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$q-$Orthogonal dualities for asymmetric particle systems*

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Abstract

We study a class of interacting particle systems with asymmetric interaction showing a self-duality property. The class includes the ASEP(q, θ), asymmetric exclusion process, with a repulsive interaction, allowing up to $\theta \in \mathbb{N}$ particles in each site, and the ASIP(q, θ), $\theta \in \mathbb{R}^+$, asymmetric inclusion process, that is its attractive counterpart. We extend to the asymmetric setting the investigation of orthogonal duality properties done in [8] for symmetric processes. The analysis leads to multivariate $q-$analogues of Krawtchouk polynomials and Meixner polynomials as orthogonal duality functions for the generalized asymmetric exclusion process and its asymmetric inclusion version, respectively. We also show how the q-Krawtchouk orthogonality relations can be used to compute exponential moments and correlations of ASEP(q, θ).

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1 Introduction

In this paper we study two models of interacting particle systems with asymmetric jump rates exhibiting a self-duality property. The first one is known in the literature as the *generalized asymmetric simple exclusion process*, ASEP(q, θ), $\theta \in \mathbb{N}$ [10]. This is a higher spin version of the asymmetric simple exclusion process ASEP(q) (corresponding to the choice $\theta = 1$) in which particles are repelled from each other and every site can

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host at most $\theta \in \mathbb{N}$ particles. The second process is the ASIP(q, θ), $\theta \in (0, \infty)$, *asymmetric simple inclusion process*, [11], where the parameter θ tunes the intensity of the attraction between particles (the smaller the θ, the higher the attraction). Particles move in a finite one-dimensional lattice and the parameter $q \in (0, 1)$ tunes the asymmetry in a certain direction. In [10, 11] a self-duality property has been shown for these models. Stochastic duality is an advantageous tool used in the study of interacting particle systems that was used for the first time in [42] for the standard symmetric exclusion process (see e.g. [24, 30, 41] for surveys on the topic). Duality relations allow to connect two Markov processes via a duality function; such function is an observable of both processes, whose expectation satisfies a specific relation. We speak of self-duality if the two Markov processes are two copies of the same process. The usefulness of (self-)duality is in the fact that it allows to study the system with a large number of particles in terms of the system initialized with a finite number of particles. For example, the study of n dual particles can give information on the n-points correlation function of the original process. Unfortunately self-duality is a property not always easy to reveal. The duality function for the standard asymmetric exclusion process, ASEP(q) (case $\theta = 1$), and its link to quantum algebras and spin chains was first revealed in [38, 40]. This discovery immediately found a vast number of applications, allowing to find for instance, combined with Bethe ansatz techniques, current fluctuations [28] and properties of the transition probabilities [27]. Among other important applications of self-duality and algebraic approach for ASEP, we mention the key role played in the study of shocks. We mention e.g. [3] for an analysis of microscopic shock dynamics, [4] for shocks in multispecies ASEP and [39] for the study of the process conditioned to low current. The self-duality function of ASEP is not given by a trivial product of 1-site duality functions (as in the symmetric case) but has a nested-product structure similar to the one exhibited by the Gärtner transform [21]. Thanks to this structure, it has played an important role in the proof of convergence to the KPZ equation, in the case of weak asymmetry (see e.g. [5, 6, 14, 15, 29]).

The partial exclusion process in its symmetric version SEP(α) appeared for the first time in [7] where the authors introduced it as a particle system version of the XXX-quantum-spin-chain, with spin higher than 1/2. Then the process, together with its attractive counterpart, SIP(α), was systematically studied in [23, 24, 25] were self-duality functions are found and used to prove correlation inequalities. These processes are not integrable (i.e. not treatable via Bethe ansatz techniques) but self-duality makes them amenable to some analytic treatment (see e.g. [9]).

The asymmetric processes ASEP(α, θ) and ASIP(α, θ) were finally introduced in [10, 11] where self-duality properties are proved. These are due to the algebraic structure of the generator that is constructed passing through the $(\alpha + 1)$-dimensional representation of a quantum Hamiltonian with $\mathcal{U}_q(\mathfrak{sl}_2)$ invariance. The self-duality function has again a nested-product structure, defining, in a sense, a generalized version of the Gärtner transform [21], that allows to compute the q-exponential moments of the current for suitable initial conditions. In the last few years, several steps forward have been done in the effort of finding suitable multispecies versions of ASEP(α, θ) showing duality properties, see e.g. [4, 12, 33, 34, 35, 36].

Most of the duality results concerning this class of processes are *triangular*, i.e. are non-zero only if the dual configuration is a subset of the original process configuration. We refer to duality functions of this type also as *classical duality functions*. Orthogonal polynomial duality functions are, on the other hand, a very recent discovery and were found, up to date, only for symmetric processes (SEP(α), SIP(α) and IRW) in a series of papers [8, 17, 18, 37]. The duality functions for these processes are products of univariate orthogonal polynomials, where the orthogonality is with respect to the reversible measures of the process itself. Knowing the expectations of orthogonal polynomial duality functions is equivalent to having all moments. The possibility to decompose polynomial functions in $L^2(\mu)$, where μ is the reversible measure of the process, in terms of orthogonal duality polynomials, is then a crucial property that has many repercussions in the study of macroscopic fields emerging as scaling limits of the particle system. See e.g. the work [1] for an application of orthogonal duality polynomials for symmetric models in the study of a generalized version of the Boltzmann-Gibbs principle. Moreover, in two recent papers [2, 13] orthogonal polynomials are at the base of the definition of the so-called higher-order fields for which the hydrodynamic limit and fluctuations are derived via duality techniques for SEP(α), SIP(α) and IRW. Finally, in a recent work [16] orthogonal duality results for this class of symmetric models have been extended to the non-equilibrium context, allowing to derive several properties of n-point correlation functions in the non-equilibrium steady state.

The families of orthogonal polynomials dualities for these processes were found for the first time in [17] by explicit computations relying on the hypergeometric structure of the polynomials. The same dualities were found in [37] via generating functions, while an algebraic approach is followed in [26] and [8], relying, respectively, on the use of unitary intertwiners and unitary symmetries. In [8] yet another approach to (orthogonal) duality is described, based on scalar products of classical duality functions.

In this paper we use this latter approach to extend the results obtained in [8] to the case of asymmetric processes. Differently from [8], the q-orthogonal duality functions for asymmetric processes are not yet known in the literature. We show that well-known families of $q-$hypergeometric orthogonal polynomials, the $q-$Krawtchouk polynomials (for exclusion processes) and $q-$Meixner polynomials (for inclusion processes), occur as 1-site duality functions for corresponding stochastic models. The q-orthogonal duality functions show again a nested-product structure, as the classical ones found in [10, 11], but, differently from the latter, they do not have a triangular form. We prove that the $q-$polynomials are orthogonal with respect to the reversible measures of our models, which, in turn, have a non-homogeneous product structure. The nested product structure and the orthogonality relations of our duality functions are very similar to the multivariate $q-$Krawtchouk and $q-$Meixner polynomials introduced in [22], but it seems that (except for the 1-variable case) they are not the same functions.

We conjecture that the orthogonal self-duality polynomials complete the picture of nested-product duality functions for ASEP(q,θ) and ASIP(q,θ), summing up to the classical or *triangular* ones, already known for these processes from [10, 11]. The strategy followed in [10, 11] to construct the so-called classical dualities relies on an algebraic approach based on the study of the symmetries of the generator. This can be written, indeed, in terms of the Casimir operator of the quantized enveloping algebras $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{U}_q(\mathfrak{su}(1,1))$. The same approach was used in [34] for the study of duality for a multi-species version of the asymmetric exclusion process, exploiting the link with a higher rank quantum algebra. In the last part of the paper we will follow this algebraic approach to write (in terms of elements of $\mathcal{U}_q(\mathfrak{su}(2))$) the symmetries of the generator yielding the $q-$polynomial dualities obtained via the scalar-product method.

1.1 Organization of the paper

The rest of the paper is organized as follows. In Section 2 we introduce the two asymmetric models of interest and their corresponding reversible measures. The dynamics takes place on a finite lattice and it is fully described by their infinitesimal generators. In particular, see Section 2.5 for a unified and comprehensive notation. In Section 3 we recall the concept of duality for Markov processes and then exhibit the main results of this work via two theorems, Theorem 3.2 for the asymmetric exclusion and Theorem 3.4 for the asymmetric inclusion. Here the families of $q-$orthogonal polynomials, that are self-duality functions for our processes, are displayed. Besides this, we also single out those symmetries which are uniquely associated to our $q-$orthogonal polynomials. In Section 4 we show that having a duality relation satisfying also an orthogonal relation considerably simplifies the computation of quantities of interest, such as the q^{-2} exponential moments of the current and their space time correlations. The rest of the paper is devoted to the proof of our main results. In Section 5 we show how to obtain functions which are biorthogonal and self-dual from construction. This is done using our general Theorem 5.1, which invokes the scalar product of classical self-duality functions. Once a biorthogonal relation is proved we show, in Section 6 for exclusion and in Section 7 for inclusion, that we can easily establish an orthogonality relation by an explicit computation of the (bi)orthogonal self-duality function. In Section 8 we explain how we find the unique symmetries which can be used to construct our $q-$orthogonal self-duality function starting from the trivial ones. This is based on the algebraic approach used in [10]-[11] and so Sections 8.1 and 8.2 are inspired by those papers in which the Markov generator is linked to the Casimir element of the algebra. In Section 8.3 we identify the symmetries which generate our $q-$orthogonal self-duality functions. Finally, in order to make some computations more readable, we created an Appendix, Section 9, where we give definitions and well-known identities regarding $q-$numbers and $q-$hypergeometric functions.

2 The models

In this paper we will study models of interacting particles moving on a finite lattice $\Lambda_L = \{1, \dots, L\}$, $L \in \mathbb{N}$, $L \geq 2$, with closed boundary conditions and an asymmetric interaction. We denote by $x = \{x_i\}_{i \in \Lambda_L}$ (or $n = \{n_i\}_{i \in \Lambda_L}$) a particle configuration where x_i (resp n_i) is the number of particles at site $i \in \Lambda_L$. We call $\Omega_L = S^L$ the state space, where $S \subseteq \mathbb{N}$ is the set where the occupancy numbers x_i take values. For $x \in \Omega_L$ and $i, \ell \in \Lambda_L$ such that $x_i > 0$, we denote by $x^{i,\ell}$ the configuration obtained from x by removing one particle from site i and putting it at site ℓ.

In this paper we will consider, in particular, two different processes: the ASEP(q, θ) Asymmetric Exclusion Process and the ASIP(q, θ) Asymmetric Inclusion Process. These processes share some algebraic properties even though they have a very different behavior. In order to define the processes and their main properties we need to introduce some notations.

2.1 Notation

The $q-$numbers

Throughout the paper we fix $q \in (0, 1)$ and, for $a \in \mathbb{R}$, define the $q-$numbers as follows:

$$\{a\}_q := \frac{1 - q^a}{1 - q} \qquad \text{and} \qquad \{a\}_{q^{-1}} := \frac{1 - q^{-a}}{1 - q^{-1}}. \tag{2.1}$$

Moreover we define

$$[a]_q := \frac{q^a - q^{-a}}{q - q^{-1}} = q^{1-a} \{a\}_{q^2}. \tag{2.2}$$

Notice that, for $q \to 1$, both $[a]_q$ and $\{a\}_{q^{\pm 1}}$ converge to a. Finally we define the $q-$factorial, for $n \in \mathbb{N}$, given by

$$[n]_q! := [n]_q \cdot [n-1]_q \cdot \dots \cdot [1]_q \quad \text{for } n \geq 1, \qquad \text{and} \qquad [0]_q! := 1. \tag{2.3}$$

For $\theta, m \in \mathbb{N}$ we define the $q-$binomial coefficient by

$$\binom{\theta}{m}_q := \frac{[\theta]_q!}{[m]_q! [\theta - m]_q!} \cdot \mathbf{1}_{\theta \geq m}, \tag{2.4}$$

and, for $m \in \mathbb{N}$ and $\theta \in (0, \infty)$,

$$\binom{m+\theta-1}{m}_q := \frac{[\theta+m-1]_q \cdot [\theta+m-2]_q \cdot \ldots \cdot [\theta]_q}{[m]_q!}. \tag{2.5}$$

The $q-$Pochhammer symbol

For $a \in \mathbb{R}$ and $m \in \mathbb{N}$ the $q-$Pochhammer symbol, or $q-$shifted factorial, $(a;q)_m$ is defined by

$$(a;q)_m = (1-a)(1-aq^1)\cdots(1-aq^{m-1}) , \quad \text{for} \quad m \geq 1 \quad \text{and} \quad (a;q)_0 = 1 \tag{2.6}$$

and furthermore,

$$(a;q)_\infty := \prod_{k=0}^{\infty} \left(1-aq^k\right) \ . \tag{2.7}$$

Most of the $q-$Pochhammer symbols we need in this paper depend on q^2 instead of q. To simplify notation we omit the dependence on q, i.e. we write

$$(a)_m := (a;q^2)_m \qquad \text{for } m \in \mathbb{N} \cup \{\infty\}.$$

In light of the above, we can rewrite the $q-$factorial and the $q-$binomial coefficient in terms of the $q-$Pochhammer symbol:

$$[n]_q! = (q^{-1}-q)^{-n} q^{-\frac{1}{2}n(n+1)} (q^2)_n, \tag{2.8}$$

so that

$$\binom{\theta}{m}_q = q^{-m(\theta-m)} \frac{(q^2)_\theta}{(q^2)_m (q^2)_{\theta-m}} \cdot \mathbf{1}_{\theta \geq m} = (-1)^m q^{m(\theta+1)} \frac{(q^{-2\theta})_m}{(q^2)_m}. \tag{2.9}$$

Similarly,

$$\binom{m+\theta-1}{m}_q = q^{-m(\theta-1)} \frac{(q^{2\theta})_m}{(q^2)_m}. \tag{2.10}$$

2.2 Particle-mass functions.

For $x \in \Omega_L$, $i \in \Lambda_L$, we introduce the functions $N_i^{\pm}(x)$ denoting the number of particles in the configuration x at the right, respectively left, of the site i:

$$N_i^+(x) := \sum_{m=i}^{L} x_m \qquad \text{and} \qquad N_i^-(x) := \sum_{m=1}^{i} x_m,$$

with the convention that $N_{L+1}^+(x) = N_0^-(x) = 0$. Moreover we denote by $N(x)$ the total number of particles in the configuration x:

$$N(x) := \sum_{m=1}^{L} x_m \ .$$

Notice that $N_L^-(x) = N_1^+(x) = N(x)$ and that these mass functions satisfy the following change of summation formula:

$$\sum_{i=1}^{L} x_i N_{i-1}^-(n) = \sum_{i=1}^{L} n_i N_{i+1}^+(x), \tag{2.11}$$

moreover the following identity holds true and will be used throughout the paper:

$$\sum_{i=1}^{L} x_i [2N_{i-1}^-(n) + n_i] + \sum_{i=1}^{L} n_i [2N_{i-1}^-(x) + x_i] = 2N(x) \cdot N(n). \tag{2.12}$$

2.3 The ASEP(q, θ)

In the generalized Asymmetric Exclusion Process particles jump with a repulsive interaction and each site can host at most θ particles, where θ is now a parameter taking values in $\mathbb{N}$. Hence, in this case $S = \{0, 1, \dots, \theta\}$ and $\Omega_L = S^L$. In the usual asymmetric simple exclusion process each site can either be empty or host one particle, while here each site can accommodate up to θ particles. Hence, by setting θ equal to 1 we recover the hard-core exclusion. The infinitesimal generator is presented in the following definition.

Definition 2.1 (ASEP(q, θ))**.** *The* ASEP(q, θ) *with closed boundary conditions is defined as the Markov process on* Ω_L *with generator* $\mathcal{L}^{ASEP} = \mathcal{L}^{ASEP}_{(L)}$ *defined on functions* $f : \Omega_L \to \mathbb{R}$ *by*

$$[\mathcal{L}^{\text{ASEP}} f](x) := \sum_{i=1}^{L-1} [\mathcal{L}^{\text{ASEP}}_{i,i+1} f](x) \qquad \textit{with}$$

$$\begin{aligned} [\mathcal{L}^{\text{ASEP}}_{i,i+1} f](x) \quad := \quad & q^{-2\theta+1} \{x_i\}_{q^2} \{\theta - x_{i+1}\}_{q^2} [f(x^{i,i+1}) - f(x)] \\ + \quad & q^{2\theta-1} \{x_{i+1}\}_{q^{-2}} \{\theta - x_i\}_{q^{-2}} [f(x^{i+1,i}) - f(x)]. \end{aligned}$$

Reversible signed measures

From Theorem 3.1 of [10] we know that the ASEP(q, θ) on Λ_L with closed boundary conditions admits a family, labeled by $\alpha \in \mathbb{R} \setminus \{0\}$, of reversible product, non-homogeneous signed measures μ_α^{ASEP} given by

$$\mu_\alpha^{\text{ASEP}}(x) = \prod_{i=1}^{L} \alpha^{x_i} \binom{\theta}{x_i}_q \cdot q^{-2\theta i x_i} \tag{2.13}$$

for $i \in \Lambda_L$. For positive values of α, (2.13) can be interpreted, after renormalization, as a probability measure. Here the normalizing constant is $\prod_{i=1}^{L} Z_{i,\alpha}^{\text{ASEP}}$, with

$$Z_{i,\alpha}^{\text{ASEP}} = \sum_{m=0}^{\theta} \binom{\theta}{m}_q \cdot \alpha^m q^{-2\theta i m} = (-\alpha q^{1-\theta(2i+1)})_\theta, \qquad \text{for} \quad \alpha \in (0, \infty)$$

where the identity follows from the $q-$binomial Theorem (9.2). In order to make sense of the constant α labelling the measure, one may e.g. compute the q-exponential moment w.r. to the normalized measure $\bar{\mu}_\alpha := \mu_\alpha / Z_\alpha$:

$$\mathbb{E}_{\bar{\mu}_\alpha}\left[q^{2x_i}\right] = \frac{(-\alpha q^2 q^{1-\theta(2i+1)})_\theta}{(-\alpha q^{1-\theta(2i+1)})_\theta} = \frac{1 + \alpha q^{1-2\theta i+\theta}}{1 + \alpha q^{1-2\theta i-\theta}} \tag{2.14}$$

where we used the identity (1.8.11) in [32].

2.4 The ASIP(q, θ)

The Asymmetric Inclusion Process is a model in which particles jump with an attractive interaction. The parameter $\theta > 0$ tunes the intensity of the interaction, the higher the attractiveness the smaller the θ. Each site of the lattice Λ_L can host an arbitrary number of particles, thus, in this case we have $S = \mathbb{N}$ and then $\Omega_L = \mathbb{N}^L$. We introduce the process by giving its generator.

Definition 2.2 (ASIP(q, θ))**.** *The* ASIP(q, θ) *with closed boundary conditions is defined as the Markov process on* Ω_L *with generator* $\mathcal{L}^{ASIP} = \mathcal{L}^{ASIP}_{(L)}$ *defined on functions* $f : \Omega_L \to \mathbb{R}$ *by*

$$[\mathcal{L}^{\text{ASIP}} f](x) := \sum_{i=1}^{L-1} [\mathcal{L}^{\text{ASIP}}_{i,i+1} f](x) \qquad \textit{with}$$

$$[\mathcal{L}^{\text{ASIP}}_{i,i+1}f](x) := q^{2\theta-1}\{x_i\}_{q^2}\{\theta+x_{i+1}\}_{q^{-2}}[f(x^{i,i+1})-f(x)] + q^{-2\theta+1}\{x_{i+1}\}_{q^{-2}}\{\theta+x_i\}_{q^2}[f(x^{i+1,i})-f(x)]. \quad (2.15)$$

Since in finite volume we always start with finitely many particles, and the total particle number is conserved, the process is automatically well defined as a finite state space continuous time Markov chain.

Reversible signed measures

It is proved in Theorem 2.1 of [11] that the ASIP(q,θ) on Λ_L with closed boundary conditions admits a family labeled by $\alpha \in \mathbb{R}\setminus\{0\}$ of reversible product non-homogeneous signed measures μ^{ASIP}_α given by

$$\mu^{\text{ASIP}}_\alpha(x) = \prod_{i=1}^L \alpha^{x_i}\binom{x_i+\theta-1}{x_i}_q \cdot q^{2\theta i x_i}, \quad (2.16)$$

for $x \in \mathbb{N}^L$. Restricting to positive values of the parameter α, this can be turned to a probability measure after renormalization that is possible only under the further restriction $\alpha < q^{-(\theta+1)}$. In order to normalize it we should divide by the constant $\prod_{i=1}^L Z^{\text{ASIP}}_{i,\alpha}$, with

$$Z^{\text{ASIP}}_{i,\alpha} = \sum_{m=0}^{+\infty}\binom{m+\theta-1}{m}_q \cdot \alpha^m q^{2\theta i m} = \frac{(\alpha q^{2\theta i+\theta+1})_\infty}{(\alpha q^{2\theta i-\theta+1})_\infty}, \qquad \text{for} \quad \alpha\in(0,q^{-(\theta+1)}) \quad (2.17)$$

where the latter identity follows from the $q-$binomial Theorem, [20, (II.3)]. However, to keep notation light we work with the non-normalized measure. Also in this case one can easily compute the q-exponential moment w.r. to $\bar\mu_\alpha := \mu_\alpha/Z_\alpha$:

$$\mathbb{E}_{\bar\mu_\alpha}\left[q^{2x_i}\right] = \frac{(\alpha q^2 q^{2\theta i+\theta+1})_\infty}{(\alpha q^{2\theta i+\theta+1})_\infty}\cdot\frac{(\alpha q^{2\theta i-\theta+1})_\infty}{(\alpha q^2 q^{2\theta i-\theta+1})_\infty} = \frac{1-\alpha q^{1+2\theta i-\theta}}{1-\alpha q^{1+2\theta i+\theta}} \quad (2.18)$$

where, for the second identity we used eq.(1.8.8) in [32].

2.5 General case

In order to simplify the notation it is convenient to introduce a parameter σ taking values in $\{-1,+1\}$, distinguishing between the two cases: $\sigma=+1$ corresponding to the inclusion process and $\sigma=-1$ corresponding to the exclusion process. In what follows, if needed, we will omit the superscripts ASIP or ASEP and simply denote by $\mathcal{L}$ the generator of one of the processes, meaning

$$\mathcal{L}_\sigma = \begin{cases} \mathcal{L}^{\text{ASEP}(q,\,\theta)} & \text{for } \sigma=-1 \\ \mathcal{L}^{\text{ASIP}(q,\,\theta)} & \text{for } \sigma=+1. \end{cases} \quad (2.19)$$

where the parameter θ takes values in $\mathbb{N}$ for $\sigma=-1$ and in $(0,\infty)$ for $\sigma=1$. Particles occupation numbers take values in

$$S_{\sigma,\theta} := \begin{cases} \{0,1,\dots,\theta\} & \text{for } \sigma=-1, \\ \mathbb{N} & \text{for } \sigma=+1, \end{cases} \quad (2.20)$$

and the state space of the process is $\Omega_L := S^L_{\sigma,\theta}$. We can then write the generator (for the bond $i,i+1\in\Lambda_L$) in the general form:

$$[\mathcal{L}_{i,i+1}f](x) := q^{\sigma(2\theta-1)}\{x_i\}_{q^2}\{\theta+\sigma x_{i+1}\}_{q^{-2\sigma}}[f(x^{i,i+1})-f(x)] + q^{-\sigma(2\theta-1)}\{x_{i+1}\}_{q^{-2}}\{\theta+\sigma x_i\}_{q^{2\sigma}}[f(x^{i+1,i})-f(x)].$$

Then, defining the function

$$\Psi_{q,\sigma}(\theta,m) := (\sigma q)^m \; q^{-\sigma\theta m} \; \frac{(q^{2\sigma\theta})_m}{(q^2)_m} = \begin{cases} \binom{\theta}{m}_q & \text{for } \sigma=-1 \\ \\ \binom{m+\theta-1}{m}_q & \text{for } \sigma=+1 \end{cases} \tag{2.21}$$

(see (2.9)-(2.10)) the reversible signed measure (2.13)-(2.16) can be rewritten in a unique expression as follows

$$\mu_{\alpha,\sigma}(x) = \prod_{i=1}^{L} \Psi_\sigma(\theta, x_i)\cdot \alpha^{x_i}\, q^{2\sigma\theta i x_i}, \qquad \text{for} \quad \alpha\in\mathbb{R}\setminus\{0\} \tag{2.22}$$

for $x\in\Omega_L = S_{\sigma,\theta}^L$.

We define a modified version $\omega_{\alpha,\sigma}$ of (2.22) that will appear in the statement of the main results in Section 3. This new signed measure differs from (2.22) only through multiplication by a function of the total number of particles $N(x)$:

$$\omega_{\alpha,\sigma}(x) = \frac{\mu_{\alpha,\sigma}(x)}{\mathcal{Z}_{\alpha,\sigma}} \cdot q^{N(x)(N(x)-1)} \cdot (-\sigma\alpha q^{1+2N(x)+\sigma\theta(2L+1)})_\infty \tag{2.23}$$

where $\mathcal{Z}_{\alpha,\sigma}$ is a constant. We remark that, as the processes conserve the total number of particles, detailed balance condition is preserved under this operation, then $\omega_{\alpha,\sigma}$ is again a reversible signed measure for the processes. In order to interpret it as a probability measure we have to restrict to the case $\alpha > 0$. This condition is sufficient for the case $\sigma = +1$, while, for $\sigma = -1$ we have to impose the further condition $\alpha < q^{-1+(2L+1)\theta}$ in order to assure the positivity of the infinite $q-$shifted factorials. Under these conditions and choosing

$$\mathcal{Z}_{\alpha,\sigma} := \sum_x \mu_{\alpha,\sigma}(x)\cdot q^{N(x)(N(x)-1)} \cdot (-\sigma\alpha q^{1+2N(x)+\sigma\theta(2L+1)})_\infty. \tag{2.24}$$

$\omega_{\alpha,\sigma}$ is a reversible probability measure for the corresponding process:

$$\begin{aligned} \omega^{\text{ASEP}(q,\,\theta)} &:= \omega_{\alpha,-1}, \quad \text{for } \alpha\in(0, q^{-1+(2L+1)\theta}) \\ \omega^{\text{ASIP}(q,\,\theta)} &:= \omega_{\alpha,+1}, \quad \text{for } \alpha\in(0, q^{-1-\theta}). \end{aligned} \tag{2.25}$$

Finally we define the function:

$$g_{\alpha,\sigma}(x) := \frac{q^{2N(x)(N(x)-1)}}{\mathcal{Z}^2_{\alpha,\sigma}} \cdot (-\sigma\alpha q^{1+2N(x)+\sigma\theta(2L+1)})_\infty \cdot (-\sigma\alpha q^{1-2N(x)+\sigma\theta})_\infty \tag{2.26}$$

that will also appear in the statement of the main results.

3 Main results

The main result of this paper is the proof of self-duality properties for the processes introduced in the previous section via $q-$hypergeometric orthogonal polynomials. For each process we show the existence of a self-duality function, D and another one, $\widetilde{D}$, that is the same modulo multiplication by a function of the total number of particles and the size of the lattice. Such duality functions can be written in terms of the $q-$Krawtchouk polynomials (respectively $q-$Meixner polynomials) for the ASEP(q,θ) (respectively for the ASIP(q,θ)). D and $\widetilde{D}$ satisfy a biorthogonality relation if one considers the scalar product with respect to the (one site) reversible measures. However, the biorthogonal relation can easily be stated as an orthogonal relation by performing the change of measure of equation (2.23) and the consequently change of norm in equation (2.23). We start by recalling below the definition of duality.

Definition 3.1. *Let $\{X_t\}_{t\geq 0}$, $\{\widehat{X}_t\}_{t\geq 0}$ be two Markov processes with state spaces Ω and $\widehat{\Omega}$ and $D:\Omega\times\widehat{\Omega}\to\mathbb{R}$ a measurable function. The processes $\{X_t\}_{t\geq 0}$, $\{\widehat{X}_t\}_{t\geq 0}$ are said to be **dual** with respect to D if*

$$\mathbb{E}_x\big[D(X_t,\widehat{x})\big]=\widehat{\mathbb{E}}_{\widehat{x}}\big[D(x,\widehat{X}_t)\big] \tag{3.1}$$

*for all $x\in\Omega,\ \widehat{x}\in\hat{\Omega}$ and $t>0$. Here $\mathbb{E}_x$ denotes the expectation with respect to the law of the process $\{X_t\}_{t\geq 0}$ started at x, while $\widehat{\mathbb{E}}_{\widehat{x}}$ denotes expectation with respect to the law of the process $\{\widehat{X}_t\}_{t\geq 0}$ initialized at $\widehat{x}$. If $\{\widehat{X}_t\}_{t\geq 0}$ is a copy of $\{X_t\}_{t\geq 0}$, we say that the process $\{X_t\}_{t\geq 0}$ is **self-dual**.*

Orthogonal polynomial dualities for ASEP(q,θ)

In this section we display the orthogonal duality function for ASEP(q,θ), namely the $q-$Krawtchouk polynomials, for which we will use the following notation

$$K_n(q^{-x};p,c;q) := {}_2\varphi_1\left(\begin{matrix} q^{-x},q^{-n} \\ q^{-c}\end{matrix};q,pq^{n+1}\right), \tag{3.2}$$

where ${}_2\phi_1$ is the $q-$hypergeometric function, $c\in\mathbb{N}$ and $n,x\in\{0,\dots,c\}$, see Section 9.4 of the Appendix for the orthogonality relations. The following theorem states that nested products of $q-$Krawtchouk polynomials form a family of self-duality functions for ASEP(q,θ).

Theorem 3.2. *The ASEP(q,θ) on Λ_L is self-dual with self-duality functions:*

$$D_\alpha^{\text{ASEP}(q,\,\theta)}(x,n) := \prod_{i=1}^{L} K_{n_i}(q^{-2x_i};p_{i,\alpha}(x,n),\theta;q^2), \qquad \alpha\in(0,q^{-1+(2L+1)\theta}),$$
$$p_{i,\alpha}(x,n) := \frac{1}{\alpha}\, q^{-2(N_{i-1}^-(x)-N_{i+1}^+(n))+\theta(2i-1)-1} \tag{3.3}$$

satisfying the following orthogonality relation

$$\langle D_\alpha^{\text{ASEP}(q,\,\theta)}(\cdot,x), D_\alpha^{\text{ASEP}(q,\,\theta)}(\cdot,n)\rangle_{\omega_\alpha^{\text{ASEP}(q,\,\theta)}} = \frac{\delta_{x,n}}{\omega_\alpha^{\text{ASEP}(q,\,\theta)}(x)}\cdot g_{\alpha,-1}(x) \tag{3.4}$$

with $\omega_\alpha^{\text{ASEP}(q,\,\theta)}$ and $g_{\alpha,-1}$ defined in (2.25)-(2.26).

Remark 3.3. For $L=1$ this gives the orthogonality relations for $q-$Krawtchouk polynomials as stated in Section 9.4, so we have obtained a family of multivariate orthogonal polynomials generalizing the $q-$Krawtchouk polynomials. Note that the restriction $\alpha\in(0,q^{-1+(2L+1)\theta})$ has been imposed in order to have a scalar product (3.4) w.r. to a (positive) reversible measure, that can be eventually turned in a probability measure, after renormalization. Note also that this is the condition required in order to have the conditions (9.10) satisfied, indeed, for $\alpha\in(0,q^{-1+(2L+1)\theta})$,

$$\theta\in\mathbb{N} \quad\text{and}\quad q^{2\theta}\cdot p_{i,\alpha}(x,n)>1 \quad\text{for all}\quad x,n\in\Omega_L,\ i\in\Lambda_L\,. \tag{3.5}$$

If we neglect this condition Theorem 3.2 holds still true with the only difference that we can not guarantee the positivity of ω_α.

Orthogonal polynomial dualities for ASIP(q,θ)

In the same spirit of the previous section we now introduce the orthogonal duality relation for ASIP(q,θ). In this case we have that the self-duality functions are a nested product of $q-$Meixner polynomials

$$M_n(q^{-x};b,c;q) := {}_2\varphi_1\left(\begin{matrix} q^{-x},q^{-n} \\ bq\end{matrix};q,-\frac{q^{n+1}}{c}\right), \qquad \text{for}\quad x,n\in\mathbb{N}, \tag{3.6}$$

see Section 9.4 in the Appendix for more details and orthogonality relations. The following theorem is the analogue of the previous one; it says that a family of nested $q-$Meixner polynomials are self-duality functions for ASIP(q,θ).

Theorem 3.4. *The* ASIP(q,θ) *on* Λ_L *is self-dual with self-duality functions*

$$\begin{aligned} D_\alpha^{\mathrm{ASIP}(q,\,\theta)}(x,n) &:= \prod_{i=1}^{L} M_{n_i}(q^{-2x_i}; q^{2(\theta-1)}, c_{i,\alpha}(x,n); q^2), \qquad \alpha>0 \\ c_{i,\alpha}(x,n) &:= \alpha q^{2(N_{i-1}^-(x)-N_{i+1}^+(n))+\theta(2i-1)+1}, \end{aligned} \tag{3.7}$$

for all $\alpha\in(0,q^{-1-\theta})$*, satisfying the following orthogonality relations*

$$\langle D_\alpha^{\mathrm{ASIP}(q,\,\theta)}(\cdot,x), D_\alpha^{\mathrm{ASIP}(q,\,\theta)}(\cdot,n)\rangle_{\omega_\alpha^{\mathrm{ASIP}(q,\,\theta)}} = \frac{\delta_{x,n}}{\omega_\alpha^{\mathrm{ASIP}(q,\,\theta)}(x)}\cdot g_{\alpha,1}(x) \tag{3.8}$$

with $\omega_\alpha^{\mathrm{ASIP}(q,\,\theta)}$ *and* $g_{\alpha,1}$ *defined in* (2.25)-(2.26).

Remark 3.5. For $L=1$ this gives the orthogonality relations for $q-$Meixner polynomials as stated in Section 9.4. The orthogonal polynomials have a similar structure as the multivariate $q-$Meixner polynomials introduced in [22], but they do not seem to be the same functions. Notice that the conditions (9.13) are satisfied, indeed

$$q^{2\theta}\in[0,1) \qquad \text{and} \qquad c_{i,\alpha}(x,n)>0 \qquad \text{for all} \quad x,n\in\Omega_L,\ i\in\Lambda_L \tag{3.9}$$

if $\alpha>0$. As in the case of ASEP, the condition $\alpha>0$ is only needed in order to assure the positivity of the measure ω_α.

Orthogonal self-dualities and symmetries

Whenever the process is reversible it has now been established that there is a one-to-one correspondence between self-duality (in the context of Markov process with countable state space) and symmetries of the Markov generator. The idea is the following: the reversible measure of our processes provides a *trivial* self-duality function (which is the inverse of the reversible measure itself). Then the action of a symmetry of the model on this trivial self-duality gives rise to a non-trivial self-duality function, see [8] (Section 2.3) or [24]. For this reason it is natural to ask which are the symmetries associated to our orthogonal self-dualities. In the context of orthogonal polynomials, we know that the symmetries must preserve the norm of the trivial self-duality function, i.e. the symmetry is unitary. Recall that a unitary operator on the space $L^2(\mu)$ is such that its adjoint corresponds to its inverse. In order to recover the unitary symmetries associated to the orthogonal dualities we first *normalize* the self-duality functions (3.3) and (3.7). At this aim we define

$$\mathcal{D}_{\alpha,\sigma}(x,n) := D_{\alpha,\sigma}(x,n)\cdot q^{\binom{N(x)}{2}-\binom{N(n)}{2}}\cdot\sqrt{\frac{(-\sigma\alpha q^{1+2N(x)+\sigma\theta(2L+1)})_\infty}{(-\sigma\alpha q^{1-2N(n)+\sigma\theta})_\infty}} \tag{3.10}$$

with

$$D_{\alpha,\sigma} = \begin{cases} D_\alpha^{\mathrm{ASEP}(q,\,\theta)} & \text{for } \sigma=-1 \\ D_\alpha^{\mathrm{ASIP}(q,\,\theta)} & \text{for } \sigma=+1 \end{cases} \tag{3.11}$$

Notice that the functions $\mathcal{D}_{\alpha,\sigma}$, with $\sigma=\pm1$ are equal to the old dualities modulo multiplication by a factor that only depends on the total number of particles in both configurations. As a consequence the functions $\mathcal{D}_{\alpha,\sigma}$ are themselves a family of self-duality functions as the dynamics conserves the mass (see e.g. Lemma 3 of [8]). After this renormalization the orthogonality relations read

$$\langle \mathcal{D}_{\alpha,\sigma}(\cdot,x), \mathcal{D}_{\alpha,\sigma}(\cdot,n)\rangle_{\mu_{\alpha,\sigma}} = \frac{\delta_{x,n}}{\mu_{\alpha,\sigma}(x)}\ . \tag{3.12}$$

We can reinterpret now the orthogonal self-duality function $\mathcal{D}_{\alpha,\sigma}$ as the result of the action of a unitary symmetry $\mathcal{S}_{\alpha,\sigma}$ of the generator on the trivial duality function constructed as the inverse of the reversible measure i.e. $\dfrac{\delta_{x,n}}{\mu_{\alpha,\sigma}(x)}$. More precisely, as a consequence of the above, defining

$$\mathcal{S}_{\alpha,\sigma}(x,n) := \mathcal{D}_{\alpha,\sigma}(x,n) \cdot \mu_{\alpha,\sigma}(n) \tag{3.13}$$

we have the following result for $\sigma = -1$.

Proposition 3.6. *For $\alpha > 0$, we have that $\mathcal{S}_{\alpha,-1}$ is a symmetry of the generator $\mathcal{L}_{-1}$ defined in* (2.19)*, i.e. $[\mathcal{S}_{\alpha,-1}, \mathcal{L}_{-1}] = 0$. Moreover it is a unitary operator in $L^2(\mu_\alpha)$ i.e. $\mathcal{S}^*_{\alpha,-1}\mathcal{S}_{\alpha,-1} = \mathcal{S}_{\alpha,-1}\mathcal{S}^*_{\alpha,-1} = I$.*

Remark 3.7. For $\sigma = +1$, both $\mathcal{L}_{+1}$ and $\mathcal{S}_{\alpha,+1}$ are unbounded operators on $L^2(\mu_\alpha)$. If we choose the set of finitely supported functions in $L^2(\mu_\alpha)$ as a dense domain for both operators, they commute on this domain. We do not have unitarity of $\mathcal{S}_{\alpha,+1}$. The relation $\mathcal{S}^*_{\alpha,+1}\mathcal{S}_{\alpha,+1} = I$ holds because this is equivalent to the orthogonality relations for D_α. But the relation $\mathcal{S}_{\alpha,+1}\mathcal{S}^*_{\alpha,+1} = I$ does not automatically follow from this as in the finite dimensional setting. In fact, the latter relation is not valid, which is a consequence of the fact that the q-Meixner polynomials do not form a complete orthogonal set in their weighted L^2-space. In the last part of Appendix 9.4 we address this issue.

In Section 8 we will give an expression of the symmetry $\mathcal{S}_{\alpha,-1}$ in terms of the generators of the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$, in the spirit of [10]-[11]. In order to do this we will pass through the construction of the generator of the processes from a quantum Hamiltonian, that is in turn built from the coproduct of the Casimir operator of $\mathcal{U}_q(\mathfrak{sl}_2)$.

Remark 3.8 (Symmetric case)**.** Performing the limiting relation as $q \to 1$ then the families of hypergeometric $q-$orthogonal polynomials converge to the classical hypergeometric orthogonal polynomials found in [8], which are families of self-duality functions for the corresponding symmetric interacting particle systems. In this limit the duality functions lose their nested-product structure and become ordinary product functions.

Remark 3.9 (Space of self-duality functions)**.** A question that naturally arises regards the space of self-duality for our asymmetric models. In the symmetric setting, it has been established in [37] that, up to constant factors, the only possible product self-duality functions are the trivial, the classical and the orthogonal ones. We conjecture that in the asymmetric case one can make a similar characterization under the assumption of a nested product form. However a rigorous proof could be an interesting subject for a future work.

4 Duality moments and correlations

In this section we show how the duality relation can be used to compute suitable moments and correlations of the process. In this section we will use the generic notation $\{x(t),\, t \ge 0\}$ and $\{n(t),\, t \ge 0\}$ to denote two copies of the process with generator $\mathcal{L}_\sigma$ defined in (2.19) with state space $\Omega_L = S^L_{\sigma,\theta}$. This process corresponds to ASEP(q,θ) for $\sigma = -1$ and to ASIP(q,θ) for $\sigma = 1$. We denote by $\mathbb{P}_x$, resp. $\mathbb{E}_x$, the probability measure, resp. expectation, of one copy of the process conditioned to the initial value $x(0) = x$. The duality relation (3.1) reads as

$$\mathbb{E}_x\left[D_{\alpha,\sigma}(x(t),n)\right] = \mathbb{E}_n\left[D_{\alpha,\sigma}(x,n(t))\right], \tag{4.1}$$

that holds true for the duality function $D_{\alpha,\sigma}$ defined in (3.11)-(3.3)-(3.7). Thinking now the original process $\{x(t),\, t \ge 0\}$ as a process with a high number of particles and the dual one $\{n(t),\, t \geq 0\}$ as a process with a few particles, and calling n-th duality moment at time t the expectation $\mathbb{E}_x\left[D_{\alpha,\sigma}(x(t),n)\right]$, relation (4.1) tells us that it is possible to compute the duality moments of the original process in terms of the dynamics of $\|n\|$ dual particles. The added value of the orthogonality relation lies in the possibility of computing the stationary two-times correlations.

Two-times correlations. For the case $\sigma = -1$, the duality functions $\{D_{\alpha,-1}(\cdot,n),\ n \in \Omega_L\}$ form a basis of $L^2(\omega_{\alpha,-1})$, where $\omega_{\alpha,-1}$ is the reversible probability measure of ASEP(q,θ) defined in (2.25). It follows that any function $f \in L^2(\omega_{\alpha,-1})$ can be expanded in terms of the duality polynomials $D_{\alpha,-1}(\cdot,n)$, $n \in \Omega_L$. The same does not hold true for $\sigma = 1$ since $\{D_{\alpha,1}(\cdot,n),\ n \in \Omega_L\}$ does not form a basis of $L^2(\omega_{\alpha,-1})$. As a consequence, for the latter case only for functions in the span of $\{D_{\alpha,1}(\cdot,n),\ n \in \Omega_L\}$ one can get a similar expansion.

In general we have that, for any fixed

$$\alpha \in (0, \alpha_\sigma^{\max}), \qquad \text{with} \qquad \alpha_\sigma^{\max} = \begin{cases} q^{-1+(2L+1)\theta} & \text{for } \sigma = -1 \\ q^{-1-\theta} & \text{for } \sigma = +1 \end{cases} \tag{4.2}$$

and for any $f \in \text{Span}\{D_{\alpha,\sigma}(\cdot,n),\ n \in \Omega_L\}$, we have

$$f = \sum_{n \in \Omega_L} C_f(n) \cdot D_{\alpha,\sigma}(\cdot,n) \tag{4.3}$$

where, from (3.4)-(3.8), C_f is given by:

$$C_f(n) = \frac{\omega_{\alpha,\sigma}(n)}{g_{\alpha,\sigma}(n)} \cdot \langle f, D_{\alpha,\sigma}(\cdot,n)\rangle_{\omega_{\alpha,\sigma}}. \tag{4.4}$$

with $g_{\alpha,\sigma}$ the function defined in (2.26). This orthogonal expansion substantially simplifies the computation of the two-times correlations as shown in the following theorem.

Theorem 4.1. *Fix $\alpha \in (0, \alpha_\sigma^{max})$ and let $f, h \in$ Span$\{D_{\alpha,\sigma}(\cdot,n),\ n \in \Omega_L\}$, then, for all $0 \leq s \leq t$,*

$$\mathbb{E}_{\omega_{\alpha,\sigma}}\left[h(x(s)) \cdot f(x(t))\right] = \sum_{n,m \in \Omega_L} \frac{\omega_{\alpha,\sigma}(n)}{g_{\alpha,\sigma}(n)} \cdot C_h(m) C_f(n) \cdot \mathbb{P}_n(n(t-s) = m). \tag{4.5}$$

Proof. In this proof we will omit the subscript σ. We have

$$\begin{aligned} &\mathbb{E}_{\omega_\alpha}\left[h(x(s)) \cdot f(x(t))\right] = \\ &= \sum_{n,m \in \Omega_L} \frac{\omega_\alpha(m)}{g_\alpha(m)} \cdot \frac{\omega_\alpha(n)}{g_\alpha(n)} \cdot C_h(m) C_f(n) \cdot \mathbb{E}_{\omega_\alpha}\left[D_\alpha(x(s),m) D_\alpha(x(t),n)\right] \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} &\mathbb{E}_{\omega_\alpha}\left[D_\alpha(x(s),m) D_\alpha(x(t),n)\right] = \sum_{x \in \Omega_L} \omega_\alpha(x) \cdot \mathbb{E}_x\left[D_\alpha(x(s),m) D_\alpha(x(t),n)\right] \\ &= \sum_{x \in \Omega_L} \omega_\alpha(x) \cdot \mathbb{E}_x\left[D_\alpha(x(s),m) \cdot \mathbb{E}_{x(s)}[D_\alpha(x(t-s),n)]\right] \\ &= \sum_{x \in \Omega_L} \omega_\alpha(x) \cdot D_\alpha(x,m) \cdot \mathbb{E}_x[D_\alpha(x(t-s),n)] \\ &= \sum_{x \in \Omega_L} \omega_\alpha(x) \cdot D_\alpha(x,m) \cdot \mathbb{E}_n[D_\alpha(x,n(t-s))] \\ &= \mathbb{E}_n\left[\sum_{x \in \Omega_L} \omega_\alpha(x) \cdot D_\alpha(x,m) \cdot D_\alpha(x,n(t-s))\right] \\ &= \frac{g_\alpha(m)}{\omega_\alpha(m)} \cdot \mathbb{P}_n(n(t-s) = m) \end{aligned}$$

where the second identity follows from the Markov property, the third one from the stationarity of ω_α, the forth one from duality, and the last one from (3.4)-(3.8). Then, using (4.6), we get (4.5). □

A similar result holds true for symmetric system, see for instance Section 3.3 of [1] where an expansion of the type of (4.3) has been used to derive an higher-order version of the Boltzmann-Gibbs principle, for a system of independent random walkers. An analogous identity holds true also for SEP(θ) and SIP(θ). In, general, for this whole class of symmetric models admitting orthogonal polynomial dualities, the symmetric version of Theorem 4.1 allows to compute the two-times correlations of the duality observables (see e.g. equation (16) in [2]). This identity has been a crucial ingredient in the definition and study of the so-called "higher-order" density fields [2, 13] for which a full characterization of the hydrodynamic and fluctuations scaling limits has been achieved thanks to orthogonal dualities.

q^{-2}-exponential moments. In order to apply Theorem 4.1 we need to detect the functions $f \in \text{Span}\{D_{\alpha,\sigma}(\cdot, n),\ n \in \Omega_L\}$ for which the coefficients $C_f(\cdot)$, or, equivalently, the projections $\langle f, D_{\alpha,\sigma}(\cdot,n)\rangle_{\omega_{\alpha,\sigma}}$ can be easily computed. The most natural example of such functions is $f := q^{-2N_i^-(\cdot)}$. Indeed one can easily check by direct computation that, choosing $n = \delta_i$ for some $i \in \Lambda_L$, one has:

$$D_\alpha(x, \delta_i) = 1 - \frac{q^{-2\sigma\theta i+1}}{(q^\theta - q^{-\theta})\,\alpha}\left[q^{-2N_{i-1}^-(x)} - q^{-2N_i^-(x)}\right] \tag{4.7}$$

and, as a consequence,

$$q^{-2N_i^-(x)} = 1 + \sigma\alpha q^{\sigma\theta-1}\left(1 - q^{2\sigma\theta i}\right) + \alpha q^{-1}\left(q^\theta - q^{-\theta}\right)\sum_{\ell=1}^{i} q^{2\sigma\theta\ell}\cdot D_\alpha(x, \delta_\ell) \tag{4.8}$$

from which it follows that

$$C_f(\,\mathbf{0}) = 1 + \sigma\alpha q^{\sigma\theta-1}\left(1 - q^{2\sigma\theta i}\right), \qquad C_f(\delta_\ell) = \alpha q^{-1}\left(q^\theta - q^{-\theta}\right)q^{2\sigma\theta\ell}\cdot \mathbf{1}_{\{\ell\le i\}},$$
$$C_f(n) = 0 \quad \text{for all } n:\ \|n\| > 1,$$

Then, using Theorem 4.1, we obtain the following formula for the space-time correlations of the q^{-2}-exponential moments of $N_i^-(x)$:

$$\begin{aligned}&\mathbb{E}_{\omega_{\alpha,\sigma}}\left[q^{-2N_i^-(x(s))}\cdot q^{-2N_j^-(x(t))}\right] = \\ &= \frac{\omega_{\alpha,\sigma}(\,\mathbf{0})}{g_{\alpha,\sigma}(\,\mathbf{0})}\left(1 + \sigma\alpha q^{\sigma\theta-1}\left(1 - q^{2\sigma\theta i}\right)\right)\left(1 + \sigma\alpha q^{\sigma\theta-1}\left(1 - q^{2\sigma\theta j}\right)\right) \\ &+\alpha^2 q^{-2}\left(q^\theta - q^{-\theta}\right)^2\sum_{\ell=1}^{j}\sum_{\kappa=1}^{i}\frac{\omega_{\alpha,\sigma}(\delta_\ell)}{g_{\alpha,\sigma}(\delta_\ell)}\cdot q^{2\sigma\theta(\ell+\kappa)}\cdot p_{t-s}(\ell,\kappa)\end{aligned}$$

where we use the notation $p_t(\kappa, \ell)$ for the one-dual particle transition probability from site κ to site ℓ at time t.

The interest of the correlations in (4.9) lies in the link between the function $N_i^-(x)$ and the total current at site i as shown in the following definition and proposition (see also section 6.2 of [11] for a more detailed treatment of the subject).

Definition 4.2 (Current)**.** *Let $\{x(s),\ s \ge 0\}$ be a càdlàg trajectory on Ω_L, then the total integrated current $J_i(t)$ in the time interval $[0, t]$ is defined as the net number of particles crossing the bond $(i-1, i)$ in the left direction. Namely, let $(t_k)_{k\in\mathbb{N}}$ be the sequence of the process jump times. Then*

$$J_i(t) = \sum_{k: t_k \in [0,t]} \left(\mathbf{1}_{\{x(t_k)=x(t_k^-)^{i,i-1}\}} - \mathbf{1}_{\{x(t_k)=x(t_k^-)^{i-1,i}\}} \right), \qquad i \in \Lambda_L \tag{4.9}$$

Lemma 4.3. *The total integrated current of a càdlàg trajectory* $\{x(s),\ 0 \le s \le t\}$ *with* $x(0) = x$ *is given by*

$$J_i(t) = N_i^-(x(t)) - N_i^-(x), \qquad i \in \Lambda_L. \tag{4.10}$$

Proof. (4.10) immediately follows from the definition of $J_i(t)$. □

As a consequence of (4.10) we have that the duality relation gives information about the q^{-2}-exponential moments and correlations of the currents. The convenient use of duality for the computation of q-exponential moments of the current has already emerged in [11]. Here the authors pointed out the link between these moments and the triangular self-duality function for the case of ASIP(q,θ). Thanks to this link an explicit formula was found for the expectation of the observable $q^{2J_i(t)}$ when the process is initialized from a deterministic configuration x. The added value of the orthogonal polynomial duality functions lies in the possibility to compute the two-times correlations of the type (4.9) by a relatively simple computation.

The form of q- Krawtchouk and q-Meixner polynomials suggests that the duality relation (4.1) is amenable to provide informations about all the q^{-2}-exponential moments of the variables $N_i^-(x)$, i.e.

$$\mathbb{E}_x \left[\prod_{k=1}^{K} q^{-2m_k N_{i_k}^-(x(t))} \right], \qquad K \in \mathbb{N}, \quad m_k \in \mathbb{N}, \quad 1 \le i_1 < \cdots < i_K \le L \tag{4.11}$$

and we expect that formulas of the type of (4.9) for the stationary space-time correlations can be obtained for any polynomials in the variable $q^{-2N_i^-(\cdot)}$, $i \in \Lambda_L$ by direct computation of the scalar product in (4.4). Computation of moments of this type will be object of future investigation.

5 Construction of the orthogonal dualities

From the analysis developed in [11] and [10] the processes ASIP(q,θ) and ASEP(q,θ) are known to be self-dual with respect to self-duality functions that have a nested-product structure and a *triangular* form, with triangular meaning that they have support contained in the set of couples $(x,n) \in \Omega_L^2$ such that $n_i \le x_i$ for all $i \in \Lambda_L$. In this section we start from these triangular duality-functions to construct new duality functions satisfying suitable orthogonality relations.

5.1 Triangular dualities

The functions

$$D_\lambda^{\mathrm{tr}}(x,n) = \prod_{i=1}^{L} \frac{\binom{x_i}{n_i}_q}{\Psi_{q,\sigma}(\theta,n_i)} \cdot q^{x_i\left(2N_{i-1}^-(n)+n_i\right)-2\sigma\theta i n_i}\, \lambda^{n_i} \tag{5.1}$$

and

$$\widehat{D}_\lambda^{\mathrm{tr}}(x,n) = \prod_{i=1}^{L} \frac{\binom{x_i}{n_i}_q}{\Psi_{q,\sigma}(\theta,n_i)} \cdot q^{-n_i\left(2N_{i-1}^-(x)+x_i\right)-2\sigma\theta i n_i}\, \lambda^{n_i} \tag{5.2}$$

with Ψ the q-binomial coefficient given in (2.21), are self-duality functions for the ASEP(q,θ) (for $\sigma=-1$), resp. for ASIP(q,θ) (for $\sigma=+1$). For the proof of the duality relation we refer to [11, Theorem 5.1] for the case $\sigma=+1$ and to [10, Theorem 3.2] for $\sigma=-1$. We notice moreover that these two functions are the same function modulo a multiplicative quantity that only depends on the total number of particles $N(x)$ and $N(n)$. More precisely, using (2.12), we have that

$$\widehat{D}_\lambda^{\mathrm{tr}}(x,n) = q^{-2N(n)N(x)} \cdot D_\lambda^{\mathrm{tr}}(x,n). \tag{5.3}$$

5.2 From triangular to orthogonal dualities

The following theorem, which is a slight generalization of [8, Proposition 4.5], will be the key ingredient needed to produce biorthogonal duality functions from the triangular ones.

Theorem 5.1 (Biorthogonal self-duality functions via scalar product)**.** *Let X be a Markov process on a countable state space Ω, with generator L. Let μ_1 and μ_2 be two reversible measures for X, and d_1, d_2, $\tilde{d}_1$ and $\tilde{d}_2$ be four self-duality functions for X. Suppose that*

$$\langle d_1(x,\cdot), d_2(\cdot,n)\rangle_{\mu_1} = \frac{\delta_{x,n}}{\mu_2(n)} \qquad \text{and} \qquad \langle \tilde{d}_2(x,\cdot), \tilde{d}_1(\cdot,n)\rangle_{\mu_2} = \frac{\delta_{x,n}}{\mu_1(n)} \tag{5.4}$$

for $x,n \in \Omega$. Here $\langle \cdot\,,\,\cdot\rangle_{\mu_i}$ denotes the scalar product corresponding to the measure μ_i. Then the functions $D, \widetilde{D} : \Omega\times\Omega \to \mathbb{R}$ given by

$$D(x,n) := \langle \tilde{d}_1(x,\cdot), d_1(n,\cdot)\rangle_{\mu_1} \qquad \widetilde{D}(x,n) := \langle \tilde{d}_2(\cdot,x), d_2(\cdot,n)\rangle_{\mu_1} \tag{5.5}$$

are self-duality functions for X. Moreover, they satisfy the biorthogonality relations

$$\langle D(\cdot,m), \widetilde{D}(\cdot,n)\rangle_{\mu_2} = \frac{\delta_{m,n}}{\mu_2(n)}, \qquad m,n\in\Omega. \tag{5.6}$$

In particular, if $\widetilde{D} = c_1(x)c_2(n)D$, where c_1 (resp. c_2) is a positive function of the total number of particles (resp. dual particles), then equation (5.6) becomes an orthogonality relation for D with respect to the weight $c_1(x)\mu_2(x)$ and with squared norm $\frac{1}{c_2(n)\mu_2(n)}$.

Proof. Since scalar products of self-duality functions are self-duality function by [8, Proposition 4.1], we have that both D and $\widetilde{D}$ are self-duality functions. For the biorthogonality relation, assuming we can interchange the order of summation, we get

$$\begin{aligned}
\langle D(\cdot,m), \widetilde{D}(\cdot,n)\rangle_{\mu_2} &= \sum_x D(x,m)\widetilde{D}(x,n)\mu_2(x) \\
&= \sum_x \left(\sum_y \tilde{d}_1(x,y)d_1(m,y)\mu_1(y)\right)\left(\sum_z \tilde{d}_2(z,x)d_2(z,n)\mu_1(z)\right)\mu_2(x) \\
&= \sum_{y,z} d_1(m,y)d_2(z,n)\mu_1(y)\mu_1(z)\sum_x \tilde{d}_2(z,x)\tilde{d}_1(x,y)\mu_2(x) \\
&= \sum_{y,z} \mu_1(y)\mu_1(z)d_1(m,y)d_2(z,n)\,\frac{\delta_{y,z}}{\mu_1(y)} \\
&= \sum_y d_1(m,y)d_2(y,n)\mu_1(y) = \frac{\delta_{m,n}}{\mu_2(m)}.
\end{aligned}$$

This proves the result. □

In order to apply this theorem to produce biorthogonal self-duality functions from the triangular ones we need to show that the triangular duality functions (5.1) and (5.2) satisfy the relations (5.4). This property is the content of proposition below.

Let μ_α, $\alpha\in\mathbb{R}\setminus\{0\}$ be the family of reversible signed measures defined in (2.22), then from now onward we will use the notation $\langle\cdot\,,\,\cdot\rangle_\alpha$ for the scalar product with respect to the reversible measure μ_α.

Proposition 5.2. *Let $D^{\rm tr}_\lambda$ and $\widehat{D}^{\rm tr}_\lambda$ the functions defined in* (5.1)-(5.2)*, then, for all $\alpha, \beta \in \mathbb{R}\setminus\{0\}$ we have*

$$\langle D^{\rm tr}_{1/\alpha q}(x,\cdot), \widehat{D}^{\rm tr}_{-q/\beta}(\cdot,n)\rangle_{-\alpha} = \frac{\delta_{x,n}}{\mu_\beta(n)} \ . \tag{5.7}$$

We will prove this result in Section 6.1 only for ASEP(q,θ) as the proof for ASIP(q,θ) is similar.

Proposition 5.2 guarantees that the two conditions in (5.4) are satisfied for the self duality functions

$$d_1 = D^{\rm tr}_{1/\alpha q}, \quad \tilde{d}_1 = \widehat{D}^{\rm tr}_{q/\alpha}, \qquad d_2 = \widehat{D}^{\rm tr}_{-q/\beta}, \quad \tilde{d}_2 = D^{\rm tr}_{-1/\beta q}$$

by taking the scalar product with respect to the measures

$$\mu_1 = \mu_{-\alpha}, \quad \mu_2 = \mu_\beta.$$

then, as a consequence of Theorem 5.1, we can deduce that the functions

$$D_\alpha(x,n) \quad := \langle \widehat{D}^{\rm tr}_{q/\alpha}(x,\cdot), D^{\rm tr}_{1/\alpha q}(n,\cdot)\rangle_{-\alpha}, \tag{5.8}$$

$$\widetilde{D}_{\alpha,\beta}(x,n) \quad := \langle D^{\rm tr}_{-1/\beta q}(\cdot,x), \widehat{D}^{\rm tr}_{-q/\beta}(\cdot,n)\rangle_{-\alpha}, \tag{5.9}$$

are again self-duality functions satisfying the following biorthogonality relation:

$$\langle D_\alpha(\cdot,m), \widetilde{D}_{\alpha,\beta}(\cdot,n)\rangle_{\mu_\beta} = \frac{\delta_{n,m}}{\mu_\beta(n)}. \tag{5.10}$$

Conclusion of the proof for ASEP(q,θ). The next step in the construction of the orthogonal dualities is the computation of the explicit expressions for the self-duality functions D_α and $\widetilde{D}_{\alpha,\beta}$ that have been implicitly defined in (5.8)-(5.9). This is the content of the next proposition where the new duality functions are identified, for the case $\sigma = -1$, in terms of $q-$Krawtchouk polynomials.

Proposition 5.3. *Let $\sigma = -1$, $\alpha, \beta \in \mathbb{R}\setminus\{0\}$, then the functions $D_\alpha(x,n)$ and $\widetilde{D}_{\alpha,\beta}(x,n)$ defined in* (5.8)-(5.9) *are given by*

$$D_\alpha(x,n) := \prod_{i=1}^L K_{n_i}(q^{-2x_i}; p_{i,\alpha}(x,n), \theta; q^2),$$

$$p_{i,\alpha}(x,n) := \frac{1}{\alpha}\ q^{-2(N^-_{i-1}(x) - N^+_{i+1}(n)) + \theta(2i-1) - 1} \tag{5.11}$$

and

$$\widetilde{D}_{\alpha,\beta}(x,n) := \frac{(\alpha q^{1+2N(x)-2\theta L-\theta})_\infty \ \ q^{N(x)(N(x)-1)}}{(\alpha q^{1-2N(n)-\theta})_\infty \ \ q^{N(n)(N(n)-1)}} \left(\frac{\alpha}{\beta}\right)^{N(x+n)} \cdot D_\alpha(x,n)\ . \tag{5.12}$$

Proposition 5.3 will be proved in Section 6.2. The function D_α emerging here is nothing else than the self-duality function $D_\alpha^{\rm ASEP(q,\,\theta)}$ defined in Theorem 3.2. Whereas $\widetilde{D}_{\alpha,\beta}$ is another self-duality function differing from D_α only via multiplication by a factor that depends only on the total number of particles in both configurations, $N(x)$ and $N(n)$. To conclude the proof of Theorem 3.2 it remains to turn the biorthogonality relation (8.22) in an orthogonality relation for D_α. This is possible by including the extra factor in (5.12) in the measure with respect to which we take the scalar product. So, at this point Theorem 3.2 follows from Theorem 5.1, (8.22) and Proposition 5.3 after choosing

$\alpha = \beta$ and switching from the scalar product with respect to μ_α to the scalar product with respect to ω_α (defined in (2.23)).

Conclusion of the proof for ASIP(q, θ). The strategy followed for the case $\sigma = -1$ does not completely work for $\sigma = 1$. In this case Theorem 5.1 can only be partially applied. More precisely we have that the scalar product (5.9) formally defining $\widetilde{D}_{\alpha,\beta}$ does not converge, as it gives rise, now, to an infinite sum. Nevertheless we have that the hypothesis (5.4) are satisfied as Proposition 5.2 holds true also for $\sigma = 1$ and the scalar product (5.8) defining D_α converges. The explicit computation of this scalar product gives rise to the multivariate $q-$Meixner polynomials $D_\alpha^{\text{ASIP}(q,\,\theta)}$ defined in (3.7). This is, due to Theorem 5.1 a self-duality function. It remains to prove, a posteriori, an orthogonality relation that can be guessed exploiting the formal similarities between ASIP and ASEP. The proof of this orthogonality relation will be the object of Section 7.

Before entering the details of our proofs, one may wonder if there is a link from the orthogonal dualities to the triangular ones. In the symmetric case this has been revealed in Remark 4.2 of [37] where the authors show that, after a proper normalization, as $\alpha \to 0$ the orthogonal dualities are precisely the triangular ones. A similar result holds true in the asymmetric context, however, the outcome of the limit is the triangular duality up to a factor that depends on the total number of (dual) particles, namely $q^{N(n)^2}\,\widehat{D}_1^{\text{tr}}(x,n)$ or $q^{-N(x)^2}\,D_1^{\text{tr}}(n,x)$ (depending on which variable we assume bigger). The constant factor converges to 1 as soon as $q \to 1$, see Remark 6.3 in the next Section.

6 Proofs for ASEP(q, θ)

In this section we will prove Theorem 3.2. In the proofs it will be convenient to write the triangular duality functions given in Section 5.1 as nested products of “1-site duality functions”. Let $\lambda, p, r \in \mathbb{R} \setminus \{0\}$. We define for $n, k \in S_{\sigma,\theta}$,

$$d_\lambda(n,k;p,r) := \frac{\binom{n}{k}_q}{\Psi_{q,\sigma}(\theta,k)}\,\lambda^k\,q^{nk}\,p^n\,r^k\,\mathbf{1}_{k\le n},$$
$$\hat{d}_\lambda(n,k;p,r) := \frac{\binom{n}{k}_q}{\Psi_{q,\sigma}(\theta,k)}\,\lambda^k\,q^{-nk}\,p^{-k}\,r^k\,\mathbf{1}_{k\le n}.$$

Then the triangular duality functions are given by

$$D_\lambda^{\text{tr}}(x,n) = \prod_{i=1}^L d_\lambda(x_i,n_i;p_i,r_i), \qquad \widehat{D}_\lambda^{\text{tr}}(x,n) = \prod_{i=1}^L \hat{d}_\lambda(x_i,n_i;\hat{p}_i,r_i), \tag{6.1}$$

where

$$p_i = p_i(n) = q^{2N_{i-1}^-(n)}, \quad \hat{p}_i = \hat{p}_i(x) = q^{2N_{i-1}^-(x)}, \qquad r_i = q^{-2i\sigma\theta}.$$

Note that the *nested* product structure comes only from the parameters p_i and $\hat{p}_i$.

Furthermore, recall that for both processes we have families of reversible measures labelled by α.

6.1 Proof of Proposition 5.2 for ASEP(q, θ)

In order to prove Proposition 5.2 we start by writing the scalar product with free parameters λ_1 for D^{tr} and λ_2 for $\widehat{D}^{\text{tr}}$ and throughout the computation the right choice will become clear. We have

$$\langle D_{\lambda_1}^{\text{tr}}(x,\cdot), \widehat{D}_{\lambda_2}^{\text{tr}}(\cdot,n)\rangle_{-\alpha} = \prod_{i=1}^L \sum_{y_i=n_i}^{x_i} d_{\lambda_1}(x_i,y_i,p_i,r_i)\hat{d}_{\lambda_2}(y_i,n_i;\hat{p}_i,r_i)\mu_{-\alpha}(y_i), \tag{6.2}$$

where both p_i and $\hat{p}_i$ depend on $N^-_{i-1}(y)$, making the display above a nested product of sums. Since the sum over y_i depends on $y_1, \ldots, y_{i-1}$ we first evaluate the sum over y_L, then the sum over y_{L-1}, and so on. Let us denote the sum over y_i by $\Sigma_i(x_i, n_i; y)$, where $y = (y_1, \ldots, y_{i-1})$ (we suppress the dependence on $r_i, \lambda_1, \lambda_2$ and α), then

$$\begin{aligned}
\Sigma_i(x_i, n_i; y) &= \sum_{y_i=n_i}^{x_i} d_{\lambda_1}(x_i, y_i, p_i, r_i)\hat{d}_{\lambda_2}(y_i, n_i; \hat{p}_i, r_i)\mu_{-\alpha}(y_i) \\
&= \sum_{y_i=n_i}^{x_i} \frac{\binom{x_i}{y_i}_q}{\binom{\theta}{y_i}_q} q^{x_i\left[2N^-_{i-1}(y)+y_i\right]+2\theta i y_i}\, \lambda_1^{y_i} \frac{\binom{y_i}{n_i}_q}{\binom{\theta}{n_i}_q} q^{-n_i\left[2N^-_{i-1}(y)+y_i\right]+2\theta i n_i}\, \lambda_2^{n_i} \cdot \\
&\quad \cdot(-\alpha)^{y_i} \binom{\theta}{y_i}_q q^{-2\theta i y_i} \\
&= q^{2\theta i n_i} \lambda_2^{n_i} \sum_{y_i=n_i}^{x_i} \frac{\binom{y_i}{n_i}_q \binom{x_i}{y_i}_q}{\binom{\theta}{n_i}_q} q^{(x_i-n_i)\left[2N^-_{i-1}(y)+y_i\right]}\, (-\alpha\lambda_1)^{y_i} \\
&= \frac{\binom{x_i}{n_i}_q}{\binom{\theta}{n_i}_q} q^{2\theta i n_i} \lambda_2^{n_i} q^{(x_i-n_i)\left[2N^-_{i-1}(y)\right]} \sum_{y_i=n_i}^{x_i} \binom{x_i - n_i}{y_i - n_i}_q q^{(x_i-n_i)y_i}\, (-\alpha\lambda_1)^{y_i},
\end{aligned}$$

where in the last equality we used the $q-$binomial identity (9.1). Performing a change of variables in the summation and setting

$$C_i(x_i, n_i; y) := \frac{\binom{x_i}{n_i}_q}{\binom{\theta}{n_i}_q}\, q^{2\theta i n_i} \lambda_2^{n_i} (-\alpha\lambda_1)^{n_i} q^{(x_i-n_i)(2N^-_{i-1}(y)+n_i)}\ ,$$

we get

$$\Sigma_i(x_i, n_i; y) = C_i(x_i, n_i; y) \sum_{z=0}^{x_i-n_i} \binom{x_i - n_i}{z}_q \cdot (-\alpha\lambda_1)^z q^{(x_i-n_i)z}\ .$$

Then the Newton formula in equation (9.2) yields

$$\Sigma_i(x_i, n_i; y) = C_i(x_i, n_i; y)(\alpha\lambda_1 q)_{x_i-n_i}\ .$$

First let us choose $\lambda_1 = \frac{q^{-1}}{\alpha}$, then the product is non-zero only for $x_i = n_i$,

$$\Sigma_i(x_i, n_i; y) = C_i(n_i, n_i; y)\, \delta_{x_i,n_i}\ ,$$

where it should be remarked that $C_i(n_i, n_i; y)$ is independent of $y = (y_1, \ldots, y_{i-1})$. Next choosing $\lambda_2 = -\frac{q}{\beta}$ we find

$$C_i(n_i, n_i) = \frac{q^{2\theta i n_i}}{\binom{\theta}{n_i}_q} \left(-\frac{\lambda_2}{q}\right)^{n_i} = \frac{1}{\mu_\beta(n_i)}\ .$$

Using this in equation (6.2), we get

$$\langle D^{\mathrm{tr}}_{1/\alpha q}(x, \cdot), \widehat{D}^{\mathrm{tr}}_{-q/\beta}(\cdot, n)\rangle_{-\alpha} = \prod_{i=1}^{L} \frac{\delta_{x_i,n_i}}{\mu_\beta(n_i)} = \frac{\delta_{x,n}}{\mu_\beta(n)}\ ,$$

which concludes the proof of the proposition. □

6.2 Proof of Proposition 5.3

The explicit expressions will follow from calculations involving $q-$binomials coefficients and $q-$hypergeometric functions. We start with the biorthogonality property.

Calculation of D. We fix $x, n \in \Omega_L$, $\alpha > 0$, and we evaluate

$$D_\alpha(x,n) = \langle \widehat{D}^{\mathrm{tr}}_{q/\alpha}(x,\cdot), D^{\mathrm{tr}}_{1/\alpha q}(n,\cdot)\rangle_{-\alpha}.$$

We make use of the product structure (6.1) again. We start with a result for the 1-site duality functions.

Lemma 6.1. *For $p, \hat{p} \in \mathbb{R}\setminus\{0\}$, $m \in \mathbb{Z}$ and $s, t \in \{0,\ldots,\theta\}$,*

$$\sum_{y=0}^{\theta} q^{2my}\, \hat{d}_{q/\alpha}(s,y;\hat{p},r_i) d_{1/\alpha q}(t,y;p,r_i)\mu_{-\alpha}(y) = \\ p^t\, {}_2\varphi_1\left(\begin{matrix} q^{-2s}, q^{-2t} \\ q^{-2\theta}\end{matrix}; q^2, \frac{1}{\alpha\hat{p}} q^{1+2t-\theta+2i\theta+2m}\right).$$

Proof. Using the explicit expressions of the 1-site duality functions we find

$$\begin{aligned}
&\sum_{y=0}^{\theta} q^{2my}\, \hat{d}_{q/\alpha}(s,y;\hat{p},r_i) d_{1/\alpha q}(t,y;p,r_i)\mu_{-\alpha}(y) \\
&= \sum_{y\le s\wedge t} \frac{\binom{s}{y}_q \binom{t}{y}_q}{\binom{\theta}{y}_q\binom{\theta}{y}_q} q^{-sy+4i\theta y+ty+2my} p^t \left(\frac{1}{\alpha\alpha\hat{p}}\right)^y (-\alpha)^y \binom{\theta}{y}_q q^{-2i\theta y} \\
&= p^t \sum_{y\le s\wedge t} \frac{\binom{s}{y}_q \binom{t}{y}_q}{\binom{\theta}{y}_q} \left(-\frac{1}{\alpha\hat{p}}\right)^y q^{y(t-s+2m+2i\theta)} \\
&= p^t \sum_{y\le s\wedge t} \frac{(q^{-2s})_y (q^{-2t})_y}{(q^2)_y (q^{-2\theta})_y} \left(\frac{1}{\alpha\hat{p}} q^{1+2t-\theta+2i\theta+2m}\right)^y ,
\end{aligned}$$

where the last equality is due to the $q-$binomial coefficient identity (2.9). The result then follows from the definition of the ${}_2\varphi_1$-function. ☐

We introduce auxiliary functions: for $i = 1,\ldots,L$,

$$A_i(y_1,\ldots,y_i) = \hat{d}_{q/\alpha}(x_i,y_i;\hat{p}_i(x),r_i) d_{1/\alpha q}(n_i,y_i;p_i(y),r_i)\mu_{-\alpha}(y_i).$$

From Lemma 6.1 with

$$s = x_i, \quad t = n_i, \quad p = p_i(y) = q^{2N^-_{i-1}(y)}, \quad \hat{p} = \hat{p}(x) = q^{2N^-_{i-1}(x)}, \quad m = n_{i+1},$$

we find the following identities.

Lemma 6.2. *Let $i \in \{1,\ldots,L\}$ and $y_1,\ldots,y_{i-1} \in \{0,\ldots,\theta\}$, then*

$$\sum_{y_i} A_i(y_1,\ldots,y_i) q^{2n_{i+1}N^-_i(y)} = \sum_{y_i} S_i(y_i;x,n) q^{2n_{i+1}N^-_i(y)+2n_iN^-_{i-1}(y)},$$

where $n_{L+1} = 0$, and

$$S_i(y_i;x,n) = \frac{(q^{-2x_i})_{y_i}(q^{-2n_i})_{y_i}}{(q^2)_{y_i}(q^{-2\theta})_{y_i}} \left(\frac{1}{\alpha} q^{1-2N^-_{i-1}(x)+2n_i-\theta+2i\theta}\right)^{y_i}.$$

Now we are ready to find an explicit expression for $D(x,n)$. We have

$$D(x,n) = \langle \widehat{D}^{\mathrm{tr}}_{q/\alpha}(x,\cdot), D^{\mathrm{tr}}_{1/\alpha q}(n,\cdot)\rangle_{-\alpha} = \sum_{y_1} A_1(y_1) \sum_{y_2} A_2(y_1,y_2) \dots \sum_{y_L} A_L(y_1,\dots,y_L).$$

From induction, using Lemma 6.2, we obtain

$$D(x,n) = \sum_y q^{2\sum_{i=1}^L n_i N^-_{i-1}(y)} \prod_{i=1}^L S_i(y_i;x,n).$$

We apply identity (2.11), then

$$D(x,n) = \sum_y q^{2\sum_{i=1}^L y_i N^+_{i+1}(n)} \prod_{i=1}^L S_i(y_i;x,n) = \prod_{i=1}^L \sum_{y_i} S_i(y_i;x,n) q^{2y_i N^+_{i+1}(n)}.$$

Finally, using the explicit expression for S and the definition (9.7) of the ${}_2\varphi_1$-function we find

$$D(x,n) = \prod_{i=1}^L \sum_{y_i=0}^{x_i\wedge n_i} \frac{(q^{-2x_i})_{y_i}(q^{-2n_i})_{y_i}}{(q^2)_{y_i}(q^{-2\theta})_{y_i}}\ \alpha^{-y_i} q^{y_i(1+2n_i+2N^+_{i+1}(n)-2N^-_{i-1}(x)-\theta+2i\theta)} = \prod_{i=1}^L {}_2\varphi_1\left(\begin{matrix} q^{-2x_i}, q^{-2n_i} \\ q^{-2\theta} \end{matrix} ; q^2, \frac{1}{\alpha} q^{1+2n_i+2N^+_{i+1}(n)-2N^-_{i-1}(x)-\theta+2i\theta}\right).$$

Comparing this with the definition of the $q-$Krawtchouk polynomials (9.9), we see that $D(x,n)$ is indeed a nested product of $q-$Krawtchouk polynomials.

Remark 6.3 (From orthogonal dualities to triangular dualities)**.** The triangular duality functions can be recoved from the duality function $D(x,n)$ by taking an appropriate limit. Indeed, note that the ${}_2\varphi_1$-function is a polynomial in α^{-1} of degree $x_i \wedge n_i$. Assuming $n_i \le x_i$ it follows that

$$\lim_{\alpha\to 0} (-\alpha)^{n_i}\, {}_2\varphi_1\left(\begin{matrix} q^{-2x_i}, q^{-2n_i} \\ q^{-2\theta} \end{matrix} ; q^2, \frac{1}{\alpha} q^{1+2n_i+2N^+_{i+1}(n)-2N^-_{i-1}(x)-\theta+2i\theta}\right) = \frac{(q^{-2x_i})_{n_i}}{(q^{-2\theta})_{n_i}} q^{-2n_iN^-_{i-1}(x)+n_i(2i-1)\theta+2n_iN^+_{i+1}(n)+n_i^2}.$$

Comparing this with the 1-site duality function $\hat{d}_1(x_i,n_i;\hat{p}_i,r_i)$ defined in the beginning of this section and the definition (6.1) of the triangular duality function, we obtain

$$\lim_{\alpha\to 0} (-\alpha)^{N(n)} D_\alpha(x,n) = \prod_{i=1}^L \hat{d}_1(x_i,n_i,q^{2N^-_{i-1}(x)},q^{2i\theta})\, q^{2n_iN^+_{i+1}(n)+n_i^2} = q^{N(n)^2}\, \widehat{D}^{\mathrm{tr}}_1(x,n), \tag{6.3}$$

assuming $n_i \le x_i$ for $i=1,\dots,n$. Here we used identity (2.12) as well as $\sum_{i=1}^L x_i N^-_{i-1}(x) = \sum_{i=1}^L x_i N^+_{i+1}(x)$.

Similarly, for $x_i \le n_i$ we obtain

$$\lim_{\alpha\to 0} (-\alpha)^{x_i}\, {}_2\varphi_1\left(\begin{matrix} q^{-2x_i}, q^{-2n_i} \\ q^{-2\theta} \end{matrix} ; q^2, \frac{1}{\alpha} q^{1+2n_i+2N^+_{i+1}(n)-2N^-_{i-1}(x)-\theta+2i\theta}\right) = \frac{(q^{-2n_i})_{x_i}}{(q^{-2\theta})_{x_i}} q^{2x_iN^+_{i+1}(n)+x_i(2i-1)\theta+2x_in_i-2x_iN^-_{i-1}(x)-x_i^2}.$$

Comparing this with the 1-site duality function $d_1(n_i, x_i; p_i, r_i)$ and the corresponding triangular duality function it follows that

$$\begin{aligned}\lim_{\alpha\to 0}(-\alpha)^{N(x)} D_\alpha(x,n) &= \prod_{i=1}^{L} d_1(n_i, x_i; q^{2N_{i-1}^-(n)}, q^{2i\theta})\, q^{-2x_i N_{i-1}^-(x) - x_i^2} \\ &= q^{-N(x)^2}\, D_1^{\mathrm{tr}}(n,x),\end{aligned}$$

provided $x_i \le n_i$ for $i = 1, \dots, n$.

Calculation of $\widetilde{D}$. The calculation of $\widetilde{D}$ is similar to the calculation for $D(x,n)$, but a bit more involved. We fix $x, n \in \Lambda_L$, $\alpha > 0$, and we evaluate

$$\widetilde{D}(x,n) = \langle D^{\mathrm{tr}}_{-1/\beta q}(\cdot, x), \widehat{D}^{\mathrm{tr}}_{-q/\beta}(\cdot, n)\rangle_{-\alpha},$$

for some $\beta \in \mathbb{R}$. We start with a result for 1-site duality functions again.

Lemma 6.4. *For $\beta, p, \hat{p} \in \mathbb{R} \setminus \{0\}$, $m \in \mathbb{N}$ and $s, t \in \{0, \dots, \theta\}$,*

$$\begin{aligned}&\sum_{y=0}^{\theta} q^{-2my} d_{-1/\beta q}(y, s; p, r_i)\hat{d}_{-q/\beta}(y, t; \hat{p}, r_i)\mu_{-\alpha}(y) = \\ &\quad \left(\frac{\alpha}{\beta}\right)^{s+t} p^{s+t}\hat{p}^{-t} q^{(s+t)(1+s-t-2m)} q^{-2s} \frac{(\alpha p q^{1+2s-2m-2i\theta-\theta})_\infty}{(\alpha p q^{1-2t-2m-2i\theta+\theta})_\infty} \\ &\quad \cdot {}_2\varphi_1\left(\begin{matrix} q^{-2s}, q^{-2t} \\ q^{-2\theta}\end{matrix}; q^2, \frac{1}{\alpha p} q^{1+2t+2i\theta-\theta+2m}\right).\end{aligned}$$

Proof. Let us denote the sum on the left hand side by Σ. From the explicit expressions of the 1-site duality functions we find

$$\begin{aligned}\Sigma &= \sum_{y=0}^{\theta} q^{-2my} d_{-1/\beta q}(y, s; p, r_i)\hat{d}_{-q/\beta}(y, t; \hat{p}, r_i)\mu_{-\alpha}(y) \\ &= \sum_{y \ge s\vee t} \frac{\binom{y}{s}_q \binom{y}{t}_q}{\binom{\theta}{s}_q\binom{\theta}{t}_q} (-\beta q)^{-s}(-q/\beta)^t q^{y(s-t-2m)} p^y \hat{p}^{-t} q^{2i\theta(s+t)} \binom{\theta}{y}_q (-\alpha)^y q^{-2i\theta y} \\ &= C_1 \sum_{y\ge s\vee t} \binom{y}{s}_q \binom{y}{t}_q \binom{\theta}{y}_q C_2^y,\end{aligned}$$

where $C_2 = -\alpha p q^{s-t-2i\theta-2m}$ and

$$\begin{aligned}C_1 &= \frac{(-\beta q^{-2i\theta})^{-(s+t)} q^{t-s}\hat{p}^{-t}}{\binom{\theta}{s}_q\binom{\theta}{t}_q} \\ &= (\beta q^{-2i\theta+2+\theta})^{-s}(\beta\hat{p} q^{-2i\theta+\theta})^{-t} \frac{(q^2)_s (q^2)_t}{(q^{-2\theta})_s (q^{-2\theta})_t}.\end{aligned}$$

We focus on the sum. Assume $s \le t$ and let C be an arbitrary constant, then we obtain from Lemma 9.2,

$$\begin{aligned}S &:= \sum_{y\ge s\vee t} \binom{y}{s}_q \binom{y}{t}_q \binom{\theta}{y}_q C^y \\ &= \frac{(-C)^t q^{t(1+\theta-s)} q^{s^2} (q^{-2\theta})_t}{(q^2)_s (q^2)_{t-s}}\, {}_2\varphi_1\left(\begin{matrix} q^{2t+2}, q^{2t-2\theta} \\ q^{2+2t-2s}\end{matrix}; q^2, -Cq^{1-s-t+\theta}\right).\end{aligned}$$

Next we transform this ${}_2\varphi_1$-series into another ${}_2\varphi_1$-series using Heine's transformation (9.8), and then we reverse the order of summation, see identity (9.9), to obtain

$$\begin{aligned}
&{}_2\varphi_1\left(\begin{matrix} q^{2t+2}, q^{2t-2\theta} \\ q^{2+2t-2s} \end{matrix}; q^2, -Cq^{1-s-t+\theta}\right) \\
&= \frac{(-Cq^{1+s+t-\theta})_\infty}{(-Cq^{1-s-t+\theta})_\infty}\, {}_2\varphi_1\left(\begin{matrix} q^{-2s}, q^{2-2s+2\theta} \\ q^{2+2t-2s} \end{matrix}; q^2, -Cq^{1+s+t-\theta}\right) \\
&= (Cq^{1+s+t-\theta})^s q^{-s-s^2} \frac{(q^{2+2\theta-2s})_s}{(q^{2+2t-2s})_s} \frac{(-Cq^{1+s+t-\theta})_\infty}{(-Cq^{1-s-t+\theta})_\infty} \\
&\quad \cdot {}_2\varphi_1\left(\begin{matrix} q^{-2s}, q^{-2t} \\ q^{-2\theta} \end{matrix}; q^2, -\frac{1}{C}q^{1+s+t-\theta}\right).
\end{aligned}$$

Using identities (9.5) and (9.6) for the $q-$Pochhammer symbols this gives us

$$S = (-Cq^{\theta+1})^{s+t} \frac{(q^{-2\theta})_s (q^{-2\theta})_t}{(q^2)_s (q^2)_t} \frac{(-Cq^{1+s+t-\theta})_\infty}{(-Cq^{1-s-t+\theta})_\infty}\, {}_2\varphi_1\left(\begin{matrix} q^{-2s}, q^{-2t} \\ q^{-2\theta} \end{matrix}; q^2, -\frac{1}{C}q^{1+s+t-\theta}\right).$$

Note that this expression is symmetric in s and t, so we can drop the condition $s \leq t$. Using this with $C = C_2$ and collecting terms gives

$$\begin{aligned}
\Sigma &= \left(\frac{\alpha}{\beta}\right)^{s+t} p^{s+t} \hat{p}^{-t} q^{(s+t)(1+s-t-2m)} q^{-2s} \frac{(\alpha p q^{1+2s-2m-2i\theta-\theta})_\infty}{(\alpha p q^{1-2t-2m-2i\theta+\theta})_\infty} \\
&\quad \cdot {}_2\varphi_1\left(\begin{matrix} q^{-2s}, q^{-2t} \\ q^{-2\theta} \end{matrix}; q^2, \frac{1}{\alpha p} q^{1+2t+2i\theta-\theta+2m}\right).
\end{aligned}$$

This proves the lemma. □

We introduce auxiliary functions again: for $i = 1, \dots, L$,

$$B_i(y_1, \dots, y_i) = d_{-1/\beta q}(y_i, x_i; p_i(x), r_i)\hat{d}_{-q/\beta}(y_i, n_i; \hat{p}_i(y), r_i)\mu_{-\alpha}(y_i).$$

Then Lemma 6.4 with

$$s = x_i, \quad t = n_i, \quad p = p_i(x) = q^{2N^-_{i-1}(x)}, \quad \hat{p} = \hat{p}_i(y) = q^{2N^-_{i-1}(y)}, \quad m = N^+_{i+1}(n),$$

gives the following identity involving the functions B_i.

Lemma 6.5. *For $i \in \{1, \dots, L\}$ and $y_1, \dots, y_{i-1} \in \mathbb{N}$,*

$$\sum_{y_i} B_i(y_1, \dots, y_i) q^{-2N^+_{i+1}(n)N^-_i(y)} = \sum_{y_i} T_i(y_i; x, n) q^{-2N^+_i(n)N^-_{i-1}(y)},$$

where $T_i(y_i; x, n) = T^{(1)}_i(x, n)T^{(2)}_i(y_i; x, n)$ with

$$\begin{aligned}
T^{(1)}_i(x, n) &= \left(\frac{\alpha}{\beta}\right)^{x_i+n_i} q^{(x_i+n_i)(1-2N^+_{i+1}(n)-n_i+2N^-_{i-1}(x)+x_i)} q^{-2x_i} \\
&\quad \cdot \frac{(\alpha q^{1+N^-_i(x)-2N^-_{i+1}(n)-\theta(2i+1)})_\infty}{(\alpha q^{1+2N^-_{i-1}(x)-2N^+_i(n)-\theta(2i-1)})_\infty}, \\
T^{(2)}_i(y_i; x, n) &= \frac{(q^{-2x_i})_{y_i}(q^{-2n_i})_{y_i}}{(q^2)_{y_i}(q^{-2\theta})_{y_i}} \left(\frac{1}{\alpha} q^{1+2n_i+2N^+_{i+1}(n)-2N^-_{i-1}(x)+2i\theta-\theta}\right)^{y_i}.
\end{aligned}$$

Now we can perform the calculation for $\widetilde{D}$. We write $\widetilde{D}(x, n)$ in terms of the auxiliary functions B_i,

$$\begin{aligned}
\widetilde{D}(x, n) &= \langle D^{\mathrm{tr}}_{-1/\beta q}(\cdot, x), \widehat{D}^{\mathrm{tr}}_{-q/\beta}(\cdot, n)\rangle_{-\alpha} \\
&= \sum_{y_1} B_1(y_1) \sum_{y_2} B_2(y_1, y_2) \dots \sum_{y_L} B_L(y_1, \dots, y_L).
\end{aligned}$$

Then from Lemma 6.5 and induction we find

$$\begin{aligned}\widetilde{D}(x,n) &= \sum_y \prod_{i=1}^L T_i(y_i;x,n) \\ &= \prod_{i=1}^L T_i^{(1)}(x,n) \sum_{y_i} T_i^{(2)}(y_i;x,n).\end{aligned}$$

Note that

$$\sum_{y_i} T_i^{(2)}(y_i;x,n) = {}_2\varphi_1\left(\begin{matrix} q^{-2x_i}, q^{-2n_i} \\ q^{-2\theta} \end{matrix}; q^2, \frac{1}{\alpha} q^{1+2n_i+2N_{i+1}^+(n)-2N_{i-1}^-(x)+2i\theta-\theta}\right),$$

and

$$\prod_{i=1}^L T_i^{(1)}(x,n) = \left(\frac{\alpha}{\beta}\right)^{N(x+n)} \frac{q^{N(x)(N(x)-1)}}{q^{N(n)(N(n)-1)}} \frac{(\alpha q^{1+2N(x)-2\theta L-\theta})_\infty}{(\alpha q^{1-2N(n)-\theta})_\infty},$$

where we used that the product of the ratio of the $q-$shifted factorials telescopes, and identities (2.12) (for $n=x$) and (2.11). So we have

$$\widetilde{D}(x,n) = \left(\frac{\alpha}{\beta}\right)^{N(x+n)} \frac{q^{N(x)(N(x)-1)}}{q^{N(n)(N(n)-1)}} \frac{(\alpha q^{1+2N(x)-2\theta L-\theta})_\infty}{(\alpha q^{1-2N(n)-\theta})_\infty} D(x,n).$$

7 Proof for ASIP(q,θ)

In this section we will prove Theorem 3.4. The proof we used for Theorem 3.2 in the previous section unfortunately does not work for ASIP. The problem lies in the computation of the function $\widetilde{D}$. To be more precise, the analogue of Lemma 6.1 in the ASIP case leads to an infinite sum that, depending on values of s, t and m, will diverge. However, the computation of the function D for ASIP is completely analogous to the computation for ASEP, and this leads to multivariate $q-$Meixner polynomials as self-duality functions. Because of the similarities between ASIP and ASEP we can make an educated guess for the explicit expression of $\widetilde{D}$ in terms of D, and then verify biorthogonality relations directly.

First we need to verify that the function D in Theorem 3.4 is a self-duality function. We can verify in exactly the same way as for ASEP that

$$D_\alpha(x,n) = \langle \widehat{D}^{\mathrm{tr}}_{q/\alpha}(x,\cdot), D^{\mathrm{tr}}_{1/\alpha q}(n,\cdot)\rangle_{-\alpha},$$

so D is indeed a self-duality function by Theorem 5.1. Note that the function $\widetilde{D}$ in Theorem 3.4 is of the form $C_1(x)C_2(n)D(x,n)$, where C_1 and C_2 only depend on the total number of particles $N(x)$ and the total number of dual particles $N(n)$. Since the total number of particles is conserved under the dynamics of ASIP, and D is a self-duality function for ASIP, it follows that $\widetilde{D}$ is also a self-duality function. It only remains to show that D and $\widetilde{D}$ are biorthogonal with respect to the measure μ_β, or equivalently, that functions $D(\,\cdot\,,n)$, $n \in \Lambda_L$, are orthogonal with respect to $C_1\mu_\beta$.

The proof of the orthogonality uses the orthogonality relations (9.14) for the $q-$Meixner polynomials $M_n(q^{-x}) := M_n(q^{-x};b,c;q)$ with $0 < b < q^{-1}$ and $c > 0$. Using identities for $q-$shifted factorials, these relations can be rewritten as follows:

$$\sum_{x=0}^\infty W(x;b,c;q) M_n(q^{-x}) M_{n'}(q^{-x}) = \delta_{n,n'} H(n;b,c;q), \tag{7.1}$$

with

$$W(x;b,c;q) = \frac{(-bcq^{x+1};q)_\infty (bq;q)_x}{(q;q)_x} c^x q^{\frac{1}{2}x(x-1)},$$
$$H(n;b,c;q) = \frac{(-cq^{-n};q)_\infty (q;q)_n}{(bq;q)_n} c^{-n} q^{\frac{1}{2}n(n-1)}.$$

Proposition 7.1. *Let $L \in \mathbb{N}$, $c > 0$ and $b_i \in (0, q^{-1})$, $i = 1, \dots, L$. Define multivariate $q-$Meixner polynomials $m_n(x) = m_n(x; b_1, \dots, b_L, c; q)$ by*

$$m_n(x) = \prod_{i=1}^{L} M_{n_i}(q^{-x_i}; b_i, cB_{i-1}q^{N_{i-1}^-(x) - N_{i+1}^+(n) + i - 1}; q), \qquad x, n \in \mathbb{N}^L,$$

where $B_i = \prod_{l=1}^{i} b_l$ (the empty product being equal to 1). Moreover, define $w(x) = w(x; b_1, \dots, b_L, c; q)$ and $h(n) = h(n; b_1, \dots, b_L, c; q)$ by

$$w(x) = q^{\frac{1}{2}N(x)(N(x)-1)} c^{N(x)} (-cB_L q^{N(x)+L}; q)_\infty \prod_{i=1}^{L} \frac{(b_i q; q)_{x_i}}{(q;q)_{x_i}} (b_i q)^{N_{i+1}^+(x)},$$
$$h(n) = q^{\frac{1}{2}N(n)(N(n)-1)} c^{-N(n)} (-cq^{-N(n)}; q)_\infty \prod_{i=1}^{L} \frac{(q;q)_{n_i}}{(b_i q; q)_{n_i}} (b_i q)^{-N_{i+1}^+(n)},$$

then

$$\sum_{x \in \mathbb{N}^L} w(x) m_n(x) m_{n'}(x) = \delta_{n,n'} h(n).$$

Proof. We use the shorthand notations

$$W_i(x,n) = W(x_i; b_i, cB_{i-1}q^{N_{i-1}^-(x) - N_{i+1}^+(n) + i - 1}; q),$$
$$H_i(x,n) = H(n_i; b_i, cB_{i-1}q^{N_{i-1}^-(x) - N_{i+1}^+(n) + i - 1}; q),$$
$$M_i(x,n) = M_{n_i}(q^{-x_i}; b_i, cB_{i-1}q^{N_{i-1}^-(x) - N_{i+1}^+(n) + i - 1}; q).$$

Note that $M_i(x,n)$ and $W_i(x,n)$ depend only on $x_1, \dots, x_i$ and *not* on $x_{i+1}, \dots, x_L$, and $H_i(x,n)$ depends only on $x_1, \dots x_{i-1}$ and not on $x_i, \dots, x_L$. Furthermore, in this notation we have

$$m_n(x) = \prod_{i=1}^{L} M_i(x,n).$$

We have a similar identity involving w, h, W_i and H_i: using identities for N_i^+, N_i^- and N from Section 2.2 and telescoping products, we obtain

$$\frac{w(x)}{h(n)} = \prod_{i=1}^{L} \frac{W_i(x,n)}{H_i(x,n)}.$$

Then, for $n, n' \in \mathbb{N}^L$,

$$\begin{aligned} \sum_{x \in \mathbb{N}^L} \frac{w(x)}{h(n)} m_n(x) m_{n'}(x) = &\sum_{x_1 \in \mathbb{N}} \frac{W_1(x,n)}{H_1(x,n)} M_1(x,n) M_1(x,n') \\ &\cdot \sum_{x_2 \in \mathbb{N}} \frac{W_2(x,n)}{H_2(x,n)} M_2(x,n) M_2(x,n') \\ &\cdots \sum_{x_L \in \mathbb{N}} \frac{W_L(x,n)}{H_L(x,n)} M_L(x,n) M_L(x,n'). \end{aligned}$$

Using the orthogonality relations (7.1) for $q-$Meixner polynomials, which imply

$$\sum_{x_i \in \mathbb{N}} \frac{W_i(x,n)}{H_i(x,n)} M_i(x,n) M_i(x,n') = \delta_{n_i,n_i'},$$

we obtain

$$\sum_{x \in \mathbb{N}^L} \frac{w(x)}{h(n)} m_n(x) m_{n'}(x) = \delta_{n,n'},$$

which is the desired orthogonality relation. □

The orthogonality relations for the duality functions D and $\widetilde{D}$ follow from the above orthogonality relations for multivariate $q-$Meixner polynomials by replacing q by q^2 and setting

$$c = \alpha q^{\theta+1}, \quad b_i = q^{2\theta-2}, \quad \text{for } i = 1, \dots, L.$$

8 Orthogonal dualities from symmetries

In this section we show the link between the self-duality functions constructed in the previous sections and the existence of symmetries of the generator. To do this we rely on the algebraic approach developed in [11]-[10] for the construction of the generator in terms of the Casimir operator of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$, where a family of finite, respectively infinite, dimensional representations are used for ASEP(q,θ) and ASIP(q,θ), respectively. The final aim will be to give an expression in terms of the generators of the algebra for the symmetry $\mathcal{S}_{\alpha,\sigma}$ connected to the orthogonal duality function $\mathcal{D}_{\alpha,\sigma}$.

8.1 The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$

For $q \in (0,1)$ we consider the complex unital algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ with generators A^+, A^-, A^0 satisfying the commutation relations

$$\begin{aligned} q^{A^0} A^+ &= q\, A^+ q^{A^0}, \\ q^{A^0} A^- &= q^{-1}\, A^- q^{A^0}, \\ [A^+, A^-] &= [2A^0]_q. \end{aligned} \tag{8.1}$$

Here $[A,B] = AB - BA$ is the usual commutator, and

$$[A]_q := \frac{q^A - q^{-A}}{q - q^{-1}}.$$

(compare to the $q-$number defined in (2.2)). In the limit $q \to 1$ the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ reduces to the enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$. The Casimir element C given by

$$C = A^+ A^- + \left[A^0\right]_q \left[A^0 - 1\right]_q \tag{8.2}$$

is in the center of $\mathcal{U}_q(\mathfrak{sl}_2)$, i.e. $[C, A] = 0$ for all $A \in \mathcal{U}_q(\mathfrak{sl}_2)$.

Co-product structure

The co-product for $\mathcal{U}_q(\mathfrak{sl}_2)$ is the map $\Delta : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$ given on the generators by

$$\begin{aligned} \Delta(A^\pm) &= A^\pm \otimes q^{-A^0} + q^{A^0} \otimes A^\pm\ , \\ \Delta(A^0) &= A^0 \otimes 1 + 1 \otimes A^0\ , \end{aligned} \tag{8.3}$$

and it is extended to $\mathcal{U}_q(\mathfrak{sl}_2)$ as an algebra homomorphism. In particular Δ preserves the commutation relations (8.1).

We also need iterated coproducts mapping from $\mathcal{U}_q(\mathfrak{sl}_2)$ to tensor products of copies of $\mathcal{U}_q(\mathfrak{sl}_2)$. We define iteratively $\Delta^n : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes(n+1)}$, i.e. higher powers of Δ, as follows:

$$\Delta^1 := \Delta, \qquad \Delta^n := (\Delta \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-1 \text{ times}})\Delta^{n-1}, \quad n \geq 2.$$

For the generators of $\mathcal{U}_q(\mathfrak{sl}_2)$ this implies, for $n \geq 2$,

$$\begin{aligned} \Delta^n(A^\pm) &= \Delta^{n-1}(A^\pm) \otimes q^{-A^0} + q^{\Delta^{n-1}(A^0)} \otimes A^\pm \ , \\ \Delta^n(A^0) &= \Delta^{n-1}(A^0) \otimes 1 + \underbrace{1 \otimes \ldots \otimes 1}_{n \text{ times}} \otimes A^0 \ . \end{aligned} \tag{8.4}$$

Representations of the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$

From here onward we use the notation $\{|n\rangle \mid n \in \mathbb{K}_\sigma\}$ for the standard orthonormal basis of $\ell^2(\mathbb{K}_\sigma)$ with $\mathbb{K}_\sigma = \{0, 1, \ldots, \theta\}$ if $\sigma = -1$ and $\mathbb{K}_\sigma = \mathbb{N}$ if $\sigma = 1$. Here and in the following, with abuse of notation, we use the same symbol for a linear operator and the matrix associated to it in a given basis.

In order to define Markov process generators from the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ we need the following two families of representations.

Infinite dimensional representations. The following ladder operators defined on the standard orthonormal basis of $\ell^2(\mathbb{N})$ define a family, labeled by $\theta \in \mathbb{R}^+$, of irreducible representations of $\mathcal{U}_q(\mathfrak{sl}_2)$:

$$\begin{cases} A^+|n\rangle &= \sqrt{[n+\theta]_q[n+1]_q}\ |n+1\rangle \\ A^-|n\rangle &= -\sqrt{[n]_q[n+\theta-1]_q}\ |n-1\rangle \\ A^0|n\rangle &= (n+\theta/2)\ |n\rangle \ . \end{cases} \tag{8.5}$$

Finite dimensional representations. There is a similar representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ on the finite dimensional Euclidian space $\mathbb{C}^{\theta+1}$, where $\theta \in \mathbb{N}$. In this case the irreducible representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ are labeled by $\theta \in \mathbb{N}$ (corresponding to the dimension of the representation) and given by $(\theta+1) \times (\theta+1)$ dimensional matrices defined by

$$\begin{cases} A^+|n\rangle &= \sqrt{[\theta-n]_q[n+1]_q}\ |n+1\rangle \\ A^-|n\rangle &= \sqrt{[n]_q[\theta-n+1]_q}\ |n-1\rangle \\ A^0|n\rangle &= (n-\theta/2)\ |n\rangle \ . \end{cases} \tag{8.6}$$

General case. It is possible to collect in a general expression the above defined representations (8.5) and (8.6). Recalling the parameter $\sigma \in \{-1, 1\}$ introduced in Section 2.5, we can write the ladder operators as

$$\begin{cases} A^+|n\rangle &= \sqrt{[\theta+\sigma n]_q[n+1]_q}\ |n+1\rangle \\ A^-|n\rangle &= -\sigma\sqrt{[n]_q[\theta+\sigma(n-1)]_q}\ |n-1\rangle \\ A^0|n\rangle &= (n+\sigma\theta/2)\ |n\rangle \ . \end{cases} \tag{8.7}$$

The Casimir element is represented by the diagonal matrix

$$C|n\rangle = [\sigma\theta/2]_q[\sigma\theta/2-1]_q|n\rangle \ .$$

The adjoints of the operators $A^\pm$ and A^0 are given by

$$(A^+)^* = -\sigma A^- \qquad \text{and} \qquad (A^0)^* = A^0. \tag{8.8}$$

It is then easily seen that $C^* = C$.

Remark 8.1. The representations we consider are irreducible $*$-representations of two real forms of $\mathcal{U}_q(\mathfrak{sl}_2)$: for $\sigma = +1$ we have the discrete series representations of the noncompact real form $\mathcal{U}_q(\mathfrak{su}(1,1))$, and for $\sigma = -1$ we have the irreducible representations of the compact real form $\mathcal{U}_q(\mathfrak{su}(2))$. Note that for $\sigma = +1$ we have a representation by unbounded operators. As a dense domain we can take the set of finite linear combinations of basis vectors.

8.2 Construction of the process from the quantum Hamiltonian

The quantum Hamiltonian

We define the algebraic version of the quantum Hamiltonian H as a sum of coproducts of the Casimir element C given by (8.2). The quantum Hamiltonian we are interested in is then the corresponding operator in the representation (8.7) plus a constant depending on the representation.

Definition 8.2 (Quantum Hamiltonian). *For $L \in \mathbb{N}$, $L \geq 2$, the element $H = H_{(L)} \in \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes L}$ is defined by*

$$H := \sum_{i=1}^{L-1} \left\{ \underbrace{1 \otimes \cdots \otimes 1}_{(i-1) \text{ times}} \otimes \Delta(C) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i-1) \text{ times}} \right\}, \tag{8.9}$$

Then the quantum Hamiltonian $\mathcal{H} = \mathcal{H}_{(L)}(\sigma\theta)$ is the operator

$$\mathcal{H} = H + c,$$

where H is the operator in the representation (8.7) *and $c = c_{(L)}(\sigma\theta)$ is a constant uniquely determined by the condition $\mathcal{H}|0\rangle^{\otimes L} = 0$.*

From here on we fix a representation, or equivalently we fix the values of σ and θ, such that $\mathcal{H} = H + c$. So by $A \in \mathcal{U}_q(\mathfrak{sl}_2)$ we mean the corresponding operator. Observe that the quantum Hamiltonian satisfies $\mathcal{H}^t = \mathcal{H}$, and that the condition $\mathcal{H}|0\rangle^{\otimes L} = 0$ uniquely determines $c \in \mathbb{R}$, because the state $|0\rangle \otimes |0\rangle$ is a right eigenvector of $\Delta(C)$. From (8.2) and (8.3) we have that

$$\begin{aligned}\Delta(C) &= \Delta(A^+)\Delta(A^-) + \Delta([A^0]_q)\Delta([A^0 - 1]_q) \\ &= (q^{A^0} \otimes 1)\left\{ A^+ \otimes A^- + A^- \otimes A^+ \right\}(1 \otimes q^{-A^0}) + A^+A^- \otimes q^{-2A^0} + q^{2A^0} \otimes A^+A^- \\ &\quad + \frac{1}{(q-q^{-1})^2}\left\{ q^{2A^0-1} \otimes q^{2A^0} + q^{1-2A^0} \otimes q^{-2A^0} - (q+q^{-1}) \right\}.\end{aligned} \tag{8.10}$$

One can check that the constant c needed to have $\mathcal{H}|0\rangle^{\otimes L} = 0$ is given by

$$c = -(L-1)[\sigma\theta]_q[\sigma\theta - 1]_q. \tag{8.11}$$

In [10] and [11] the ASIP(q,θ) and ASEP(q,θ) have been constructed from the quantum Hamiltonian via a ground-state transformation. It is possible to produce a symmetry of the processes by applying the same ground state transformation to a symmetry of the Hamiltonian. The strategy is contained in the following result that has been proven in Section 2.1 of [10].

Theorem 8.3 (Positive ground state transformation)**.** *Let $\mathcal{H}$ be a $|\Omega| \times |\Omega|$ matrix with non-negative off diagonal elements. Suppose there exists a column vector $g \in \mathbb{R}^{|\Omega|}$ with strictly positive entries and such that $\mathcal{H}g = 0$. Let us denote by G the diagonal matrix with entries $G(x,x) = g(x)$ for $x \in \Omega$. Then we have the following*

a) The matrix

$$\mathcal{L} = G^{-1}\mathcal{H}G$$

with entries

$$\mathcal{L}(x,y) = \frac{\mathcal{H}(x,y)g(y)}{g(x)}, \qquad x,y \in \Omega \times \Omega \tag{8.12}$$

is the generator of a Markov process $\{X_t : t \geq 0\}$ taking values on Ω.

b) S commutes with $\mathcal{H}$ if and only if $G^{-1}SG$ commutes with $\mathcal{L}$.

c) If $\mathcal{H} = \mathcal{H}^t$, where ^{t} denotes transposition, then the probability measure μ on Ω

$$\mu(x) = \frac{(g(x))^2}{\sum_{x \in \Omega}(g(x))^2} \tag{8.13}$$

is reversible for the process with generator $\mathcal{L}$.

The constructive procedure to obtain a suitable ground state matrix G as in Theorem 8.3 is explained in [10] and [11]. In this paper, as we already know the target processes and corresponding generators $\mathcal{L}^{\text{ASIP}}$ and $\mathcal{L}^{\text{ASEP}}$, we restrict ourselves to noticing that, using item c) of Theorem 8.3, the entries of the ground-state vector g can be written in terms of the reversible measures μ_α^{ASIP} and μ_α^{ASEP} given by (2.16) and (2.13).

Ground state transformation

Let $\mu_\alpha = \mu_{\alpha,\sigma}$, $\alpha \in \mathbb{R} \setminus \{0\}$ be the reversible signed measure defined in (2.22) (in this section we will often omit the dependence on σ). Then the vectors

$$g_\alpha(x) = \sqrt{\mu_\alpha(x)} \tag{8.14}$$

are ground states for $\mathcal{H}$. Notice that, for negative values of α, the vector g_α has entries taking values in $\mathbb{C}$. The diagonal matrix G_α represented by a diagonal matrix whose coefficients in the standard basis are given by (8.14), i.e.

$$G_\alpha(x,n) = \sqrt{\mu_\alpha(x)} \cdot \delta_{x,n} \ , \tag{8.15}$$

yields a ground state transformation as in Theorem 8.3. For simplicity we denote by G the matrix obtained for the choice $\alpha = 1$, $G = G_1$, in which case Theorem 8.3 applies since the measure μ_1 is finite and strictly positive. We have, as a consequence of item a) of Theorem 8.3, that the operator $\mathcal{L}$ conjugated to $\mathcal{H}$ via G^{-1}, i.e.

$$\mathcal{L} = G^{-1}\mathcal{H}G \tag{8.16}$$

is the generator of a Markov jump process $x(t) = (x_1(t), \ldots, x_L(t))$ describing particles jumping on the chain Λ_L. In [10] and [11] it has been proved that the operator $\mathcal{L}$ is the generator of the ASIP(q,θ) and ASEP(q,θ), respectively, depending on the choice of σ. As a consequence of item b) of Theorem 8.3, if S is a symmetry of $\mathcal{H}$ (i.e. $[\mathcal{H}, S] = 0$), then $G^{-1}SG$ is a symmetry of $\mathcal{L}$.

The following proposition, proven in [11], allows to construct a duality function for ASIP and ASEP starting from a symmetry of the Hamiltonian.

Proposition 8.4. *If S is a symmetry of $\mathcal{H}$ then*

- *$G^{-1}SG$ is a symmetry for $\mathcal{L}$,*
- *$D_{1,\alpha} := G_\alpha^{-1} S G_\alpha^{-1}$ is a self-duality function for $\mathcal{L}$,*
- *$D_{2,\alpha} := G_\alpha^{-1} (S^t)^{-1} G_\alpha^{-1}$ is a self-duality function for $\mathcal{L}$,*
- *$D_{1,\alpha}$ and $D_{2,\alpha}$ are orthogonal with respect to the measure $G_\alpha^2(x)$, i.e. $D_{1,\alpha} G_\alpha^2 D_{2,\alpha}^t = G_\alpha^{-2}$.*

Symmetries

At this aim we need a non-trivial symmetry which yields a non-trivial ground state. Starting from the basic symmetries of H and inspired by the analysis of the symmetric case ($q \to 1$), it will be convenient to consider the *exponential* of those symmetries.

8.3 Symmetries associated to the self-duality functions

We use the following $q-$exponential functions:

$$E_{q^2}(z) := (-z)_\infty = \sum_{n=0}^{\infty} q^{n(n-1)} \frac{z^n}{(q^2)_n}, \qquad e_{q^2}(z) := \frac{1}{(z)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q^2)_n} \quad \text{for } |z| < 1.$$

These satisfy $e_{q^2}(z) E_{q^2}(-z) = 1$, and the following factorization rules: if x and y satisfy $xy = q^2 yx$, then

$$E_{q^2}(x+y) = E_{q^2}(x) \cdot E_{q^2}(y) \quad \text{and} \quad e_{q^2}(x+y) = e_{q^2}(y) \cdot e_{q^2}(x). \tag{8.17}$$

With the $q-$exponential functions we define the following operators: for $\alpha > 0$

$$S_\alpha^{\rm tr} := e_{q^2}\left(\sqrt{\alpha}(1-q^2) \cdot \Delta^{L-1}(q^{A^0} A^+)\right),$$
$$\widehat{S}_\alpha^{\rm tr} := E_{q^2}\left(\sqrt{\alpha} q^{\frac{\sigma\theta}{2}(1+2L)} (1-q^2) \Delta^{L-1}(q^{-A^0} A^+)\right).$$

In case we work in an infinite dimensional representation, i.e. $\sigma = +1$, we should be careful with convergence of the series obtained from applying these operators to functions. If we apply these operators only to finitely supported functions there are no convergence issues. We have the following lemma.

Lemma 8.5. *For all $\alpha > 0$, $\widehat{S}_\alpha^{\rm tr}$ and $S_\alpha^{\rm tr}$ are symmetries of $\mathcal{H}$, i.e.*

$$[\mathcal{H}, S_\alpha^{\rm tr}] = 0, \qquad [\mathcal{H}, \widehat{S}_\alpha^{\rm tr}] = 0. \tag{8.18}$$

Proof. This follows from the fact that A^+ and A^0 commute with the Casimir operator C, and then $\Delta^{L-1}(q^{\pm A^0} A^+)$ commutes with $1 \otimes \ldots \otimes 1 \otimes \Delta(C) \otimes 1 \otimes \ldots \otimes 1$. See Section 4 of [10] for more details. □

Triangular dualities

In the spirit of Section 4 of [10], the following proposition shows that we can write the triangular dualities in terms of the symmetries $S_\alpha^{\rm tr}$ and $\widehat{S}_\alpha^{\rm tr}$ given in Lemma 8.5. We first define two diagonal matrices by

$$A(x,n) := q^{N^2(x)} \delta_{x,n}, \qquad B(x,n) := q^{N(x)} \delta_{x,n} \tag{8.19}$$

Proposition 8.6. *Let $D_\lambda^{\rm tr}$ and $\widehat{D}_\lambda^{\rm tr}$ be the triangular self-duality functions defined in* (5.1)*, then we have:*

$$D_{1/\alpha q}^{\rm tr} = B^{-1} G_\alpha^{-1} S_\alpha^{\rm tr} G_\alpha^{-1} A \tag{8.20}$$

and

$$\widehat{D}_{q/\alpha}^{\rm tr} = B G_\alpha^{-1} \widehat{S}_\alpha^{\rm tr} G_\alpha^{-1} A^{-1}. \tag{8.21}$$

Here we consider a duality function D as the matrix with elements $D(x,n)$, while we denote D^t the transpose matrix. The proof of Proposition 8.6 is given in Section 8.4.

Orthogonal dualities

Now we fix $\alpha > 0$ and use Proposition 8.6 and the expression (5.8) to write the orthogonal dualities and the associated symmetries in terms of the symmetries S_α^{tr} and $\widehat{S}_\alpha^{\mathrm{tr}}$. We first define the following diagonal operators:

$$\begin{aligned} M(x,n) &:= (-1)^{N(x)}\,\delta_{x,n}, \\ R_\alpha(x,n) &:= (-\sigma\alpha q^{1+2N(x)+\sigma\theta(2L+1)})_\infty\,\delta_{x,n}, \\ T_\alpha(x,n) &:= (-\sigma\alpha q^{1-2N(x)+\sigma\theta})_\infty\,\delta_{x,n}. \end{aligned}$$

Let $\mathcal{D}_{\alpha,\sigma}$ be the normalized orthogonal self-duality function defined in equation (3.10) and $\mathcal{S}_{\alpha,\sigma}$ its associated symmetry (3.13) then we have

$$\mathcal{D}_\alpha = G_\alpha^{-1}(ABR_\alpha)^{\frac{1}{2}}\cdot\widehat{S}_\alpha^{\mathrm{tr}}M(S_\alpha^{\mathrm{tr}})^t\cdot(ABT_\alpha)^{-\frac{1}{2}}G_\alpha^{-1} \tag{8.22}$$

and

$$\mathcal{S}_\alpha = G_\alpha^{-1}(ABR_\alpha)^{\frac{1}{2}}\cdot\widehat{S}_\alpha^{\mathrm{tr}}M(S_\alpha^{\mathrm{tr}})^t\cdot(ABT_\alpha)^{-\frac{1}{2}}G_\alpha. \tag{8.23}$$

Proof. From (5.8) we have that D_α can be given in terms of scalar products of the triangular dualities. In matrix form this reads

$$D_\alpha = \widehat{D}_{q/\alpha}^{\mathrm{tr}}G_{-\alpha}^2(D_{1/\alpha q}^{\mathrm{tr}})^t, \tag{8.24}$$

then, using the expressions in Proposition 8.6, it follows that

$$D_\alpha = BG_\alpha^{-1}\widehat{S}_\alpha^{\mathrm{tr}}M\left(S_\alpha^{\mathrm{tr}}\right)^tG_\alpha^{-1}B^{-1}. \tag{8.25}$$

Then (8.22) follows from

$$\mathcal{D}_\alpha = (AB^{-1}R_\alpha)^{\frac{1}{2}}D_\alpha(A^{-1}BT_\alpha^{-1})^{\frac{1}{2}} \tag{8.26}$$

and (8.23) follows from (8.22) and the fact that

$$\mathcal{S}_\alpha = \mathcal{D}_\alpha G_\alpha^2. \tag{8.27}$$

This concludes the proof. □

Remark 8.7. Notice that we can rewrite the orthogonality relation (3.12) of $\mathcal{D}_\alpha$ as

$$\mathcal{D}_\alpha^tG_\alpha^2\mathcal{D}_\alpha = G_\alpha^{-2} \tag{8.28}$$

and the unitarity property of $\mathcal{S}_\alpha$ as follows:

$$\mathcal{S}_\alpha^tG_\alpha^2\mathcal{S}_\alpha = G_\alpha^2. \tag{8.29}$$

These identities imply relations between q-exponentials of generators of $\mathcal{U}_q(\mathfrak{sl}_2)$. Such relations have been exploited in e.g. [31],[19] to obtain orthogonality relations for specific q-hypergeometric functions.

Remark 8.8. In the infinite dimensional setting, $\sigma = +1$, this should be interpreted as a formal identity; as this is an identity involving unbounded operators, the above calculation is not all rigorous.

8.4 Proof of Proposition 8.6.

We first compute the action of the symmetries associated to the triangular dualities.

Action of S_α^{tr}.

We have

$$S_\alpha^{\mathrm{tr}} = e_{q^2}(\sqrt{\alpha}(1-q^2)\cdot\Delta^{L-1}(q^{A^0}A^+)),$$

where

$$\Delta^{L-1}(q^{+A^0}A^+) = q^{A_1^0}A_1^+ + q^{2A_1^0+A_2^0}A_2^+ + ... + q^{2\sum_{i=1}^{L-1}A_i^0+A_L^0}A_L^+.$$

From (8.1) we know that

$$q^{2A^0}q^{A^0}A^+ = q^2q^{A^0}A^+q^{2A^0}, \tag{8.30}$$

then from (8.17) we have

$$S_\alpha^{\mathrm{tr}} = S_1^+S_2^+\dots S_L^+$$

with

$$S_i^+ = e_{q^2}(\sqrt{\alpha}(1-q^2)q^{2\sum_{m=1}^{i-1}A_m^0+A_i^0}A_i^+).$$

Then, for $\sigma=1$,

$$S_\alpha^{\mathrm{tr}}|n\rangle = \sum_{\ell_1,\dots,\ell_L}\prod_i\sqrt{\binom{n_i+\ell_i}{\ell_i}_q\cdot\binom{n_i+\ell_i+\theta-1}{\ell_i}_q}\cdot q^{\ell_i(n_i+\theta/2+1)+2\ell_iN_{i-1}^-(n+\theta/2)}\alpha^{\frac{\ell_i}{2}}|n+\ell\rangle,$$

so that

$$S_\alpha^{\mathrm{tr}}(x,n) = \prod_i\sqrt{\binom{x_i}{n_i}_q\cdot\binom{x_i+\theta-1}{n_i+\theta-1}_q}\cdot q^{(x_i-n_i)[(n_i+\theta/2+1)+2N_{i-1}^-(n+\theta/2)]}\alpha^{\frac{x_i-n_i}{2}}\cdot\mathbf{1}_{x_i\ge n_i}.$$

For $\sigma=-1$,

$$S_\alpha^{\mathrm{tr}}(x,n) = \prod_i\sqrt{\binom{x_i}{n_i}_q\cdot\binom{\theta-n_i}{\theta-x_i}_q}\cdot q^{(x_i-n_i)[(n_i-\theta/2+1)+2N_{i-1}^-(n-\theta/2)]}\alpha^{\frac{x_i-n_i}{2}}\cdot\mathbf{1}_{x_i\ge n_i}.$$

Action of $\widehat{S}_\alpha^{\mathrm{tr}}$.

We have

$$\widehat{S}_\alpha^{\mathrm{tr}} := E_{q^2}(\sqrt{\alpha}q^{\frac{\sigma\theta}{2}(1+2L)}(1-q^2)\Delta^{L-1}(q^{-A^0}A^+)),$$

where

$$\Delta^{L-1}(q^{-A^0}A^+) = q^{-A_1^0}A_1^+ + q^{-2A_1^0-A_2^0}A_2^+ + ... + q^{-2\sum_{i=1}^{L-1}A_i^0-A_L^0}A_L^+.$$

From (8.1) we know that

$$q^{-2A^0}q^{-A^0}A^+ = q^{-2}q^{-A^0}A^+q^{-2A^0}, \tag{8.31}$$

then, from (8.17) we have

$$\widehat{S}_\alpha^{\mathrm{tr}} = \widehat{S}_L^+\widehat{S}_{L-1}^+\dots\widehat{S}_1^+,$$

with

$$\widehat{S}_i^+ = E_{q^2}(\sqrt{\alpha} q^{\frac{\sigma\theta}{2}(1+2L)}(1-q^2)q^{-2\sum_{m=1}^{i-1} A_m^0 - A_i^0} A_i^+).$$

Then it follows that

$$\widehat{S}_\alpha^{\rm tr}(x,n) = \prod_i \sqrt{\binom{x_i}{n_i}_q \cdot \binom{x_i+\theta-1}{n_i+\theta-1}_q} \alpha^{\frac{x_i-n_i}{2}} \cdot q^{-(x_i-n_i)[2N_{i+1}^+(n+\theta/2)+(n_i-\theta L+1)]} \cdot \mathbf{1}_{x_i \geq n_i}$$

for $\sigma = +1$ and

$$\widehat{S}_\alpha^{\rm tr}(x,n) = \prod_i \sqrt{\binom{x_i}{n_i}_q \cdot \binom{\theta-n_i}{\theta-x_i}_q} \alpha^{\frac{x_i-n_i}{2}} \cdot q^{-(x_i-n_i)[2N_{i+1}^+(n-\theta/2)+(n_i+\theta L+1)]} \cdot \mathbf{1}_{x_i \geq n_i}$$

for $\sigma = -1$.

To complete the proof we will make use of the following Lemma:

Lemma 8.9. *For $n \geq m$,*

$$\sqrt{\frac{\binom{n}{m}_q \cdot \binom{n+\theta-1}{m+\theta-1}_q}{\binom{m+\theta-1}{m}_q \cdot \binom{n+\theta-1}{n}_q}} = \frac{\binom{n}{m}_q}{\binom{m+\theta-1}{m}_q} \quad \text{and} \quad \sqrt{\frac{\binom{n}{m}_q \cdot \binom{\theta-m}{\theta-n}_q}{\binom{\theta}{m}_q \cdot \binom{\theta}{n}_q}} = \frac{\binom{n}{m}_q}{\binom{\theta}{m}_q}. \tag{8.32}$$

Now we can conclude the proof of Proposition 8.6.

Proof of (8.20)

Using (8.32) and (8.15) we find that the corresponding triangular duality is given by

$$G_\alpha^{-1} S_\alpha^{\rm tr} G_\alpha^{-1}(x,n) = \prod_i \frac{\binom{x_i}{n_i}_q}{\Psi_{q,\sigma}(\theta,n_i)} \; q^{(x_i-n_i)[(n_i+\theta/2+1)+2N_{i-1}^-(n+\theta/2)]-\theta i(n_i+x_i)} \cdot \alpha^{-n_i}.$$

Now, using that

$$\sum_i (x_i-n_i)[(n_i+\theta/2)+2N_{i-1}^-(n+\theta/2)-\theta i] = -N^2(n) + \sum_i x_i(n_i+2N_{i-1}^-(n)) \tag{8.33}$$

we get

$$G_\alpha^{-1} S_\alpha^{\rm tr} G_\alpha^{-1}(x,n) = q^{-N^2(n)+N(x)} \cdot \prod_i \frac{\binom{x_i}{n_i}_q}{\Psi_{q,\sigma}(\theta,n_i)} \cdot q^{x_i(n_i+2N_{i-1}^-(n))} q^{-2\theta i n_i} \cdot \left(\frac{1}{\alpha q}\right)^{n_i}.$$

Comparing this with (5.1) we obtain

$$G_\alpha^{-1} S_\alpha^{\rm tr} G_\alpha^{-1}(x,n) = q^{-N^2(n)+N(x)} \cdot D_{1/\alpha q}^{\rm tr}(x,n)$$

from which the statement follows.

Proof of (8.21)**.**

Using (8.32) and (8.15) we obtain

$$G_\alpha^{-1} \widehat{S}_\alpha^{\rm tr} G_\alpha^{-1}(x,n) = \prod_i \frac{\binom{x_i}{n_i}_q}{\Psi_{q,\sigma}(\theta,n_i)} q^{-(x_i-n_i)[2N_{i+1}^+(n+\theta/2)+(n_i-\theta L+1)]-\theta i(x_i+n_i)} \alpha^{-n_i}.$$

We use that

$$\sum_i (x_i - n_i)[2N^+_{i+1}(n+\theta/2) + (n_i+\theta/2) + \theta i] =$$

$$- N^2(n) + \theta/2(1+2L)N(x-n) + \sum_i n_i(2N^-_{i-1}(x) + x_i)$$

to get

$$G_\alpha^{-1}\widehat{S}_\alpha^{\mathrm{tr}}G_\alpha^{-1}(x,n) = q^{N^2(n)-N(x)} \cdot \prod_i \frac{\binom{x_i}{n_i}_q}{\Psi_{q,\sigma}(\theta,n_i)} \cdot q^{-n_i[2N^-_{i-1}(x)+x_i]-2\theta i n_i} \cdot (\tfrac{q}{\alpha})^{n_i}.$$

Then comparing with (5.2) we obtain

$$G_\alpha^{-1}\widehat{S}_\alpha^{\mathrm{tr}}G_\alpha^{-1}(x,n) = q^{N^2(n)-N(x)} \cdot \widehat{D}^{\mathrm{tr}}_{q/\alpha}(x,n)\ ,$$

which is the desired result. □

9 Appendix

See Section 2.1 for the definition of the $q-$binomial coefficients and $q-$Pochhammer symbols. We refer to Appendix I of [20] for the formulas involving $q-$Pochhammer symbols. Identity (9.1) follows directly from the definition of the $q-$binomial coefficient, and (9.2) is (a special case of) the $q-$binomial formula [20, (II.4)].

9.1 Identities for $q-$binomial coefficients

For $n, x, y \in \mathbb{N}$,

$$\binom{y}{n}_q \binom{x}{y}_q = \binom{x}{n}_q \binom{x-n}{y-n}_q , \tag{9.1}$$

moreover, for $N \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\sum_{\kappa=0}^{N} \binom{N}{\kappa}_q q^{\kappa N} \left(tq^{-1}\right)^\kappa = \prod_{\kappa=1}^{N} (1 + tq^{2(\kappa-1)}) = (-t)_N. \tag{9.2}$$

9.2 Identities for $q-$Pochhammer symbols

For $n, m \in \mathbb{N}$ and $a \neq 0$ we have

$$(a)_{n+m} = (a)_m (aq^{2m})_n \tag{9.3}$$

$$(a)_{m-n} = \frac{(a)_m}{(q^{2-2m}/a)_n} \left(-\frac{q^2}{a}\right)^n q^{n(n-1)-2mn}\ , \tag{9.4}$$

moreover, for $b \neq 0$, $c \neq 0$,

$$\frac{(bq^{-2n})_n}{(cq^{-2n})_n} = \left(\frac{b}{c}\right)^n \frac{(b^{-1}q^2)_n}{(c^{-1}q^2)_n}\ , \tag{9.5}$$

finally, for $n, m \in \mathbb{N}$, $n \geq m$,

$$\frac{(q^2)_n}{(q^2)_{n-m}(q^{-2n})_m} = (-1)^m q^{2mn-m(m-1)}. \tag{9.6}$$

9.3 Identities for $q-$hypergeometric functions

We refer to the book [20] for theory on $q-$hypergeometric functions. Here we only use the $q-$hypergeometric function

$$ {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(c;q)_k} \frac{z^k}{(q;q)_k} . \tag{9.7} $$

where, as before, $(a;q)_k = \prod_{i=0}^{k-1}(1-aq^i)$. We always assume that $c \notin q^{-\mathbb{N}}$, so that the denominator never equals zero. The series converges absolutely for $|z|<1$. Note that for $a = q^{-n}$, $n \in \mathbb{N}$, the series terminates after the $(n+1)$-th term; in this case the series is a polynomial of degree n in b.

The ${}_2\varphi_1$-functions we encounter in this paper will depend on q^2 instead of q. We need the following two transformation formulas for ${}_2\varphi_1$-functions. The first is one of Heine's transformation formulas, see [20, (III.3)], which is valid as long as the series on both sides converge. The second one is only valid for a terminating ${}_2\varphi_1$-series, and is obtained from reversing the order of summation.

Heine's transformation:

$$ {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q^2, z\right) = \frac{(\frac{abz}{c})_\infty}{(z)_\infty}\, {}_2\varphi_1\left(\begin{matrix} c/a, c/b \\ c \end{matrix}; q^2, \frac{abz}{c}\right). \tag{9.8} $$

Transformation for terminating series:

$$ {}_2\varphi_1\left(\begin{matrix} q^{-2n}, b \\ c \end{matrix}; q^2, z\right) = \frac{(b)_n}{(c)_n} q^{-n-n^2}(-z)^n\, {}_2\varphi_1\left(\begin{matrix} q^{-2n}, q^{2-2n}c^{-1} \\ q^{2-2n}b^{-1} \end{matrix}; q^2, \frac{cq^{2+2n}}{bz}\right). $$

Lemma 9.1. *For $n, x, y \in \mathbb{N}$, $x \le n$, $m \in \mathbb{R}^+$ and $|C| < q^{x+m+n-1}$ we have*

$$ \sum_{y=n}^{\infty} \binom{y}{x}_q \binom{y}{n}_q \binom{y+m-1}{y}_q C^y = \frac{C^n q^{n(1-m-x)}(q^{2m})_n q^{x^2}}{(q^2)_x (q^2)_{n-x}}\, {}_2\varphi_1\left(\begin{matrix} q^{2(n+1)}, q^{2(m+n)} \\ q^{2(1+n-x)} \end{matrix}; q^2, Cq^{1-x-n-m}\right). $$

Proof. By some algebraic manipulation of the $q-$numbers and changing the variable in the summation we get

$$ \begin{aligned} &\sum_{y=n}^{\infty} \binom{y}{x}_q \binom{y}{n}_q \binom{y+m-1}{y}_q C^y \\ &= \frac{q^{x^2+n^2}}{(q^2)_x (q^2)_n} \sum_{y=n}^{\infty} \frac{(q^2)_y (q^{2m})_y}{(q^2)_{y-x}(q^2)_{y-n}} C^y q^{-y(x+n+m-1)} \\ &= \frac{C^n q^{n(1-m-x-n)} q^{x^2+n^2}}{(q^2)_x (q^2)_n} \sum_{r=0}^{\infty} \frac{(q^2)_{r+n}(q^{2m})_{r+n}}{(q^2)_{r+n-x}(q^2)_r} C^r q^{r(1-m-x-n)} \\ &= \frac{C^n q^{n(1-m-x)}(q^{2m})_n q^{x^2}}{(q^2)_x (q^2)_{n-x}} \sum_{r=0}^{\infty} \frac{(q^{2(n+1)})_r (q^{2(m+n)})_r}{(q^{2(1+n-x)})_r (q^2)_r} C^r q^{r(1-m-x-n)} \\ &= \frac{C^n q^{n(1-m-x)}(q^{2m})_n q^{x^2}}{(q^2)_x (q^2)_{n-x}}\, {}_2\varphi_1\left(\begin{matrix} q^{2(n+1)}, q^{2(m+n)} \\ q^{2(1+n-x)} \end{matrix}; q^2, Cq^{1-x-n-m}\right). \end{aligned} $$

That concludes the proof. ∎

Lemma 9.2. *For $x, n, m \in \mathbb{N}$ with $x \le n \le m$ and $C \in \mathbb{R}$ we have*

$$\sum_{y=n}^{m} \binom{y}{x}_q \binom{y}{n}_q \binom{m}{y}_q C^y$$
$$= \frac{(-C)^n q^{n(1+m-x)} (q^{-2m})_n q^{x^2}}{(q^2)_x (q^2)_{n-x}} \, {}_2\varphi_1 \left(\begin{matrix} q^{2(n+1)}, q^{-2(m-n)} \\ q^{2(1+n-x)} \end{matrix} ; q^2, -Cq^{1+m-x-n} \right).$$

We omit the proof of this identity which is similar to that of Lemma 9.1.

9.4 $q-$Orthogonal polynomials

$q-$Krawtchouk polynomials

The $q-$Krawtchouk polynomials in the q-hypergeometric representation are given by:

$$K_n(q^{-x}; p, c; q) := {}_2\varphi_1 \left(\begin{matrix} q^{-x}, q^{-n} \\ q^{-c} \end{matrix} ; q, pq^{n+1} \right), \qquad \text{for} \quad c \in \mathbb{N}, \quad n, x \in \{0, \dots, c\} \tag{9.9}$$

and ${}_2\phi_1$ as in definition (9.7). We remark that in the literature [32] they are known as quantum $q-$Krawtchouk polynomials.

Orthogonality relations. Under the condition

$$pq^c > 1 \qquad \text{and} \qquad c \in \mathbb{N} \tag{9.10}$$

these polynomials are orthogonal with respect to a positive measure on $\{0, 1, \dots, c\}$, see [32, §14.14]. The orthogonality relations for $q-$Krawtchouk polynomials read as follows:

$$\sum_{x=0}^{c} \frac{(pq; q)_{c-x} \, (-1)^{c-x}}{(q; q)_x (q; q)_{c-x}} \; q^{\binom{x}{2}} \cdot K_m(q^{-x}; p, c; q) \cdot K_n(q^{-x}; p, c; q)$$
$$= \frac{(-1)^n \, p^c \, (q; q)_{c-n} (q; q)_n (pq; q)_n}{((q; q)_c)^2} \; \cdot q^{\binom{c+1}{2} - \binom{n+1}{2} + cn} \cdot \delta_{m,n}. \tag{9.11}$$

$q-$Meixner polynomials

The $q-$Meixner polynomials in the q-hypergeometric representation are given by

$$M_n(q^{-x}; b, c; q) := {}_2\varphi_1 \left(\begin{matrix} q^{-x}, q^{-n} \\ bq \end{matrix} ; q, -\frac{q^{n+1}}{c} \right), \qquad \text{for} \quad x, n \in \mathbb{N}, \tag{9.12}$$

where ${}_2\phi_1$ is the $q-$hypergeometric function defined in (9.7). Note that $M_n(q^{-x}; b, c; q)$ is a polynomial in q^{-x} of degree n, but it is also a polynomial in c^{-1} of degree n. We remark the similarity with the $q-$Krawtchouk polynomials: for $c \in \mathbb{N}$ we have $K_n(q^{-x}; p, c; q) = M_n(q^{-x}; q^{-1-c}, -p^{-1}; q)$.

Orthogonality relations. Under the conditions

$$bq \in [0, 1) \qquad \text{and} \qquad c > 0 \tag{9.13}$$

these polynomials are orthogonal with respect to a positive measure on $\mathbb{N}$, see [32, §14.13]. The orthogonality relations for $q-$Meixner polynomials read as follows:

$$\sum_{x=0}^{\infty} \frac{(bq; q)_x \, c^x}{(q; q)_x (-cbq; q)_x} \; q^{\binom{x}{2}} \cdot M_m(q^{-x}; b, c; q) \cdot M_n(q^{-x}; b, c; q)$$
$$= \frac{(-c; q)_\infty (q; q)_n (-c^{-1}q; q)_n}{(-cbq; q)_\infty (bq; q)_n} \; \cdot q^{-n} \cdot \delta_{m,n}. \tag{9.14}$$

The function $M_n(q^{-x};b,c;q)$, $x\in\mathbb{N}$, is also a polynomial in q^n of degree x. It can be considered as an instance of a rescaled big $q-$Laguerre polynomial, see [32, §14.11],

$$M_n(q^{-x};b,c;q)=(-q^{-x}/bc;q)_x P_n(bq^{1+n};b,-bc;q).$$

The big $q-$Laguerre polynomials $P_m(y;\alpha,\beta;q)$ with $0<\alpha q<1$ and $\beta<0$ satisfy orthogonality relations of the form

$$\sum_{k=0}^{\infty} P_m(\beta q^{1+k};\alpha,\beta;q)P_n(\beta q^{1+k};\alpha,\beta;q)w(\beta q^{k+1}) + \sum_{k=0}^{\infty} P_m(\alpha q^{1+k};\alpha,\beta;q)P_n(\alpha q^{1+k};\alpha,\beta;q)w(\alpha q^{k+1})=\delta_{m,n}N_n,$$

where the weight function w and the squared norm N_n are known explicitly. We see that the big q-Laguerre polynomials are orthogonal with respect to a measure supported on the finite interval $[\beta q,\alpha q]$, hence they form a complete orthogonal basis for the corresponding weighted L^2-space. It follows that they also satisfy the dual orthogonality relation

$$\sum_{n=0}^{\infty} P_n(y;\alpha,\beta;q)P_n(y';\alpha,\beta;q)N_n^{-1}=\frac{\delta_{y,y'}}{w(y)}, \qquad y,y'\in\beta q^{1+\mathbb{N}}\cup\alpha q^{1+\mathbb{N}},$$

and the set $\{n\mapsto P_n(y;\alpha,\beta;q) \mid y\in\beta q^{1+\mathbb{N}}\cup\alpha q^{1+\mathbb{N}}\}$ is a complete orthogonal basis for the weighted L^2-space $\ell^2(\mathbb{N};N_n^{-1})$. The orthogonality relations for the q-Meixner polynomials are equivalent to the dual orthogonality relations of the big $q-$Laguerre polynomials for $y\in\alpha q^{1+\mathbb{N}}$, so the q-Meixner polynomials do not form a complete basis for their weighted L^2-space.

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