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FORCE SCHEME APPLIED TO FLUX RECONSTRUCTION IN THE NUMERICAL SIMULATION OF NONLINEAR ACOUSTIC PROBLEMS

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Abstract. *Nonlinear acoustic waves have many applications in very different fields of industry. These problems are governed by differential equations of the form (see [7]):*

$$\rho_0 \frac{\partial^{(2)} u(x,t)}{\partial t^2} = -\rho_0 \frac{c_0^2}{\gamma} \frac{\partial u(x,t)}{\partial x} \left(1 + \frac{\partial u(x,t)}{\partial x}\right)^{-\gamma} + \rho_0 \nu b \frac{\partial^{(3)} u(x,t)}{\partial x^2 \partial t} \tag{1}$$

$$+ \text{ initial and boundary conditions.}$$

The unknown, u(x,t), represents gas particles displacements, as a function of the spatial coordinate x and time t.

The numerical technique we have applied to solve this nonlinear acoustics problem is a third order finite volume method with ENO reconstruction, combined with a third order TVD version of Runge-Kutta method for time integration, as described in references [1,3]. We point out that the technique applied in [5] shows stability limitations when the nonlinearity of the problem is more severe.

The aim of this paper is to improve the stability of numerical solution by applying a flux reconstruction in the finite volume interfaces, via FORCE scheme. A description of the FORCE technique may be found in [4,5,6]. Results of the application of this technique are shown and they are compared with the ones obtained in references [1].

1 INTRODUCTION

Nonlinear ultrasonic waves may be applied in very different fields, such as: submarine signals, aerosols, foams separation, metal welding, seismic studies, medicine, etc.

The design of devices to carry out these applications requires a detailed knowledge of the high frequence wave propagation. In this kind of problems, it is very interesting to develop efficient numerical methods to solve the nonlinear phenomena associated to these waves.

In particular high order methods based on finite volumes are specially suited to this purpose. In order to obtain an accurate computation of intercell fluxes, even for very high frequencies, it is useful to write the governing differential equation as a system of conservation laws, and to apply appropriate techniques to obtain the solution.

2 MATHEMATICAL FORMULATION

The mathematical model describing nonlinear acoustics problems may be obtained, as described in references [1,6], from the fluid state equation, the continuity equation and the momentum conservation equation, and reads:

$$\rho_0 \frac{\partial^{(2)} u(x,t)}{\partial t^2} = -\rho_0 \frac{c_0^2}{\gamma} \frac{\partial u(x,t)}{\partial x} \left(1 + \frac{\partial u(x,t)}{\partial x}\right)^{-\gamma} + \rho_0 \nu b \frac{\partial^{(3)} u(x,t)}{\partial x^2 \partial t} \tag{2}$$

The coefficients in equation (2) represent: $x \in [0, L]$ is the spatial coordinate, $t > 0$ is the time, γ is the specific heats relation of the fluid (at constant pressure and constant volume), c_0 is the sound speed, ν is the fluid kinematic shear viscosity, ρ_0 is the ambient density, $u(x,t)$ is the displacements function, while b is viscosity number whose value is obtained from the formula: $b = \frac{4}{3} + \frac{\mu_B}{\mu}$, where μ is the dynamic shear viscosity and μ_B is the dynamic compressive viscosity.

The problem considered in this work is defined by equation (2) with the following initial conditions:

$$u(x,0) = u^{(0)}(x) \;\; 0 < x < L \tag{3}$$

$$\frac{\partial u}{\partial t}(x,0) = v^{(0)}(x) \;\; 0 < x < L \tag{4}$$

and, also, the following boundary conditions, for the left boundary:

$$u(0,t) = u_0(t) \;\; 0 \le t \le T \tag{5}$$

$$\frac{\partial u}{\partial t}(0,t) = u_0'(t) \;\; 0 \le t \le T \tag{6}$$

For the right boundary:

$$u(L,t) = u_L(t) \;\; 0 \le t \le T \tag{7}$$

$$\frac{\partial u}{\partial t}(L,t) = u_L'(t) \;\; 0 \le t \le T \tag{8}$$

We can also obtain the acoustic pressure in the fluid, given by the expression:

$$p(x,t) = p_0 \left[\left(1 + \frac{\partial u}{\partial x}\right)^{-\gamma} - 1 \right] \tag{9}$$

where $p_0 = \rho_0 \frac{c_0^2}{\gamma}$ is the ambient pressure.

3 DISCRETIZATION IN SPACE AND TIME

3.1 Discretization in space

As is shown in reference [1], the equation (2) may be transformed into the following system of equations:

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = v(x,t) \\ \\ \frac{\partial}{\partial t} v(x,t) = -\frac{c_0^2}{\gamma} \frac{\partial}{\partial x} \left(1 + \frac{\partial u(x,t)}{\partial x}\right)^{-\gamma} + \nu b \frac{\partial^2 v}{\partial x^2}(x,t) \end{cases} \tag{10}$$

Let us consider, within the interval [0,L], a set of (n+1) nodes, assumed uniformly distributed for simplicity, obtained as: $x_i = (i-1)h$, $(i = 1, 2, \cdots, n+1)$ where the distance between two consecutive points is defined as: $h = \frac{L}{n}$.

We will also consider a partition in intervals (control volumes) $S_j = [z_{j-1}, z_j]$, $(j = 2, \cdots, n)$, where: $z_j = \frac{1}{2}(x_{j+1} + x_j)$.

If we now integrate equations of system (10) over each one of the control volumes, and divide by the length of the control volume (h) we have:

$$\begin{cases} \frac{\partial \bar{u}_i}{\partial t}(t) = \bar{v}_i(x,t) \\ \frac{\partial \bar{v}_i}{\partial t}(t) = A \left[\left(1 + \frac{\partial u}{\partial x}(z_i,t)\right)^{-\gamma} - \left(1 + \frac{\partial u}{\partial x}(z_{i-1},t)\right)^{-\gamma} \right] + B \left(\frac{\partial v}{\partial x}(z_i,t) - \frac{\partial v}{\partial x}(z_{i-1},t) \right) \end{cases} \tag{11}$$

where we have denoted by:

$$\bar{u}_i(t) = \frac{1}{h} \int_{zi-1}^{z_i} u(x,t)dx,\ \bar{v}_i(t) = \frac{1}{h} \int_{zi-1}^{z_i} v(x,t)dx,\ A = -\frac{c_0^2}{h\gamma} \text{ and } B = \frac{\nu b}{h}.$$

We can simplify the notation used in (11), writing it as:

$$\begin{aligned} \frac{\partial \bar{u}_i}{\partial t}(t) &= \bar{v}_i(x,t) \\ \frac{\partial \bar{v}_i}{\partial t}(t) &= \Lambda_i(u(t), v(t)) \end{aligned} \tag{12}$$

where: $\Lambda_i(u(t), v(t)) = A \left[\left(1 + \frac{\partial u}{\partial x}(z_i,t)\right)^{-\gamma} - \left(1 + \frac{\partial u}{\partial x}(z_{i-1})\right)^{-\gamma} \right] + B \left(\frac{\partial v}{\partial x}(z_i,t) - \frac{\partial v}{\partial x}(z_{i-1},t) \right)$.

3.2 Discretization in time

For time discretization we use a third order Runge-Kutta method verifying the TVD property (see[3] for a description of the method and [1] for its application to nonlinear acoustics problem). To apply this method for time integration, we start from the initial conditions given in (3)-(4), so that cell averages may be obtained over each control volume $(\bar{u}_i^0, \bar{v}_i^0)$.
Therefore, it can be assumed that at a particular time t^k, the approached values of the cell averages are known, so we can compute values for the next time step $(t^{k+1} = t^k + \Delta t)$ by means of the following process:

$$\begin{array}{l} \bar{u}_i^{k,1} = \bar{u}_i^k + \Delta t \bar{v}_i^k,\ \bar{v}_i^{k,1} = \bar{v}_i^k + \Delta t L_i\left(\bar{u}^k, \bar{v}^k\right) \\ \bar{u}_i^{k,2} = \frac{3}{4}\bar{u}_i^k + \frac{1}{4}\bar{u}_i^{k,1} + \frac{\Delta t}{4}\bar{v}_i^{k,1}, \bar{v}_i^{k,2} = \frac{3}{4}\bar{v}_i^k + \frac{1}{4}\bar{v}_i^{k,1} + \frac{\Delta t}{4} L_i\left(\bar{u}^{k,1}, \bar{v}^{k,1}\right) \\ \bar{u}_i^{k+1} = \frac{1}{3}\bar{u}_i^{k+\frac{2}{3}}\bar{u}_i^{k,2} + \frac{2\Delta t}{3}\bar{v}_i^{k,2},\ \bar{v}_i^{k+1} = \frac{1}{3}\bar{v}_i^{k+\frac{2}{3}}\bar{v}_i^{k,2} + \frac{2\Delta t}{3} L_i\left(\bar{u}^{k,2}, \bar{v}^{k,2}\right) \end{array} \tag{13}$$

where it has been denoted as $L_i\left(\bar{u}^{k,j}, \bar{u}^{k,j}\right)$ an approach to $\Lambda_i\left(u(t^{k,j}, v(t^{k,j})\right)$ using the cell average values. This process, called reconstruction, consists of approximating the intercell derivatives in each step of Runge-Kutta method:

$$\frac{\partial u}{\partial x}\left(z_{i-1}, t^{k,j}\right), \frac{\partial u}{\partial x}\left(z_i, t^{k,j}\right), \frac{\partial v}{\partial x}\left(z_{i-1}, t^{k,j}\right), \frac{\partial v}{\partial x}\left(z_i, t^{k,j}\right)$$

by the intercell values taken by the derivatives of an interpolating function obtained from cell-averages.

This process is described in [1]. The technique we have used is based on Harten's "primitive function" (see [2] for the description of this technique, and [1] for its application in nonlinear acoustics problem), and the interpolating polynomials are obtained by the E.N.O. method. By this method, a piecewise interpolating polynomial function is obtained, and it defines, over each control volume, one different polynomial.

4 INTERCELL FLUX RECONSTRUCTION

When calculating intercell derivatives, discontinuities may be present (since the derivatives have been computed by means of, in general, different polynomials) which are necessary to avoid. We have used and compared three methods for this purpose:

- Lax-Friedrichs scheme for a system of conservation laws (LF).
- Two step Richtmyer version of Lax-Wendroff scheme for a system of conservation laws (LW2).
- FORCE scheme.

Comparison of these three schemes is made in terms of experimental stability limit. This stability limit is experimentally obtained according to numerical schemes.

All of them require system (10) to be written as a system of conservation laws with source term:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U}) \tag{14}$$

where,

$$\begin{aligned} \mathbf{U} &= \{u, v\}^T \\ \mathbf{F}(\mathbf{U}) &= \left\{0, -\frac{c_0^2}{\gamma}\left(1+\frac{\partial u}{\partial x}\right)^{-\gamma} + \nu b \frac{\partial v}{\partial x}\right\}^T \\ \mathbf{S}(\mathbf{U}) &= \{v, 0\}^T \end{aligned} \tag{15}$$

The time integration in system (14) is carried out by means of the Runge-Kutta TVD scheme (using expressions (13)), while fluxes are computed by the methods mentioned above. In the next subsections, the methods applied are briefly described.

4.1 Lax-Friedrichs scheme for a system of conservation laws (LF)

Lax-Friedrichs flux for a system of conservation laws (see [4]) at the intercell z_i, is given by the expression:

$$\mathbf{F}^{LF}(z_i, t^k) = \frac{1}{2}\Big(\mathbf{F}(\mathbf{U}(x_i, t^k)) + \mathbf{F}(\mathbf{U}(x_{i+1}, t^k))\Big) + \frac{h}{2\Delta t}\Big(\mathbf{U}(x_i, t^k) - \mathbf{U}(x_{i+1}, t^k)\Big) \tag{16}$$

We can apply this flux to the nonlinear acoustics problem, obtaining:

$$\begin{aligned} F_1^{LF}(z_i, t^k) &= \frac{1}{2}\Big[F_1(u(x_i, t^k)) + F_1(u(x_{i+1}, t^k))\Big] - \frac{h}{2\Delta t}\Big(u(x_{i+1}, t^k) - u(x_i, t^k)\Big) \\ F_2^{LF}(z_i, t^k) &= \frac{1}{2}\Big[F_2(v(x_i, t^k)) + F_2(v(x_{i+1}, t^k))\Big] - \frac{h}{2\Delta t}\Big(v(x_{i+1}, t^k) - v(x_i, t^k)\Big) \end{aligned} \tag{17}$$

where,

$$\begin{aligned} F_1(u(x_j, t^k)) &= 0 \\ F_2(v(x_j, t^k)) &= \frac{A}{2}N_u + \frac{B}{2}M_v, \;\; (j = i, i+1) \end{aligned} \tag{18}$$

where it has been used the notation:

$$\begin{aligned} N_u &= \left(1 + \frac{\partial u}{\partial x}(x_j, t^k)\right)^{-\gamma} \\ M_v &= \frac{\partial v}{\partial x}(x_j, t^k), \;\; (j = i, i+1) \end{aligned} \tag{19}$$

4.2 Two step Richtmyer version of Lax-Wendroff scheme for a system of conservation laws (LW2)

This method consists of two steps (as can be found in [4]):
First step: Evaluate the intermediate solution:

$$\mathbf{U}(z_i, t^{k+\frac{1}{2}}) = \frac{1}{2}\left(\mathbf{U}_i^k + \mathbf{U}_{i+1}^k\right) + \frac{\Delta t}{2h}\left(\mathbf{F}_i^k - \mathbf{F}_{i+1}^k\right) \tag{20}$$

The expression (20) corresponds to the solution at the point z_i at the time $t^{k+\frac{1}{2}}$, which is the half time in the interval $[t^k, t^{k+1}]$.
Second step: Compute the flux as:

$$\mathbf{F}^{LW2}(z_i, t^k) = \mathbf{F}\left(\mathbf{U}^{k+\frac{1}{2}}(z_i)\right) \tag{21}$$

When applying the formulation to nonlinear acoustics problem, we start by applying formula (20) to obtain the intermediate solution :

$$\begin{aligned} u(z_i, t^{k+\frac{1}{2}}) &= \frac{1}{2}\left(u(x_i, t^k) + u(x_{i+1}, t^k)\right) \\ v(z_i, t^{k+\frac{1}{2}}) &= \frac{1}{2}\left(v(x_i, t^k) + v(x_{i+1}, t^k)\right) + \frac{\Delta t}{2}\left(F_2(x_i, t^k) - F_2(x_{i+1}, t^k)\right) \end{aligned} \tag{22}$$

where $F_2(x_i, t^k) = A\left(1 + \frac{\partial u}{\partial x}(x_i, t^k)\right)^{-\gamma} + B\frac{\partial v}{\partial x}(x_i, t^k))$.

From the intermediate solution, calculated above, we can finally obtain the final expresion ef the LW2 flux ($\mathbf{F}^{LW}$):

$$\begin{aligned} F_1^{LW2}(z_i, t^{k+\frac{1}{2}}) &= 0 \\ F_2^{LW2}(z_i, t^{k+\frac{1}{2}}) &= A\left(1 + \frac{\partial u}{\partial x}(z_i, t^{k+\frac{1}{2}})\right)^{-\gamma} + B\frac{\partial v}{\partial x}(z_i, t^{k+\frac{1}{2}}) \end{aligned} \tag{23}$$

4.3 FORCE scheme for systems of conservation law

FORCE scheme, as described in [4,5,6] is just an arithmetic mean between the Lax-Friedrichs scheme and the Lax-Wendroff one. Therefore, we can write the FORCE flux as follows:

$$\mathbf{F}^{FORCE}(z_i, t^k) = \frac{1}{2}\left(\mathbf{F}^{LF}(z_i, t^k) + \mathbf{F}^{LW2}(z_i, t^k)\right) \tag{24}$$

5 NUMERICAL EXAMPLE

We have solved the same problem as in [1] but trying to reach stability in the solution for higher frequencies.

It is considered a horizontal 1-D tube, filled with air, and with a length L. In its left boundary there is a piston hitting the gas for a particular frequency. The maximum displacement allowed for the piston is $3 \times 10^{-7} m$. The air density $1.29 kg/m^3$ and the speed of sound in air is considered to be $c_0 = 340 m/s$. The value of the coefficient of specific heat relation is taken as $\gamma = 1.6$. The maximum simulation time is obtained as: $t_{fin} = \frac{100\pi}{\omega}$, where $\omega = 2\pi\nu$. The number of time steps is 5000. To achieve a correct discretization for good stability conditions, the length L is equivalent to half the wave-length of sound, that is: $L = \frac{c_0}{2\nu}$. The number of spatial nodes, in which numerical solution is computed, that have been used in calculations is 100.

The initial conditions considered are:

$$u(x, 0) = 0, \; 0 < x < L \tag{25}$$

$$\frac{\partial u}{\partial t}(x, 0) = 0 \; 0 < x < L \tag{26}$$

And, also, the following boundary conditions, for the left boundary:

$$u(0, t) = 3 \times 10^{-7} sin(\omega t), \; 0 \leq t \leq T \tag{27}$$

$$\frac{\partial u}{\partial t}(0, t) = 3 \times 10^{-7} cos(\omega t), \; 0 \leq t \leq T \tag{28}$$

And for the right boundary:

$$u(L, t) = 0, \; 0 \leq t \leq T \tag{29}$$

$$\frac{\partial u}{\partial t}(L, t) = 0, \; 0 \leq t \leq T \tag{30}$$

In the following examples, we compare LW2 and FORCE schemes. LF scheme is just used in order to build the FORCE one.

Results are shwon for frequencies: $\nu = 20kHz$ (Figures 1-4) and $\nu = 100kHz$ (Figures 5-12).

The results for $\nu = 20kHz$ show a perfect agreement with the ones obtained in [1], regarding both gas particles displacement and pressure. There is also a perfect agreement whether using LW2 scheme or FORCE scheme.

The numerical solution for $\nu = 100kHz$ is also stable, although it shows oscillations for both schemes considered. For this frequence, it is also shown pressure and displacement in the middle point of the tube.

Experiments carried out for higher frequencies have shown a higher stability limit for FORCE scheme than for LW2 one.

6 CONCLUSIONS

- It has been developed a scheme in high order finite volumes to solve the $1 - D$ nonlinear acoustics problem.
- The intercell flux reconstruction have been carried out, writing the mathematical formulation of the problem as a system of conservation laws with source term, and solving

it by three different schemes for conservation laws: Lax-Friedrichs, the two step Richtmyer version of Lax-Wendroff method and FORCE sheme.

- The use of E.N.O. interpolation in the finite volume formulation do a very good job to eliminate oscillations appearing.

- Numerical examples show that stability for increasing frequencies depends clearly on the reconstruction process, being the FORCE the one with better stability conditions, among the ones considered.

- Further work will consider a theoretical stability study, other types of non-oscillatory interpolation (such as, for example WENO), and application to bi and tridimensional problems.

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7 FIGURES

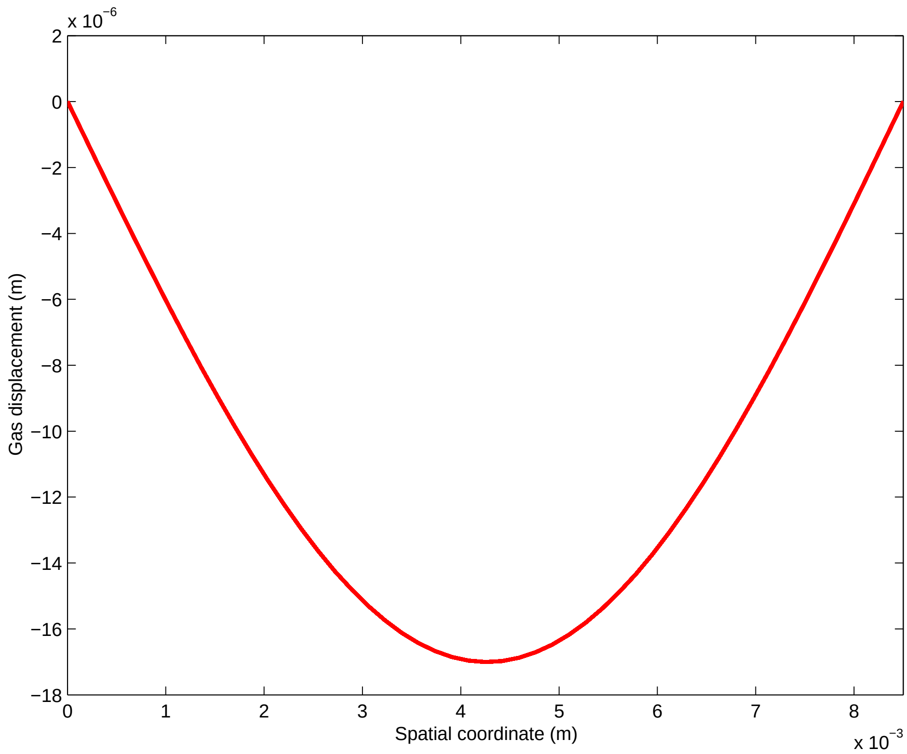

Figure 1: Displacement of gas particles for t=0.01 s. Scheme LW2. ν =20kHz

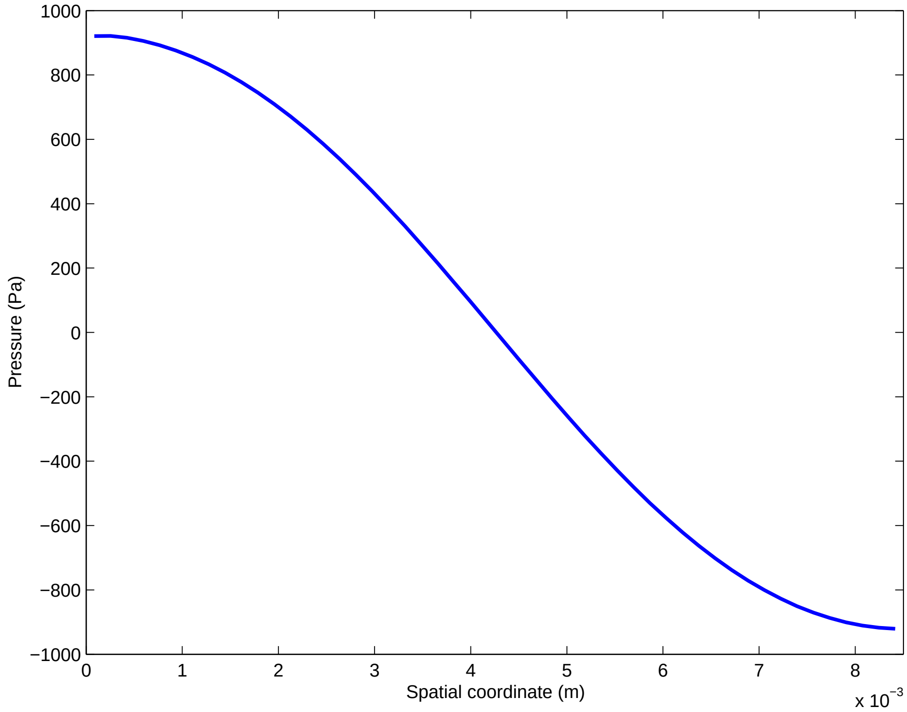

Figure 2: Pressure for t=0.01 s. Scheme LW2. ν =20kHz

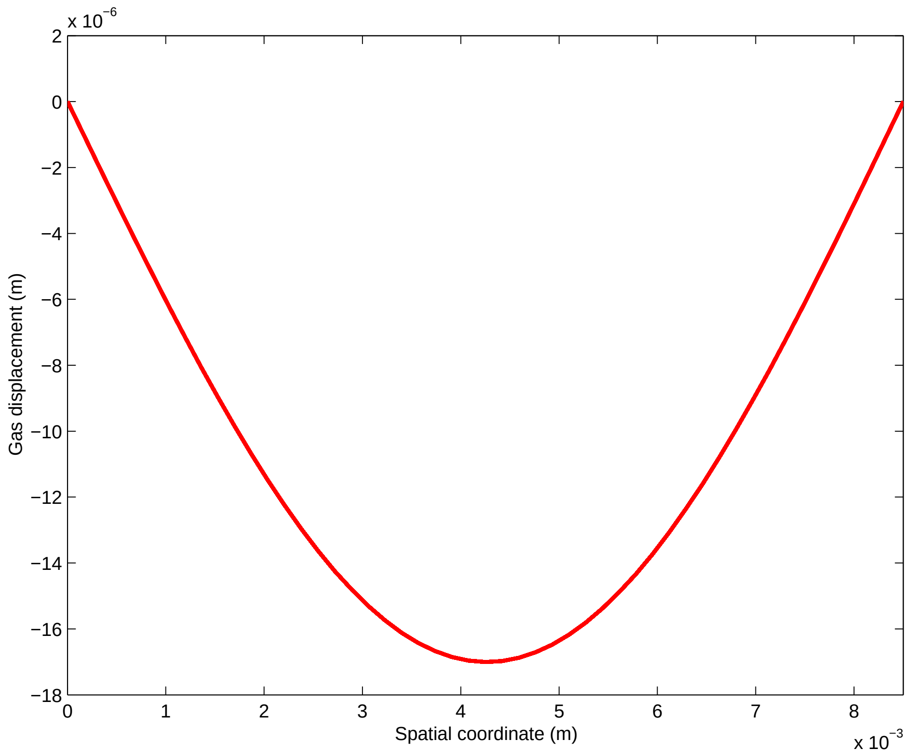

Figure 3: Displacement of gas particles for t=0.01 s. Scheme FORCE. ν =20kHz

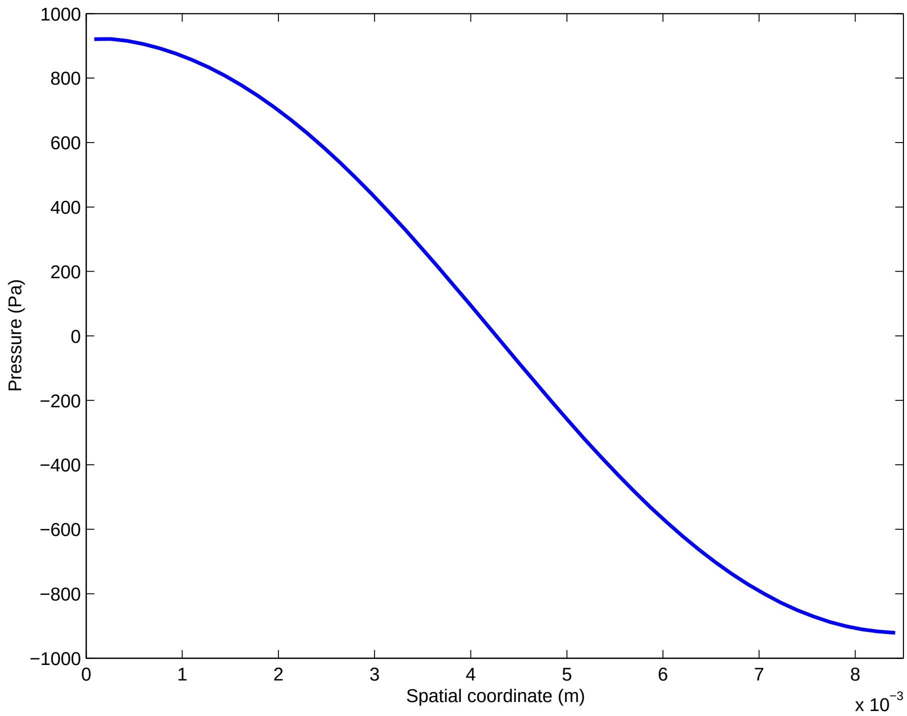

Figure 4: Pressure for t=0.01 s. Scheme FORCE. ν =20kHz

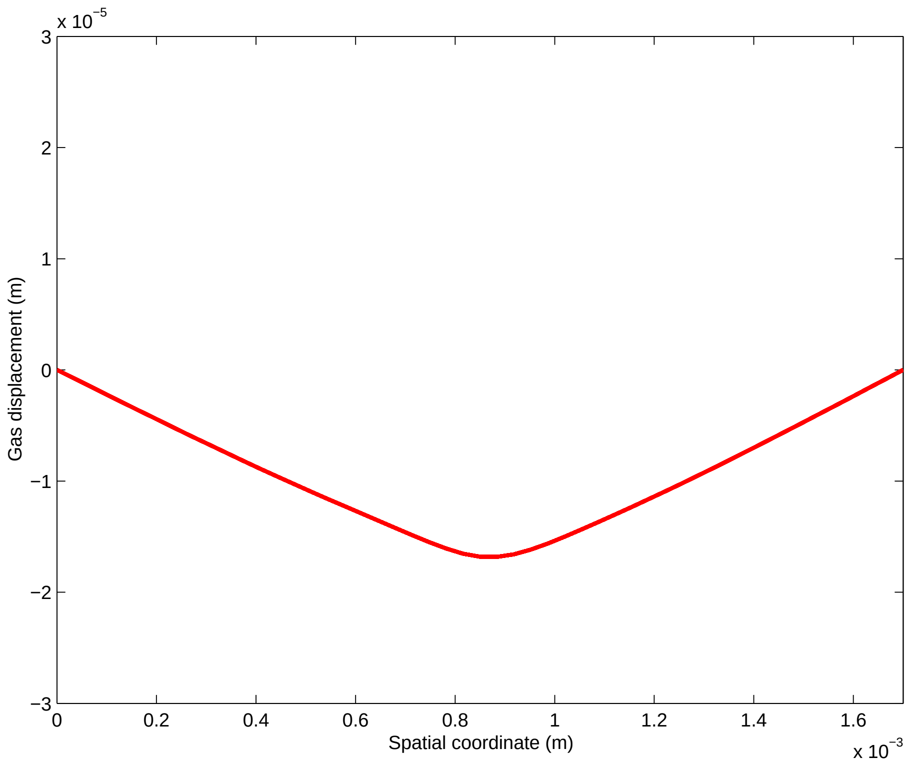

Figure 5: Displacement of gas particles for t=0.01 s. Scheme LW2. ν =100kHz

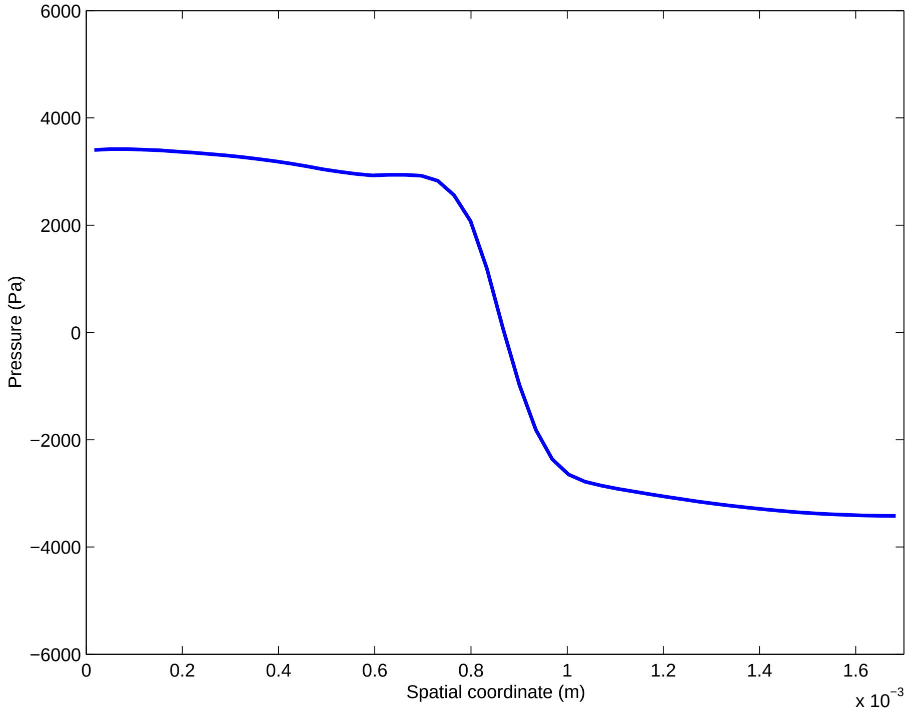

Figure 6: Pressure for t=0.01 s. Scheme LW2. ν =100kHz

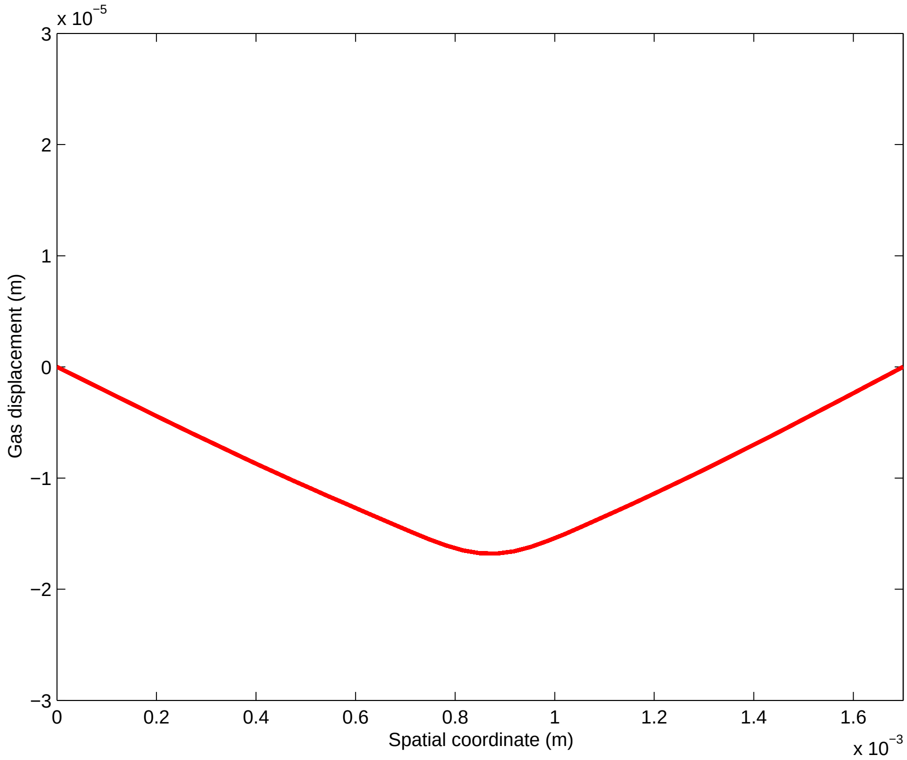

Figure 7: Displacement of gas particles for t=0.01 s. Scheme FORCE. ν =100kHz

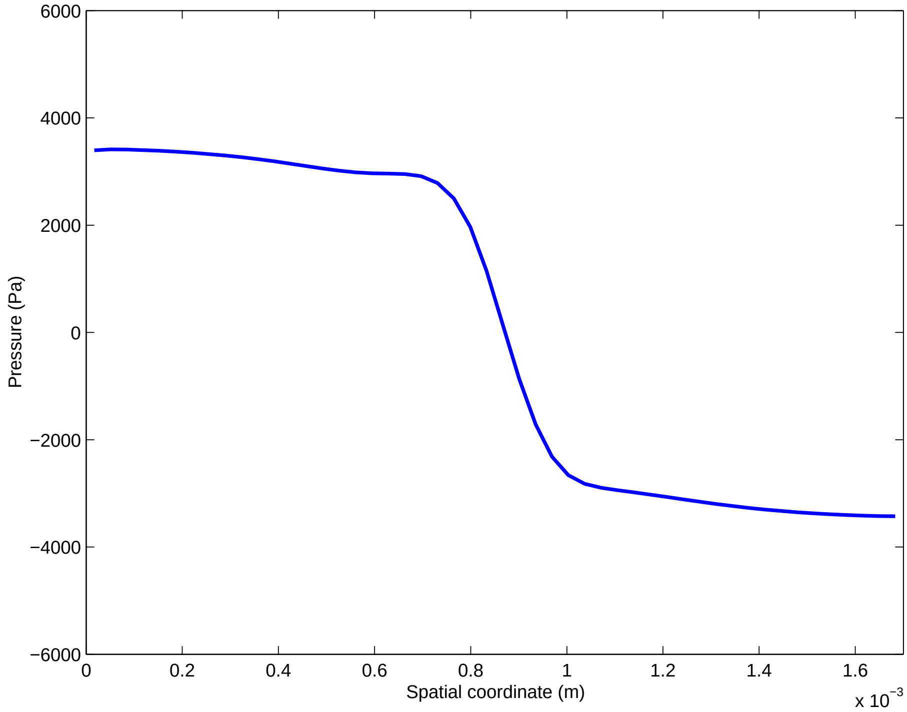

Figure 8: Pressure for t=0.01 s. Scheme FORCE. ν =100kHz

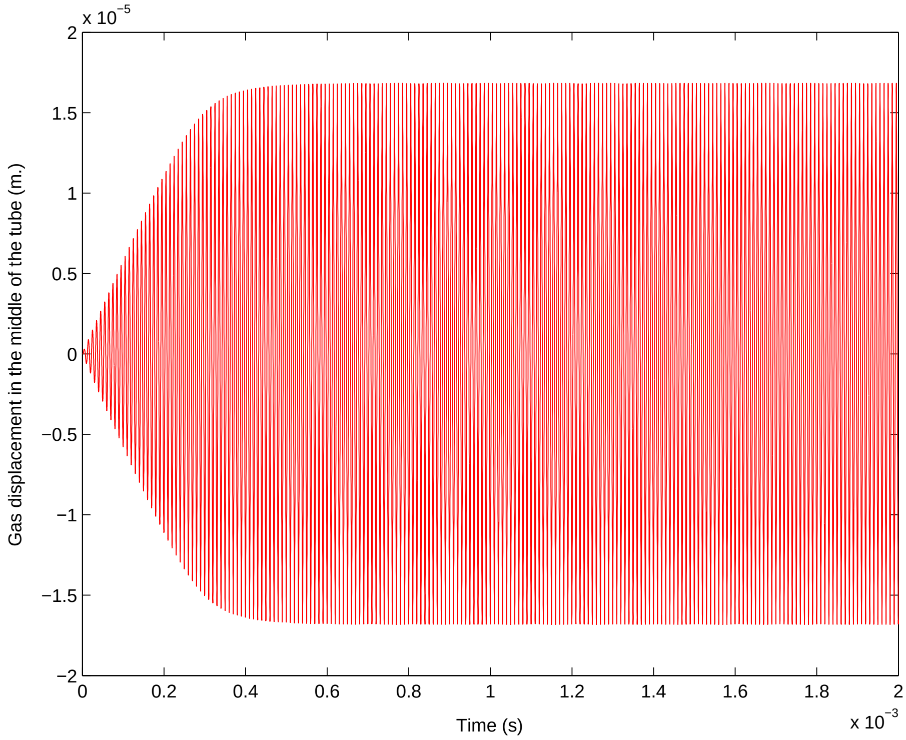

Figure 9: Displacement of gas particles in $x = L/2$. Scheme LW2. $\nu = 100kHz$

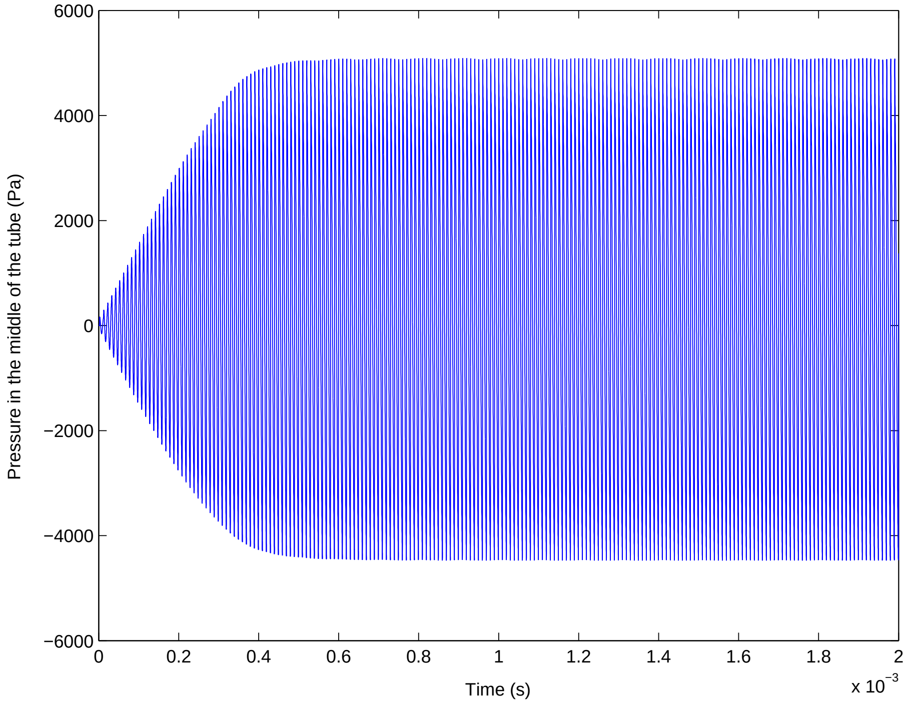

Figure 10: Pressure in $x = L/2$. Scheme LW2. $\nu = 100kHz$

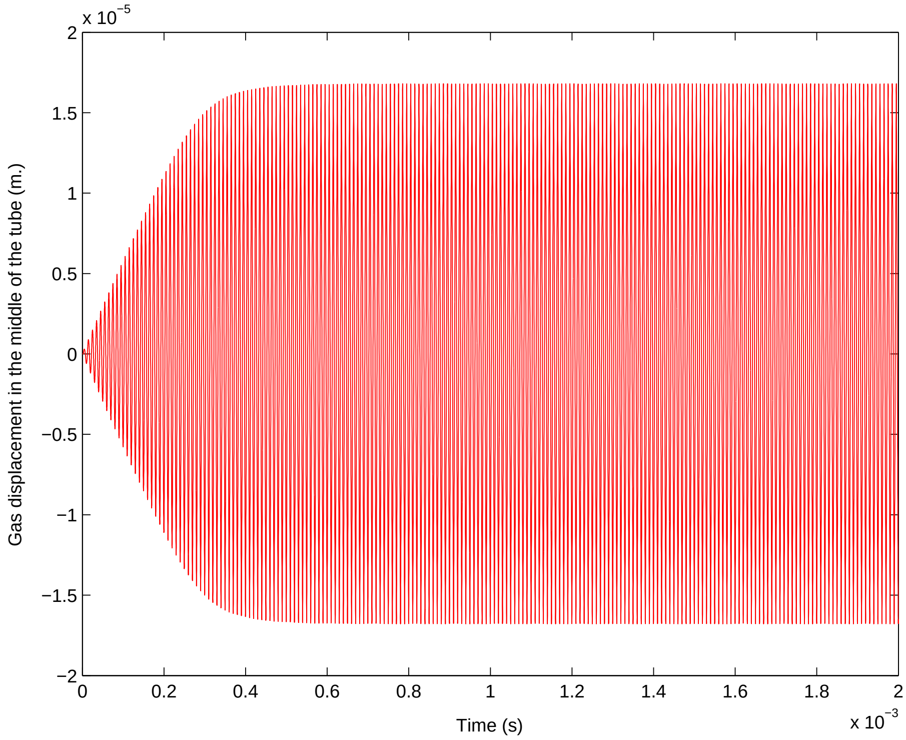

Figure 11: Displacement of gas particles in $x = L/2$. Scheme FORCE. $\nu = 100kHz$

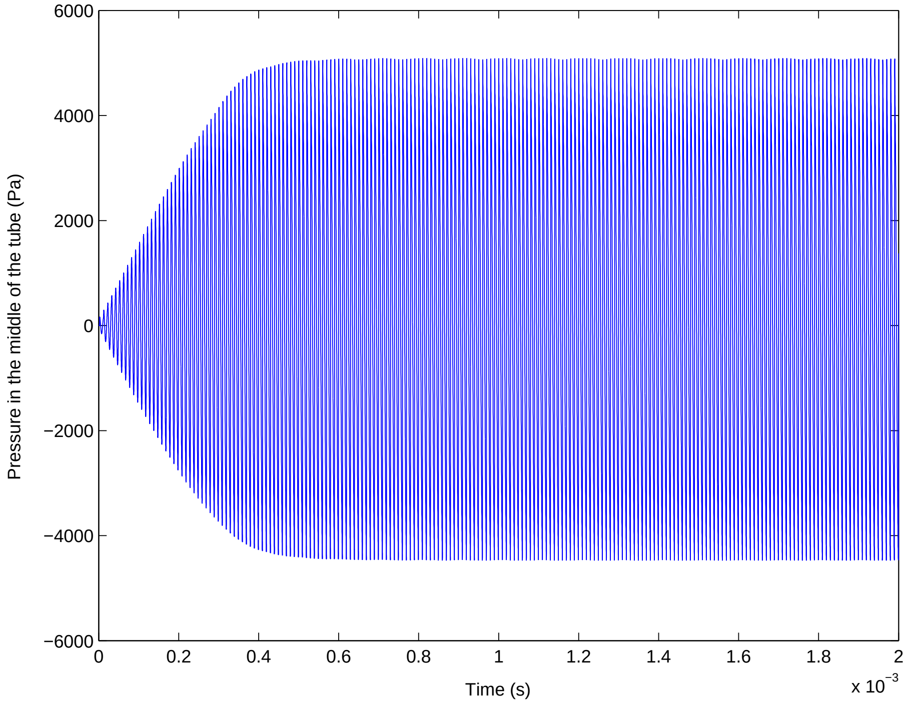

Figure 12: Pressure in $x = L/2$. Scheme FORCE. $\nu = 100kHz$