel with the original one, the problem is defined by

$$\frac{\partial z}{\partial t} - D \frac{\partial^2 z}{\partial x^2} = 0$$

$$z\ (x,o) = 0$$
$$z\ (o,t) = z_o \qquad (3\text{-}7)$$

The solution derived in the Appendix reads

$$z = z_o \text{ erfc } \left(- \frac{x}{2\sqrt{Dt}}\right) \qquad (3\text{-}8)$$

with

$$\text{erfc } y = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-\xi^2} d\xi$$

The time at which 50 % of the final lowering of the bed has been reached is defined by

$$\text{erfc } \left(- \frac{x_{50}}{2\sqrt{Dt_{50}}}\right) = \tfrac{1}{2} \qquad (3\text{-}9)$$

or

$$x_{50} = -\ 0.96 \sqrt{Dt_{50}} \qquad (3\text{-}10)$$

thus

$$z = \tfrac{1}{2}\, z_o \quad \text{for} \quad x_{50} \approx -\sqrt{D t_{50}} \tag{3-11}$$

In the derivation of the parabolic model, Eq. (3-5), no assumption is made concerning the variation of the discharge q with time. Therefore the equation is also valid for transient flow, of course within the assumption of uniform flow. To take this into account, the linearization should be restricted to the consequences of bed-level changes, whereas the variation of the coefficient of diffusion D with time remains. With the conditions (3-7) then a very simple solution results:

$$z = z_o \;\operatorname{erfc} \left\{ -\tfrac{1}{2}\, x \left(\int_0^t D\,(t')\, dt' \right)^{-\frac{1}{2}} \right\} \tag{3-12}$$

which degenerates into Eq. (3-8) for a constant discharge. A similar simple extension of the hyperbolic equation is not possible.

4 Comparison of linear models

For the schematic case of a sudden drop z_o in the bottom level the solutions of the hyperbolic and parabolic models, as derived in the Appendix, are

hyperbolic model

$$\frac{z\,(x,t)}{z_o} = e^{-\tau_o} + \tau_o \int_{\tau_o}^{\tau_o^{\,2}/\theta + \tau_o} e^{-w}\, \frac{I_1\left(\sqrt{w^2 - \tau_o^{\,2}}\right)}{\sqrt{w^2 - \tau_o^{\,2}}}\, dw \tag{4-1}$$

parabolic model

$$\frac{z\,(x,t)}{z_o} = \operatorname{erfc}\left(\sqrt{\tfrac{1}{2}\,\theta}\right) \tag{4-2}$$

where $\theta = x^2/(2\,Dt)$ and $\tau_o = -\,cx/D$. These expressions are shown in Fig. 1 as a function of θ with τ_o as a parameter. For small values of θ (large values of time) the results of both models approach each other. This effect is stronger for larger values of τ_o, i.e. for larger distances from the origin of the erosion. A quantitative estimate of the rate at which the two models approach each other can be obtained from asymptotic expansions of Eqs. (4-1) and (4-2) for large values of time. In the Appendix the following expansions are derived:

hyperbolic model:

$$\frac{z\,(x,t)}{z_o} \approx 1 - \sqrt{\frac{2\theta}{\pi}}\left[1 - \theta\left(\frac{1}{8\tau_o^{\,2}} - \frac{1}{2\tau_o} + \frac{1}{6}\right) + \ldots\right] \qquad (4\text{-}3)$$

parabolic model:

$$\frac{z\,(x,t)}{z_o} \approx 1 - \sqrt{\frac{2\theta}{\pi}}\left(1 - \frac{1}{6}\theta + \ldots\right) \qquad (4\text{-}4)$$

These two expressions will be almost identical if θ is small (smaller than, say, 0.25 or 0.1) and/or if τ_o is large (larger than, say, 10). The latter condition can be given a physical interpretation by introducing the values of c and D from chapter 2. For small values of the Froude number F

$$c \approx b\,s/h \quad \text{and} \quad D \approx \frac{1}{3}\, b\,\frac{s}{i_o}$$

so $\tau_o > 10$ means

$$|\,x\,| > 10\,D/c \approx 3\,h/i_o$$

The parabolic model therefore is certainly a good approximation at distances where the river bottom is more than a few times the waterdepth higher than its downstream level. The approximation may also be good at smaller distances after a sufficient interval of time (θ small). The preceding is also illustrated by the time at which 50 % of the final lowering of the riverbed has been reached. From Eq. (3-10) this time for the parabolic model is given by

$$x_{50} = 0.96 \sqrt{Dt_{50}} \text{ or } \theta_{50} = 0.46$$

It is not possible to derive an analytical expression for the hyperbolic model. Values can be read, however, from Fig. 1 and the results are given in Fig. 2. It can also be seen from Fig. 1, that the asymptotic expansion for the hyperbolic model is not very suitable to estimate θ_{50}.

5 Comparison to non-linear model*

The numerical solution of the hyperbolic model without the linearization of equation (2-21) gives additional insight into the validity of the analytical approaches.

In order to solve the same problem of a degradating river with an initial bottom discontinuity, a numerical "pseudo-viscosity" method was chosen. This method is especially suitable for avoiding "shock-fitting" complications when discontinuities in the dependent variables exist [4] .

The numerical procedure starts from a known steady state and approximates the differential equations by difference equations. Conditions at each new time-step are derived from results of previous time-steps with additional information at the boundaries. By the numerical procedure, solving Eqs. (2-3), (2-4) and (2-6), an artificial "viscosity" is introduced which means that to a first approximation Eq. (2.3) is replaced by [6]

$$\frac{\partial z}{\partial t} + \frac{\partial s}{\partial x} = (\beta + \mu^2_{max} - \mu^2) \frac{\Delta x^2}{2\Delta t} \frac{\partial^2 z}{\partial x^2} \qquad (5\text{-}1)$$

*Material for this chapter has been contributed by Mr. C. Parra where $\mu = c\Delta t/\Delta x$ with the time step Δt, the mesh width Δx and the velocity of propagation c defined by Eq. (2-16). The maximal value of μ at a certain time level is denoted by μ_{max}; further β is an adjustable parameter. Although the presence of the diffusive term limits the accuracy, it is essential to take care of possible shocks. The difference scheme is formed by equally-spaced points. Three locations of special interest can be recognized: the upstream boundary, the bottom-drop location and the downstream boundary.

The upstream boundary presents no problems. At this point a constant bottom-level and sand-transport were specified as boundary conditions. Although this is not an exact assumption this point was chosen so far upstream that the region of interest is not influenced.

The sudden drop in the river bottom, used as an example in the preceding chapters, has been treated in two different ways, shown in Figs. 3A and B. The schematization conforming as closely as possible to the analytical case is given in Fig. 3A. The initial discontinuity is (necessarily) spread over one mesh width. The downstream boundary condition to be used for the backwater equation was found from two considerations. Within the most downstream mesh the equation of continuity for sand (2-3) should be satisfied and also the backwater equation (2-6) is valid. Expressing both in finite differences it is found that the water depth at the downstream boundary point can be solved. As the bottom level at that point is kept constant also the water level is known. Some results are shown in Fig. 4, using data given in Fig. 3. Dimensionless values τ and θ were computed using the local values of D and c. In all numerical computations the coefficient β in the numerical viscosity was taken 0.01.

A second approach, shown in Fig. 3B, was used to represent the phenomenon in a physically more realistic way without fixing the bottom level at point B. Instead of this the bottom level was kept constant at a point so far downstream that backwater effects due to this assumption do not reach the region of interest. The initial conditions downstream of the bottom discontinuity (again represented by a steep slope over one mesh width) were chosen equal to the equilibrium conditions for the bottom level with uniform flow at point B.

As the drop in the bottom level at point B (Fig. 3) is not constant in time in this schematization, the asymtotic value reached after a long time has been used for z_o in the comparison.

Figs. 5 and 6 show results corresponding to this second schematization. If the bottom discontinuity z_o is small (e.g. smaller than 5 $^o/o$ of the normal depth of flow), results are very similar to those of the analytical hyperbolic model. This is also true for small values of τ_o (points near the bottom discontinuity), where the parabolic model presents only a rough approximation. For larger values of τ_o and for small values of θ (large values of time) the results of the three models approach each other. For larger values of z_o, however, the numerical model presents ever greater discrepancies from the linearized results. This non-linearity influences mainly those points near the bottom-discontinuity.

Therefore, the linearized approximations must be used carefully. When non-linear effects are negligible, the linearized hyperbolic model presents a good approximation to the phenomenon. The parabolic model gives adequate information only in those points far-in space and/or in time - from the initial disturbance.

6 Conclusions

The above given mathematical approach of the degradation problem leads to the following practical conclusions.

1 The non-linear hyperbolic model gives of course via its numerical solutions quantitatively the best answers to the problem.

2 The parabolic model provides easily analytical solutions for particular problems. The errors introduced in the differential equations via the rather crude assumptions have been indicated. Nevertheless the parabolic model can give useful results when used with care.

3 The linear hyperbolic model, though principally better than the parabolic one, is hampered by the fact that analytical solutions cannot easily be obtained.

4 An interesting aspect of the parabolic model is that the river-regime can easily be introduced. It is possible that this aspect enlarges the applicability of this model significantly. The introduction of the river-regime in the linear hyperbolic model is not possible. For the non-linear hyperbolic model the regime can of course be introduced in the numerical computations.

REFERENCES

1 Ashida, K. and M. Michiue - An investigation of riverbed degradation downstream of a dam, I.A.H.R. Congress Paris 1971, Paper C 30.

2 Carslow, H.S. and J.C. Jaeger - Operational methods in applied mathematics, Dover Publ. New York 1963.

3 Culling, W.E.H. - Analytical theory of erosion, J. of Geol., 68, 3, 1960.

4 Vreugdenhil, C.B. and M. de Vries - Computations on non-steady bedload transport by a pseudo-viscosity method, I.A.H.R. Congress Fort Collins 1967, also Delft Hydraulics Laboratory Publ. no. 45.

5 Vries, M. de - Considerations about non-steady bedload transport in open channels, I.A.H.R. Congress Leningrad 1963, also Delft Hydraulics Laboratory Publ. no. 36.

6 Vries, M. de - Solving river problems by hydraulic and mathematical models, Delft Hydraulics Laboratory Publ. no. 76-II, (1969).

NOTATION

a, b	coefficients in power-law transport formula
c	velocity of propagation
C	Chézy coefficient
D	coefficient of diffusion
f	sediment transport function
F	Froude number
g	acceleration due to gravity
h	water depth
i_o	initial bottom slope
q	discharge per unit width
s	sediment transport per unit width
t	time
v	mean velocity
x	coordinate
z	bottom level
α	$\partial z/\partial t$
β	$\partial z/\partial x$
θ	$x^2\ (2\ Dt)^{-1}$
τ	$2c^2t/D$
τ_o	$-cx/D$

Solution of the linearized equations

The linearized differential equations (2-21) and (3-5) are most conveniently solved by means of Laplace-transforms (e.g. [2]).
Denoting the Laplace transform of a quantity by a bar over the symbol, the transformed equation (2-21) becomes

$$p\bar{z} - D \frac{\partial^2 \bar{z}}{\partial x^2} - \frac{D}{c} p \frac{\partial \bar{z}}{\partial x} = 0 \qquad \text{(A-1)}$$

where p is the transformed time variable. Boundary conditions are:

$$\bar{z}\ (o,\ p) = \frac{z_o}{p}$$

and for mathematical completeness

$$\bar{z}\ (x,\ p) \rightarrow o \quad \text{if} \quad x \rightarrow -\infty$$

The solution of Eq (A-1) using these boundary conditions is straightforward and yields

$$\bar{z} = \frac{z_o}{p} \exp \left\{ bp - b\sqrt{p^2 + 2ap} \right\} \qquad \text{(A-2)}$$

where $b = -x/2c$ and $a = 2c^2/D$.

To find the original function of this transformed function the following series of relations is noted, where arrow indicates the inverse Laplace transform [2] .

1 $\quad \exp\ (-b\sqrt{p^2-a^2}) - \exp\ (-bp) \rightarrow ab \dfrac{I_1\ (a\sqrt{t^2-b^2})}{\sqrt{t^2-b^2}}\ H\ (t-b)$

2 $\quad \exp\ (-bp) \rightarrow \delta\ (t-b)$

3 from 1 and 2:

$$\exp\left(-b\sqrt{p^2-a^2}\right) \rightarrow \delta\,(t-b) + ab\,\frac{I_1\left(a\sqrt{t^2-b^2}\right)}{\sqrt{t^2-b^2}}\;H\,(t-b)$$

4 $$\exp\left(-b\sqrt{(p+a)^2-a^2}\right) \rightarrow e^{-at}\,\delta\,(t-b) + abe^{-at}\,\frac{I_1\left(a\sqrt{t^2-b^2}\right)}{\sqrt{t^2-b^2}}\;H\,(t-b)$$

The factor 1/p in Eq (a-2) indicates an integration with respect to time and the factor exp (bp) indicates a displacement over a time interval -b. Consequently

$$z\,(x,\,t) = z_o\,H\,(t+b)\int_0^{t+b} e^{-au}\left\{\delta(u-b) + ab\,\frac{I_1\left(a\sqrt{u^2-b^2}\right)}{\sqrt{u^2-b^2}}\,H\,(u-b)\right\}du$$

or

$$\frac{z\,(x,\,t)}{z_o} = e^{-ab} + ab\int_b^{t+b} e^{-au}\,\frac{I_1\left(a\sqrt{u^2-b^2}\right)}{\sqrt{u^2-b^2}}\;du \qquad \text{(A-3)}$$

which should be compared to Eq (3.8). In the above formulae the following definitions apply.

$\delta(t)$ Dirac- or delta function, defined by

$$\int_{-\infty}^{+\infty} \delta(t)\,f(t)\,dt = f(o)$$

for an arbitrary function f(t)

H(t) unit-step function $H(t) = 1 \quad t \geq 0$

$= 0 \quad t < 0$

$I_1(t)$ modified Bessel function of the first kind and first order, defined as $-iJ_1(it)$ where J_1 is an "ordinary" Bessel function of the first order and i is the imaginary unit.

Further simplification of Eq (A-3) does not appear to be possible. It therefore has to be evaluated numerically. In this respect it is important to note that only two dimensionless parameters are involved. By introduction of

$$\tau = at = \frac{2\,c^2\,t}{D} \quad \text{and} \quad \tau_o = ab = -\frac{cx}{D} \tag{A-4}$$

Eq. (A-3) becomes

$$\frac{z\,(x,\,t)}{z_o} = e^{-\tau_o} + \tau_o \int_{\tau_o}^{\tau+\tau_o} e^{-w} \frac{I_1(\sqrt{w^2-\tau_o^2})}{\sqrt{w^2-\tau_o^2}}\, dw \; . \tag{A-5}$$

The solution of the parabolic equation (3-5) is found in the same way. The Laplace transform of this equation reads

$$p\bar{z} - D\frac{d^2\bar{z}}{dx^2} = 0 \tag{A-6}$$

$\bar{z}\,(o,\,p) = z_o/p$
$\bar{z}\,(x,\,p) \rightarrow 0 \quad \text{if } x \rightarrow -\infty$

with the solution $\dfrac{\bar{z}}{z_o} = \dfrac{1}{p}\exp\,(x\sqrt{p/D})$ (A-7)

The original function of this can be found in [2]:

$$\frac{z\,(x,\,t)}{z_o} = \operatorname{erfc}\,\left(-\frac{x}{2\sqrt{Dt}}\right) \tag{A-8}$$

or in the dimensionless quantities defined before:

$$\frac{z\,(x,\,t)}{z_o} = \operatorname{erfc}\,\left(\frac{\tau_o}{\sqrt{2\tau}}\right) \tag{A-9}$$

A comparison of Eqs (A-5) and (A-9) is difficult because of the complicated form of the former. Some insight can be gained by means of asymptotic expansions for large values of time, derived from the Laplace transforms (A-2) and (A-7). For the relevant theory cf. [2]. It is noted that both expressions have singularities only at p = o. Developing them into power series with respect to p yields

<u>hyperbolic case</u>

$$\frac{\bar{z}}{z_o} = \frac{1}{p}\left[1 + \text{integer powers of } p - b\sqrt{2ap}\left\{1 + \left(\frac{1}{4a} - b + \frac{1}{3}b^2a\right)p + \ldots\right\}\right] \tag{A-10}$$

<u>parabolic case</u>

$$\frac{\bar{z}}{z_o} = \frac{1}{p}\left[1 + \text{integer powers of } p + x\sqrt{\frac{p}{D}}\left\{1 + \frac{1}{6}x\frac{p}{D} + \ldots\right\}\right] \tag{A-11}$$

The corresponding asymptotic expansions of the original functions are then

<u>hyperbolic case</u>

$$\frac{z(x,\ t)}{z_o} \sim 1 - \frac{1}{\sqrt{\pi}}\left\{b\sqrt{\frac{2a}{t}} - \frac{1}{2}\frac{b\sqrt{2a}}{t^{3/2}}\left(\frac{1}{4a} - b + \frac{1}{3}ab^2\right) + 0\left(t^{-5/2}\right)\right\}$$

or

$$\frac{z(x,\ t)}{z_o} \sim 1 + \frac{x}{\sqrt{\pi Dt}}\left[1 - \frac{1}{4Dt}\left\{\frac{1}{4}\left(\frac{D}{c}\right)^2 + x\frac{D}{c} + \frac{1}{3}x^2\right\} + \ldots\right]$$

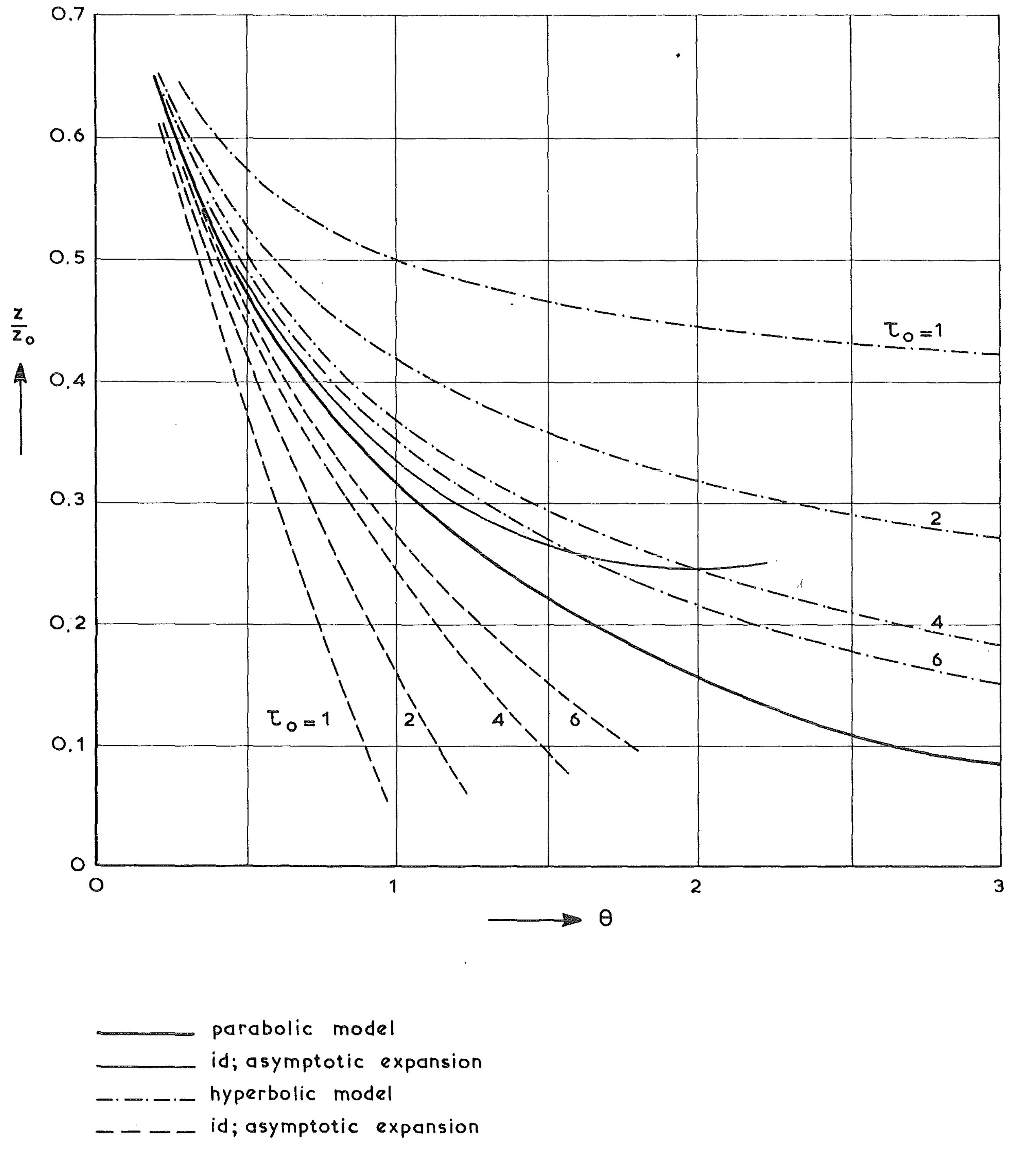
$$\sim 1 - \sqrt{\frac{2}{\pi}}\frac{\tau_o}{\sqrt{\tau}}\left[1 - \frac{1}{\tau}\left\{\frac{1}{8} - \frac{1}{2}\tau_o + \frac{1}{6}\tau_o^2\right\} + \ldots\right] \tag{A-12}$$

<u>parabolic case</u>

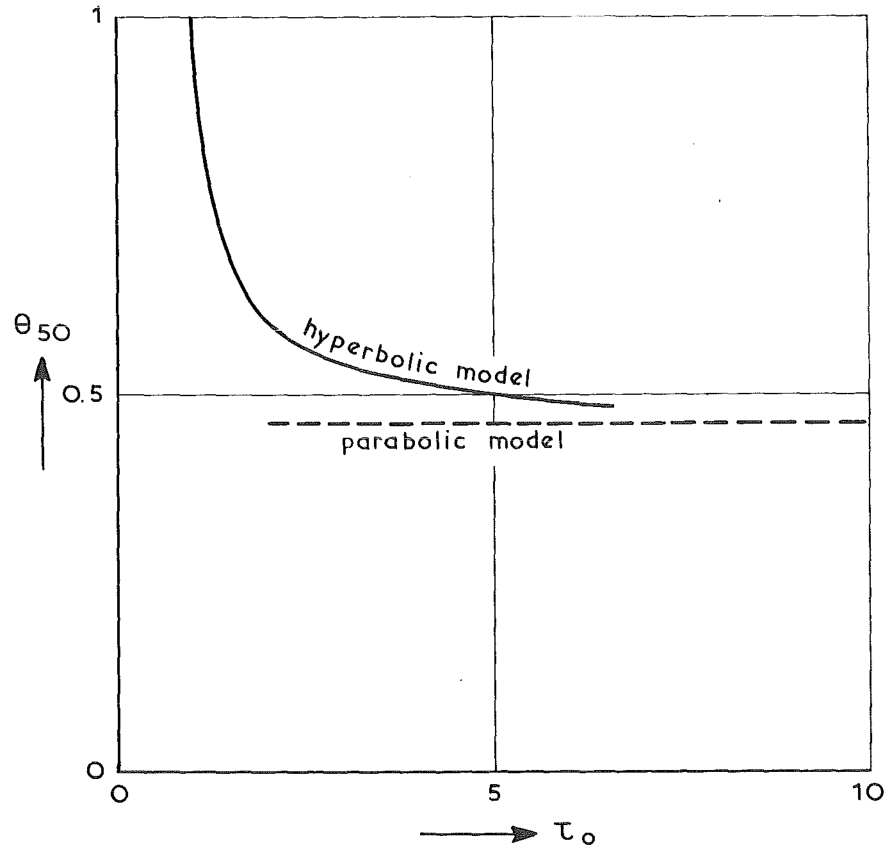
$$\frac{z\ (x,\ t)}{z_o} \sim 1 + \frac{x}{\sqrt{\pi D t}} \left[1 - \frac{x^2}{12Dt} \right] + 0\ (t^{-5/2})$$

$$\sim 1 - \frac{\tau_o \sqrt{2}}{\sqrt{\pi \tau}} \left\{ 1 - \frac{\tau_o^{\ 2}}{6\tau} + \ldots. \right\} \qquad \text{(A-13)}$$

In the above formulae it is convenient to replace the variable τ by $\theta = \tau_o^{\ 2}/\tau$. The parabolic case then will depend only on $\theta = x^2/(2Dt)$.

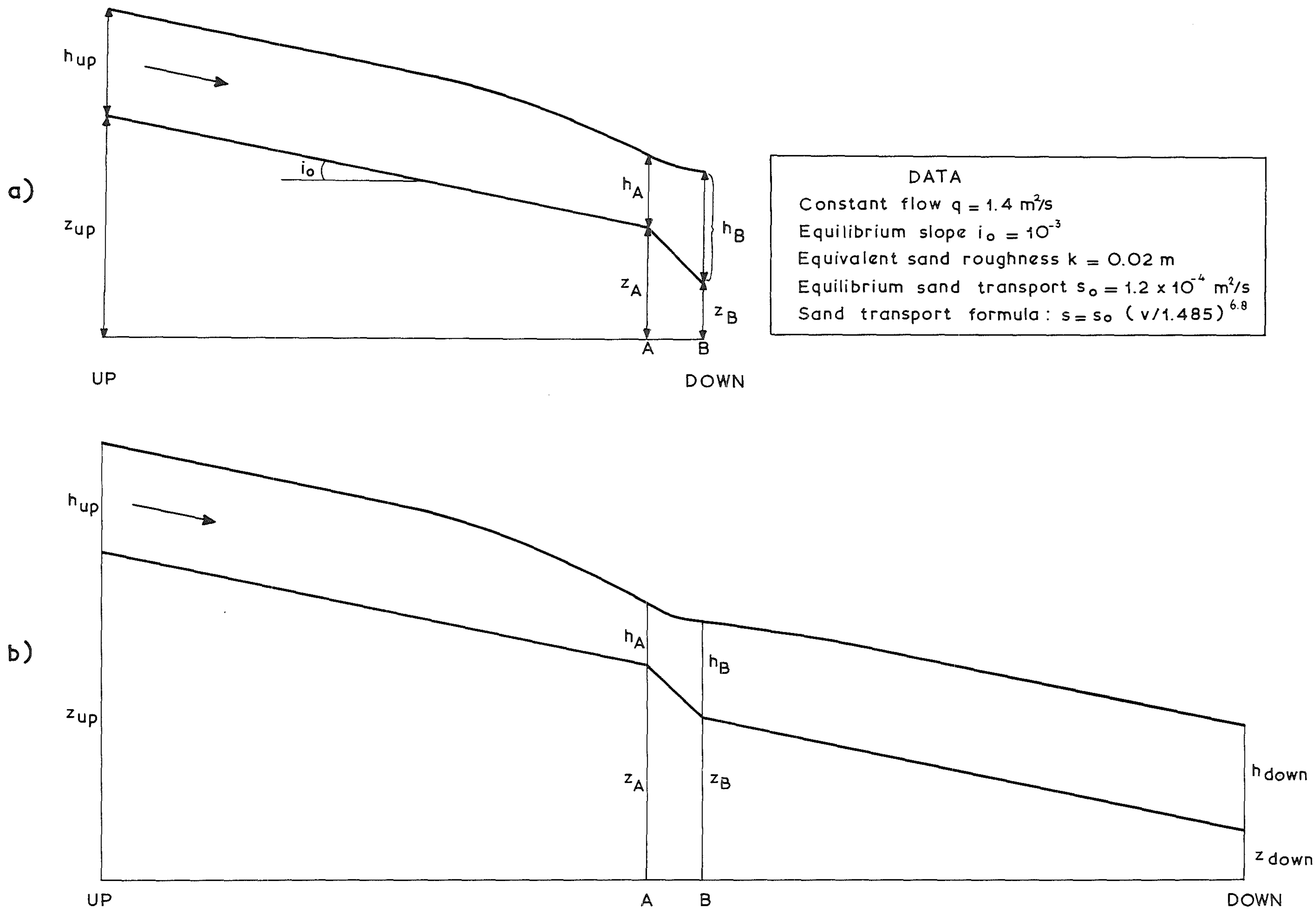


DIMENSIONLESS RIVER-BED DEGRADATION ACCORDING TO LINEARIZED HYPERBOLIC AND PARABOLIC MODELS		
WATERLOOPKUNDIG LABORATORIUM	S. 78-IV	FIG. 1



DIMENSIONLESS TIME AT WHICH DEGRADATION REACHES 50 % OF ITS FINAL VALUE

WATERLOOPKUNDIG LABORATORIUM | S. 78-IV | FIG. 2

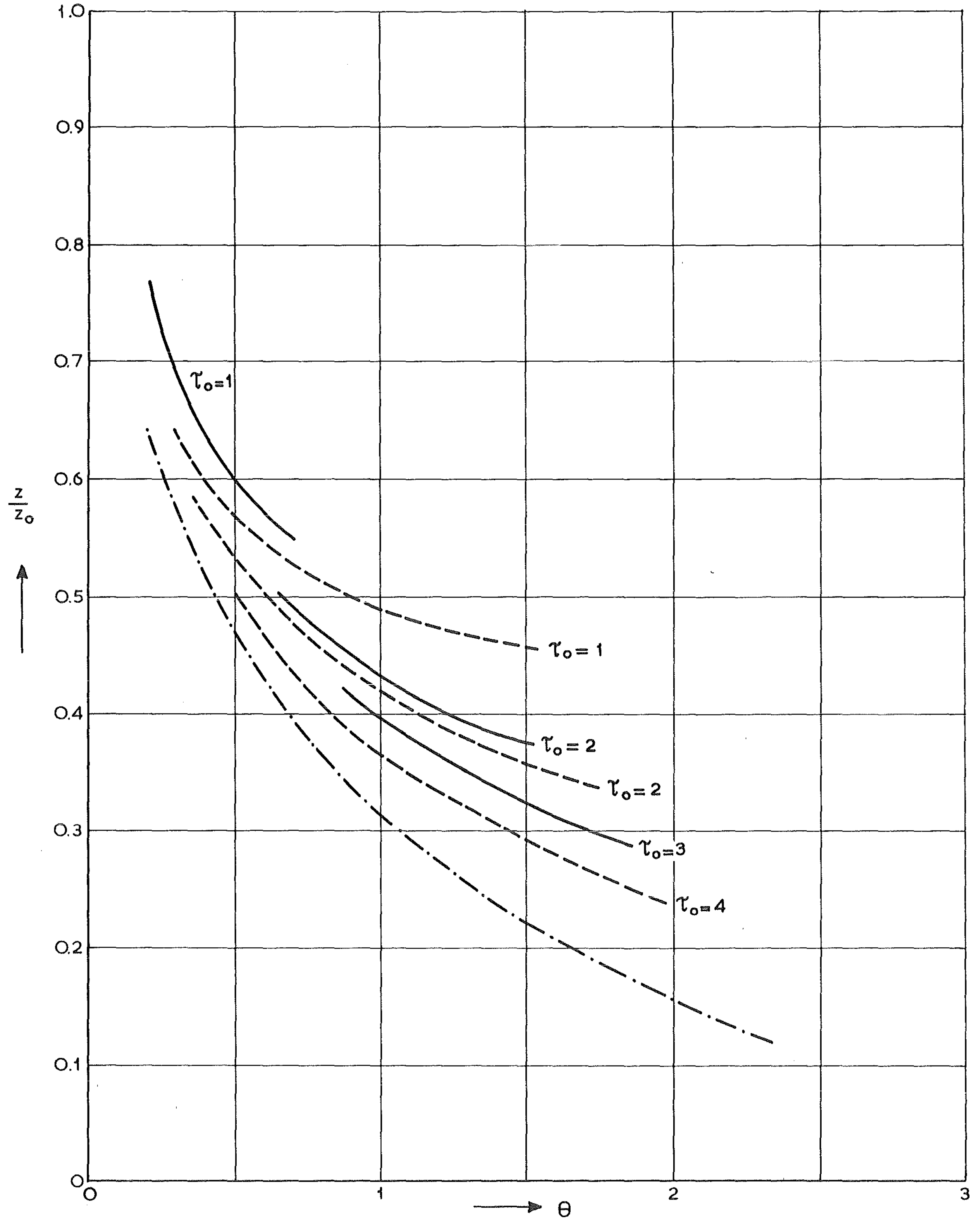


SCHEMATIZATIONS, NUMERICAL MODEL

WATERLOOPKUNDIG LABORATORIUM

S. 78-IV

FIG. 3



—·—·—·— parabolic model

– – – – – – analytic hyperbolic model

———— numerical solution with drop at downstream boundary ($z_o = 0.10$ m)

COMPARISON OF NUMERICAL AND ANALYTICAL RESULTS, SCHEMATIZATION A

WATERLOOPKUNDIG LABORATORIUM | S.78-IV | FIG. 4

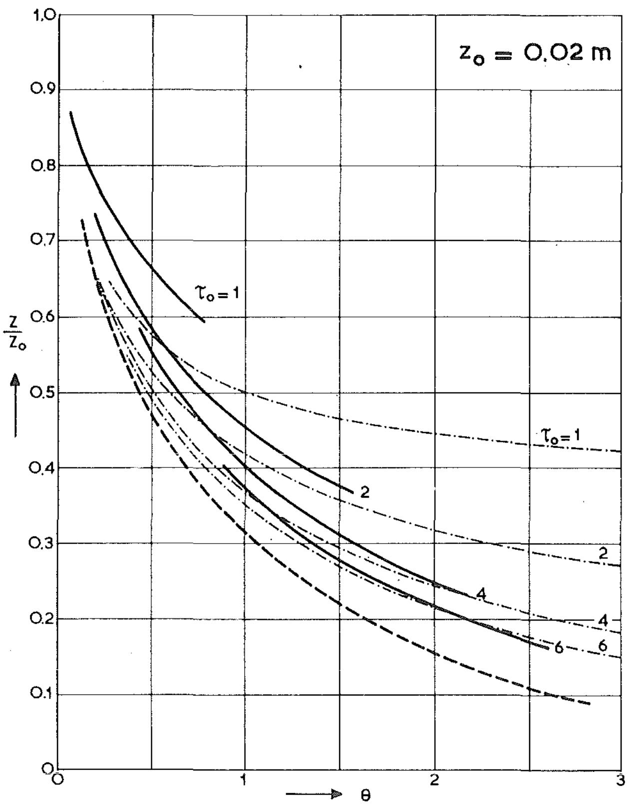


Fig. 5A

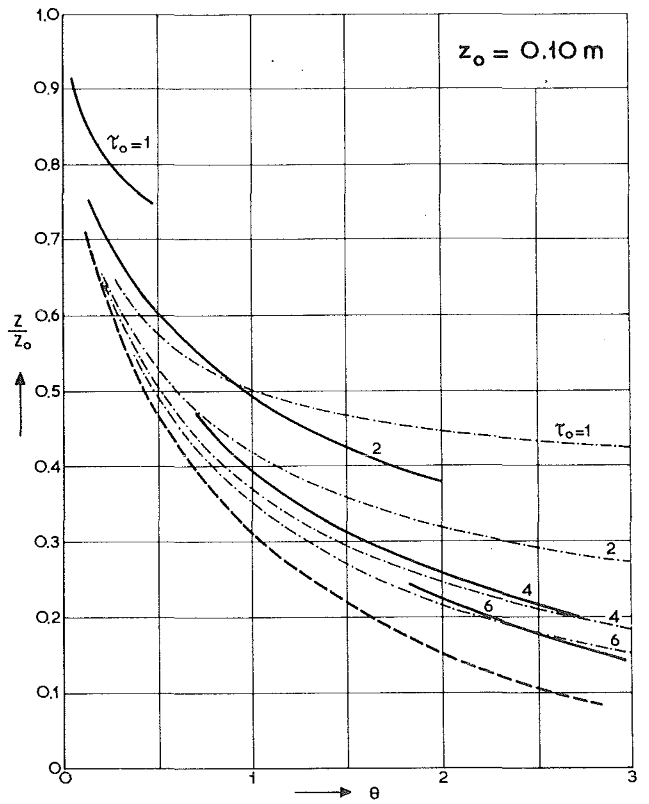


Fig. 5B

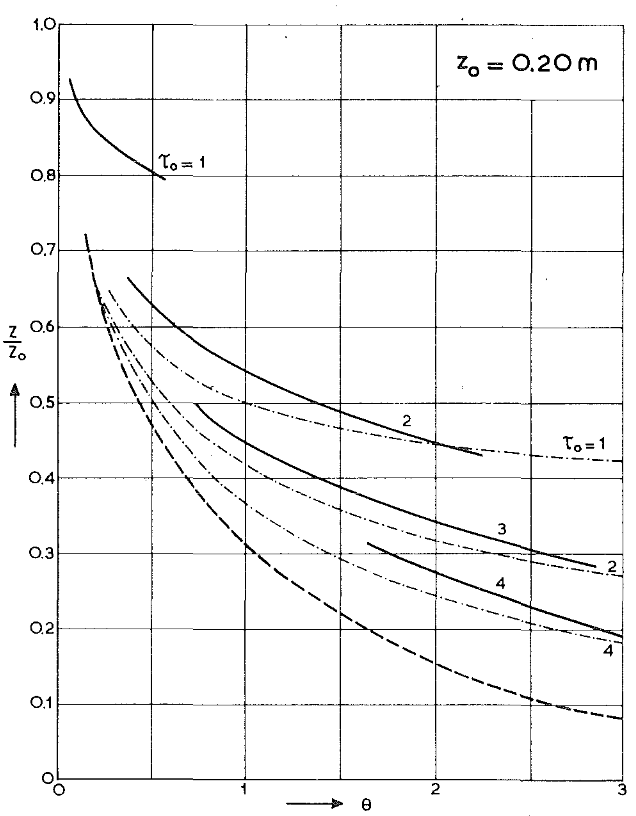
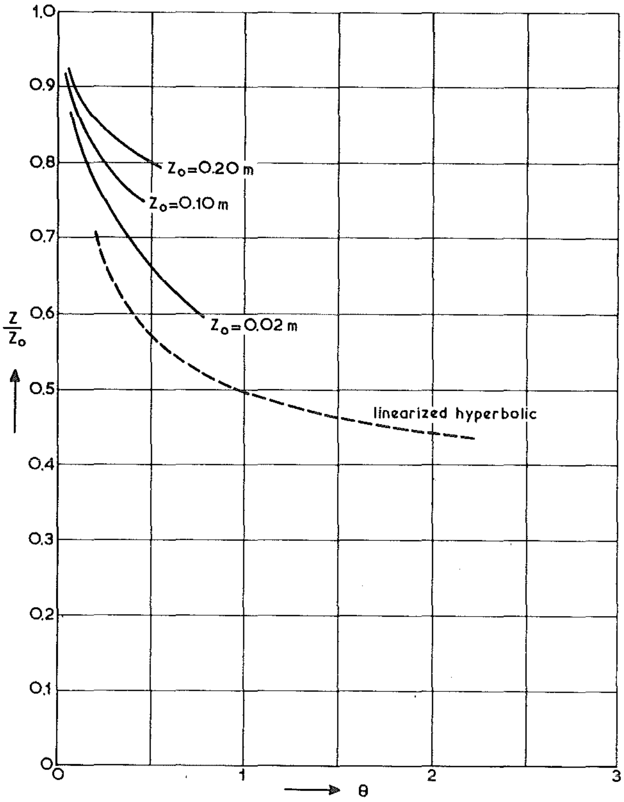


Fig. 5C

parabolic model
numerical solution by hyperbolic model without linearization
hyperbolic model, linearized

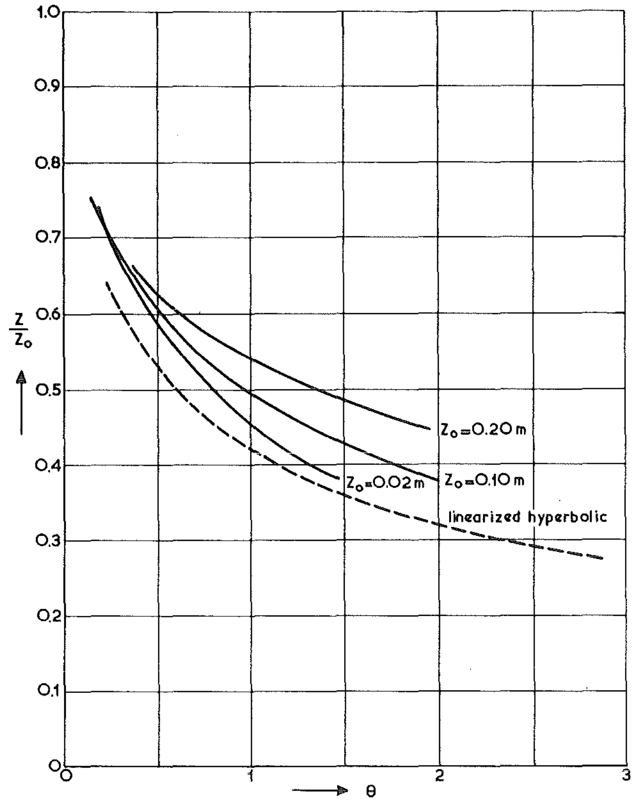
COMPARISON OF NUMERICAL AND ANALYTICAL RESULTS, SCHEMATIZATION B

WATERLOOPKUNDIG LABORATORIUM | S.78 IV | FIG. 5



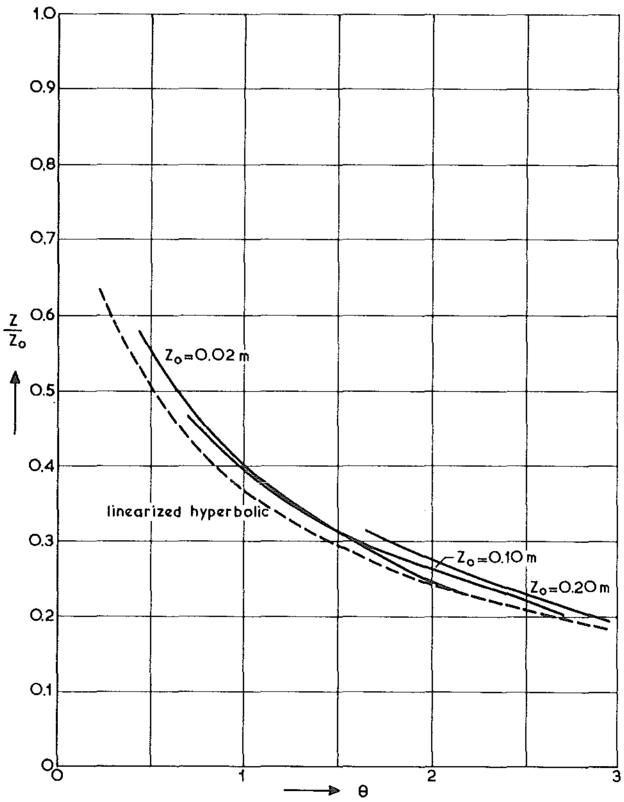
$\tau_o = 1$

Fig. 6A



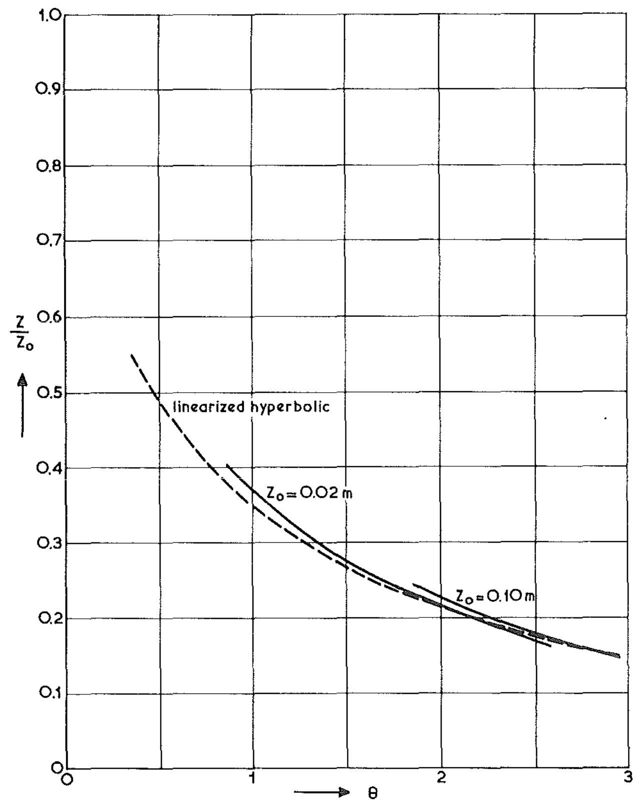
$\tau_o = 2$

Fig. 6B



$\tau_o = 4$

Fig. 6C



$\tau_o = 6$

Fig. 6D

COMPARISON OF NUMERICAL AND ANALYTICAL RESULTS
SCHEMATIZATION B. EFFECT OF NON LINEARITY

WATERLOOPKUNDIG LABORATORIUM

S. 78-IV

FIG. 6