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Guo, Meichen; De Persis, Claudio; Tesi, Pietro

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Chapter 12
Data-Driven Stabilization of Nonlinear Systems via Taylor's Expansion

Meichen Guo, Claudio De Persis, and Pietro Tesi

Abstract Lyapunov's indirect method is one of the oldest and most popular approaches to model-based controller design for nonlinear systems. When the explicit model of the nonlinear system is unavailable for designing such a linear controller, finite-length off-line data is used to obtain a data-based representation of the closed-loop system, and a data-driven linear control law is designed to render the considered equilibrium locally asymptotically stable. This work presents a systematic approach for data-driven linear stabilizer design for continuous-time and discrete-time general nonlinear systems. Moreover, under mild conditions on the nonlinear dynamics, we show that the region of attraction of the resulting locally asymptotically stable closed-loop system can be estimated using data.

12.1 Introduction

Most control approaches of nonlinear systems are based on well-established models of the system constructed by prior knowledge or system identification. When the models are not explicitly constructed, nonlinear systems can be directly controlled using input–output data. The problem of controlling a system via input–output data without explicitly identifying the model has been gaining more and more attentions for both linear and nonlinear systems. An early survey of data-driven control

M. Guo (✉)
Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, The Netherlands
e-mail: m.guo@tudelft.nl

C. De Persis
Engineering and Technology Institute Groningen, University of Groningen, Nijenborgh 4, 9747 Groningen, The Netherlands
e-mail: c.de.persis@rug.nl

P. Tesi
Department of Information Engineering, University of Florence, 50139 Florence, Italy
e-mail: pietro.tesi@unifi.it

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methods can be found in [17]. More recently, the authors of [32] developed an online control approach, the work [18] utilized the dynamic linearization data models for discrete-time non-affine nonlinear systems, the authors of [13, 30] considered feedback linearizable systems, and the works [2, 20] designed data-driven model predictive controllers. Inspired by Willems et al.'s fundamental lemma, [11] proposed data-driven control approaches for linear and nonlinear discrete-time systems. Using a matrix Finsler's lemma, [34] applied data-driven control to Lur'e systems. The authors of [10] used state-dependent representation and proposed an online optimization method for data-driven stabilization of nonlinear dynamics. For polynomial systems, [14] designed global stabilizers using noisy data, and [23] synthesized data-driven safety controllers. The recent work [22] investigated dissipativity of nonlinear systems based on polynomial approximation.

Related works. Some recent works related to nonlinear data-driven control and the region of attraction (RoA) estimation are discussed in what follows.

Deriving a data-based representation of the dynamics is one of the important steps in data-driven control of unknown nonlinear systems. If the controlled systems are of certain classes, such as polynomial systems having a known degree, the monomials of the state can be chosen as basis functions to design data-driven controllers such as presented in [14, 15]. By integrating noisy data and side information, [1] showed that unknown polynomial dynamics can be learned via semi-definite programming. When the nonlinearities satisfy quadratic constraints, data-driven stabilizer was developed in [21]. With certain knowledge and assumptions on the nonlinear basis functions, systems containing more general types of nonlinearities have also been studied in recent works. For instance, under suitable conditions, some nonlinear systems can be lifted into polynomial systems in an extended state for control, such as the results shown in [29, Sect. IV] and [19, Sect. 3.2]. Using knowledge of the basis functions, [24] designed data-driven controllers by (approximate) cancellation of the nonlinearity. When the system nonlinearities cannot be expressed as a combination of known functions, [24] presented data-driven local stabilization results by choosing basis functions carefully such that the neglected nonlinearities are small in a known set of the state. On the other hand, if the knowledge on the basis functions is not available, approximations of the nonlinear systems are often involved. The previous work [11] tackled the nonlinear data-driven control problem by linearizing the dynamics around the known equilibrium and obtaining a local stability result. According to these existing results, it is clear that the efficiency and the performance of data-driven controllers can be improved via prior knowledge such as specific classes of the systems or the nonlinear basis functions. Nonetheless, there is still a lack of comprehensive investigation of the more general case where the nonlinear basis functions cannot be easily and explicitly obtained.

The RoA estimation is another relevant topic in nonlinear control. For general nonlinear systems, it is common that the designed controllers only guarantee local stability. Hence, it is of importance to obtain the RoA of the closed-loop systems for the purpose of theoretical analysis as well as engineering applications. Unfortunately, for general nonlinear systems, it is extremely difficult to derive the exact RoA even when the model is explicitly known. A common solution is to estimate the RoA based on Lyapunov functions. Using Taylor's expansion and considering the worst-case remainders, [7] estimated the RoA of uncertain non-polynomial systems via linear matrix inequality (LMI) optimizations. RoA analysis for polynomial systems was presented in [31] using polynomial Lyapunov functions and sum of squares (SOS) optimizations. The authors of [33] studied uncertain nonlinear systems subject to perturbations in certain forms and used the SOS technique to compute invariant subsets of the RoA. It is noted that, in these works, the RoA estimation winds up in solving bilinear optimization problems, and techniques such as bisection or special bilinear inequality solver are required to find the solutions. For nonlinear systems without explicit models, there are also efforts devoted to learning the RoA by various approaches. The authors of [6] developed a sampling-based approach for a class of piecewise continuous nonlinear systems that verifies stability and estimates the RoA using Lyapunov functions. Based on the converse Lyapunov theorem, [9] processed system trajectories to lift a Lyapunov function whose level sets lead to an estimation of the RoA. Using the properties of recurrent sets, [27] proposed an approach that learns an inner approximation of the RoA via finite-length trajectories. It should be pointed out that, all the aforementioned works focus on stability analysis of autonomous nonlinear systems, i.e., the control design is not considered.

Contributions. For general nonlinear systems, this chapter presents a data-driven approach to simultaneously obtaining a Lyapunov function and designing a state feedback stabilizer that renders the known equilibrium locally asymptotically stable. Specifically, the unknown dynamics are approximated by linear dynamics with an approximation error. Then, linear stabilizers are designed for the approximated models using finite-length input-state data collected in an off-line experiment. To handle the approximation error we conduct the experiment close to the known equilibrium such that the approximation error is small with a known bound. An over-approximation of all the feasible dynamics is then found using the collected data, and Petersen's lemma [25] is used for the controller design. The data-driven stabilizer design can be seen as a generalization of the nonlinear control result in the previous work [11, Sect. V.B]. Specifically, this work considers the linear approximations of both continuous-time and discrete-time systems.

For estimating the RoA with the designed data-driven controller, we first derive an estimation of the approximation error by assuming a known bound on the derivatives of the unknown functions. With the help of the Positivstellensatz [28] and the SOS relaxations, we derive data-driven conditions that verify whether a given sublevel set of the obtained Lyapunov function is an invariant subset of the RoA. This is achieved by finding a sufficient condition for the negativity of the derivative of the Lyapunov function based on the estimation of the approximation error. The conditions are derived via data and some prior knowledge on the dynamics, and can be easily solved by software such as MATLAB®. The estimated RoA gives insights to the closed-loop system under the designed data-driven controller and is relevant for both theoretical and practical purposes. Note that alternatively, the RoA can be estimated based on other data-driven methods, such as the ones developed in [6, 9, 27]. Simulations results on the inverted pendulum show the applicability of the design and estimation approach.

The rest of the chapter is arranged as follows. Section 12.2 formulates the nonlinear data-driven stabilization problem. Data-based descriptions of feasible dynamics consistent with the collected data and the over-approximation of the feasible set is presented in Sect. 12.3. Data-driven control designs for both continuous-time and discrete-time systems are presented in Sect. 12.4. The data-driven characterization of the RoA is derived in Sect. 12.5. Numerical results and analysis on an example are illustrated in Sect. 12.6. Finally, Sect. 12.7 summarizes this chapter.

Notation. Throughout the chapter, $A \succ (\succeq) 0$ denotes that matrix A is positive (semi-) definite, and $A \prec (\preceq) 0$ denotes that matrix A is negative (semi-) definite. For vectors $a, b \in \mathbb{R}^n$, $a \preceq b$ means that $a_i \leq b_i$ for all $i = 1, \ldots, n$. $\|\cdot\|$ denotes the Euclidean norm.

12.2 Data-Driven Nonlinear Stabilization Problem

Consider a general nonlinear continuous-time system

$$\dot{x} = f(x, u) \tag{12.1}$$

or a nonlinear discrete-time system

$$x^+ = f(x, u), \tag{12.2}$$

where the state $x \in \mathbb{R}^n$ and the input $u \in \mathbb{R}^m$. Assume that (x_e, u_e) is a known equilibrium of the system to be stabilized. For simplicity and without loss of generality, in this chapter we let $(x_e, u_e) = (0, 0)$, as any known equilibrium can be converted to the origin by a change of coordinates.

To gather information regarding the nonlinear dynamics, E experiments are performed on the system, where E is an integer satisfying $1 \leq E \leq T$ and T is the total number of collected samples. On one extreme, one could perform 1 single experiment during which a total of T samples are collected. On the other extreme, one could perform T independent experiments, during each one of which a single sample is collected. The advantage of short multiple experiments is that they allow information collection about the system at different points in the state space without incurring problems due to the free evolution of the system.

Either way, a data set $\mathcal{DS}_c := \{\dot{x}(t_k); x(t_k); u(t_k)\}_{k=0}^{T-1}$ for the continuous-time system, or $\mathcal{DS}_d := \{x(t_k); u(t_k)\}_{k=0}^{T}$ for the discrete-time system can be obtained. Organize the data collected in the experiment(s) as

$$
\begin{aligned}
X_0 &= \left[x(t_0) \ \ldots \ x(t_{T-1}) \right], \\
U_0 &= \left[u(t_0) \ \ldots \ u(t_{T-1}) \right], \\
X_1 &= \left[\dot{x}(t_0) \ \ldots \ \dot{x}(t_{T-1}) \right] \text{ for continuous-time systems,} \\
X_1 &= \left[x(t_1) \ \ldots \ x(t_T) \right] \text{ for discrete-time systems.}
\end{aligned}
$$

Remark 12.2.1 (*On the experimental data*) To avoid further complicating the problem and focus on the unknown general nonlinear dynamics, we do not consider external disturbances in the dynamics or measurement noise in the data. In the case of disturbed dynamics or noisy data, if some bounds on the disturbance/noise data are known then we can follow the same approach used in this work to design a data-driven controller, *cf.* [24].

Data-driven Stabilization Problem

Use the data set $\mathcal{DS}_c$ ($\mathcal{DS}_d$) to design a feedback controller $u = Kx$ for the system (12.1) ((12.2)) such that the origin is locally asymptotically stable for the closed-loop system, and an estimation of the RoA with respect to the origin is derived.

12.3 Data-Based Feasible Sets of Dynamics

As limited information is available for the nonlinear dynamics, a natural way to deal with the unknown model is to approximate it as a linear model in a neighborhood of the considered equilibrium. Due to the approximation error, true dynamics of the linearized models cannot be uniquely determined by the collected data. Instead of explicitly identifying the linearized model from the data set $\mathcal{DS}_c$ or $\mathcal{DS}_d$, in this section, we find feasible sets that contain all linear dynamics that are consistent with the collected data. To achieve this, we will use the approach proposed in [3], where an over-approximation of the feasible set is found by solving an optimization problem depending on the data set and the bound of the approximation error during the experiment.

In what follows, we will address the linear approximations of the continuous-time system (12.1) and the discrete-time system (12.2), and the over-approximation of the feasible sets of dynamics.

For the system (12.1), denote each element of f as f_i, and let $f_i \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$, $i = 1, \ldots, n$. The linear approximation of (12.1) at the origin is

$$\dot{x} = Ax + Bu + R(x, u), \tag{12.3}$$

where $R(x, u)$ denotes the approximation error and

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x,u)=(0,0)}, \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x,u)=(0,0)}.$$

Using the same definitions, the linear approximation of the discrete-time nonlinear systems are given as

$$x^+ = Ax + Bu + R(x,u). \tag{12.4}$$

For both types of systems, one can treat the approximation error $R(x,u)$ as a disturbance that affects the data-driven characterization of the unknown dynamics, and focus on controlling the approximated linear dynamics to obtain a locally stabilizing controller. Then, by tackling the impact of $R(x,u)$, the RoA of the closed-loop system can also be characterized.

Assumption 12.3.1 (*Bound on the approximation error*) For $k = 0, \ldots, T$ and a known γ,

$$R(x(t_k), u(t_k))^\top R(x(t_k), u(t_k)) \le \gamma^2. \tag{12.5}$$

Remark 12.3.2 (*Bound on the approximation error*) Assumption 12.3.1 gives an *instantaneous* bound on the maximum amplitude of the approximation error during the experiment. The bound can be obtained by prior knowledge of the model, such as the physics of the system. If such knowledge is unavailable, one may resort to an over-estimation of γ.

Next, we determine the feasible set of dynamics that are consistent with the data sets.

Denote $S = [B \;\; A]$. Based on the dynamics (12.3) and (12.4), at each time t_k, $k = 0, \ldots, T-1$, the collected data satisfies

$$\dot{x}(t_k) = S\begin{bmatrix} u(t_k) \\ x(t_k) \end{bmatrix} + R\big(x(t_k), u(t_k)\big)$$

for continuous-time systems and

$$x(t_{k+1}) = S\begin{bmatrix} u(t_k) \\ x(t_k) \end{bmatrix} + R\big(x(t_k), u(t_k)\big)$$

for discrete-time systems. Under Assumption 12.3.1, at each time t_k, $k = 0, \ldots, T-1$, the matrices $\widehat{S} = [\widehat{B} \;\; \widehat{A}]$ consistent with the data belongs to the set

$$\mathcal{C}_k = \left\{\widehat{S} : \mathbf{C}_k + \widehat{S}\mathbf{B}_k + \mathbf{B}_k^\top \widehat{S}^\top + \widehat{S}\mathbf{A}_k\widehat{S}^\top \preceq 0\right\}, \tag{12.6}$$

where

$$\mathbf{A}_k = l(t_k)l(t_k)^\top, \quad \mathbf{B}_k = -l(t_k)\dot{x}(t_k)^\top,$$
$$\mathbf{C}_k = \dot{x}(t_k)\dot{x}(t_k)^\top - \gamma^2 I, \quad l(t_k) = \begin{bmatrix} u(t_k) \\ x(t_k) \end{bmatrix}$$

for continuous-time systems, and the same definitions except for

$$\mathbf{B}_k = -l(t_k)x(t_{k+1})^\top, \quad \mathbf{C}_k = x(t_{k+1})x(t_{k+1})^\top - \gamma^2 I$$

hold for discrete-time systems. Then, the feasible set of matrices $\widehat{S}$ that is consistent with all data collected in the experiment(s) is the intersection of all the sets $\mathcal{C}_k$, i.e., $\mathcal{I} = \bigcap_{k=0}^{T-1} \mathcal{C}_k$.

Though the exact set $\mathcal{I}$ is difficult to obtain, an over-approximation of $\mathcal{I}$ in the form of a matrix ellipsoid and of minimum size can be computed. Denote the over-approximation set as

$$\overline{\mathcal{I}} := \left\{\widehat{S} : \overline{\mathbf{C}} + \widehat{S}\overline{\mathbf{B}} + \overline{\mathbf{B}}^\top \widehat{S}^\top + \widehat{S}\overline{\mathbf{A}}\widehat{S}^\top \preceq 0\right\}, \tag{12.7}$$

where $\overline{\mathbf{A}} = \overline{\mathbf{A}}^\top \succ 0$, $\overline{\mathbf{C}}$ is set as $\overline{\mathbf{C}} = \overline{\mathbf{B}}^\top \overline{\mathbf{A}}^{-1} \overline{\mathbf{B}}^\top - \delta I$ and $\delta > 0$ is a constant fixed arbitrarily. Following [3, Sect. 5.1], the set $\overline{\mathcal{I}}$ can be found by solving the optimization problem

$$\begin{aligned}
\underset{\overline{\mathbf{A}},\overline{\mathbf{B}},\overline{\mathbf{C}}}{\text{minimize}} \quad & -\log\det(\overline{\mathbf{A}}) \\
\text{subject to} \quad & \overline{\mathbf{A}} = \overline{\mathbf{A}}^\top \succ 0 \\
& \tau_k \geq 0, \; k = 0, \ldots, T-1 \\
& \begin{bmatrix} -\delta I - \sum_{k=0}^{T-1} \tau_k \mathbf{C}_k & \overline{\mathbf{B}}^\top - \sum_{k=0}^{T-1} \tau_k \mathbf{B}_k^\top & \overline{\mathbf{B}}^\top \\ \overline{\mathbf{B}} - \sum_{k=0}^{T-1} \tau_k \mathbf{B}_k & \overline{\mathbf{A}} - \sum_{k=0}^{T-1} \tau_k \mathbf{A}_k & 0 \\ \overline{\mathbf{B}} & 0 & -\overline{\mathbf{A}} \end{bmatrix} \preceq 0.
\end{aligned} \tag{12.8}$$

Proposition 12.3.3 (Over-approximated feasible set) *Consider the data set $\mathcal{DS}_c$ ($\mathcal{DS}_d$) collected from the dynamics* (12.3) *((12.4)), which satisfies Assumption 12.3.1. If the optimization problem* (12.8) *is feasible for $\mathbf{A}_k$, $\mathbf{B}_k$, and $\mathbf{C}_k$ defined in* (12.6)*, then the set $\overline{\mathcal{I}}$ defined in* (12.7) *is an over-approximation set of all dynamics that is consistent with the data set $\mathcal{DS}_c$ ($\mathcal{DS}_d$).*

Remark 12.3.4 (*Persistency of excitation*) As pointed out in [3, Sect. 3.1], if the collected data is rich enough, i.e., $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank, then the intersection set $\mathcal{I}$ is bounded, which allows the optimization problem (12.8) to have a solution. Hence, $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ having full row rank implies the feasibility of (12.8).

12.4 Data-Driven Controller Design

Stabilizing the linear approximation of the unknown system renders the origin locally asymptotically stable as the approximation error $R(x, u)$ contains higher-order terms and converges to the origin faster than the linear part in a neighborhood of the origin. Hence, the objective of the controller design is to stabilize the origin for all dynamics belonging to the over-approximation set $\overline{\mathcal{I}}$. This can be achieved in the same manner as done in [4] via Petersen's lemma; see Appendix 12.8.1 for background material. For the completeness of this work, we include the following results on designing data-driven local stabilizers using Petersen's lemma.

Theorem 12.4.1 (Data-driven controller design for continuous-time systems) *Under Assumption 12.3.1, given a constant $w > 0$, if there exist matrices Y and $P = P^\top$ such that*

$$\begin{bmatrix} wP - \overline{\boldsymbol{C}} & \overline{\boldsymbol{B}}^\top - \begin{bmatrix} Y \\ P \end{bmatrix}^\top \\ \overline{\boldsymbol{B}} - \begin{bmatrix} Y \\ P \end{bmatrix} & -\overline{\boldsymbol{A}} \end{bmatrix} \preceq 0 \qquad (12.9)$$
$$P \succ 0,$$

then the origin is a locally asymptotically stable equilibrium for the closed-loop system composed of (12.1) and the control law $u = YP^{-1}x$.

Proof To make the origin a locally asymptotically stable, we look for a control gain K such that the linear part of the closed-loop dynamics $(\widehat{A} + \widehat{B}K)$ is Hurwitz for all $\begin{bmatrix}\widehat{A} & \widehat{B}\end{bmatrix} \in \overline{\mathcal{I}}$. For this purpose, it suffices to find a $P = P^\top \succ 0$ such that

$$P^{-1}(\widehat{A} + \widehat{B}K) + (\widehat{A} + \widehat{B}K)^\top P^{-1} + wP^{-1} \preceq 0 \quad \forall \begin{bmatrix}\widehat{B} & \widehat{A}\end{bmatrix} \in \overline{\mathcal{I}} \qquad (12.10)$$

for any fixed $w > 0$. Recalling that $\widehat{S} = \begin{bmatrix}\widehat{B} & \widehat{A}\end{bmatrix}$, and left- and right- multiplying P to both sides of the inequality gives

$$\widehat{S}\begin{bmatrix} K \\ I \end{bmatrix} P + P \begin{bmatrix} K \\ I \end{bmatrix}^\top \widehat{S}^\top + wP \preceq 0 \quad \forall \widehat{S} \in \overline{\mathcal{I}}. \qquad (12.11)$$

Following the description (12.7) of set $\overline{\mathcal{I}}$ and the definition of $\overline{\mathbf{C}}$, it holds that

$$\begin{aligned} &\overline{\mathbf{C}} + \widehat{S}\overline{\mathbf{B}} + \overline{\mathbf{B}}^\top \widehat{S}^\top + \widehat{S}\overline{\mathbf{A}}\widehat{S}^\top \\ &= \left(\widehat{S}^\top + \overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right)^\top \overline{\mathbf{A}} \left(\widehat{S}^\top + \overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right) - \overline{\mathbf{B}}^\top \overline{\mathbf{A}}^{-1}\overline{\mathbf{B}} + \overline{\mathbf{C}} \\ &= \left(\widehat{S}^\top + \overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right)^\top \overline{\mathbf{A}} \left(\widehat{S}^\top + \overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right) - \delta I \preceq 0. \end{aligned}$$

Define $\Delta = \overline{\mathbf{A}}^{1/2}\left(\widehat{S}^\top + \overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right)$, and it follows that $\Delta^\top\Delta \preceq \delta I$ and

$$\widehat{S}^\top = -\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}} + \overline{\mathbf{A}}^{-1/2}\Delta.$$

Defining $Y = KP$ we can rewrite (12.11) as

$$\begin{aligned}&\widehat{S}\begin{bmatrix}K\\I\end{bmatrix}P + P\begin{bmatrix}K\\I\end{bmatrix}^\top\widehat{S}^\top + wP\\ &= \left(-\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right)^\top\begin{bmatrix}Y\\P\end{bmatrix} + \left(\overline{\mathbf{A}}^{-1/2}\Delta\right)^\top\begin{bmatrix}Y\\P\end{bmatrix} + (\star)^\top + wP \preceq 0 \quad \forall\, \Delta^\top\Delta \preceq \delta I.\end{aligned}$$

By Petersen's lemma, the inequality above holds if and only if there exists $\epsilon > 0$ such that

$$\left(-\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right)^\top\begin{bmatrix}Y\\P\end{bmatrix} + \begin{bmatrix}Y\\P\end{bmatrix}^\top\left(-\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right) + wP + \epsilon\begin{bmatrix}Y\\P\end{bmatrix}^\top\overline{\mathbf{A}}^{-1}\begin{bmatrix}Y\\P\end{bmatrix} + \epsilon^{-1}\delta I \preceq 0, \tag{12.12}$$

which is equivalent to

$$\left(-\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right)^\top\begin{bmatrix}\epsilon Y\\\epsilon P\end{bmatrix} + \begin{bmatrix}\epsilon Y\\\epsilon P\end{bmatrix}^\top\left(-\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right) + \epsilon wP + \begin{bmatrix}\epsilon Y\\\epsilon P\end{bmatrix}^\top\overline{\mathbf{A}}^{-1}\begin{bmatrix}\epsilon Y\\\epsilon P\end{bmatrix} + \delta I \preceq 0.$$

As Y and P are unknown variables, we can neglect ϵ and obtain the inequality

$$\begin{aligned}&\left(-\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right)^\top\begin{bmatrix}Y\\P\end{bmatrix} + \begin{bmatrix}Y\\P\end{bmatrix}^\top\left(-\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}\right) + wP + \begin{bmatrix}Y\\P\end{bmatrix}^\top\overline{\mathbf{A}}^{-1}\begin{bmatrix}Y\\P\end{bmatrix} + \delta I\\ &= \left(\overline{\mathbf{B}} - \begin{bmatrix}Y\\P\end{bmatrix}\right)^\top\overline{\mathbf{A}}^{-1}\left(\overline{\mathbf{B}} - \begin{bmatrix}Y\\P\end{bmatrix}\right) + wP - \overline{\mathbf{C}} \preceq 0.\end{aligned}$$

By the Schur complement, the inequality above is equivalent to (12.9), and the matrices Y and P satisfies (12.9) renders $(\widehat{A} + \widehat{B}YP^{-1})$ Hurwitz.

Recall that the closed-loop dynamics is $\dot{x} = (\widehat{A} + \widehat{B}YP^{-1})x + R(x, u)$, where the approximation error $R(x, u)$ contains high-order terms and converges to the origin faster than the linear part for all x in a neighborhood of the origin. Therefore, the origin is a locally asymptotically stable equilibrium for the closed-loop system under the designed linear controller. □

A similar result is obtained for discrete-time systems as shown in the following theorem. The proof follows that of Theorem 12.4.1 and thus is omitted.

Theorem 12.4.2 (Data-driven controller design for discrete-time systems) *Under Assumption 12.3.1, given a* $w \in (0, 1)$*, if there exist matrices* Y *and* $P = P^\top$ *such that*

$$\begin{bmatrix} -(1-w)P & 0 & \begin{bmatrix} Y \\ P \end{bmatrix}^\top \\ 0 & -P-\overline{\boldsymbol{C}} & -\overline{\boldsymbol{B}}^\top \\ \begin{bmatrix} Y \\ P \end{bmatrix} & -\overline{\boldsymbol{B}} & -\overline{\boldsymbol{A}} \end{bmatrix} \preceq 0 \tag{12.13}$$

$$P \succ 0, \tag{12.14}$$

then the origin is a locally asymptotically stable equilibrium for the closed-loop system composed of (12.2) *and the control law* $u = YP^{-1}x$.

Remark 12.4.3 (*Enforcing a decay rate via* w) Consider the Lyapunov function $V(x) = x^\top P^{-1}x$. The controller designed by Theorem 12.4.1 guarantees that the derivative of $V(x)$ along the trajectory of the closed-loop system $\dot{x} = \widehat{A}x + \widehat{B}u$ with the designed control law $u = YP^{-1}x$ satisfies that

$$\dot{V}(x) \leq -wV(x)$$

for any given $w > 0$ and any $\begin{bmatrix} \widehat{B} & \widehat{A} \end{bmatrix} \in \overline{\mathcal{I}}$. Hence, by choosing the value of w, a certain decay rate of the closed-loop solution is enforced. Similarly, for the discrete-time system, Theorem 12.4.2 leads to a closed-loop system $x^+ = \widehat{A}x + \widehat{B}u$ with the designed control law $u = YP^{-1}x$ such that

$$V(x^+) - V(x) \leq -wV(x)$$

for any given $w \in (0, 1)$ and $\begin{bmatrix} \widehat{B} & \widehat{A} \end{bmatrix} \in \overline{\mathcal{I}}$.

Remark 12.4.4 (*Data-driven control design via high-order approximation*) Besides approximating the nonlinear dynamics (12.1) and (12.2) as linear systems with approximated errors, one can also perform high-order approximation using Taylor's expansion. In particular, consider the continuous-time input-affine system

$$\dot{x} = f(x) + g(x)u, \tag{12.15}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f(0) = 0$. Under certain continuity assumptions on the functions in f and g, one can write the nonlinear dynamics into a polynomial system having a linear-like form, i.e.,

$$\dot{x} = AZ(x) + BW(x)u + R(x, u), \tag{12.16}$$

where $Z(x)$ and $W(x)$ are a vector and a matrix containing monomials in x, respectively, and $R(x, u)$ is the approximation error. If $R(x, u)$ satisfies Assumption 12.3.1, then an over-approximated set of the feasible dynamics can also be found in a similar manner as shown in Sect. 12.3. Based on the over-approximation of the feasible set and using Petersen's lemma, a *nonlinear* data-driven control law can be designed by solving SOS conditions. This approach is also applicable to discrete-time nonlinear dynamics. Detailed results on nonlinear data-driven control design via high-order approximation for continuous-time and discrete-time systems can be found in our work [16].

12.5 RoA Estimation

In the previous section, we have shown that data-driven stabilizers can be designed for unknown nonlinear systems using linear approximations. The resulting controllers make the origin locally asymptotically stable. Besides this property, it is of paramount importance to estimate the RoA of the closed-loop system. The definition of the RoA is given as follows.

Definition 12.5.1 (*Region of attraction*) For the system $\dot{x} = f(x)$ or $x^+ = f(x)$, if for every initial condition $x(t_0) \in \mathcal{R}$, it holds that $\lim_{t\to\infty} x(t) = 0$, then $\mathcal{R}$ is a region of attraction of the system with respect to the origin. If there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ and a positive constant c such that

$$\Omega_c := \{x \in \mathbb{R}^n : V(x) \le c\}$$

is bounded and

$$\begin{aligned}
&V(0) = 0, \quad V(x) > 0 \ \ \forall x \in \mathbb{R}^n \\
&\{x \in \mathbb{R}^n : V(x) \le c, x \neq 0\} \subseteq \{x \in \mathbb{R}^n : \dot{V}(x) < 0\} \text{ for the continuous system, or} \\
&\{x \in \mathbb{R}^n : V(x) \le c, x \neq 0\} \subseteq \{x \in \mathbb{R}^n : V(x^+) - V(x) < 0\}
\end{aligned}$$

for the discrete-time system, then Ω_c is an invariant subset, or called an estimation, of the RoA.

In this section, for the designed data-driven controllers in Sect. 12.4, we derive data-driven conditions to determine whether a given sublevel set of the Lyapunov function is an invariant subset of the RoA. The derived conditions are data-driven because they are obtained using the over-approximated set $\overline{\mathcal{I}}$. We note that once the controller is computed, it is possible to use any other data-driven method to estimate the RoA, see for example [6, 9, 27].

By the controller design method in Sect. 12.4, the Lyapunov function $V(x)$, and thus the set Ω_c with a given c, are available for analysis. To characterize the set $\{x \in \mathbb{R}^n : \dot{V}(x) < 0\}$, we need a bound on the approximation error for all x in a neighborhood of the origin. This is achievable by posing the following assumption on the partial derivative of each f_i.

Assumption 12.5.2 For all $z \in \mathbb{D} \subseteq \mathbb{R}^{n+m}$, where $\mathbb{D}$ is a star-convex neighborhood of the origin, f_i is continuously differentiable and

$$\left| \frac{\partial f_i}{\partial z_j}(z) - \frac{\partial f_i}{\partial z_j}(0) \right| \le L_i \|z\| \quad \forall j = 1, \ldots, m+n \tag{12.17}$$

for $i = 1, \ldots, n$ with known $L_i > 0$.

Under Assumption 12.5.2, a bound on the approximation error can be obtained by the following lemma.

Lemma 12.5.3 *Under Assumption 12.5.2, the approximation errors $R(x, u)$ in (12.3) and (12.4) satisfy*

$$|R_i(x, u)| \le \frac{\sqrt{n+m}L_i}{2} \|(x, u)\|^2 \quad \forall (x, u) \in \mathbb{D}, \tag{12.18}$$

where $R_i(x, u)$, $i = 1, \ldots, n$, is such that $R(x, u) = \begin{bmatrix} R_1(x, u) & \cdots & R_n(x, u) \end{bmatrix}^\top$.

The proof of Lemma 12.5.3 can be found in Appendix 12.8.2.

Remark 12.5.4 (*Existence of L_i*) Assumption 12.5.2 is the weakest condition needed for deriving a bound on the approximation error using Lemma 12.5.3. A stronger condition, such as the Lipschitz continuity of $\frac{\partial f_i}{\partial z_j}$, guarantees the existence of L_i. It is also noted that L_i can be estimated using a data-based bisection procedure as shown in [22, Sect. III.C].

Remark 12.5.5 (*On Assumptions 12.3.1 and 12.5.2*) Under Assumption 12.5.2, using Lemma 12.5.3, the bound γ^2 in Assumption 12.3.1 can be derived for the experimental data. During the experiment(s), suppose that the smallest ball containing $(x(t), u(t))$ has radius R_e, i.e., $\|(x(t_k), u(t_k))\| \le R_e$ for all $k = 0, \ldots, T$. Then, for $k = 0, \ldots, T$,

$$\begin{aligned} & R(x(t_k), u(t_k))^\top R(x(t_k), u(t_k)) \\ &= \sum_{i=1}^{n} R_i(x(t_k), u(t_k))^2 \\ &\le \sum_{i=1}^{n} \frac{(m+n)L_i^2}{4} \|x(t_k), u(t_k)\|^4 \\ &\le \sum_{i=1}^{n} \frac{(m+n)L_i^2}{4} R_e^4. \end{aligned}$$

Hence, if some prior knowledge on the dynamics is known such that Assumption 12.5.2 holds, γ^2 can be chosen as $\sum_{i=1}^{n} \frac{(m+n)L_i^2}{4} R_e^4$ to satisfy Assumption 12.3.1.

12.5.1 Continuous-Time Systems

We now characterize the time derivative of $V(x)$ along the trajectory of the continuous-time closed-loop system with the designed controller, and analyze the RoA of the closed-loop system.

Lemma 12.5.6 *Consider system (12.1) and the linear controller $u = YP^{-1}x$, where Y and P are designed to satisfy (12.9) with any given constant $w > 0$. Under Assumption 12.5.2, the derivative of the Lyapunov function $V(x) = x^\top P^{-1}x$ along the trajectory of the closed-loop system with the controller $u = YP^{-1}x$ satisfies, for all $x \in \mathbb{D}$,*

$$\dot{V}(x) \leq -wx^\top P^{-1}x + 2\kappa(x)\rho(x), \tag{12.19}$$

where

$$\kappa(x) := \left[x^\top Q_1 \|(x, Kx)\|^2 \cdots x^\top Q_n \|(x, Kx)\|^2\right], \tag{12.20}$$

Q_i is the ith column of P^{-1}, and the vector $\rho(x)$ is contained in the polytope

$$\mathcal{H} := \{\varrho : -\bar{h} \preceq \varrho \preceq \bar{h}\} \tag{12.21}$$

with

$$\bar{h} = \left[\bar{h}_1 \cdots \bar{h}_n\right]^\top = \left[\tfrac{\sqrt{m+n}L_1}{2} \cdots \tfrac{\sqrt{m+n}L_n}{2}\right]^\top.$$

The proof of Lemma 12.5.6 can be found in Appendix 12.8.3.

Denote the number of distinct vertices of $\mathcal{H}$ as ν and each vertex of $\mathcal{H}$ as h_k, $k = 1, \ldots, \nu$. Using SOS techniques (see Appendix 12.8.4 for background material) and the Positivstellensatz (Appendix 12.8.5), we have the following result.

Proposition 12.5.7 *Suppose that the controller $u = Kx$ renders the origin a locally asymptotically stable equilibrium for (12.1) with the Lyapunov function $V(x) = x^\top P^{-1}x$. Under Assumption 12.5.2, given a $c > 0$ such that $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\} \subseteq \mathbb{D}$, if there exist SOS polynomials s_{1k}, s_{2k} in x, $k = 1, \ldots, \nu$ such that*

$$-\left[s_{1k}(c - V(x)) + s_{2k}\left(-wx^\top P^{-1}x + 2\kappa(x)h_k\right) + x^\top x\right] \tag{12.22}$$

is SOS, where $\kappa(x)$ is defined as in (12.20) and h_k are the distinct vertices of the polytope $\mathcal{H}$ defined in (12.21), then Ω_c is an invariant subset of the RoA of the system $\dot{x} = f(x, Kx)$ relative to the equilibrium $x = 0$.

Proof According to [5, p. 87], the polytope $\mathcal{H}$ can be expressed as

$$\mathcal{H} = \left\{\varrho = \sum_{k=1}^{\nu} \lambda_k(x)h_k, \ \sum_{k=1}^{\nu} \lambda_k(x) = 1, \ \lambda_k(x) \geq 0\right\}$$

for any fixed $x \in \Omega_c$. Then, the derivative of the Lyapunov function satisfies that

$$\begin{aligned}\dot{V}(x) &\le -wx^\top P^{-1}x + 2\kappa(x)\rho(x)\\ &= -wx^\top P^{-1}x + 2\kappa(x)\sum_{k=1}^{\nu}\lambda_k(x)h_k\\ &= \sum_{k=1}^{\nu}\lambda_k(x)(-wx^\top P^{-1}x) + \sum_{k=1}^{\nu}\lambda_k(x)2\kappa(x)h_k\\ &= \sum_{k=1}^{\nu}\lambda_k(x)\left(-wx^\top P^{-1}x + 2\kappa(x)h_k\right).\end{aligned}$$

As $\lambda_k(x) \ge 0$ and $\sum_{k=1}^{\nu}\lambda_k(x) = 1$, if

$$-wx^\top P^{-1}x + 2\kappa(x)h_k < 0$$

holds for all $k = 1, \dots, \nu$, then $\dot{V}(x) < 0$.

By Lemma 12.8.4 in Appendix 12.8.5, for each $k = 1, \dots, \nu$, if there exist SOS polynomials s_{1k}, s_{2k} such that

$$-\left[s_{1k}(c - V(x)) + s_{2k}\left(-wx^\top P^{-1}x + 2\kappa(x)h_k\right) + x^\top x\right]$$

is SOS, then the set inclusion condition

$$\{x \in \mathbb{R}^n : V(x) \le c,\ x \ne 0\} \subseteq \{x \in \mathbb{R}^n : -wx^\top P^{-1}x + 2\kappa(x)h_k < 0\}$$

holds. This leads to the set inclusion condition

$$\{x \in \mathbb{R}^n : V(x) \le c,\ x \ne 0\} \subseteq \{x \in \mathbb{R}^n : \dot{V}(x) < 0\},$$

and hence Ω_c is an inner estimate of the RoA. □

12.5.2 Discrete-Time Systems

Similarly, for discrete-time systems satisfying Assumption 12.5.2, we can describe the difference of the Lyapunov functions of the closed-loop system with the designed data-driven controller.

Lemma 12.5.8 *Consider system (12.2) and the linear controller $u = YP^{-1}x$, where Y and P are designed to satisfy (12.13) and (12.14) with any given $w \in (0, 1)$. Under Assumption 12.5.2, the difference of the Lyapunov function $V(x) = x^\top P^{-1}x$ along the trajectory of the closed-loop system with the controller $u = YP^{-1}x$ satisfies that, for all $x \in \mathbb{D}$ and any $\varepsilon > 0$,*

$$V(x^+) - V(x) \le -x^\top\left(wP^{-1} - \varepsilon^{-1}r_1^2 I\right)x + \left(\varepsilon + \|P^{-1}\|\right)\widehat{\kappa}(x)\widehat{\rho}(x), \quad (12.23)$$

where

$$r_1 := \left\| -P^{-1}\overline{\boldsymbol{B}}^\top \overline{\boldsymbol{A}}^{-1} \begin{bmatrix} K \\ I \end{bmatrix} \right\| + \sqrt{\delta}\, \|P^{-1}\| \left\| \overline{\boldsymbol{A}}^{-1/2} \begin{bmatrix} K \\ I \end{bmatrix} \right\|, \tag{12.24}$$

$$\widehat{\kappa}(x) := \left[x^\top Q_1 \|(x, Kx)\|^2 \cdots x^\top Q_n \|(x, Kx)\|^2 \right], \tag{12.25}$$

Q_i is the ith column of P^{-1}, and the vector $\widehat{\rho}(x)$ is contained in the polytope

$$\widehat{\mathcal{H}} := \{\varrho : 0 \preceq \varrho \preceq \widehat{h}\} \tag{12.26}$$

with

$$\widehat{h} = \begin{bmatrix} \frac{(m+n)L_1^2}{4} & \dots & \frac{(m+n)L_n^2}{4} \end{bmatrix}^\top.$$

The proof of Lemma 12.5.8 can be found in Appendix 12.8.6.

Let $\widehat{\nu}$ denote the number of distinct vertices of $\widehat{\mathcal{H}}$, and let $\widehat{h}_k$, $k = 1, \dots, \widehat{\nu}$, denote a vertex of $\widehat{\mathcal{H}}$. It can be proved that if the set inclusion condition

$$\begin{aligned} &\{x \in \mathbb{R}^n : V(x) \le c,\ x \ne 0\} \\ \subseteq &\left\{ x \in \mathbb{R}^n : -x^\top \left(wP^{-1} - \varepsilon^{-1} r_1^2 I \right) x + \left(\varepsilon + \|P^{-1}\| \right) \widehat{\kappa}(x) \widehat{h}_k < 0 \right\} \end{aligned}$$

holds for $k = 1, \dots, \widehat{\nu}$, then the origin is a locally asymptotically stable equilibrium of the closed-loop system and Ω_c is an invariant subset of the RoA. Using Lemma 12.8.4, we derive a result similar to Proposition 12.5.7 for discrete-time systems.

Proposition 12.5.9 *Suppose that the $u = Kx$ renders the origin a locally asymptotically stable equilibrium for (12.2) with the Lyapunov function $V(x) = x^\top P^{-1} x$. Under Assumption 12.5.2, given a $c > 0$ such that $\Omega_c \subseteq \mathbb{D}$, if there exist SOS polynomials s_{1k}, s_{2k} in x, $k = 1, \dots, \nu$ such that*

$$-\left\{ s_{1k}(c - V(x)) + s_{2k} \left[-x^\top \left(wP^{-1} - \varepsilon^{-1} r_1^2 I \right) x + \left(\varepsilon + \|P^{-1}\| \right) \widehat{\kappa}(x) \widehat{h}_k \right] + x^\top x \right\}$$

is SOS, where $\widehat{\kappa}(x)$ is defined as in (12.25) and $\widehat{h}_k$ are the distinct vertices of polytope $\widehat{\mathcal{H}}$ defined in (12.26), then Ω_c is an invariant subset of the RoA of the system $x^+ = f(x, Kx)$ relative to the equilibrium $x = 0$.

Remark 12.5.10 (*Alternative methods for RoA estimation*) In [24], the RoA is estimated by a numerical method, i.e., a sufficient condition of $V(x^+) - V(x) < 0$ is found using data, and the grids in a compact region are tested to see whether the sufficient condition is satisfied so that the sublevel sets of $V(x)$ can be found as the RoA estimation. In this work, we derive SOS conditions for the RoA estimation that is an alternative to the mesh method used in [24].

Remark 12.5.11 (*Data-driven RoA estimation via high-order approximation*) When the data-driven controller is designed via high-order approximation, the RoA of the resulting closed-loop system can also be estimated using data. In the case of high-order approximation, an assumption on the high-order derivative similar to Assumption 12.5.2 is needed to characterize the time derivative or difference of the Lyapunov function via data. For details on the RoA estimation via high-order approximation, the reader is referred to our work [16].

12.6 Example

In this section, we illustrate the proposed nonlinear data-driven stabilizing design by presenting simulation results on a system with $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{R}^2$ and $u \in \mathbb{R}$. The true dynamics of the system is the inverted pendulum given in Appendix 12.8.7, on which an open-loop experiment is conducted to obtain the data matrices.

12.6.1 Continuous-Time Systems

To collect the data, an experiment is conducted with $x(0) = \begin{bmatrix} 0.01 & -0.01 \end{bmatrix}^\top$ and $u = 0.1\sin(t)$ during the time interval [0, 0.5]. The data is sampled with fixed sampling period $T_s = 0.05$s. We collect the data and arrange them into a data set with length $T = 10$.

We assume that the bound on $R(x, u)$ is over-approximated by 100%; in other words, the bound γ is twice the largest instantaneous norm of $R(x, u)$ during the experiment. Then, for the experimental data, Assumption 12.3.1 holds with

$$\gamma = 3.3352 \cdot 10^{-6}.$$

Setting $\delta = 0.01$, we first solve the optimization problem (12.8) to find $\overline{\mathcal{I}}$, and then apply Theorem 12.4.1 with $w = 1$. The solution found by CVX is

$$\begin{aligned} P &= 10^3 \cdot \begin{bmatrix} 1.0152 & -1.3289 \\ -1.3289 & 1.7727 \end{bmatrix}, \\ u &= -12.0432x_1 - 8.887x_2. \end{aligned} \tag{12.27}$$

To estimate the RoA for the closed-loop system with the designed data-driven controller, we need to first find L_1 and L_2 satisfying Assumption 12.5.2. From physical considerations we can argue that the x_1-subsystem is linear, which leads to $L_1 = 0$. As for L_2, by over-estimating the true bound we let $L_2 = 1.4697$.

By applying Proposition 12.5.7, the largest c found for the controller in (12.27) is $c^* = 7.58 \cdot 10^{-4}$.

Figure 12.1 illustrates the estimated RoA using different approaches. In particular, the light gray area is obtained by checking every point in an meshed area, the medium

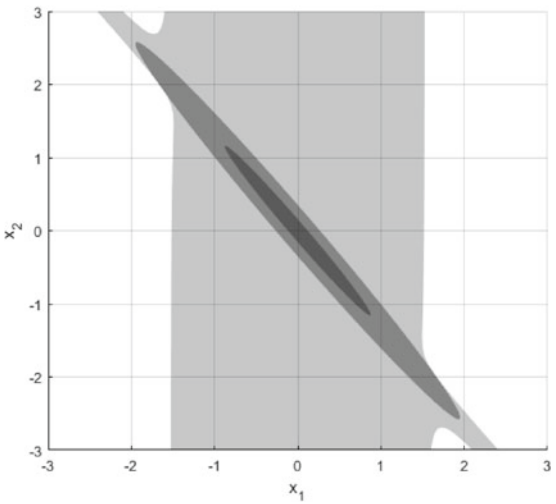

Fig. 12.1 Estimations of the RoA using the Lyapunov function by linear approximation. The light gray area is the estimated RoA by checking the sign of the derivative of the Lyapunov function for the grids in a compact region with explicit known dynamics; the medium gray area is the largest sublevel set of the Lyapunov function contained in the light gray estimated RoA; the dark gray area is the estimated RoA obtained by Proposition 12.5.7

gray area is the largest sublevel set of the obtained Lyapunov function contained in the light gray RoA, and the dark gray area is the estimated RoA found using Proposition 12.5.7.

12.6.2 Discrete-Time Systems

For the discrete-time system, we consider the Euler discretization of the true dynamics used for the continuous-time case with the sampling time $T_s = 0.1$. The true dynamics used for data generation can be found in Appendix 12.8.7. The initial condition and the input are randomly chosen in the interval $[-1, 1]$. In particular, the initial condition and input used in the experiment is $x(0) = \begin{bmatrix}-0.196 \; 0.0395\end{bmatrix}$ and

$$U_0 = [0.0724 \quad -0.0960 \quad 0.0720 \quad 0.0118 \quad -0.0194 \quad 0.0517 \quad 0.0434 \quad 0.0975 \quad -0.0444 \quad -0.0992].$$

Same as the continuous-time case, we collect and arrange the data into a data set with length $T = 10$. To obtain a reasonable estimation of the bound on $R(x, u)$ in Assumption 12.3.1, we compare the data generated by the true dynamics and the data generated by the linear approximation and over-estimate the difference by 100%. Then, Assumption 12.3.1 holds for this example with

$$\gamma^2 = 3.3646 \cdot 10^{-8}.$$

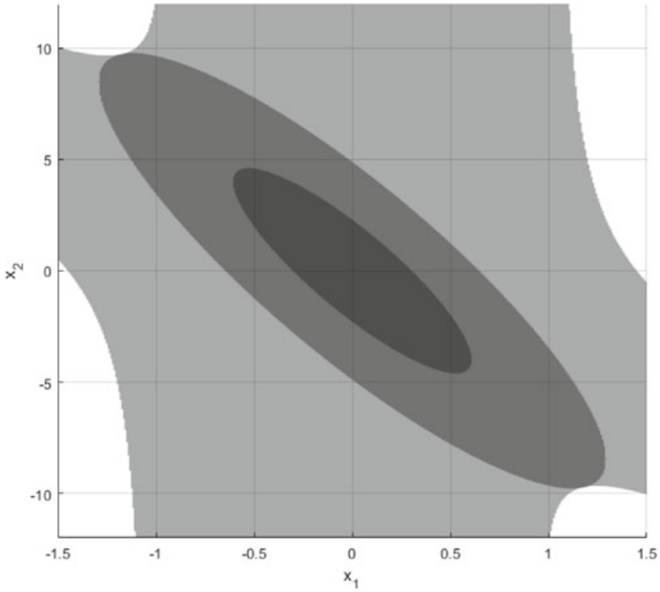

Fig. 12.2 Estimations of the RoA using the Lyapunov function by linear approximation. The light gray area is the estimated RoA by checking the sign of the derivative of the Lyapunov function for the grids in a compact region with explicit known dynamics; the medium gray area is the largest sublevel set of the Lyapunov function contained in the light gray estimated RoA; the dark gray area is the estimated RoA obtained by Proposition 12.5.9

The over-estimated bound is used to obtain the over-approximation of the feasible set by Proposition 12.3.3. Then, applying Theorem 12.4.2 with $\delta = 10^{-2}$ gives the solution

$$P = \begin{bmatrix} 5.9860 & -45.3511 \\ -45.3511 & 448.1767 \end{bmatrix},$$
$$u = -9.2787x_1 - 1.8095x_2.$$

To estimate the RoA of the closed-loop system under the designed controllers, we choose L_1 and L_2 satisfying Assumption 12.5.2 as $L_1 = 0$ and $L_2 = 2.1213$. These parameters are chosen by similar arguments made for the continuous-time case. Applying Proposition 12.5.9, the largest c found for the controller is $c^* = 2.3 \cdot 10^{-3}$.

To evaluate the RoA estimation, we plot in Fig. 12.2 the RoA estimation using Proposition 12.5.9 (darkest area), the estimated RoA by checking point-by-point of a mesh of initial conditions using explicit dynamics (light gray area), and the largest sublevel set of the Lyapunov function contained in the estimated RoA (medium dark area).

12.7 Summary

Without any model information on the unknown general nonlinear dynamics, this work proposes data-driven stabilizer designs and RoA analysis by approximating the unknown functions using linear approximation. Using finite-length input-state data, linear stabilizers are designed for both continuous-time and discrete-time systems that make the known equilibrium locally asymptotically stable. Then, by estimating a bound on the approximation error, data-driven conditions are given to find an invariant subset of the RoA. Simulation results on the inverted pendulum illustrate the designed data-driven controllers and the RoA estimations. Topics such as enlarging the RoA estimation and case studies on more complicated nonlinear benchmarks are all interesting directions to be considered in future works.

12.8 Appendix

12.8.1 Petersen's Lemma

In the section for data-driven controller design, Petersen's lemma is essential for deriving the sufficient condition characterizing the controller. Due to the space limit, the proof of the lemma is omitted and one may refer to works such as [4, 25, 26] for more details.

Lemma 12.8.1 (Petersen' s lemma [25]) *Consider matrices* $\mathcal{G} = \mathcal{G}^\top \in \mathbb{R}^{n\times n}$, $\mathcal{M} \in \mathbb{R}^{n\times m}$, $\mathcal{M} \neq 0$, $\mathcal{N} \in \mathbb{R}^{p\times n}$, $\mathcal{N} \neq 0$, *and a set* F *defined as*

$$F = \{\mathcal{F} \in \mathbb{R}^{m\times p} : \mathcal{F}^\top \mathcal{F} \preceq \overline{\mathcal{F}}\},$$

where $\overline{\mathcal{F}} = \overline{\mathcal{F}}^\top \succeq 0$. *Then, for all* $\mathcal{F} \in F$,

$$\mathcal{G} + \mathcal{M}\mathcal{F}\mathcal{N} + \mathcal{N}^\top \mathcal{F}^\top \mathcal{M}^\top \preceq 0$$

if and only if there exists $\mu > 0$ *such that*

$$\mathcal{G} + \mu \mathcal{M}\mathcal{M}^\top + \mu^{-1}\mathcal{N}^\top \overline{\mathcal{F}}\mathcal{N} \preceq 0.$$

12.8.2 Proof of Lemma 12.5.3

Let $z = \begin{bmatrix} x^\top & u^\top \end{bmatrix}^\top$. Each element $f_i(z)$, $i = 1, \ldots, n$ in $f(z)$ can be written as

$$f_i(z) = \sum_{j=1}^{n+m} \frac{\partial f_i}{\partial z_j}(0) z_j + R_i(z).$$

On the other hand, as shown in [12], the function $f_i(z)$ can be expressed as

$$f_i(z) = \sum_{j=1}^{n+m} z_j \int_0^1 \frac{\partial f_i}{\partial z_j}(tz) dt.$$

As a consequence, one can write $R_i(z)$ as

$$\begin{aligned} R_i(z) &= \sum_{j=1}^{n+m} z_j \int_0^1 \frac{\partial f_i}{\partial z_j}(tz) dt - \sum_{j=1}^{n+m} \frac{\partial f_i}{\partial z_j}(0) z_j \\ &= \sum_{j=1}^{n+m} z_j \int_0^1 \left(\frac{\partial f_i}{\partial z_j}(tz) - \frac{\partial f_i}{\partial z_j}(0) \right) dt. \end{aligned}$$

Under Assumption 12.5.2, one has

$$\left| \frac{\partial f_i}{\partial z_j}(tz) - \frac{\partial f_i}{\partial z_j}(0) \right| \le L_i \|tz\|, \quad t \in (0, 1).$$

Then, it holds that

$$\begin{aligned} |R_i(z)| &\le \sum_{j=1}^{n+m} |z_j| \int_0^1 L_i \|z\| \cdot |t| dt \\ &= L_i \|z\| \int_0^1 |t| dt \cdot \sum_{j=1}^{n+m} |z_j|. \end{aligned}$$

By the fact that $\int_0^1 |t| dt = \frac{1}{2}$ and $|z_1 + \cdots z_{n+m}| \le \sqrt{n+m} \|z\|$, it holds that

$$|R_i(z)| \le \frac{\sqrt{n+m} L_i}{2} \|z\|^2.$$

The proof is complete.

12.8.3 *Proof of Lemma 12.5.6*

For the closed-loop system with the controller $u = Kx$ designed via Theorem 12.4.1, the derivative of the Lyapunov function $V(x) = x^\top P^{-1} x$ satisfies

$$\begin{aligned}\dot{V}(x) &= x^\top P^{-1}(A+BK)x + x^\top (A+BK)^\top P^{-1}x + 2x^\top P^{-1}R(x, Kx)\\ &\le -wx^\top P^{-1}x + 2x^\top P^{-1}R(x, Kx).\end{aligned}$$

Under Assumption 12.5.2, for all $x \in \mathbb{D}$ and $i = 1, \ldots, n$, the bounds on the approximation error can be found as

$$|R_i(x, Kx)| \le \frac{\sqrt{m+n}L_i}{2}\|(x, Kx)\|^2.$$

Hence, for all $x \in \mathbb{D}$, there exists a continuous $\rho_i(x)$ for each $i = 1, \ldots, n$ such that for each $x \in \mathbb{D}$

$$\begin{aligned}R_i(x, Kx) &= \rho_i(x)\|(x, Kx)\|^2,\\ \rho_i(x) &\in \left[-\frac{\sqrt{m+n}L_i}{2}, \frac{\sqrt{m+n}L_i}{2}\right].\end{aligned}$$

Define $\rho(x) = [\rho_1(x) \;\ldots\; \rho_n(x)]^\top$. By the definition of polytope [5, Definition 3.21], the vector $\rho(x)$ belongs to the polytope

$$\mathcal{H} = \{\varrho : -\bar{h} \preceq \varrho \preceq \bar{h}\},$$

where

$$\bar{h} = [\bar{h}_1 \;\cdots\; \bar{h}_n]^\top = \left[\tfrac{\sqrt{m+n}L_1}{2} \;\ldots\; \tfrac{\sqrt{m+n}L_n}{2}\right]^\top.$$

Denote Q_i as the ith column of P^{-1}. It holds that

$$\begin{aligned}&2x^\top P^{-1}R(x, Kx)\\ &= 2\left[x^\top Q_1 \cdots x^\top Q_n\right]\begin{bmatrix}\rho_1(x)\|(x, Kx)\|^2\\ \vdots\\ \rho_n(x)\|(x, Kx)\|^2\end{bmatrix}\\ &= 2\sum_{i=1}^{n} x^\top Q_i \rho_i(x)\|(x, Kx)\|^2\\ &= 2\sum_{i=1}^{n} x^\top Q_i\|(x, Kx)\|^2 \cdot \rho_i(x)\\ &= 2\left[x^\top Q_1\|(x, Kx)\|^2 \cdots x^\top Q_n\|(x, Kx)\|^2\right]\rho(x).\end{aligned}$$

Denote

$$\kappa(x) = \left[x^\top Q_1\|(x, Kx)\|^2 \cdots x^\top Q_n\|(x, Kx)\|^2\right].$$

Then, the derivative of the Lyapunov function satisfies for all $x \in \mathbb{D}$

$$\dot{V}(x) \leq -wx^\top P^{-1}x + 2\kappa(x)\rho(x),$$

where $\rho(x) \in \mathcal{H}$.

12.8.4 Sum of Squares Relaxation

As solving positive conditions of multivariable polynomials is in general NP-hard, the SOS relaxations are often used to obtain sufficient conditions that are tractable. The SOS polynomial matrices are defined as follows.

Definition 12.8.2 (*SOS polynomial matrix* [8]) $M : \mathbb{R}^n \to \mathbb{R}^{\sigma\times\sigma}$ is an SOS polynomial matrix if there exist $M_1, \ldots, M_k : \mathbb{R}^n \to \mathbb{R}^{\sigma\times\sigma}$ such that

$$M(x) = \sum_{i=1}^{k} M_i(x)^\top M_i(x) \;\; \forall x \in \mathbb{R}^n. \tag{12.28}$$

Note that when $\sigma = 1$, $M(x)$ becomes a scalar SOS polynomial.

It is straightforward to see that if a matrix $M(x)$ is an SOS polynomial matrix, then it is positive semi-definite, i.e., $M(x) \succeq 0\, \forall x \in \mathbb{R}^n$. Relaxing the positive polynomial conditions into SOS polynomial conditions makes the conditions tractable and easily solvable by common software.

12.8.5 Positivstellensatz

In the RoA analysis, we need to characterize polynomials that are positive on a semialgebraic set, and the Positivstellensatz plays an important role in this characterization.

Let $p_1, \ldots, p_k$ be polynomials. The multiplicative monoid, denoted by $\mathcal{S}_M(p_1, \ldots, p_k)$, is the set generated by taking finite products of the polynomials $p_1, \ldots, p_k$. The cone $\mathcal{S}_C(p_1, \ldots, p_k)$ generated by the polynomials is defined as

$$\begin{aligned} &\mathcal{S}_C(p_1, \ldots, p_k) \\ &= \{s_0 + \sum_{i=1}^{j} s_i q_i : s_0, \ldots, s_j \text{ are SOS polynomials}, q_1, \ldots, q_j \in \mathcal{S}_M(p_1, \ldots, p_k)\}. \end{aligned}$$

The ideal $\mathcal{S}_I(p_1, \ldots, p_k)$ generated by the polynomials is defined as

$$\mathcal{S}_I(p_1, \ldots, p_k) = \left\{ \sum_{i=1}^{k} r_i p_i : r_1, \ldots, r_k \text{ are polynomials} \right\}.$$

Stengle's Positivstellensatz [28] is presented as follows in [8].

Theorem 12.8.3 (Positivstellensatz) *Let $f_1, \ldots, f_k$, $g_1, \ldots, g_l$, and $h_1, \ldots, h_m$ be polynomials. Define the set*

$$\begin{aligned}\mathcal{X} = \{x \in \mathbb{R}^n :\ & f_1(x) \geq 0, \ldots, f_k(x) \geq 0,\\ & g_1(x) = 0, \ldots, g_l(x) = 0,\\ & \text{and } h_1(x) \neq 0, \ldots, h_m(x) \neq 0\}.\end{aligned}$$

Then, $\mathcal{X} = \emptyset$ if and only if

$$\exists f \in \mathcal{S}_C(f_1, \ldots, f_k), \quad g \in \mathcal{S}_I(g_1, \ldots, g_l), \quad h \in \mathcal{S}_M(h_1, \ldots, h_m)$$

such that

$$f(x) + g(x) + h(x)^2 = 0.$$

For the subsequent RoA analysis, we will use the following result derived from the Positivstellensatz.

Lemma 12.8.4 *Let φ_1 and φ_2 be polynomials in x. If there exist SOS polynomials s_1 and s_2 in x such that*

$$-\left(s_1\varphi_1(x) + s_2\varphi_2(x) + x^\top x\right) \text{ is SOS } \forall x \in \mathbb{R}^n \tag{12.29}$$

then the set inclusion condition

$$\{x \in \mathbb{R}^n : \varphi_1(x) \geq 0, x \neq 0\} \subseteq \{x \in \mathbb{R}^n : \varphi_2(x) < 0\} \tag{12.30}$$

holds.

Proof The set inclusion condition (12.30) can be equivalently written as

$$\{x \in \mathbb{R}^n : \varphi_1(x) \geq 0, \varphi_2(x) \geq 0, x \neq 0\} = \emptyset.$$

By Theorem 12.8.3, we know that this is true if and only if there exist $\varphi(x) \in \mathcal{S}_C(\varphi_1, \varphi_2)$ and $\zeta(x) \in \mathcal{S}_M(x)$, such that

$$\varphi(x) + \zeta(x)^2 = 0. \tag{12.31}$$

Let

$$\varphi = s_0 + s_1\varphi_1 + s_2\varphi_2,$$

where s_j, $j = 0, 1, 2$ are SOS polynomials. By the definition of the cone $\mathcal{S}_C$, one has that $\varphi \in \mathcal{S}_C(\varphi_1, \varphi_2)$. Choosing $\zeta(x)^2 = x^\mathsf{T} x$, we write the condition (12.31) as

$$s_0 + s_1\varphi_1 + s_2\varphi_2 + x^\mathsf{T} x = 0. \tag{12.32}$$

As $s_0 = -(s_1\varphi_1 + s_2\varphi_2 + x^\mathsf{T} x)$ from (12.32), if there exist SOS polynomials s_1 and s_2 such that the SOS condition (12.29) holds, then there exist SOS polynomials s_j, $j = 0, 1, 2$ such that (12.32) is true, and hence the set inclusion condition (12.30) holds. □

12.8.6 Proof of Lemma 12.5.8

For the closed-loop system with the controller $u = Kx$ designed via Theorem 12.4.2, the difference between the Lyapunov functions $V(x^+) = (x^+)^\top P^{-1} x^+$ and $V(x) = x^\top P^{-1} x$ is

$$\begin{aligned}
&V(x^+) - V(x)\\
&= [(A + BK)x + R(x, Kx)]^\top P^{-1}[(A + BK)x + R(x, Kx)] - x^\top P^{-1} x\\
&= x^\top \left[(A + BK)^\top P^{-1}(A + BK) - P^{-1}\right] x + 2R(x, Kx)^\top P^{-1}(A + BK)x\\
&\quad + R(x, Kx)^\top P^{-1} R(x, Kx).
\end{aligned}$$

Observe that

$$A + BK = \left(-\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}} + \overline{\mathbf{A}}^{-1/2}\Delta\right)^\top \begin{bmatrix} K \\ I \end{bmatrix}$$

with $\Delta\Delta^\top \preceq \delta I$. Then, it holds that

$$\left\|P^{-1}(A + BK)\right\| \leq \left\|-P^{-1}\overline{\mathbf{B}}^\top\overline{\mathbf{A}}^{-1}\begin{bmatrix} K \\ I \end{bmatrix}\right\| + \sqrt{\delta}\left\|P^{-1}\right\|\left\|\overline{\mathbf{A}}^{-1/2}\begin{bmatrix} K \\ I \end{bmatrix}\right\| = r_1.$$

For any $\varepsilon > 0$, it holds that

$$\begin{aligned}
&2R(x, Kx)^\top P^{-1}(A + BK)x\\
&\leq \varepsilon R(x, Kx)^\top R(x, Kx) + \varepsilon^{-1}\left\|P^{-1}(A + BK)\right\|^2 x^\top x\\
&\leq \varepsilon \|R(x, Kx)\|^2 + \varepsilon^{-1} r_1^2 x^\top x.
\end{aligned}$$

Recall that, by Theorem 12.4.2, $(A + BK)^\top P^{-1}(A + BK) - P^{-1} \preceq -wP^{-1}$. Hence, one has that

$$V(x^+) - V(x) = -x^\top\left(wP^{-1} - \varepsilon^{-1} r_1^2 I\right)x + \left(\varepsilon + \|P^{-1}\|\right)\|R(x, Kx)\|^2.$$

Under Assumption 12.5.2,

$$\|R(x, Kx)\|^2 = \sum_{i=1}^{n} R_i(x, Kx)^2 \leq \sum_{i=1}^{n} \frac{(m+n)L_i^2}{4} \|(x, Kx)\|^4.$$

If we write $\|R(x, Kx)\|^2$ as

$$\|R(x, Kx)\|^2 = \begin{bmatrix} \|(x, Kx)\|^4 & \cdots & \|(x, Kx)\|^4 \end{bmatrix} \begin{bmatrix} \widehat{\rho}_1(x) \\ \vdots \\ \widehat{\rho}_n(x) \end{bmatrix},$$

then the scalars $\widehat{\rho}_i(x)$ are such that $\widehat{\rho}_i(x) \in \left[0, \frac{(m+n)L_i^2}{4}\right]$, $i = 1, \ldots, n$ for all $x \in \mathbb{D}$. Defining

$$\widehat{\kappa}(x) = \begin{bmatrix} \|(x, Kx)\|^4 & \cdots & \|(x, Kx)\|^4 \end{bmatrix} \quad \text{and} \quad \widehat{\rho}(x) = \begin{bmatrix} \widehat{\rho}_1(x) & \cdots & \widehat{\rho}_n(x) \end{bmatrix}^\top$$

gives $\|R(x, Kx)\|^2 = \widehat{\kappa}(x)\widehat{\rho}(x)$, and for any $x \in \mathbb{D}$ the vector $\widehat{\rho}(x)$ is contained in the polytope $\widehat{\mathcal{H}}$ defined in (12.26).

12.8.7 Dynamics Used for Data Generation in the Example

The dynamics used for data generation in Sect. 12.6 is the inverted pendulum written as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{mgl}{J}\sin(x_1) - \frac{r}{J}x_2 + \frac{l}{J}\cos(x_1)u, \end{aligned} \tag{12.33}$$

where $m = 0.1$, $g = 9.8$, $r = l = J = 1$. For the discrete-time case, consider the Euler discretization of the inverted pendulum, i.e.,

$$\begin{aligned} x_1^+ &= x_1 + T_s x_2, \\ x_2^+ &= \frac{T_s g}{l}\sin(x_1) + \left(1 - \frac{T_s r}{ml^2}\right) x_2 + \frac{T_s}{ml^2}\cos(x_1)u, \end{aligned}$$

where $m = 0.1$, $g = 9.8$, $T_s = 0.1$, $l = 1$, and $r = 0.01$.

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