Technische Universiteit Delft
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Delft Institute of Applied Mathematics

On the Rain-Wind Induced Vibrations of a Mass-Spring System

Verslag ten behoeve van het
Delft Institute of Applied Mathematics
als onderdeel ter verkrijging

van de graad van

BACHELOR OF SCIENCE
in
TECHNISCHE WISKUNDE

door

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Delft, Nederland
Augustus 2016

BSc verslag TECHNISCHE WISKUNDE

"On the Rain-Wind Induced Vibrations of a Mass-Spring System"

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Augustus, 2016 | Delft

Abstract

In this report, the rain-wind induced vibrations of cables are studied. This is done by modeling the cable cross-section as a mass-spring system with two time-varying masses. Thereafter, the solution of this model is approximated using a multiple timescale perturbation method. Lastly, for some choices of the time-varying masses the eigenfrequencies are analyzed, stability properties are derived, and approximations of the solutions are given.

Contents

1 Introduction

In September 1996, the Erasmus bridge was officially opened, but it had to be closed in October of the same year due to large vibrations of the cables. These vibrations were present during rainfall and wind speeds of around ten kilometers an hour. The rain caused two rivulets to form on the cable, which changed the aerodynamic properties of the cable. The low wind speed together with these new aerodynamic properties caused a lift force to act on the cable. This force made the cable vibrate with a low frequency, but with a high amplitude. Such vibrations are cause rain-wind induced vibrations.

To solve this problem, stronger dampers were installed to reduce the vibrations' amplitude. ([1]). The Erasmus bridge was not the only bridge to have this problem, other bridges and transmission cables also suffer from this problem. However, installing stronger dampers costs money and is not always an option. Therefore, the cause of these vibrations must be analyzed, so other methods of reducing them can be found. This cause can be analyzed by constructing a model of the cable and its environment.

In this paper, such a model is derived and analyzed. The derivation starts in section two by looking at a cross-section of and the environmental effects on the cable: the cross-section is modeled as a mass-spring system; the effect of the rain is implemented by adding two small time-varying masses, representing the rivulets, to the mass of the cable. In section three, the model is constructed using Newton's second law of motion with four forces: gravity, tension, drag, and lift. The last two depend on the locations of the rivulets, which are such that the instability criterion of den Hartog applies, meaning that the cable is unstable.

There have been other papers in which a similar model is constructed. For example, [2] models an ice ridge on transmission cables, and [3] models a single rivulet on a cable. The first paper mainly used its model as an example of an application of a theory, and did analyze the relation between the vibrations and cable properties in depth. The second paper models the entire cable as a beam, whereas this paper models a cross-section of the cable as a mass-spring system.

For the analysis the multiple time-scale perturbation method is used. This is possible as the rivulets' masses and sizes are small. This method is explained in short in section four and is illustrated using the Duffing equation. In section five, this method is applied to the present model and some situations are examined. Finally, in section six conclusions are drawn and in section seven recommendations are given.

2 Case

In this section, the case of the cable will be outlined. This will be done by looking at the environmental effects that act on the cable: rain and wind. Before this can be done, the coordinate system must be defined.

2.1 Coordinate system

In figure (2.1.1), the cross-section of the cable can be seen. Here, the y- and the z-axis are the vertical and the horizontal axis parallel to the cross-section, respectively. Additionally, up and to the right are defined ad positive. Furthermore, the displacement of this cross-section's center's from its center position will be described by the function $u(t)$. It is assumed that the cable only moves along the y-axis, thus $u(t)$ describes the vertical displacement of the cable.

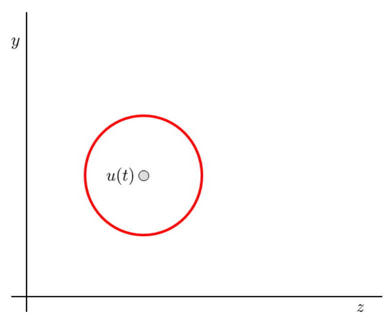


Figure 2.1.1: Coordinate system with the cross-section of the cable.

2.2 Effect of the rain

One of the environmental effects acting on the cable is rain (see figure (2.2.1)). Due to this rain, rivulets may form on the cable. In general, there will be two rivulets; one on the upper half, and one on the lower half. Their locations, α_1 and α_2 respectively, are assumed to be static ([4]).

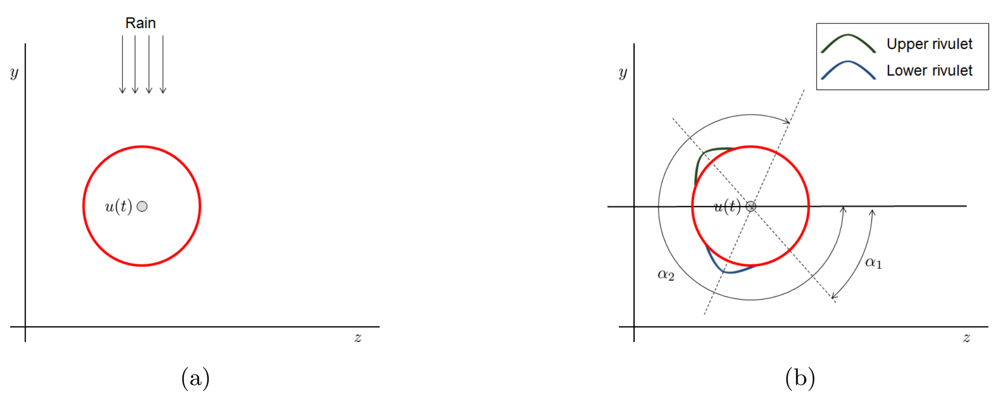


Figure 2.2.1: Cross sections of the cable. (a) With rain. (b) With upper and lower rivulets.

The sizes of these rivulets on the cross-section change periodically in time, due to the periodic water rivulets along the cable. Therefore, the masses of the rivulets can be modeled by:

$$\begin{aligned} m_1(t) &= M_1(1 + A_1 \sin(\omega_1 t + \beta_1)), \quad |A_1| \ll 1, \quad |M_1| \ll 1, \\ m_2(t) &= M_2(1 + A_2 \sin(\omega_2 t + \beta_2)), \quad |A_2| \ll 1, \quad |M_1| \ll 1, \end{aligned} \tag{2.2.1}$$

where M_1 and M_2 are the average masses; A_1 and A_2 the relative amplitudes; ω_1 and ω_2 the wave frequencies; and β_1 and β_2 the phase shifts of the upper and the lower rivulet respectively. If M_0 is the mass of the cable, the combined mass of the cable and rivulets can be modeled as:

$$\begin{aligned} m(t) &= M \cdot \left(1 + \tilde{A}_1 \sin(\omega_1 t + \beta_1) + \tilde{A}_2 \sin(\omega_2 t + \beta_2)\right), \\ M &= M_0 + M_1 + M_2, \\ \tilde{A}_1 &= \frac{M_1}{M} A_1, \\ \tilde{A}_2 &= \frac{M_2}{M} A_2. \end{aligned} \tag{2.2.2}$$

The interesting parts of these equations are the latter:

$$\begin{aligned} r_1(t) &= (1 + A_1 \sin(\omega_1 t + \beta_1)), \\ r_2(t) &= (1 + A_2 \sin(\omega_2 t + \beta_2)), \\ r(t) &= (1 + \tilde{A}_1 \sin(\omega_1 t + \beta_1) + \tilde{A}_2 \sin(\omega_2 t + \beta_2)), \end{aligned} \tag{2.2.3}$$

where r_1, r_2, and r are the relative amplitudes of the upper, the lower, and the combined rivulets respectively.

2.3 Effect of the wind

Another environmental effect acting on the cable is the wind (see figure (2.3.1)).

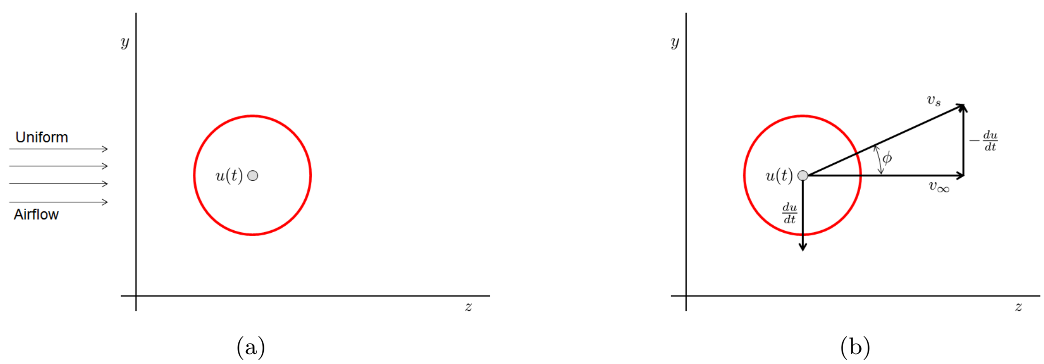


Figure 2.3.1: Cross-section of the cable. (a) With wind. (b) With relative wind-speed.

where v_∞ is the wind-speed. This is not the wind-speed relative to the cable, however, because the cable is moving with its own speed:

$$v = \frac{du}{dt}, \tag{2.3.1}$$

thus, the relative wind speed is:

$$v_s(t) = v_\infty - \frac{du}{dt}(t). \tag{2.3.2}$$

or

$$(v_s(t))^2 = (v_\infty)^2 + \left(\frac{du}{dt}(t)\right)^2. \tag{2.3.3}$$

Furthermore, ϕ is the angle between v_s and the horizontal, so:

$$\phi(t) = \arctan\left(\frac{-\frac{du}{dt}(t)}{v_\infty}\right). \tag{2.3.4}$$

As galloping is a low frequency oscillation, it is assumed that $|u'| \ll v_\infty$, thus $\left|\frac{u'}{v_\infty}\right| \ll 1$. Now, the previous equation can be expanded around $\frac{u'}{v_\infty} = 0$:

$$\phi(t) \approx \arctan\left(\frac{-u'}{v_\infty}\right) = -\left(\frac{u'(t)}{v_\infty}\right) + \frac{1}{3}\left(\frac{u'(t)}{v_\infty}\right)^3. \tag{2.3.5}$$

3 Constructing the Mathematical Model

In this section, a model for the cable will be derived. This will be done using the case described in the previous section and Newton's second law of motion:

$$\frac{d(vm)}{dt}(t) = \sum_i F_i(t), \tag{3.0.1}$$

where v is the vertical velocity; m the mass; and F_i are the forces acting along the vertical axis of the system. The functions v and m are already described in equations (2.3.1) and (2.2.1), but the forces still have to be determined.

3.1 Determining the Forces

There are four forces acting on the cable cross-section: gravity, tension, drag, and lift.

Firstly, gravity acts upon the mass of the cross section. As gravity pulls the cross-section down, and as an up-force is defined to be positive, the force due to gravity is:

$$F_g(t) = -gm(t), \tag{3.1.1}$$

where g is the gravitational acceleration.

Secondly, the tension of the cable pulls the cross-section to the opposite side of the displacement u. From Hooke's law it follows that:

$$F_t(t) = ku(t), \tag{3.1.2}$$

where k is a constant depending on the tension of the cable.

Thirdly, a drag force acts upon the cable due to the wind v_s in the direction of v_s. Because only vertical forces are of interest, the vertical component of the drag is derived. Furthermore, the amount of drag depends on the size and location of the rivulets:

$$F_d(t) = D(t)\sin(\phi(t)), \tag{3.1.3}$$

where D is a function depending on the on the cable, rivulets, and wind properties; and where ϕ is given by equation (2.3.5).

Lastly, due to the rivulets, the cable may become wing shaped. This results in a lift force acting upon the cross-section in direction perpendicular to v_s. Again, only the vertical component of the force is of interest, which is derived using $\cos(\phi)$. This force also depends on the size and location of the rivulets:

$$F_l(t) = L(t)\cos(\phi(t)), \tag{3.1.4}$$

where L is a function depending on the on the cable, rivulets, and wind properties; and where ϕ is given by equation (2.3.5).

3.2 Aerodynamics

The functions D and L are determined empirically, and are:

$$\begin{aligned} D(t) &= \frac{1}{2}\rho_a d v_s^2 \cdot C_D(t), \\ L(t) &= \frac{1}{2}\rho_a d v_s^2 \cdot C_L(t), \end{aligned} \tag{3.2.1}$$

where ρ_a is the air density, and d the diameter of the cross-section (without rivulets). The constants C_D and C_L are the quasi-steady drag and lift coefficients which, can be approximated, for θ_1 and θ_2 in a certain interval, by:

$$\begin{aligned} C_D(t) =& C_{D0} \cdot (\kappa_1 r_1(t) + \kappa_2 r_2(t)), \\ C_L(t) =& C_{L1} \cdot ((\phi(t) + \alpha_1 - \theta_1) \cdot \tilde{\kappa}_1 r_1(t) + (\phi(t) + \alpha_2 - \theta_2) \cdot \tilde{\kappa}_2 r_2(t)) \\ &+ C_{L3} \cdot \left((\phi(t) + \alpha_1 - \theta_1)^3 \cdot \tilde{\tilde{\kappa}}_1 r_1(t) + (\phi(t) + \alpha_2 - \theta_2)^3 \cdot \tilde{\tilde{\kappa}}_2 r_2(t)\right), \end{aligned} \tag{3.2.2}$$

where:

$$\begin{aligned} &C_{D0} > 0, \qquad C_{L1} < 0, \qquad C_{L3} > 0, \\ &C_{D0} + C_{L1} < 0, \\ \frac{1}{2}&C_{D0} + \frac{1}{6}C_{L1} + C_{L3} > 0, \end{aligned} \tag{3.2.3}$$

are experimentally derived constants ([2]). Furthermore, Remember that r_1, and r_2 are the relative changes of the mass, which are directly linked to the size of the rivulets and thus the amount of drag and lift. The κ_i are introduced, as the change in mass may have a different effect on each constant C, so $|\kappa_i - 1| \ll 1$. Furthermore, it is useful to change order and gather terms:

$$\begin{aligned} C_D(t) =& r_1(t) \cdot \kappa_1 C_{D0}, \\ &+ r_2(t) \cdot \kappa_2 C_{D0}, \\ C_L(t) =& r_1(t) \cdot \left((\phi(t) + \alpha_1 - \theta_1) \cdot \tilde{\kappa}_1 C_{L1} + (\phi(t) + \alpha_1 - \theta_1)^3 \cdot \tilde{\tilde{\kappa}}_1 C_{L3} \right) \\ &+ r_2(t) \cdot \left((\phi(t) + \alpha_2 - \theta_2) \cdot \tilde{\kappa}_2 C_{L1} + (\phi(t) + \alpha_2 - \theta_2)^3 \cdot \tilde{\tilde{\kappa}}_2 C_{L3} \right) \\ =& r_1(t) \cdot G_1(\phi(t)) \\ &+ r_2(t) \cdot G_2(\phi(t)). \end{aligned} \tag{3.2.4}$$

Substituting equations (2.2.2), (2.3.1) and the just determined forces into equation (3.0.1), the model becomes:

$$\frac{d\left(\frac{du}{dt}m\right)}{dt}(t) = -m(t)g - ku(t) + D(t)\sin(\phi(t)) + L(t)\cos(\phi(t)).$$

By expanding the left hand side, the above equation becomes:

$$\frac{d^2u}{dt^2}(t)m(t) + \frac{du}{dt}(t)\frac{dm}{dt}(t) = -m(t)g - ku(t) + D(t)\sin(\phi(t)) + L(t)\cos(\phi(t)).$$

This is equal to:

$$mu'' + mg = -ku - m'u' + D\sin(\phi) + L\cos(\phi),$$

where $'$ represents the derivative with respect to t. Substituting equations (3.2.1) and (3.2.4) into the equation above gives:

$$\begin{aligned} mu'' + mg =& -ku - m'u' \\ &+ \frac{\rho_a d v_s^2}{2} \left(r \cdot C_{D0}\sin(\phi) + r_1 \cdot G_1(\phi)\cos(\phi) + r_2 \cdot G_2(\phi)\cos(\phi) \right). \end{aligned}$$

This is a second order non-linear differential equation, which needs two initial values in order to be well-posed, so let:

$$\begin{aligned} u(0) &= u_0, \\ \frac{du}{dt}(0) &= u'_0. \end{aligned} \tag{3.2.5}$$

The current form of the model is not desirable, as it depends on an unknown function $\phi(t)$. In order to solve this problem, a Taylor expansion is used.

3.3 Taylor Expansion of $\phi(t)$

As $|\phi| \ll 1$, $C_{D0}\sin(\phi)$, $G_1(\phi)\cos(\phi)$, and $G_2(\phi)\cos(\phi)$ in this equation can now be expanded near $\phi = 0$ using a Taylor series of order three. This results in:

$$mu'' + mg = -ku - m'u' + \frac{\rho_a d v_\infty^2}{2}\left(a_0 + \frac{a1}{v_\infty}(u') + \frac{a_2}{v_\infty^2}(u')^2 + \frac{a_3}{v_\infty^3}(u')^3 \right), \tag{3.3.1}$$

where equations (2.3.3) and (2.3.5) are used, and:

$$
\begin{aligned}
a_0(t) &= r_1(t)\cdot\Big(\quad (\alpha_1-\gamma_1)\tilde{\kappa}_1 C_{L1} \quad +(\alpha_1-\gamma_1)^3\tilde{\tilde{\kappa}}_1 C_{L3}\Big) \\
&\quad +r_2(t)\cdot\Big(\quad (\alpha_2-\gamma_2)\tilde{\kappa}_2 C_{L1} \quad +(\alpha_2-\gamma_2)^3\tilde{\tilde{\kappa}}_2 C_{L3}\Big), \\
a_1(t) &= r_1(t)\cdot\Big(-\kappa_1 C_{D0} \quad -\tilde{\kappa}_1 C_{L1} \quad -3(\alpha_1-\gamma_1)^2\tilde{\tilde{\kappa}}_1 C_{L3}\Big) \\
&\quad +r_2(t)\cdot\Big(-\kappa_2 C_{D0} \quad -\tilde{\kappa}_2 C_{L1} \quad -3(\alpha_2-\gamma_2)^2\tilde{\tilde{\kappa}}_2 C_{L3}\Big), \\
a_2(t) &= r_1(t)\cdot\Big(\quad \tfrac{1}{2}(\alpha_1-\gamma_1)\tilde{\kappa}_1 C_{L1} \quad +\left(3(\alpha_1-\gamma_1)+\tfrac{1}{2}(\alpha_1-\gamma_1)^3\right)\tilde{\tilde{\kappa}}_1 C_{L3}\Big) \\
&\quad +r_2(t)\cdot\Big(\quad \tfrac{1}{2}(\alpha_2-\gamma_2)\tilde{\kappa}_2 C_{L1} \quad +\left(3(\alpha_2-\gamma_2)+\tfrac{1}{2}(\alpha_2-\gamma_2)^3\right)\tilde{\tilde{\kappa}}_2 C_{L3}\Big), \\
a_3(t) &= r_1(t)\cdot\Big(-\tfrac{1}{2}\kappa_1 C_{D0} \quad -\tfrac{1}{6}\tilde{\kappa}_1 C_{L1} \quad -\left(1+\tfrac{1}{2}(\alpha_1-\gamma_1)^2\right)\tilde{\tilde{\kappa}}_1 C_{L3}\Big) \\
&\quad +r_2(t)\cdot\Big(-\tfrac{1}{2}\kappa_2 C_{D0} \quad -\tfrac{1}{6}\tilde{\kappa}_2 C_{L1} \quad -\left(1+\tfrac{1}{2}(\alpha_2-\gamma_2)^2\right)\tilde{\tilde{\kappa}}_2 C_{L3}\Big).
\end{aligned}
\tag{3.3.2}
$$

Rewrite $a_i(t)$ as:

$$
a_i(t) = r_1(t)a_{i1} + r_2(t)a_{i2} = (1 + A_1\sin(\omega_1 t+\beta_1))a_{i1} + (1 + A_2\sin(\omega_2 t+\beta_2))a_{i2}, \tag{3.3.3}
$$

in order to shorten the equations to come. Substituting equations (2.2.2), (2.2.3) into equation (3.3.1) gives:

$$
\begin{aligned}
u'' + \frac{k}{M}u = &-g\Big[1 + \tilde{A}_1\sin(\omega_1 t+\beta_1) + \tilde{A}_2\sin(\omega_2 t+\beta_2)\Big] \\
&+\frac{S_1}{M}\Big[a_{01}(1 + A_1\sin(\omega_1 t+\beta_1)) + a_{02}(1 + A_2\sin(\omega_2 t+\beta_2))\Big] \\
&+\frac{S_1}{M}\Big[a_{11}(1 + A_1\sin(\omega_1 t+\beta_1)) + a_{12}(1 + A_2\sin(\omega_2 t+\beta_2))\Big]\left(\frac{u'}{v_\infty}\right) \\
&+\frac{S_1}{M}\Big[a_{21}(1 + A_1\sin(\omega_1 t+\beta_1)) + a_{22}(1 + A_2\sin(\omega_2 t+\beta_2))\Big]\left(\frac{u'}{v_\infty}\right)^2 \\
&+\frac{S_1}{M}\Big[a_{31}(1 + A_1\sin(\omega_1 t+\beta_1)) + a_{32}(1 + A_2\sin(\omega_2 t+\beta_2))\Big]\left(\frac{u'}{v_\infty}\right)^3 \\
&-\Big[\omega_1\tilde{A}_1\cos(\omega_1 t+\beta_1) + \omega_2\tilde{A}_2\cos(\omega_2 t+\beta_2)\Big](u') \\
&-\Big[\tilde{A}_1\sin(\omega_1 t+\beta_1) + \tilde{A}_2\sin(\omega_2 t+\beta_2)\Big](u''),
\end{aligned}
\tag{3.3.4}
$$

where:

$$
S_1 = \frac{\rho_a d v_\infty^2}{2}, \tag{3.3.5}
$$

is introduced to simplify the equation. This form of the model is already more desirable, however, it is not in a general form. In the next subsection, it is generalized by scaling u and t so that they become dimensionless.

3.4 Nondimensionalization

By substituting:

$$\bar{u} = \frac{1}{v_\infty}\sqrt{\frac{k}{M}}S_2^{-1}\left(u + \frac{gM}{k}\right), \qquad \bar{t} = \sqrt{\frac{k}{M}}t,$$
$$\Longleftrightarrow \tag{3.4.1}$$
$$u = v_\infty\sqrt{\frac{M}{k}}S_2\bar{u} - \frac{gM}{k}, \qquad t = \sqrt{\frac{M}{k}}\bar{t},$$

and multiplying by:

$$\frac{1}{v_\infty}\sqrt{\frac{M}{k}}S_2^{-1},$$

where:

$$S_2 = \sqrt{-\frac{a_{11}+a_{12}}{a_{31}+a_{32}}}, \tag{3.4.2}$$

. equation (3.3.4) becomes dimensionless:

$$\begin{aligned}
&\ddot{\bar{u}} + \bar{u} = \\
&-\frac{g}{v_\infty}\sqrt{\frac{M}{k}}S_2^{-1}\left[\tilde{A}_1\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right) + \tilde{A}_2\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right)\right] \\
&+\frac{S_1}{v_\infty}\sqrt{\frac{1}{kM}}S_2^{-1}\left[a_{01}\left(1+A_1\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)\right) + a_{02}\left(1+A_2\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right)\right)\right] \\
&+\frac{S_1}{v_\infty}\sqrt{\frac{1}{kM}}\left[a_{11}\left(1+A_1\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)\right) + a_{12}\left(1+A_2\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right)\right)\right](\dot{\bar{u}}) \\
&+\frac{S_1}{v_\infty}\sqrt{\frac{1}{kM}}S_2\left[a_{21}\left(1+A_1\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)\right) + a_{22}\left(1+A_2\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right)\right)\right](\dot{\bar{u}})^2 \\
&+\frac{S_1}{v_\infty}\sqrt{\frac{1}{kM}}S_2^2\left[a_{31}\left(1+A_1\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)\right) + a_{32}\left(1+A_2\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right)\right)\right](\dot{\bar{u}})^3 \\
&-\sqrt{\frac{M}{k}}\left[\omega_1\tilde{A}_1\cos\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right) + \omega_2\tilde{A}_2\cos\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right)\right](\dot{\bar{u}}) \\
&-\left[\tilde{A}_1\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right) + \tilde{A}_2\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right)\right](\ddot{\bar{u}}),
\end{aligned}$$

where $\dot{}$ denotes the derivative with respect to $\bar{t}$. This equation can be rewritten as:

$$\ddot{\bar{u}} + \dot{\bar{u}} = \varepsilon f(\bar{t}, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}), \tag{3.4.3}$$

where:

$$\begin{aligned}
f(\bar{t}, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}) = {} & S_2^{-1}(C_0 + P_0(\bar{t})) \\
&+(C_1 + P_1(\bar{t}))(\dot{\bar{u}}) \\
&+S_2(C_2 + P_2(\bar{t}))(\dot{\bar{u}})^2 \\
&+S_2^2(C_3 + P_3(\bar{t}))(\dot{\bar{u}})^3 \\
&+(C_4 + P_4(\bar{t}))(\ddot{\bar{u}}), \\
\varepsilon = {} & \frac{1}{v_\infty}\sqrt{\frac{1}{kM}},
\end{aligned} \tag{3.4.4}$$

where C_i are constants, and $P_i(\bar{t})$ are periodic functions:

$$\begin{aligned}
C_0 &= S_1\left(a_{01}+a_{02}\right), \\
C_1 &= S_1\left(a_{11}+a_{12}\right), \\
C_2 &= S_1\left(a_{21}+a_{22}\right), \\
C_3 &= S_1\left(a_{31}+a_{32}\right), \\
C_4 &= 0,
\end{aligned}$$

and:

$$\begin{aligned}
P_0(\bar{t}) =& \left(S_1 a_{01} A_1 - 2Mg\tilde{A}_1\right)\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right) \\
&+\left(S_1 a_{02} A_2 - 2Mg\tilde{A}_2\right)\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right) \\
=& p_{01}\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)+p_{02}\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right), \\
P_1(\bar{t}) =& S_1 A_1\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)-v_\infty M\omega_1\tilde{A}_1\cos\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right) \\
&+S_1 A_2\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right)-v_\infty M\omega_2\tilde{A}_2\cos\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right) \\
=& p_{11}\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)+p_{13}\cos\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right) \\
&+p_{12}\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right)+p_{14}\cos\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right), \\
P_2(\bar{t}) =& S_1\left(a_{21}A_1\right)\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right) \\
&+S_1\left(a_{22}A_2\right)\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right) \\
=& p_{21}\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)+p_{22}\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right), \\
P_3(\bar{t}) =& S_1\left(a_{31}A_1\right)\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right) \\
&+S_1\left(a_{32}A_2\right)\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right) \\
=& p_{31}\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)+p_{32}\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right), \\
P_4(\bar{t}) =& -v_\infty\sqrt{kM}\tilde{A}_1\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right) \\
&-v_\infty\sqrt{kM}\tilde{A}_2\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right) \\
=& p_{41}\sin\left(\omega_1\sqrt{\frac{M}{k}}\bar{t}+\beta_1\right)+p_{42}\sin\left(\omega_2\sqrt{\frac{M}{k}}\bar{t}+\beta_2\right).
\end{aligned} \tag{3.4.5}$$

By assuming that the locations α_1 and α_2 of the rivulets are such that they satisfy the instability criterion of den Hartog:

$$\alpha_1 = \gamma_1 + O(\varepsilon),$$
$$\alpha_2 = \gamma_2 + O(\varepsilon),$$

the coefficients a_{ij} and p_{ij} can be simplified. The a_{ij} in equations (3.3.2) become:

$$\begin{aligned}
a_{01} &= O(\varepsilon),\\
a_{02} &= O(\varepsilon),\\
a_{11} &= -\kappa_1 C_{D0} - \tilde{\kappa}_1 C_{L1} + O(\varepsilon),\\
a_{12} &= -\kappa_2 C_{D0} - \tilde{\kappa}_2 C_{L1} + O(\varepsilon),\\
a_{21} &= O(\varepsilon),\\
a_{22} &= O(\varepsilon),\\
a_{31} &= -\frac{1}{2}\kappa_1 C_{D0} - \frac{1}{6}\tilde{\kappa}_1 C_{L1} - \tilde{\tilde{\kappa}}_1 C_{L3} + O(\varepsilon),\\
a_{32} &= -\frac{1}{2}\kappa_2 C_{D0} - \frac{1}{6}\tilde{\kappa}_2 C_{L1} - \tilde{\tilde{\kappa}}_2 C_{L3} + O(\varepsilon).
\end{aligned}$$

From the assumptions about the aerodynamic constants (3.2.3), it then follows that:

$$\begin{aligned}
C_0 &= 0,\\
C_1 &> 0,\\
C_2 &= 0,\\
C_3 &< 0,\\
C_4 &= 0.
\end{aligned} \tag{3.4.6}$$

Note S_2 can also be written using these constants:

$$S_2 = \sqrt{-\frac{C_1}{C_3}} \tag{3.4.7}$$

The constants p_{ij} become:

$$\begin{aligned}
p_{01} &= -2M_1 A_1 g,\\
p_{02} &= -2M_2 A_2 g,\\
p_{11} &= S_1 A_1,\\
p_{12} &= S_2 A_2,\\
p_{13} &= -v_\infty \omega_1 M_1 A_1,\\
p_{13} &= -v_\infty \omega_2 M_2 A_2,\\
p_{21} &= 0,\\
p_{22} &= 0,\\
p_{31} &= S_1 A_1 a_{31},\\
p_{32} &= S_1 A_1 a_{32},\\
p_{41} &= v_\infty \sqrt{\frac{k}{M}} M_1 A_1,\\
p_{42} &= v_\infty \sqrt{\frac{k}{M}} M_2 A_2 l,
\end{aligned} \tag{3.4.8}$$

where equation (2.2.2) is used.

Furthermore, the initial values for $\bar{u}$ follow by substituting equation (3.4.1) into (3.2.5):

$$\begin{aligned}
\bar{u}(0) &= \bar{u}_0 = u_0 + \frac{g}{v_\infty}\sqrt{\frac{M}{k}} S_2^{-1},\\
\dot{\bar{u}}(0) &= \dot{\bar{u}}_0 = u_0'.
\end{aligned} \tag{3.4.9}$$

Now that the mathematical model has been derived, a method is needed to solve or approximate its solution. In the next section, the multiple time-scale perturbation method for approximating the model is explained.

4 Multiple time-scale perturbation method

In order to approximate the solution of (3.4.3), the multiple timescale perturbation method is used. This method is explained in depth in *Problems in Perturbation* by Ali Hasan Nayfeh [5]. In this section, it will be explained in short.

4.1 Assumptions

Let:

$$u' + u = \varepsilon f(u, u', u'), \tag{4.1.1}$$

be a differential equation with dimensionless variables and small ε, which is to be approximated using the multiple time-scale perturbation method. According to the method, the solution $u(t)$ can be written as:

$$\begin{aligned} u(t;\varepsilon) &= \hat{u}(t, \varepsilon t, \varepsilon^2 t, \ldots; \varepsilon) \\ &= \hat{u}\left(T_0, T_1, T_2, \ldots; \varepsilon\right) \\ &= \hat{u}_0\left(T_0, T_1, T_2, \ldots\right) + \varepsilon \hat{u}_1\left(T_0, T_1, T_2, \ldots\right) + \varepsilon^2 \hat{u}_2\left(T_0, T_1, T_2, \ldots\right) + \ldots, \end{aligned} \tag{4.1.2}$$

where:

$$T_i = \varepsilon^i t,$$

are called time-scales. Furthermore, using the chain-rule, it follows that:

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \ldots. \tag{4.1.3}$$

4.2 Procedure

In order to find an $O(\varepsilon^n)$ approximation of u, substitute equation (4.1.2) and (4.1.3) into equation (4.1.1) and disregard all powers of ε higher than n. Or equivalently, substitute u and $\frac{d}{dt}$ for:

$$\begin{aligned} u(t;\varepsilon) &= \hat{u}_0\left(T_0, T_1, \ldots, T_{n-1}\right) + \ldots + \varepsilon^{n-1} \hat{u}_{n-1}\left(T_0, T_1, \ldots, T_{n-1}\right) + O(\varepsilon^n) \\ \frac{d}{dt} &= \frac{\partial}{\partial T_0} + \ldots + \varepsilon^{n-1} \frac{\partial}{\partial T_{n-1}} + O(\varepsilon^n). \end{aligned}$$

Next, equate all coefficients of like powers of ε. This results in a set of $n + 1$ differential equations; one for each $\hat{u}_i$. Furthermore, the dependence on $T_0, \ldots, T_n$ should be determined by requiring that each u_k ($1 \le k \le n$) is free of secular terms.

This procedure is made more clear in the next subsection, where it is illustrated using the Duffing equation.

4.3 Example: the Duffing Equation

For example, in order to find a 1^{st}-order approximation of the Duffing equation:

$$\begin{aligned} &u'' + u + \varepsilon u^3 = 0, \qquad (f(u, \dot{u}, \ddot{u}) = -u^3), \\ &u(0) = \alpha, \\ &u'(0) = \beta, \end{aligned} \tag{4.3.1}$$

u and $\frac{d}{dt}$ should be substituted with:

$$u = \hat{u}_0 + \varepsilon \hat{u}_1 + O(\varepsilon^2), \tag{4.3.2}$$

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + O(\varepsilon^2). \tag{4.3.3}$$

By equation coefficients of like powers of ε, a set of two equations follows:

$$\frac{\partial^2 \hat{u}_0}{\partial T_0^2} + \hat{u}_0 = 0,$$
$$\frac{\partial^2 \hat{u}_1}{\partial T_0^2} + \hat{u}_1 = -\hat{u}_0^3 - 2\frac{\partial^2 \hat{u}_0}{\partial T_0 T_1},$$

and initial values:

$$\hat{u}_0(0,0) = \alpha, \ \left.\frac{\partial \hat{u}_0(T_0,T_1)}{\partial T_0}\right|_{T_0=0,T_1=0} = \beta,$$
$$\hat{u}_1(0,0) = 0, \ \left.\frac{\partial \hat{u}_1(T_0,T_1)}{\partial T_0}\right|_{T_0=0,T_1=0} = 0.$$

The solution of the first equation is:

$$\hat{u}_0 = A(T_1)\sin(T_0) + B(T_1)\cos(T_0) \tag{4.3.4}$$

where A and B are unknown functions of T_1. Due to the initial values, it follows that:

$$A(0) = \alpha, \ B(0) = \beta. \tag{4.3.5}$$

Substituting the solution for $\hat{u}_0$ in the second equation, it becomes:

$$\begin{aligned}\frac{\partial^2 \hat{u}_1}{\partial T_0^2} + \hat{u}_1 =& - A^3\sin^3(T_0) - 3A^2B\sin^2(T_0)\cos(T_0) - 3AB^2\sin(T_0)\cos^2(T_0) - B^3\cos^3(T_0) \\ & - 2A'\cos(T_0) + 2B'\sin(T_0) \\ =& \left(\frac{dB}{dT_1} - \frac{3}{4}A^3 - \frac{3}{4}AB^2\right)\sin(T_0) - \left(\frac{dA}{dT_1} + \frac{3}{4}B^3 + \frac{3}{4}A^2B\right)\cos(T_0) \\ & \left(\frac{1}{4}A^3 - \frac{3}{4}AB^2\right)\sin(3T_0) - \left(\frac{1}{4}B^3 + \frac{3}{4}A^2B\right)\cos(3T_0),\end{aligned}$$

where:

$$\left(\frac{dA}{dT_1} + \frac{3}{4}B^3 + \frac{3}{4}A^2B\right)\cos(T_0),$$
$$\left(\frac{dB}{dT_1} - \frac{3}{4}A^3 - \frac{3}{4}AB^2\right)\sin(T_0),$$

produce secular terms, as they contain terms of equation (4.3.4). In order to remove these terms, the following equations must apply:

$$\begin{aligned}\frac{dA}{dT_1} &= -\frac{3}{4}(A^2+B^2)B, \\ \frac{dB}{dT_1} &= \frac{3}{4}(B^2+A^2)A.\end{aligned} \tag{4.3.6}$$

From these, A and B can be solved. Multiplying the first equation by A, the second by B, and adding them gives:

$$\begin{aligned} & \frac{dA}{dT_1}A + \frac{dB}{dT_1}B = 0, \\ \iff & \quad AdA + BdB = 0, \\ \iff & \quad A^2 + B^2 = R_1.\end{aligned}$$

Due to the initial values of A and B (4.3.5), it follows that $R_1 = \alpha^2 + \beta^2$. Substituting this into equation (4.3.6) gives:

$$\frac{dA}{dT_1} = -\frac{3}{4}\left(\alpha^2+\beta^2\right)B, \quad \wedge \quad \frac{dB}{dT_1} = \frac{3}{4}\left(\alpha^2+\beta^2\right)A. \tag{4.3.7}$$

Differentiating the left equation with respect to T_1 and inserting the right equation gives:

$$\frac{d^2A}{dT_1^2} = -\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)\right)^2 A,$$

which has solution:

$$A(T_1) = C_1 \sin\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right) + C_2\cos\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right).$$

From the first equation of system (4.3.7), it follows that:

$$\begin{aligned}B(T_1) &= -\frac{4}{3\left(\alpha^2+\beta^2\right)}\frac{dA}{dT_1}\\ &= C_1\cos\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right) - C_2\sin\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right).\end{aligned}$$

From the initial values (4.3.5), it follows that $C_1 = \beta$ and $C_2 = \alpha$, so A and B become:

$$\begin{aligned}A(T_1) &= \beta\sin\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right) + \alpha\cos\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right),\\ B(T_1) &= -\alpha\sin\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right) + \beta\cos\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right).\end{aligned}$$

The 1^{st}-order solution becomes:

$$\begin{aligned}\hat{u}_0(T_0,T_1) =& \left(\beta\sin\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right) + \alpha\cos\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right)\right)\sin(T_0)\\ &+\left(-\alpha\sin\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right) + \beta\cos\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1\right)\right)\cos(T_0),\\ =&\beta\cos\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1 - T_0\right) - \alpha\sin\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)T_1 - T_0\right).\end{aligned}$$

It follows that:

$$\begin{aligned}u(t,\varepsilon) &= \hat{u}_0(t,\varepsilon t) + O(\varepsilon)\\ &= \beta\cos\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)\varepsilon t - t\right) - \alpha\sin\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)\varepsilon t - t\right) + O(\varepsilon)\\ &= \beta\cos\left(\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)\varepsilon - 1\right)t\right) - \alpha\sin\left(\left(\frac{3}{4}\left(\alpha^2+\beta^2\right)\varepsilon - 1\right)t\right) + O(\varepsilon).\end{aligned}$$

In order to check the accuracy of this approximation, it needs to be compared to the analytical solution. However, as the entire analytical solution is difficult to calculate, the periods of the analytical and the approximated solution, T_{an} and T_{ap} respectively, are compared instead.

It is easily seen that the approximation $u(t,\varepsilon)$ has period:

$$T_{ap} = \frac{2\pi}{1-\frac{3}{4}\varepsilon}, \tag{4.3.8}$$

where the initial values (4.3.5) have been used. In order to show that the analytical solutions period T_{an} exists, equation (4.3.1) is multiplied by $\frac{du}{dt}$:

$$\begin{aligned}&\frac{1}{2}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}u^2 + \frac{\varepsilon}{4}u^4 = \frac{1}{2}\beta^2 + \frac{1}{2}\alpha^2 + \frac{\varepsilon}{4}\alpha^4,\\ \iff\quad &\left(\frac{du}{dt}\right)^2 + u^2\left(1+\frac{\varepsilon}{2}u^2\right) = 1 + \frac{\varepsilon}{2},\\ \iff\quad &\frac{du}{dt} = \pm\sqrt{1+\frac{\varepsilon}{2} - u^2 - \frac{\varepsilon}{2}u^4},\\ \iff\quad &\frac{1}{\sqrt{1+\frac{\varepsilon}{2}-u^2-\frac{\varepsilon}{2}u^4}}\frac{du}{dt} = \pm 1.\end{aligned}$$

where the initial values (4.3.5) have been used. From the second equation, it follows that the parametric plot of u and $\frac{du}{dt}$ is elliptical, and thus T_{an} exists.

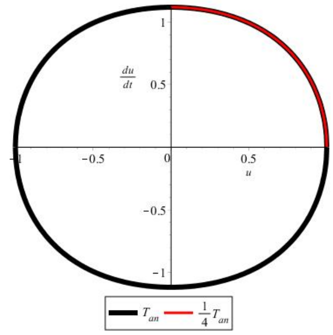


Figure 4.3.1: Parametric plot of u and $\frac{u}{dt}$ with $\varepsilon = 0.5$.

In order to calculate this period, the ellipse is integrated from $t = 0$ to $t = \frac{1}{4}T_{an}$ (see figure (4.3.1)):

$$\int_0^{\frac{T_{an}}{4}} \frac{1}{\sqrt{1 + \frac{\varepsilon}{2} - u^2 - \frac{\varepsilon}{2}u^4}} \frac{du}{dt} dt = \pm \int_0^{\frac{T_{an}}{4}} 1dt,$$

$$\iff \quad \int_0^{\frac{T_{an}}{4}} \frac{1}{\sqrt{1 + \frac{\varepsilon}{2} - u^2 - \frac{\varepsilon}{2}u^4}} \frac{du}{dt} dt = \pm \frac{T_{an}}{4}.$$

From figure (4.3.1), it follows that $u(t) = 1$ at $t = 0$ and $u(t) = 0$ at $t = \frac{T_{an}}{4}$. Using this information, the above equation becomes:

$$T_{an} = 4 \int_0^1 \frac{1}{\sqrt{1 + \frac{\varepsilon}{2} - u^2 - \frac{\varepsilon}{2}u^4}} du,$$

where the sign has been chosen to make T_{an} positive. This is a complete elliptic integral of the first kind ([6]):

$$T_{an} = \frac{4}{\sqrt{1+\varepsilon}} K(k),$$

$$k = \frac{\sqrt{2\varepsilon}}{2\sqrt{1+\varepsilon}}.$$

The elliptic integral $K(k)$ can be calculated using the arithmetic-geometric mean (agm) ([7]):

$$K(k) = \frac{\frac{\pi}{2}}{agm(1, \sqrt{1-k^2})}.$$

In the table below, the analytical and the approximated period and their difference can be seen for several values of ε, which were calculated using Appendix B:

ε	k	T_{an}	T_{ap}	$\lvert T_{an} - T_{ap} \rvert$
0.1	0.2132007163	6.060656736	6.792632765	0.731976029
0.01	0.0703597544	6.259762304	6.330665297	0.070902993
0.001	0.0223495078	6.280830511	6.287901234	0.007070723
0.0001	0.0070707142	6.282949703	6.283656581	0.000706878

Table 1: Period of the analytical and approximated solution of the Duffing equation for several values of ε.

As the difference between T_{an} and T_{ap} is bounded by 8ε, it follows that the approximation of the solution, at least its period, is of $O(\varepsilon)$.

5 Approximating the Solution of the Mathematical Model

Now that the multiple time-scale perturbation analysis method is explained, it is possible to find an approximation of the solution of the mathematical model (3.4.3):

$$\ddot{\bar{u}} + \dot{\bar{u}} = \varepsilon f(\bar{t}, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}),$$

with initial values (3.4.9):

$$\begin{aligned} \bar{u}(0) &= \frac{g}{v_\infty}\sqrt{\frac{M}{k}} S_2^{-1}, \\ \dot{\bar{u}}(0) &= 0. \end{aligned}$$

5.1 Multiple Time-Scale Perturbation Analysis

In accordance with the method, let:

$$\begin{aligned} \bar{u}(t, \varepsilon) &= z_0(T_0, T_1, \varepsilon) + \varepsilon z_1(T_0, T_1, \varepsilon) + O(\varepsilon^2), \\ \frac{d}{dt} &= \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1}. \end{aligned} \tag{5.1.1}$$

Inserting this into the model and initial values results in two coupled differential equations:

$$\frac{\partial^2 z_0}{\partial T_0^2} + z_0 = 0, \tag{5.1.2}$$

$$\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = f\left(T_0, z_0, \frac{\partial z_0}{T_0}, \frac{\partial^2 z_0}{T_0^2}\right) - 2\frac{\partial^2 z_0}{\partial T_1 T_0}, \tag{5.1.3}$$

and initial values:

$$\begin{aligned} z_0(0,0) &= \frac{g}{v_\infty}\sqrt{\frac{M}{k}} S_2^{-1}, \qquad && \left.\frac{\partial z_0(T_0, T_1)}{\partial T_0}\right|_{T_0=0, T_1=0} = 0, \\ z_1(0,0) &= 0, && \left.\frac{\partial z_1(T_0, T_1)}{\partial T_0}\right|_{T_0=0, T_1=0} = 0. \end{aligned}$$

The first order equation will always have a solution of the form:

$$z_0(T_0, T_1) = K(T_1)\sin(T_0) + L(T_1)\cos(T_0), \tag{5.1.4}$$

where the functions K and L may differ for different choices of ω_1 and ω_2.

Furthermore, from the initial values, it follows that:

$$\begin{aligned} K(0) &= 0, \\ L(0) &= \frac{g}{v_\infty}\sqrt{\frac{M}{k}} S_2^{-1}. \end{aligned} \tag{5.1.5}$$

In order to find K and L, the $O(\varepsilon)$ equation must be computed,

5.2 The $O(\varepsilon)$ Equation

In order to find the $O(1)$ approximation, the function f (3.4.4) must be substituted into the ε-order equation (5.1.3):

$$\begin{aligned}
\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = S_2^{-1}&(C_0 + P_0(T_0)) \\
&+(C_1 + P_1(T_0)) \left[\frac{\partial z_0}{\partial T_0}\right] \\
&+S_2(C_2 + P_2(T_0)) \left[\frac{\partial z_0}{\partial T_0}\right]^2 \\
&+S_2^2(C_3 + P_3(T_0)) \left[\frac{\partial z_0}{\partial T_0}\right]^3 \\
&+(C_4 + P_4(T_0)) \left[\frac{\partial^2 z_0}{\partial T_0^2}\right] \\
&-2\frac{\partial^2 z_0}{\partial T_1 T_0}.
\end{aligned}$$

Inserting the 1^{st}-order solution (5.1.4); rewriting powers and products of sines and cosines as sines and cosines; and splitting into terms independent and dependent of ω_1 and ω_2; substituting (3.4.5); again rewriting products of sines and cosines as sines and cosines; gathering terms; and finally inserting C_0, C_2, and C_4 from (3.4.6) and p_{21}, and p_{22} from (3.4.8) gives:

$$
\begin{aligned}
\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = &- \left[C_1 L - 2\frac{dL}{dT_1} + 0.75 S_2^2 C_3 (L^3 + K^2 L)\right] \sin(T_0) \\
&+ \left[C_1 K - 2\frac{dK}{dT_1} + 0.75 S_2^2 C_3 (K^3 + K L^2)\right] \cos(T_0) \\
&- \left[S_2^2 C_3 (0.75 K^2 L + 0.25 L^3)\right] \sin(3T_0) \\
&+ \left[S_2^2 C_3 (0.75 K L^2 - 0.25 K^3)\right] \cos(3T_0) \\
&+ \left[S_2^{-1} p_{01}\right] \cos(\beta_1) \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0\right) \\
&+ \left[S_2^{-1} p_{01}\right] \sin(\beta_1) \cos\left(\omega_1 \sqrt{\frac{M}{k}} T_0\right) \\
&+ \left[S_2^{-1} p_{02}\right] \cos(\beta_2) \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0\right) \\
&+ \left[S_2^{-1} p_{02}\right] \sin(\beta_2) \cos\left(\omega_2 \sqrt{\frac{M}{k}} T_0\right) \\
&+ 0.5\Big\{[p_{11} K - p_{41} L + 0.75 S^2 p_{31} (K^3 + K L^2) + p_{13} L] \sin(\beta_1) \\
&\qquad - [p_{11} L + p_{41} K + 0.75 S^2 p_{31} (L^3 + K^2 L) - p_{13} K] \cos(\beta_1)\Big\} \cdot \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
&+ 0.5\Big\{[p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L) - p_{13} K] \sin(\beta_1) \\
&\qquad + [p_{11} K - p_{41} L + 0.75 S_2^2 p_{31} (K^3 + K L^2) + p_{13} L] \cos(\beta_1)\Big\} \cdot \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
&+ 0.5\Big\{[p_{11} K - p_{41} L + 0.75 S_2^2 p_{31} (K^3 + K L^2) - p_{13} L] \sin(\beta_1) \\
&\qquad + [p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L) + p_{13} K] \cos(\beta_1)\Big\} \cdot \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
&+ 0.5\Big\{[p_{11} K - p_{41} L + 0.75 S_2^2 p_{31} (K^3 + L^2 K) - p_{13} L] \cos(\beta_1) \\
&\qquad - [p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L) + p_{13} K] \sin(\beta_1)\Big\} \cdot \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
&+ 0.5\Big\{[p_{12} K - p_{42} L + 0.75 S_2^2 p_{32} (K^3 + K L^2) + p_{14} L] \sin(\beta_2) \\
&\qquad - [p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L) - p_{14} K] \cos(\beta_2)\Big\} \cdot \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
&+ 0.5\Big\{[p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L) - p_{14} K] \sin(\beta_2) \\
&\qquad + [p_{12} K - p_{42} L + 0.75 S_2^2 p_{32} (K^3 + K L^2) + p_{14} L] \cos(\beta_2)\Big\} \cdot \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
&+ 0.5\Big\{[p_{12} K - p_{42} L + 0.75 S_2^2 p_{32} (K^3 + K L^2) - p_{14} L] \sin(\beta_2) \\
&\qquad + [p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L) + p_{14} K] \cos(\beta_2)\Big\} \cdot \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
&+ 0.5\Big\{[p_{12} K - p_{42} L + 0.75 S_2^2 p_{32} (K^3 + K L^2) - P_{14} L] \cos(\beta_2) \\
&\qquad - [p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L) + p_{14} K] \sin(\beta_2)\Big\} \cdot \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
&- 0.5\Big\{[S_2^2 p_{31} (0.75 K L^2 - 0.25 K^3)] \sin(\beta_1) + [S_2^2 p_{31} (0.75 K^2 L + 0.25 L^3)] \cos(\beta_1)\Big\} \cdot \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 3\right) T_0\right) \\
&+ 0.5\Big\{-[S_2^2 p_{31} (0.75 K L^2 - 0.25 K^3)] \cos(\beta_1) + [S_2^2 p_{31} (0.75 K^2 L + 0.25 L^3)] \sin(\beta_1)\Big\} \cdot \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 3\right) T_0\right) \\
&+ 0.5\Big\{-[S_2^2 p_{31} (0.75 K L^2 - 0.25 K^3)] \sin(\beta_1) + [S_2^2 p_{31} (0.75 K^2 L + 0.25 L^3)] \cos(\beta_1)\Big\} \cdot \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 3\right) T_0\right) \\
&- 0.5\Big\{[S_2^2 p_{31} (0.75 K L^2 - 0.25 K^3)] \cos(\beta_1) + [S_2^2 p_{31} (0.75 K^2 L + 0.25 L^3)] \sin(\beta_1)\Big\} \cdot \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 3\right) T_0\right) \\
&- 0.5\Big\{[S_2^2 p_{32} (0.75 K L^2 - 0.25 K^3)] \sin(\beta_2) + [S_2^2 p_{32} (0.75 K^2 L + 0.25 L^3)] \cos(\beta_2)\Big\} \cdot \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 3\right) T_0\right) \\
&+ 0.5\Big\{-[S_2^2 p_{32} (0.75 K L^2 - 0.25 K^3)] \cos(\beta_2) + [S_2^2 p_{32} (0.75 K^2 L + 0.25 L^3)] \sin(\beta_2)\Big\} \cdot \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 3\right) T_0\right) \\
&+ 0.5\Big\{-[S_2^2 p_{32} (0.75 K L^2 - 0.25 K^3)] \cos(\beta_2) + [S_2^2 p_{32} (0.75 K^2 L + 0.25 L^3)] \sin(\beta_2)\Big\} \cdot \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 3\right) T_0\right) \\
&- 0.5\Big\{[S_2^2 p_{32} (0.75 K L^2 - 0.25 K^3)] \cos(\beta_2) + [S_2^2 p_{32} (0.75 K^2 L + 0.25 L^3)] \sin(\beta_2)\Big\} \cdot \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 3\right) T_0\right).
\end{aligned}
$$

Intermediate steps of this calculation can be seen in Appendix A.

In this equation, the terms with $\sin(T_0)$ and $\cos(T_0)$ produce secular terms, as these are already solutions of the 1^{st}-order equation. However, the coefficients of these temrs depend on ω_1, ω_2, M, and k, as will be explained in the next subsection.

5.3 Resonance frequencies

For certain relations between the frequencies ω_i and system properties M and k, the system will resonate. Frequencies which satisfy such a relation are called resonance frequencies. For example, if:

$$\omega_1\sqrt{\frac{M}{k}} = 1,$$

then:

$$\sin\left(\omega_1\sqrt{\frac{M}{k}}T_0\right) = \sin(T_0), \tag{5.3.1}$$

and so, the coefficient of $\sin(T_0)$ becomes:

$$2\frac{dL}{dT_1} + S_2^{-1}p_{01} - C_1L + -0.75S_2^2C_3L(L^2+K^2).$$

There are many relations between ω_i, and M and k which produces resonances. All of them are listed here:

$$\begin{aligned}
\omega_1\sqrt{\frac{M}{k}} &= 0, & \omega_2\sqrt{\frac{M}{k}} &= 0,\\
\omega_1\sqrt{\frac{M}{k}} &= 1, & \omega_2\sqrt{\frac{M}{k}} &= 1,\\
\omega_1\sqrt{\frac{M}{k}} &= 2, & \omega_2\sqrt{\frac{M}{k}} &= 2,\\
\omega_1\sqrt{\frac{M}{k}} &= 4, & \omega_2\sqrt{\frac{M}{k}} &= 4.
\end{aligned}$$

Note that it is possible that ω_1 and ω_2 satisfy such a relation at the same time, and each combination will result in a different solution.

5.4 Detuning Frequencies

It can be imagined, that resonance still occurs when, for example, a frequency ω_i is close to a resonance frequency:

$$\omega_1\sqrt{\frac{M}{k}} = 1 + \gamma\varepsilon, \qquad \gamma \in \mathbb{R},$$

this is called detuning. This will also alter the terms producing secular terms, as:

$$\sin\left(\omega_1\sqrt{\frac{M}{k}}T_0\right) = \sin\left(T_0 + \gamma\varepsilon T_0\right) = \sin\left(T_0 + \gamma T_1\right) = \cos(\gamma T_1)\sin(T_0) + \sin(\gamma T_1)\cos(T_0).$$

If $\gamma = 0$, the above equation is the same as the one for the resonance frequency (5.3.1), as ω_1 is now a resonance frequency.

If $\gamma = O(\frac{1}{\varepsilon})$, then $\gamma T_1 = O(T_0)$, and thus the above equation becomes:

$$\sin\left(\omega_1\sqrt{\frac{M}{k}}T_0\right) = \sin\left(T_0 + \gamma\varepsilon T_0\right) = \sin((\gamma+1)T_0),$$

which will not result in changes for the coefficients of $\sin(T_0)$ and $\cos(T_0)$, as ω_1 is too far away from the resonance frequency.

If $\gamma \neq 0$ and $|\gamma| \ll \frac{1}{\varepsilon}$, the coefficients of $\sin(T_0)$ becomes:

$$2\frac{dL}{dT_1} + \cos(\gamma T_1) S_2^{-1} p_{01} - C_1 L + -0.75 S_2^2 C_3 L(L^2 + K^2).$$

Note that in the case of a detuning frequency, the coefficients will contain $\sin(\gamma T_1)$ or $\cos(\gamma T_1)$, which results in a non-autonomous system of differential equations.

For each resonance frequency, there is one detuning frequency:

$$\begin{aligned}
\omega_1\sqrt{\frac{M}{k}} &= 0 + \gamma_1\varepsilon, & \omega_2\sqrt{\frac{M}{k}} &= 0 + \gamma_2\varepsilon,\\
\omega_1\sqrt{\frac{M}{k}} &= 1 + \gamma_1\varepsilon, & \omega_2\sqrt{\frac{M}{k}} &= 1 + \gamma_2\varepsilon,\\
\omega_1\sqrt{\frac{M}{k}} &= 2 + \gamma_1\varepsilon, & \omega_2\sqrt{\frac{M}{k}} &= 2 + \gamma_2\varepsilon,\\
\omega_1\sqrt{\frac{M}{k}} &= 4 + \gamma_1\varepsilon, & \omega_2\sqrt{\frac{M}{k}} &= 4 + \gamma_2\varepsilon.
\end{aligned}$$

Altogether, there are 4 choices for ω_1 and 4 for ω_2 which are resonance frequencies; there are 4 choices for both which are detuning frequencies; and 1 which is neither. This results in a total of $9 \cdot 9 = 81$ combinations of either or both resonance and detuning frequencies, which all result in a different solution. The calculation of all of these is not interesting for the extend of this paper. So some interesting combination will be chosen.

5.5 Case 1: No Resonance or Detuning Frequencies

In order to be able the analyze the effect of resonance or detuning on the cable, a baseline is needed, so, the first case is that neither ω_1 and ω_2 are resonance or detuning frequencies:

$$\begin{aligned}
\omega_1\sqrt{\frac{M}{k}} &\neq \pm n + \gamma_1\varepsilon, & \forall\, n \in \{0,1,2,4\}, \quad \forall|\gamma| \ll \frac{1}{\varepsilon},\\
\omega_2\sqrt{\frac{M}{k}} &\neq \pm n + \gamma_2\varepsilon, & \forall\, n \in \{0,1,2,4\}, \quad \forall|\gamma| \ll \frac{1}{\varepsilon}.
\end{aligned}$$

Note that the negative frequencies are not omitted as these do change the solution. The coefficients of $\sin(T_0)$ and $\cos(T_0)$ in this case are:

$$\begin{aligned}
-C_1 L + 2\frac{dL}{dT_1} - 0.75 S_2^2 C_3 (L^3 + K^2 L),\\
C_1 K - 2\frac{dK}{dT_1} + 0.75 S_2^2 C_3 (K^3 + K L^2),
\end{aligned}$$

respectively. These coefficients must be equal to zero in order to remove the secular terms, resulting in two coupled differential equations:

$$\begin{aligned}
\frac{dL}{dT_1} &= \frac{1}{2} C_1 L + \frac{3}{8} S_2^2 C_3 (L^2 + K^2) L,\\
\frac{dK}{dT_1} &= \frac{1}{2} C_1 K + \frac{3}{8} S_2^2 C_3 (K^2 + L^2) K,
\end{aligned} \tag{5.5.1}$$

with initial values given by equation (5.1.5). Note that $K(T_1) = L(T_1) = 0$ is the trivial solution, with $z_0(t,\varepsilon) = 0$ for all t. From now on, it will be assumed that $z_0(t,\varepsilon) \neq 0$ for at least one t.

This system can be more easily solved by switching to polar coordinates:

$$\begin{aligned}
K(T_1) &= R(T_1)\sin(\Phi(T_1)),\\
L(T_1) &= R(T_1)\cos(\Phi(T_1)).
\end{aligned} \tag{5.5.2}$$

Substituting this into the first order solution (5.1.4) gives:

$$\begin{aligned} z_0(T_0, T_1) &= K(T_1)\sin(T_0) + L(T_1)\cos(T_0) \\ &= R(T_1)\left(\sin(\Phi(T_1))\sin(T_0) + \cos(\Phi(T_1))\cos(T_0)\right) \\ &= R(T_1)\cos(\Phi(T_1) + T_0). \end{aligned} \tag{5.5.3}$$

Furthermore, from the change of coordinates (5.5.2), it follows that:

$$\begin{aligned} R(T_1)^2 &= K(T_1)^2 + L(T_1)^2, \\ \Phi(T_1) &= \arctan(K(T_1), L(T_1)). \end{aligned}$$

To transform the system to polar coordinates, two steps are needed. Firstly, in order to get an equation for R, multiply the first equation by L and the second by K, and add them, which gives:

$$\frac{dL}{dT_1}L + \frac{dK}{dT_1}K = \frac{1}{2}C_1(L^2 + K^2) + \frac{3}{8}S_2^2 C_3 (K^2 + L^2)^2.$$

By differentiating both sides of $R(T_1)^2 = L(T_1)^2 + K(T_1)^2$, it follows that $2\frac{dR}{dT_1}R = 2\frac{dL}{dT_1}L + 2\frac{dK}{dT_1}K$. So the above equation becomes:

$$\begin{aligned} \frac{dR}{dT_1} &= \frac{1}{2}C_1 R + \frac{3}{8}S_2^2 C_3 R^3 \\ &= \frac{1}{2}C_1 R + \frac{3}{8}C_1 R^3 \\ &= \frac{1}{2}C_1 R\left(1 - \frac{3}{4}R^2\right), \end{aligned}$$

where S_2 has been substituted (3.4.2). This equation has two equilibrium points, namely $R = 0$ and $R = \frac{2}{3}\sqrt{3}$. For $R \neq 0$ and $R \neq \frac{2}{3}\sqrt{3}$, it is a separable differential equation, and has solution:

$$R(T_1) = \sqrt{\frac{\frac{1}{2}C_1 R_0 e^{C_1 T_1}}{1 + \frac{3}{8}C_1 R_0 e^{C_1 T_1}}},$$

where $C_1 > 0$ from equation (3.4.6) is used, and R_0 is the integration constant.

Secondly, in order to get an equation for Φ, multiply the first and the second equation of the original system (5.5.1) by K and L respectively, and then subtract the first from the second, which gives:

$$\frac{dK}{dT_1}L - \frac{dL}{dT_1}K = 0.$$

Dividing both sides by $(L^2 + K^2)$:

$$\frac{\frac{dK}{dT_1}L - \frac{dL}{dT_1}K}{K^2 + L^2} = 0.$$

Note that the left hand side of this equation is the derivative of $\arctan(K, L)$. Using $\Phi(T_1) = \arctan(K(T_1), L(T_1)$ gives:

$$\frac{d\Phi}{dT_1} = 0,$$

which has solution:

$$\Phi(T_1) = \Phi_0,$$

where Φ_0 the integration constant.

Now that the system has been solved using polar coordinates, it must be transformed back. Inserting R and Φ into the first order solution in polar form (5.5.3) gives:

$$z_0(T_0, T_1) = \sqrt{\frac{\frac{1}{2}C_1 R_0 e^{C_1 T_1}}{1 + \frac{3}{8}C_1 R_0 e^{C_1 T_1}}} \cos\left(\Phi_0 + T_0\right),$$

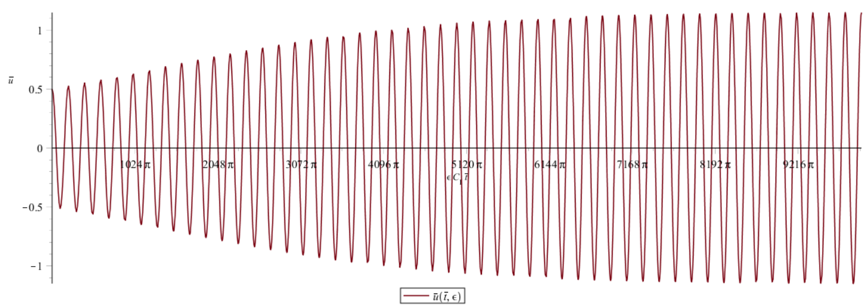
and thus:

$$\begin{aligned} \bar{u}(t, \varepsilon) &= z_0(t, \varepsilon t) + O(\varepsilon) \\ &= \sqrt{\frac{\frac{1}{2}C_1 R_0 e^{C_1 \varepsilon t}}{1 + \frac{3}{8}C_1 R_0 e^{C_1 \varepsilon t}}} \cdot \cos\left(t + \Phi_0\right) + O(\varepsilon). \end{aligned}$$

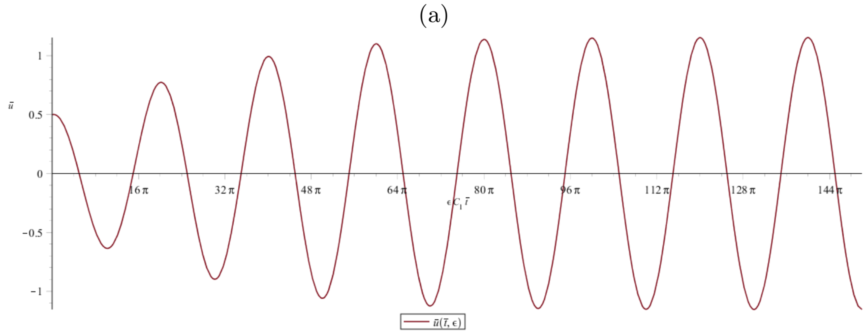
In the next subsection, this solution is analyzed and plotted for different values of the constants.

5.5.1 Analysis

In the figures below, the solution $\bar{u}$ is plotted for certain values of C_1, and ε:



(a)



(b)

Figure 5.5.1: Solution $\bar{u}$ with $C_1 = 1$ for different values of ε. (a) $\varepsilon = 0.01$. (b) $\varepsilon = 0.1$.

where R_0 was chosen so that $\bar{u}(0) = \frac{1}{2}$.

The interesting part of this solution is its amplitude, as the frequency is quite low and will not damage the cable. The amplitude $R(\varepsilon t)$ converges when $t \to \infty$:

$$\begin{aligned} \lim_{t\to\infty} R(\varepsilon t) &= \sqrt{\frac{\frac{1}{2}}{\frac{3}{8}}} \\ &= \frac{2}{3}\sqrt{3} \\ &\approx 1.154700539, \end{aligned}$$

as can also be seen in the figure below:

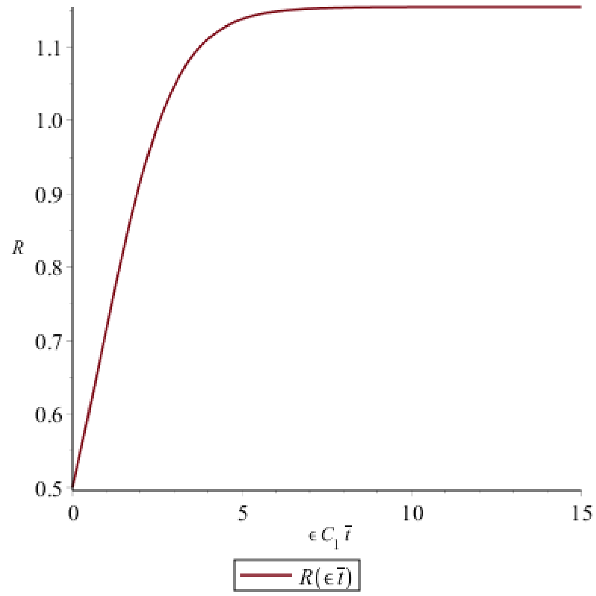


Figure 5.5.2: Amplitude R over time.

Notice that this amplitude is independent of wind speed and cable properties. The rate of convergence depends on the small parameter ε and C_1, which depends on the aerodynamic parameters. The higher ε and C_1 are, the faster the solution converges.

In order to be able to compare the coming cases, a phase plane is made:

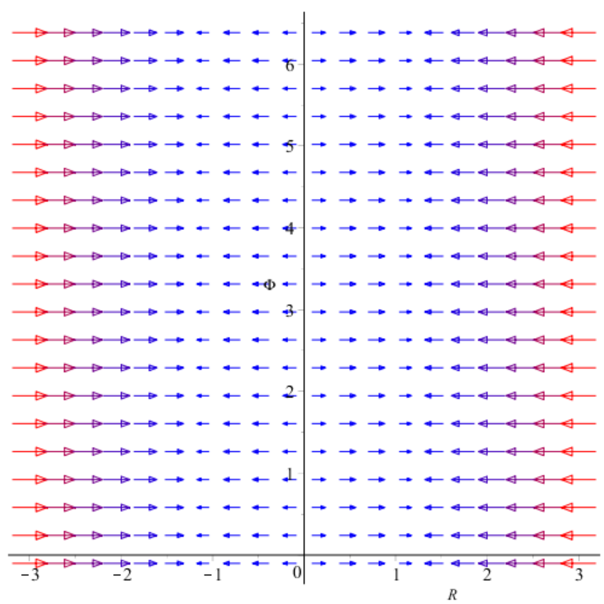


Figure 5.5.3: Phase plane of R and Φ.

It can be seen that Φ is always stable, and all arrows point to $R = \frac{2}{3}\sqrt{3}$, as expected.

In conclusion, if the rivulets frequencies are not resonance frequencies, the cable will either stay static or vibrate with an amplitude of $\frac{2}{3}\sqrt{3}$.

5.6 Case 2: One Resonance Frequency

The second case is that ω_1 is a resonance frequency, but ω_2 is neither a resonance nor a detuning frequency. For instance, let:

$$\omega_1\sqrt{\frac{M}{k}} = 1,$$
$$\omega_2\sqrt{\frac{M}{k}} \neq \pm n + \gamma\varepsilon, \qquad \forall\, n \in \{0,1,2,4\}, \quad \forall|\gamma| \ll \frac{1}{\varepsilon}.$$

The coefficients of $\sin(T_0)$ and $\cos(T_0)$ in this case are:

$$-C_1L + 2\frac{dL}{dT_1} - 0.75S_2^2C_3(L^3 + K^2L) + \cos(\beta_1)S_2^{-1}p_{01},$$
$$C_1K - 2\frac{dK}{dT_1} + 0.75S_2^2C_3(K^3 + KL^2) + \sin(\beta_1)S_2^{-1}p_{01}.$$

These coefficients must be equal to zero in order to remove the secular terms, resulting in two coupled differential equations:

$$\frac{dL}{dT_1} = \frac{1}{2}C_1L + (K^2 + L^2)\frac{3}{8}S_2^2C_3L - \frac{1}{2}p_{01}\cos(\beta_1)S_2^{-1},$$
$$\frac{dK}{dT_1} = \frac{1}{2}C_1K + (K^2 + L^2)\frac{3}{8}S_2^2C_3K + \frac{1}{2}p_{01}\sin(\beta_1)S_2^{-1}.$$

with initial values given by equation (5.1.5).

This system can be more easily solved by switching to polar coordinates (5.5.2); performing the same operations as in in Case 1 (Section 5.5) to get equations for R and Φ, gives:

$$\frac{dR}{dT_1} = R\left(\frac{3}{8}S_2^2C_3R^2 + \frac{1}{2}C_1\right) - \frac{1}{2R}p_{01}S_2^{-1}(\cos(\beta_1)L - \sin(\beta_1)K),$$
$$\frac{d\Phi}{dT_1} = \frac{1}{2}\frac{(\sin(\beta_1)L + \cos(\beta_1)K)\left(p_{01}S_2^{-1}\right)}{K^2 + L^2}.$$

Also changing K and L to polar coordinates (5.5.2) and using angle sum and difference identities:

$$\frac{dR}{dT_1} = \left(\frac{3}{8}S_2^2C_3R^2 + \frac{1}{2}C_1\right)R - \frac{1}{2}p_{01}S_2^{-1}\cos(\Phi + \beta_1),$$
$$\frac{d\Phi}{dT_1} = \frac{1}{2}\sin(\Phi + \beta_1)p_{01}S_2^{-1}.$$

By applying:

$$T_1 = \frac{\tau_1}{C_1}, \tag{5.6.1}$$

it follows that:

$$\frac{dR}{d\tau_1} = \left(-\frac{3}{8}R^2 + \frac{1}{2}\right)R - \frac{p_{01}}{2\sqrt{-\frac{C_1^3}{C_3}}}\cos(\Phi + \beta_1),$$
$$\frac{d\Phi}{d\tau_1} = \frac{p_{01}}{2\sqrt{-\frac{C_1^3}{C_3}}}\sin(\Phi + \beta_1).$$

Letting:

$$S_3 = -\frac{C_1^3}{C_3}, \tag{5.6.2}$$

gives:

$$\frac{dR}{d\tau_1} = \left(-\frac{3}{8}R^2 + \frac{1}{2}\right)R - \frac{p_{01}}{2\sqrt{S_3}}\cos(\Phi + \beta_1),$$
$$\frac{d\Phi}{d\tau_1} = \frac{p_{01}}{2\sqrt{S_3}}\sin(\Phi + \beta_1).$$

From this form, it follows that for large values of S_3 it is similar to the system in Case 1.

In the next section, this system is analyzed and approximations of solutions are plotted for several different values of the constants.

5.6.1 Stability Analysis

As this system is difficult to solve, its stability is analyzed instead. This is done by first calculating and analyzing its equilibrium points, then evaluating the Jacobian of the system in these points, and finally analyzing their eigenvalues.

From Maple, it follows that there are six equilibrium points:

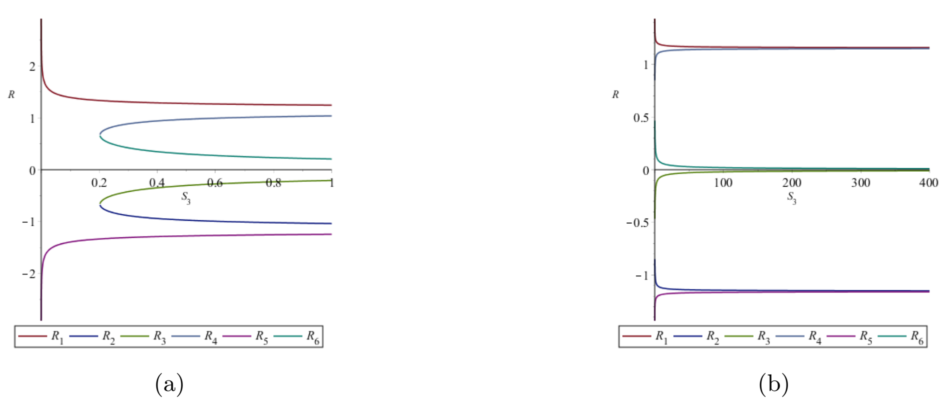
#	Φ	R
1	$-\beta_1$	R_1
2	$-\beta_1$	R_2
3	$-\beta_1$	R_3
4	$-\beta_1 + \pi$	R_4
5	$-\beta_1 + \pi$	R_5
6	$-\beta_1 + \pi$	R_6

with R_i as in Appendix B. These points consist of an amplitude and an angle. As the angle is constant, only the amplitude is analyzed.

The amplitudes depend only on p_{01} and S_3. The constant p_{01}, see equation (3.4.8), depends only on the characteristics of the upper rivulet:

$$p_{01} = -2M_1A_1g \geq -0.02g \approx -0.2,$$

as both M_1 and A_1 can be assumed to be lower than 0.1 in equation (2.2.1). For the remainder of this analysis, and the other analyses, it will be assumed that $p_{01} = -0.2$. The constant S_3 depends on the ratio between the aerodynamic constants, which are unknown. The amplitudes for different values of S_3 can be seen in the figure below:



(a) (b)

Figure 5.6.1: (a) Amplitudes for different low values of S_3. (b) Amplitude for different high values of S_3.

The amplitudes converge to three values: the first and fourth to $\frac{2}{3}\sqrt{3}$; the second and fifth to $-\frac{2}{3}\sqrt{3}$; and the third and sixth to 0. These three values are the same as in Case 1, as $-\frac{2}{3}\sqrt{3}$ results in the same solution as $\frac{2}{3}\sqrt{3}$ up to a phase-shift. Thus, as expected, this case is similar to the first for large values of S_3. Furthermore, note that the second up-to fifth point are only real for $S_3 \geq -p_{01}$.

The stability of the equilibrium points can be analyzed by evaluating the Jacobian of the system in these points. As β_1 only shifts the solution, it is chosen to be 0. In the figures below, the eigenvalues of these equilibriums are plotted against S_3:

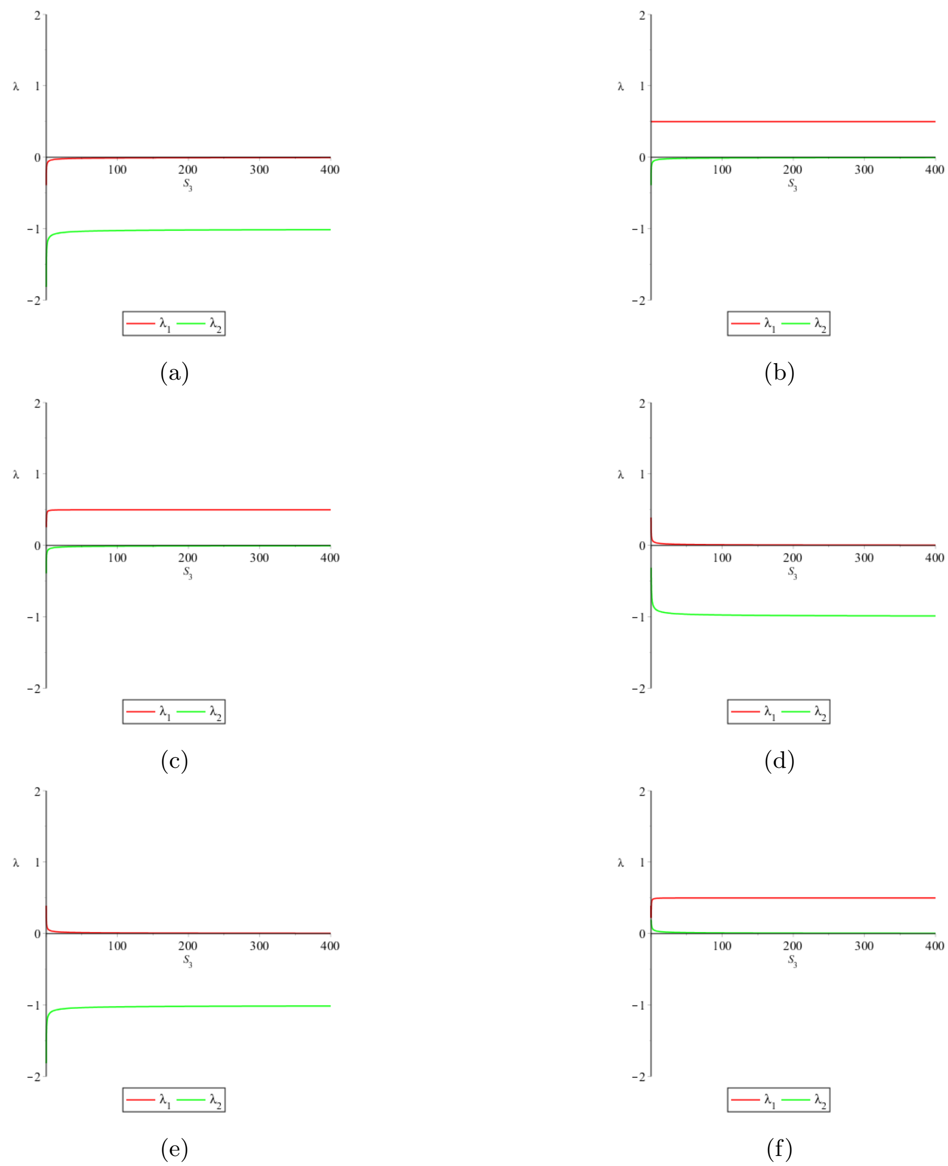


Figure 5.6.2: Eigenvalues of the equilibrium. (a) First equilibrium. (b) Second equilibrium. (c) Third equilibrium. (d) Fourth equilibrium. (e) Fifth equilibrium. (f) Sixth equilibrium.

From these figures, it follows that the first equilibrium is stable; the second, third, fourth and fifth saddle points; and the sixth unstable. For large S_3, this is again the same as in the first case.

In the figures below, the phase plane of R and Φ is plotted for several values of S_3.

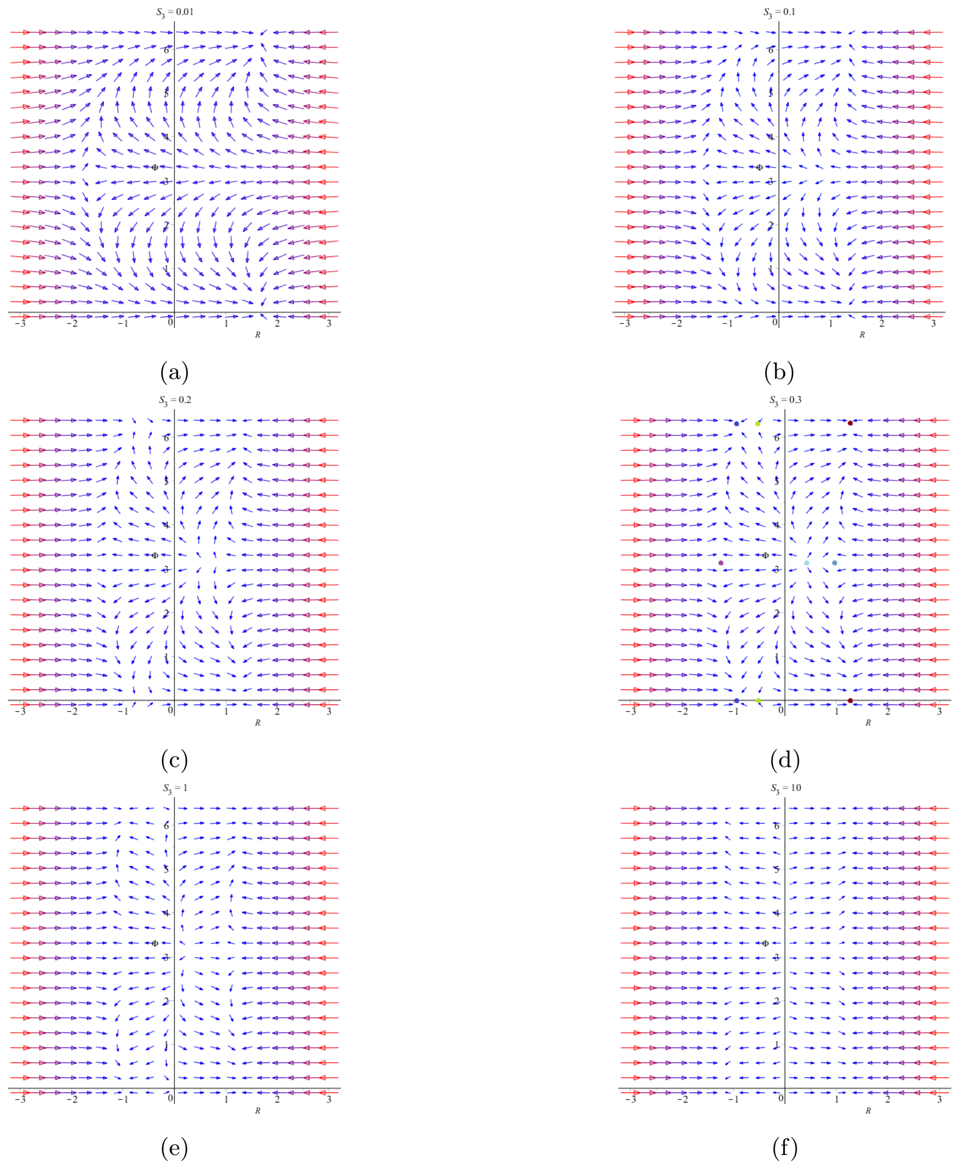
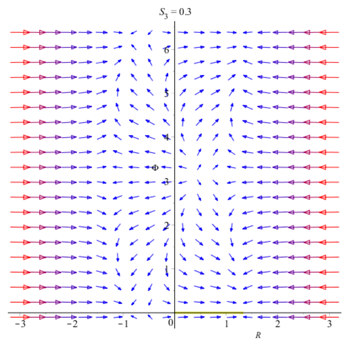


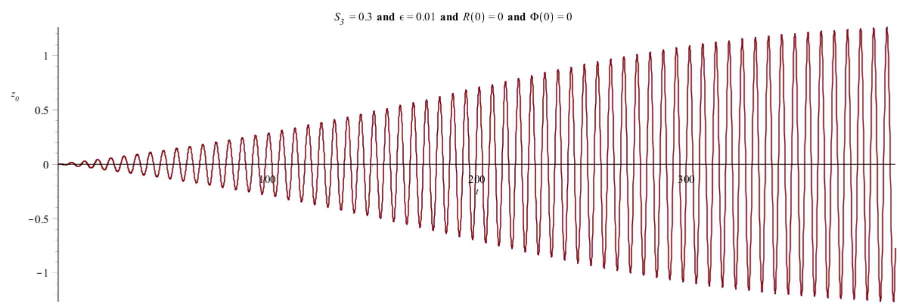
Figure 5.6.3: Phase planes for different values of S_3. (a) $S_3 = 0.01$. (b) $S_3 = 0.1$. (c) $S_3 = 0.2$. (d) $S_3 = 0.3$. (e) $S_3 = 1$. (f) $S_3 = 10$.

Note that this plot is 2π periodic in Φ, so the first equilibrium point can both be seen at $\Phi = 0$ and $\Phi = 2\pi$. The equilibriums are also marked in (d): the first in red, the second in blue, the third in green, the fourth in grey-blue, the fifth in purple, and the sixth in turquoise. The second, third, fourth and fifth are only present in $(d) - (f)$, as here $S_3 \geq -p_{01} = 0.2$. Furthermore, it can be seen that these are saddle points. The sixth point is present in all figures, and is an unstable point. For large values of S_3, the phase plane indeed looks like the one from Case 1 (5.5.3).

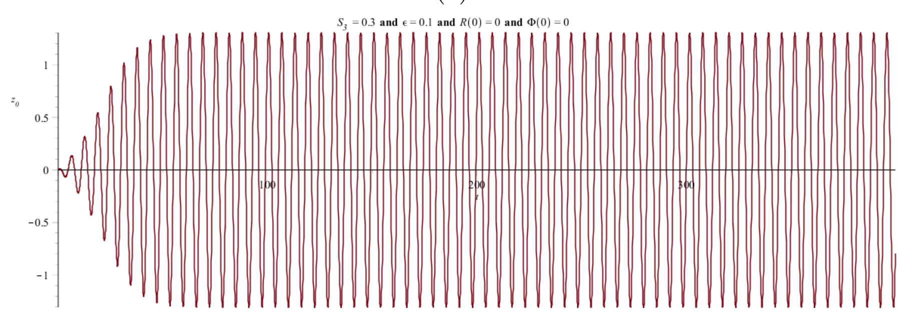
Using Maple, the first order solution z_0 can be approximated if initial values are given. For example, letting $R(0) = 0$ and $\Phi(0) = 0$ with $S_3 = 0.3$ gives the figures below:



(a)



(b)

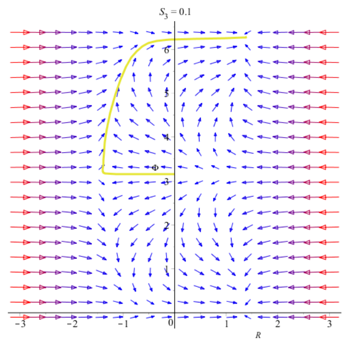


(c)

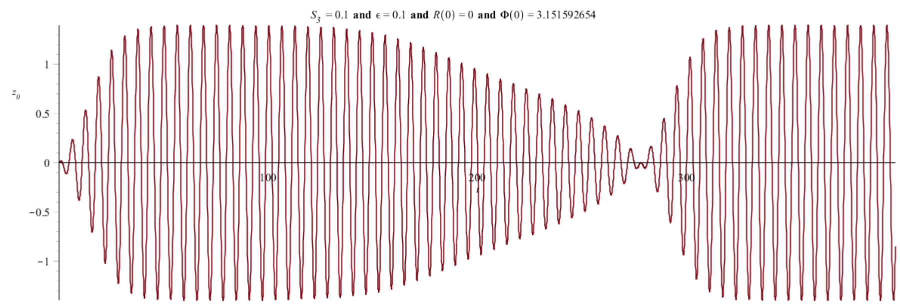
Figure 5.6.4: (a) Phase plane with solution for $R(0) = 0$, $\Phi(0) = 0$ and $S_3 = 0.3$. (b) Approximated solution for $R(0) = 0$, $\Phi(0) = 0$, $S_3 = 0.3$ and $\varepsilon = 0.01$. (c) Approximated solution for $R(0) = 0$, $\Phi(0) = 0$, $S_3 = 0.3$ and $\varepsilon = 0.1$.

From (a), it follows that the amplitude converges to around $\frac{2}{3}\sqrt{3}$. This can also be seen in (b) and (c). However, the rate of convergence in (b) is much lower than in (c), due to the lower ε value.

In the figure below, the initial values $R(0) = 0$ and $\Phi(0) = \pi + 0.01$ are used together with $S_3 = 0.1$:



(a)

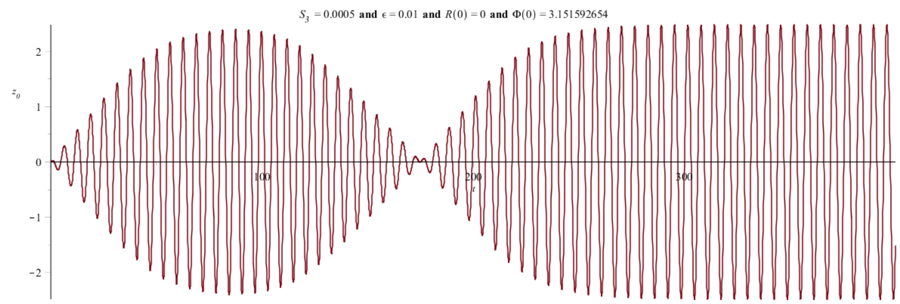


(b)

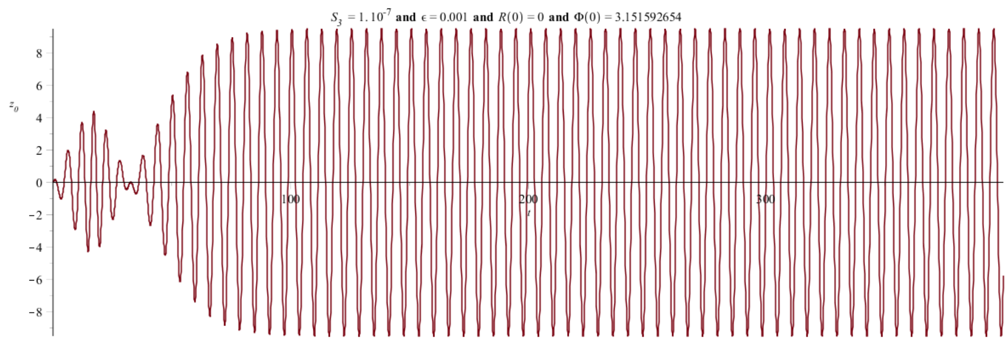
Figure 5.6.5: (a) Phase plane with solution for $R(0) = 0$, $\Phi(0) = \pi + 0.01$, and $S_3 = 0.1$. (b) Approximated solution for $R(0) = 0$, $\Phi(0) = \pi + 0.01$, $S_3 = 0.1$, and $\varepsilon = 0.01$.

In (a), it can be seen that $|R|$ becomes larger, then smaller, and then larger again, moves past one equilibrium point, into the other. The switch from positive to negative can be seen clearly too, at around $t = 280$.

For lower S_3, the equilibrium amplitudes become higher, as can be seen in the figures below:



(a)



(b)

Figure 5.6.6: (a) Approximated solution for $R(0) = 0$, $\Phi(0) = \pi + 0.01$, $S_3 = 0.0005$ and $\varepsilon = 0.01$. (b) Approximated solution for $R(0) = 0$, $\Phi(0) = \pi + 0.01$, $S_3 = 0.0000001$ and $\varepsilon = 0.001$.

Here, the general solutions are the same as in the previous figure, however, the amplitude is larger. Note also the small ε value in the second figure: due to the small S_3 value, the equilibrium is reached much faster for the same ε.

In conclusion, there is only one stable equilibrium, namely the first. So the cable will vibrate with the first equilibriums resolution, which depends on the value of S_3. For low values of S_3 the amplitude can very be high. For large values of S_3 the amplitude converges to the amplitude from the previous case: $\frac{2}{3}\sqrt{3}$.

5.7 Case 3: One Detuning Frequency

It can be imagined that ω_1 being exactly a resonance frequency is nearly impossible, therefore, the third case is that ω_1 is very close to a resonance frequency. Thus let ω_1 be a detuning frequency and ω_2 neither a resonance nor a detuning frequencies. For instance, let:

$$\begin{aligned} \omega_1\sqrt{\frac{M}{k}} &= 1 + \gamma_1\varepsilon, \qquad |\gamma_1| \ll \frac{1}{\varepsilon}, \\ \omega_2\sqrt{\frac{M}{k}} &\neq \pm n + \gamma\varepsilon, \qquad \forall\, n \in \{0, 1, 2, 4\}, \quad \forall|\gamma| \ll \frac{1}{\varepsilon}. \end{aligned}$$

The coefficients of $\sin(T_0)$ and $\cos(T_0)$ in this case are:

$$\begin{aligned} -C_1L + 2\frac{dL}{dT_1} - 0.75S_2^2C_3(L^3 + K^2L) + S_2^{-1}p_{01}\cos(\gamma_1T_1 + \beta_1), \\ C_1K - 2\frac{dK}{dT_1} + 0.75S_2^2C_3(K^3 + KL^2) + S_2^{-1}p_{01}\sin(\gamma_1T_1 + \beta_1), \end{aligned}$$

respectively. Note that these coefficients are the same as in the previous case if $\gamma = 0$. These coefficients must be equal to zero in order to remove the secular terms, resulting in two coupled differential equations:

$$\frac{dL}{dT_1} = \frac{1}{2}C_1 L + (K^2+L^2)\frac{3}{8}S_2^2 C_3 L - \frac{1}{2}p_{01}\cos(\gamma_1 T_1 + \beta_1)S_2^{-1},$$
$$\frac{dK}{dT_1} = \frac{1}{2}C_1 K + (K^2+L^2)\frac{3}{8}S_2^2 C_3 K + \frac{1}{2}p_{01}\sin(\gamma_1 T_1 + \beta_1)S_2^{-1}.$$

with initial values given by equation (5.1.5). This system is the same as the system in the previous case (Case 2), except for the last term which has become dependent on T_1, making it a non-autonomous system.

Changing to polar coordinates by performing the same operations as in the previous case gives:

$$\frac{dR}{d\tau_1} = \left(-\frac{3}{8}R^2 + \frac{1}{2}\right)R - \frac{p_{01}}{2\sqrt{S_3}}\cos(\Phi + \frac{\gamma_1}{C_1}\tau_1 + \beta_1),$$
$$\frac{d\Phi}{d\tau_1} = \frac{p_{01}}{2\sqrt{S_3}}\sin(\Phi + \frac{\gamma_1}{C_1}\tau_1 + \beta_1).$$

Changing variables to:

$$\Psi(\tau_1) = \Phi(\tau_1) + \frac{\gamma_1}{C_1}\tau_1, \tag{5.7.1}$$

results in the next autonomous system:

$$\frac{dR}{d\tau_1} = \left(-\frac{3}{8}R^2 + \frac{1}{2}\right)R - \frac{p_{01}}{2\sqrt{S_3}}\cos(\Psi + \beta_1),$$
$$\frac{d\Phi}{d\tau_1} = \frac{p_{01}}{2\sqrt{S_3}}\sin(\Psi + \beta_1) + \frac{\gamma_1}{C_1}.$$

In the next section, this system's stability is analyzed and approximated solutions are plotted.

5.7.1 Stability Analysis

Again, like in the previous case, this system is difficult to solve, so its stability is analyzed instead. This is done by first calculating and analyzing the equilibrium points, then using the Jacobian of the system to analyze their stability, and finally constructing phase planes to create an idea of how the solution behaves.

The equilibrium points are calculated using Maple, and are:

#	Ψ	R
1	$\arcsin\left(\frac{2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right) - \beta_1$	R_1
2	$\arcsin\left(\frac{2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right) - \beta_1$	R_2
3	$\arcsin\left(\frac{2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right) - \beta_1$	R_3
4	$\arcsin\left(\frac{-2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right) - \beta_1 + \pi$	R_4
5	$\arcsin\left(\frac{-2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right) - \beta_1 + \pi$	R_5
6	$\arcsin\left(\frac{-2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right) - \beta_1 + \pi$	R_6

with the R_i as in (Appendix B). These points consist of an amplitude R and an angle Ψ. First, the amplitude is analyzed.

The amplitudes depend only on S_3, $\frac{\gamma_1}{C_1}$, and p_{01}. Like before, $p_{01} = -0.2$. Furthermore, these amplitudes are only real for $S_3 \leq \frac{p_{01}^2 C_1^2}{4\gamma_1^2}$. In the figures below, they can be seen for $0 \leq S_3 \leq 400$ and $0 \leq \gamma_1 C_1 \leq 0.01$:

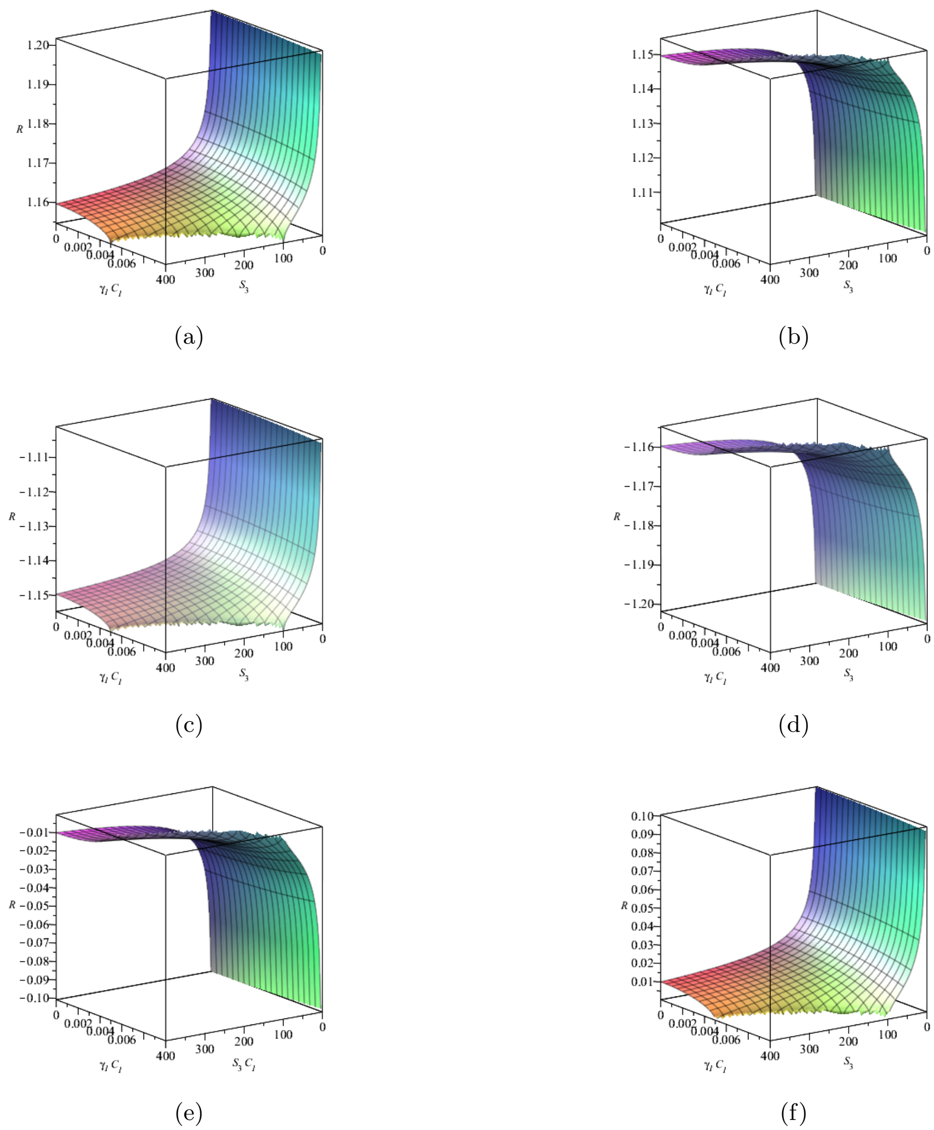


Figure 5.7.1: Equilibrium amplitudes plotted against $\gamma_1 C_1$ and $S_3 C_1$. (a) First amplitude. (b) Fourth amplitude. (c) Second amplitude. (d) Fifth amplitude. (e) Third amplitude. (f) Sixth amplitude.

Just as in the previous case, there are three positive and three negative amplitudes, which converge to $\frac{2}{3}\sqrt{3}$, $-\frac{2}{3}\sqrt{3}$ and 0 for large values of S_3. For larger values of γ_1, the amplitudes R_1, R_4, R_5, and R_6 get closer to their limit, whereas the other two move away from it. Note, however, that not all combinations of S_3 and γ_1 result in real values of the amplitudes, as was discussed before. This concludes the amplitude analysis, next comes the angle analysis.

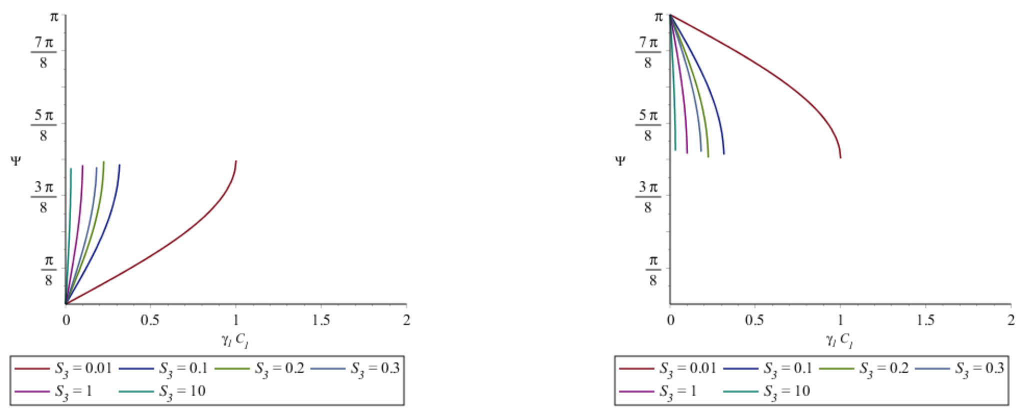
First of all, these equilibrium points for Ψ are not equilibrium points of the original system. For Φ, these points are in fact lines, as:

$$\Phi(T_1) = \Psi(T_1) - \frac{\gamma_1}{C_1} T_1,$$

so, the equilibrium lines of Φ are:

$$\arcsin\left(\frac{2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right) - \beta_1 - \frac{\gamma_1}{C_1}T_1,$$
$$\arcsin\left(\frac{-2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right) - \beta_1 - \frac{\gamma_1}{C_1}T_1 + \pi.$$

Additionally, as $\arcsin(x)$ is only defined for $-1 \leq x \leq 1$, it follows that $S_3 \leq \frac{p_{01}^2 C_1^2}{4\gamma_1^2}$ in order for the equilibrium points to be real. In the figure below, the equilibrium angles are plotted against γC_1 for several values of S_3:



(a) (b)

Figure 5.7.2: (a) Equilibrium angle $\arcsin\left(\frac{2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right)$ for different values of S_3. (b) Equilibrium angle $\arcsin\left(-\frac{2\gamma_1\sqrt{S_3}}{p_{01}C_1}\right) + \pi$ for different values of S_3.

It can be seen that the angles converge to $\frac{\pi}{2}$ for higher values of $\gamma_1 C_1$, and vanish if it becomes to high. For lower values of S_3, this convergence is slower.

Now, the stability of these four equilibrium points can be analyzed by evaluating the Jacobian of the system in these points. As β_1 only shifts the solution, it is chosen to be zero. In the figures below, the eigenvalues of these equilibriums are plotted against S_3 and $\gamma_1 C_1$:

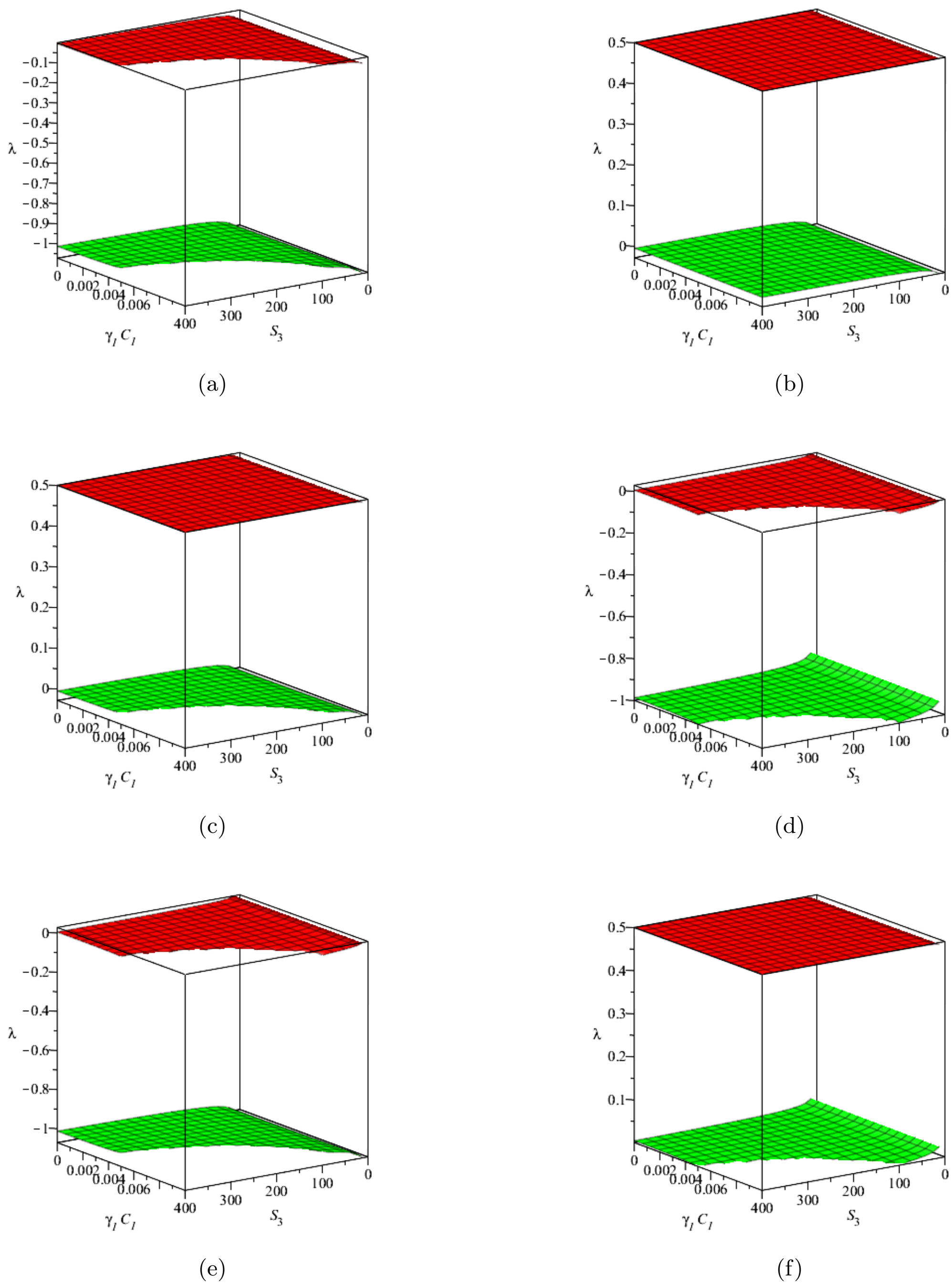


Figure 5.7.3: Eigenvalues of equilibriums. (a) First equilibrium. (b) Second equilibrium. (c) Third equilibrium. (d) Fourth equilibrium. (e) Fifth equilibrium. (f) Sixth equilibrium.

These figures look like the figures in Case 2 (figure 5.6.2). The equilibriums also have the same stability as in Case 2: the first is stable, the sixth unstable, and the others are saddle points. The value of $\gamma_1 C_1$ is of little influence to the eigenvalues, however, if it becomes to large, they become imaginary. This has no impact on the stability of the equilibriums, as they are imaginary for the same values.

In the figures below, the phase plane of R and Ψ is plotted for $S_3 = 0.3$ and several values of γ_1 (with $C_1 = 1$):

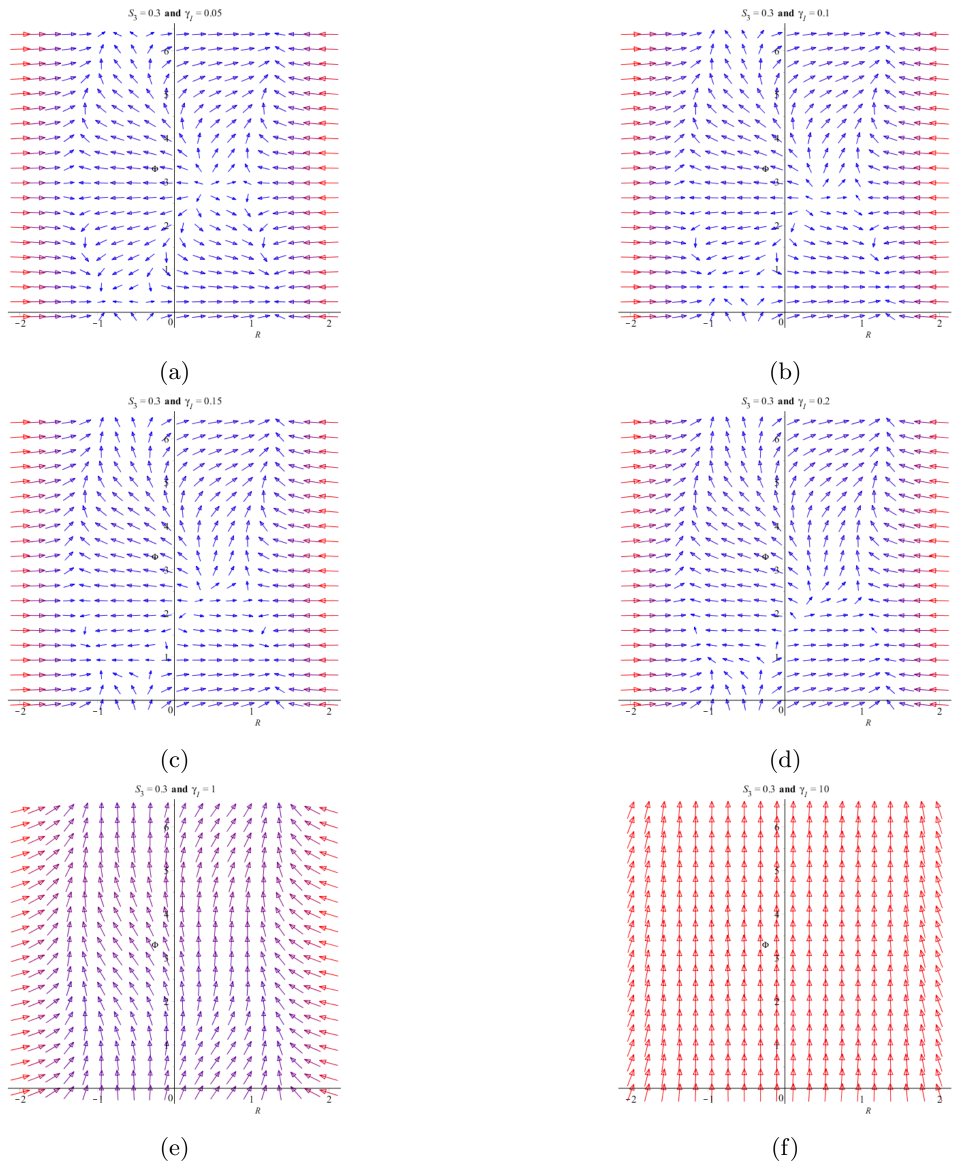


Figure 5.7.4: Phase planes for different values of γ_1. (a) $\gamma_1 C_1 = 0.05$. (b) $\gamma_1 C_1 = 0.1$. (c) $\gamma_1 C_1 = 0.15$. (e) $\gamma_1 C_1 = 0.2$. (e) $\gamma_1 C_1 = 1$. (f) $\gamma_1 C_1 = 10$.

In these figures, it can be seen that the equilibriums move towards $\frac{\pi}{2}$. Furthermore, for small γ_1 the phase planes look like in the previous case. This can be seen more clearly in the figures below, in which z_0 is plotted for $S_3 = 0.3$, $\varepsilon = 0.1$ and several values of γ_1:

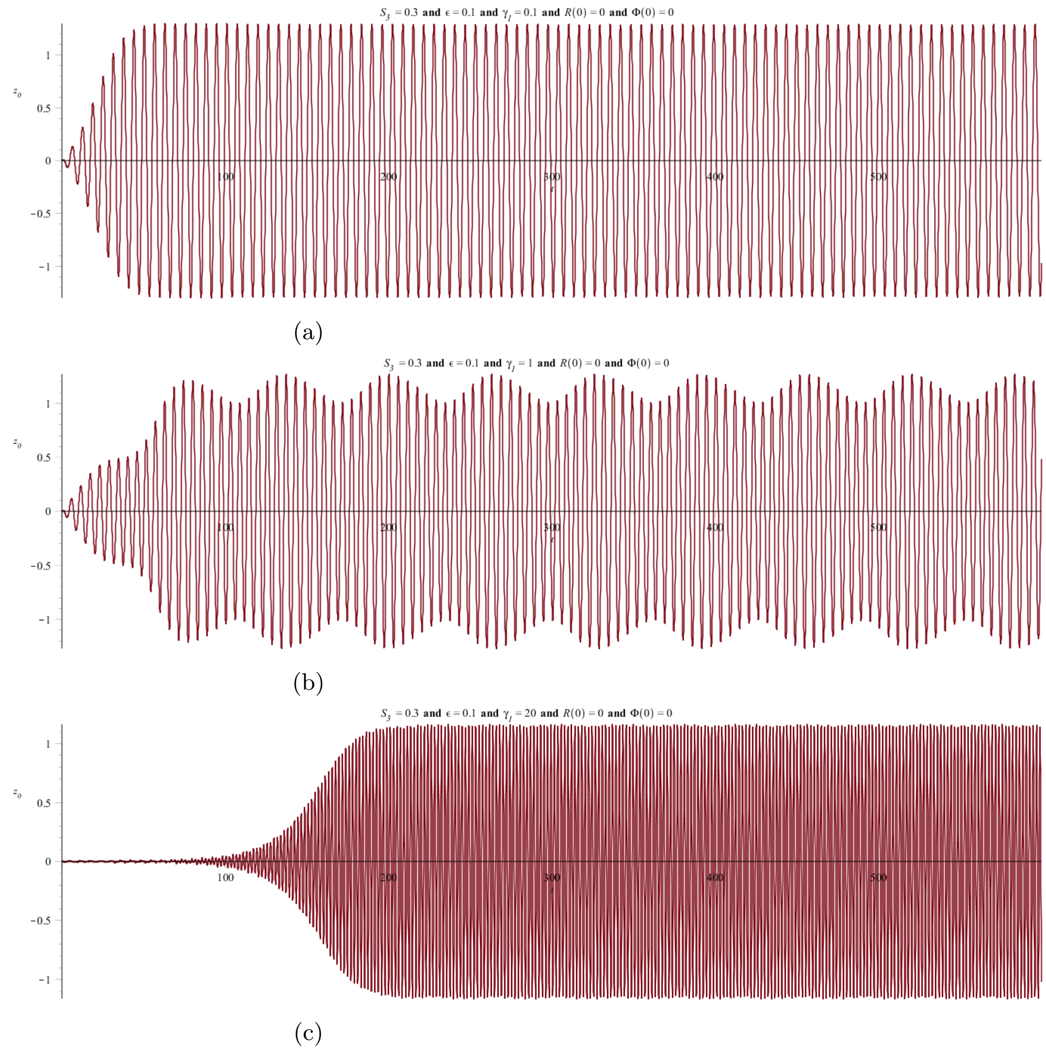


Figure 5.7.5: Solutions z_0 for different values of γ_1. (a) $\gamma_1 = 0.1$. (b) $\gamma_1 = 1$. (c) $\gamma_1 = 20$.

Indeed, (a) looks a lot like figure (5.6.4(c)), whereas (c) looks like figure (5.5.1(a). The most important difference between (a) and (c) is the amplitude: the amplitude in (a) is around 0.15 larger than in (c). Figure (b) is also interesting, as here the amplitude alternates between the one from (a) and the one from (c). This is oscillation is due to the detuning frequency being alternately in and out of phase with the resonance frequency.

In conclusion, a detuning frequency results in almost the same solution as a resonance frequency if γ_1 is small enough. For large γ_1, the solution is the same as in the case of no resonance or detuning frequency. For γ_1 in between small and large, the solution alternates between the other two options.

5.8 Case 4: Two Resonance Frequencies

The fourth case is that both ω_1 and ω_2 are resonance frequencies. This is interesting, as the one may amplify the other. For instance, let:

$$\omega_1\sqrt{\frac{M}{k}} = 1,$$
$$\omega_2\sqrt{\frac{M}{k}} = 2.$$

The coefficients of $\sin(T_0)$ and $\cos(T_0)$ in this case are:

$$
\begin{aligned}
&-C_1L + 2\frac{dL}{dT_1} - \frac{3}{4}S_2^2C_3(L^3+K^2L) + \cos(\beta_1)S_2^{-1}p_{01} \\
&\quad + \frac{1}{2}\cos(\beta_2)\left(p_{11}K - p_{41}L + \frac{3}{4}S^2p_{31}(K^3+KL^2) + p_{13}L\right) \\
&\quad + \frac{1}{2}\sin(\beta_2)\left(p_{11}L + p_{41}K + \frac{3}{4}S^2p_{31}(L^3+K^2L) - p_{13}K\right), \\
&C_1K - 2\frac{dK}{dT_1} + \frac{3}{4}S_2^2C_3(K^3+KL^2) + \sin(\beta_1)S_2^{-1}p_{01} \\
&\quad + \frac{1}{2}\sin(\beta_2)\left(p_{11}K - p_{41}L + 0.75S^2p_{31}(K^3+KL^2) + p_{13}L\right) \\
&\quad - \frac{1}{2}\cos(\beta_2)\left(p_{11}L + p_{41}K + 0.75S^2p_{31}(L^3+K^2L) - p_{13}K\right).
\end{aligned}
$$

These coefficients must be equal to zero in order to remove the secular terms, resulting in two coupled differential equations:

$$
\begin{aligned}
\frac{dL}{dT_1} &= \frac{1}{2}C_1L + (K^2+L^2)\frac{3}{8}S_2^2C_3L - \frac{1}{2}p_{01}\cos(\beta_1)S_2^{-1} \\
&\quad - \frac{1}{4}\cos(\beta_2)\left(p_{11}K - p_{41}L + \frac{3}{4}S^2p_{31}(K^2+L^2)K + p_{13}L\right) \\
&\quad - \frac{1}{4}\sin(\beta_2)\left(p_{11}L + p_{41}K + \frac{3}{4}S^2p_{31}(L^2+K^2)L - p_{13}K\right), \\
\frac{dK}{dT_1} &= \frac{1}{2}C_1K + (K^2+L^2)\frac{3}{8}S_2^2C_3K + \frac{1}{2}p_{01}\sin(\beta_1)S_2^{-1} \\
&\quad + \frac{1}{4}\sin(\beta_2)\left(p_{11}K - p_{41}L + \frac{3}{4}S^2p_{31}(K^2+L^2)K + p_{13}L\right) \\
&\quad - \frac{1}{4}\cos(\beta_2)\left(p_{11}L + p_{41}K + \frac{3}{4}S^2p_{31}(L^2+K^2)L - p_{13}K\right),
\end{aligned}
$$

with initial values given by equation (5.1.5).

Changing to polar coordinates (5.5.2) by performing the same operations as in the previous case and using angle sum and difference identities gives:

$$
\begin{aligned}
\frac{dR}{dT_1} &= \frac{1}{16}\big(4\cos(\beta_2)p_{13}R - 4\cos(\beta_2)p_{41}R + 3\sin(\beta_2)p_{31}S_2^2R^3 \\
&\quad + 4\sin(\beta_2)p_{11}R + 8R - 6C_1R^3 - 8p_{01}\cos(\Phi+\beta_1)S_2^{-1}\big), \\
\frac{d\Phi}{dT_1} &= \frac{1}{16}\left(-4\sin(\beta_2)p_{41}R - 4\cos(\beta_2)p_{11}R + 4\sin(\beta_2)p_{13}R - 3\cos(\beta_2)p_{31}S_2^2R^3 + 8p_{01}\sin(\Phi+\beta_1)S_2^{-1}\right).
\end{aligned}
$$

Due the many unknown constants, this system is difficult to analyze. Therefore, it will be omitted in this paper.

6 Conclusion

In this paper, it was found that wind and rain may cause resonance of cables. The effect of the wind and rain depends on the properties of the cable and water rivulets. These cable properties consist of mass and tension, whereas the water rivulet properties depend on wind-speed and material, diameter, and roughness of the cable.

In most situations, the cable will be static or vibrating with an amplitude of $\frac{2}{3}\sqrt{3}$. For certain combinations of cable properties, the static option vanishes and the cable will vibrate with a larger amplitude. This amplitude depends on the aerodynamic properties of the cable and rivulets. For smaller values of S_3 this amplitude is larger, whereas it converges to $\frac{2}{3}\sqrt{3}$ for large values of S_3.

7 Discussion

For future study of this problem, more information about the cable and the rivulets are needed. For example, the relation between material, diameter, and roughness of the cable and rivulet frequency; the size and mass of the rivulets; the tension on and the mass of the cable; and much more. For example, the cable tension k was assumed to be constant. This results in less damping, and may have resulted in larger amplitudes than in reality. Additionally, the analysis of the type of equilibrium points was difficult due to the eigenvalues being dependent on many of these unknown characteristics. Research was done to find this information, but not much general data was found. Therefore, it may be better to focus on a single cable for which much information is available.

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List of Variables

Symbol	Definition	First use
	Time	
t	Time	Section (2.1)
$\bar{t}$	Dimensionless t	Equation (3.4.1)
T_i	Time scales	Equation (5.1.1)
τ_1	Transformed time scale	Equation (5.6.1)
	Displacement	
$u(t)$	Vertical displacement of cross-section from center	Section (2.1)
$\bar{u}(\bar{t})$	Dimensionless u	Equation (3.4.1)
$z_i(T_i)$	Approximated displacement u_i	Equation (5.1.1)
$K(T_1)$	Function coefficient	Equation 5.1.4)
$L(T_1)$	Function coefficient	Equation 5.1.4)
$R(T_1)$	Displacement amplitude	Equation (5.5.3)
$\Phi(T_1)$	Displacement angle	Equation (5.5.3)
$\Phi(T_1)$	Transformed displacement angle	Equation (5.7.1)
	Cable characteristics	
M_0	Cross-section mass	Equation (2.2.2)
k	Spring stiffness (Cable tension)	Equation (3.1.2)
d	Cable diameter	Equation (3.2.1)
	Rivulet characteristics	
α_1	Upper rivulet position	Section (2.2)
α_2	Lower rivulet position	Section (2.2)
M_1	Average upper rivulet mass	Equation (2.2.1)
M_2	Average lower rivulet mass	Equation (2.2.1)
A_1	Relative upper rivulet change in mass (to M_1)	Equation (2.2.2)
A_2	Relative lower rivulet change in mass (to M_2)	Equation (2.2.2)
ω_1	Upper rivulet wave frequency	Equation (2.2.1)
ω_2	Lower rivulet wave frequency	Equation (2.2.1)
β_1	Upper rivulet wave phase shift	Equation (2.2.1)
β_2	Lower rivulet wave phase shift	Equation (2.2.1)
$\tilde{A}_1$	Relative upper rivulet change in mass (to M)	Equation (2.2.2)
$\tilde{A}_2$	Relative lower rivulet change in mass (to M)	Equation (2.2.2)
$r_1(t)$	Relative lower rivulet mass (to M_1)	Equation (2.2.3)
$r_2(t)$	Relative upper rivulet mass (to M_2)	Equation (2.2.3)
	System characteristics	
M	Average total mass	Equation (2.2.2)
$r(t)$	Relative combined system mass (to M)	Equation (2.2.3)
	Wind characteristics	
v_∞	Wind speed	Section (2.3)
v_s	Relative wind speed	Equation (2.3.2)
$\phi(t)$	Angle of attack of wind	Equation (2.3.4)
	Environmental characteristics	
g	Gravitational acceleration	Equation (3.1.1)
ρ_a	Air density	Equation (3.2.1)

	Drag and lift	
$D(t)$	Vertical drag function	Equation (3.1.3)
$C_D(t)$	Drag coefficient	Equation (3.2.1)
C_{D0}	Drag coefficient	Equation (3.2.2)
$L(t)$	Vertical lift function	Equation (3.1.4)
$C_L(t)$	Lift coefficient	Equation (3.2.1)
C_{L1}	Lift coefficient	Equation (3.2.2)
C_{L3}	Lift coefficient	Equation (3.2.2)
θ_1	Lift coefficient	Equation (3.2.2)
θ_2	Lift coefficient	Equation (3.2.2)
κ_i	Drag and lift coefficients	Equation (3.2.2)

	Gathered terms	
$a_0(t)$	Taylor expansion constant coefficient	Equation (3.3.1)
$a_1(t)$	Taylor expansion linear term coefficient	Equation (3.3.1)
$a_2(t)$	Taylor expansion quadratic term coefficient	Equation (3.3.1)
$a_3(t)$	Taylor expansion cubic term coefficient	Equation (3.3.1)
a_{ij}	Gathered constants	Equation (3.3.3)
S_1	Gathered constants	Equation (3.3.5)
S_2	Gathered constants	Equation (3.4.2)
S_2	Gathered constants	Equation (5.6.2)
C_0	Gathered terms	Equation (3.4.4)
C_1	Gathered terms	Equation (3.4.4)
C_2	Gathered terms	Equation (3.4.4)
C_3	Gathered terms	Equation (3.4.4)
C_4	Gathered terms	Equation (3.4.4)
$P_0(t)$	Gathered terms	Equation (3.4.4)
$P_1(t)$	Gathered terms	Equation (3.4.4)
$P_2(t)$	Gathered terms	Equation (3.4.4)
$P_3(t)$	Gathered terms	Equation (3.4.4)
$P_4(t)$	Gathered terms	Equation (3.4.4)
p_{ij}	Gathered constants	Equation (3.4.5)
ε	Gathered terms and small parameter	Equation (3.4.3)

	*Multiple time-scale analysis**	
$\hat{u}_i$	Approximated solutions	Equation (4.3.3)
T_i	Time scales	Equation (4.3.3)
α	Initial value	Equation (4.3.1)
β	Initial value	Equation (4.3.1)
$A(T_1)$	Function coefficient	Equation (4.3.4)
$B(T_1)$	Function coefficient	Equation (4.3.4)
T_{an}	Analytical period Duffing solution	Equation (4.3.8)
T_{ap}	Approximated period Duffing solution	Section (4.3)

* Only in Section (4).

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Appendices

A O(ε) equation calculation

Inserting the first order solution (5.1.4):

$$\begin{aligned}\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = {} & S_2^{-1}(C_0 + P_0(T_0)) \\ & +(C_1 + P_1(T_0))\Big[K\cos(T_0) - L\sin(T_0)\Big] \\ & +S_2(C_2 + P_2(T_0))\Big[K^2\cos(T_0)^2 - 2KL\cos(T_0)\sin(T_0) + L^2\sin(T_0)^2\Big] \\ & +S_2^2(C_3 + P_3(T_0))\Big[K^3\cos(T_0)^3 - 3K^2L\cos(T_0)^2\sin(T_0) + 3KL^2\cos(T_0)\sin(T_0)^2 - L^3\sin(T_0)^3\Big] \\ & -(C_4 + P_4(T_0))\Big[K\sin(T_0) + L\cos(T_0)\Big] \\ & -2\Big[\frac{dK}{dT_1}\cos(T_0) - \frac{dL}{dT_1}\sin(T_0)\Big]\end{aligned}$$

Rewriting powers and products of sines and cosines as sines and cosines:

$$\begin{aligned}\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = {} & S_2^{-1}(C_0 + P_0(T_0)) \\ & +(C_1 + P_1(T_0))\Big[K\cos(T_0) - L\sin(T_0)\Big] \\ & +S_2(C_2 + P_2(T_0))\Big[0.5K^2(1 + \cos(2T_0)) - KL\sin(2T_0) + 0.5L^2(1 - \cos(2T_0))\Big] \\ & +S_2^2(C_3 + P_3(T_0))\Big[K^3(0.75\cos(T_0) + 0.25\cos(3T_0)) - 3K^2L(0.25\sin(T_0) + 0.25\sin(3T_0)) \\ & \qquad + 3KL^2(0.25\cos(T_0) - 0.25\cos(3T_0)) - L^3(0.75\sin(T_0) - 0.25\sin(3T_0))\Big] \\ & -(C_4 + P_4(T_0))\Big[K\sin(T_0) + L\cos(T_0)\Big] \\ & -2\Big[\frac{dK}{dT_1}\cos(T_0) - \frac{dL}{dT_1}\sin(T_0)\Big]\end{aligned}$$

Gathering sines and cosines of equal angles, and splitting into factors dependent on sines and cosines of different angles:

$$\begin{aligned}\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = {} & \Big[S_2^{-1}(C_0 + P_0(T_0)) + \frac{1}{2}S_2(C_2 + P_2(T_0))(K^2 + L^2)\Big] \\ & - \Big[(C_1 + P_1(T_0))L - 2\frac{dL}{dT_1} + (C_4 + P_4(T_0))K + 0.75S_2^2(C_3 + P_3(T_0))(L^3 + K^2L)\Big]\sin(T_0) \\ & + \Big[(C_1 + P_1(T_0))K - 2\frac{dK}{dT_1} - (C_4 + P_4(T_0))L + 0.75S_2^2(C_3 + P_3(T_0))(K^3 + KL^2)\Big]\cos(T_0) \\ & - \Big[S_2(C_2 + P_2(T_0))KL\Big]\sin(2T_0) \\ & + \Big[0.5S_2(C_2 + P_2(T_0))(K^2 + L^2)\Big]\cos(2T_0) \\ & - \Big[S_2^2(C_3 + P_3(T_0))(0.75K^2L + 0.25L^3)\Big]\sin(3T_0) \\ & - \Big[S_2^2(C_3 + P_3(T_0))(0.75KL^2 - 0.25K^3)\Big]\cos(3T_0).\end{aligned}$$

Splitting C and P:

$$
\begin{aligned}
\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = & \left[S_2^{-1} C_0 + 0.5 S_2 C_2 (K^2 + L^2)\right] \\
& - \left[C_1 L - 2\frac{dL}{dT_1} + C_4 K + 0.75 S_2^2 C_3 (L^3 + K^2 L)\right] \sin(T_0) \\
& + \left[C_1 K - 2\frac{dK}{dT_1} - C_4 L + 0.75 S_2^2 C_3 (K^3 + K L^2)\right] \cos(T_0) \\
& - \left[S_2 C_2 K L\right] \sin(2T_0) \\
& + \left[0.5 S_2 C_2 (K^2 + L^2)\right] \cos(2T_0) \\
& - \left[S_2^2 C_3 (0.75 K^2 L + 0.25 L^3)\right] \sin(3T_0) \\
& - \left[S_2^2 C_3 (0.75 K L^2 - 0.25 K^3)\right] \cos(3T_0) \\
& + \left[S_2^{-1} P_0(T_0) + 0.5 S_2 P_2(T_0)(K^2 + L^2)\right] \\
& - \left[P_1(T_0) L + P_4(T_0) K + 0.75 S_2^2 P_3(T_0)(L^3 + K^2 L)\right] \sin(T_0) \\
& + \left[P_1(T_0) K - P_4(T_0) L + 0.75 S_2^2 P_3(T_0)(K^3 + K L^2)\right] \cos(T_0) \\
& - \left[S_2 P_2(T_0) K L\right] \sin(2T_0) \\
& + \left[0.5 S_2 P_2(T_0)(K^2 + L^2)\right] \cos(2T_0) \\
& - \left[S_2^2 P_3(T_0)(0.75 K^2 L + 0.25 L^3)\right] \sin(3T_0) \\
& - \left[S_2^2 P_3(T_0)(0.75 K L^2 - 0.25 K^3)\right] \cos(3T_0).
\end{aligned}
$$

Inserting P:

$$
\begin{aligned}
\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = & \left[S_2^{-1} C_0 + 0.5 S_2 C_2 (K^2 + L^2)\right] \\
& - \left[C_1 L - 2\frac{dL}{dT_0} + C_4 K + 0.75 S_2^2 C_3 (L^3 + K^2 L)\right] \sin(T_0) \\
& + \left[C_1 K - 2\frac{dK}{dT_0} - C_4 L + 0.75 S_2^2 C_3 (K^3 + K L^2)\right] \cos(T_0) \\
& - \left[S_2 C_2 K L\right] \sin(2T_0) \\
& + \left[0.5 S_2 C_2 (K^2 + L^2)\right] \cos(2T_0) \\
& - \left[S_2^2 C_3 (0.75 K^2 L + 0.25 L^3)\right] \sin(3T_0) \\
& - \left[S_2^2 C_3 (0.75 K L^2 - 0.25 K^3)\right] \cos(3T_0) \\
& + \left[S_2^{-1} p_{01} + 0.5 S_2 p_{21} (K^2 + L^2)\right] \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& + \left[S_2^{-1} p_{02} + 0.5 S_2 p_{22} (K^2 + L^2)\right] \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right) \\
& - \left[p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L)\right] \sin(T_0) \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& - \left[p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L)\right] \sin(T_0) \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right) \\
& - \left[p_{13} L\right] \sin(T_0) \cos\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& - \left[p_{14} L\right] \sin(T_0) \cos\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right) \\
& + \left[p_{11} K - p_{41} L + 0.75 S_2^2 p_{31} (K^3 + K L^2)\right] \cos(T_0) \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& + \left[p_{12} K - p_{42} L + 0.75 S_2^2 p_{32} (K^3 + K L^2)\right] \cos(T_0) \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right) \\
& + \left[p_{13} K\right] \cos(T_0) \cos\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& + \left[p_{14} K\right] \cos(T_0) \cos\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right) \\
& - \left[S_2 p_{21} K L\right] \sin(2T_0) \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& - \left[S_2 p_{22} K L\right] \sin(2T_0) \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right) \\
& + \left[0.5 S_2 p_{21} (K^2 + L^2)\right] \cos(2T_0) \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& + \left[0.5 S_2 p_{22} (K^2 + L^2)\right] \cos(2T_0) \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right) \\
& - \left[S_2^2 p_{31} (0.75 K^2 L + 0.25 L^3)\right] \sin(3T_0) \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& - \left[S_2^2 p_{32} (0.75 K^2 L + 0.25 L^3)\right] \sin(3T_0) \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right) \\
& - \left[S_2^2 p_{31} (0.75 K L^2 - 0.25 K^3)\right] \cos(3T_0) \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& - \left[S_2^2 p_{32} (0.75 K L^2 - 0.25 K^3)\right] \cos(3T_0) \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right).
\end{aligned}
$$

Again, rewriting products of sines and cosines as sines and cosines:

$$
\begin{aligned}
\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = & \Big[S_2^{-1} C_0 + 0.5 S_2 C_2 (K^2 + L^2)\Big] \\
& - \Big[C_1 L - 2\frac{dL}{dT_1} + C_4 K + 0.75 S_2^2 C_3 (L^3 + K^2 L)\Big] \sin(T_0) \\
& + \Big[C_1 K - 2\frac{dK}{dT_1} - C_4 L + 0.75 S_2^2 C_3 (K^3 + K L^2)\Big] \cos(T_0) \\
& - \Big[S_2 C_2 K L\Big] \sin(2T_0) \\
& + \Big[0.5 S_2 C_2 (K^2 + L^2)\Big] \cos(2T_0) \\
& - \Big[S_2^2 C_3 (0.75 K^2 L + 0.25 L^3)\Big] \sin(3T_0) \\
& - \Big[S_2^2 C_3 (0.75 K L^2 - 0.25 K^3)\Big] \cos(3T_0) \\
& + \Big[S_2^{-1} p_{01} + 0.5 S_2 p_{21} (K^2 + L^2)\Big] \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0 + \beta_1\right) \\
& + \Big[S_2^{-1} p_{02} + 0.5 S_2 p_{22} (K^2 + L^2)\Big] \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0 + \beta_2\right) \\
& - 0.5\Big[p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L)\Big] \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0 + \beta_1\right) \\
& + 0.5\Big[p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L)\Big] \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0 + \beta_1\right) \\
& - 0.5\Big[p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L)\Big] \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0 + \beta_2\right) \\
& + 0.5\Big[p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L)\Big] \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 1\right) T_0 + \beta_2\right) \\
& + 0.5\Big[p_{13} L\Big] \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0 + \beta_1\right) \\
& - 0.5\Big[p_{13} L\Big] \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0 + \beta_1\right) \\
& + 0.5\Big[p_{14} L\Big] \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0 + \beta_2\right) \\
& - 0.5\Big[p_{14} L\Big] \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 1\right) T_0 + \beta_2\right) \\
& + 0.5\Big[p_{11} K - p_{41} L + 0.75 S_2^2 p_{31} (K^3 + K L^2)\Big] \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0 + \beta_1\right) \\
& + 0.5\Big[p_{11} K - p_{41} L + 0.75 S_2^2 p_{31} (K^3 + K L^2)\Big] \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0 + \beta_1\right) \\
& + 0.5\Big[p_{12} K - p_{42} L + 0.75 S_2^2 p_{32} (K^3 + K L^2)\Big] \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0 + \beta_2\right) \\
& + 0.5\Big[p_{12} K - p_{42} L + 0.75 S_2^2 p_{32} (K^3 + K L^2)\Big] \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 1\right) T_0 + \beta_2\right) \\
& + 0.5\Big[p_{13} K\Big] \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0 + \beta_1\right) \\
& + 0.5\Big[p_{13} K\Big] \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0 + \beta_1\right) \\
& + 0.5\Big[p_{14} K\Big] \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0 + \beta_2\right) \\
& + 0.5\Big[p_{14} K\Big] \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 1\right) T_0 + \beta_2\right)
\end{aligned}
$$

$$- 0.5\Big[Sp_{21}KL\Big]\cos\left(\left(\omega_1\sqrt{\frac{M}{k}} - 2\right)T_0 + \beta_1\right)$$
$$+ 0.5\Big[Sp_{21}KL\Big]\cos\left(\left(\omega_1\sqrt{\frac{M}{k}} + 2\right)T_0 + \beta_1\right)$$
$$- 0.5\Big[Sp_{22}KL\Big]\cos\left(\left(\omega_2\sqrt{\frac{M}{k}} - 2\right)T_0 + \beta_2\right)$$
$$+ 0.5\Big[Sp_{22}KL\Big]\cos\left(\left(\omega_2\sqrt{\frac{M}{k}} + 2\right)T_0 + \beta_2\right)$$
$$+ 0.5\Big[0.5S_2p_{21}(K^2 + L^2)\Big]\sin\left(\left(\omega_1\sqrt{\frac{M}{k}} - 2\right)T_0 + \beta_1\right)$$
$$+ 0.5\Big[0.5S_2p_{21}(K^2 + L^2)\Big]\sin\left(\left(\omega_1\sqrt{\frac{M}{k}} + 2\right)T_0 + \beta_1\right)$$
$$+ 0.5\Big[0.5S_2p_{22}(K^2 + L^2)\Big]\sin\left(\left(\omega_2\sqrt{\frac{M}{k}} - 2\right)T_0 + \beta_2\right)$$
$$+ 0.5\Big[0.5S_2p_{22}(K^2 + L^2)\Big]\sin\left(\left(\omega_2\sqrt{\frac{M}{k}} + 2\right)T_0 + \beta_2\right)$$
$$- 0.5\Big[S_2^2p_{31}(0.75K^2L + 0.25L^3)\Big]\cos\left(\left(\omega_1\sqrt{\frac{M}{k}} - 3\right)T_0 + \beta_1\right)$$
$$+ 0.5\Big[S_2^2p_{31}(0.75K^2L + 0.25L^3)\Big]\cos\left(\left(\omega_1\sqrt{\frac{M}{k}} + 3\right)T_0 + \beta_1\right)$$
$$- 0.5\Big[S_2^2p_{32}(0.75K^2L + 0.25L^3)\Big]\cos\left(\left(\omega_2\sqrt{\frac{M}{k}} - 3\right)T_0 + \beta_2\right)$$
$$+ 0.5\Big[S_2^2p_{32}(0.75K^2L + 0.25L^3)\Big]\cos\left(\left(\omega_2\sqrt{\frac{M}{k}} + 3\right)T_0 + \beta_2\right)$$
$$- 0.5\Big[S_2^2p_{31}(0.75KL^2 - 0.25K^3)\Big]\sin\left(\left(\omega_1\sqrt{\frac{M}{k}} - 3\right)T_0 + \beta_1\right)$$
$$- 0.5\Big[S_2^2p_{31}(0.75KL^2 - 0.25K^3)\Big]\sin\left(\left(\omega_1\sqrt{\frac{M}{k}} + 3\right)T_0 + \beta_1\right)$$
$$- 0.5\Big[S_2^2p_{32}(0.75KL^2 - 0.25K^3)\Big]\sin\left(\left(\omega_2\sqrt{\frac{M}{k}} - 3\right)T_0 + \beta_2\right)$$
$$- 0.5\Big[S_2^2p_{32}(0.75KL^2 - 0.25K^3)\Big]\sin\left(\left(\omega_2\sqrt{\frac{M}{k}} + 3\right)T_0 + \beta_2\right)$$

Rewriting sines and cosines in order to split $\sin(\beta_i)$ and $\cos(\beta_i)$:

$$
\begin{aligned}
\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = & \Big[S_2^{-1} C_0 + 0.5 S_2 C_2 (K^2 + L^2)\Big] \\
& - \Big[C_1 L - 2\frac{dL}{dT_1} + C_4 K + 0.75 S_2^2 C_3 (L^3 + K^2 L)\Big] \sin(T_0) \\
& + \Big[C_1 K - 2\frac{dK}{dT_1} - C_4 L + 0.75 S_2^2 C_3 (K^3 + K L^2)\Big] \cos(T_0) \\
& - \Big[S_2 C_2 K L\Big] \sin(2T_0) \\
& + \Big[0.5 S_2 C_2 (K^2 + L^2)\Big] \cos(2T_0) \\
& - \Big[S_2^2 C_3 (0.75 K^2 L + 0.25 L^3)\Big] \sin(3T_0) \\
& + \Big[S_2^2 C_3 (0.75 K L^2 - 0.25 K^3)\Big] \cos(3T_0) \\
& + \Big[S_2^{-1} p_{01} + 0.5 S_2 p_{21} (K^2 + L^2)\Big] \cos(\beta_1) \sin\left(\omega_1 \sqrt{\frac{M}{k}} T_0\right) \\
& + \Big[S_2^{-1} p_{01} + 0.5 S_2 p_{21} (K^2 + L^2)\Big] \sin(\beta_1) \cos\left(\omega_1 \sqrt{\frac{M}{k}} T_0\right) \\
& + \Big[S_2^{-1} p_{02} + 0.5 S_2 p_{22} (K^2 + L^2)\Big] \cos(\beta_2) \sin\left(\omega_2 \sqrt{\frac{M}{k}} T_0\right) \\
& + \Big[S_2^{-1} p_{02} + 0.5 S_2 p_{22} (K^2 + L^2)\Big] \sin(\beta_2) \cos\left(\omega_2 \sqrt{\frac{M}{k}} T_0\right) \\
& - 0.5\Big[p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L)\Big] \cos(\beta_1) \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
& + 0.5\Big[p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L)\Big] \sin(\beta_1) \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
& + 0.5\Big[p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L)\Big] \cos(\beta_1) \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
& - 0.5\Big[p_{11} L + p_{41} K + 0.75 S_2^2 p_{31} (L^3 + K^2 L)\Big] \sin(\beta_1) \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
& - 0.5\Big[p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L)\Big] \cos(\beta_2) \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
& + 0.5\Big[p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L)\Big] \sin(\beta_2) \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
& + 0.5\Big[p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L)\Big] \cos(\beta_2) \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
& - 0.5\Big[p_{12} L + p_{42} K + 0.75 S_2^2 p_{32} (L^3 + K^2 L)\Big] \sin(\beta_2) \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
& + 0.5\Big[p_{13} L\Big] \cos(\beta_1) \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
& + 0.5\Big[p_{13} L\Big] \sin(\beta_1) \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
& - 0.5\Big[p_{13} L\Big] \cos(\beta_1) \sin\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
& - 0.5\Big[p_{13} L\Big] \sin(\beta_1) \cos\left(\left(\omega_1 \sqrt{\frac{M}{k}} + 1\right) T_0\right) \\
& + 0.5\Big[p_{14} L\Big] \cos(\beta_2) \sin\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0\right) \\
& + 0.5\Big[p_{14} L\Big] \sin(\beta_2) \cos\left(\left(\omega_2 \sqrt{\frac{M}{k}} - 1\right) T_0\right)
\end{aligned}
$$

$$- 0.5\Big[p_{14}L\Big]\cos(\beta_2)\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$- 0.5\Big[p_{14}L\Big]\sin(\beta_2)\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$+ 0.5\Big[p_{11}K - p_{41}L + 0.75S_2^2p_{31}(K^3+KL^2)\Big]\cos(\beta_1)\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}-1\right)T_0\right)$$
$$+ 0.5\Big[p_{11}K - p_{41}L + 0.75S_2^2p_{31}(K^3+KL^2)\Big]\sin(\beta_1)\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}-1\right)T_0\right)$$
$$+ 0.5\Big[p_{11}K - p_{41}L + 0.75S_2^2p_{31}(K^3+KL^2)\Big]\cos(\beta_1)\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$+ 0.5\Big[p_{11}K - p_{41}L + 0.75S_2^2p_{31}(K^3+KL^2)\Big]\sin(\beta_1)\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$+ 0.5\Big[p_{12}K - p_{42}L + 0.75S_2^2p_{32}(K^3+KL^2)\Big]\cos(\beta_2)\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}-1\right)T_0\right)$$
$$+ 0.5\Big[p_{12}K - p_{42}L + 0.75S_2^2p_{32}(K^3+KL^2)\Big]\sin(\beta_2)\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}-1\right)T_0\right)$$
$$+ 0.5\Big[p_{12}K - p_{42}L + 0.75S_2^2p_{32}(K^3+KL^2)\Big]\cos(\beta_2)\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$+ 0.5\Big[p_{12}K - p_{42}L + 0.75S_2^2p_{32}(K^3+KL^2)\Big]\sin(\beta_2)\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$+ 0.5\Big[p_{13}K\Big]\cos(\beta_1)\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}-1\right)T_0\right)$$
$$- 0.5\Big[p_{13}K\Big]\sin(\beta_1)\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}-1\right)T_0\right)$$
$$+ 0.5\Big[p_{13}K\Big]\cos(\beta_1)\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$- 0.5\Big[p_{13}K\Big]\sin(\beta_1)\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$+ 0.5\Big[p_{14}K\Big]\cos(\beta_2)\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}-1\right)T_0\right)$$
$$- 0.5\Big[p_{14}K\Big]\sin(\beta_2)\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}-1\right)T_0\right)$$
$$+ 0.5\Big[p_{14}K\Big]\cos(\beta_2)\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$- 0.5\Big[p_{14}K\Big]\sin(\beta_2)\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}+1\right)T_0\right)$$
$$- 0.5\Big[S_2p_{21}KL\Big]\cos(\beta_1)\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}-2\right)T_0\right)$$
$$+ 0.5\Big[S_2p_{21}KL\Big]\sin(\beta_1)\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}-2\right)T_0\right)$$
$$+ 0.5\Big[S_2p_{21}KL\Big]\cos(\beta_1)\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}+2\right)T_0\right)$$
$$- 0.5\Big[S_2p_{21}KL\Big]\sin(\beta_1)\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}+2\right)T_0\right)$$
$$- 0.5\Big[S_2p_{22}KL\Big]\cos(\beta_2)\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}-2\right)T_0\right)$$
$$+ 0.5\Big[S_2p_{22}KL\Big]\sin(\beta_2)\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}-2\right)T_0\right)$$

$$+ 0.5\Big[S_2 p_{22} KL\Big] \cos(\beta_2) \cos\left(\left(\omega_2\sqrt{\frac{M}{k}} + 2\right) T_0\right)$$

$$- 0.5\Big[S_2 p_{22} KL\Big] \sin(\beta_2) \sin\left(\left(\omega_2\sqrt{\frac{M}{k}} + 2\right) T_0\right)$$

$$+ 0.5\Big[0.5 S_2 p_{21}(K^2 + L^2)\Big] \cos(\beta_1) \sin\left(\left(\omega_1\sqrt{\frac{M}{k}} - 2\right) T_0\right)$$

$$+ 0.5\Big[0.5 S_2 p_{21}(K^2 + L^2)\Big] \sin(\beta_1) \cos\left(\left(\omega_1\sqrt{\frac{M}{k}} - 2\right) T_0\right)$$

$$+ 0.5\Big[0.5 S_2 p_{21}(K^2 + L^2)\Big] \cos(\beta_1) \sin\left(\left(\omega_1\sqrt{\frac{M}{k}} + 2\right) T_0\right)$$

$$+ 0.5\Big[0.5 S_2 p_{21}(K^2 + L^2)\Big] \sin(\beta_1) \cos\left(\left(\omega_1\sqrt{\frac{M}{k}} + 2\right) T_0\right)$$

$$+ 0.5\Big[0.5 S_2 p_{22}(K^2 + L^2)\Big] \cos(\beta_2) \sin\left(\left(\omega_2\sqrt{\frac{M}{k}} - 2\right) T_0\right)$$

$$+ 0.5\Big[0.5 S_2 p_{22}(K^2 + L^2)\Big] \sin(\beta_2) \cos\left(\left(\omega_2\sqrt{\frac{M}{k}} - 2\right) T_0\right)$$

$$+ 0.5\Big[0.5 S_2 p_{22}(K^2 + L^2)\Big] \cos(\beta_2) \sin\left(\left(\omega_2\sqrt{\frac{M}{k}} + 2\right) T_0\right)$$

$$+ 0.5\Big[0.5 S_2 p_{22}(K^2 + L^2)\Big] \sin(\beta_2) \cos\left(\left(\omega_2\sqrt{\frac{M}{k}} + 2\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{31}(0.75K^2 L + 0.25L^3)\Big] \cos(\beta_1) \cos\left(\left(\omega_1\sqrt{\frac{M}{k}} - 3\right) T_0\right)$$

$$+ 0.5\Big[S_2^2 p_{31}(0.75K^2 L + 0.25L^3)\Big] \sin(\beta_1) \sin\left(\left(\omega_1\sqrt{\frac{M}{k}} - 3\right) T_0\right)$$

$$+ 0.5\Big[S_2^2 p_{31}(0.75K^2 L + 0.25L^3)\Big] \cos(\beta_1) \cos\left(\left(\omega_1\sqrt{\frac{M}{k}} + 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{31}(0.75K^2 L + 0.25L^3)\Big] \sin(\beta_1) \sin\left(\left(\omega_1\sqrt{\frac{M}{k}} + 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{32}(0.75K^2 L + 0.25L^3)\Big] \cos(\beta_2) \cos\left(\left(\omega_2\sqrt{\frac{M}{k}} - 3\right) T_0\right)$$

$$+ 0.5\Big[S_2^2 p_{32}(0.75K^2 L + 0.25L^3)\Big] \sin(\beta_2) \sin\left(\left(\omega_2\sqrt{\frac{M}{k}} - 3\right) T_0\right)$$

$$+ 0.5\Big[S_2^2 p_{32}(0.75K^2 L + 0.25L^3)\Big] \cos(\beta_2) \cos\left(\left(\omega_2\sqrt{\frac{M}{k}} + 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{32}(0.75K^2 L + 0.25L^3)\Big] \sin(\beta_2) \sin\left(\left(\omega_2\sqrt{\frac{M}{k}} + 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{31}(0.75KL^2 - 0.25K^3)\Big] \cos(\beta_1) \sin\left(\left(\omega_1\sqrt{\frac{M}{k}} - 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{31}(0.75KL^2 - 0.25K^3)\Big] \sin(\beta_1) \cos\left(\left(\omega_1\sqrt{\frac{M}{k}} - 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{31}(0.75KL^2 - 0.25K^3)\Big] \cos(\beta_1) \sin\left(\left(\omega_1\sqrt{\frac{M}{k}} + 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{31}(0.75KL^2 - 0.25K^3)\Big] \sin(\beta_1) \cos\left(\left(\omega_1\sqrt{\frac{M}{k}} + 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{32}(0.75KL^2 - 0.25K^3)\Big] \cos(\beta_2) \sin\left(\left(\omega_2\sqrt{\frac{M}{k}} - 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{32}(0.75KL^2 - 0.25K^3)\Big] \sin(\beta_2) \cos\left(\left(\omega_2\sqrt{\frac{M}{k}} - 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{32}(0.75KL^2 - 0.25K^3)\Big] \cos(\beta_2) \sin\left(\left(\omega_2\sqrt{\frac{M}{k}} + 3\right) T_0\right)$$

$$- 0.5\Big[S_2^2 p_{32}(0.75KL^2 - 0.25K^3)\Big] \sin(\beta_2) \cos\left(\left(\omega_2\sqrt{\frac{M}{k}} + 3\right) T_0\right)$$

And gathering terms:

$$
\begin{aligned}
\frac{\partial^2 z_1}{\partial T_0^2} + z_1 = & \left[S^{-1}C_0 + 0.5S_2C_2(K^2 + L^2)\right] \\
& - \left[C_1L - 2\frac{dL}{dT_1} + C_4K + 0.75S^2C_3(L^3 + K^2L)\right]\sin(T_0) \\
& + \left[C_1K - 2\frac{dK}{dT_1} - C_4L + 0.75S^2C_3(K^3 + KL^2)\right]\cos(T_0) \\
& - \left[SC_2KL\right]\sin(2T_0) \\
& + \left[0.5SC_2(K^2 + L^2)\right]\cos(2T_0) \\
& - \left[S^2C_3(0.75K^2L + 0.25L^3)\right]\sin(3T_0) \\
& + \left[S^2C_3(0.75KL^2 - 0.25K^3)\right]\cos(3T_0) \\
& + \left[S^{-1}p_{01} + 0.5S_2p_{21}(K^2 + L^2)\right]\cos(\beta_1)\sin\left(\omega_1\sqrt{\frac{M}{k}}T_0\right) \\
& + \left[S^{-1}p_{01} + 0.5S_2p_{21}(K^2 + L^2)\right]\sin(\beta_1)\cos\left(\omega_1\sqrt{\frac{M}{k}}T_0\right) \\
& + \left[S^{-1}p_{02} + 0.5S_2p_{22}(K^2 + L^2)\right]\cos(\beta_2)\sin\left(\omega_2\sqrt{\frac{M}{k}}T_0\right) \\
& + \left[S^{-1}p_{02} + 0.5S_2p_{22}(K^2 + L^2)\right]\sin(\beta_2)\cos\left(\omega_2\sqrt{\frac{M}{k}}T_0\right) \\
& + 0.5\Big\{[p_{11}K - p_{41}L + 0.75S^2p_{31}(K^3 + KL^2) + p_{13}L]\sin(\beta_1) \\
& \quad - [p_{11}L + p_{41}K + 0.75S^2p_{31}(L^3 + K^2L) - p_{13}K]\cos(\beta_1)\Big\}\cdot\cos\left(\left(\omega_1\sqrt{\frac{M}{k}} - 1\right)T_0\right) \\
& + 0.5\Big\{[p_{11}L + p_{41}K + 0.75S^2p_{31}(L^3 + K^2L) - p_{13}K]\sin(\beta_1) \\
& \quad + [p_{11}K - p_{41}L + 0.75S^2p_{31}(K^3 + KL^2) + p_{13}L]\cos(\beta_1)\Big\}\cdot\sin\left(\left(\omega_1\sqrt{\frac{M}{k}} - 1\right)T_0\right) \\
& + 0.5\Big\{[p_{11}K - p_{41}L + 0.75S^2p_{31}(K^3 + KL^2) - p_{13}L]\sin(\beta_1) \\
& \quad + [p_{11}L + p_{41}K + 0.75S^2p_{31}(L^3 + K^2L) + p_{13}K]\cos(\beta_1)\Big\}\cdot\cos\left(\left(\omega_1\sqrt{\frac{M}{k}} + 1\right)T_0\right) \\
& + 0.5\Big\{[p_{11}K - p_{41}L + 0.75S^2p_{31}(K^3 + L^2K) - p_{13}L]\cos(\beta_1) \\
& \quad - [p_{11}L + p_{41}K + 0.75S^2p_{31}(L^3 + K^2L) + p_{13}K]\sin(\beta_1)\Big\}\cdot\sin\left(\left(\omega_1\sqrt{\frac{M}{k}} + 1\right)T_0\right) \\
& + 0.5\Big\{[p_{12}K - p_{42}L + 0.75S^2p_{32}(K^3 + KL^2) + p_{14}L]\sin(\beta_2) \\
& \quad - [p_{12}L + p_{42}K + 0.75S^2p_{32}(L^3 + K^2L) - p_{14}K]\cos(\beta_2)\Big\}\cdot\cos\left(\left(\omega_2\sqrt{\frac{M}{k}} - 1\right)T_0\right) \\
& + 0.5\Big\{[p_{12}L + p_{42}K + 0.75S^2p_{32}(L^3 + K^2L) - p_{14}K]\sin(\beta_2) \\
& \quad + [p_{12}K - p_{42}L + 0.75S^2p_{32}(K^3 + KL^2) + p_{14}L]\cos(\beta_2)\Big\}\cdot\sin\left(\left(\omega_2\sqrt{\frac{M}{k}} - 1\right)T_0\right) \\
& + 0.5\Big\{[p_{12}K - p_{42}L + 0.75S^2p_{32}(K^3 + KL^2) - p_{14}L]\sin(\beta_2) \\
& \quad + [p_{12}L + p_{42}K + 0.75S^2p_{32}(L^3 + K^2L) + p_{14}K]\cos(\beta_2)\Big\}\cdot\cos\left(\left(\omega_2\sqrt{\frac{M}{k}} + 1\right)T_0\right) \\
& + 0.5\Big\{[p_{12}K - p_{42}L + 0.75S^2p_{32}(K^3 + KL^2) - P_{14}L]\cos(\beta_2) \\
& \quad - [p_{12}L + p_{42}K + 0.75S^2p_{32}(L^3 + K^2L) + p_{14}K]\sin(\beta_2)\Big\}\cdot\sin\left(\left(\omega_2\sqrt{\frac{M}{k}} + 1\right)T_0\right)
\end{aligned}
$$

$$+ 0.5\Big\{[0.5Sp_{21}(K^2+L^2)]\sin(\beta_1) - [Sp_{21}KL]\cos(\beta_1)\Big\}\cdot\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}-2\right)T_0\right)$$
$$+ 0.5\Big\{[Sp_{21}KL]\sin(\beta_1) + [0.5Sp_{21}(K^2+L^2)]\cos(\beta_1)\Big\}\cdot\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}-2\right)T_0\right)$$
$$+ 0.5\Big\{[0.5Sp_{21}(K^2+L^2)]\sin(\beta_1) + [Sp_{21}KL]\cos(\beta_1)\Big\}\cdot\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}+2\right)T_0\right)$$
$$+ 0.5\Big\{[0.5Sp_{21}(K^2+L^2)]\cos(\beta_1) - [Sp_{21}KL]\sin(\beta_1)\Big\}\cdot\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}+2\right)T_0\right)$$
$$+ 0.5\Big\{[0.5Sp_{22}(K^2+L^2)]\sin(\beta_2) - [Sp_{22}KL]\cos(\beta_2)\Big\}\cdot\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}-2\right)T_0\right)$$
$$+ 0.5\Big\{[Sp_{22}KL]\sin(\beta_2) + [0.5Sp_{22}(K^2+L^2)]\cos(\beta_2)\Big\}\cdot\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}-2\right)T_0\right)$$
$$+ 0.5\Big\{[0.5Sp_{22}(K^2+L^2)]\sin(\beta_2) + [Sp_{22}KL]\cos(\beta_2)\Big\}\cdot\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}+2\right)T_0\right)$$
$$+ 0.5\Big\{[0.5Sp_{22}(K^2+L^2)]\cos(\beta_2) - [Sp_{22}KL]\sin(\beta_2)\Big\}\cdot\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}+2\right)T_0\right)$$
$$- 0.5\Big\{[S^2p_{31}(0.75KL^2-0.25K^3)]\sin(\beta_1) + [S^2p_{31}(0.75K^2L+0.25L^3)]\cos(\beta_1)\Big\}\cdot\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}-3\right)T_0\right)$$
$$+ 0.5\Big\{-[S^2p_{31}(0.75KL^2-0.25K^3)]\cos(\beta_1) + [S^2p_{31}(0.75K^2L+0.25L^3)]\sin(\beta_1)\Big\}\cdot\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}-3\right)T_0\right)$$
$$+ 0.5\Big\{-[S^2p_{31}(0.75KL^2-0.25K^3)]\sin(\beta_1) + [S^2p_{31}(0.75K^2L+0.25L^3)]\cos(\beta_1)\Big\}\cdot\cos\left(\left(\omega_1\sqrt{\frac{M}{k}}+3\right)T_0\right)$$
$$- 0.5\Big\{[S^2p_{31}(0.75KL^2-0.25K^3)]\cos(\beta_1) + [S^2p_{31}(0.75K^2L+0.25L^3)]\sin(\beta_1)\Big\}\cdot\sin\left(\left(\omega_1\sqrt{\frac{M}{k}}+3\right)T_0\right)$$
$$- 0.5\Big\{[S^2p_{32}(0.75KL^2-0.25K^3)]\sin(\beta_2) + [S^2p_{32}(0.75K^2L+0.25L^3)]\cos(\beta_2)\Big\}\cdot\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}-3\right)T_0\right)$$
$$+ 0.5\Big\{-[S^2p_{32}(0.75KL^2-0.25K^3)]\cos(\beta_2) + [S^2p_{32}(0.75K^2L+0.25L^3)]\sin(\beta_2)\Big\}\cdot\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}-3\right)T_0\right)$$
$$+ 0.5\Big\{-[S^2p_{32}(0.75KL^2-0.25K^3)]\cos(\beta_2) + [S^2p_{32}(0.75K^2L+0.25L^3)]\sin(\beta_2)\Big\}\cdot\cos\left(\left(\omega_2\sqrt{\frac{M}{k}}+3\right)T_0\right)$$
$$- 0.5\Big\{[S^2p_{32}(0.75KL^2-0.25K^3)]\cos(\beta_2) + [S^2p_{32}(0.75K^2L+0.25L^3)]\sin(\beta_2)\Big\}\cdot\sin\left(\left(\omega_2\sqrt{\frac{M}{k}}+3\right)T_0\right).$$

B Maple code

B.1 Duffing equation: Period approximation

```
>
> for n from 1 by 1 to 4 do
  alpha := 1 :
  beta := 0 :
  assume(0 ≤ u ≤ alpha) :
  epsilon := evalf(10^(-n)) :
```

$$fun := u \rightarrow \frac{1}{\mathrm{sqrt}\left(\beta^2 + \alpha^2 + \frac{\mathrm{epsilon}}{2}\alpha^4 - u^2 - \frac{\mathrm{epsilon}}{2}u^4\right)};$$

$$Tpn := 4 \cdot integrate(fun(u), u = 0 \,..\mathrm{alpha}) :$$

$$Tpa := \mathrm{abs}\left(\frac{2 \cdot \mathrm{Pi}}{\frac{3}{4}\left(\alpha^2 + \beta^2\right) \epsilon - 1}\right);$$

```
  print("epsilon:", epsilon);
  print("Ongesplitste integraal: ", abs(Tpn));
  print("Approx: ", Tpa);
  print("Verschil: ", abs(abs(Tpa) - abs(Tpn)));
  print( )
  restart :

  end do:
```

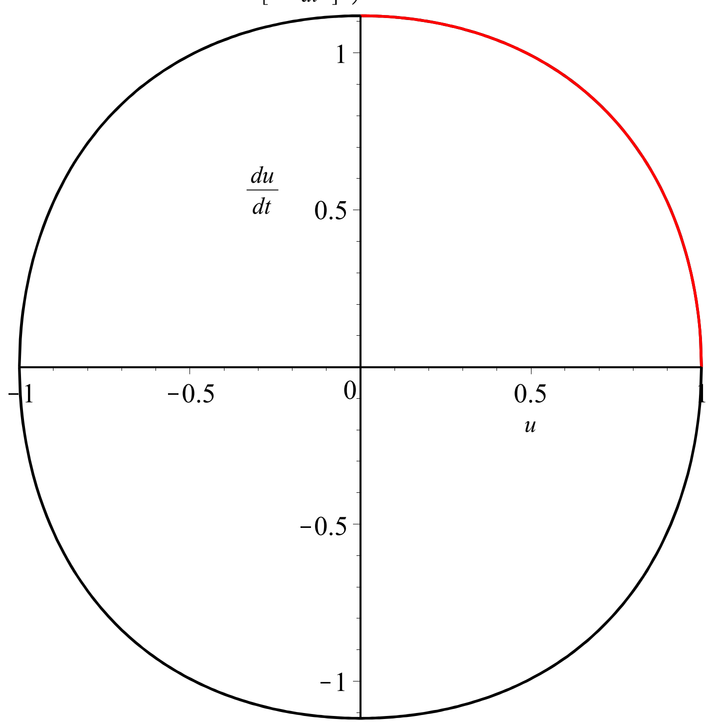
```
"epsilon:", 0.1000000000
"Ongesplitste integraal: ", 6.060656736
"Approx: ", 6.792632764
"Verschil: ", 0.731976028

"epsilon:", 0.01000000000
"Ongesplitste integraal: ", 6.259762304
"Approx: ", 6.330665298
"Verschil: ", 0.070902994

"epsilon:", 0.001000000000
"Ongesplitste integraal: ", 6.280830512
"Approx: ", 6.287901234
"Verschil: ", 0.007070722

"epsilon:", 0.0001000000000
"Ongesplitste integraal: ", 6.282949700
"Approx: ", 6.283656584
"Verschil: ", 0.000706884
```

(1)

> epsilon := 0.5

$$\epsilon := 0.5 \tag{2}$$

> $plot\left(\left\{\left[u, \mathrm{sqrt}\left(1+\frac{\mathrm{epsilon}}{2}-u^2-\frac{\mathrm{epsilon}}{2}\cdot u^4\right), u=-1..1\right], \left[u, -\mathrm{sqrt}\left(1+\frac{\mathrm{epsilon}}{2}-u^2-\frac{\mathrm{epsilon}}{2}\cdot u^4\right), u=-1..1\right], \left[u, \mathrm{sqrt}\left(1+\frac{\mathrm{epsilon}}{2}-u^2-\frac{\mathrm{epsilon}}{2}\cdot u^4\right), u=0..1\right]\right\}, color = [black, red, black], labels = \left[u, \frac{du}{dt}\right], \right)$



B.2 Case 1: Solution

> restart;

> $R := T1 \rightarrow \mathrm{sqrt}\left(\frac{0.5 \cdot C1 \cdot R0 \cdot \exp(C1 \cdot T1)}{1 + \frac{3}{8} C1 \cdot R0 \cdot \exp(C1 \cdot T1)}\right)$

$$R := T1 \rightarrow \sqrt{\frac{0.5\, C1\, R0\, e^{C1\, T1}}{1 + \frac{3}{8}\, C1\, R0\, e^{C1\, T1}}} \quad \textbf{(1)}$$

> $u := t \rightarrow R(\mathrm{epsilon} \cdot t) \cdot \cos(t + Phi0)$

$$u := t \rightarrow R(\epsilon\, t)\, \cos(t + \Phi0) \quad \textbf{(2)}$$

(3)

> $C1 := 1;$
> $g := 9.81;$
> $M := 10.14;$
> $k := 331.5780000;$
> $v := 10;$

$$C1 := 1$$
$$g := 9.81$$
$$M := 10.14$$
$$k := 331.5780000$$
$$v := 10 \quad \textbf{(4)}$$

> $R0 := \mathrm{solve}\left(R(0) = \frac{1}{2}, R0\right);$
> $R(0);$
> $Phi0 := 0;$
> $\mathrm{epsilon} := 0.01$

$$R0 := 0.6153846154$$
$$0.5000000000$$
$$\Phi0 := 0$$
$$\epsilon := 0.01 \quad \textbf{(5)}$$

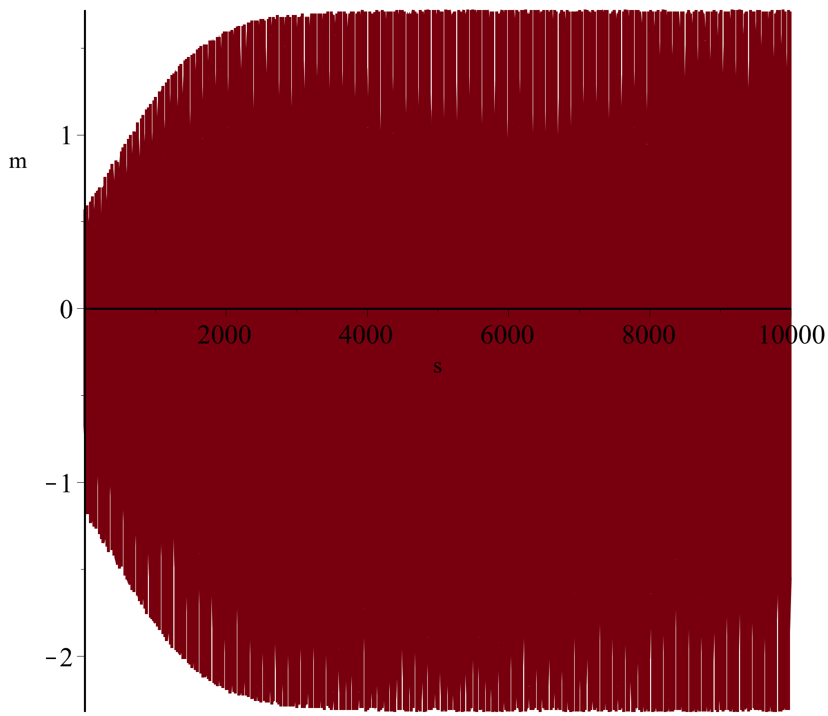
> $realu := t \rightarrow v \cdot \mathrm{sqrt}\left(\frac{M}{k}\right) \cdot u(t) - \frac{g \cdot M}{k}$

$$realu := t \rightarrow v \sqrt{\frac{M}{k}}\, u(t) - \frac{g\, M}{k} \quad \textbf{(6)}$$

> $realu(10)$

$$0.9102705240 \cos(10) - 0.3000000000 \quad \textbf{(7)}$$

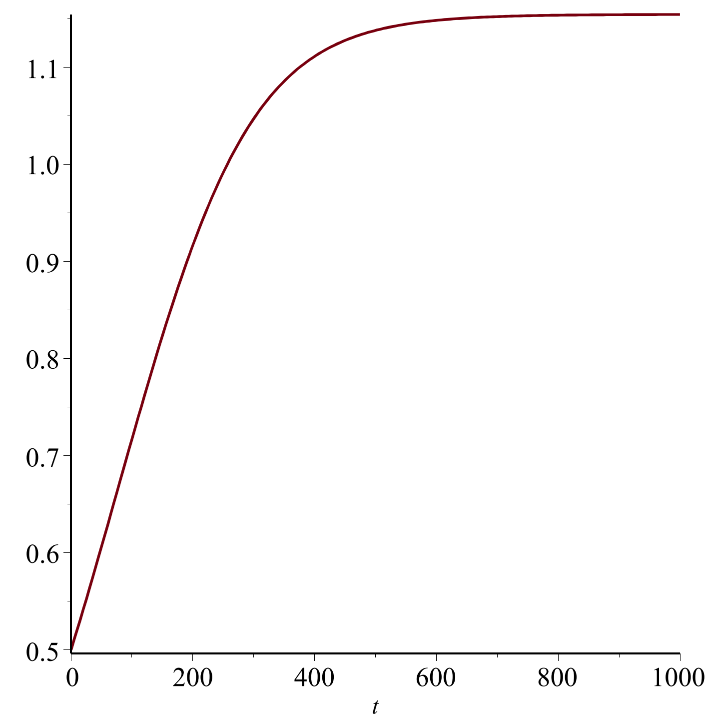
> $plot\left(realu\left(\mathrm{sqrt}\left(\frac{M}{k}\right) \cdot t\right), t = 0..10000, labels = [\text{"s"}, \text{"m"}]\right)$



> $limit(R(\text{epsilon}\cdot t),\ t = infinity)$

$$1.154700538 \qquad (8)$$

> $plot(R(\text{epsilon}\cdot t),\ t = 0\ ..1000)$



1.1
1.0
0.9
0.8
0.7
0.6
0.5
0
200
400
600
800
1000
t

>

>

B.3 Case 1: Phase plane

```
> restart;
> with(DEtools) :
> p01 := - 1/5;
  beta1 := 0;
```

$$p01 := -\frac{1}{5}$$

$$\beta 1 := 0 \qquad \textbf{(1)}$$

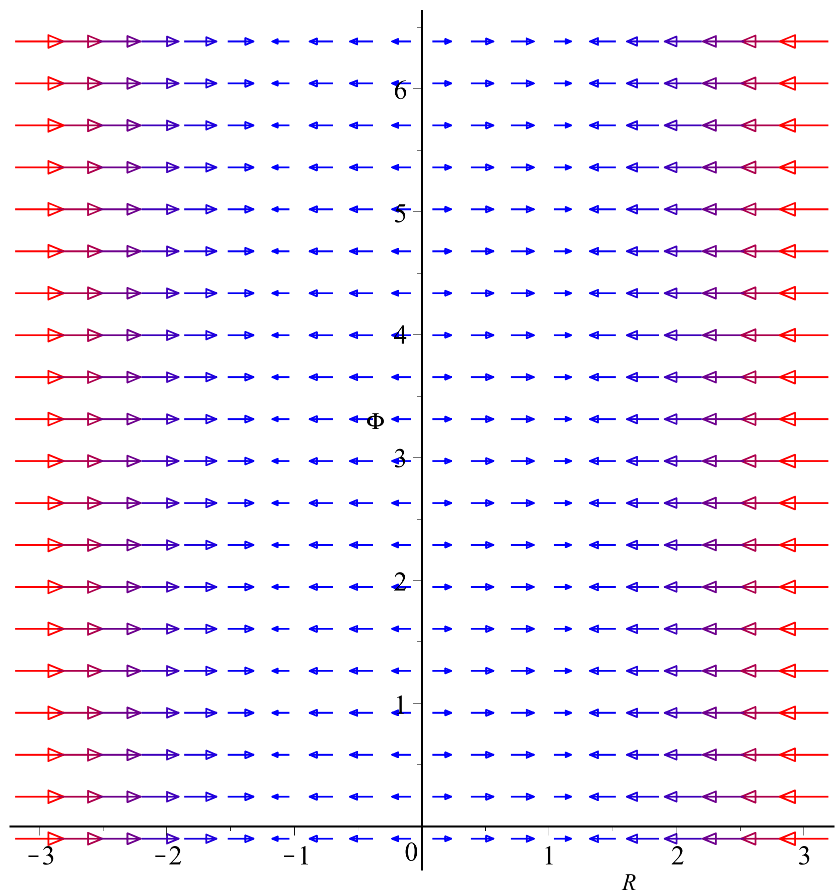
```
> DER := diff(R(T1), T1) = (-3/8 R(T1)^2 + 1/2) R(T1)
```

$$DER := \frac{d}{dT1} R(T1) = \left(-\frac{3}{8} R(T1)^2 + \frac{1}{2}\right) R(T1) \qquad \textbf{(2)}$$

```
> DEPhi := diff(Phi(T1), T1) = 0
```

$$DEPhi := \frac{d}{dT1} \Phi(T1) = 0 \qquad \textbf{(3)}$$

```
> DEplot([DER, DEPhi], [R(T1), Phi(T1)], T1 = 0..100, R = -3..3, Phi = -0.1..2·Pi + 0.1, arrows
      = medium, arrowsize = magnitude, size = [500, 500], color = magnitude, )
```



6
5
4
Φ
3
2
1
0
−3
−2
−1
1
2
3
R

>

>

B.4 Case 2: Equilibriums and eigenvalues

```
> restart;
> p01 := -1/5;
  beta1 := 0;
```

$$p01 := -\frac{1}{5}$$

$$\beta 1 := 0 \qquad (1)$$

```
> eqR := (R, P) -> (-3/8·R^2 + 1/2)·R - 1/2·(p01·cos(P + beta1))/sqrt(S3)
```

$$eqR := (R, P) \rightarrow \left(-\frac{3}{8} R^2 + \frac{1}{2}\right) R - \frac{1}{2} \frac{p01 \cos(P + \beta 1)}{\sqrt{S3}} \qquad (2)$$

```
> eqP := (R, P) -> (0.5· sin(P + beta1)·(p01))/sqrt(S3)
```

$$eqP := (R, P) \rightarrow \frac{0.5 \sin(P + \beta 1)\, p01}{\sqrt{S3}} \qquad (3)$$

```
> r := solve([eqP(R, P) = 0, 0 ≤ P, P ≤ 2·Pi], P, allsolutions)
```

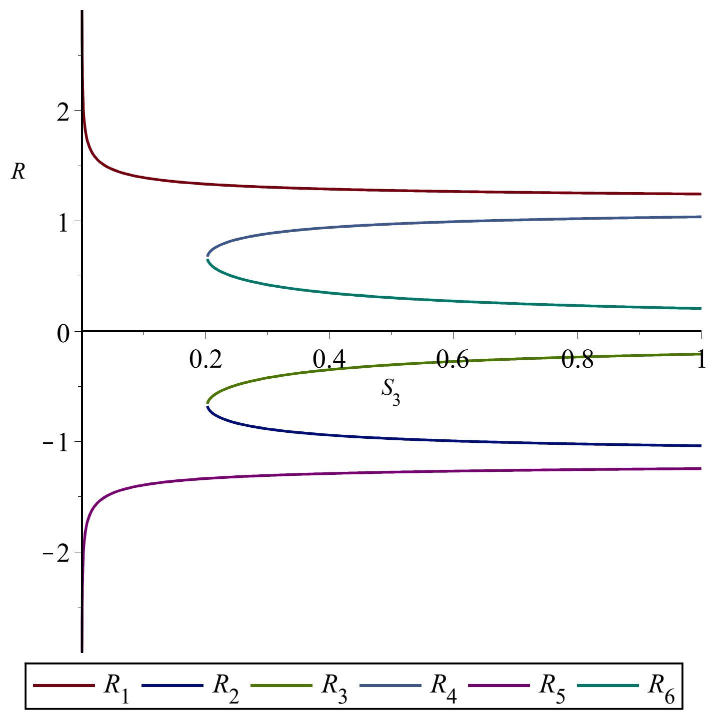
$$r := \{P = 3.141592654\ \_Z2\sim\}, \{P = 3.141592654\ \_Z1\sim\} \qquad (4)$$

```
> P1 := -beta1;
  P2 := -beta1 + Pi;
```

$$P1 := 0$$

$$P2 := \pi \qquad (5)$$

```
> RsP1 := evalf(solve(simplify(subs(P = P1 , eqR(R, P)) = 0), R)) :
  RsP2 := evalf(solve(simplify(subs(P = P2 , eqR(R, P)) = 0), R)) :
> R1P1 := unapply(simplify(RsP1[1]), S3) :
  R2P1 := unapply(simplify(RsP1[2]), S3) :
  R3P1 := unapply(simplify(RsP1[3]), S3) :
  R1P2 := unapply(simplify(RsP2[1]), S3) :
  R2P2 := unapply(simplify(RsP2[2]), S3) :
  R3P2 := unapply(simplify(RsP2[3]), S3) :
> plot([R1P1(S3), R2P1(S3), R3P1(S3), R1P2(S3), R2P2(S3), R3P2(S3)], S3 = 0..1, legend
      = [R_1, R_2, R_3, R_4, R_5, R_6], labels = [S_3, R])
```

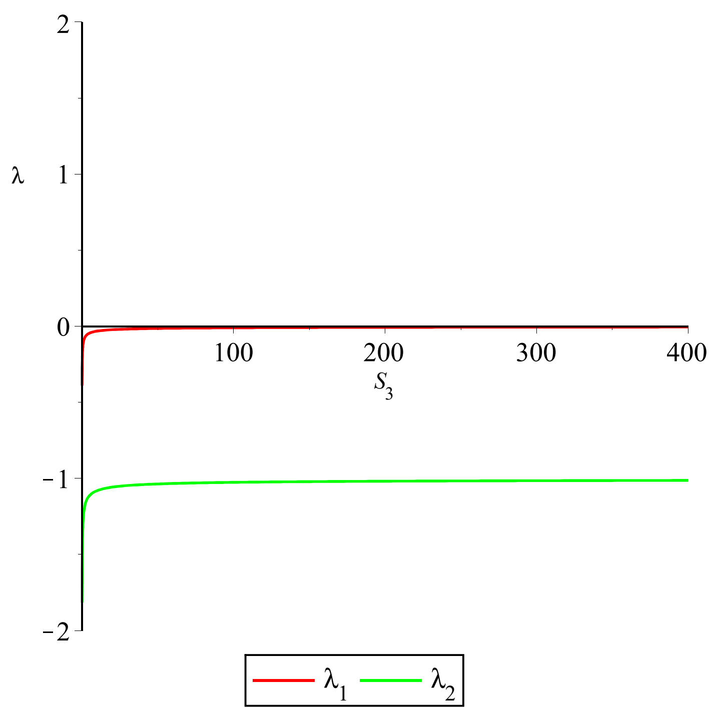



```
> limit(R1P1(S3), S3 = infinity);
  limit(R1P2(S3), S3 = infinity);
  limit(R2P1(S3), S3 = infinity);
  limit(R2P2(S3), S3 = infinity);
  limit(R3P1(S3), S3 = infinity);
  limit(R3P2(S3), S3 = infinity);
```

$$1.154700538 + 0.\,I$$
$$1.154700538 + 0.\,I$$
$$-1.154700539 + 0.\,I$$
$$-1.154700539 + 0.\,I$$
$$6.\,10^{-10} + 0.\,I$$
$$6.\,10^{-10} + 0.\,I \quad \textbf{(6)}$$

```
> R1P1 := R1P1(S3) :
  R2P1 := R2 P1(S3) :
  R3P1 := R3P1(S3) :
  R1P2 := R1P2(S3) :
  R2P2 := R2P2(S3) :
  R3P2 := R3P2(S3) :
> with(linalg) : J := jacobian([eqR(R, P), eqP(R, P)], [R, P]) :
```

```
> Jstar := map(x → subs(R = R1P1, P = P1, x), J) :
> chareq := collect(charpoly(Jstar, lambda), lambda) :
> a := coeff(chareq, lambda, 2) :
  b := coeff(chareq, lambda, 1) :
  c := coeff(chareq, lambda, 0) :
  D1 := sqrt(b^2 − 4·a·c) :
> l11 := evalf((−b + D1)/(2·a)) :
  l12 := (−b − D1)/(2·a) :
> plot([l11, l12], S3 = 0 ..400, y = −2 ..2, labels = [S_3, lambda], legend = [lambda_1, lambda_2], color
      = [red, green])
```




```
>
```

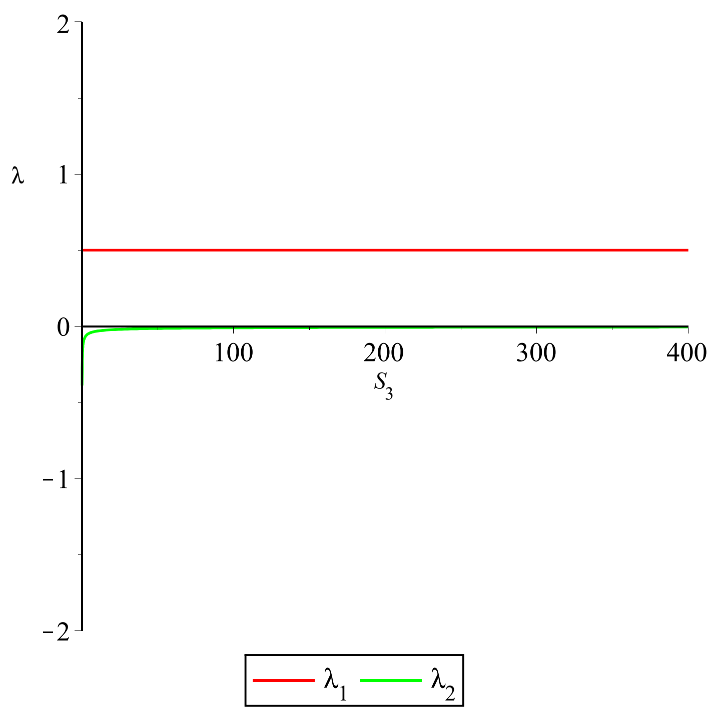
> $Jstar := map(x \rightarrow subs(R = R2P1, P = P1, x), J)$:

> $chareq := collect(charpoly(Jstar, \text{lambda}), \text{lambda})$:

> $a := coeff(chareq, \text{lambda}, 2)$:
$b := coeff(chareq, \text{lambda}, 1)$:
$c := coeff(chareq, \text{lambda}, 0)$:
$D1 := \text{sqrt}(b^2 - 4 \cdot a \cdot c)$:

> $l11 := evalf\left(\frac{(-b + D1)}{2 \cdot a}\right)$:
$l12 := \frac{(-b - D1)}{2 \cdot a}$:

> $plot([l11, l12], S3 = 0 ..400, y = -2 ..2, labels = [S_3, \text{lambda}], legend = [\text{lambda}_1, \text{lambda}_2], color = [red, green])$



>

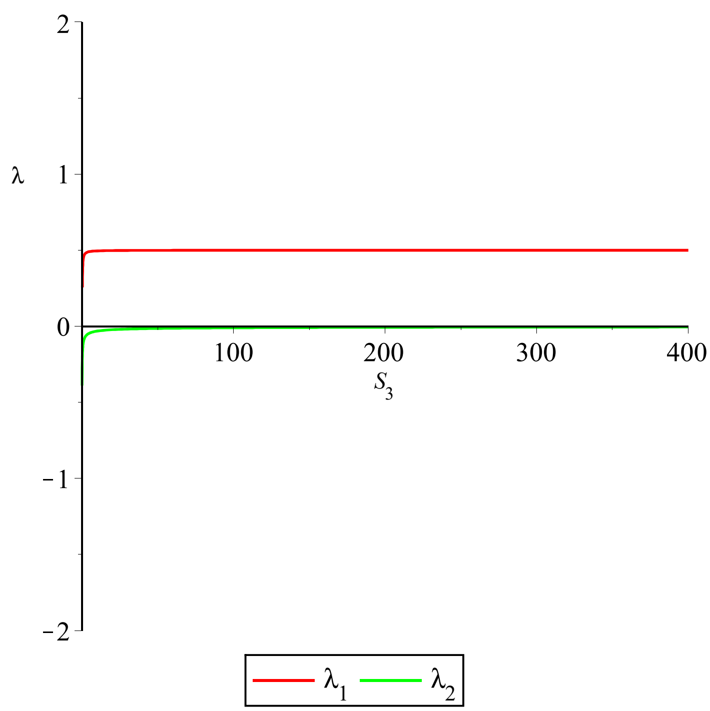
> $Jstar := map(x \rightarrow subs(R = R3P1, P = P1, x), J)$:

> $chareq := collect(charpoly(Jstar, \text{lambda}), \text{lambda})$:

> $a := coeff(chareq, \text{lambda}, 2)$:
$b := coeff(chareq, \text{lambda}, 1)$:
$c := coeff(chareq, \text{lambda}, 0)$:
$D1 := \text{sqrt}(b^2 - 4 \cdot a \cdot c)$:

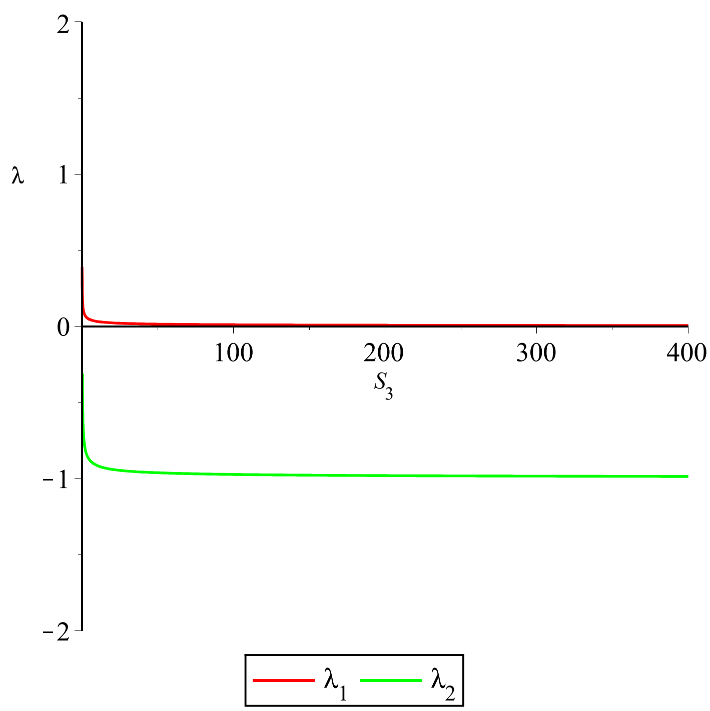
> $l11 := evalf\left(\frac{(-b + D1)}{2 \cdot a}\right)$:
$l12 := \frac{(-b - D1)}{2 \cdot a}$:

> $plot([l11, l12], S3 = 0..400, y = -2..2, labels = [S_3, \text{lambda}], legend = [\text{lambda}_1, \text{lambda}_2], color = [red, green])$



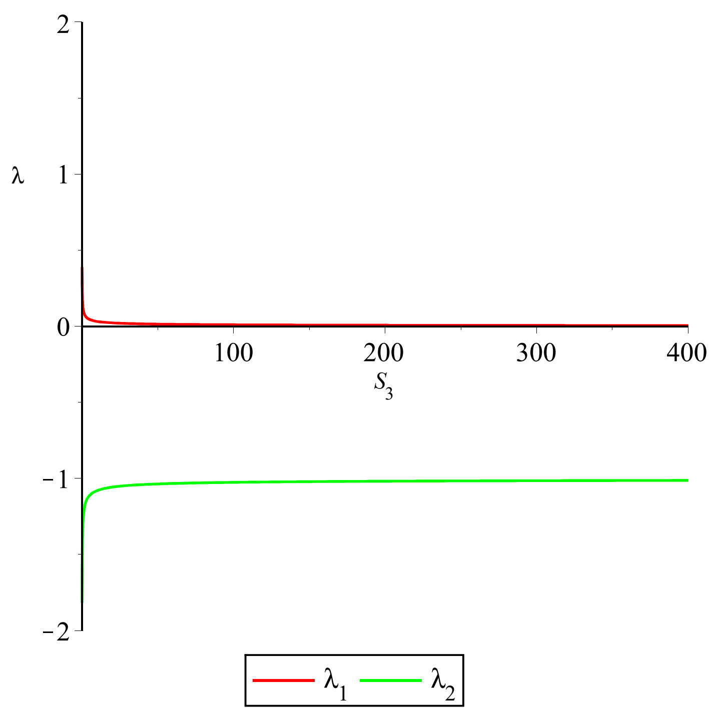
>

```
> Jstar := map(x → subs(R = R1P2, P = P2, x), J) :
> chareq := collect(charpoly(Jstar, lambda), lambda) :
> a := coeff(chareq, lambda, 2) :
  b := coeff(chareq, lambda, 1) :
  c := coeff(chareq, lambda, 0) :
  D1 := sqrt(b^2 - 4·a·c) :
> l11 := evalf((-b + D1)/(2·a)) :
  l12 := (-b - D1)/(2·a) :
> plot([l11, l12], S3 = 0 ..400, y = -2 ..2, labels = [S_3, lambda], legend = [lambda_1, lambda_2], color = [red, green])
```



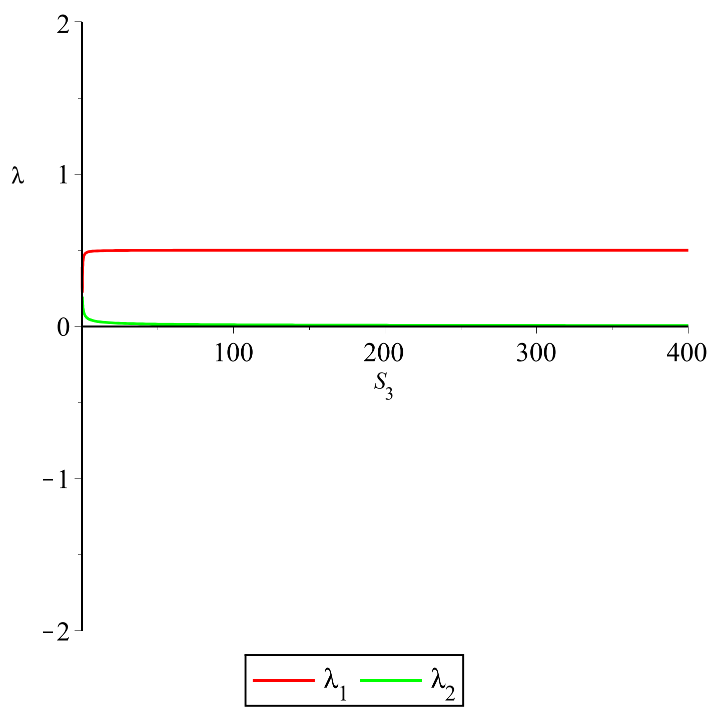

```
>
```

```
> Jstar := map(x → subs(R = R2P2, P = P2, x), J) :

> chareq := collect(charpoly(Jstar, lambda), lambda) :

> a := coeff(chareq, lambda, 2) :
  b := coeff(chareq, lambda, 1) :
  c := coeff(chareq, lambda, 0) :
  D1 := sqrt(b^2 - 4·a·c) :

> l11 := evalf((-b + D1)/(2·a)) :
  l12 := (-b - D1)/(2·a) :

> plot([l11, l12], S3 = 0..400, y = -2..2, labels = [S_3, lambda], legend = [lambda_1, lambda_2], color
      = [red, green])
```



>

```
> Jstar := map(x → subs(R = R3P2, P = P2, x), J) :
> chareq := collect(charpoly(Jstar, lambda), lambda) :
> a := coeff(chareq, lambda, 2) :
  b := coeff(chareq, lambda, 1) :
  c := coeff(chareq, lambda, 0) :
  D1 := sqrt(b^2 - 4·a·c) :
> l11 := evalf((-b + D1)/(2·a)) :
  l12 := (-b - D1)/(2·a) :
> plot([l11, l12], S3 = 0 ..400, y = -2 ..2, labels = [S_3, lambda], legend = [lambda_1, lambda_2], color
    = [red, green])
```




```
>
```

B.5 Case 2: Phase plane and solution

```
> restart;
> with(DEtools) :
  with(LinearAlgebra) :
>
> p01 := -1/5;
  beta1 := 0;
```

$$p01 := -\frac{1}{5}$$

$$\beta 1 := 0 \qquad (1)$$

```
> DER := diff(R(T1), T1) = (-3/8 R(T1)^2 + 1/2) R(T1) - (p01 cos(β1 + Phi(T1)))/(2·sqrt(S3))
```

$$DER := \frac{d}{dT1} R(T1) = \left(-\frac{3}{8} R(T1)^2 + \frac{1}{2}\right) R(T1) + \frac{1}{10} \frac{\cos(\Phi(T1))}{\sqrt{S3}} \qquad (2)$$

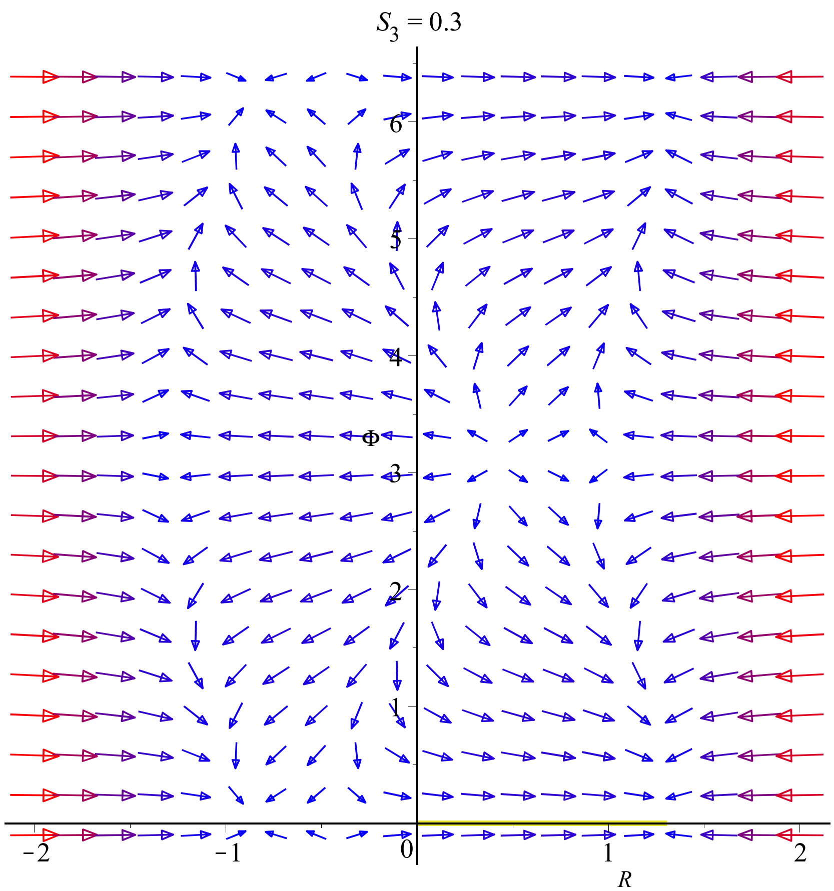
```
> DEPhi := diff(Phi(T1), T1) = 1/2 · (sin(Phi(T1) + beta1)·(p01))/sqrt(S3)
```

$$DEPhi := \frac{d}{dT1} \Phi(T1) = -\frac{1}{10} \frac{\sin(\Phi(T1))}{\sqrt{S3}} \qquad (3)$$

```
> S3 := 0.3
```

$$S3 := 0.3 \qquad (4)$$

```
> DEplot([DER, DEPhi], [R(T1), Phi(T1)], T1 = 0..100, R = -2..2, Phi = -0.1..2·Pi + 0.1,
    arrows = medium, arrowsize = magnitude, size = [500, 500], color = magnitude, title = [S_3
    = S3], [[R(0) = 0, Phi(0) = 0]], numpoints = 10000)
```



> $dt := \frac{1}{10}$:

> $N := \frac{800}{dt}$

$$N := 8000 \quad (5)$$

> $T := Transpose(Array(1..N, (i) \rightarrow dt \cdot (i-1)))$:

> epsilon := 0.1

$$\epsilon := 0.1 \quad (6)$$

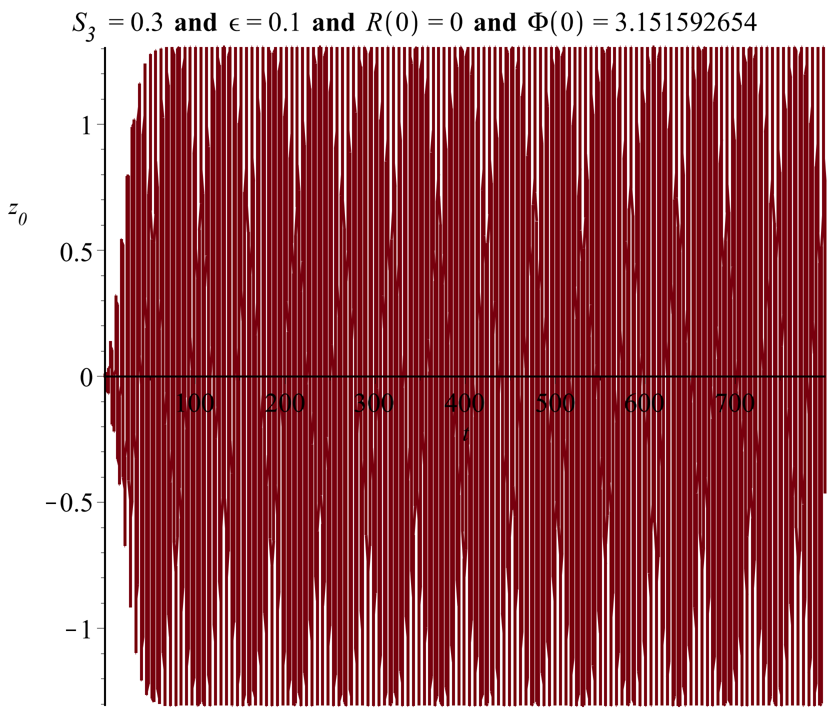
> $sol := dsolve(\{DER, DEPhi, R(0) = 0, \Phi(0) = 0\}, numeric, output = Array(1..N, (i) \rightarrow dt \cdot \text{epsilon} \cdot (i-1)))$:

> $solM := sol[2][1]$:

> $RM := solM[.., 3]$:

```
PhiM := solM[ .., 2] :
```

```
> u := RM·~map(evalf@cos, PhiM + T) :
> pair := (T, u) → [T, u] :
  P := zip(pair, T, u) :
  plot(P, labels = [t, z_0], legend = [z_0(t, epsilon)], title = [S_3 = S3 and 'epsilon'= epsilon and R(0)
      = 0 and Phi(0) = π + 0.01]);
```




```
>
>
>
>
>
>
>
```

B.6 Case 3: Equilibriums and eigenvalues

```
> restart;
> beta1 := 0;
  p01 := -1/5;
```

$$\beta 1 := 0$$
$$p01 := -\frac{1}{5} \qquad \textbf{(1)}$$

```
> eqR := (R, P) -> (-3/8*R^2 + 1/2)*R - 1/2*(p01*cos(beta1 + P))/sqrt(S3)
```

$$eqR := (R, P) \rightarrow \left(-\frac{3}{8}R^2 + \frac{1}{2}\right)R - \frac{1}{2}\,\frac{p01\cos(\beta 1 + P)}{\sqrt{S3}} \qquad \textbf{(2)}$$

```
> eqP := (R, P) -> 1/2*(sin(P + beta1)*(p01))/sqrt(S3) + ga/C1
```

$$eqP := (R, P) \rightarrow \frac{1}{2}\,\frac{\sin(\beta 1 + P)\,p01}{\sqrt{S3}} + \frac{ga}{C1} \qquad \textbf{(3)}$$

```
> r := solve([eqP(R, P) = 0, 0 <= P, P <= 2*Pi], P, allsolutions)
```

$$r := \left\{P = -2\arcsin\left(\frac{10\,ga\sqrt{S3}}{C1}\right)\_B4\sim + \pi\,\_B4\sim + 2\,\pi\,\_Z4\sim + \arcsin\left(\frac{10\,ga\sqrt{S3}}{C1}\right)\right\}, \left\{P = -2\arcsin\left(\frac{10\,ga\sqrt{S3}}{C1}\right)\_B3\sim + \pi\,\_B3\sim + 2\,\pi\,\_Z3\sim + \arcsin\left(\frac{10\,ga\sqrt{S3}}{C1}\right)\right\}, \left\{P = -2\arcsin\left(\frac{10\,ga\sqrt{S3}}{C1}\right)\_B2\sim + \pi\,\_B2\sim + 2\,\pi\,\_Z2\sim + \arcsin\left(\frac{10\,ga\sqrt{S3}}{C1}\right)\right\}, \left\{P = -2\arcsin\left(\frac{10\,ga\sqrt{S3}}{C1}\right)\_B1\sim + \pi\,\_B1\sim + 2\,\pi\,\_Z1\sim + \arcsin\left(\frac{10\,ga\sqrt{S3}}{C1}\right)\right\} \qquad \textbf{(4)}$$

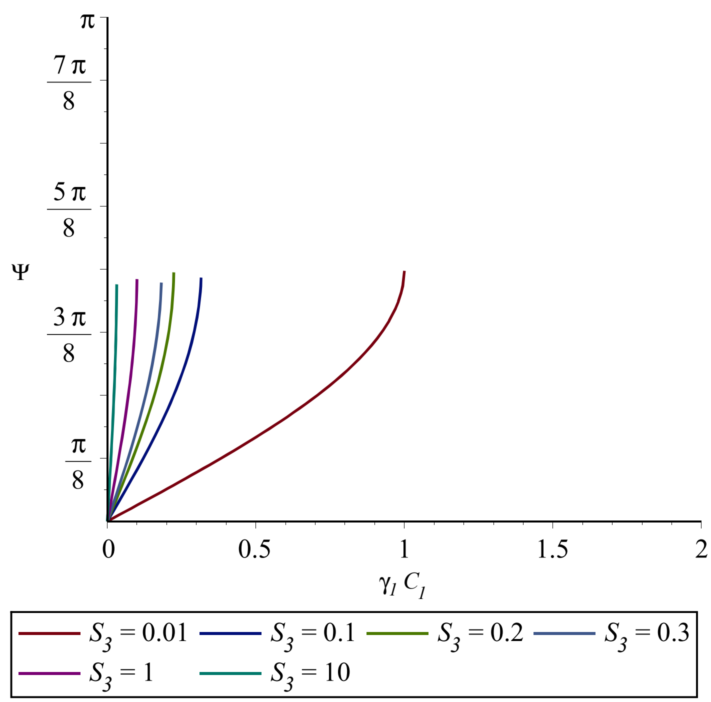
```
> P1 := -beta1 + arcsin(-(2*ga)/(p01*C1)*sqrt(S3));
  P2 := -beta1 + arcsin((2*ga)/(p01*C1)*sqrt(S3)) + Pi;
```

$$P1 := \arcsin\left(\frac{10\,ga\sim\sqrt{S3\sim}}{C1\sim}\right)$$
$$P2 := -\arcsin\left(\frac{10\,ga\sim\sqrt{S3\sim}}{C1\sim}\right) + \pi \qquad \textbf{(5)}$$

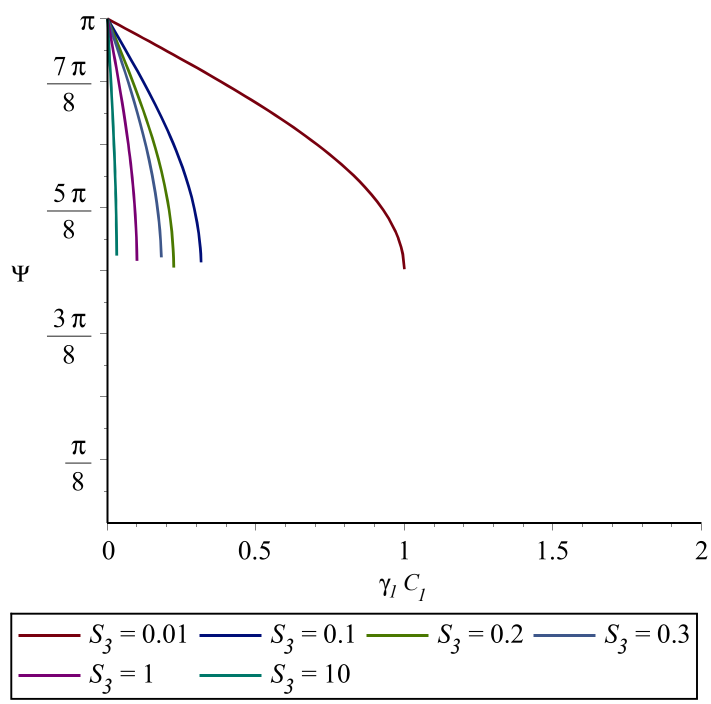
```
> P1 := unapply(P1, C1, S3, ga);
  P2 := unapply(P2, C1, S3, ga);
```

$$P1 := (C1\sim, S3\sim, ga\sim) \rightarrow \arcsin\left(\frac{10\,ga\sim\sqrt{S3\sim}}{C1\sim}\right)$$
$$P2 := (C1\sim, S3\sim, ga\sim) \rightarrow -\arcsin\left(\frac{10\,ga\sim\sqrt{S3\sim}}{C1\sim}\right) + \pi \qquad \textbf{(6)}$$

> $plot([P1(1, 0.01, ga), P1(1, 0.1, ga), P1(1, 0.2, ga), P1(1, 0.3, ga), P1(1, 1, ga), P1(1, 10, ga)], ga = 0..2, y = 0..\text{Pi}, numpoints = 100, labels = [\gamma_1 \cdot C_1, \Psi], legend = [S_3 = 0.01, S_3 = 0.1, S_3 = 0.2, S_3 = 0.3, S_3 = 1, S_3 = 10])$



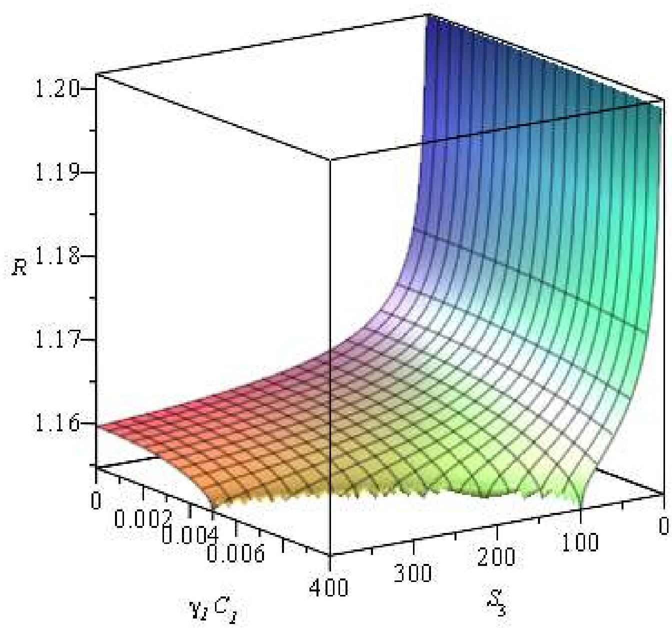
> $plot([P2(1, 0.01, ga), P2(1, 0.1, ga), P2(1, 0.2, ga), P2(1, 0.3, ga), P2(1, 1, ga), P2(1, 10, ga)], ga = 0..2, y = 0..\text{Pi}, numpoints = 100, labels = [\gamma_1 C_1, \Psi], legend = [S_3 = 0.01, S_3 = 0.1, S_3 = 0.2, S_3 = 0.3, S_3 = 1, S_3 = 10])$



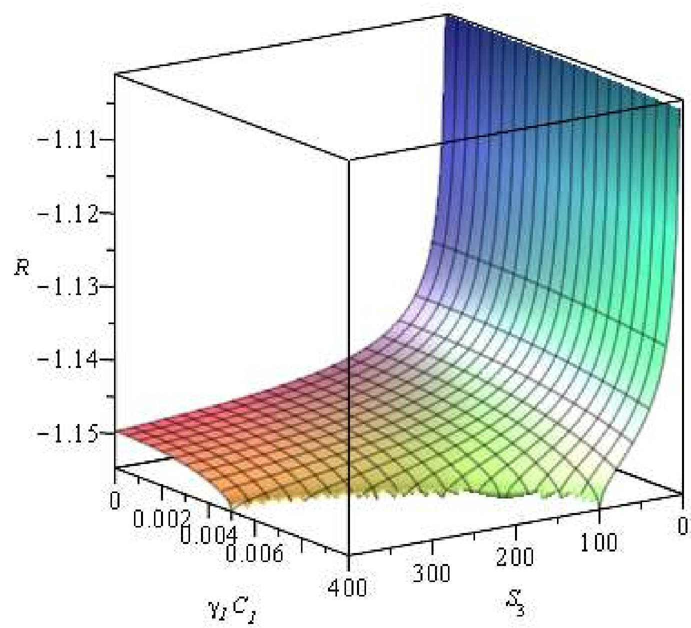

```
> P1 := P1(C1, S3, ga) :
  P2 := P2(C1, S3, ga) :

> RsP1 := solve(simplify(subs(P = P1 , eqR(R, P)) = 0), R) :
  RsP2 := solve(simplify(subs(P = P2 , eqR(R, P)) = 0), R) :

> R1P1 := unapply(simplify(RsP1[1]), C1, S3, ga) :
  R2P1 := unapply(simplify(RsP1[2]), C1, S3, ga) :
  R3P1 := unapply(simplify(RsP1[3]), C1, S3, ga) :
  R1P2 := unapply(simplify(RsP2[1]), C1, S3, ga) :
  R2P2 := unapply(simplify(RsP2[2]), C1, S3, ga) :
  R3P2 := unapply(simplify(RsP2[3]), C1, S3, ga) :
```

(7)

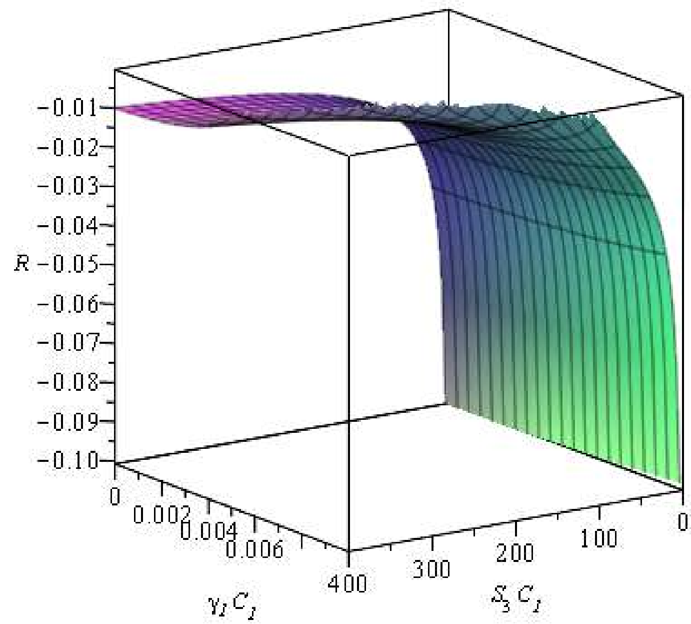
> $plot3d\left(R1P1(1, S3, ga), S3 = 0 ..400, ga = 0 ..0.01, labels = \left[S_3, '\gamma_1' \cdot C_1, R\right], numpoints = 10000, orientation = [55, 75, 0]\right);$



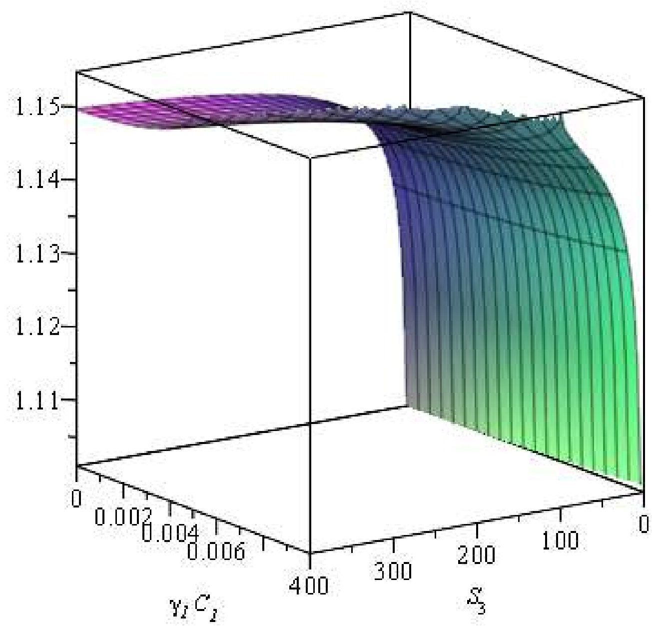
> $plot3d\left(R2P1(1, S3, ga), S3 = 0 ..400, ga = 0 ..0.01, labels = \left[S_3,'\gamma_1'\cdot C_1, R\right], numpoints = 10000, orientation = [55, 75, 0]\right);$



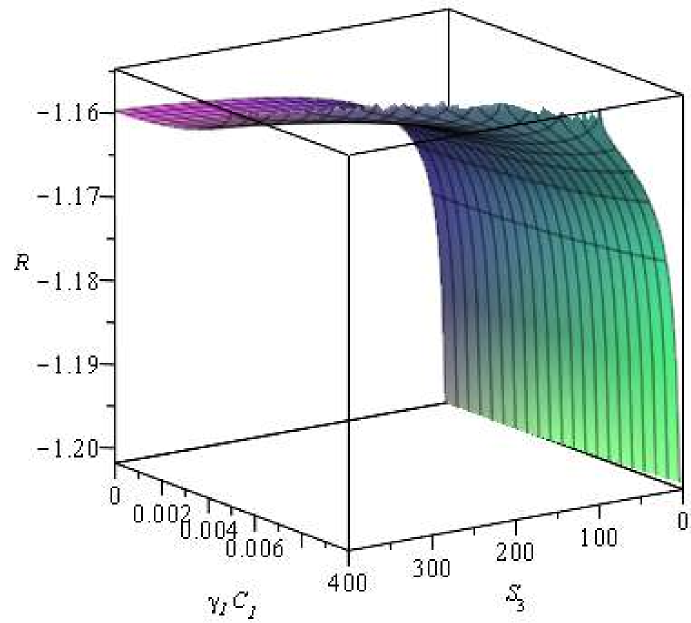
> $plot3d\left(R3P1(1, S3, ga), S3 = 0..400, ga = 0..0.01, labels = \left[S_3 \cdot C_1, 'γ_1' \cdot C_1, R\right], numpoints = 10000, orientation = [55, 75, 0]\right);$



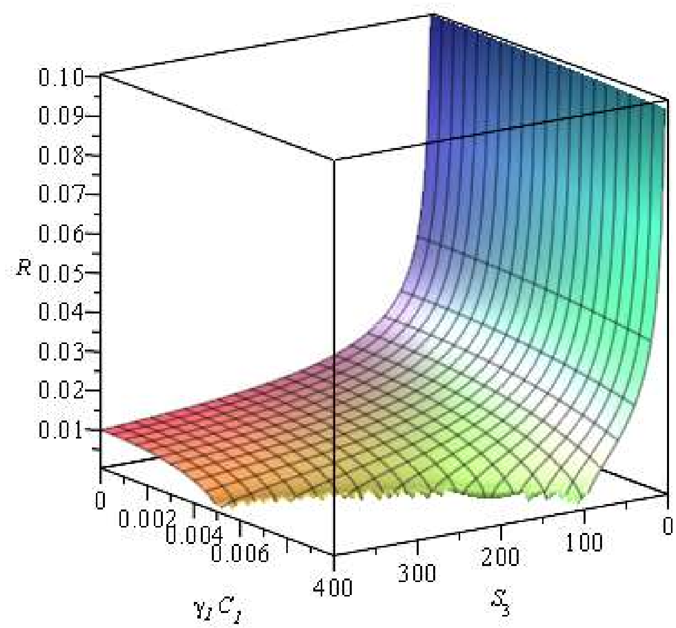
> $plot3d\left(R1P2(1, S3, ga), S3 = 0..400, ga = 0..0.01, labels = \left[S_3, '\gamma_1' \cdot C_1, R\right], numpoints = 10000, orientation = [55, 75, 0]\right);$



> $plot3d\left(R2P2(1, S3, ga), S3 = 0 ..400, ga = 0 ..0.01, labels = \left[S_3, '\gamma_1' \cdot C_1, R\right], numpoints = 10000, orientation = [55, 75, 0]\right);$

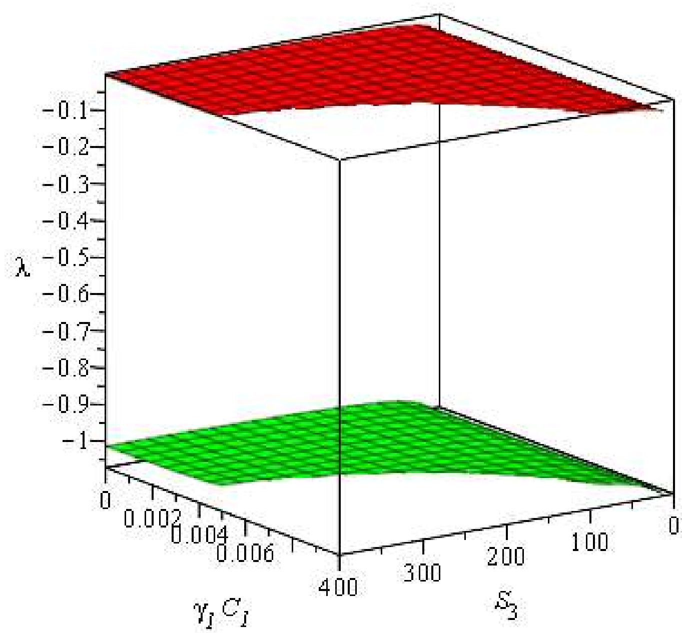


> $plot3d\Big(R3P2(1, S3, ga), S3 = 0..400, ga = 0..0.01, labels = \big[S_3, '\gamma_1' \cdot C_1, R\big], numpoints = 10000, orientation = [55, 75, 0]\Big);$



```
>
> R1P1 := R1P1(1, S3, ga) :
  R2P1 := R2 P1(1, S3, ga) :
  R3P1 := R3P1(1, S3, ga) :
  R1P2 := R1P2(1, S3, ga) :
  R2P2 := R2P2(1, S3, ga) :
  R3P2 := R3P2(1, S3, ga) :

> with(linalg) :
> J := jacobian( [eqR(R, P), eqP(R, P)], [R, P]) :
```

```
> Jstar := map(x → subs(R = R1P1, P = P1, x), J) :

> chareq := collect(charpoly(Jstar, lambda), lambda) :

> a := (coeff(chareq, lambda, 2)) :
  b := (coeff(chareq, lambda, 1)) :
  c := (coeff(chareq, lambda, 0)) :
  D1 := sqrt(b^2 - 4·a·c) :

> l11 := unapply((-b + D1)/(2·a), C1, S3, ga) :
  l12 := unapply((-b - D1)/(2·a), C1, S3, ga) :

> plot3d([l11(1, S3, ga), l12(1, S3, ga)], S3 = 0 ..400, ga = 0 ..0.01, labels = [S_3, γ_1·C_1, λ], color
      = [red, green], numpoints = 1000, orientation = [55, 75, 0])
```




```
>
>
```

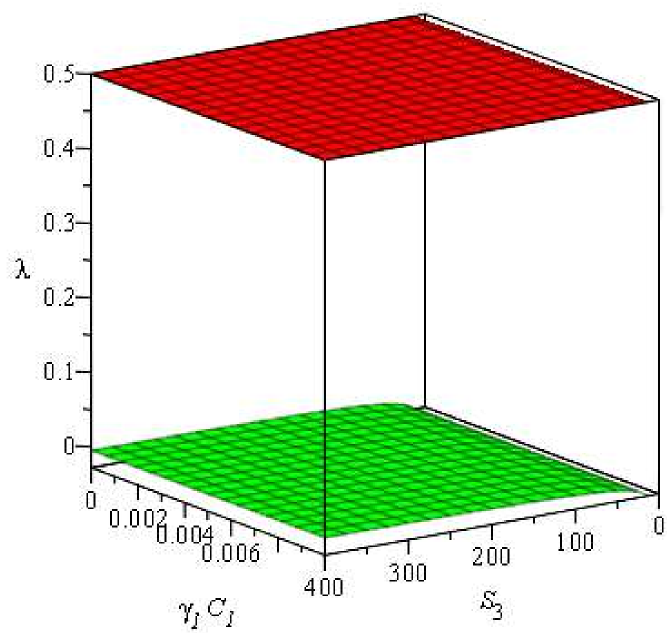
> $Jstar := map(x \rightarrow subs(R = R2P1, P = P1, x), J)$:

> $chareq := collect(charpoly(Jstar, \text{lambda}), \text{lambda})$:

> $a := (coeff(chareq, \text{lambda}, 2))$:
$b := (coeff(chareq, \text{lambda}, 1))$:
$c := (coeff(chareq, \text{lambda}, 0))$:
$D1 := \text{sqrt}(b^2 - 4 \cdot a \cdot c)$:

> $l11 := unapply\left(\frac{(-b + D1)}{2 \cdot a}, C1, S3, ga\right)$:
$l12 := unapply\left(\frac{(-b - D1)}{2 \cdot a}, C1, S3, ga\right)$:

> $plot3d\left([l11(1, S3, 0), l12(1, S3, 0)], S3 = 0..400, ga = 0..0.01, labels = \left[S_3, \gamma_1 \cdot C_1, \text{lambda}\right],\right.$
$\left. color = [red, green], numpoints = 1000, orientation = [55, 75, 0]\right)$



>

>

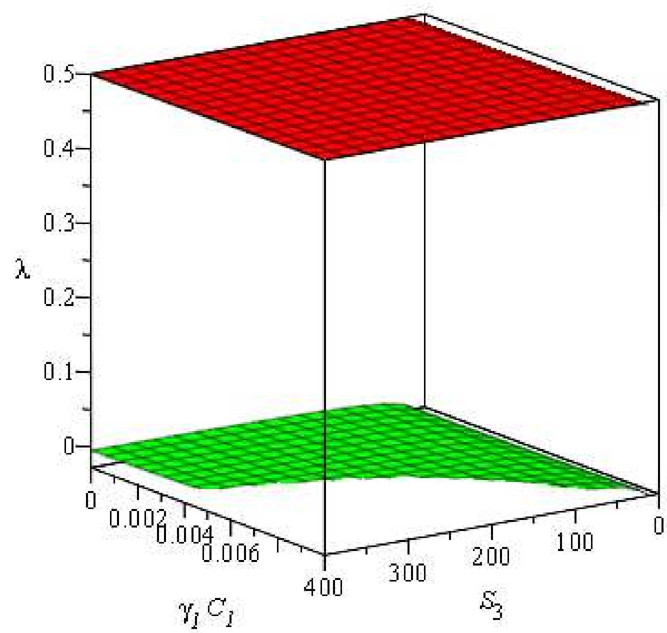
> $Jstar := map(x \rightarrow subs(R = R3P1, P = P1, x), J)$:

> $chareq := collect(charpoly(Jstar, \text{lambda}), \text{lambda})$:

> $a := (coeff(chareq, \text{lambda}, 2))$:
$b := (coeff(chareq, \text{lambda}, 1))$:
$c := (coeff(chareq, \text{lambda}, 0))$:
$D1 := \text{sqrt}(b^2 - 4 \cdot a \cdot c)$:

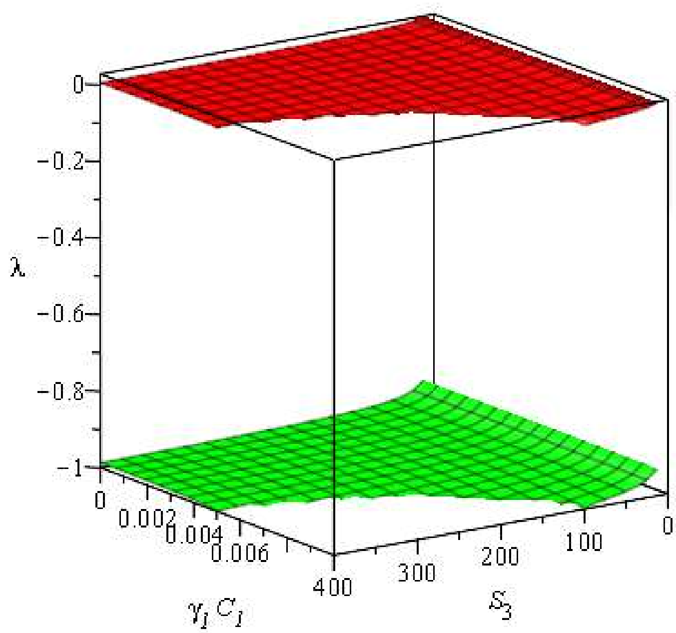
> $l11 := unapply\left(\frac{(-b + D1)}{2 \cdot a}, C1, S3, ga\right)$:
$l12 := unapply\left(\frac{(-b - D1)}{2 \cdot a}, C1, S3, ga\right)$:

> $plot3d\left([l11(1, S3, ga), l12(1, S3, ga)], S3 = 0..400, ga = 0..0.01, labels = \left[S_3, \gamma_1 \cdot C_1, \text{lambda}\right], color = [red, green], numpoints = 1000, orientation = [55, 75, 0]\right)$



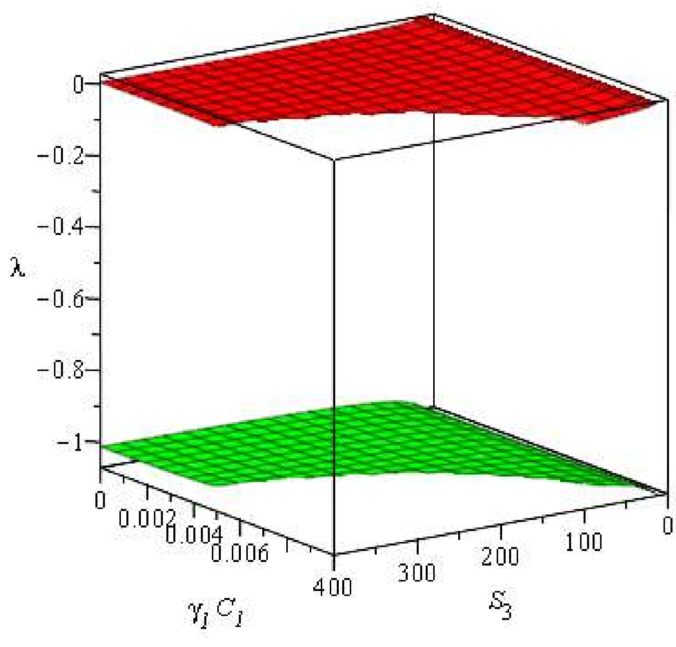
>
>

```
> Jstar := map(x → subs(R = R1P2, P = P2, x), J) :
> chareq := collect(charpoly(Jstar, lambda), lambda) :
> a := (coeff(chareq, lambda, 2)) :
  b := (coeff(chareq, lambda, 1)) :
  c := (coeff(chareq, lambda, 0)) :
  D1 := sqrt(b^2 - 4·a·c) :
> l11 := unapply((-b + D1)/(2·a), C1, S3, ga) :
  l12 := unapply((-b - D1)/(2·a), C1, S3, ga) :
> plot3d([l11(1, S3, ga), l12(1, S3, ga)], S3 = 0 ..400, ga = 0 ..0.01, labels = [S_3, γ_1·C_1, lambda],
     color = [red, green], numpoints = 1000, orientation = [55, 75, 0])
```




```
>
```

```
> Jstar := map(x → subs(R = R2P2, P = P2, x), J) :

> chareq := collect(charpoly(Jstar, lambda), lambda) :

> a := (coeff(chareq, lambda, 2)) :
  b := (coeff(chareq, lambda, 1)) :
  c := (coeff(chareq, lambda, 0)) :
  D1 := sqrt(b^2 − 4·a·c) :

> l11 := unapply((−b + D1)/(2·a), C1, S3, ga) :
  l12 := unapply((−b − D1)/(2·a), C1, S3, ga) :

> plot3d([l11(1, S3, ga), l12(1, S3, ga)], S3 = 0 ..400, ga = 0 ..0.01, labels = [S_3, γ_1·C_1, lambda],
      color = [red, green], numpoints = 1000, orientation = [55, 75, 0])
```




```
>
>
```

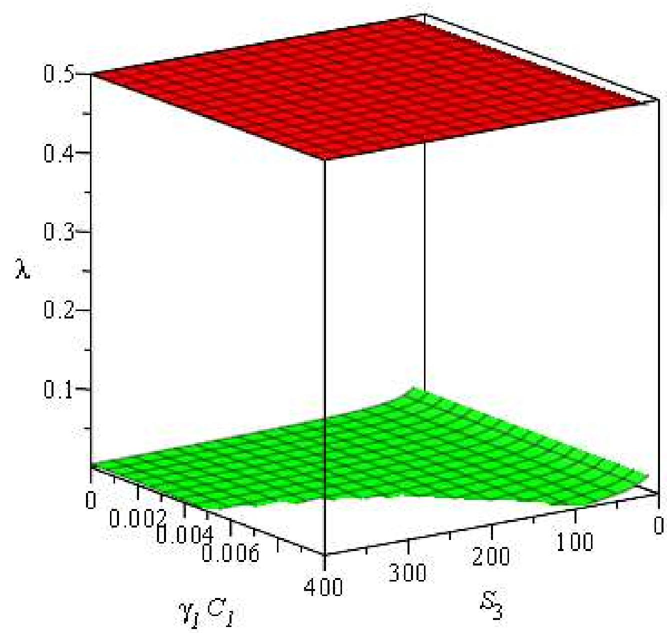
> $Jstar := map(x \rightarrow subs(R = R3P2, P = P2, x), J)$:

> $chareq := collect(charpoly(Jstar, \text{lambda}), \text{lambda})$:

> $a := (coeff(chareq, \text{lambda}, 2))$:
$b := (coeff(chareq, \text{lambda}, 1))$:
$c := (coeff(chareq, \text{lambda}, 0))$:
$D1 := \text{sqrt}(b^2 - 4 \cdot a \cdot c)$:

> $l11 := unapply\left(\frac{(-b + D1)}{2 \cdot a}, C1, S3, ga\right)$:
$l12 := unapply\left(\frac{(-b - D1)}{2 \cdot a}, C1, S3, ga\right)$:

> $plot3d\left([l11(1, S3, ga), l12(1, S3, ga)], S3 = 0..400, ga = 0..0.01, labels = \left[S_3, \gamma_1 \cdot C_1, \text{lambda}\right], color = [red, green], numpoints = 1000, orientation = [55, 75, 0]\right)$



>

B.7 Case 3: Phase plane and solution

```
> restart;
> with(DEtools) : with(LinearAlgebra) :
> p01 := -1/5;
  beta1 := 0;
```

$$p01 := -\frac{1}{5}$$

$$\beta 1 := 0 \qquad (1)$$

```
> DER := diff(R(T1), T1) = (-3/8 R(T1)^2 + 1/2) R(T1) - p01 cos(β1 + Phi(T1)) / (2 · sqrt(S3))
```

$$DER := \frac{d}{dT1} R(T1) = \left(-\frac{3}{8} R(T1)^2 + \frac{1}{2}\right) R(T1) + 0.1825741858 \cos(\Phi(T1)) \qquad (2)$$

```
> DEPhi := diff(Phi(T1), T1) = 1/2 · sin(Phi(T1) + beta1) · (p01) / sqrt(S3) + ga/C1
```

$$DEPhi := \frac{d}{dT1} \Phi(T1) = -0.1825741858 \sin(\Phi(T1)) + 0.1 \qquad (3)$$

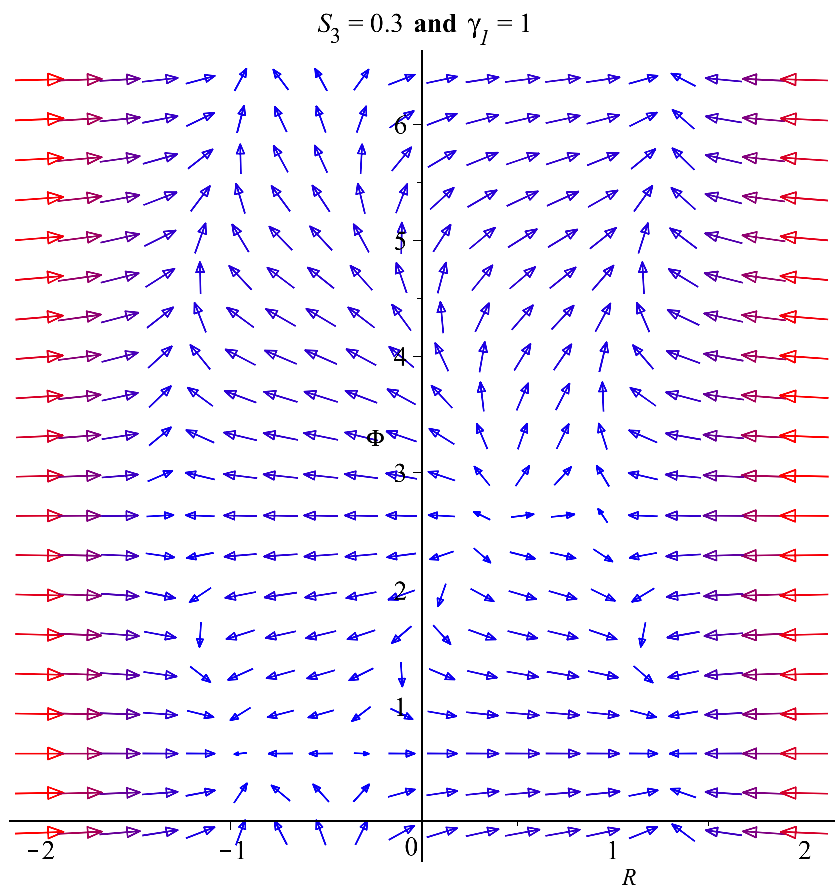
```
> C1 := 1;
  ga := 1;
  S3 := 0.3
```

$$C1 := 1$$

$$ga := 1$$

$$S3 := 0.3 \qquad (4)$$

```
> DEplot([DER, DEPhi], [R(T1), Phi(T1)], T1 = 0..8, R = -2..2, Phi = -0.1..2 · Pi + 0.1, arrows
    = medium, arrowsize = magnitude, size = [500, 500], color = magnitude, title = [S_3 = S3
    and γ_1 = ga])
```



> $C1 := 1;$
> $ga := 1;$
> $S3 := 0.3;$

$$C1 := 1$$
$$ga := 1$$
$$S3 := 0.3 \quad \textbf{(5)}$$

> $dt := \frac{1}{10}:$

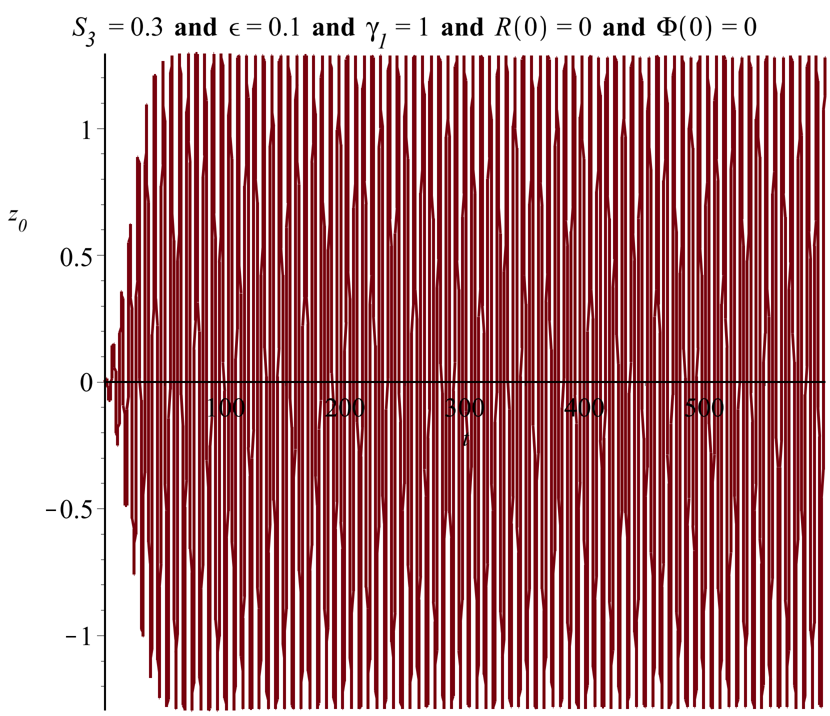
> $N := \frac{600}{dt}$

$$N := 6000 \quad \textbf{(6)}$$

```
> T := Transpose(Array(1 ..N, (i) → dt·(i − 1))) :
> epsilon := 0.1
```

$\epsilon := 0.1$ **(7)**

```
> sol := dsolve({DER, DEPhi, R(0) = 0, Phi(0) = 0}, numeric, output = Array(1 ..N, (i) → dt
      ·epsilon·(i − 1))) :
> solM := sol[2][1] :
> RM := solM[.., 3] :
  PsiM := solM[.., 2] :

> PhiM := PsiM − ga·epsilon·T :
> u := RM·~map(evalf@cos, PhiM + T) :
> pair := (T, u) → [T, u] :
  P := zip(pair, T, u) :
  plot(P, labels = [t, z0], legend = [z0(t, epsilon)], title = [S3 = S3 and 'epsilon'= epsilon and γ1
      = ga and R(0) = 0 and Phi(0) = 0]);
```




```
>
>
```