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Representation theory and the regular representation

Splitting the regular representation into its group invariant subspaces

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Representatietheorie en de reguliere representatie

Het splitsen van de reguliere representatie in zijn groepsinvariante deelruimtes

Preface

This thesis was written as part of the bachelor Applied Mathematics at the Delft University of Technology. The last quarter of the academic year is intended to set up a research project, to attain better research skills, to become better at academic writing and to become familiar with a mathematical concept. After finishing the thesis, a thesis defence is held.

This thesis is written for peers, i.e. third year students in applied mathematics. It is assumed that the reader of this thesis has prior knowledge in linear algebra as well as algebra on an undergraduate level.

Readers who are interested in the two methods to construct group invariant subspaces of $\mathbb{C}^{|G|}$, where G is a group, can read chapters 3 and 4. Chapter 3 uses change of basis matrices whereas chapter 4 uses the grand orthogonalisation method. If the reader is not yet familiar with representation theory it is recommended to read chapter 1.

I would like to thank my supervisors dr. Jeroen Spandaw and dr. Paul Visser for their enormous help in this project. In the first weeks of this project Jeroen Spandaw has provided extensive notes covering the basis of representation theory and an abundance of worked out examples. Moreover, I could always ask questions if I were to be stuck and Jeroen Spandaw and Paul Visser patiently gave me a fountain of feedback and explanations in our weekly online meetings, which I am sincerely thankful for. I am grateful that I have had the privilege and opportunity to work together with the both of them. Thanks for accompanying me in this project. Thank you Jeroen Spandaw. Thank you Paul Visser.

Delft, 26 August 2021
Quirijn van Gulik

Summary

Representation theory is a branch in mathematics that studies group homomorphisms between a group and the automorphism group of a vector space. A special representation that every group has is the regular representation. This representation permutes all elements of the group in a vector space which dimension is equal to the order of the group. Within this vector space there are group invariant subspaces. There are several methods to finding representation invariant subspaces of this vector space.

This thesis aims to do two things: First of all, this thesis aims to give the reader an introduction to representation theory, presenting various key concepts, definitions and theorems. Moreover, ways to construct character tables are presented along with multiple worked out examples. Second, the regular representations of D_4 and Q_8 are decomposed into representation invariant subspaces of $\mathbb{C}^8$. To this end, two methods were used.

The first method (A), proposed by dr. Jeroen Spandaw, works out all the possible decompositions of the vector space a regular representation acts on. This is quite a laborious process in which change of basis matrices, expressed in several parameters, will have to be made for all generators of the considered group. The second methods (B), proposed by dr. Paul Visser, is known as the grand orthogonalization method and makes use of an extended version of the character table of the considered group.

Both methods are perfectly fine to make a desired decomposition. However, method A takes a lot more computational effort than method B to come up with the desired result. The benefit of using method A over method B is that method A considers all possible decompositions, whereas method B only considers one of the infinitely many that are possible.

Summary for the general audience

Representation theory is a combination of two fundamental branches of mathematics, namely group theory, i.e. theories that describe symmetry, and linear algebra. Symmetries can be found all throughout mathematics, physics and other disciplines. In physics for example, conservation laws follow from symmetries in the laws of nature. Moreover, the heat capacity of materials depends strongly on the amount of symmetry of the molecules that make up the material. In mathematics, laying oranges in a basket forms a certain symmetrical grid. A final example is the fact that the well-known *abc*-formula cannot be extended to an *abcdef*-formula for equation $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ due to restrictions given by symmetries. Symmetries found in different objects might be

classified as the same symmetry and vice versa, certain symmetries can be found in a wild variety of locations and concepts.

An introduction to the branch of mathematics called 'representation theory' will be given. Along with worked out examples, definitions and ensuing results, the reader will get familiar with the key concepts of representation theory. After that, two methods were presented to break-up a higher-dimensional space into smaller dimensional spaces. One of these methods takes more time to apply, but covers all cases. The other method is a lot faster, especially for higher dimensions, but does cover only one of the infinitely many cases.

List of Symbols

Symbol	Description
G	group
g	element of G
χ	character
V	vector space
$\mathrm{Aut}(V)$	the group of automorphisms of V
$\mathbf{v}$	vector in V
h	positive definite Hermitian inner product
H	representation invariant positive definite Hermitian inner product
k	the class number of G, i.e. the number of conjugacy classes
N	the order of G, i.e. $\|G\|$
ρ	representation
f	homomorphism of representations
C_n	the cyclic group of order n
D_n	the dihedral group of order $2n$
Q_8	the quaternion group
S_n	the permutation group on n elements
A_n	the alternating group on n elements
$SO(3)$	the group of orthonormal 3×3-matrices with det $= 1$
$SU(2)$	the group of unitary 2×2-matrices with det $= 1$
T	the tetrahedral group
$2T$	the binary tetrahedral group
O	the octahedral group
$2O$	the binary octahedral group
I	the icosahedral group
$2I$	the binary icosahedral group
$\mathbb{R}^n$	the group of vectors consisting of real numbers
$\mathbb{C}^n$	the group of vectors consisting of complex numbers
$\mathbb{H}$	the group of quaternions
$\mathbb{H}_1$	the group of quaternions of length 1

Please note that the symbol descriptions mentioned in the List of Symbols stand unless stated otherwise.

Introduction

Symmetries can be found all throughout mathematics, physics and other disciplines. In physics for example, conservation laws follow from symmetries in the laws of nature. Moreover, the heat capacity of materials depends strongly on the amount of symmetry of the molecules that make up the material. In mathematics, placing as many spheres as possible in a space forms a certain symmetrical lattice. A final example, "which played an important role in the historical development of group theory" [13], is the fact that there does not exist an '$abcdef$-formula' to find all the complex roots of a given fifth degree equation, $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$. This was proven again using symmetries. Representations, which are symmetries in the realm of linear algebra, found in different objects might be classified as the same symmetry and vice versa, certain symmetries can be found in a wild variety of locations and concepts.

Elementary particles are described by representation theory. Different types of particles use different irreducible representations of the same overall symmetry group. Their study is of great importance in understanding particle physics on a fundamental mathematical basis. Maybe the regular representation of the icosians, $2I$, can be used to describe the Standard Model of particles, because each family of particles seems to require irreducible representations of the same size. This is a conjecture by dr. Paul Visser.

The aim of this thesis is to build up to ways a vector space a regular representation acts on can be split into representation invariant subspaces. Moreover, the representation theory needed to get to this result is extensively treated to make the process understandable for undergraduates in applied mathematics. Treating representation theory is a goal of this thesis on its own and more emphasis goes to the making of group tables. Several standard books, theses and unpublished notes were consulted to study the representation theory. For the splitting of the vector space in representation invariant subspaces two methods were suggested, one by dr. Jeroen Spandaw and one by dr. Paul Visser. These methods were described and compared to each other.

This thesis will be structured in the following way. In chapter 1 an extensive description of the relevant information concerning representation theory will be presented. Chapter 2 will provide character tables of some elementary groups. More importantly, the methods to make group tables that were presented in chapter 1 will be used. The images of the irreducible representations of the groups D_4 and Q_8 will also be presented with respect to a chosen basis. These two groups will have a role in chapters 3 and 4. Chapter 3 and chapter 4 will each present a method to split a vector space a regular representation acts on. Moreover, in chapter 3 the groups D_4 and Q_8, which have the same order, will be compared as to how their only multidimensional (4-dimensional) isotypical vectorspaces can be split.

This thesis is for a large part based on personal communication with dr. Jeroen Spandaw and dr. Paul Visser. Moreover, several standard books were consulted. These books are [3], [4], [6], [8], [9], [11], [14], [15] and [16]. Most content in this thesis is already well established in the field of representation theory. If no source is explicitly stated upon a statement, either this statement is backed up by several of the standard books mentioned or this statement is of my own work. If a source is cited, either this information is less standard, which happens in section 4, or it is cited or paraphrased from that particular standard source.

Lastly, to avoid some unnecessary repetition, a short list of symbols is given.

Chapter 1

Representation theory

This chapter aims to state and explain some of the key concepts in representation theory. These concepts are well known in the literature, but may not be familiar to the reader. Chapter 1 offers an introduction to representation theory.

First the concept of a representation will be given (section 1.1). Then the direct sum of representations and accompanying theorems will be treated (section 1.2). In the following sections Schur's lemma (section 1.3) and Schur orthogonality (section 1.4) will be illustrated. After that, the dual representation (section 1.5) and tensor products are treated (section 1.6). In section 1.7 the regular representation will be presented. Lastly, in section 1.8, methods that help to construct a character table are presented.

1.1 What is a representation?

In this section the framework in which a representation is defined is given, accompanied with some other definitions. Furthermore, an example regarding representations will be presented.

1.1.1 The definition of a representation

A group G can be said to act on a set X. The following definition defines this concept:

Definition 1.1. Let X be a set. G is said to act on X if there exists a map $f : G \times X \to X$, given by $(g, x) \mapsto g \bullet x$ such that:

Identity:	$e \bullet x = x$ for all $x \in X$
Compatibility:	$(g_1 g_2) \bullet x = g_1 \bullet (g_2 \bullet x)$ for all $g_1, g_2 \in G$ and $x \in X$

Here e is the identity of G. If G acts on X, then the map f is an action of G on X. [5, p. 97]

In representation theory such a group action is slightly modified to form a group homomorphism. Instead of $f : G \times X \to X$, a map $\rho : G \to X^X$ is defined. In representation theory, the set X has to be a vector space V. The definition of a representation is given:

Definition 1.2. A representation of G on V is a group homomorphism ρ from G to the group of automorphisms of V:

$$\rho : G \to \mathrm{Aut}(V).$$

It is said that the dimension of the representation ρ is the dimension of V. [10, p. 24]

These two interpretations of a representation are interchangeable by setting $g \bullet \mathbf{v} = \rho(g)(\mathbf{v})$ for all $g \in G$, $\mathbf{v} \in V$, for $g \bullet \mathbf{v}$ as defined in definition 1.1 and for ρ a representation as defined in definition 1.2. Indeed, let f be an action of G on V and let $\rho : G \to \mathrm{Aut}(V)$ such that $\rho(g)(\mathbf{v}) = g \bullet \mathbf{v}$ for all $g \in G$ and $\mathbf{v} \in V$. Now

$$\rho(g_1 g_2)(\mathbf{v}) = (g_1 g_2) \bullet \mathbf{v} = g_1 \bullet (g_2 \bullet \mathbf{v}) =$$
$$g_1 \bullet (\rho(g_2)(\mathbf{v})) = \rho(g_1)(\rho(g_2)(\mathbf{v})) = (\rho(g_1) \circ \rho(g_2))(\mathbf{v}),$$

making ρ a group homomorphism.

Conversely, let ρ be a representation and let $f : G \times V \to V$ be given by $(g, \mathbf{v}) \mapsto g \bullet \mathbf{v}$ such that $g \bullet \mathbf{v} = \rho(g)(\mathbf{v})$ for all $g \in G$ and $\mathbf{v} \in V$. Now

$$e \bullet \mathbf{v} = \rho(e)(\mathbf{v}) = \mathbf{v}$$

and

$$(g_1 g_2) \bullet \mathbf{v} = \rho(g_1 g_2)(\mathbf{v}) = (\rho(g_1) \circ \rho(g_2))(\mathbf{v}) = \rho(g_1)(\rho(g_2)(\mathbf{v})) = g_1 \bullet (g_2 \bullet \mathbf{v}).$$

Remark 1.3. Sometimes, instead of stating that ρ acts on V, it is stated that G acts on V.

From now on all vector spaces that are considered will be *finite-dimensional*. Some of the definitions and theorems defined later in this thesis are also properly defined for infinite-dimensional vector spaces. However, in this thesis, we are only interested in finite-dimensional vector spaces. After choosing a basis for V, the following function is obtained:

$$\rho : G \to \mathrm{GL}(n, K).$$

Changing the basis of V sets:

$$\rho' : g \mapsto P\rho(g)P^{-1}, \tag{1.1}$$

where P is a change of basis matrix. Such a change of basis should not alter the essence of a representation ρ. ρ and ρ' are said to be equivalent. From now on, all vector spaces that are considered will be over the complex field, $K = \mathbb{C}$. To this end, a representation is a group homomorphism

$$\rho : G \mapsto \mathrm{GL}(n, \mathbb{C}).$$

Thus, depending on whether or not a basis for V is chosen, either a representation is a homomorphism

$$\rho : G \to \mathrm{Aut}(V),$$

or a representation is a homomorphism

$$\rho : G \to \mathrm{GL}(n, \mathbb{C}).$$

Now that a representation is properly defined, another important definition is given:

Definition 1.4. The character χ of ρ is the map

$$\chi : G \to K,\ g \mapsto \chi(g) := \mathrm{tr}(\rho(g)).$$

[12, p. 11]

Indeed, the choice of basis vectors for V does not affect $\chi_\rho(g)$, since

$$\chi_{\rho'}(g) = \mathrm{tr}(\rho'(g)) = \mathrm{tr}(P\rho(g)P^{-1}) = \mathrm{tr}(P^{-1}P\rho(g)) = \mathrm{tr}(\rho(g)) = \chi_\rho(g).$$

Here P is a change of basis matrix as in equation 1.1.

Moreover, all elements of G within the same conjugacy class have the same character. Indeed,

$$\chi_\rho(gg_0g^{-1}) = \mathrm{tr}(\rho(gg_0g^{-1})) = \mathrm{tr}(\rho(g)\rho(g_0)\rho(g^{-1})) = \mathrm{tr}(\rho(g)\rho(g^{-1})\rho(g_0)) =$$
$$\mathrm{tr}(\rho(gg^{-1})\rho(g_0)) = \mathrm{tr}(\rho(eg_0)) = \mathrm{tr}(\rho(g_0)) = \chi_\rho(g_0)$$

for any $g \in G$, which is shown in lemma 1.5.

Lemma 1.5. All elements of G within the same conjugacy class have the same character.

Lastly, three other definitions are given:

Definition 1.6. A vector space $W \subseteq V$ is said to be representation invariant if for a representation $\rho : G \to \mathrm{Aut}(V)$, $\rho(g)(\mathbf{w}) \in W$ for any $g \in G$ and $\mathbf{w} \in W$. In particular, W is said to be representation invariant under ρ or ρ-invariant if this holds for that particular ρ.

Definition 1.7. ρ is said to be faithful if it is injective.

Definition 1.8. The degree of ρ, noted as $\deg(\rho)$, is the dimension of the vector space V that ρ acts on.

Remark 1.9. Sometimes, instead of the degree of a representation, we speak of the dimension of a representation.

Since ρ is a group homomorphism, it holds that $\rho(e_1) = e_2$. Here e_1 is the identity element of the group G and e_2 is the identity element of $\mathrm{Aut}(V)$, which is, chosen any basis, I_n, where n is the dimension of V. Hence

Corollary 1.10. $\chi_\rho(e_1) = n =: \mathrm{Dim}(V) = \deg(\rho)$ for any group G and vector space V. Here e_1 is the identity element of G.

1.1.2 Example

Take for example $(G, V) = (S_3, \mathbb{C}^3)$. The group S_3 can act on $\mathbb{C}^3$ by permuting the complex axes. That is, taking f as in definition 1.1, f acts on $\mathbb{C}^3$ as follows:

$$\begin{aligned}
f((1),(x_1,x_2,x_3)^\top) &= (x_1,x_2,x_3)^\top \quad , \quad & f((12),(x_1,x_2,x_3)^\top) &= (x_2,x_1,x_3)^\top\\
f((13),(x_1,x_2,x_3)^\top) &= (x_3,x_2,x_1)^\top \quad , \quad & f((23),(x_1,x_2,x_3)^\top) &= (x_1,x_3,x_2)^\top\\
f((123),(x_1,x_2,x_3)^\top) &= (x_3,x_1,x_2)^\top \quad , \quad & f((132),(x_1,x_2,x_3)^\top) &= (x_2,x_3,x_1)^\top.
\end{aligned}$$

Or, equivalently, regarding definition 1.2

$$\begin{aligned}
\rho((1)) : (x_1, x_2, x_3)^\top &\mapsto (x_1, x_2, x_3)^\top \quad , \quad & \rho((12)) : (x_1, x_2, x_3)^\top &\mapsto (x_2, x_1, x_3)^\top \\
\rho((13)) : (x_1, x_2, x_3)^\top &\mapsto (x_3, x_2, x_1)^\top \quad , \quad & \rho((23)) : (x_1, x_2, x_3)^\top &\mapsto (x_1, x_3, x_2)^\top \\
\rho((123)) : (x_1, x_2, x_3)^\top &\mapsto (x_3, x_1, x_2)^\top \quad , \quad & \rho((132)) : (x_1, x_2, x_3)^\top &\mapsto (x_2, x_3, x_1)^\top .
\end{aligned}$$

Due to the linearity of $\text{Aut}(V)$, only the basis vectors for V have to be considered to define how ρ acts on V. In general, let $\rho : G \to \text{Aut}(V)$ be a representation of G on V. Let $\{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ be a basis for V, where the vector space V has dimension n. Suppose $\rho(g)(\mathbf{b_j}) = a_{1,j}\mathbf{b_1} + a_{2,j}\mathbf{b_2} + \cdots + a_{n,j}\mathbf{b_n}$, then

$$\rho(g) = \begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,n} \end{bmatrix}.$$

The standard basis for $\mathbb{C}^3$, $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, is taken. Now

$$\begin{aligned}
\rho((12))(\mathbf{e_1}) &= \mathbf{e_2} \quad , \quad & \rho((23))(\mathbf{e_1}) &= \mathbf{e_1} \\
\rho((12))(\mathbf{e_2}) &= \mathbf{e_1} \quad , \quad & \rho((23))(\mathbf{e_2}) &= \mathbf{e_3} \\
\rho((12))(\mathbf{e_3}) &= \mathbf{e_3} \quad , \quad & \rho((23))(\mathbf{e_3}) &= \mathbf{e_2}.
\end{aligned}$$

Hence,

$$\rho((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{1.2}$$

Due to the fact that a representation is a group homomorphism, only the generators of G have to be considered. All function values of ρ for other group elements $g \in G$ follow from the fact that $\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)$. For completeness, the function values $\rho(g)$ for the other $g \in G$ are given as well:

$$\rho((1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\rho((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \rho((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

1.2 Direct sum

1.2.1 The definition of the direct sum of representations

A direct sum of two functions is defined as follows: Let $f : A \to B$ and $g : C \to D$ be two arbitrary functions. Then:

$$(f \oplus g)(a, c) := (f(a), g(c)),$$

with $(a, c) \in (A, C)$. However, the direct sum domain of two representation is taken as G instead of (G, G). The definition of the direct sum of representations is given (definition 1.11):

Definition 1.11. let $\rho_1 : G \to \mathrm{Aut}(V_1)$ and $\rho_2 : G \to \mathrm{Aut}(V_2)$ be two representations of the same group G. Now,

$$(\rho_1 \oplus \rho_2) := g \mapsto (\rho_1(g), \rho_2(g)).$$

The direct sum of two representations $\rho_1 : G \to \mathrm{Aut}(V_1)$ and $\rho_2 : G \to \mathrm{Aut}(V_2)$ as defined in definition 1.11 is again a representation. The codomain of $\rho_1 \oplus \rho_2$ is the automorphism group of the well known direct sum of vector spaces $V_1 \oplus V_2$: $\mathrm{Aut}(V_1 \oplus V_2)$.

Given bases for V_1 and V_2, let the matrices A_{ρ_1} and A_{ρ_2} represent $\rho_1(g)$ and $\rho_2(g)$ respectively. Then

$$A_{\rho_1 \oplus \rho_2} = \begin{bmatrix} A_{\rho_1} & 0 \\ 0 & A_{\rho_2} \end{bmatrix} \tag{1.3}$$

represents $\rho_1(g) \oplus \rho_2(g)$.

The definition of (ir)reducibility of representations is given (definition 1.12):

Definition 1.12. If ρ is the direct sum of two or more representations, then ρ is said to be reducible. Any of the representations taking part in the direct sum of ρ is a subrepresentation of ρ. If ρ is not reducible, ρ is said to be irreducible.

Irreducible representations will have an essential role in representation theory as they are the 'building blocks' of all the representations of a group.

Remark 1.13. Let $\rho_1 : G \to \mathrm{Aut}(V_1)$ and $\rho_2 : G \to \mathrm{Aut}(V_2)$ be two representations. Sometimes, instead of noting that ρ_1 is a subrepresentation of ρ_2, it is stated that V_1 or χ, where $\chi = \chi_{\rho_1}$, is a subrepresentation of V_2 or χ', where $\chi' = \chi_{\rho_2}$. Moreover, sometimes, instead of noting that ρ is (ir)reducible, V_1 or χ is said to be (ir)reducible instead. Any alike statement can also be made. Whenever this is done, there is no ambiguity regarding which representations are meant.

1.2.2 Essential theorems on representation theory regarding direct sums

Corollary 1.14. Let ρ be the direct sum of two subrepresentations: $\rho = \rho_1 \oplus \rho_2$. Then $\deg(\rho) = \deg(\rho_1) + \deg(\rho_2)$

Corollary 1.14 follows directly from the definition of the direct sum of representations.

Theorem 1.15. Let $\rho_1 : G \to \mathrm{Aut}(V_1)$ and $\rho_2 : G \to \mathrm{Aut}(V_2)$ be representations. Let $\rho = \rho_1 \oplus \rho_2$. Then $\chi_\rho(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$ for all $g \in G$.

Regarding theorem 1.15, it is said that the character of the (direct) sum is the sum of the characters.

Yet another definition is given, definition 1.16, as well as a theorem (theorem 1.17):

Definition 1.16. Let $\rho : G \to V$ be a representation. An inner product $\langle \cdot \,|\, \cdot \rangle$ on V is said to be representation invariant under ρ if $\langle \mathbf{v_1} \,|\, \mathbf{v_2} \rangle = \langle \rho(g)(\mathbf{v_1}) \,|\, \rho(g)(\mathbf{v_2}) \rangle$ for all $g \in G$ and $\mathbf{v_1}, \mathbf{v_2} \in V$.

Theorem 1.17. Let $\rho : G \to V$ be a representation, where V is a finite-dimensional inner product space, equipped with a ρ-invariant positive definite Hermitian inner product $\langle \cdot \,|\, \cdot \rangle$. If $W \subseteq V$ is a ρ-invariant subspace of V, then $W^\perp$ is a ρ-invariant subspace of V as well. Moreover, $\rho = \rho_1 \oplus \rho_2$, where ρ_1 acts on W and ρ_2 acts on $W^\perp$, and $V = W \oplus W^\perp$.[12, p. 14]

In order to apply theorem 1.17, a representation invariant positive definite Hermitian inner product is needed. Such an inner product can be made of a positive definite Hermitian inner product $h : V \times V \to \mathbb{C}$, as is stated in lemma 1.18. The proof of lemma 1.18 is clearly treated by B. Steinberg ([14, pp. 21,22]).

Lemma 1.18. Let $h : V \times V \to \mathbb{C}$ be a positive definite Hermitian inner product. Then the inner product $H : V \times V \to \mathbb{C}$ defined as

$$H(\mathbf{v_1}, \mathbf{v_2}) := \frac{1}{|G|} \sum_{g \in G} h(\rho(g)(\mathbf{v_1}), \rho(g)(\mathbf{v_2})) \tag{1.4}$$

is a representation invariant, under the chosen ρ, positive definite Hermitian inner product.

Using lemma 1.18, the following corollary is obtained.

Corollary 1.19. Any representation $\rho : G \to \mathrm{Aut}(V)$ has a basis $\mathbf{b_1} \ldots \mathbf{b_n}$ for V such that the matrix $\rho(g)$ with respect to this basis is unitary for all $g \in G$. Here a matrix A is unitary if $A\overline{A}^T = I$. Moreover, $\chi(g^{-1}) = \overline{\chi(g)}$. [12, p. 13]

As follows from theorem 1.17, a representation ρ on a vector space V that has a representation invariant subspace W under ρ can be written as the direct sum of two subrepresentations. These subrepresentations act on a lower dimensional vector space than $\deg(\rho)$. The vector spaces these representations act on can have representation invariant subspaces themselves. If such a vector space does not have a representation invariant subspace it must be irreducible. Finally, since V is finite-dimensional, the following corollary, corollary 1.20, holds:

Corollary 1.20. Any reducible representation decomposes as the direct sum of irreducible representations. [3, p. 7][12, p. 14]

1.3 Schur's lemma

Most of this section is based on hints given by dr. Jeroen Spandaw. Again, for nuances, inspiration and theory, the mentioned standard books were also consulted.

1.3.1 The definition of a homomorphism of representations

Definition 1.21. Let $\rho_1 : G \to \mathrm{Aut}(V_1)$ and $\rho_2 : G \to \mathrm{Aut}(V_2)$ be two representations. A homomorphism between representations ρ_1 and ρ_2 is a linear map $f : V_1 \to V_2$ such that

$$f(\rho_1(g)(\mathbf{v})) = \rho_2(g)(f(\mathbf{v})), \tag{1.5}$$

for all $g \in G$ and $\mathbf{v} \in V_1$.

If such a homomorphism f between representations as described in definition 1.21 is bijective, then f is said to be an isomorphism of representations and ρ_1 and ρ_2 are said to be isomorphic, denoted by $\rho_1 \cong \rho_2$. This implies that V_1 and V_2 are isomorphic vector spaces, denoted by $V_1 \cong V_2$. In order for f to be an isomorphism of representations, $V_1 \cong V_2$ is necessary, but not sufficient. [12, p. 37]

An extension of this definition of a homomorphism between representations is obtained by setting $\rho_1 : G_1 \to \mathrm{Aut}(V_1)$ and $\rho_2 : G_2 \to \mathrm{Aut}(V_2)$, where $G_1 \neq G_2$. A homomorphism between ρ_1 and ρ_2 is then defined as a linear map $f : V_1 \to V_2$ such that

$$f(\rho_1(g)(\mathbf{v})) = \rho_2(h(g))(f(\mathbf{v})),$$

where $h(g)$ is a group homomorphism from G_1 to G_2.

1.3.2 Properties of homomorphisms of representations

Suppose such a homomorphism f between representations as described in section 1.3.1 exists. We are interested in determining which homomorphisms between representations are possible. At the end of this section Schur's lemma will be presented.

Schur's lemma part 1

let $K \subseteq V_1$ be the kernel of a homomorphism f of representations $\rho_1 : G \to \mathrm{Aut}(V_1)$ and $\rho_2 : G \to \mathrm{Aut}(V_2)$. Let $\mathbf{k} \in K$. By definition, $f(\mathbf{k}) = \mathbf{0}$. This implies that $\rho_2(g)(f(\mathbf{k})) = \mathbf{0}$ for all $g \in G$. This in turn implies that $f(\rho_1(g)\mathbf{k}) = \mathbf{0}$. Hence, $\rho_1(g)\mathbf{k} \in K$. Thus, K is a representation invariant, under ρ_1, vector space. Therefore, K is a subrepresentation of V_1.

By definition $f(\mathbf{v}) \in \mathrm{Im}(f)$ for any $\mathbf{v} \in V_1$. We have $\rho_1(g)\mathbf{v} \in V_1$, since ρ_1 is a representation, thus $f(\rho_1(g)\mathbf{v}) \in \mathrm{Im}(f)$. Hence $\rho_2(g)(f(\mathbf{v})) \in \mathrm{Im}(f)$ and $\mathrm{Im}(f)$ is a subrepresentation of V_2.

Suppose ρ_1 is irreducible. Since K is a subrepresentation of V_1 and ρ_1 is irreducible, we have either $K = \{\mathbf{0}\}$ or $K = V_1$. If $K = V_1$, then $f : \mathbf{v} \mapsto \mathbf{0}$ for all $\mathbf{v} \in V_1$. If however, $K = \mathbf{0}$, then f is injective. Thus, if V_1 is irreducible, then $f = \mathbf{0}$ or f is injective.

Now suppose instead that ρ_2 is irreducible. Since $\mathrm{Im}(f)$ is a subrepresentation of V_2, but ρ_2 is irreducible, we have either $f = \mathbf{0}$ or $\mathrm{Im}(f) = V_2$. If $f = \mathbf{0}$, then $f : \mathbf{v} \mapsto \mathbf{0}$ for all $\mathbf{v} \in V_1$. If however, $\mathrm{Im}(f) = V_2$ then f is surjective. Thus, if V_2 is irreducible, then $f = \mathbf{0}$ or f is surjective.

Now suppose both V_1 as well as V_2 are irreducible. Suppose $f \neq \mathbf{0}$. Then, f is injective, since V_1 is irreducible and f is surjective, since V_2 is irreducible. Hence, $f = \mathbf{0}$

or f is bijective. In other words,

$$\text{Im}(f) = \{\mathbf{0}\} \text{ or } f \text{ is an isomorphism of representations.} \tag{1.6}$$

Now, suppose V_1 and V_2 are both irreducible and $\rho_1 \ncong \rho_2$. f can not be an isomorphism, thus f is the zero-map.

Schur's lemma part 2

Now we consider the case $\rho_1 = \rho_2$. We denote this representation $\rho_1 = \rho_2$ as ρ, which is still assumed to be irreducible, and the corresponding vector space $V_1 = V_2$ as V. Now, suppose again that V is irreducible. Clearly, if f is a multiple of the identity map, then f is a homomorphism of representations. Next it will be shown that these are the only options for a homomorphism of representations f in the above mentioned circumstances.

$f : V \to V$ is a linear map of complex vector spaces, thus f has complex eigenvalues. That is, there exists $\lambda \in \mathbb{C}$ and $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ such that $f(\mathbf{v}) = \lambda\mathbf{v}$. Let λ_0 be such a λ. Indeed,

$$(f - \lambda_0 \cdot \text{id})\mathbf{v} = \mathbf{0}, \tag{1.7}$$

where id is the identity map.

As mentioned, $\lambda_0 \cdot \text{id}$ is a homomorphism of representations. f is a homomorphism of representations as well. Hence, their difference, $f - \lambda_0 \cdot \text{id}$ is also a homomorphism of representations. Since equation 1.7 holds for both the zero vector, $\mathbf{0}$, and at least one other vector $\mathbf{v} \in V$, $\ker(f - \lambda_0 \cdot \text{id}) \neq \mathbf{0}$. Thus this kernel equals V itself. Therefore, $f(\mathbf{v}) = \lambda \cdot \mathbf{v}$ for all $\mathbf{v} \in V$. In conclusion:

$$f = \lambda \cdot \text{id} \tag{1.8}$$

Last of all, suppose again that V_1 and V_2 are irreducible. Moreover, suppose that $\rho_1 \cong \rho_2$, but ρ_1 and ρ_2 are not necessarily identical. Since ρ_1 and ρ_2 are isomorphic there exists an invertible function h, such that $h(\rho_1(g)(\mathbf{v}) = \rho_2(g)(h(\mathbf{v}))$.

Let $f : V_1 \to V_2$ be an arbitrary homomorphism of representations. Since h and f are both homomorphisms of representations, $F := f \circ h^{-1}$ is also a homomorphism of representations. F is a homomorphism of representations from V_2 to V_2, where ρ_2 acts twice on V_2. Indeed $\rho_2(g)((f \circ h^{-1})(\mathbf{v})) = \rho_2(g)(f(h^{-1}(\mathbf{v}))) = f(\rho_2(g)(h^{-1}(\mathbf{v}))) = f(h^{-1}(\rho_2(g)(\mathbf{v}))) = (f \circ h^{-1})(\rho_2(g)(\mathbf{v}))$, where $v \in V_2$. Thus $F = \lambda \cdot \mathbf{v}$. But then $f = F \circ h = (\lambda \cdot \text{id}) \circ h = \lambda h$, which makes f either an isomorphism or the zero map.

Schur's lemma complete

Equations 1.6 and 1.8 give exactly Schur's lemma, which is given in lemma 1.22.

Lemma 1.22 (Schur's lemma)**.** Let $\rho_1 : G \to \text{Aut}(V_1)$ and $\rho_2 : G \to \text{Aut}(V_2)$ be two irreducible representations of a group G. Then,

1. If $f : V \to W$ is a homomorphism of representations, then either $f = \mathbf{0}$ or f is an isomorphism of representations.

2. If $f : V \to V$ is an isomorphism of representations where the same representation ρ works on both vector spaces V, then f is a scalar multiple of the identity isomorphism of representations, i.e. for a particular $c \in \mathbb{C} \setminus \{0\}$: $f(\mathbf{v}) = c\mathbf{v}$ for all $\mathbf{v} \in V$.

Schur's lemma is at the very basis of representation theory as numerous sources, see the introduction, show. In the following sections, 1.4, 1.5, 1.6 and 1.7, more important theorems in representation theory will be presented. It goes too far to prove all of them, hence, for a proof of some theorems there is referred to other sources, again, see the introduction.

In the following sections, the irreducible representations of a group will be described in great detail. Moreover, a method to decompose a representation into its irreducible parts will be shown. Irreducible representations take an important role in the process of grand orthogonalization.

1.4 Schur orthogonality

In this section a method to decompose a representation into its irreducible parts will be shown, accompanied with elementary and powerful theorems. Most important, the Schur orthogonality theorem will be presented.

1.4.1 The decomposition of a representation into irreducible representations is unique

Corollary 1.20 in subsection 1.2.2 can be extended to theorem 1.23, which reads as follows.

Theorem 1.23. The decomposition of a reducible representation into irreducibles is unique. [12, p. 14]

Theorem 1.23 is of great importance in representation theory. Treating any of the infinitely many representations, indeed one can always make a new representation by taking the direct sum of two other representations, can be reduced to treating the irreducible representations it is made up of.

Moreover, we are interested in the exact decomposition of such a reducible representation. To this end, in the next subsection the Schur orthogonality theorem, theorem 1.25, and several corollaries are presented.

1.4.2 Schur orthogonality

First an inner product is defined:

Definition 1.24. Let f_1, f_2 be functions on G. Then,

$$\langle f_1 \mid f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}. \tag{1.9}$$

This inner product will be called the Schur inner product. [12, p. 14]

This inner product is Hermitian. Notice that inner product 1.9 is conjugate linear in the second argument whereas inner product 1.4 is conjugate linear in the first argument. Now in theorem 1.25 the Schur orthogonality theorem is presented.

Theorem 1.25 (Schur orthogonality)**.** Let G be a group with class number k. Then,

1. G has exactly k irreducible representations, $\rho_1, \rho_2, \ldots, \rho_k$.
2. The characters of the irreducible representations, $\chi_1, \chi_2, \ldots \chi_k$, are orthonormal with respect to the Schur inner product.[12, p. 14][9, p. 680][11, p. 15]

Corollary 1.26. An abelian group G of order N has precisely N one-dimensional irreducible representations.

Corollary 1.26 follows directly from theorem 1.25 and the fact that the class number, i.e. the amount of conjugacy classes, of an abelian group equals its order.

The following two very useful corollaries, corollary 1.27 and corollary 1.28, follow from the Schur orthogonality theorem.

Corollary 1.27. The decomposition of any representation ρ of G into irreducible representations, $\rho_1, \rho_2, \ldots, \rho_k$ is given by

$$\rho = m_1\rho_1 \oplus m_2\rho_2 \oplus \cdots \oplus m_k\rho_k.$$

Here, the multiplicities m_j are unique and are determined by

$$m_j = \langle \chi_\rho \,|\, \chi_{\rho_j} \rangle.$$

[12, p. 15]

Corollary 1.28. Let χ be the character of a representation ρ. ρ is irreducible if and only if $\langle \chi_\rho \,|\, \chi_\rho \rangle = 1$, where $\langle \cdot \,|\, \cdot \rangle$ is the Schur inner product. [13]

Corollary 1.28 gives us a criterion that works two ways to check whether or not a representation is (ir)reducible.

Lastly, one more corollary will be presented, corollary 1.29.

Corollary 1.29. Two representations are isomorphic if and only if they have the same character. [12, p. 14]

Indeed, representations are entirely characterized by their character! The study of irreducible representations and their characters is therefore of the highest interest. Indeed, this is where the focus is on in the remaining part of this thesis.

1.4.3 Isotypical vector spaces are orthogonal

Let $\rho = m_1\rho_1 \oplus m_2\rho_2 \oplus \cdots \oplus m_k\rho_k$ be a representation acting on a vector space W. For each $i = 1, 2, \ldots, k$, $m_i\rho_i$ is said to be an isotypical component or isotypical representation of ρ. Let the isotypical component $m_0\rho_0$ of ρ act on the vector space $V_0 \subseteq W$. V_0 is said to be an isotypical vector space of W, ρ or $m_0\rho_0$.

The following theorem, theorem 1.30, holds. This theorem will be proven in this subsection. The proof is made by dr. Jeroen Spandaw.

Theorem 1.30. Let $\rho : G \to \mathrm{Aut}(V)$. Isotypical subspaces of V are orthogonal when considering a ρ-invariant positive definite Hermitian inner product.

To prove theorem 1.30, let ρ_1 be an irreducible subrepresentation of a representation ρ, which acts on V. Let S be the sum of all ρ-invariant subspaces of V that ρ_1 acts on, so S is ρ-invariant itself, and let U be any irreducible subspace of S that ρ_1 acts on. Let ρ_2 be an irreducible subrepresentation of ρ which is not isomorphic to ρ_1. Let S' be the sum of all ρ-invariant subspaces of V that ρ_2 acts on, so S' is ρ-invariant itself, and let U' be any irreducible subspace of S' that ρ acts on.

S and S' are indeed isotypical vector spaces: Suppose there exists such a vector space U in V. If U is unique, then $S = U$. Otherwise there exists another such vector space in V, say $\tilde{U}$. By irreducibility of U and $\tilde{U}$, $U \cap \tilde{U} = \{\mathbf{0}\}$, so $U \oplus \tilde{U} \subseteq S$. If $U \oplus \tilde{U} \subset S$, then there exists yet another such vector space U. This process terminates as S is finite-dimensional. Hence $S = \bigoplus \rho_1$. The same argument holds for S'. Now, $V = S \oplus S' \oplus \ldots$.

It has to be proven that any non-isomorphic irreducible subrepresentations of V, here noted as U and U', are H-orthogonal on V. Here orthogonality is taken with respect to any chosen ρ-invariant positive definite Hermitian inner product $H : V \times V \to \mathbb{C}$.

To this end, let $\pi : V \to S$ be the H-orthogonal projection from V on S. In other words, if $\mathbf{v} \in V$, $\mathbf{v}$ can be uniquely decomposed as $\mathbf{v} = \mathbf{v_1} + \mathbf{v_2}$, where $\mathbf{v_1} \in S$ and $\mathbf{v_2} \in S^{\perp}$. Now, $\pi(\mathbf{v}) = \mathbf{v_1}$. Let $p : U' \to S$ be the same function as π, but with a smaller domain.

It will be proven that the function π is a homomorphism of representations, i.e. that $\pi(\rho(g)\mathbf{v}) = \rho(g)(\pi(\mathbf{v}))$ for all $v \in V$. This is the same as proving that $\pi(\rho(g)(\mathbf{v_1} + \mathbf{v_2})) = \rho(g)(\mathbf{v_1})$, which is, due to the linearity of ρ, the same as proving that $\pi(\rho(g)(\mathbf{v_1}) + \rho(g)(\mathbf{v_2})) = \rho(g)(\mathbf{v_1})$. By the linearity of π, this is the same as proving that

$$\pi(\rho(g)(\mathbf{v_1})) + \pi(\rho(g)(\mathbf{v_2})) = \rho(g)(\mathbf{v_1}).$$

Since $\mathbf{v_1} \in S$ and since S is ρ-invariant, $\rho(g)\mathbf{v_1}$ is also in S. Hence $\pi(\rho(g)(\mathbf{v_1})) = \rho(g)(\mathbf{v_1})$. Thus is has to be proven that $\pi(\rho(g)(\mathbf{v_2})) = 0$. This is true: Indeed, let $\mathbf{v_2} \in S^{\perp}$, then $\rho(g)\mathbf{v_2} \in S^{\perp}$ for all $g \in G$, since $S^{\perp}$ is ρ-invariant (see theorem 1.17).

In conclusion, $\pi : V \to S$ is a homomorphism of representations. Then the restriction of V to the ρ-invariant subspace U' of V, $p : U' \to S$ is a homomorphism of representations as well.

It will be shown that the only homomorphism of representations between U' and S is the zero-map. To this end let q be an arbitrary homomorphism of representations between U' and S. Indeed, due to the irreducibility of U', either $\ker(q) = \{0\}$ or $\ker(q) = U'$. Suppose $\ker(q) = \{0\}$. Then q is injective. Hence, $q' : U' \to \mathrm{Im}(U')$ is bijective and $\mathrm{Im}(U') \subseteq S$ is isomorphic to U'. However, by uniqueness of decomposition of representations, see corollary 1.27, the only irreducible subrepresentations of S are ρ_1, not ρ_2, which acts on U'. Hence $\mathrm{Im}(U')$ can not be isomorphic to a subrepresentation of S.

The only possibility is that $\ker(q) = U'$, or, in other words, that $q : U' \to S$ is the zero-map. Indeed, $p : U' \to S$ must be the zero-map as well. So every $v \in U'$ splits as $v = 0 + v_2$ with $v_2 \in S^{\perp}$. Hence, all elements of U' are H-orthogonal to S. Therefore $S \perp_H S'$. We conclude that any non-isomorphic isotypical vector spaces are orthogonal under H, which is what had to be proven.

1.5 Dual representation

Every representation ρ has a so-called dual representation. Its definition is given as follows:

Definition 1.31. Let $\rho : G \to \mathrm{Aut}(V)$ be a representation and let $f : V \to \mathbb{C}$ be an element of $\mathrm{Hom}(V, \mathbb{C})$. The dual representation of ρ, denoted as ρ^*, acts on $\mathrm{Hom}(V, \mathbb{C})$. Given $g \in G$ and $f \in \mathrm{Hom}(V, \mathbb{C})$, the dual representation is defined by:

$$\rho^*(g) \bullet f : \mathbf{v} \to f(\rho(g^{-1})(\mathbf{v})).$$

The vector space $\mathrm{Hom}(V, K)$ is also called the dual vector space and is noted as V^*.

The definition of the dual representation is of the same form as mentioned in definition 1.1. When denoting the dual representation as in definition 1.2, we get $\rho^* : G \to \mathrm{Aut}(\mathrm{Hom}(V, \mathbb{C}))$. For any representation $\rho : G \to \mathrm{Aut}(V)$, such a dual representation ρ^* exists.

It is true that this definition actually gives rise to a representation. To this end,

$$\begin{aligned}
\rho^*(g_2 g_1) \bullet f &= \mathbf{v} \to f(\rho^*((g_2 g_1)^{-1})(\mathbf{v})) \\
&= \mathbf{v} \to f(\rho^*(g_1^{-1} g_2^{-1})(\mathbf{v})) \\
&= \mathbf{v} \to f((\rho^*(g_1^{-1})\rho^*(g_2^{-1}))(\mathbf{v})) \\
&= \mathbf{v} \to f((\rho^*(g_1^{-1})(\rho^*(g_2^{-1})(\mathbf{v})) \\
&= \mathbf{v} \to (\rho^*(g_1) \bullet f)((\rho^*(g_2^{-1})(\mathbf{v})) \\
&= \rho^*(g_2) \bullet (\rho^*(g_1) \bullet f)
\end{aligned}$$

The following theorem can be attained (theorem 1.32):

Theorem 1.32. Let ρ be a representation with character χ. The character of its dual, $\rho*$, equals $\overline{\chi}$. [12, p. 26]

1.6 Tensor products

The tensor product can be seen, somewhat analogously to the direct sum, as a direct product. In this section the tensor product will be properly defined. Then an important theorem will be stated, namely that characters can be multiplied, forming again a representation.

In order to achieve this, first definition 1.33 will be given.

Definition 1.33. Let $F : V \times W \to Z$ be a bilinear map, where V, W and Z are vector spaces. (Z, F) is said to be universal if for any $G : V \times W \to U$, there is a unique linear map $\phi : Z \to U$ such that $G = \phi \circ F$. [3, p. 471][12, p. 28]

Such a universal bilinear map as defined in definition 1.33 can be shown to exist [12, p. 28] and be unique up to a unique linear isomorphism. This gives rise to definition 1.34.

Definition 1.34. Let V be a n-dimensional vector space and let W be an a m-dimensional vector spaces. The tensor product of V and W is the unique, up to unique linear isomorphisms, vector space denoted as $V \otimes W$ such that there is a universal bilinear mapping $F : V \times W \to V \otimes W, \mathbf{v} \times \mathbf{w} \mapsto \mathbf{v} \otimes \mathbf{w}$, with $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Moreover, let $\mathbf{b_1}.\mathbf{b_2} \dots \mathbf{b_n}$ be a basis for V and let $\mathbf{c_1}.\mathbf{c_2} \dots \mathbf{c_m}$ be a basis for W. Also, let $V \otimes W := \mathbb{C}^{nm}$ be the vector space that consists of $n \times m$-matrices and where $\mathbf{e_{i,j}}$ is the matrix with value 1 for the element on the i-th row and j-th columns and all zeros elsewhere. Now the unique universal bilinear map $F : V \times W \to V \otimes W$ is given by: $F : V \times W \to V \otimes W, F(\mathbf{b_i}, \mathbf{c_j}) = \mathbf{e_{i,j}}$. Extending this function bilinearly gives:

$$F\left(\sum_{i=1}^{n} x_i \mathbf{b_j}, \sum_{j=1}^{m} y_j \mathbf{c_j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_i \mathbf{e_{i,j}}.$$

Indeed $V \otimes W$ has dimension nm. $F(v, w)$ with $v \in V$ and $w \in W$ is denoted as $v \otimes w$. [12, p. 29]

Corollary 1.35. if $\mathbf{b_1}.\mathbf{b_2} \dots \mathbf{b_n}$ be a basis for V and if $\mathbf{c_1}.\mathbf{c_2} \dots \mathbf{c_m}$ be a basis for W. Then $\mathbf{b_1} \otimes \mathbf{c_1}, \dots, \mathbf{b_i} \otimes \mathbf{c_j}, \dots, \mathbf{b_n} \otimes \mathbf{c_m} = \mathbf{e_{1,1}}, \dots, \mathbf{e_{i,j}}, \dots, \mathbf{e_{n,m}}$ is a basis for $V \otimes W$. [3, p. 471][12, p. 29]

Somewhat analogous to the definition of direct sums of functions and representations of the same group in subsection 1.2.1, tensor products for functions whose domain is a vector space and for representations of the same group will be defined.

Definition 1.36. Let $f : A \to V_1$ and $g : C \to V_2$ be two arbitrary functions. Then:

$$(f \otimes g)(a, c) := f(a) \otimes g(c),$$

with $(a, c) \in (A, C)$.

However, in contrast to definition 1.36, the direct sum domain of two representation is taken as G instead of (G, G). Now:

Definition 1.37. let $\rho_1 : G \to \mathrm{Aut}(V_1)$ and $\rho_2 : G \to \mathrm{Aut}(V_2)$ be two representations of the same group G. Now,

$$(\rho_1 \otimes \rho_2) := g \mapsto \rho_1(g) \otimes \rho_2(g),$$

or equivalently:

$$(\rho_1 \otimes \rho_2) = g \mapsto \rho_1(g) \otimes \rho_2(g).$$

The tensor product of two representations $\rho_1 : G \to \mathrm{Aut}(V_1)$ and $\rho_2 : G \to \mathrm{Aut}(V_2)$ as defined in definition 1.37 is again a representation. The codomain of $\rho_1 \otimes \rho_2$ is the automorphism group of $V \otimes W$.

The following useful theorem, theorem 1.38, that is analogous to theorem 1.15, holds:

Theorem 1.38. Let $\rho_1 : G \to \text{Aut}(V_1)$ and $\rho_2 : G \to \text{Aut}(V_2)$ be representations. Let $\rho = \rho_1 \otimes \rho_2$. Then $\chi_\rho(g) = \chi_{\rho_1}(g) \cdot \chi_{\rho_2}(g)$ for all $g \in G$. [9, p. 668]

Regarding theorem 1.38, it is said that the character of the (tensor) product is the product of the characters. To this end, the multiplying of characters gives rise to a new, possibly reducible, representation.

1.7 The regular representation

Let $G = \{g_1, g_2, \ldots g_N\}$, $N = |G|$, be a group. There exists a faithful representation of G of dimension $|G|$, named the regular representation (ρ). Let $\{\mathbf{e_1}, \mathbf{e_2}, \ldots \mathbf{e_N}\}$ be the standard basis for the $|G|$-dimensional vector space C^N. The regular representation permutes all basis vectors, with respect to the corresponding group operations. In other words, let $g_i, g_j, g_k \in G$. If $g_i \circ g_j = g_k$, then we define

$$\rho(g_i)(\mathbf{e_j}) := \mathbf{e_k}$$

. The word 'regular' means 'transitive' as well as 'free', where 'transitive' means:

$$\text{For all}\, x, y \in X, \text{ there exists a}\, g \in G\, \text{such that}\, g \bullet x = y.$$

And were 'free' means:

$$\text{If}\, g \bullet x = x, \text{ then}\, g = e.$$

[10, p. 23] Here, X is the set of the standard basis vectors.

The character of the regular representation is as follows. Since ρ permutes the standard basis vectors and since, if $g \neq e$, $\rho(g)(\mathbf{e_i}) \neq \mathbf{e_i}$, all diagonal elements of $\rho(g)$ equal 0, hence $\chi_\rho(g) = 0$ for all $g \in G \backslash \{e\}$. If $g = e$, then $\chi_\rho(g) = \deg(\rho) = N$ by corollary 1.10. This is stated in lemma 1.39.

Lemma 1.39. Let ρ be the regular representation of G. The character of ρ equals $(N, 0, 0, \ldots, 0)$. Here $N = |G|$.

Using lemma 1.39, the decomposition of the regular representation into irreducible components will be obtained. The multiplicities of the irreducible representations $\rho_i, i = 1, 2, \ldots, k$ where k is the class number of G are obtained by taking the Schur inner product of ρ_i with ρ, by corollary 1.27. To this end:

$$\begin{aligned} m_i = \langle \chi_\rho \,|\, \chi_{\rho_j} \rangle &= \frac{1}{N} \sum_{g \in G} \chi_\rho(g) \overline{\chi_{\rho_i}(g)} \\ &= \frac{1}{N} (N \cdot \chi_{\rho_i}(e) + 0 \cdot \chi_{\rho_i}(g_2) + \cdot + 0 \cdot \chi_{\rho_i}(g_N)) \\ &= \frac{1}{N} (N \cdot \chi_{\rho_i}(e)) \\ &= \deg(\rho_i). \end{aligned}$$

This results in corollary 1.40.

Corollary 1.40. The multiplicity m_i of the i-th irreducible representation ρ_i in the regular representation ρ is equal to $\deg(\rho_i)$, i.e.

$$\rho = d_1\rho_1 \oplus d_2\rho_2 \oplus \cdots \oplus d_k\rho_k.$$

Here $d_i = \deg(\rho_i)$ and k is the class number of G. Moreover,

$$\chi_\rho = d_1\chi_{\rho_1} + d_2\chi_{\rho_2} + \cdots + d_k\chi_{\rho_k}.$$

Evaluating this in $g = e$ gives

$$|G| = \chi_\rho(e) = \sum_{i=1}^{k} d_i\chi_{\rho_i}(e) = \sum_{i=1}^{k} d_i^2. \tag{1.10}$$

[12, p. 18]

The regular representation will take a prominent role in the remaining part of this thesis. In chapters 3 and 4, the vector space $\mathbb{C}^N$, $N = |G|$, G acts on will be decomposed as the direct sum of ρ-invariant vector spaces. Indeed, $\mathbb{C}^N$ will be splitted into $\sum_{i=1}^{k} d_i =: S$ irreducible representation invariant vector spaces. In each chapter of chapters 3 and 4 a method in order to achieve this will be described.

In lemma 1.18 an ρ-invariant inner product H was defined. When considering the regular representation (ρ), the standard complex inner product is ρ-invariant as well. This is stated in theorem 1.41. From now on, when referring to orthogonal vectors, this is meant in the context of the standard complex inner product.

Theorem 1.41. The standard complex inner product is representation invariant for the regular representation.

1.8 Ways to construct a character table

In subsection 1.8.1 will be defined what a character table is. Calculating character tables can be quite cumbersome. So far several helpful ways to find new characters were discussed. In this section, even more methods will be presented in order to find new characters.

1.8.1 What is a character table?

As noted in subsection 1.4.2, the study of irreducible representations and their characters is of the highest interest. A character table is a way to present all characters of all irreducible representations in a convenient way. A character table is of the following form:

Table 1.1: The general form of a character table. Here '$\chi_{\rho_i}(\text{Conjugacy class}_j)$' means the value of $\chi_{\rho_i}(g)$ for some g in conjugacy class j.

	Conjugacy class$_1$	Conjugacy class$_2$	...	Conjugacy class$_k$
ρ_1	χ_{ρ_1}(Conjugacy class$_1$)	χ_{ρ_1}(Conjugacy class$_2$)	...	χ_{ρ_1}(Conjugacy class$_k$)
ρ_2	χ_{ρ_2}(Conjugacy class$_1$)	χ_{ρ_2}(Conjugacy class$_2$)	...	χ_{ρ_2}(Conjugacy class$_k$)
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ρ_k	χ_{ρ_k}(Conjugacy class$_1$)	χ_{ρ_k}(Conjugacy class$_2$)	...	χ_{ρ_k}(Conjugacy class$_k$)

As, by lemma 1.5, all elements of a conjugacy class have the same character, the different conjugacy classes are considered in a character table rather than the individual group elements $g \in G$. Moreover, since the number of irreducible representations of G is the same as its class number, a character table is square.

A character table can be expanded by adding for example the orders of the elements in a conjugacy class, which are the same for all elements in the same conjugacy class. Moreover, the order of the conjugacy class itself can be added. In the rest of this chapter several methods that can help to construct a character table will be presented.

1.8.2 Orthogonality of the columns of a character table

As already seen in theorem 1.25, the rows of a character table are orthonormal with respect to the Schur inner product. However, the columns are also orthogonal, as stated in the following corollary (1.42).

Corollary 1.42. The columns of a character table are orthogonal considering the standard complex inner product. The columns are even orthonormal considering the following inner product for two columns c_i and c_j:

$$\langle c_i \,|\, c_j \rangle = \frac{\sqrt{|C_i||C_j|}}{|G|} \sum_{l=1}^{k} a_{l,i}\overline{a_{l,j}}.$$

Here $a_{i,j}$ is an entry of the matrix A which represents the character table of a group G.

1.8.3 Basic methods to construct a character table

First of all, equation 1.10 in section 1.7 is of great use. Squaring the first column of the character table component-wise and adding all these squares gives the order of G. This rule reduces the amount of possible dimensions the irreducible representations can have considerably. This method is used extensively and is a great way to start any character table.

The trivial representation is the representation that sends all elements of G to e_2, where e_2 is the identity element of V G acts on. Choosing a basis for V, the trivial representation sends all elements of G to 1. For any group G, the trivial representation is always the first row in the character table.

When considering one-dimensional representations, the characters of these representations are the same as the 1×1-dimensional matrices $\rho(g)$, with $g \in G$, no matter the chosen basis for V ρ acts on. Every $\rho(g)$ only has one degree of freedom. It is worth trying to make equations that enormously reduce the possible one-dimensional representations to a treatable amount of options.

Corollary 1.19 stated that $\chi(g^{-1}) = \overline{\chi(g)}$. This fact can help in making a character table.

In section 1.6 we have seen that characters can be multiplied. When multiplying characters new, possible reducible, representation might be formed. In practice, this method proves to be very useful.

Suppose $\rho_A = m_1\rho_1 \oplus m_2\rho_2 \oplus \cdots \oplus m_k\rho_k$ and $\rho_B = n_1\rho_1 \oplus n_2\rho_2 \oplus \cdots \oplus n_k\rho_k$ are two representations of which the vector spaces $V_1 \subseteq W$ and $V_2 \subseteq W$ they act on respectively are known. The decomposition of ρ_A and ρ_B might not be known. It can be easier to decompose the representation that works on $\rho_A \cap \rho_B$ (see corollary **??**), provided that $\rho_A \cap \rho_B \neq \{\mathbf{0}\}$.

1.8.4 Filtering out known irreducible representations

Given is a representation $\rho = m_1\rho_1 \oplus \cdots \oplus m_l\rho_l$ where $\rho_1, \ldots, \rho_l$ are irreducible representations. Suppose we know the character of $l-1$ of these irreducible representations. Beforehand, it is not known what the value of l is, whether or not these $l-1$ characters correspond to irreducible representations and whether or not they partake in the direct sum of ρ.

In order to determine whether or not such a representation ρ_i, $i = 1, \ldots, l-1$ is irreducible, we calculate the Schur inner product of ρ_i with itself, i.e $\langle \chi_{\rho_i}, \chi_{\rho_i} \rangle$. As seen in corollary 1.28, χ_{ρ_i} is irreducible if and only if $\langle \chi_{\rho_i}, \chi_{\rho_i} \rangle = 1$.

As we have seen in corollary 1.27 in subsection 1.4.2, we can check the multiplicity of an irreducible representations ρ_i within the representation ρ by taking their Schur inner product $\langle \chi_{\rho_i}, \chi_\rho \rangle$. Suppose this irreducible representation ρ_i is in fact part of the decomposition of ρ into irreducible representations, then $\chi_\rho - m_i\chi_{\rho_i}$ is again a representation.

Check whether or not $\chi_\rho - m_i\chi_{\rho_i}$ is irreducible. If this representation is irreducible, then the obtained character might be new and can be added to the character table. If this representation is reducible, then take another known irreducible representation and check whether or not this irreducible representation is part of the decomposition of the newly obtained reducible representation. This can be done until a newly obtained representation is irreducible. If a newly obtained representation is indeed irreducible, we have successfully 'filtered out' all the other irreducible representations in ρ with respect to their multiplicities.

Remark 1.43. In fact, any representation that is a subrepresentation of ρ can be 'filtered out'.

1.8.5 Taking the orthogonal complement of W in V

Suppose a vector space W is a representation invariant subspace of V, where ρ acts on. By theorem 1.17, $W^\top$ in V is a subrepresentation of V. Suppose moreover that the images of ρ and the subrepresentation of ρ acting on W are known. The character of the subrepresentation of ρ acting on $W^\top$ might be new and can be added to the character table. Indeed, this is the same as 'filtering out' the representation acting on W from ρ. However, now the actual representation acting on $W^\top$ is known, not just the character.

1.8.6 The pulling back method

Let G be a group of which we want to state the character table. Let N be a normal subgroup of G. Hence, G/N is a quotient group. Suppose we know a representation of G/N, that is, we have a homomorphism $\rho : G/N \to \text{Aut}(V)$, where V a vector space. There also exists at least one group homomorphism from G to G/N, namely the canonical (or natural) homomorphism: $\phi : G \to G/N, \phi(a) = aN$.

Consider lemma 1.44.

Lemma 1.44. Let G_1, G_2 be two groups and let $f : G_1 \to G_2$ be a group homomorphism. Then all elements in a conjugacy class of G_1 are mapped to one conjugacy class of G_2.

Now, $\rho_2 = \rho \circ \phi$ is again a homomorphism. Let C be a conjugacy class of G. $\rho(C)$ is, or is part of, a conjugacy class in G/N by lemma 1.44, which has a single character in the character table. Hence, only by looking at the function ρ, the characters in the character table of G/N can be transferred to the character table of G, taking into account which conjugacy classes of G are mapped to which conjugacy classes of G/N by ρ. We call this process 'pulling back'.

Chapter 2

The groups C_n, V_4, S_3, Q_8, D_4, T and $2T$

In this chapter several group tables will be made using the theorems and methods described in section 1. The groups that will be treated are C_n, V_4, S_3, D_4, Q_8 and $T \cong A_4$. Certain elementary properties of these groups will be regarded as known, such as their conjugacy classes and normal subgroups. The other rotation groups, besides T, of the Platonic solids as well as their double covers will briefly be introduced. Moreover, the explicit images of the irreducible representations of D_4 and Q_8 will be determined. In chapter 3 and chapter 4 two methods to find representation invariant subspaces of $\mathbb{C}^N$, where N is the order of a group G, will be presented.

2.1 The character table of C_n

As described in theorem 1.25 in subsection 1.4.2, the number of irreducible representations of a group equals it's class number, i.e. the amount of conjugacy classes G has. As C_n is an abelian group, it has n conjugacy classes, thus n irreducible representations. Intuitively, as C_n denotes circular operations, rotations of the plane will be a good starting point. Indeed, multiplying with one-dimensional complex matrices, given a basis for V with $\dim(V) = 1$, of the form

$$\exp\left(ai\frac{2\cdot\pi}{n}\right), \tag{2.1}$$

with $a = 1, 2 \ldots n$, agrees with the representation requirements.

To this end, the group elements of C_n are numbered agreeing to their power. g_1 generates all of C_n. g_1 can be any element of expression 2.1. Hence all n irreducible representations are already given. The character table is given in table 2.1.

Table 2.1: Character table of C_n

Conjugacy class	$\{e\}$	$\{c^1\}$	$\dots$	$\{c^{n-1}\}$
χ_1	1	$\exp\left(i\frac{2\cdot\pi}{n}\right)$	$\dots$	$\exp\left((n-1)i\frac{2\cdot\pi}{n}\right)$
χ_2	1	$\exp\left(2i\frac{2\cdot\pi}{n}\right)$	$\dots$	$\exp\left((n-2)i\frac{2\cdot\pi}{n}\right)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_n	1	$\exp\left((n-1)i\frac{2\cdot\pi}{n}\right)$	$\dots$	$\exp\left(i\frac{2\cdot\pi}{n}\right)$

2.2 The character table of V_4

As V_4 is abelian, it's class number is the order of the group itself, which is four. Let the elements of V_4 be denoted as e, a, b and c, where e is the identity element. As these representations are all one-dimensional, the following equations hold:

$$\chi_\rho(g)^2 = \chi_\rho(e) = 1, \tag{2.2}$$

for $g \in \{a, b, c\}$. Hence, $a, b, c \in \{1, -1\}$. Moreover, the following equation holds:

$$\chi_\rho(a) \cdot \chi_\rho(b) = \chi_\rho(c) \tag{2.3}$$

These two equations, 2.2 and 2.3, result in the character table in table 2.2.

Table 2.2: Character table of V_4

Conjugacy class	$\{e\}$	$\{a\}$	$\{b\}$	$\{c\}$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

2.3 The character table of S_3

More often than not there are several ways in which a character table can be obtained. In this section three ways to obtain the character table of S_3 will be shown. In subsection 2.3.1 equations related to one-dimensional representations and Schur orthogonality are used. In subsection 2.3.2 a three-dimensional interpretation and a check for reducibility are used. Lastly, in subsection 2.3.3 the same 3-dimensional interpretation as in subsection 2.3.2 is used, however instead of checking for reducibility, an orthogonal complement is taken.

There are more ways to find the character table of S_3 than the ones presented here. Multiple ways are shown to give the reader a view on the ways several lemma's, theorems and corollaries described in chapter 1 can be used. What is more, in certain situations certain methods work better than others. By presenting several ways, the process of making character tables and all that can come with it is treated in a more comprehensive way.

The conjugacy classes of S_3 and the dimension of its representations

The conjugacy classes of S_3 are: $\{(1)\}, \{(12), (13), (23)\}$ and $\{(123), (132)\}$. Hence, S_3 has three different irreducible representations. By corollary 1.40 $|S_3| = \sum_{i=1}^{3} d_i^2$. Hence, the representations ρ_1, ρ_2, ρ_3 have degree one, one and two respectively.

2.3.1 One-dimensional representations and Schur orthogonality

The first representation is again the standard representation. The second representation is one-dimensional, so we try to make equations that result in the character, and even image, of ρ_2. The following equations hold:

$$\chi_{\rho_2}(1) = \chi_{\rho_2}((123)(123)(123)) = \chi_{\rho_2}(((123))^3$$
$$\chi_{\rho_2}(1) = \chi_{\rho_2}((123)(132)) = \chi_{\rho_2}((123))^2$$

Hence, $\chi_{\rho_2}((123)) = 1$. Now, since $\chi_{\rho_2}((12)(12)) = \chi_{\rho_2}(e)$, either $\chi_{\rho_2}((12)) = 1$ or $\chi_{\rho_2}((12)) = -1$. The first option gives the standard representation, hence the second option gives the second representation.

Now using Schur orthogonality, the last character is obtained, which makes the character table of S_3 complete, see table 2.3.

Table 2.3: Character table of S_3

Conjugacy class	$\{e\}$	$\{(12)\}$	$\{(123)\}$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Using equations for one-dimensional representations does not always complete a character table. That is why another way to make the character table of S_3 will be presented.

2.3.2 Filtering out irreducible representations

Let S_3 act on $\mathbb{C}^3$ as described in subsection 1.1.2. The matrices $\rho((12))$ and $\rho((123))$ are then as in equation 1.2. The following character is obtained: $\chi : (3, 1, 0)$. This character is reducible since $3^2 > 6$ or alternatively, since $\langle \chi \,|\, \chi \rangle \neq 1$.

The trivial character, χ_{trivial} is a subcharacter of χ. Indeed, $\langle \chi_{\text{trivial}} \,|\, \chi \rangle = 1 \neq 0$. Therefore, $\chi_{\text{new}} = \chi - \chi_{\text{trivial}}$ is again a character of a representation of S_3. This character, however, is irreducible. Indeed $\langle \chi_{\text{new}} \,|\, \chi_{\text{new}} \rangle = 1$. Again, for example, Schur orthogonality can be used to obtain the last character of S_3.

2.3.3 Taking an orthogonal complement

Again, let S_3 act on $\mathbb{C}^N$ as described in subsection 1.1.2. Notice that the trivial representation acts on the space $W = \text{Span}(1, 1, 1)^\top \subset \mathbb{C}^3$. By theorem 1.17, $W^\perp$ in V is also acted on by a representation, say ρ', of S_3.

Vectors $(1,-1,0)^\top$ and $(1,0,-1)^\top$ form a basis for $W^\perp$. Making matrices $\rho'((12))$ and $\rho'((23))$ with respect to this basis gives:

$$\rho((12)) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad \rho((23)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Again, the character $\chi = (2,0,1)$ is obtained. Moreover, all matrices $\rho'(g)$, for all $g \in S_3$ are obtained for a particular basis. Again, for example, Schur orthogonality can be used to obtain the last character of S_3.

2.4 The character tables of Q_8 and D_4

The group Q_8 consists of the quaternion elements

$$1, -1, i, -i, j, -j, k, -k$$

and the group D_4 consists of all rotations (r) and reflections (s) of the square, i.e.

$$e, r, r^2, r^3, s, r \circ s, r^2 \circ s, r^3 \circ s,$$

where e is the identity element. Both of these groups have the same order, and, as we will see, the character tables of Q_8 and D_4 are the same. Hence, they are treated in the same section.

We start by stating the conjugacy classes of both Q_8 and D_4. The conjugacy classes of Q_8 are

$$(\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}).$$

The conjugacy classes of D_4 are

$$(\{e\}, \{r^2\}, \{r, r^3\}, \{s, r^2 \circ s\}, \{r \circ s, r^3 \circ s\}).$$

Now we take a look at the dimensions d_i of the irreducible representations of D_4. Since D_4 has 5 conjugacy classes, it also has 5 irreducible representations, see theorem 1.25. We know that by corollary 1.40, $|D_4| = \sum_{i=1}^{5} d_i^2$. All d_i are in $\mathbb{N}$, so the only possible option based on this theorem is: $(d_1, d_2, d_3, d_4, d_5) = (1,1,1,1,2)$. The same holds for the group Q_8.

Note that e, r^2 is a normal subgroup of D_4. Hence $D_4/\{e, r^2\} \cong C_2 \times C_2 \cong V_4$ is a quotient group of D_4. Using the pulling back technique as described in section 1.8.6 and knowing the characters of the irreducible representations of $C_2 \times C_2 \cong V_4$ beforehand, we already know the following four characters of D_4: $(1,1,1,1,1), (1,1,1,-1,-1), (1,1,-1,1,-1)$ and $(1,1,-1,-1,1)$.

Now, by using Schur orthogonality of irreducible representations, the last character of the character table of D_4 can be calculated: $(2,-2,0,0,0)$. Hence, we know the characters of all irreducible representations of D_4. The character table is given in table 2.4.

Table 2.4: Character table of D_4

Conjugacy class	$\{e\}$	$\{r^2\}$	$\{r, r^3\}$	$\{s, r^2 \circ s\}$	$\{r \circ s, r^3 \circ s\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

The group Q_8 has $\{1, -1\}$ as normal subgroup. We have $Q_8/\{1, -1\} \cong V_4$. Again, using the pulling back method as described in section 1.8.6 and using Schur orthogonality, we find the same character table for Q_8 as for D_4, see table 2.5.

Table 2.5: Character table of Q_8

Conjugacy class	$\{1\}$	$\{-1\}$	$\{i, -i\}$	$\{j, -j\}$	$\{k, -k\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

2.5 The Platonic solids

Platonic solids are the three dimensional objects that consist of regular polygons. Subsections 2.5.1 and 2.5.2 will be mainly based on the thesis of G.M.C. van Ittersum.

2.5.1 The groups $SO(3)$ and $SU(2)$

The special orthogonal group, denoted by $SO(3)$, is the group of linear rotations of a three-dimensional real vector space. Its elements can be represented by all orthonormal 3×3-matrices. The group operation is defined by composition of linear mappings. The group $SO(3)$ is at the basis of the rotation groups of the Platonic solids as soon will be shown.

The group $SU(2)$ consists of all 2×2-dimensional unitary matrices with $\det = 1$. Each of these matrices can be represented by a unit quaternion, also known as a versor. The group of unit quaternions, which is isomorphic to $SU(2)$, is denoted as $\mathbb{H}_1$. [2]

A rotation of a vector in three-dimensional space can be interpreted as a quaternion multiplication. To this end, given an orthonormal basis, let $\mathbf{x} = (x_1, x_2, x_3)$ be a vector in a three-dimensional vector space V. To apply quaternion multiplication, with $q \in \mathbb{H}_1$, the vector $\mathbf{x}$ is denoted as the quaternion $0 + x_1 i + x_2 j + x_3 k$. Indeed, qxq^{-1} is again of the form $0+ai+bj+ck$, which will be the vector (a, b, c) in V with respect to the chosen basis.

In fact the mapping $SU(2) \to SO(3), \rho(q) : x \to qxq^{-1}$ is a group homomorphism. This homomorphism is two-to-one [17, p. 113],[13]. This two-to-one map can be used to form so called binary groups of the rotation groups of the Platonic solids. These binary groups are denoted as $2T$, $2O$ and $2I$. Their order is twice the order of their rotational counterpart. $2T$, $2O$ and $2I$ are said to be double covers of T, O and I respectively. They are the pre-image of, respectively, the groups T, O and I under the given two-to-one group homomorphism. These six groups will be further illustrated in subsection 2.5.2.

2.5.2 The groups T, $2T$, O, $2O$, I and $2I$

The groups T, O and I are the rotation groups of the Platonic solids in $\mathbb{R}^3$. The group T rotates the tetrahedron, the group O rotates the cube and its dual, the octahedron, and the group I rotates the icosahedron as well as its dual, the dodecahedron. T, O and I are respectively called the tetrahedral group, the octahedral group and the icosahedral group. As these groups rotate the three-dimensional Platonic solids such that the Platonic solids are mapped onto themselves in three-dimensional space and preserve orientation, these groups are finite subgroups of $SO(3)$ [17, p. 141], [8, p. 3]. In fact, T, O and I are, besides the cyclic groups C_n and the dihedral groups D_n, the only subgroups of $SO(3)$[17, p. 114]. Moreover, by the two-to-one mapping from $SU(2)$ to $SO(3)$, the groups $2T$, $2O$ and $2I$ are subgroups of $SU(2)$.

2.5.3 The character table of $T \cong A_4$

The tetrahedral group T is isomorphic to A_4 [7, p. 16]. To this end, we consider the group A_4. The conjugacy classes of A_4 are

$$\begin{aligned} 1 &= \{e\} \\ 2 &= \{(12)(34), (13)(24), (14)(23)\} \\ 3a &= \{(123), (134), (142), (243)\} \\ 3b &= \{(132), (143), (124), (234)\}. \end{aligned}$$

Here $1, 2$ and 3 stand for the order of the elements in that particular conjugacy class.

Since A_4 has 4 conjugacy classes, it also has 4 irreducible representations, see theorem 1.25. We know that by corollary 1.40, $|A_4| = \sum_{i=1}^{4} d_i^2$. All d_i are in $\mathbb{N}$, so the only possible option based on this theorem is: $(d_1, d_2, d_3, d_4) = (1, 1, 1, 3)$.

Note that V_4 is a normal subgroup of A_4. Hence $A_4/\{V_4\} \cong C_3$ is a quotient group of A_4. Using the pulling back technique as described in section 1.8.6 and knowing the characters of the irreducible representations of C_3 beforehand, we know already the following three characters of A_4: $(1, 1, 1, 1), (1, 1, \exp\left(i\frac{2\cdot\pi}{3}\right), \exp\left(i\frac{4\cdot\pi}{3}\right))$ and $(1, 1, \exp\left(i\frac{4\cdot\pi}{3}\right), \exp\left(i\frac{2\cdot\pi}{3}\right))$.

Now, by using Schur orthogonality of irreducible representations, the last character of the character table of A_4 can be calculated: $(3, -1, 0, 0)$. Hence, we know the characters of all irreducible representations of A_4. The character table is given in table 2.6.

Table 2.6: Character table of T

Conjugacy class	1	2	$3a$	$3b$
χ_1	1	1	1	1
χ_2	1	1	$\exp\left(i\frac{2\cdot\pi}{3}\right)$	$\exp\left(i\frac{4\cdot\pi}{3}\right)$
χ_3	1	1	$\exp\left(i\frac{4\cdot\pi}{3}\right)$	$\exp\left(i\frac{2\cdot\pi}{3}\right)$
χ_4	3	-1	0	0

2.5.4 The character table of 2T

This subsection is based on pp.51-56 of [12]. Since $2T$ is a subgroup of $\mathbb{H}_1$, its elements can be represented by unit quaternions. The conjugacy classes of 2T are the following

$$\begin{aligned}
1 &= \{1\} \\
2 &= \{-1\} \\
3a &= \left\{-\frac{1}{2}(1 \pm i \pm j \pm k)|\text{ 0 or 2 minuses}\right\} \\
3b &= \left\{-\frac{1}{2}(1 \pm i \pm j \pm k)|\text{ 1 or 3 minuses}\right\} \\
4 &= \{(\pm i \pm j \pm k)|\} \\
6a &= \left\{\frac{1}{2}(1 \pm i \pm j \pm k)|\text{ 0 or 2 minuses}\right\} \\
6b &= \left\{\frac{1}{2}(1 \pm i \pm j \pm k)|\text{ 1 or 3 minuses}\right\}.[12, p.\ 54]
\end{aligned}$$

Again, by corollary 1.40, $|2T| = \sum_{i=1}^{7} d_i^2$. All d_i are in $\mathbb{N}$ and the only possible option is: $(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = (1, 1, 1, 2, 2, 2, 3)$.

The group $\{1, -1\}$ is a normal subgroup of $2T$. Hence $2T/\{1, -1\} \cong T$ is a quotient group of $2T$. Using the pulling back technique and knowing the characters of the irreducible representations of T beforehand, we know already the following four characters of $2T$:

$$\begin{aligned}
\chi_1 &= (1, 1, 1, 1, 1, 1, 1) \\
\chi_2 &= \left(1, 1, \exp\left(i\frac{2\cdot\pi}{3}\right), \exp\left(i\frac{4\cdot\pi}{3}\right), 1, \exp\left(i\frac{2\cdot\pi}{3}\right), \exp\left(i\frac{4\cdot\pi}{3}\right)\right) \\
\chi_3 &= \left(1, 1, \exp\left(i\frac{4\cdot\pi}{3}\right), \exp\left(i\frac{2\cdot\pi}{3}\right), 1, \exp\left(i\frac{4\cdot\pi}{3}\right), \exp\left(i\frac{2\cdot\pi}{3}\right)\right) \\
\chi_7 &= (3, 3, 0, 0, -1, 0, 0).
\end{aligned}$$

Now we define a two-dimensional representation, say ρ_4, of $2T$. Since all elements of $2T$ are unit quaternions, as are all elements of Q_8, the same representation as in subsection 3.2.2 will be described is taken. Again, as will be done in subsection 3.2.2, $\{1, j\}$ is chosen as basis for $\mathbb{H} \cong \mathbb{C}^2$.

To obtain χ_4, $\rho_4(a+bi+cj+dk)$ with respect to the basis $\{1,j\}$ is determined. To this end,

$$\rho_4(a+bi+cj+dk)(1) = a-bi-cj-dk = a-bi+(-c-di)j$$
$$\rho_4(a+bi+cj+dk)(j) = j(a-bi-cj-dk) = c-di+aj+bk = c-di+(a+bi)j.$$

Hence,

$$\rho_4(a+bi+cj+dk) = \begin{bmatrix} a-bi & c-di \\ -c-di & a+bi \end{bmatrix}.$$

Now, $\chi_4(a+bi+cj+dk) = a-bi+a+bi = 2a$. Taking the real part of an element of each of the seven conjugacy classes an multiplying with two gives $chi_4 : (2,-2,-1,-1,0,1,1)$.

In order to obtain χ_5 and χ_6, theorem 1.38 is used. First, the product of χ_2 and χ_4 is taken. This gives

$$\chi_2\chi_4 = \left(2, -2, -\exp\left(i\frac{2\cdot\pi}{3}\right), -\exp\left(i\frac{4\cdot\pi}{3}\right), 0, \exp\left(i\frac{2\cdot\pi}{3}\right), \exp\left(i\frac{4\cdot\pi}{3}\right)\right).$$

The Schur inner product of $\chi_2\chi_4$ with itself equals one, thus $\chi_2\chi_4$ is a newly obtained irreducible character of $2T$, say χ_5.

In the same way χ_6,

$$\chi_3\chi_4 = \left(2, -2, -\exp\left(i\frac{4\cdot\pi}{3}\right), -\exp\left(i\frac{2\cdot\pi}{3}\right), 0, \exp\left(i\frac{4\cdot\pi}{3}\right), \exp\left(i\frac{2\cdot\pi}{3}\right)\right),$$

is obtained as the product of χ_3 and χ_4.

The complete character table of $2T$ is given in table 2.7.

Table 2.7: Character table of $2T$

Conjugacy class	1	2	$3a$	$3b$	4	$6a$	$6b$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	$\exp\left(i\frac{2\cdot\pi}{3}\right)$	$\exp\left(i\frac{4\cdot\pi}{3}\right)$	1	$\exp\left(i\frac{2\cdot\pi}{3}\right)$	$\exp\left(i\frac{4\cdot\pi}{3}\right)$
χ_3	1	1	$\exp\left(i\frac{4\cdot\pi}{3}\right)$	$\exp\left(i\frac{2\cdot\pi}{3}\right)$	1	$\exp\left(i\frac{4\cdot\pi}{3}\right)$	$\exp\left(i\frac{2\cdot\pi}{3}\right)$
χ_4	2	-2	-1	-1	0	1	1
χ_5	2	-2	$-\exp\left(i\frac{2\cdot\pi}{3}\right)$	$-\exp\left(i\frac{4\cdot\pi}{3}\right)$	0	$\exp\left(i\frac{2\cdot\pi}{3}\right)$	$\exp\left(i\frac{4\cdot\pi}{3}\right)$
χ_6	2	-2	$-\exp\left(i\frac{4\cdot\pi}{3}\right)$	$-\exp\left(i\frac{2\cdot\pi}{3}\right)$	0	$\exp\left(i\frac{4\cdot\pi}{3}\right)$	$\exp\left(i\frac{2\cdot\pi}{3}\right)$
χ_7	3	3	0	0	-1	0	0

Chapter 3

Comparing the groups Q_8 and D_4

In this chapter we will compare the groups Q_8 and D_4 as a worked out example for how the first method, proposed by dr. Jeroen Spandaw, splits $\mathbb{C}^8$, where the regular representation (ρ) acts on, into ρ-invariant subspaces. Doing this for both Q_8 as well as for D_4 is especially useful as these two groups have the same character table. Here Q_8 consists of the quaternion elements

$$1, -1, i, -i, j, -j, k, -k$$

and D_4, consists of all rotations and reflections of the square, i.e.

$$e, r, r^2, r^3, s, r \circ s, r^2 \circ s, r^3 \circ s,$$

where e is the identity element.

First of all, $\mathbb{C}^8$ will be split into ρ-invariant isotypical components for D_4 as well as for Q_8 (section 3.1). After that, the isotypcial components of higher dimensions (≥ 2) will be split into its representation invariant parts (section 3.2). Last of all, the regular representation will be shown in block diagonal form for both Q_8 and D_4.

3.1 Splitting the regular representations of Q_8 and D_4 into isotypical subrepresentations

In this section $\mathbb{C}^8$ will be split into ρ-invariant isotypical components. This is done for the group D_4 in subsection 3.1.1 and for the group Q_8 in subsection 3.1.2.

3.1.1 Splitting the regular representation of D_4 into isotypical subrepresentations

First the regular representation (ρ) is defined by numbering the group elements of D_4 and making a basis for $\mathbb{C}^8$. To this end, let $g_1 = e, g_2 = r^2, g_3 = r, g_4 = r^3, g_5 = s, g_6 =$

$r^2 \circ s, g_7 = r \circ s, g_8 = r^3 \circ s$ and let $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_8}$ be the standard basis of $\mathbb{C}^8$. As stated in section 1.7, if $g_i \circ g_j = g_k$, with $g_i, g_j, g_k \in G$, then the regular representation ρ is defined as $\rho(g_i)e_j := e_k$.

Now that the character tables for Q_8 as well as D_4 are made, we are interested in an actual decomposition of the regular representation (ρ) into all subrepresentations. By corollary 1.27, the regular representation ρ splits in all the irreducible representations as follows:

$$\rho = d_1\rho_1 \oplus d_2\rho_2 \oplus \ldots \oplus d_k\rho_k,$$

where k is the number of irreducible representations and d_i is the dimension of the irreducible representation ρ_i for all i. Hence:

$$\begin{aligned} D_4 : \rho &= \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4 \oplus 2\rho_5 \\ Q_8 : \rho &= \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4 \oplus 2\rho_5. \end{aligned}$$

Since the regular representation can be written as a direct sum of irreducible representations, there exist matrices corresponding to elements of $\text{Aut}(\mathbb{C}^8)$ in which these matrices are in blockdiagonal form. Different irreducible representations will be pairwise orthogonal, see theorem 1.30. To this end, we will find vectors in $\mathbb{C}^8$ corresponding to exactly one irreducible subrepresentation of ρ.

From the character table it is known that D_4 has four representations of dimension one. This means that one of such a representation is the span of a single vector, say $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^\top$. For such a one-dimensional representation, say $\rho_i, i = 1, 2, 3, 4$, the following equations hold:

$$\begin{aligned} \rho(g_1)\mathbf{x} &= \chi_{\rho_i}(g_1)\mathbf{x} \\ \rho(g_2)\mathbf{x} &= \chi_{\rho_i}(g_2)\mathbf{x} \\ &\vdots \\ \rho(g_8)\mathbf{x} &= \chi_{\rho_i}(g_8)\mathbf{x}. \end{aligned}$$

For instance, for the trivial representation (ρ_1) of ρ the following equations hold:

$$\begin{aligned} \rho(g_1)\mathbf{x} &= \mathbf{x} \\ \rho(g_2)\mathbf{x} &= \mathbf{x} \\ &\vdots \\ \rho(g_8)\mathbf{x} &= \mathbf{x}. \end{aligned}$$

Equivalently,

$$
\begin{aligned}
x_1\mathbf{e_1}+x_2\mathbf{e_2}+x_3\mathbf{e_3}+x_4\mathbf{e_4}+x_5\mathbf{e_5}+x_6\mathbf{e_6}+x_7\mathbf{e_7}+x_8\mathbf{e_8} &= x_1\mathbf{e_1}+x_2\mathbf{e_2}+\cdots+x_8\mathbf{e_8}\\
x_2\mathbf{e_1}+x_1\mathbf{e_2}+x_4\mathbf{e_3}+x_3\mathbf{e_4}+x_6\mathbf{e_5}+x_5\mathbf{e_6}+x_8\mathbf{e_7}+x_7\mathbf{e_8} &= x_1\mathbf{e_1}+x_2\mathbf{e_2}+\cdots+x_8\mathbf{e_8}\\
x_4\mathbf{e_1}+x_3\mathbf{e_2}+x_2\mathbf{e_3}+x_1\mathbf{e_4}+x_8\mathbf{e_5}+x_7\mathbf{e_6}+x_5\mathbf{e_7}+x_6\mathbf{e_8} &= x_1\mathbf{e_1}+x_2\mathbf{e_2}+\cdots+x_8\mathbf{e_8}\\
x_3\mathbf{e_1}+x_4\mathbf{e_2}+x_1\mathbf{e_3}+x_2\mathbf{e_4}+x_7\mathbf{e_5}+x_8\mathbf{e_6}+x_6\mathbf{e_7}+x_5\mathbf{e_8} &= x_1\mathbf{e_1}+x_2\mathbf{e_2}+\cdots+x_8\mathbf{e_8}\\
x_5\mathbf{e_1}+x_6\mathbf{e_2}+x_8\mathbf{e_3}+x_7\mathbf{e_4}+x_1\mathbf{e_5}+x_2\mathbf{e_6}+x_4\mathbf{e_7}+x_3\mathbf{e_8} &= x_1\mathbf{e_1}+x_2\mathbf{e_2}+\cdots+x_8\mathbf{e_8}\\
x_6\mathbf{e_1}+x_5\mathbf{e_2}+x_7\mathbf{e_3}+x_8\mathbf{e_4}+x_2\mathbf{e_5}+x_1\mathbf{e_6}+x_3\mathbf{e_7}+x_4\mathbf{e_8} &= x_1\mathbf{e_1}+x_2\mathbf{e_2}+\cdots+x_8\mathbf{e_8}\\
x_7\mathbf{e_1}+x_8\mathbf{e_2}+x_5\mathbf{e_3}+x_6\mathbf{e_4}+x_3\mathbf{e_5}+x_4\mathbf{e_6}+x_1\mathbf{e_7}+x_2\mathbf{e_8} &= x_1\mathbf{e_1}+x_2\mathbf{e_2}+\cdots+x_8\mathbf{e_8}\\
x_8\mathbf{e_1}+x_7\mathbf{e_2}+x_6\mathbf{e_3}+x_5\mathbf{e_4}+x_4\mathbf{e_5}+x_3\mathbf{e_6}+x_2\mathbf{e_7}+x_1\mathbf{e_8} &= x_1\mathbf{e_1}+x_2\mathbf{e_2}+\cdots+x_8\mathbf{e_8}.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) &= (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\\
(x_2,x_1,x_4,x_3,x_6,x_5,x_8,x_7) &= (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\\
(x_4,x_3,x_2,x_1,x_8,x_7,x_5,x_6) &= (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\\
(x_3,x_4,x_1,x_2,x_7,x_8,x_6,x_5) &= (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\\
(x_5,x_6,x_8,x_7,x_1,x_2,x_4,x_3) &= (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\\
(x_6,x_5,x_7,x_8,x_2,x_1,x_3,x_4) &= (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\\
(x_7,x_8,x_5,x_6,x_3,x_4,x_1,x_2) &= (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\\
(x_8,x_7,x_6,x_5,x_4,x_3,x_2,x_1) &= (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8).
\end{aligned}
$$

Equivalently,

$$x_1=x_2=x_3=x_4=x_5=x_6=x_7=x_8.$$

Hence, the subrepresentation ρ_1 of ρ is generated by the vector $(1,1,1,1,1,1,1,1)^\top =: \mathbf{a_1}$.

We can do this for any one-dimensional representation. We get the following subspaces of $\mathbb{C}^8$, representing all the one-dimensional representations of D_4:

The subrepresentation ρ_1 of ρ is generated by the vector $(1,1,1,1,1,1,1,1)^\top =: \mathbf{a_1}$ (3.1a)

The subrepresentation ρ_2 of ρ is generated by the vector $(1,1,1,1,-1,-1,-1,-1)^\top =: \mathbf{a_2}$ (3.1b)

The subrepresentation ρ_3 of ρ is generated by the vector $(1,1,-1,-1,1,1,-1,-1)^\top =: \mathbf{a_3}$ (3.1c)

The subrepresentation ρ_4 of ρ is generated by the vector $(1,1,-1,-1,-1,-1,1,1)^\top =: \mathbf{a_4}$. (3.1d)

Now, only subrepresentation $2\rho_5$ of ρ still needs to be found. This representation is orthogonal to $\rho_1\oplus\rho_2\oplus\rho_3\oplus\rho_4$ in $\mathbb{C}^8$. A basis for

$$\text{Null}\begin{bmatrix}1&1&1&1&1&1&1&1\\1&1&1&1&-1&-1&-1&-1\\1&1&-1&-1&1&1&-1&-1\\1&1&-1&-1&-1&-1&1&1\end{bmatrix} =: V_5$$

is for instance:

$$B = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right), \tag{3.2}$$

with $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}$, the respective basis vectors.

However,

$$\begin{aligned}
\rho(s)(\mathbf{b_1}) &= \rho(s)(\mathbf{e_1} - \mathbf{e_2}) = \mathbf{e_5} - \mathbf{e_6} = \mathbf{b_3} \\
\rho(s)(\mathbf{b_2}) &= \rho(s)(\mathbf{e_3} - \mathbf{e_4}) = \mathbf{e_8} - \mathbf{e_7} = -\mathbf{b_4} \\
\rho(s)(\mathbf{b_3}) &= \rho(s)(\mathbf{e_5} - \mathbf{e_6}) = \mathbf{e_1} - \mathbf{e_2} = \mathbf{b_1} \\
\rho(s)(\mathbf{b_4}) &= \rho(s)(\mathbf{e_7} - \mathbf{e_8}) = \mathbf{e_4} - \mathbf{e_3} = -\mathbf{b_2}.
\end{aligned}$$

Hence,

$$\rho_{V_5}(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

with respect to $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}$. This matrix is not in block diagonal form. Thus $z_1\mathbf{b_1} + z_2\mathbf{b_2}$, with $z_1, z_2 \in \mathbb{C}$ is not a representation invariant subspace. If we let $\mathbf{b_2'} := \mathbf{b_3}$ and $\mathbf{b_3'} := \mathbf{b_2}$ however, we get:

$$\rho_{V_5}(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

which is block diagonal. But then,

$$\rho_{V_5}(r) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

which is not block diagonal.

Unfortunately, no two vectors out of $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}$ will form a representation invariant subspace of V_5. However, such a representation invariant subspace, and therefore block diagonal matrices $\rho_{v_5}(g)$ for all $g \in G$ exists as $2\rho_5$ is reducible. We write $V_5 = V_5^1 \oplus V_5^2$, where V_5^1 and V_5^2 are two two-dimensional representation invariant subspaces of V_5. Notice that $\rho_{v_5}(g)$ is block diagonal for all $g \in G$ if $\rho_{v_5}(g)$ is block diagonal for all generators of G. In the next subsection 3.1.2 we will find all possible basis vectors of V_5 such that two representation invariant subspaces, V_5^1 and V_5^2, are formed. That is, half of the found basis vectors for V_5 will be a basis for V_5^1 and the other half will be a basis for V_5^2.

3.1.2 Splitting the regular representation of Q_8 into isotypical subrepresentations

Again, the group elements are numbered, this time of group Q_8. To this end, let $g_1 = 1, g_2 = -1, g_3 = i, g_4 = -i, g_5 = j, g_6 = -j, g_7 = k, g_8 = -k$ and let $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_8}$ be the standard basis for $\mathbb{C}^8$.

The four one-dimensional representations within the regular representation of Q_8, ρ, are found analogously to the ones of D_4. They are precisely as in equations 3.1x.

The subspace V_5 as mentioned in subsection 3.1.1 and the basis B for V_5 as in equation 3.2 can also be completely copied. However, the representation acting on this space must be different, as Q_8 is different from D_4.

i and j are generators for Q_8. ρ of these generators on the basis vectors of B result in:

$$\begin{aligned}
\rho(i)(\mathbf{b_1}) &= \rho(i)(\mathbf{e_1} - \mathbf{e_2}) = \mathbf{e_3} - \mathbf{e_4} = \mathbf{b_2} \\
\rho(i)(\mathbf{b_2}) &= \rho(s)(\mathbf{e_3} - \mathbf{e_4}) = \mathbf{e_2} - \mathbf{e_1} = -\mathbf{b_1} \\
\rho(i)(\mathbf{b_3}) &= \rho(s)(\mathbf{e_5} - \mathbf{e_6}) = \mathbf{e_7} - \mathbf{e_8} = \mathbf{b_4} \\
\rho(i)(\mathbf{b_4}) &= \rho(s)(\mathbf{e_7} - \mathbf{e_8}) = \mathbf{e_6} - \mathbf{e_5} = -\mathbf{b_3} \\
\rho(j)(\mathbf{b_1}) &= \rho(s)(\mathbf{e_1} - \mathbf{e_2}) = \mathbf{e_5} - \mathbf{e_6} = \mathbf{b_3} \\
\rho(j)(\mathbf{b_2}) &= \rho(s)(\mathbf{e_3} - \mathbf{e_4}) = \mathbf{e_8} - \mathbf{e_7} = -\mathbf{b_4} \\
\rho(j)(\mathbf{b_3}) &= \rho(s)(\mathbf{e_5} - \mathbf{e_6}) = \mathbf{e_2} - \mathbf{e_1} = -\mathbf{b_1} \\
\rho(j)(\mathbf{b_4}) &= \rho(s)(\mathbf{e_7} - \mathbf{e_8}) = \mathbf{e_3} - \mathbf{e_4} = \mathbf{b_2}.
\end{aligned}$$

Hence,

$$\rho_{V_5}(i) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$\rho_{V_5}(j) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

which are not both block diagonal.

3.2 Splitting irreducible representations of D_4 and Q_8 of dimension ≥ 2 into their irreducible components

In section 3.1 $\mathbb{C}^8$ is split into representation invariant, with respect to the regular representation, isotypical components. In this section these isotypical components will be further decomposed. In subsection 3.2.1 this will be done for the group D_8 and in subsection 3.2.2 this is done for the group Q_8.

3.2.1 Splitting irreducible representations of D_8 of dimension ≥ 2 into its irreducible components

Consider the elements of D_4 to act on a square (in $\mathbb{R}^2$), where r is a rotation of $\frac{\pi}{2}$ radians and s is a reflection in the x-axis. This gives the unique irreducible two-dimensional representation of D_4 (ρ_5). Take $\mathbf{e_1}$ and $\mathbf{e_2}$ as basis vectors for $\mathbb{C}^2$. Now,

$$\rho_5(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$\rho_5(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The mappings of the other six group elements of D_4 follow from these two generators of D_4 (r and s).

Hence, the block matrices of $2\rho_5 = \rho_5 \oplus \rho_5$ of the generators of D_4 are given by:

$$(2\rho_5)(r) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$(2\rho_5)(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

which are both block diagonal.

In order to find all possible basis vectors $C = (\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4})$ of $2\rho_5$ such that $(2\rho_5)(g)$ with respect to the basis C are block diagonal for all $g \in G$, all possible change of basis matrices L must be found such that

$$L \circ (2\rho_5)(g) \circ L^{-1} = \rho_{V_5}(g) \tag{3.3}$$

for all $g \in G$, or equivalently, for all generators of G. Here ρ_{v_5} is taken with respect to the basis B. In this way, The matrices L that agree with equation 3.3 for the generators of D_4, i.e. r and s,

$$g = r : \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} L = L \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$g = s : \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} L = L \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

are of the form:

$$L = \begin{bmatrix} p & q & r & s \\ -q & p & -s & r \\ p & -q & r & -s \\ q & p & s & r \end{bmatrix}.$$

We have $\det(L) = 4(ps - qr)^2$. Thus L will be invertible if and only if $ps \neq qr$.

Now, the vectors of the basis C are formed by expressing the columns of P in the basis B. To this end, let $B' = \begin{bmatrix} \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} & \mathbf{b_4} \end{bmatrix}$. Now, let $C' = B'L$. The columns of C', expressed in B form a basis for $2\rho_5$ in $\mathbb{C}^8$. This results in:

$$\begin{aligned} \mathbf{c_1} &= p\mathbf{b_1} - q\mathbf{b_2} + p\mathbf{b_3} + q\mathbf{b_4} \\ \mathbf{c_2} &= q\mathbf{b_1} + p\mathbf{b_2} - q\mathbf{b_3} + p\mathbf{b_4} \\ \mathbf{c_3} &= r\mathbf{b_1} - s\mathbf{b_2} + r\mathbf{b_3} + s\mathbf{b_4} \\ \mathbf{c_4} &= s\mathbf{b_1} + r\mathbf{b_2} - s\mathbf{b_3} + r\mathbf{b_4}, \end{aligned}$$

which gives

$$\begin{aligned} \mathbf{c_1} &= p(\mathbf{e_1} - \mathbf{e_2}) - q(\mathbf{e_3} - \mathbf{e_4}) + p(\mathbf{e_5} - \mathbf{e_6}) + q(\mathbf{e_7} - \mathbf{e_8}) \\ \mathbf{c_2} &= q(\mathbf{e_1} - \mathbf{e_2}) + p(\mathbf{e_3} - \mathbf{e_4}) - q(\mathbf{e_5} - \mathbf{e_6}) + p(\mathbf{e_7} - \mathbf{e_8}) \\ \mathbf{c_3} &= r(\mathbf{e_1} - \mathbf{e_2}) - s(\mathbf{e_3} - \mathbf{e_4}) + r(\mathbf{e_5} - \mathbf{e_6}) + s(\mathbf{e_7} - \mathbf{e_8}) \\ \mathbf{c_4} &= s(\mathbf{e_1} - \mathbf{e_2}) + r(\mathbf{e_3} - \mathbf{e_4}) - s(\mathbf{e_5} - \mathbf{e_6}) + r(\mathbf{e_7} - \mathbf{e_8}). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{c_1} &= (p, -p. - q, q, p, -p, q, -q)^\top \\ \mathbf{c_2} &= (q, -q, p, -p, -q, q, p, -p)^\top \\ \mathbf{c_3} &= (r, -r, -s, s, r, -r, s, -s)^\top \\ \mathbf{c_4} &= (s, -s, r, -r, -s, s, r, -r)^\top, \end{aligned}$$

where, indeed, $C = (\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4})$ is a basis for $2\rho_5$ acting on $V_5 \subset \mathbb{C}^8$.

Moreover, $\{\mathbf{c_1}, \mathbf{c_2}\}$ and $\{\mathbf{c_3}, \mathbf{c_4}\}$ both form a basis for V_5^1 and $V_5^2 \subset \mathbb{C}^8$ respectively. V_5^1 and V_5^2 are not fixed, as they depend on the parameters p, q, r, s. These specific basis vectors, although dependent on their parameters, define all possible subspaces V_5^1 and $V_5^2 \subset \mathbb{C}^8$, but are by no means the only possible basis vectors for V_5^1 and V_5^2, since a basis transformation can always be performed. Notice that $z_1\mathbf{c_1} + z_2\mathbf{c_2}$ and $z_3\mathbf{c_3} + z_4\mathbf{c_4}$, with $z_1, z_2, z_3, z_4 \in \mathbb{C}$ do not need to be orthogonal, but are allowed to.

3.2.2 Splitting irreducible representations of Q_8 of dimension ≥ 2 into its irreducible components

First, we define the two-dimensional representation ρ_5 of Q_8. All elements of Q_8 can be seen as unit quaternions. Hence, these elements might act on the one-dimensional vector space $\mathbb{H}^1$, where $\mathbb{H}$ is the set of quaternions. Setting $\rho_5(g) : v \mapsto g \cdot v$, where $g \in Q_8$ and $v \in \mathbb{H}^1$, will not work as the complex linearity is violated. Indeed,

$$\begin{aligned} \rho_5(g)(z \cdot v) &= g \cdot z \cdot v \\ z \cdot \rho_5(g)(v) &= z \cdot g \cdot v, \end{aligned}$$

which are not equal in general for $g \in G$ and $z \in \mathbb{C}$.

Multiplication from the right will resolve this problem. However, ρ_5 is not a group homomorphism anymore. Indeed,

$$\begin{aligned} \rho_5(g)(z \cdot v) &= z \cdot v \cdot g \\ z \cdot \rho_5(g)(v) &= z \cdot v \cdot g. \end{aligned}$$

However,

$$\begin{aligned} \rho_5(g_1 g_2)(v) &= v \cdot g_1 g_2 \\ (\rho_5(g_1) \cdot \rho_5(g_2))(v) &= v \cdot g_2 g_1. \end{aligned}$$

Finally, setting $\rho_5(g) : v \mapsto v \cdot g^{-1}$ is a proper representation:

$$\begin{aligned} \rho_5(g)(z \cdot v) &= z \cdot v \cdot g^{-1} \\ z \cdot \rho_5(g)(v) &= z \cdot v \cdot g^{-1} \\ \rho_5(g_1 g_2)(v) &= v \cdot g_2^{-1} g_2^{-1} \\ (\rho_5(g_1) \cdot \rho_5(g_2))(v) &= v \cdot g_2^{-1} g_2^{-1}. \end{aligned}$$

Now, this one-dimensional quaternionic vector space, with multiplication from the left, can be seen as a two-dimensional complex vector space. That is, every element $a + bi + cj + dk \in \mathbb{H}^1$ will be written as $(a + bi) + (c + di)j$, with $a + bi$ and $c + di \in \mathbb{C}$. As a basis for this vector space we take $\{1, j\}$. Now,

$$\begin{aligned} \rho_5(i)(1) &= 1 \cdot i^{-1} = -i \\ \rho_5(i)(j) &= j \cdot i^{-1} = i \cdot j \\ \rho_5(j)(1) &= 1 \cdot j^{-1} = -j \\ \rho_5(j)(j) &= j \cdot j^{-1} = 1. \end{aligned}$$

Hence,

$$\begin{aligned} \rho_5(i) &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\ \rho_5(j) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned}$$

Again, we find all matrices L such that the following two equations hold:

$$i : \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \circ L = L \circ \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix},$$

$$j : \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \circ L = L \circ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

The matrices L that agree with these two equations are of the form:

$$L = \begin{bmatrix} p & q & r & s \\ pi & -qi & ri & -si \\ q & -p & s & -r \\ qi & pi & si & ri \end{bmatrix}.$$

$\det(P) = -4(ps - qr)^2$. If and only if $ps \neq qr$, P will be invertible.

Lastly, finding basis vectors for V_5^1 and V_5^2 as seen in the previous subsection 3.2.1 gives:

$$\begin{aligned} \mathbf{d_1} &= p(\mathbf{e_1} - \mathbf{e_2}) + pi(\mathbf{e_3} - \mathbf{e_4}) + q(\mathbf{e_5} - \mathbf{e_6}) + qi(\mathbf{e_7} - \mathbf{e_8}) \\ \mathbf{d_2} &= q(\mathbf{e_1} - \mathbf{e_2}) - qi(\mathbf{e_3} - \mathbf{e_4}) - p(\mathbf{e_5} - \mathbf{e_6}) + pi(\mathbf{e_7} - \mathbf{e_8}) \\ \mathbf{d_3} &= r(\mathbf{e_1} - \mathbf{e_2}) + ri(\mathbf{e_3} - \mathbf{e_4}) + s(\mathbf{e_5} - \mathbf{e_6}) + si(\mathbf{e_7} - \mathbf{e_8}) \\ \mathbf{d_4} &= s(\mathbf{e_1} - \mathbf{e_2}) - si(\mathbf{e_3} - \mathbf{e_4}) - r(\mathbf{e_5} - \mathbf{e_6}) + ri(\mathbf{e_7} - \mathbf{e_8}) \end{aligned}$$

Hence,

$$\mathbf{d_1} = (p, -p.pi, -pi, q, -q, qi, -qi)^\top \tag{3.8a}$$

$$\mathbf{d_2} = (q, -q, -qi, qi, -p, p, pi, -pi)^\top \tag{3.8b}$$

$$\mathbf{d_3} = (r, -r, ri, -ri, s, -s, si, -si)^\top \tag{3.8c}$$

$$\mathbf{d_4} = (s, -s, -si, si, -r, r, ri, -ri)^\top \tag{3.8d}$$

$\{\mathbf{d_1}, \mathbf{d_2}\}$ is a basis for V_5^1 and $\{\mathbf{d_3}, \mathbf{d_4}\}$ is a basis for V_5^2. Although the character tables of D_4 and Q_8 are the same, the spaces V_5^1 and V_5^1 found for Q_8 are different from these spaces found for D_4.

3.3 The regular representation of D_4 in block diagonal form

In order to let V_5^1 and V_5^2 be orthogonal, the following must be true:

$$\begin{aligned} \mathbf{c_1} &\perp \mathbf{c_3} \\ \mathbf{c_1} &\perp \mathbf{c_4} \\ \mathbf{c_2} &\perp \mathbf{c_3} \\ \mathbf{c_2} &\perp \mathbf{c_4}. \end{aligned}$$

This results in the following equations:

$$\langle \mathbf{c_1}, \mathbf{c_3} \rangle = 0$$
$$\langle \mathbf{c_1}, \mathbf{c_4} \rangle = 0$$
$$\langle \mathbf{c_2}, \mathbf{c_3} \rangle = 0$$
$$\langle \mathbf{c_2}, \mathbf{c_4} \rangle = 0,$$

which comes down to $pr + qs = 0$.

However, this criterion is not necessary in order to make $\rho(g)$ a block diagonal matrix for all $g \in G$. To this end, we conclude:

$$\rho(r) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
$$\rho(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

with basis $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}, \mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4})$ where the choice of $p, q, r, s \in \mathbb{C}$ for the basis C is free. $\rho(g)$ for other $g \in G$ follow from multiplication of the generator matrices $\rho(r)$ and $\rho(s)$ above.

Chapter 4

Grand orthogonalization

As we have seen in chapter 3, finding the representation invariant subspaces of the regular representation of a group G can be quite cumbersome. There is a faster way however: grand orthogonalization. The advantage of using this method over the method described in chapter 3 is that it is faster. However, not all possible representation invariant subspaces are found, but only one instance. First the goal of grand orthogonalization will be briefly stated in section 4.1. Then the way the grand orthogonalization method works will be described in section 4.2.

4.1 The goal of grand orthogonalization

Let ρ be the regular representation of a group G on $\mathbb{C}^N$ where $N = |G|$. $\mathbb{C}^N$ can be split up in $\sum_i^k d_i =: S$ transversal representation invariant vector spaces. Here transversal means the following: If V_1 and V_2 are two vector spaces such that $V_1 \cap V_2 = \{\mathbf{0}\}$, then V_1 and V_2 are said to be transversal vector spaces. Here k is the number of irreducible representations, and thus conjugacy classes, of G and d_i is the dimension of the i-th irreducible representation of G. To this end let $\mathbb{C}^N = \bigoplus_{i=1}^S V_i$. This splitting is not unique as can be seen in chapter 3. Grand orthogonalization is a method in which the goal is to make one of the infinitely many possible bases for $\mathbb{C}^N$ such that all vector spaces $V_i \subseteq \mathbb{C}^N$ are the span of a subset of these basis vectors.

In order to apply the grand orthogonalization method, not only the characters of $\rho_i(g)$ for each irreducible representation ρ_i and each $g \in G$ are needed, but also the actual matrices of $\rho_i(g)$ are needed, each vector space V_i accompanied with a basis. These requirements were also applicable in the process executed in chapter 3.

4.2 The way the grand orthogonalization method works

The matrices which represent elements of $\mathrm{Aut}(V_i)$ have dimension $d_i \times d_i$. Hence, such a matrix has d_i^2 individual entries. For one element $g \in G$, take all corresponding matrices $\rho_i(g)$, where ρ_i is the i-th irreducible representation. All these matrices have to be unitary. This is always possible by corollary 1.19. Now, summing over all these matrices, the total number of entries of these k matrices is

$$\sum_{i=1}^{k} d_i^2 = N,$$

as seen in corollary 1.40.

Since the above mentioned equation holds, taking all entries of the described matrices forms a vector in $\mathbb{C}^N$. These vectors are exactly the vectors which split $\mathbb{C}^N$ in the desired way. The way this works is best explained in an example:

Take the group Q_8. We make the character table, but instead of characters $\chi_i(g)$, we take the corresponding matrices given the basis $\{1, j\}$ analogous to chapter 3. Since $\rho_i(g_1) \neq \rho_i(g_2)$ necessarily, although g_1 and g_2 are in the same conjugacy class, the 'character table' is also extended horizontally, displaying all elements $g \in G$ individually. We call this 'character table' the extended character table. Indeed,

Table 4.1: Extended character table of $\mathbb{Q}_8$

Conjugacy class	1	-1	i	$-i$	j	$-j$	k	$-k$
ρ_1	1	1	1	1	1	1	1	1
ρ_2	1	1	1	1	-1	-1	-1	-1
ρ_3	1	1	-1	-1	1	1	-1	-1
ρ_4	1	1	-1	-1	-1	-1	1	1
ρ_5	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

Now each column in table 4.1 has 8 entries. These entries will form the coordinates of vectors in $\mathbb{C}^8$. These vectors are formed by placing the columns of a matrix from left to right beneath each other. Then, these vectors will be placed in a matrix L^{-1}. This results in:

$$L^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -i & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -i & i \\ 0 & 0 & 0 & 0 & 1 & -1 & -i & i \\ 1 & -1 & i & -i & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix L^{-1} is the matrix as described in equation 3.3! Now the matrix L is given by:

$$L = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{4}i & 0 & 0 & -\frac{1}{4}i \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{4}i & 0 & 0 & \frac{1}{4}i \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{4}i & \frac{1}{4}i & 0 \\ \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & 0 & -\frac{1}{4}i & -\frac{1}{4}i & 0 \end{bmatrix}.$$

These columns can then be expressed in the chosen basis for $\mathbb{C}^N$. However, this chosen basis was $(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_8})$. Thus the wanted basis vectors are exactly the columns of L.

Indeed, these vectors were already found in chapter 3 in equations 3.1 and 3.8. In equation 3.8, take

$$\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}.$$

In conclusion, the grand orthogonalization method gives a basis for $\mathbb{C}^N$ such that all vector spaces $V_i \subseteq S$ are the span of a subset of these basis vectors. In particular,

$$\mathbb{C}^8 = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus 2V_5,$$

where:

$$V_1 = \text{Span}\left\{ \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \end{bmatrix} \right\} \qquad V_2 = \text{Span}\left\{ \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ -\frac{1}{8} \\ -\frac{1}{8} \\ -\frac{1}{8} \\ -\frac{1}{8} \end{bmatrix} \right\}$$

$$V_3 = \text{Span}\left\{ \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ -\frac{1}{8} \\ -\frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ -\frac{1}{8} \\ -\frac{1}{8} \end{bmatrix} \right\} \qquad V_4 = \text{Span}\left\{ \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ -\frac{1}{8} \\ -\frac{1}{8} \\ -\frac{1}{8} \\ -\frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \end{bmatrix} \right\}$$

$$V_5^1 = \text{Span}\left\{ \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4}i \\ -\frac{1}{4}i \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4}i \\ -\frac{1}{4}i \end{bmatrix} \right\} \qquad V_5^2 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4}i \\ -\frac{1}{4}i \end{bmatrix}, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4}i \\ \frac{1}{4}i \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

A proof of the grand orthogonalisation method is given by V. Bharati [1, pp.24-27]. The proof first states that the rows of L are orthonormal, hence the name 'grand orthogonalisation'. It also states that there are no zero vectors as rows of L. After that, the theorem states that the rows of L form a basis for $\mathbb{C}^N$, which then implies that the columns of L form a desired basis.

Conclusion

The purpose of this thesis was to build up to ways to split a vector space that the regular representation of a finite group G acts on into representation invariant subspaces. In order to do this, two methods were described and used. The first method is suggested by dr. Jeroen Spandaw and the second method, which is called the grand orthogonalisation method, is suggested by dr. Paul Visser. Both methods make use of matrices which form the images of the irreducible representations that make up the regular representation, under a certain, arbitrary, basis. Moreover, the first method makes use of change of basis matrices. In order to treat both methods, representation theory was treated. This was a purpose of this thesis on its own.

Comparing both methods, none of the two can be considered the best. Both methods have advantages and disadvantages. The first method constructs all possible bases that split a vector space that a regular representation acts on into representation invariant subspaces. The second method, grand orthogonalisation, only constructs one such basis, making method two less complete than method one. However, method one makes use of change of basis matrices which are $|G| \times |G|$-dimensional, where $|G|$ is the order of a group G. To do calculations with these matrices takes time. Moreover, basis vectors for vectorspaces consisting of isotypical components of the regular representations are needed for this method as well, which can be hard to obtain. This work and time is not needed to apply the second method.

This thesis reviewed two methods, but did not optimise them. In that regard, two recommendations are given. First, other methods can be compared to the two methods considered in this thesis. Second, the two considered methods can be further optimised. The first method might be a better candidate for this, as it could be that the construction of the change of basis matrices can be done faster due to properties of representations.

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Appendices

Appendix A

Proofs

Proof theorem 1.15. Let $\rho_1 : G \to \mathrm{Aut}(V_1)$ and $\rho_2 : G \to \mathrm{Aut}(V_2)$ be two representations of the same group G to finite-dimensional vector spaces V_1 and V_2. Let $\rho = \rho_1 \oplus \rho_2$. Now,

$$\chi_\rho(g) = \mathrm{tr}(\rho(g)) = \mathrm{tr}(\rho_1(g) \oplus \rho_2(g)) = \mathrm{tr}(\rho_1(g)) + \mathrm{tr}(\rho_2(g)) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g).$$

□

The proof of theorem 1.17 is based on a proof given by dr. Jeroen Spandaw ([12, p. 14]).

Proof theorem 1.17. Suppose $W \subseteq V$ is a representation invariant subspace of V. Let $g \in G$ and let $\mathbf{u} \in W^\perp$, i.e. $< \mathbf{w} \,|\, \mathbf{u} >= 0$ for all $\mathbf{w} \in W$. For $W^\perp$ to be a group invariant vector space it has to be proven that $\rho(g)(\mathbf{u})$ is also in $W^\perp$, i.e. $< \mathbf{w} \,|\, \rho(g)(\mathbf{u}) >= 0$ for all $\mathbf{w} \in W$. Since W is representation invariant and since every $\rho(g)$ is invertible, for all $g \in G$, there exists a $\mathbf{w'} \in W$ such that $\rho(g)(\mathbf{w'}) = \mathbf{w}$. Hence,

$$< \mathbf{w} \,|\, \rho(g)(\mathbf{u}) >=< \rho(g)(\mathbf{w'}) \,|\, \rho(g)(\mathbf{u}) >=< \mathbf{w'} \,|\, \mathbf{u} >= 0$$

W and $W^\perp$ are both representation invariant vector spaces. $\rho_1 : G \to W$ and $\rho_2 : G \to W^\perp$ are both representations. Indeed, $\rho = \rho_1 \oplus \rho_2$, where ρ_1 acts on W and ρ_2 acts on $W^\perp$, and $V = W \oplus W^\perp$. [12, p. 14] □

Proof corollary 1.19. Let $\rho : G \to \mathrm{Aut}(V)$ be a representation. Let H be a positive definite Hermitian inner product that is ρ-invariant. Such an inner product exists by lemma 1.18. Let $\mathbf{b_1} \ldots \mathbf{b_n}$ be an orthonormal basis for V with respect to the inner product H. Let $A(g)$ be the matrix of $\rho(g)$ for $g \in G$ with respect to the given orthonormal basis.

The following relation holds:

$$\rho(g_0)(\mathbf{b_j}) = \sum_{i=1}^{n} a_{i,j}\mathbf{b_i}$$

. By orthonormality of the basis vectors this implies

$$\begin{aligned} A(g)_{i,j} &= H(\mathbf{b_i}, \sum_{i=1}^{n} A(g)_{i,j}\mathbf{b_i}) \\ &= H(\mathbf{b_i}, A(g)_{i,j}(\mathbf{b_j})) \end{aligned}$$

This gives

$$\begin{aligned} A^{-1}(g)_{i,j} &= A(g^{-1})_{i,j} \\ &= H(\mathbf{b_i}, A(g^{-1})_{i,j}\mathbf{b_j}) \\ &= H(A(g)_{i,j}\mathbf{b_i}, \mathbf{b_j})) \\ &= \overline{H(\mathbf{b_j}, A(g)_{i,j}\mathbf{b_i})} \\ &= \overline{A(g)_{j,i}}. \end{aligned}$$

Hence $A^{-1}(g) = \overline{A^{\top}(g)}$ for all $g \in G$ and $A(g)$ is unitary for all $g \in G$.

Moreover, since $A(g)$ is unitary for all $g \in G$,

$$\chi(g^{-1}) = \mathrm{Tr}(A(g^{-1})) = \mathrm{Tr}(A(g)^{-1}) = \mathrm{Tr}(\overline{A(g)^{top}}) = \\ \mathrm{Tr}(\overline{A(g)}) = \overline{\mathrm{Tr}(A(g))} = \overline{\chi(g)}[12][p.\ 13][18]$$

□

Proof Corollary 1.27. By theorem 1.23 ρ decomposes uniquely into irreducible representations. By the linearity in the first argument of the Schur inner product, see equation 1.9, it holds that:

$$\langle \chi_\rho \,|\, \chi_{\rho_j} \rangle = \langle m_1\chi_{\rho_1} + m_2\chi_{\rho_2} \cdots + m_k\chi_{\rho_k} \,|\, \chi_{\rho_j} \rangle = \sum_{i=1}^{k} m_i \langle \chi_{\rho_i} \,|\, \chi_{\rho_j} \rangle.$$

By theorem 1.25, Schur orthogonality,

$$\langle \chi_{\rho_i} \,|\, \chi_{\rho_j} \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

But then,

$$\sum_{i=1}^{k} m_i \langle \chi_{\rho_i} \,|\, \chi_{\rho_j} \rangle = m_j$$

and we conclude $m_j = \langle \chi_\rho \,|\, \chi_{\rho_j} \rangle$. □

Proof Corollary 1.28. ($\Rightarrow$): Suppose ρ is irreducible. Then χ_ρ is orthonormal to the characters of all the irreducible representations by theorem 1.25. Hence $\langle \chi_\rho \,|\, \chi_\rho \rangle = 1$. ($\Leftarrow$): Let ρ be a representation. By corollary 1.27 and theorem 1.25,

$$\langle \chi_\rho \,|\, \chi_\rho \rangle = \sum_{i=1}^{k} m_i, \tag{A.1}$$

where i, k and m_i are as in corollary 1.27. Suppose $\langle \chi_\rho \,|\, \chi_\rho \rangle = 1$. Since all m_i in equation A.1 are greater or equal to 0 and are integral, only 1 m_i is equal to 1 whereas the rest is 0. Hence, ρ is irreducible. □

Proof theorem 1.41. When considering the standard basis for $\mathbb{C}^N$, $\rho(g)$ is a permutation matrix for all $g \in G$, where ρ is the regular representation. A permutation matrix is always unitary. Hence, the standard complex inner product is ρ-invariant. □

Proof corollary 1.42. Take the character table of any group G. We see this character table as a matrix A where the rows are the characters and the columns are the characters of particular conjugacy classes. By theorem 1.25, the rows of a character table are orthonormal with respect to the Schur inner product. Divide all entries in the character table A by $\sqrt{|G|}$ and multiply the entries in column j by $\sqrt{|C_j}$ where $|C_j|$ is the order of the j-th conjugacy class, which is in the j-th column of A. $C(g)$ is defined as the conjugacy class g belongs to. Let B be this modified character table. Now, the rows in the modified character table are orthonormal under the standard complex inner product. Indeed, take two rows of the modified character table, say r_i and r_j. Now

$$\begin{aligned}
\langle r_i \,|\, r_j \rangle_{\text{standard}} &= \sum_{l=1}^{k} b_{i,l}\overline{b_{j,l}} \\
&= \sum_{l=1}^{k} \frac{\sqrt{|C_l|}}{\sqrt{|G|}} a_{i,l} \overline{\frac{\sqrt{|C_l|}}{\sqrt{|G|}} a_{j,l}} \\
&= \sum_{g\in G} \frac{1}{|C(g)|} \frac{\sqrt{|C(g)|}}{\sqrt{|G|}} \chi_i(g) \overline{\frac{\sqrt{|C(g)|}}{\sqrt{|G|}}} \chi_j(g) \\
&= \frac{1}{|G|} \sum_{g\in G} \chi_i(g)\overline{\chi_j(g)} \\
&= \begin{cases} 1 & \text{if}\, i = j \\ 0 & \text{if}\, i \neq j. \end{cases}
\end{aligned}$$

Since the modified character table, seen as a matrix B, is unitary, it's columns are orthonormal as well. Now, multiplying each entry $b_{i,j}$ in the modified character table by $\frac{\sqrt{|G|}}{|C_j|}$ gives the original character table again. Since each column of B is only multiplied by a factor, the columns of matrix A, which is the original character table is also, are still orthogonal. In fact, the columns of a character table are even orthonormal with the following inner product defined for two columns c_i and c_j:

$$\langle c_i \,|\, c_j \rangle = \frac{\sqrt{|C_i||C_j|}}{|G|} \sum_{l=1}^{k} a_{l,i}\overline{a_{l,j}}.$$

□

Proof lemma 1.44. Let G_1, G_2 be two groups and let $f : G_1 \to G_2$ be a group homomorphism. Now consider C to be a conjugacy class of G_1. Take $a, b \in C$ arbitrary. Since a and b are in the same conjugacy class, there exists $g \in G$, such that $gag^{-1} = b$. Now

$f(b) = f(gag^{-1}) = f(g) \circ f(a) \circ f(g^{-1}) = f(g) \circ f(a) \circ f(g)^{-1}$, which is in the same conjugacy class as $f(a)$. Hence, all elements of a conjugacy class of the domain of f are mapped by a group homomorphism to one conjugacy class of the codomain of f. $\square$