STOCHASTIC SIMULATION OF NON-STATIONARY CONTINUOUS MULTIFRACTAL TIME SERIES

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Abstract Intermittency is an ubiquitous property of fully developed turbulence, for Eulerian and Lagrangian fields, and for velocity, passive and active scalars. Intermittency corresponds to multi-scale high fluctuations, with some underlying long-range correlations. Such property is usually characterized using scaling approaches, verified using experimental or numerical data. However there are only few studies devoted to the generation of continuous stochastic processes having non-stationary multifractal properties, able to mimic Eulerian or Lagrangian velocity or passive scalar time series. Here we review recent works on this topic, and we provide stochastic simulations in order to verify the theoretical predictions. In the lognormal framework we provide a $h-\mu$ plane expressing the scale invariant properties of these simulations.

NON-STATIONARY MULTIFRACTAL TIME SERIES

We consider here the scaling properties of a time series $X(t)$, assumed to have a Fourier spectrum of the form $E_X(f) = Cf^{-\beta}$, where C is a constant and $\beta > 0$ is the spectral exponent. We consider the moments of order $q > 0$ of fluctuations at scale ℓ, $M_\ell(q) = \langle |X(t+\ell) - X(t)|^q \rangle$ of $X(t)$ (called structure functions), where ℓ is the scale belonging to a given range between a larger and a small scale. Since we are dealing with scaling processes that have stationary increments, we expect the following scaling behaviour:

$$M_\ell(q) = A_q \ell^{\zeta(q)} \tag{1}$$

where A_q is a parameter independent from the scale, and $\zeta(q)$ the scaling moment function, with $\beta = 1 + \zeta(2)$. The knowledge of the full $\zeta(q)$ function then provides much more information than the single parameter β. Some completely different stochastic processes may possess the same spectral exponent, showing that, when doing data analysis and model assessment, estimation of $\zeta(q)$ on a full range of values is much better than only estimating the single parameter β. Figure 1a shows the $\zeta(q)$ function for several classical linear stochastic processes. A continuous multifractal time series is a scaling process with a nonlinear $\zeta(q)$ function.

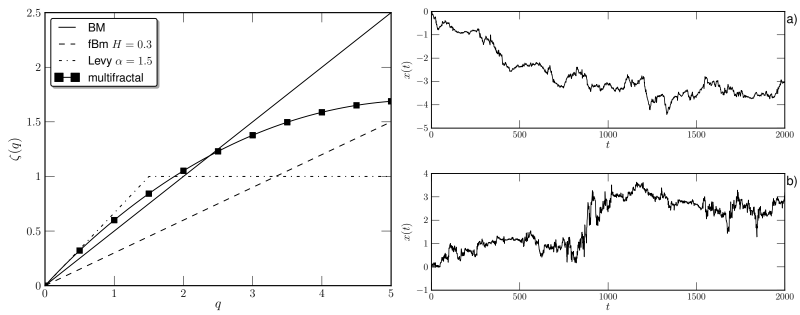

Figure 1. Left: Typical moment functions $\zeta(q)$ for several cases, Brownian motion, fractional Brownian motion for $H = 0.3$, Levy stable motion with $\alpha = 1.5$ and a multifractal time series with a nonlinear moment function. Right: examples of moving average simulations of a multifractal process, with a lognormal multiplicative cascade with $\mu = 0.2$, and values of $h = 0.3$ (above), and $h = 0.5$ (below)

MOVING AVERAGE GENERATION OF A NON-STATIONARY MULTIFRACTAL TIME SERIES

A few recent papers have considered the generation of a non-stationary multifractal time series (also called multiaffine or multifractal random walk), using a moving average representation of the form [1, 2, 3]:

$$X(t) = \int_0^t \epsilon(u) dY_h(u) \tag{2}$$

where ϵ is a kernel having multiplicative scaling properties and $Y_h(t)$ is a self-similar process of parameter h, independent of ϵ. In the last decade, only a few papers have considered the mathematical construction, existence, and convergence of such a process. When taking Y_h as a fractional Brownian motion, the first point is to construct a stochastic integral with respect to fBm, and show that it is well defined and not diverging. This was done in [2] for $h > 1/2$. Abry et al. [1] further explored the case $h > 1/2$ using fractional Wiener integrals. In such a situation, the process generated is shown to be converging and different from that previously produced. Let us note $K(q)$ the scaling moment function of ϵ (at scale ℓ the moments of ϵ_ℓ write $\langle \epsilon_\ell^q \rangle \sim \ell^{-K(q)}$) and $\mu = K(2)$. The result of [1], with our notations, is the following (for $h > 1/2$ only):

$$\zeta_X(q) = \begin{cases} qh - K(q); & \mu \leq 2h - 1 \\ \frac{\mu+1}{2}q - K(q); & 2h < \mu + 1 \end{cases} \tag{3}$$

where $K(q)$ is the scaling exponent for the kernel function. When $\mu \leq 2h-1$, we have the Hurst exponent $H = \zeta_X(1) = h$, whereas for $2h < \mu + 1$, $H = \zeta_X(1) = \frac{\mu+1}{2}$ and is not related to h.
The case $h < 1/2$ has only been considered, up to now, in one paper. In the case $h < 1/2$, Perpete [3] has shown that the process is well defined and, using a different method, found the following scaling exponents:

$$\zeta_X(q) = \frac{q}{2} - K(q) \tag{4}$$

with the following conditions:

$$\begin{cases} 0 < h < 1/4 \ \& \ K(\frac{1}{2h}) < \frac{1}{2h} - 1 \\ 1/4 < h < 1/2 \end{cases} \tag{5}$$

This result is surprising since there is no h dependence in the value of $\zeta_X(q)$. We have here in both cases $H = \zeta_X(1) = 1/2$. The different H values are shown in Figure 2a in a $h - \mu$ plane.
In the lognormal case, since $K(q) = \frac{\mu}{2}(q^2 - q)$, the condition $K(\frac{1}{2h}) < \frac{1}{2h} - 1$ becomes $\mu < f(h)$ with $f(x) = 2x(1-2x)/(1-x)$. Since $0 < h < 1$ and $0 < \mu < 1$, we can plot in the $h - \mu$ plane the result in Figure 2b, following the calculations of [1] and [3], given in Equations (3) and (5). There are four zones in this $h - \mu$ quadrant: from left to right, there is no result in the zone left blank; for the next region we have $\zeta_X(q) = \frac{q}{2} - \Psi(q)$; then $\zeta_X(q) = \frac{\mu+1}{2}q - \Psi(q)$ and finally $\zeta_X(q) = qh - \Psi(q)$.
The $h - \mu$ planes provided here (see [4]) can be generalized when introducing a moment of order $a > 0$ of the kernel process. Stochastic simulations are performed in order to test these theoretical results, and to show for a given desired value of the parameters H and μ, with the following moment function: $\zeta(q) = qH - \frac{\mu}{2}(q^2 - q)$, how to generate a stochastic time series with such scaling moment function.

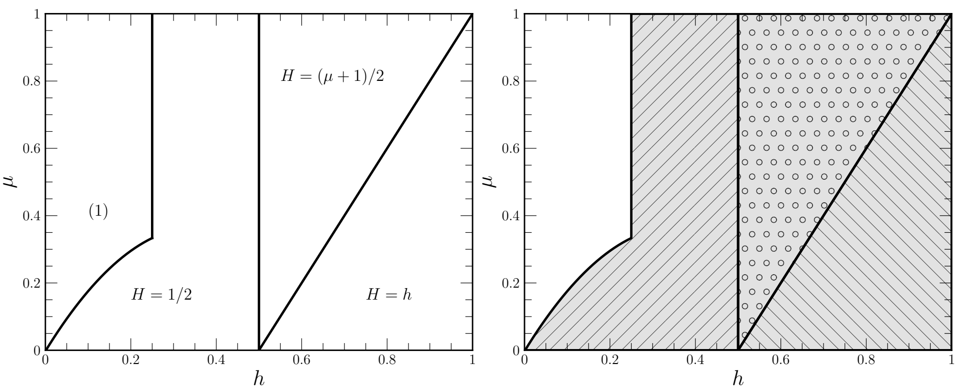

Figure 2. Left: Different values of $H = \zeta_X(1)$ obtained in a $h - \mu$ plane. The region (1) corresponds to a region with unknown value for H. The nonlinear curve which is at the bottom left corresponds to the condition $K(1/2h) < 1/2h - 1$. Right: The plane $h - \mu$ giving the value of the scaling exponent $\zeta_X(q)$ for a lognormal process. For the zone which is left in blank, there is no result for the moment. Increasing diagonal: $\zeta_X(q) = \frac{q}{2} - K(q)$; open dots: $\zeta_X(q) = \frac{\mu+1}{2}q - K(q)$ and decreasing diagonals $\zeta_X(q) = qh - K(q)$.

References

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