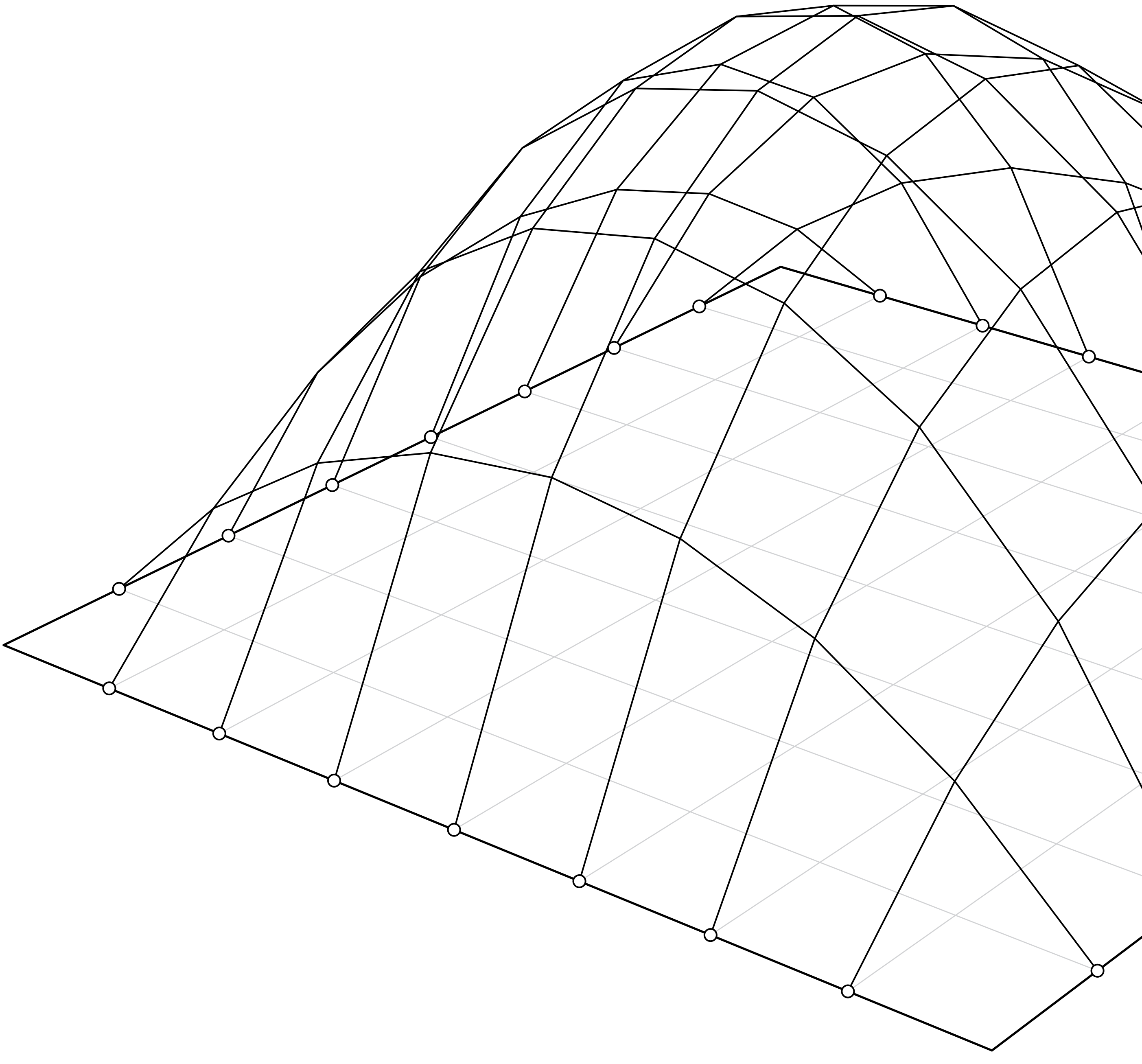
Optimizing shell structures

By calculating the minimum complementary energy



Graduation thesis
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Part I
Introduction

1 Introduction

1.1 Problem statement

The design of funicular structures, or compression only structures, has been an inspiration for many architects and structural engineers.

The most efficient way to transfer loads is through axial forces instead of by bending. Funicular systems act solely in compression or tension for a given loading (Block, 2009), see figure 1.1.

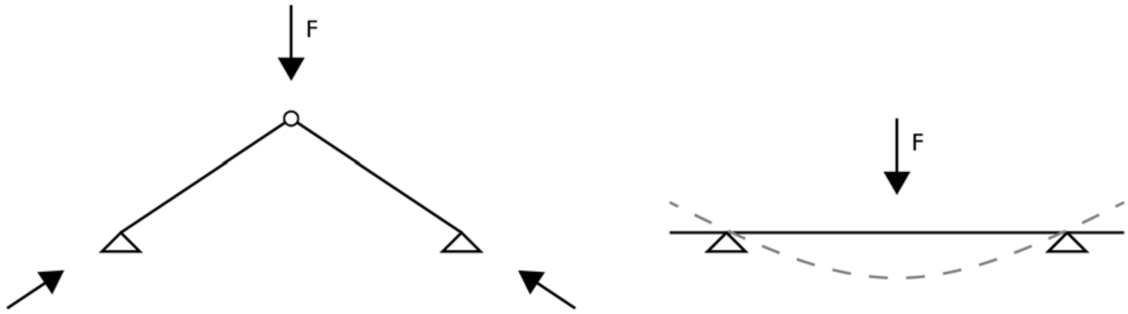


Figure 1.1 Load transfer by axial forces versus bending moments

The beginning of the research of shell structures can be seen in the catenary. A famous person who was working a lot with catenaries was Antoni Gaudí. He made different models of hanging chains, and after gluing the chains he turned the upside down and the result is a stable catenary shaped construction, see figure 1.2.

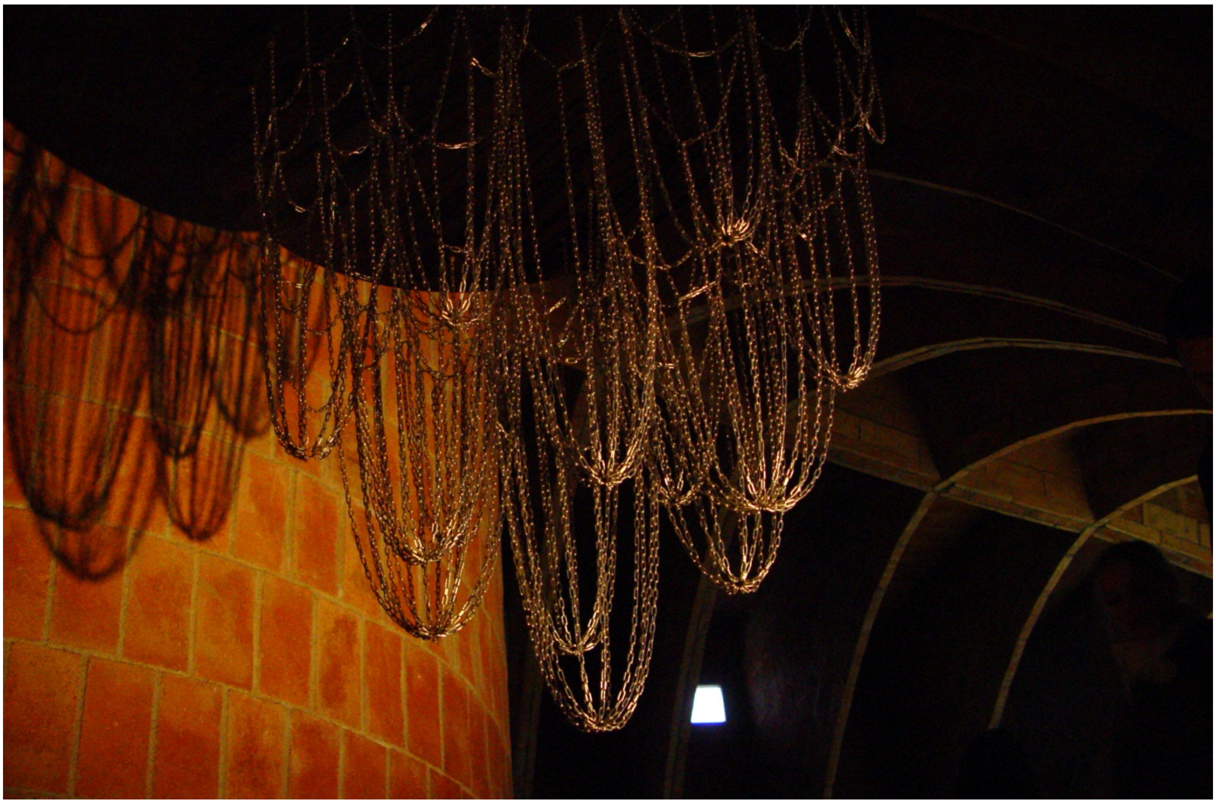


Figure 1.2 Gaudi's catenary model at Casa Milà; Etan J. Tal, Wikipedia.org

From the catenary structure the step to a shell structure is only a small step. Shell structures can be seen as catenaries with infinite small members, see figure 1.3.

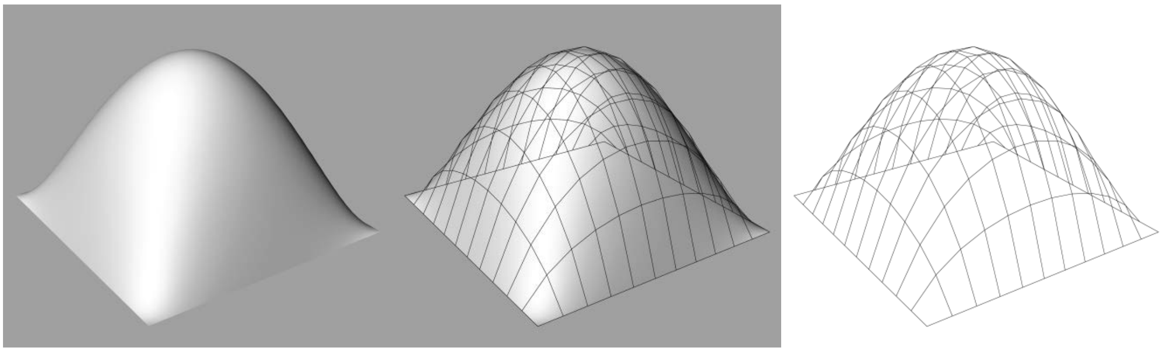


Figure 1.3 From shell structure to the funicular structure

With more than 1,400 planned and realized shell structures, Heinz Isler (1926 – 2009) is considered the world's largest shell builder, for an example see figure 1.4.

A shell structure has a great benefit; it is possible to build a very slender shell, which is strong enough to carry the dead weight including other imposed loads.

Because the forces in a shell structure are mainly axial forces, the structure can be dimensioned much more slender than non-funicular structures of the same size which are designed on bending.

In a shell structure shape follows force.

Figure 1.4 Heinz Isler, 1968: Laboratory and research facility for the Gips Union; panoramio.com

To calculate and design a shell structure, there are many variables. The most existing calculation methods are like a black box with a certain outcome, or are too complicated to use as a design tool.

So this thesis describes a methodology for generating shells with the least complementary energy on a faster way, compared with the traditional methods. In this case only with vertical loads.

There are methods to solve such a problems, but in my opinion it must be possible to do it on a smarter en faster way.

1.2 Goal of this thesis

The goal of this graduation project is to develop an easy tool that calculates the optimum shell structure loaded by forces in the direction of gravity.

Besides an introduction about the mechanics of shell structures, a description of each calculation step will be given to show the functionality of this tool. This tool makes it possible (with the input of loadings on shells), to calculate the optimum height of the structure.

For visualization purpose Grasshopper has been used while changing loads and shape of the structure is very easy using Grasshopper as a graphic part.

From the previous discussions the following assignment can be formulated:

Design a tool to optimize the shape of a shell with a minimum of complementary energy.

This assignment will serve as a guide for this thesis. To develop the tool, a good background knowledge of the calculation methods is needed. This will be the first part to investigate.

1.3 Thesis outline

This thesis comprised from four parts:

Part I Introduction

This part introduces the thesis with some background information. It also introduces the goal of the research.

Part II Mechanics

This part contains the research part about the calculation methods. Basically it contains two parts. First there is a theoretical framework with more back ground information and the basics of the calculations.

The second part contains the actual research from the mechanics. First there is explanation about the complementary energy and how it can be calculated. After that, the calculation of the height in the shell structures will be explained.

Part III Informatics

The informatics part contains the explanation of the informatics part of the designed tool. It gives a step by step overview of the calculation and the working of the tool itself.

Part IV Conclusions

In this part there is the overall conclusion of the thesis and some recommendations for the future.

Part II
Mechanics

2 Theoretical framework

In this chapter there is more background information about the theoretical framework of the research. In four paragraphs there is a closer look on graphic statics, how to calculate the complementary energy and what it is, the thrust network method and the calculation of the force density .

2.1 Graphic statics

In this paragraph there is more information about graphic statics; what it is and what is possible with this method.

2.1.1 Force polygons

Forces acting on a structure could be drawn in a force diagram, this is called graphic statics. The most well-known aspect of graphic statics is the head-to-tail method. Changing the force diagram or the force polygons, will directly effect on another.

The external forces on each structure are plotted to a scale of length to force on a load line. Working from the load line, the forces in the members of the structure are determined by scaling the lengths of lines constructed parallel to the members. The diagram of forces that results from this process is called the force polygon. For an example see figure 2.1.

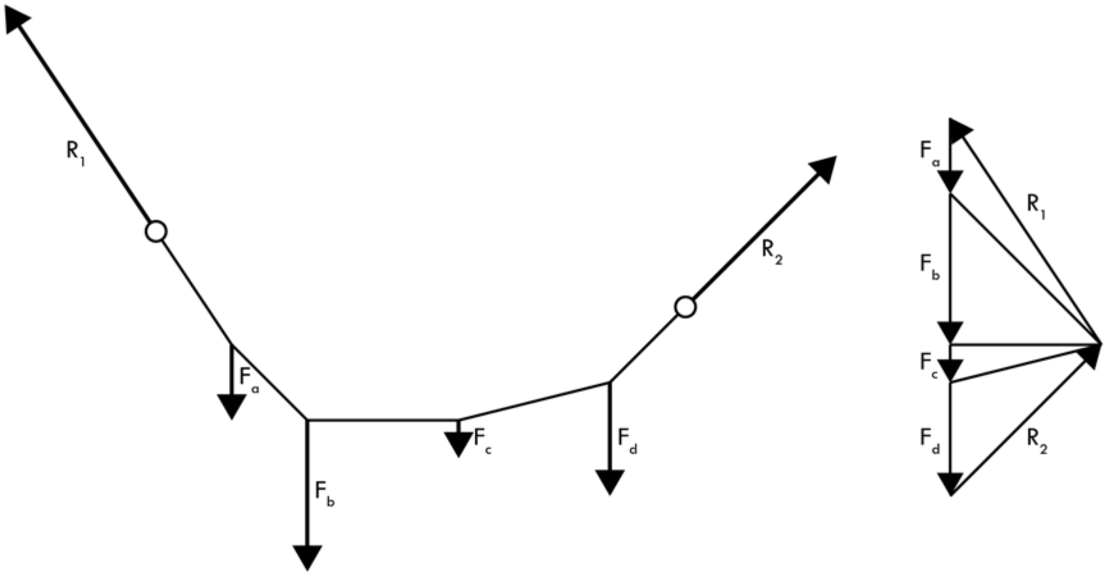


Figure 2.1 Force polygon in a simple catenary

When three forces are acting on a point; the magnitude of the forces can be graphically determined when the directions of the forces and the magnitude of one of the forces is known. We can use the head-to-tail method using a scaled vector diagram. This is a statically determinate situation.

When there are more than 3 forces acting on a point, the situation will be statically indeterminate; there are may be infinite solutions to determine, see figure 2.2. In this case it is possible to draw infinite possible polygons. To solve a statically indeterminate problem, it can be analysed by the direct stiffness method, the flexibility method, the finite element method or by using complementary energy. (Borgart & Liem, Force network analysis using complementary energy, 2011)

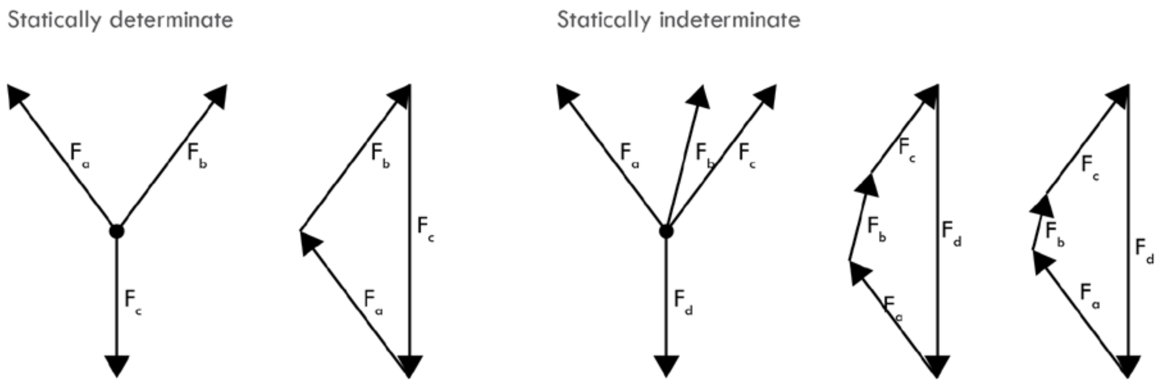


Figure 2.2 Difference in statically determinate and indeterminate structures

2.1.2 Thrust lines

When a force polygon is drawn, also the thrust line of a certain beam loaded by an uniform distributed load can be drawn.

By reducing the uniform distributed load (q) to point loads (F_a – F_d) on the beam, a beam loaded by point loads is drawn, see figure 2.3.

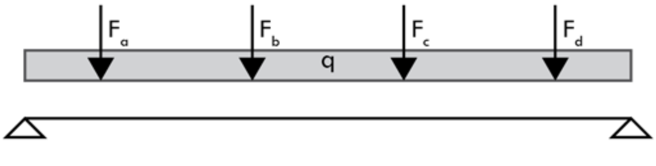


Figure 2.3 Beam loaded by an uniform distributed load, converted to point loads

From those point loads a force polygon can be made, see for example figure 2.4.

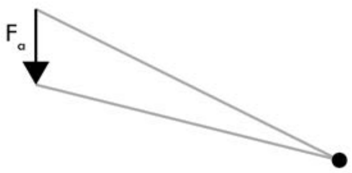


Figure 2.4 Force polygon

When the forces of the point loads are added together, with the use of the head-to-tail method, the result is a polyline of the resulting force. Then the polar coordinate has to be added. This is a point outside the resulting force polyline, to which coordinate the start and the end of the individual forces can be connected. So the result is a collection of connected force polygons, see figure 2.5.

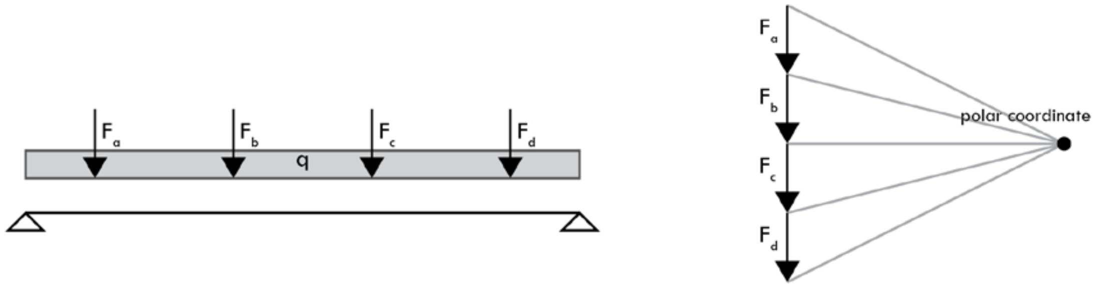


Figure 2.5 Beam with a corresponding force polygon

With the slope of the lines from the polar coordinate to the start point and the end point of the forces we can describe the thrust line of the beam, see figure 2.6.

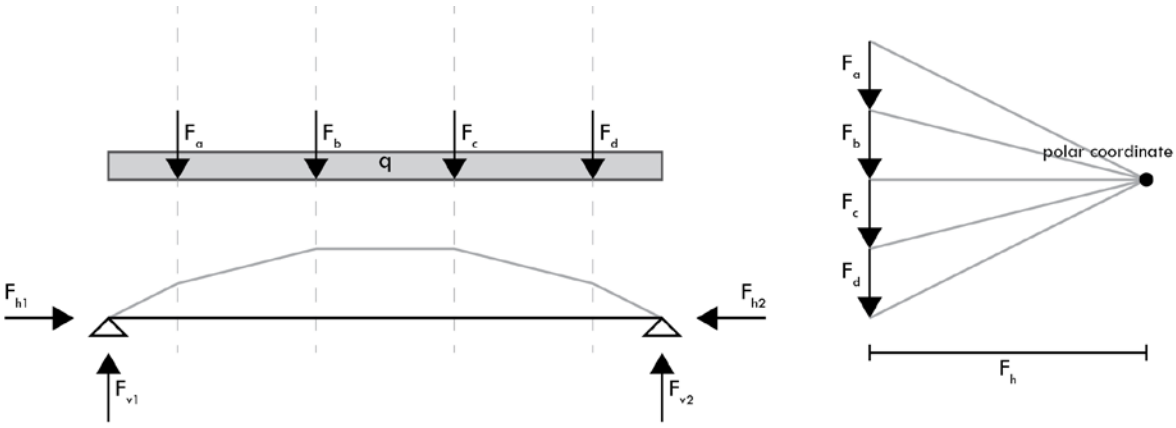


Figure 2.6 Beam with the thrust line derived from the force polygon

The horizontal distance from the polar coordinate to the forces is the horizontal reaction force on the supporting points; $F_{h1} = F_{h2} = F_h$, see figure 2.6. Figure 2.6 gives also the vertical equilibrium between the loads by a span of l:

$$F_a + F_b + F_c + F_d = F_{v1} + F_{v2} = ql$$

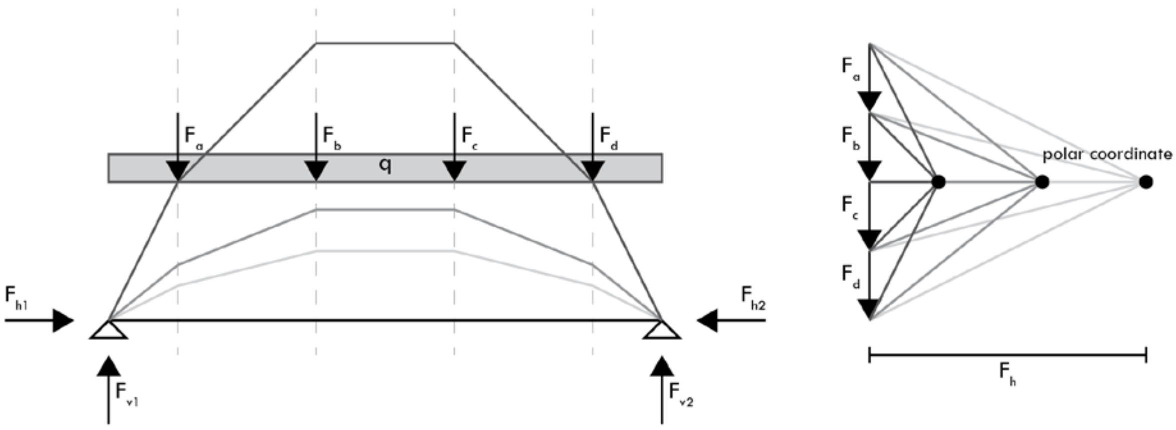


Figure 2.7 Different thrust line by the corresponding force polygons

When the distance of the polar coordinate changes, the thrust line also changes, see figure 2.7. By a higher thrust line the horizontal reaction force is also lower.

2.1.3 Summary

The graphic static method is very powerful for defining the thrust lines for arches or hanging cables. The only disadvantage of this method is that it is not applicable for three dimensional systems, only for two dimensional systems.

2.2 Complementary energy

As result of forces acting on a bar, the bar will deform by elongating or shortening.

When the material behaves linear elastically, Hooke's law will apply. The stress level will be proportional to the elongation. So, when the stress increases the elongation will also increase.

The potential energy accumulated in an elastic body is called strain energy. In figure 2.8 the stress (σ) and elongation (ε) are plotted. The area below the stress-strain curve is the strain energy (E_v), the area above the curve is the complementary energy (E_c), see figure 2.8. (Blaauwendraad, 2004)

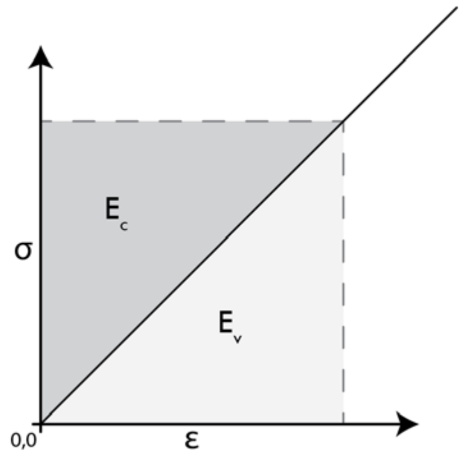


Figure 2.8 Elongation as a function of the stress (Blaauwendraad, 2004)

So in this case:

$$E_{compl} = \frac{1}{2}\sigma\varepsilon = E_v$$

Since

$$\varepsilon = \frac{\sigma}{E}$$

And since

$$\sigma = \frac{N}{A}$$

The complementary energy per unit bar length equals to:

$$E_{compl,} = \int_v \frac{1}{2}\frac{\sigma^2}{E}dV = \frac{1}{2}\frac{N^2}{EA}$$

The complementary energy can be expressed in stresses:

$$E_{compl,} = \frac{1}{2}\frac{\sigma^2}{E}$$

When a bar with length l is used:

$$E_{compl,} = \frac{1}{2}\frac{\sigma^2}{E} l$$

When the total complementary energy of a polygon is calculated, the following formula can be used:

$$E_{compl,tot} = \sum_{i=1}^{n} F_i^2 l_i$$

To solve the statically indeterminate problem in figure 2.9, the solution with the least complementary energy has to be found. So:

$$E_{compl,tot} = \sum_{i=1}^{n} F_i^2 l_i = minimum$$

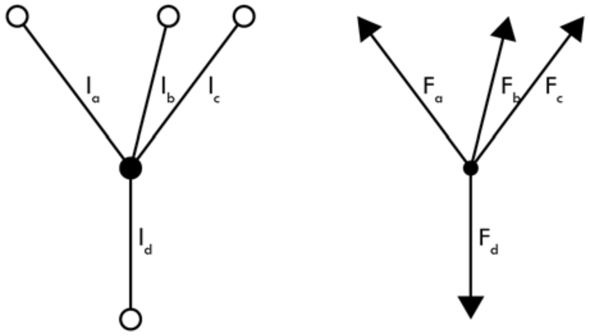


Figure 2.9 Example of a statically indeterminate structure with the corresponding forces

Next the equation below can be derived:

$$EC = F_a^2 l_a + F_b^2 l_b + F_c^2 l_c + F_d^2 l_d$$

For this case it can be said that the member lengths are known, and that F_d is also known. So the normal forces acting on the bars *a*, *b* and *c* are unknown.

Now every possible solution can be filled in and the complementary energy can be calculated. A powerful method to solve such a problem is to use a Generalized Reduced Gradient (GRG). (Borgart & Liem, Force network analysis using complementary energy, 2011) For example Microsoft Excel has a solver function that can use this method to solve such problems.

For this thesis this formula has been leading:

$$E_{compl,tot} = \sum_{i=1}^{n} F_i^2 l_i$$

2.3 Thrust network analysis

Thrust network analysis is a three dimensional version of the thrust line analysis, using reciprocal diagrams. Like graphic statics, thrust network analysis uses a graphical representation of the forces in a system, using force polygons. Reciprocal figures are introduced to relate the geometry of the three-dimensional systems to their internal forces. (Block, 2009)

A planar projection of the surface is necessary to describe the force network; this is the reciprocal diagram of the force polygon. Because it is a planar projection, the primal grid contains only the x and y coordinates of the nodes, see figure 2.10.

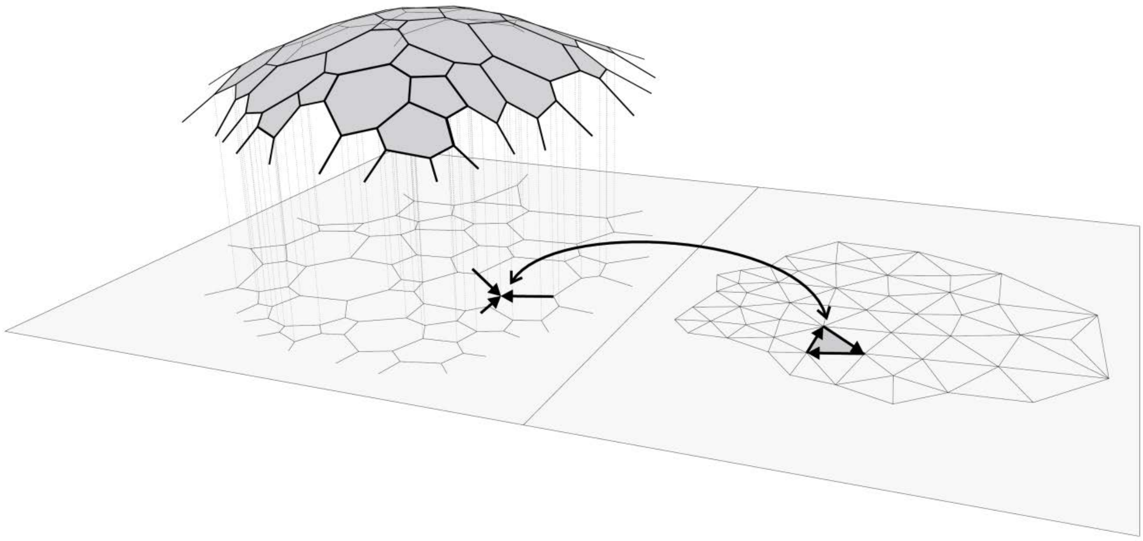


Figure 2.10 Thrust network analysis considers a planar projection of a structure (P. Block, 2009)

The forces acting on a node in the primal grid are described as a closed force polygon in the reciprocal figure and vice versa, see figure 2.11.

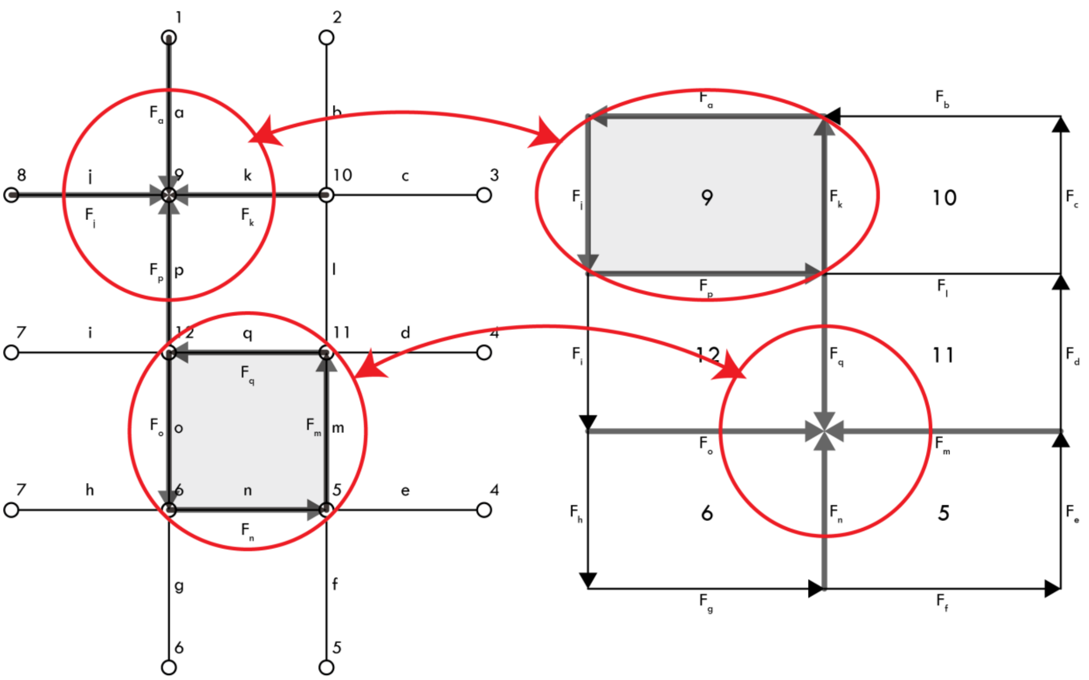


Figure 2.11 Relation of the primal grid and the reciprocal figure

When several force polygons are constructed for one statically indeterminate system, they are usually not directly comparable. If one solution gives equal forces for all four members in a four-valent system, and a second solution appoints a larger force to one of the members, the forces in the other members have to become smaller, in to get a solution for the same load in the polygons.

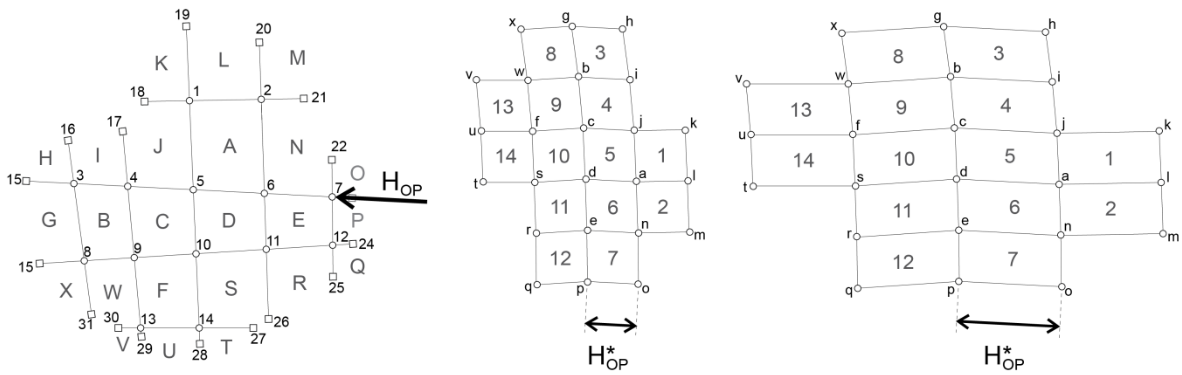


Figure 2.12 For an indeterminate primal grid (left) multiple reciprocal grids coresponding to different internal distribution of the horizontal forces are possible (P. Block, 2009)

In figure 2.12, a four-valent system with a vertical load is considered. Two force polygons are constructed to describe the horizontal components of the axial forces in the members.

Since the dimensions of the system are known, using the horizontal forces, the support reactions and the magnitude of the vertical force can be determined.

For the two solutions of horizontal force equilibrium, a different vertical force is achieved. The two force polygons are therefore no solutions for the same problem, and cannot be compared directly.

By changing the scale of the reciprocal figure, the height of the structure changes accordingly. There is a relation in the horizontal support forces and the height of the structure, as in the two dimensional graphic statics. The different horizontal supporting forces provide different heights of the structure, see figure 2.13.

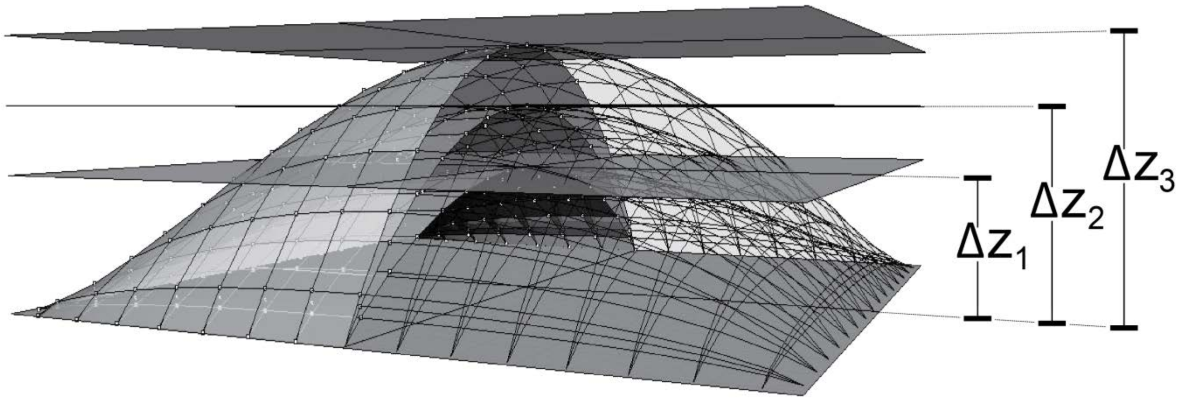


Figure 2.13 Effect of changing the scale factor of the reciprocal grid (P. Block, 2009)

2.4 Force density method

The force density method is a powerful method for form finding for shells and membranes.

The force density method was introduced by J.H. Shek in 1974. This method is commonly used in engineering to find the equilibrium shape of a structure consisting of a network of cables with different elasticity properties when stress is applied. (Southern, 2011)

While shape analysis of tensile structures is a geometrically non–linear problem, the force density method linearizes the form-fitting equations analytically, by using the force density ratio for each cable element. The method uses $q = F/l$, where F and l are the force and length of a cable element in the network. (Yang, Southern, & Zhang, 2009)

The network consists of a number of points (nodes) which are connected by lines (bars), see figure 2.14.

There are two types of nodes: fixed nodes, and free nodes. For the fixed nodes, the x, y and z coordinates are known. For the free nodes, the x, y and z nodes are unknown. The x, y and z components of the external forces acting on each node are also known.

In figure 2.14 a projection of a network can be seen. The nodes 1-6 are the fixed nodes; these nodes act as support nodes. The nodes 7 and 8 are free nodes.

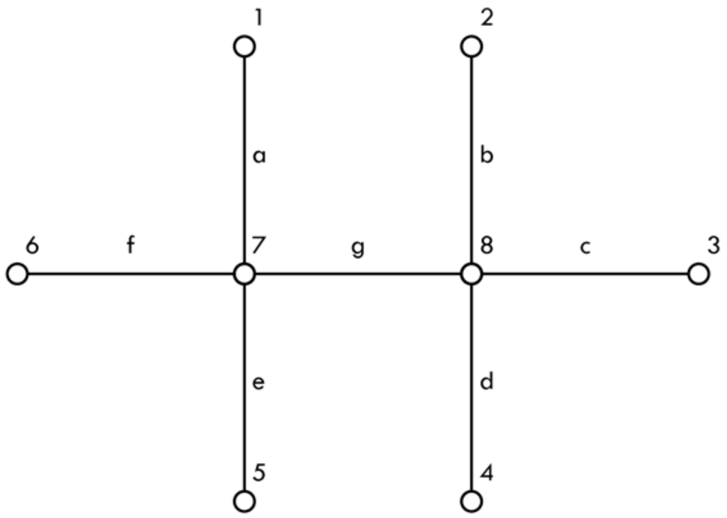


Figure 2.14 Planar projection of a network, the primal grid, with six fixed nodes and two free nodes

In order the construction to be stable, there should be force equilibrium in each of the free nodes. This force equilibrium can be described for every node, as equilibrium in the x, y and z direction. (Liem, 2011)

This is presented in the following equations:

Node 7

Equilibrium in x-direction:

$$\sum x_7: (x_7 - x_1)\frac{F_a}{l_a} + (x_7 - x_8)\frac{F_g}{l_g} + (x_7 - x_5)\frac{F_e}{l_e} + (x_7 - x_6)\frac{F_f}{l_f} - F_{x;7} = 0$$

Equilibrium in y-direction:

$$\sum y_7: (y_7 - y_1)\frac{F_a}{l_a} + (y_7 - y_8)\frac{F_g}{l_g} + (y_7 - y_5)\frac{F_e}{l_e} + (y_7 - y_6)\frac{F_f}{l_f} - F_{y;7} = 0$$

Equilibrium in z-direction:

$$\sum z_7: (z_7 - z_1)\frac{F_a}{l_a} + (z_7 - z_8)\frac{F_g}{l_g} + (z_7 - z_5)\frac{F_e}{l_e} + (z_7 - z_6)\frac{F_f}{l_f} - F_{z;7} = 0$$

Node 8

Equilibrium in x-direction:

$$\sum x_8: (x_8 - x_2)\frac{F_b}{l_b} + (x_8 - x_3)\frac{F_c}{l_c} + (x_8 - x_4)\frac{F_d}{l_d} + (x_8 - x_7)\frac{F_g}{l_g} - F_{x;8} = 0$$

Equilibrium in y-direction:

$$\sum y_8: (y_8 - y_2)\frac{F_b}{l_b} + (y_8 - y_3)\frac{F_c}{l_c} + (y_8 - y_4)\frac{F_d}{l_d} + (y_8 - y_7)\frac{F_g}{l_g} - F_{y;8} = 0$$

Equilibrium in z-direction:

$$\sum z_8: (z_8 - z_2)\frac{F_b}{l_b} + (z_8 - z_3)\frac{F_c}{l_c} + (z_8 - z_4)\frac{F_d}{l_d} + (z_8 - z_7)\frac{F_g}{l_g} - F_{z;8} = 0$$

In this case only the equilibriums in z-direction are needed, so 2 equations are left:

$$\sum z_7: (z_7 - z_1)\frac{F_a}{l_a} + (z_7 - z_8)\frac{F_g}{l_g} + (z_7 - z_5)\frac{F_e}{l_e} + (z_7 - z_6)\frac{F_f}{l_f} - F_{z;7} = 0$$

$$\sum z_8: (z_8 - z_2)\frac{F_b}{l_b} + (z_8 - z_3)\frac{F_c}{l_c} + (z_8 - z_4)\frac{F_d}{l_d} + (z_8 - z_7)\frac{F_g}{l_g} - F_{z;8} = 0$$

For this subject the forces and the lengths are known, so there are two unknown and two equation left, which is solvable.

In general, it can be concluded that the following shortened formula can be used:

$$\left(\sum \Delta z * \frac{F_n}{l_n}\right) - F_{z,tot} = 0$$

This can be rewritten to:

$$\sum \Delta z * \frac{F_n}{l_n} = F_{z,tot}$$

Because the force density equals to:

$$q = \frac{F}{l}$$

It can be said that:

$$\sum \Delta z * \frac{F_n}{l_n} = \sum \Delta z * q_n = F_{z,tot}$$

3 Calculation – Complementary energy

In this chapter more information about the calculation of the optimal reciprocal figure by an orthogonal primal grid will be given. There is also more about the calculation of the complementary energy itself.

To compare different possible solutions, the perimeter of the reciprocal figure has to be set to constant.

3.1 Calculation complementary energy

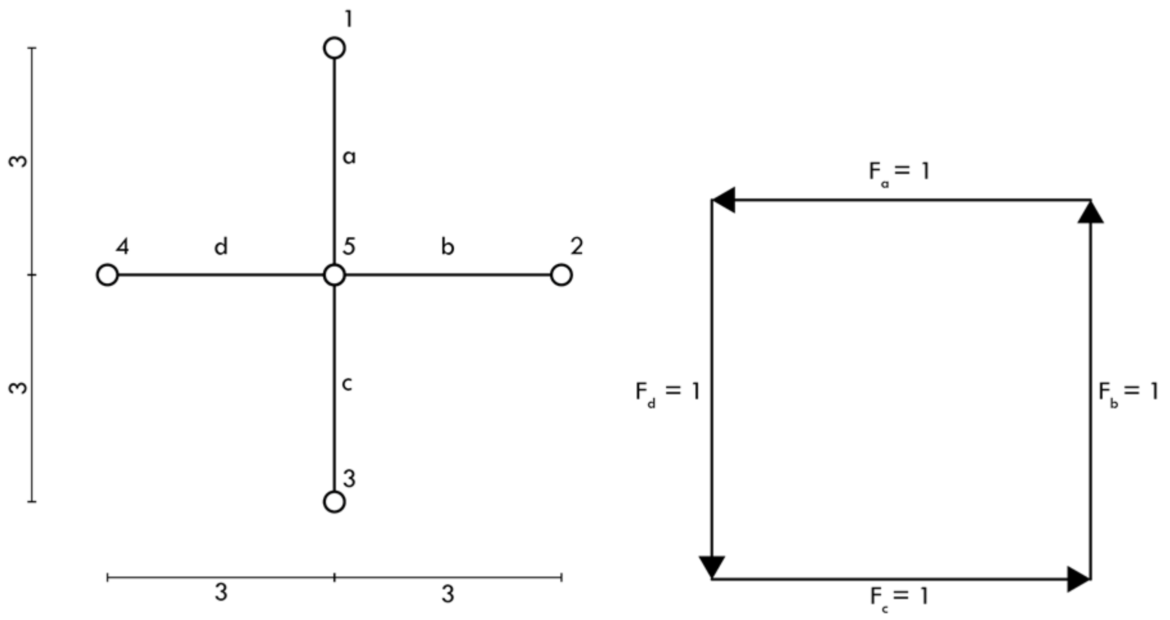


Figure 3.1 A basic primal grid with four bars and a corresponding reciprocal figure

Consider figure 3.1 to calculate the complementary energy of a structure.

The variables are in this case:

$l_a = l_b = l_c = l_d = 3$

$F_a = F_b = F_c = F_d = 1$

When the formula of paragraph 2.2 to calculate the complementary energy is used;

$$E_{compl,tot} = \sum_{i=1}^{n} F_i^2 l_i$$

And scribed out;

$$E_{compl,tot} = F_a^2 l_a + F_b^2 l_b + F_c^2 l_c + F_d^2 l_d = 1^2 * 3 + 1^2 * 3 + 1^2 * 3 + 1^2 * 3 = 12$$

So in this case the complementary energy has a value of 12.

3.2 Calculating the optimal reciprocal figure

When searching for the reciprocal figure with the least complementary energy, a link is found in the ratio between the height and the length of the grid and the size of the force in the reciprocal figure with the least complementary energy.

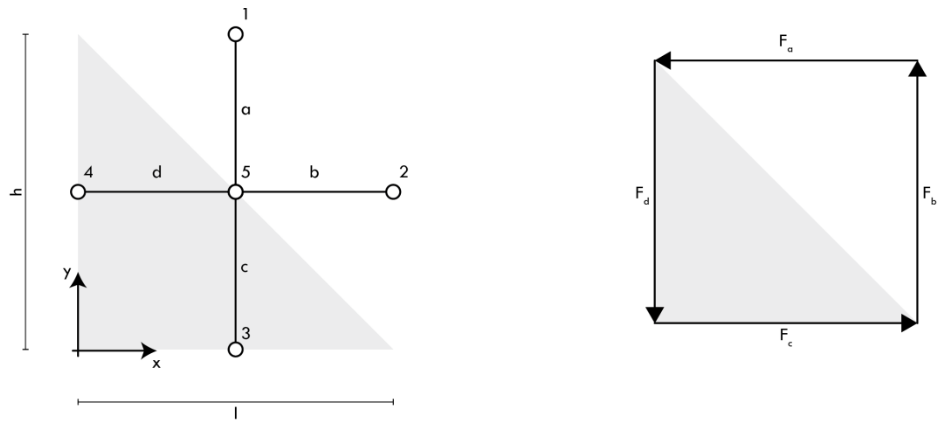


Figure 3.2 A primal grid with four equal members and the corresponding reciprocal figure

If the perimeter of the reciprocal figure is set as P, we can describe P as:

$P = F_a + F_b + F_c + F_d$

The forces acting on member a and c act in the y-direction, so $F_a = F_c = F_y$. The forces acting on b and d work in the x-direction, so $F_b = F_d = F_x$. Because of the reciprocal figure, the x- and y-axes will be changed. The relation between the dimensions of the grid and the dimensions of the reciprocal figure can be described as:

$l: h = F_y: F_x$

The optimal reciprocal figure by a square shaped primal grid is also a square, see figure 3.2. It doesn't matter for this relation what the length of the members is in the grid, see figure 3.3. The dimensions of the primal grid determine the ratio of the reciprocal figure.

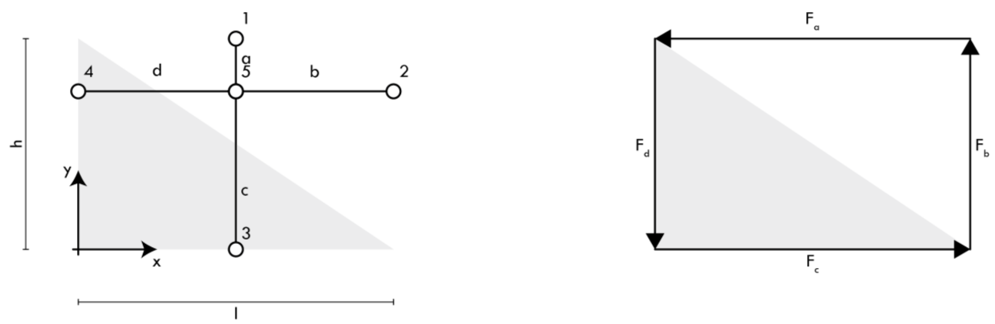


Figure 3.3 A primal grid with four members and the corresponding reciprocal figure

In case of an orthogonal grid with more than 4 bars, the relation will be maintained, see figure 3.4.

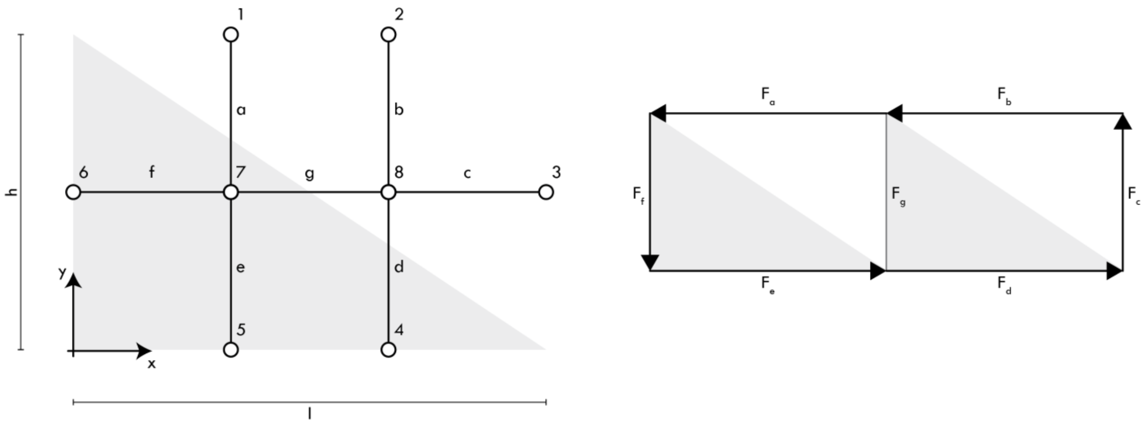


Figure 3.4 A primal grid with seven equal members and the corresponding reciprocal figure

But instead of the ratio $l:h = F_y:F_x$, further information has to be added; the perimeter of the reciprocal grid is based on more than 4 forces. The relation of the ratio is not for the reciprocal figure as a whole, but only for the area of a node.

The length of the reciprocal figure is half of the forces acting in the y-direction on the grid. The amount of forces acting on the y-direction in the grid can be called n_y. The same can be done for the forces in x-direction, this amount is n_x. For example in figure 3.3 n_y is 2 (F_a and F_b) and n_y is 1 (F_c).

So the ratio $l:h = F_y:F_x$ will be:

$$l:h = \frac{F_{y;tot}}{n_y} : \frac{F_{x;tot}}{n_x}$$

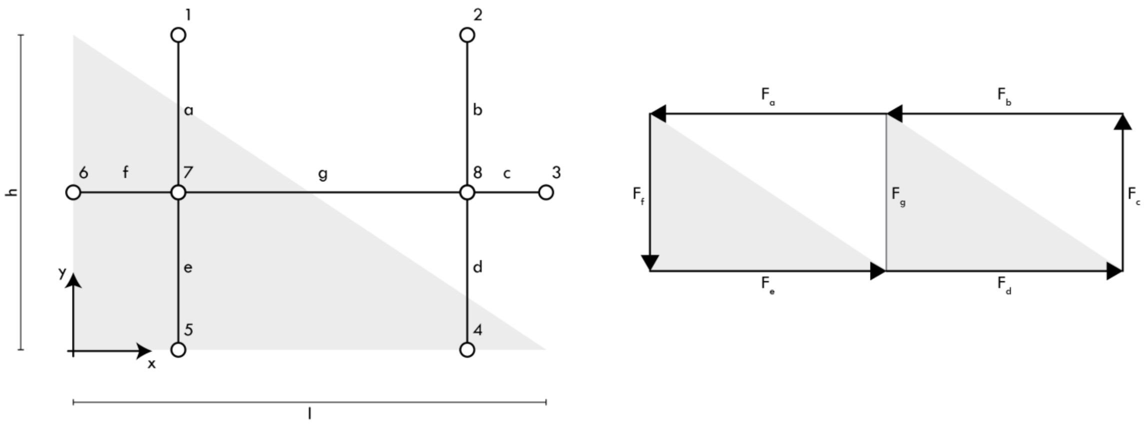


Figure 3.5 A bigger primal grid with seven members and the corresponding reciprocal figure

As can be seen in figure 3.5, also for those kind of orthogonal grids the relation of the dimensions of the grid and the reciprocal figure remains.

3.3 Conclusion

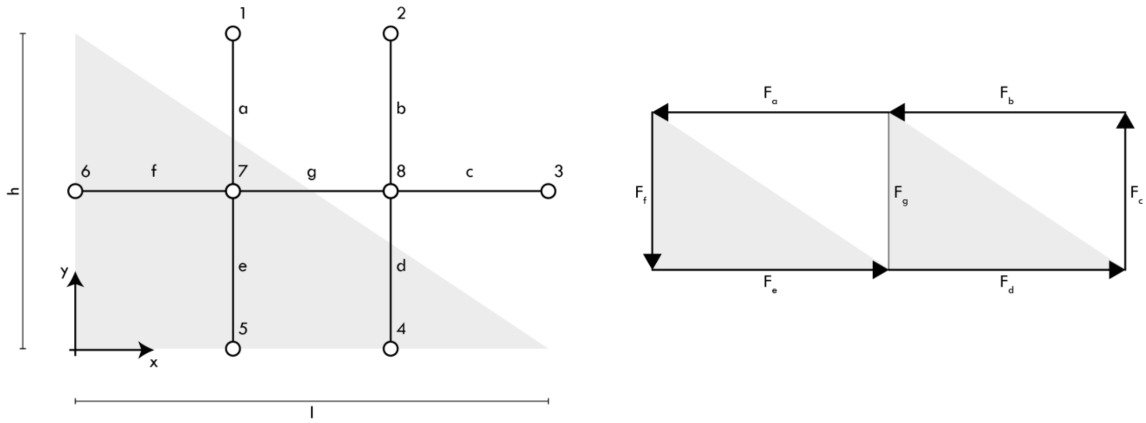


Figure 3.6 A primal grid with seven equal members and the corresponding reciprocal figure

The most important thing in this chapter is the relation in the dimensions of the primal grid and the reciprocal figure. The length of the reciprocal figure is half of the forces acting in the y-direction on the grid. The amount of forces acting on the y-direction in the grid is n_y and the amount of forces in x-direction is n_x.

So the ratio $l{:}\,h = F_y{:}\,F_x$ will be:

$$l{:}\,h = \frac{F_{y;tot}}{n_y} : \frac{F_{x;tot}}{n_x}$$

4 Calculation – The height

This chapter contains more information how to calculate the height of the free nodes as mentioned in paragraph 2.4.

4.1 The height of a node

In this paragraph the calculation of the height of the free nodes related to the force on the node, will be described.

Consider the primal grid as in figure 4.1.

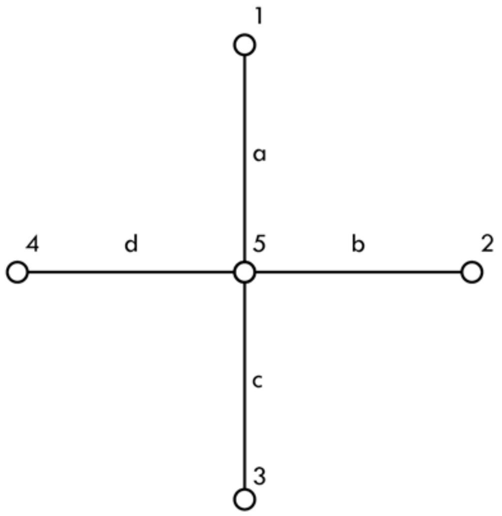


Figure 4.1 Primal grid with four equal members

To calculate the height of node 5 in the primal grid, the equation as mentioned in paragraph 2.4 can be used:

$$\sum z_5 : (z_5 - z_1)\frac{F_a}{l_a} + (z_5 - z_2)\frac{F_b}{l_b} + (z_5 - z_3)\frac{F_c}{l_c} + (z_5 - z_4)\frac{F_d}{l_d} - F_{z;5} = 0$$

Because nodes 1-4 are fixed nodes, with a height equal to 0, this can be rewritten to:

$$(z_5 - 0)\frac{F_a}{l_a} + (z_5 - 0)\frac{F_b}{l_b} + (z_5 - 0)\frac{F_c}{l_c} + (z_5 - 0)\frac{F_d}{l_d} - F_{v5} = 0 \rightarrow$$

$$z_5\frac{F_a}{l_a} + z_5\frac{F_b}{l_b} + z_5\frac{F_c}{l_c} + z_5\frac{F_d}{l_d} - F_{v5} = 0 \rightarrow$$

$$z_5\frac{F_a}{l_a} + z_5\frac{F_b}{l_b} + z_5\frac{F_c}{l_c} + z_5\frac{F_d}{l_d} = F_{v5}$$

This can be rewritten to:

$$z_5\left(\frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d}\right) = F_{v5}$$

The part between the parentheses is the force density:

$$q = \frac{F}{l}$$

So the equation can be shortened to:

$$\left(\frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d}\right) = \sum q$$

So:

$$z_5 * \sum q = F_{v5}$$

For example the primal grid in figure 4.2; the vertical force on node 5 (F_{v5}) has a size of 3 and the forces F_a, F_b, F_c and F_d, are 1:

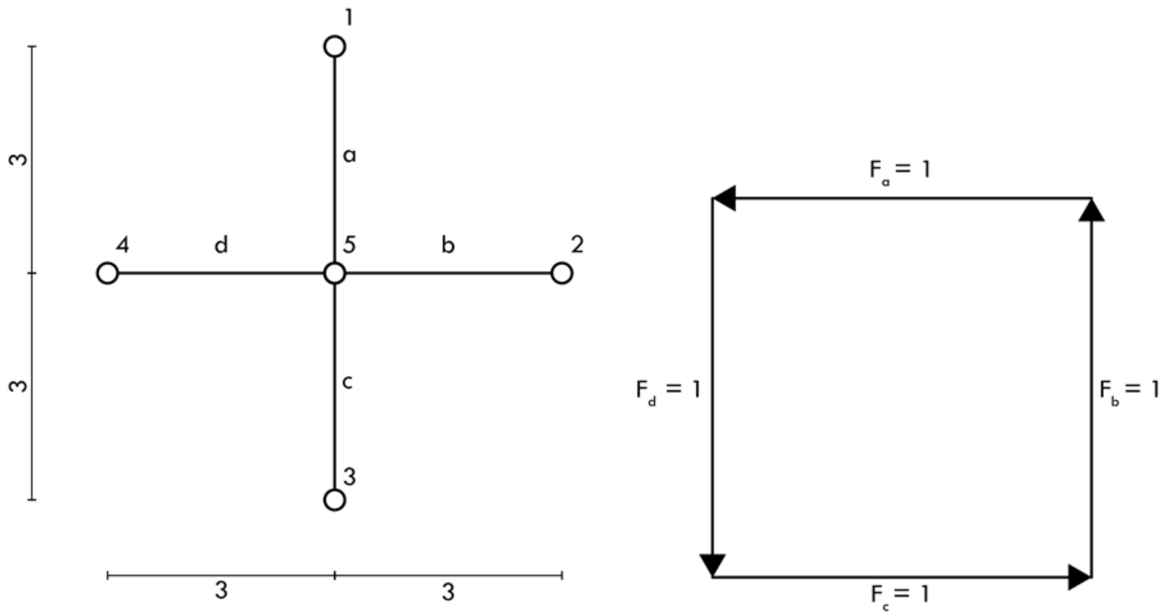


Figure 4.2 Primal grid with four equal members and the reciprocal figure

$$F_{v5} = z_5\left(\frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d}\right) = z_5\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) = \frac{4}{3} * z_5 = 3 \rightarrow z = \frac{9}{4}$$

According to this formula, $z(q) = F_{v5}$, the forces acting on a point can be reduced to a total resulting force at the cross of the working line of the members, see figure 4.3.

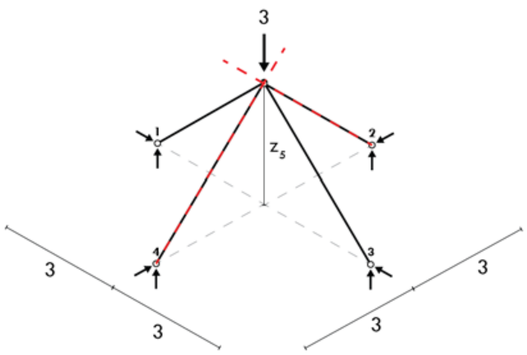


Figure 4.3 Resulting force on the cross of the working lines in the structure

When the same is done for a more complex primal grid (see figure 4.4), with a unit load of 3 on nodes 7 and 8 and the reciprocal as in figure 4.5, we get:

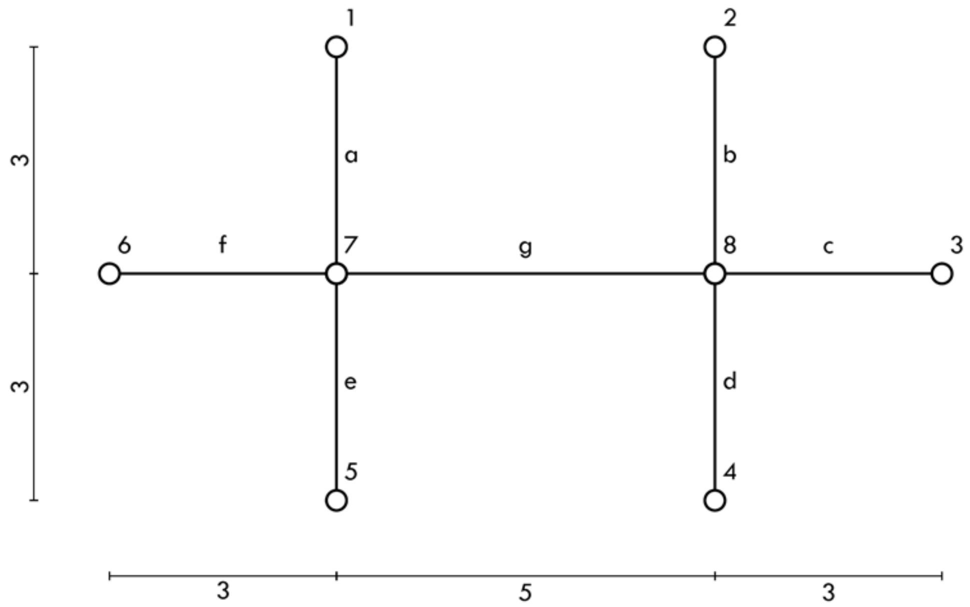


Figure 4.4 A more complex primal grid with seven members

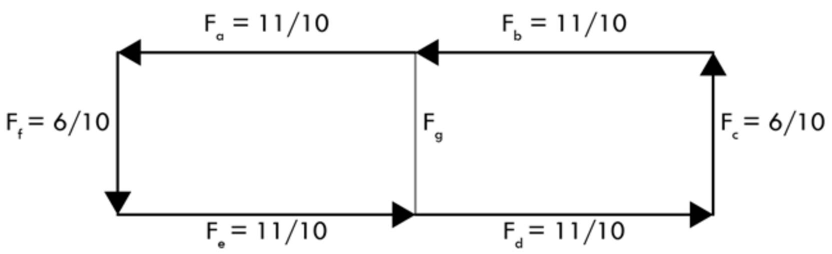


Figure 4.5 The reciprocal figure by figure 4.4

$$F_{v;tot} = z_7\left(\frac{F_a}{l_a}+\frac{F_g}{l_g}+\frac{F_e}{l_e}+\frac{F_f}{l_f}\right)+z_8\left(\frac{F_b}{l_b}+\frac{F_c}{l_c}+\frac{F_d}{l_d}+\frac{F_g}{l_g}\right)=$$

$$z_7\left(\frac{\frac{11}{10}}{3}+\frac{\frac{6}{10}}{5}+\frac{\frac{11}{10}}{3}+\frac{\frac{6}{10}}{3}\right)+z_8\left(\frac{\frac{11}{10}}{3}+\frac{\frac{6}{10}}{3}+\frac{\frac{11}{10}}{3}+\frac{\frac{6}{10}}{5}\right)=\frac{79}{75}z_7+\frac{79}{75}z_8=6$$

Because the reciprocal is symmetrical, the height of node 7 is equal to node 8, so:

$$z_7 = z_8 \rightarrow \frac{79}{75}z_7 + \frac{79}{75}z_7 = \frac{158}{75}z_7 = 6 \rightarrow z_7 = \frac{225}{79}$$

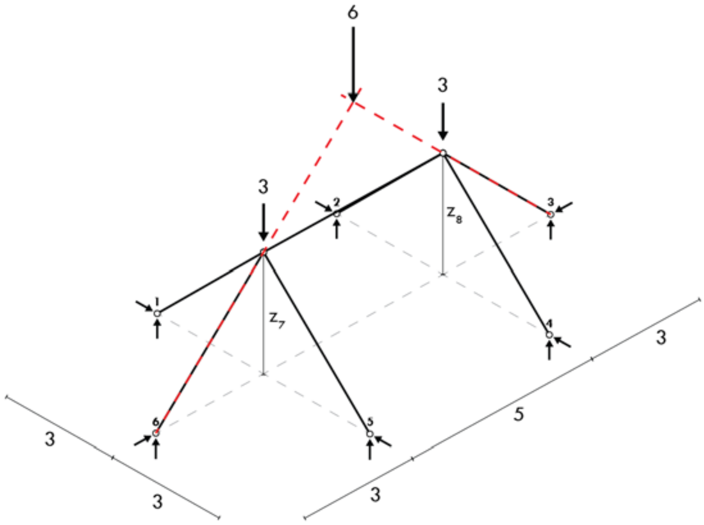


Figure 4.6 Resulting force on the cross of the working lines in the more complex structure

This gives a similarity to the force polygon of a funicular shape; there is also a resulting force at the cross of the working lines of the 'members' at the support points, see figure 4.7

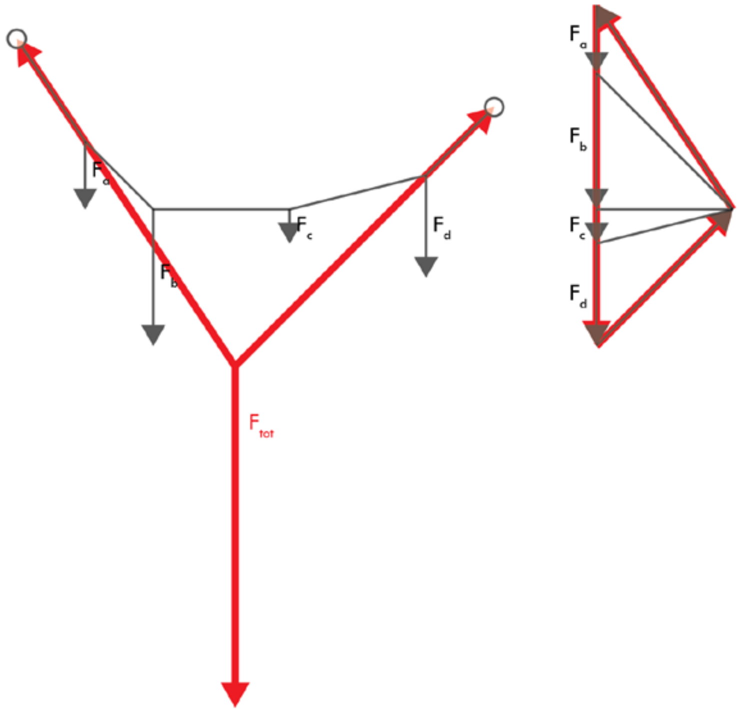


Figure 4.7 Hanging cable with forces and the corresponding force polygon

From figure 4.7 it can deduce that the individual forces, F_a, F_b, F_c and F_d, have no influence on the outline of the force polygon. So if the forces are changed, the shape of the funicular will change, but the lines at the support points will remain the same.

4.2 Elimination bars in the middle

Consider the primal grid as in figure 4.8 and the reciprocal figure as in 4.9.

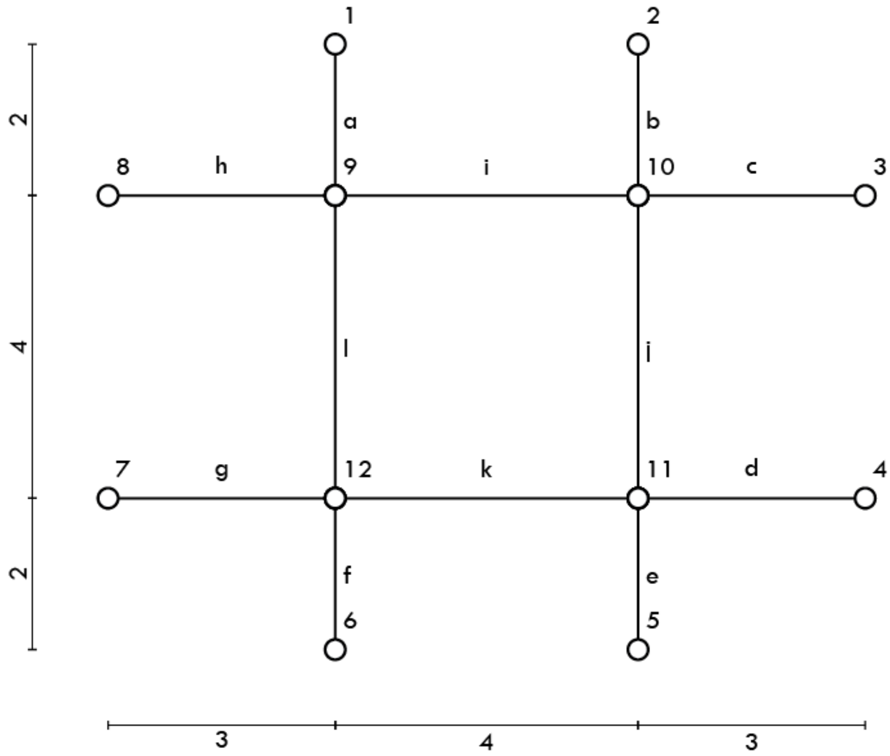


Figure 4.8 Primal grid with 12 members

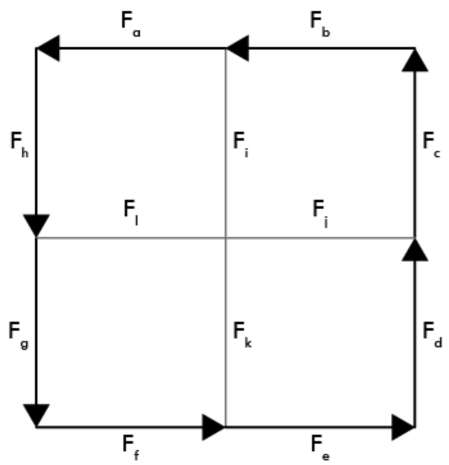


Figure 4.9 The reciprocal figure by the primal grid in figure 4.8

To calculate the height of the nodes 9, 10, 11 and 12, the following equations can be used:

$$(z_9 - z_1)\frac{F_a}{l_a} + (z_9 - z_{10})\frac{F_i}{l_i} + (z_9 - z_{12})\frac{F_l}{l_l} + (z_9 - z_8)\frac{F_h}{l_h} - F_{z9} = 0$$

$$(z_{10} - z_2)\frac{F_b}{l_b} + (z_{10} - z_3)\frac{F_c}{l_c} + (z_{10} - z_{11})\frac{F_j}{l_j} + (z_{10} - z_9)\frac{F_i}{l_i} - F_{z10} = 0$$

$$(z_{11} - z_{10})\frac{F_j}{l_j} + (z_{11} - z_4)\frac{F_d}{l_d} + (z_{11} - z_5)\frac{F_e}{l_e} + (z_{11} - z_{12})\frac{F_k}{l_k} - F_{z11} = 0$$

$$(z_{12} - z_9)\frac{F_l}{l_l} + (z_{12} - z_{11})\frac{F_k}{l_k} + (z_{12} - z_6)\frac{F_f}{l_f} + (z_{12} - z_7)\frac{F_g}{l_g} - F_{z12} = 0$$

When the brackets we get are eliminated, the following equations are derived:

$$z_9\frac{F_a}{l_a} + z_9\frac{F_i}{l_i} - z_{10}\frac{F_i}{l_i} + z_9\frac{F_l}{l_l} - z_{12}\frac{F_l}{l_l} + z_9\frac{F_h}{l_h} = F_{z9}$$

$$z_{10}\frac{F_b}{l_b} + z_{10}\frac{F_c}{l_c} + z_{10}\frac{F_j}{l_j} - z_{11}\frac{F_j}{l_j} + z_{10}\frac{F_i}{l_i} - z_9\frac{F_i}{l_i} = F_{z10}$$

$$z_{11}\frac{F_j}{l_j} - z_{10}\frac{F_j}{l_j} + z_{11}\frac{F_d}{l_d} + z_{11}\frac{F_e}{l_e} + z_{11}\frac{F_k}{l_k} - z_{12}\frac{F_k}{l_k} = F_{z11}$$

$$z_{12}\frac{F_l}{l_l} - z_9\frac{F_l}{l_l} + z_{12}\frac{F_k}{l_k} - z_{11}\frac{F_k}{l_k} + z_{12}\frac{F_f}{l_f} + z_{12}\frac{F_g}{l_g} = F_{z12}$$

When those equations are added, some items can be erased, because they are added in one equation and subtracted in the other. So:

$$z_9\frac{F_a}{l_a} \cancel{+ z_9\frac{F_i}{l_i} - z_{10}\frac{F_i}{l_i} + z_9\frac{F_l}{l_l} - z_{12}\frac{F_l}{l_l}} + z_9\frac{F_h}{l_h} = F_{z9}$$

$$z_{10}\frac{F_b}{l_b} + z_{10}\frac{F_c}{l_c} \cancel{+ z_{10}\frac{F_j}{l_j} - z_{11}\frac{F_j}{l_j} + z_{10}\frac{F_i}{l_i} - z_9\frac{F_i}{l_i}} = F_{z10}$$

$$\cancel{z_{11}\frac{F_j}{l_j} - z_{10}\frac{F_j}{l_j}} + z_{11}\frac{F_d}{l_d} + z_{11}\frac{F_e}{l_e} \cancel{+ z_{11}\frac{F_k}{l_k} - z_{12}\frac{F_k}{l_k}} = F_{z11}$$

$$\cancel{z_{12}\frac{F_l}{l_l} - z_9\frac{F_l}{l_l} + z_{12}\frac{F_k}{l_k} - z_{11}\frac{F_k}{l_k}} + z_{12}\frac{F_f}{l_f} + z_{12}\frac{F_g}{l_g} = F_{z12}$$

____+

$$z_9\frac{F_a}{l_a} + z_9\frac{F_h}{l_h} + z_{10}\frac{F_b}{l_b} + z_{10}\frac{F_c}{l_c} + z_{11}\frac{F_d}{l_d} + z_{11}\frac{F_e}{l_e} + z_{12}\frac{F_f}{l_f} + z_{12}\frac{F_g}{l_g} = F_{z,tot}$$

It is considered that there is only influence of the forces and the lengths (= the force density) of the members connected to the fixed nodes on the .

From this it can concluded that the force density in the members which are only connected to the free nodes, have no influence on the z coordinate, see figure 4.10.

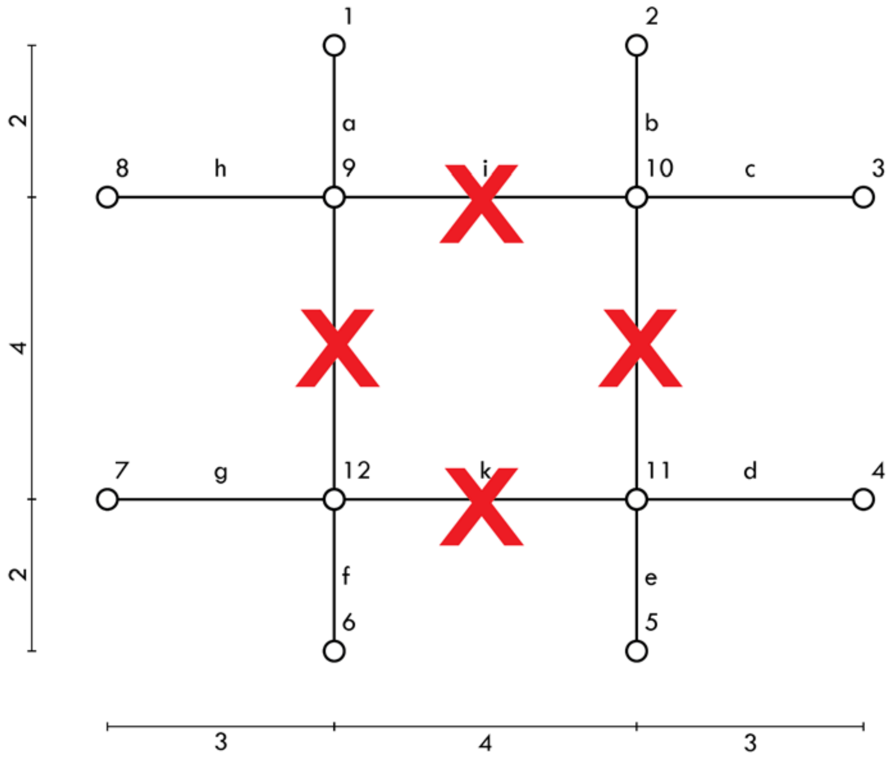


Figure 4.10 Primal grid with the unneeded members

4.3 Conclusion

To calculate the height of a node the structure the sum of the force density of a member times the difference in height of the start point and end point has to be equal to the vertical load on the node.
For example see figure 4.11.

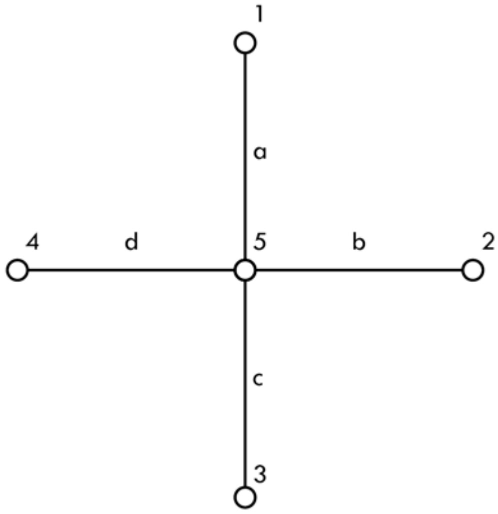


Figure 4.11 Primal grid with four equal members

To calculate the height of node 5 in the primal grid of figure 4.11, the following equation can be used:

$$\sum z_5: (z_5 - z_1)\frac{F_a}{l_a} + (z_5 - z_2)\frac{F_b}{l_b} + (z_5 - z_3)\frac{F_c}{l_c} + (z_5 - z_4)\frac{F_d}{l_d} - F_{z;5} = 0$$

By a more complex primal grid it has to be done for every node.

When we work out the formula for the primal grid in figure 4.11 we get:

$$(z_5 - 0)\frac{F_a}{l_a} + (z_5 - 0)\frac{F_b}{l_b} + (z_5 - 0)\frac{F_c}{l_c} + (z_5 - 0)\frac{F_d}{l_d} - F_{v5} = 0 \ \rightarrow$$

$$z_5\frac{F_a}{l_a} + z_5\frac{F_b}{l_b} + z_5\frac{F_c}{l_c} + z_5\frac{F_d}{l_d} = F_{v5}$$

This can be rewritten to:

$$z_5\left(\frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d}\right) = F_{v5}$$

The part between the parentheses is the force density:

$$q = \frac{F}{l}$$

So the equation can be shortened to:

$$z_5\left(\frac{F_a}{l_a}+\frac{F_b}{l_b}+\frac{F_c}{l_c}+\frac{F_d}{l_d}\right)=z_5*\sum q$$

So:

$$z_5*\sum q=F_{v5}$$

In general can be said that:

$$\sum \Delta z*q_n=F_{z,tot}$$

In paragraph 4.2 can be found that the members connected only to the free nodes, have no influence on the slope of the members connected to the fixed nodes. They are only needed to solve the equations for the z coordinate.

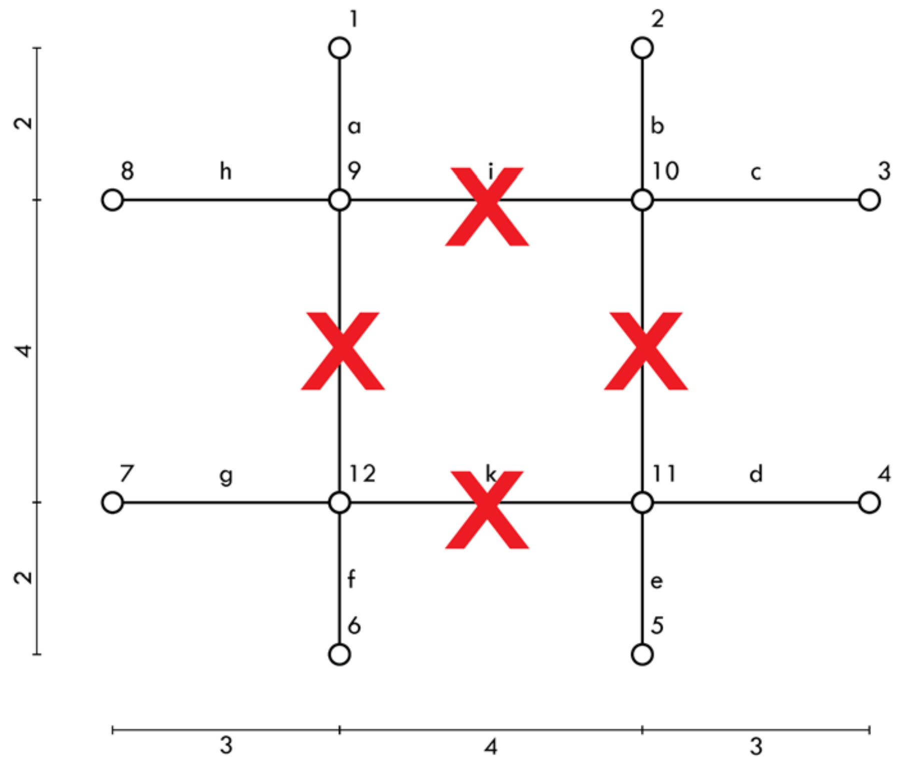


Figure 4.12 Primal grid with 12 members

So the resulting formula by figure 4,12 is:

$$z_9\frac{F_a}{l_a}+z_9\frac{F_h}{l_h}+z_{10}\frac{F_b}{l_b}+z_{10}\frac{F_c}{l_c}+z_{11}\frac{F_d}{l_d}+z_{11}\frac{F_e}{l_e}+z_{12}\frac{F_f}{l_f}+z_{12}\frac{F_g}{l_g}=F_{z,tot}$$

5 Calculation - Overview

5.1 Calculation examples

In this chapter some examples are presented, to compare the influence between the force density including the perimeter of the reciprocal figure and the complementary energy.

For all the examples the primal grid as in figure 5.1 is used.

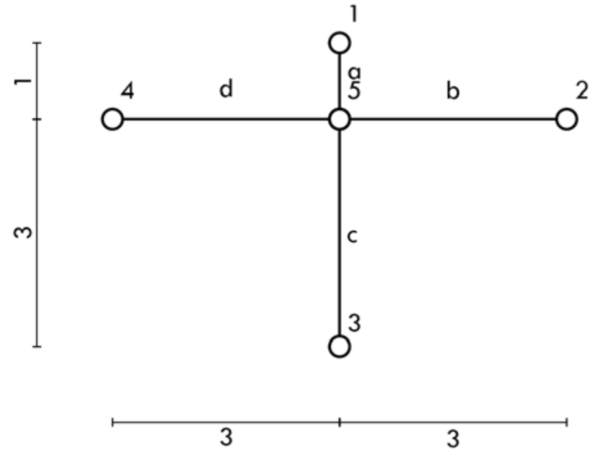


Figure 5.1 Primal grid with four members for the examples to be calculated

The sizes of the bars in the primal grid are:

$l_a = 1$; $l_b = l_c = l_d = 3$

5.1.1 Example 1

In this example the force size for all bars is set on 1, see figure 5.2.

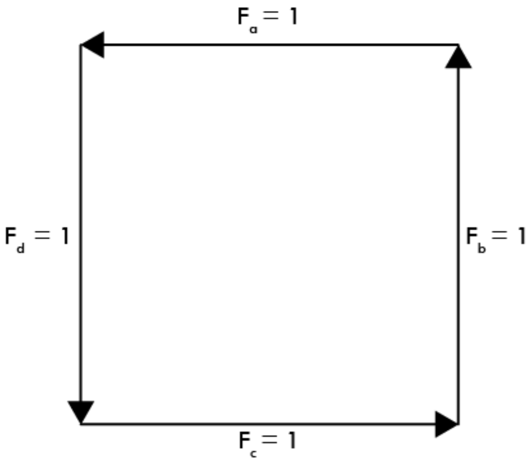


Figure 5.2 The reciprocal figure for example 1

$$F_a = F_b = F_c = F_d = 1$$

$$EC = F_a^2 l_a + F_b^2 l_b + F_c^2 l_c + F_d^2 l_d = 1^2 * 1 + 1^2 * 3 + 1^2 * 3 + 1^2 * 3 = 10$$

$$FD = \frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d} = \frac{1}{1} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 2$$

$$z\left(\frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d}\right) - F = 0 \rightarrow z\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) - F = 0 \rightarrow 2z = F$$

If $z = 3$; $\quad 2 * 3 = F = 6$

Perimeter: $\quad F_a + F_b + F_c + F_d = 1 + 1 + 1 + 1 = 4$

Area: $\quad F_a * F_b = 1 * 1 = 1$

Summary:

EC:	10	**Perimeter:**	4
FD:	2	**Area:**	1
z:	3	**Perimeter/area:**	4
F:	6		

5.1.2 Example 2 - least complementary energy

In paragraph 3.2 it is found that there is a relation between the ratio of the size of the primal grid and the reciprocal:

$l: h = F_y: F_x$

See also figure 5.3.

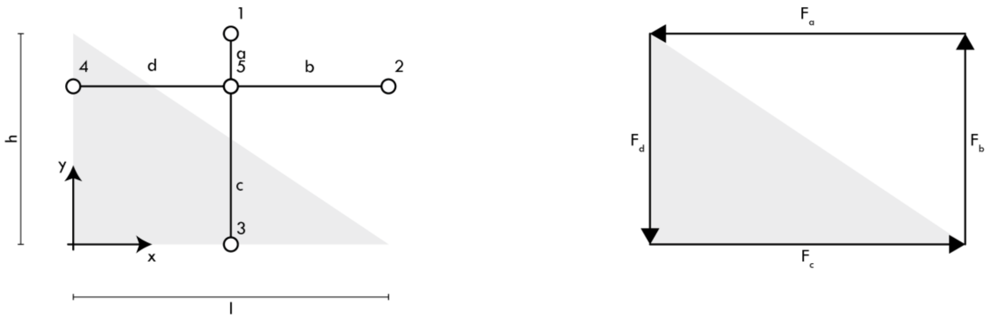


Figure 5.3 Primal grid with four members and the corresponding reciprocal figure

In this example this knowledge will be used to solve the problem and generate the reciprocal with the least complementary energy.

With this information the reciprocal figure can be calculated with the minimum complementary energy.

$l: h = 6: 4 \rightarrow 4F_y: 6F_x \rightarrow \frac{3}{2}F_x: F_y$

The perimeter of the reciprocal figure remains constant at 4, so:

$P = F_a + F_b + F_c + F_d = 2F_x + 2F_y = 4$

$2F_x + 2F_y = 2F_x + 2 * \frac{3}{2}F_x = 5F_x = 4 \rightarrow F_x = \frac{4}{5}$

$F_y = \frac{3}{2}F_x = \frac{3}{2} * \frac{4}{5} = \frac{6}{5}$

$F_x = F_b = F_d = \frac{4}{5}$

$F_y = F_a = F_c = \frac{6}{5}$

The reciprocal will be according figure 5.4.

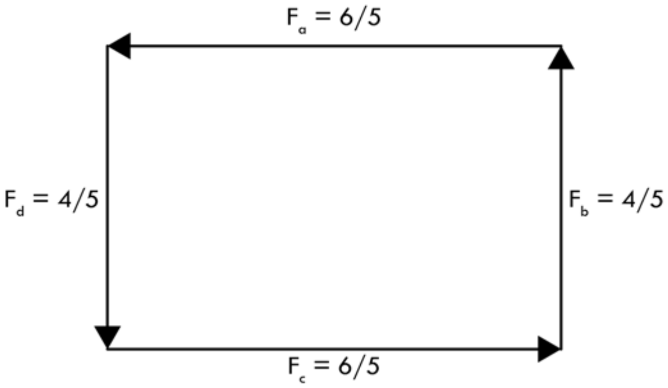


Figure 5.4 The reciprocal figure for example 2

$$EC = F_a^2 l_a + F_b^2 l_b + F_c^2 l_c + F_d^2 l_d = \frac{6^2}{5} * 1 + \frac{4^2}{5} * 3 + \frac{6^2}{5} * 3 + \frac{4^2}{5} * 3 = 9{,}6$$

$$FD = \frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d} = \frac{\frac{6}{5}}{1} + \frac{\frac{4}{5}}{3} + \frac{\frac{6}{5}}{3} + \frac{\frac{4}{5}}{3} = \frac{32}{15}$$

$$z\left(\frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d}\right) - F = 0 \rightarrow z\left(\frac{\frac{6}{5}}{1} + \frac{\frac{4}{5}}{3} + \frac{\frac{6}{5}}{3} + \frac{\frac{4}{5}}{3}\right) - F = 0 \rightarrow \frac{32}{15} z = F$$

If $z = 3$; $\frac{32}{15} * 3 = F = 6{,}4$

Perimeter: $F_a + F_b + F_c + F_d = 1 + 1 + 1 + 1 = 4$

Area: $F_a * F_b = 1 * 1 = 1$

Summary:

EC:	9,6	**Perimeter:**	4
FD:	2,13	**Area:**	1
z:	3	**Perimeter/area:**	4
F:	6,4		

5.1.3 Example 3

In this example there are some values for the forces in the reciprocal figure are chosen in such a way that the perimeter stays the same as in example 1 and 2 on 4.

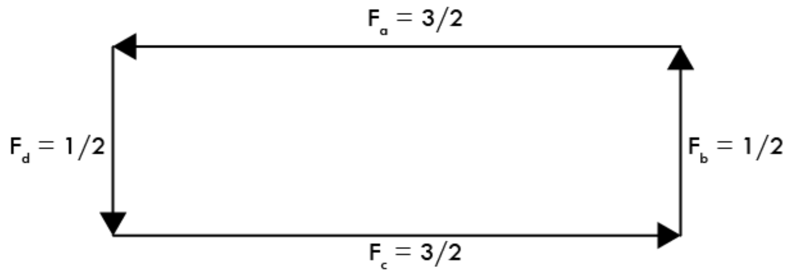


Figure 5.5 The reciprocal figure for example 3

$F_a = 3F_b = \frac{3}{2};\qquad F_b = \frac{1}{3}F_a = \frac{1}{2};\qquad F_c = F_a = \frac{3}{2};\qquad F_d = \frac{1}{3}F_a = \frac{1}{2}$

$$EC = F_a^2 l_a + F_b^2 l_b + F_c^2 l_c + F_d^2 l_d = \frac{3}{2}^2 * 1 + \frac{1}{2}^2 * 3 + \frac{3}{2}^2 * 3 + \frac{1}{2}^2 * 3 = 10\frac{1}{2}$$

$$FD = \frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d} = \frac{\frac{3}{2}}{1} + \frac{\frac{1}{2}}{3} + \frac{\frac{3}{2}}{3} + \frac{\frac{1}{2}}{3} = 2\frac{1}{3}$$

$$z\left(\frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d}\right) - F = 0 \rightarrow z\left(\frac{\frac{3}{2}}{1} + \frac{\frac{1}{2}}{3} + \frac{\frac{3}{2}}{3} + \frac{\frac{1}{2}}{3}\right) - F = 0 \rightarrow 2\frac{1}{3}z = F$$

If $F = 6$; $\qquad 6/2\frac{1}{3} = z = \frac{18}{7}$

Perimeter: $\qquad F_a + F_b + F_c + F_d = \frac{3}{2} + \frac{1}{2} + \frac{3}{2} + \frac{1}{2} = 4$

Area: $\qquad F_a * F_b = \frac{3}{2} * \frac{1}{2} = \frac{3}{4}$

Summary:

EC:	10,5	**Perimeter:**	4
FD:	2,33	**Area:**	0,75
z:	2,57	**Perimeter/area:**	5,33
F:	6		

5.1.4 Example 4

In this example the force density is set the same as in example 2 with the least complementary energy; 2,13 (= 32/15). The perimeter of the reciprocal figure is a variable.

The difference with example is that the forces are expressed in x times F_a. So:

$F_a = ?$; $F_b = xF_a$; $F_c = F_a$; $F_d = xF_a$

When the variable x equals 3, the following forces are the result:

$F_a = ?$; $F_b = 3F_a$; $F_c = F_a$; $F_d = 3F_a$

Because the force density is known, we can calculate F_a:

$$FD = \frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d}$$

Al the force can be expressed in F_a, so:

$$\frac{F_a}{1} + \frac{3F_a}{3} + \frac{F_a}{3} + \frac{3F_a}{3} = \frac{32}{15} \rightarrow F_a = \frac{16}{25}$$

The resulting forces are:

$F_a = \frac{16}{25}$; $F_b = \frac{48}{25}$; $F_c = \frac{16}{25}$; $F_d = \frac{48}{25}$

When the related reciprocal figure is drawn, it results in figure 5.6.

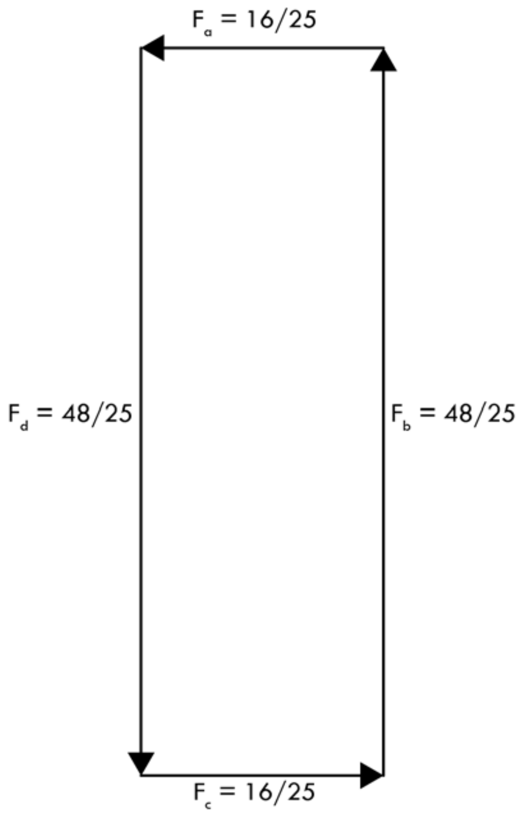


Figure 5.6 The reciprocal figure for example 4

$$EC = F_a^2 l_a + F_b^2 l_b + F_c^2 l_c + F_d^2 l_d = \frac{16}{25}^2 * 1 + \frac{48}{25}^2 * 3 + \frac{16}{25}^2 * 3 + \frac{48}{25}^2 * 3 = 23{,}76$$

$$z\left(\frac{F_a}{l_a} + \frac{F_b}{l_b} + \frac{F_c}{l_c} + \frac{F_d}{l_d}\right) - F = 0 \rightarrow z\left(\frac{\frac{16}{25}}{1} + \frac{\frac{48}{25}}{3} + \frac{\frac{16}{25}}{3} + \frac{\frac{48}{25}}{3}\right) - F = 0 \rightarrow \frac{32}{15} z = F$$

If $z = 3$; $\frac{32}{15} * 3 = F = 6{,}4$

Perimeter: $F_a + F_b + F_c + F_d = \frac{16}{25} + \frac{48}{25} + \frac{16}{25} + \frac{48}{25} = 5{,}12$

Area: $F_a * F_b = \frac{16}{25} * \frac{48}{25} = \frac{768}{625}$

Summary:

EC:	23,76	**Perimeter:**	5,12
FD:	2,13	**Area:**	1,23
z:	3	**Perimeter/area:**	4,17
F:	6,4		

5.2 Conclusion

When the results of the previous four examples are compared, the following summary can be made, see the table in figure 5.7.

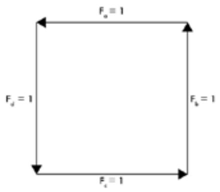
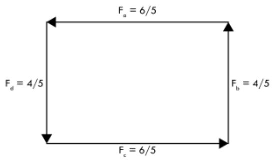
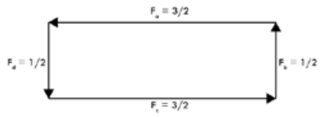
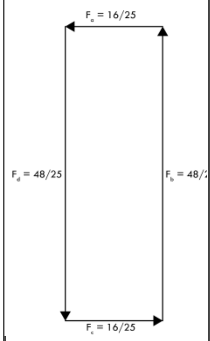
	Example 1	Example 2	Example 3	Example 4
Reciprocal				
EC	10	9,6	10,5	23,76
FD	2	2,13	2,33	2,13
Perimeter	4	4	4	5,12

Figure 5.7 Comparison of the results of the different examples

The perimeter of the reciprocal figure should be the lowest for the least complementary energy. The value of the force density is not necessary to find the solution with the least complementary energy or to compare the different solutions.

Part III
Informatics

6 Informatics

6.1 Introduction

Informatics has an important role in the calculation of the items described before.

In the very early phase some solver sheets in Excel were made to calculate the reciprocal figure with the least complementary energy.

Rhino with Grasshopper are very important in this informatics part. Rhino is a powerful three dimensional drawing program. A major advantage of Rhino is the availability of various plugins. Grasshopper for example is such a plugin. Grasshopper is a graphical algorithm editor tightly integrated with Rhino's three dimensional modelling tools.

6.2 Workflow

The workflow of the tool can be summarized as in figure 6.1.

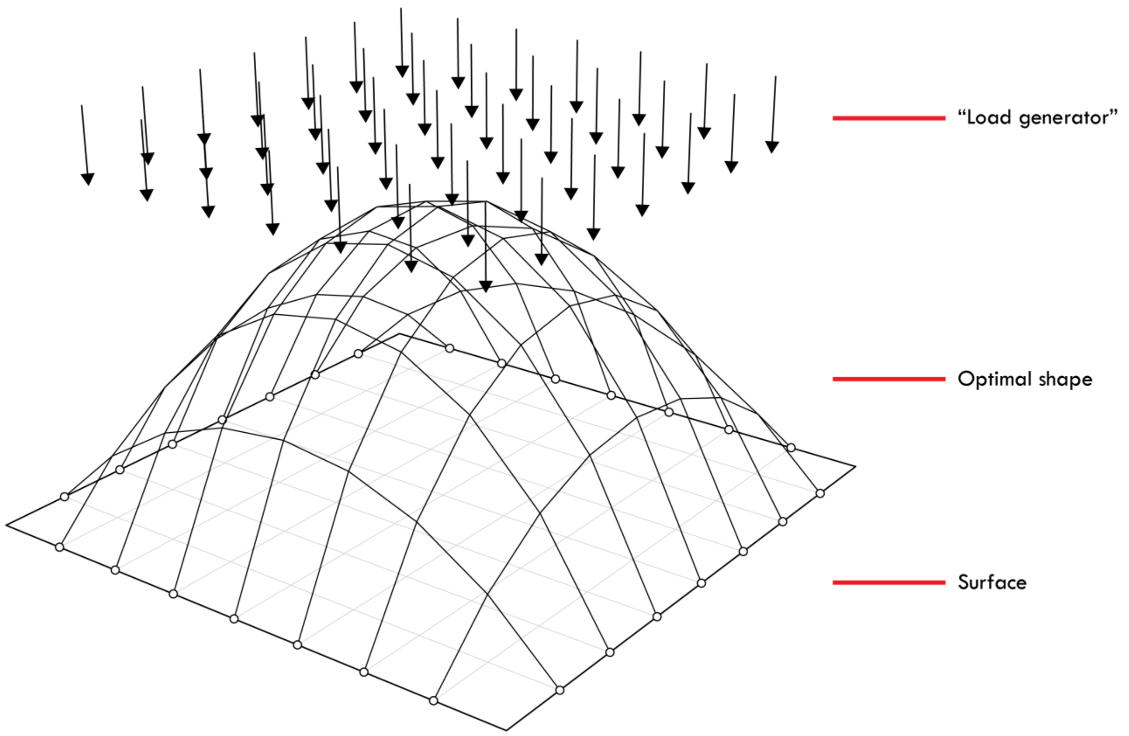


Figure 6.1 Resume of the calculating process

There are three layer in the whole process. It starts with the 'load generator'. With this generator it is possible to determine the magnitude of the loads on the nodes. The second layer ('Optimal shape' in figure 6.1) is actually the last step in the process; the result of the calculations. After calculating the structure with the minimum complementary energy, it is possible to draw this structure. The layer 'Surface' in figure 6.1 is along with the 'Load generator' part of the input variables.

In this paragraph the workflow of the tool is described. Each (calculation) step will be explained. Figure 6.2 shows the total Grasshopper workflow. In Appendix 1 there is a larger image of the model. The numbering of each frame corresponds to the numbering of the paragraphs with the explanation of that step.

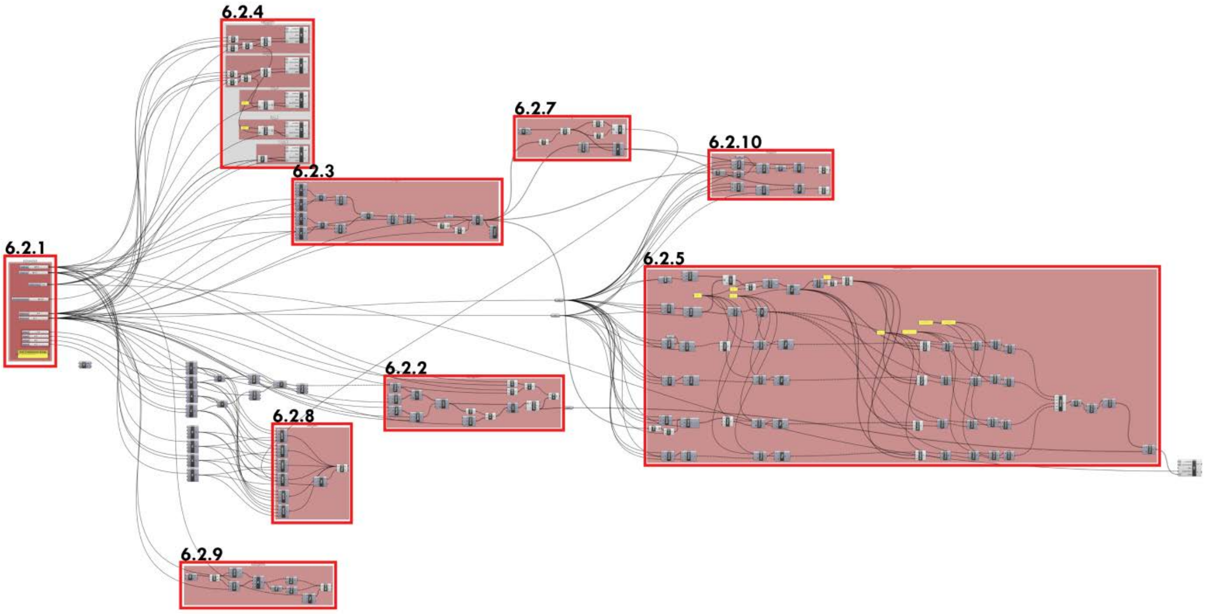


Figure 6.2 Total Grasshopper model

6.2.1 Inputs

The first step in the process is the panel with the sliders for the input for the calculation, see figure 6.3.

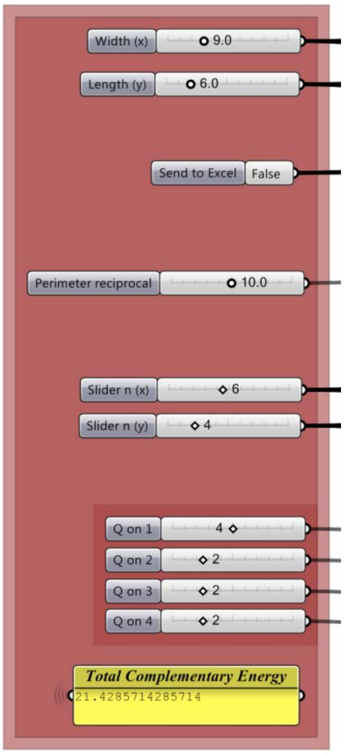


Figure 6.3 Grasshopper: input sliders

Width (x):	With this slider the width (in the x direction) of the grid can be set.
Length (y):	With this slider the length (in the y direction) of the grid can be set.
Send to Excel:	When this Boolean is set to 'True', Grasshopper updates continuously the values of sliders to Excel. The disadvantage of this continue updating process is that the response of the sliders in Grasshopper is very slow. It is recommended first to enter the values with the Boolean on 'False' and after that to toggle it to 'True'. Grasshopper then exports everything needed to calculate to Excel. After that the Boolean can be set back to 'False'.
Perimeter reciprocal:	With this value the perimeter of the reciprocal figure can be set. The value of this perimeter has influence on the height of the structure.
Slider n (x):	The value of this slider determines in how many segments the structure is distributed, in the x direction. The higher this values, the higher the accuracy of the model.
Slider n (y):	This slider does the same as 'Slider n (x)', but in the y direction.
Q on 1-4:	With those sliders the size of the load can be set. Each slider corresponds to a corner, see also paragraph 6.1.2.
Total Complementary Energy:	This is the total complementary energy of the calculated structure.

6.2.2 Generating forces

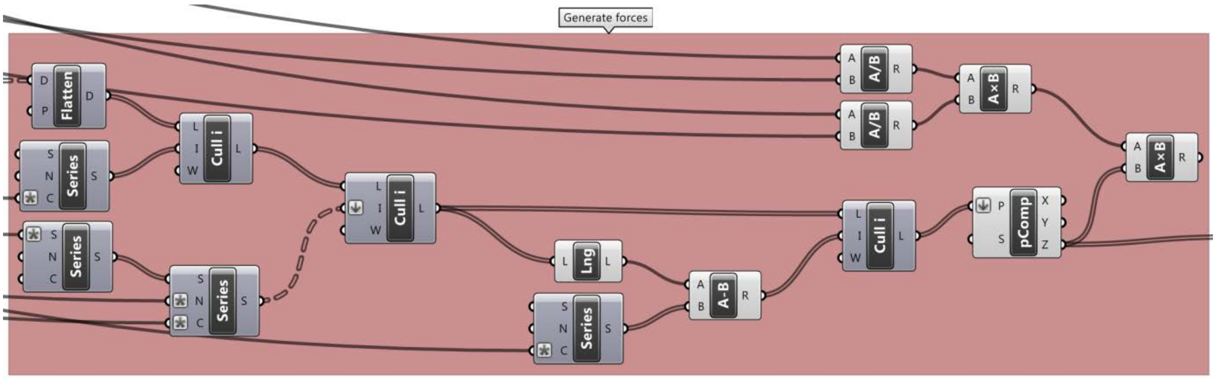


Figure 6.4 Grasshopper: generating the magnitude of the forces

This step generates the forces on the nodes of the structure. By setting the force size of the corners in the input panel (see 6.1.1), Grasshopper generates a volume of the force. This ‘volume’ of the load is converted to point loads on the nodes of the structure. By doing this, it is possible to use no equally distributed loads on the structure, see figure 6.5.

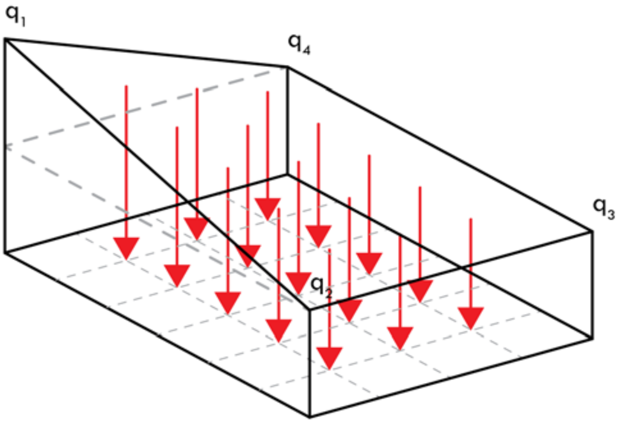


Figure 6.5 From distributed load to point loads

6.2.3 Defining the primal grid

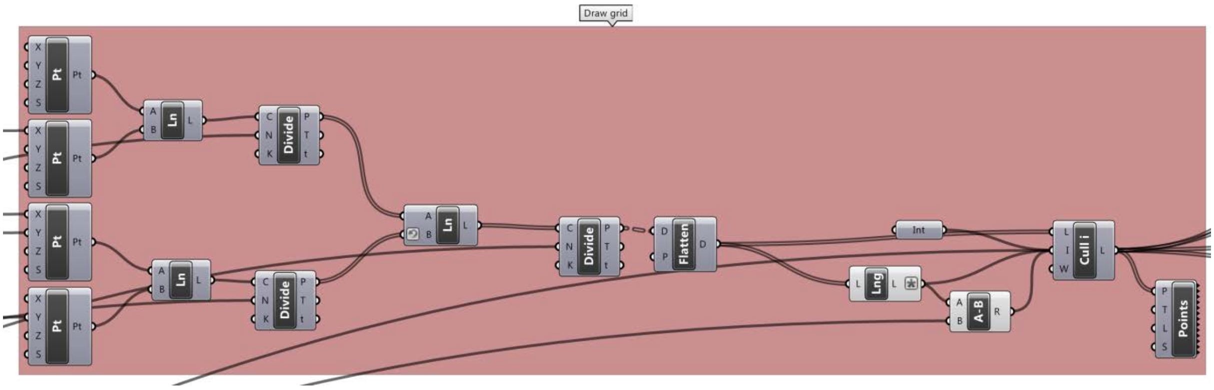


Figure 6.6 Grasshopper: defining the projected grid: the primal grid

In this part the dimensions and the amount of steps (‘Slider n’) set in the input part, will be converted to the primal grid. In this grid there is no height (z coordinate) used.

6.2.4 Export to Excel

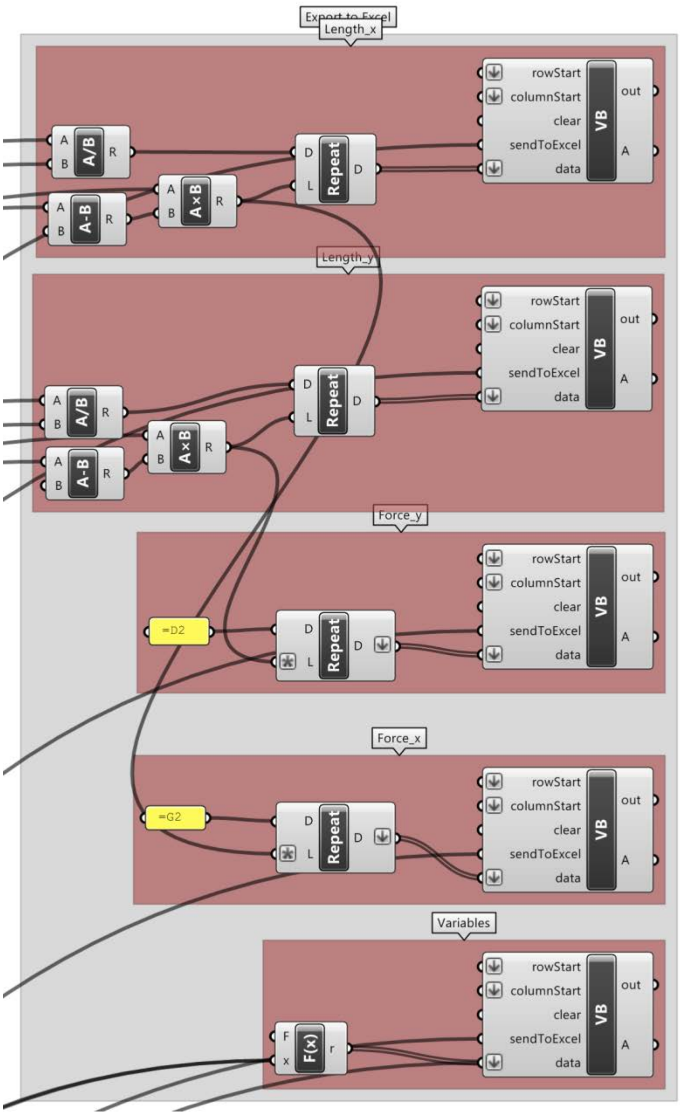


Figure 6.7 Grasshopper: export to Excel

With this part of the tool all the needed values to calculate the least complementary energy is generated. Those values will be exported to Excel when the ‘Send to Excel’ toggle (see 6.1.1) is set to ‘True’. To export the values to Excel, the tool uses some Visual Basic scripts, see Appendix 2. Those scripts provide a direct link between Grasshopper and Excel, so it is possible to get the right values on the right places in Excel. To calculate the least complementary energy, Excel needs the lengths of the members in the x and y direction, the two red areas at the top in figure 6.7. In the third and the fourth red areas there is a cell reference for Excel. With this reference Excel ‘knows’ how many forces there are. In the bottom red area the values for the ‘Slider n (x)’, ‘Slider n (y)’ and the ‘Perimeter reciprocal’ are collected.

In earlier versions of the tool the stream function of a panel in Grasshopper was used. A disadvantage of this was that the file path of this stream cannot be relative. On each computer the path has to be set.

6.2.5 Generating equations for calculating the height

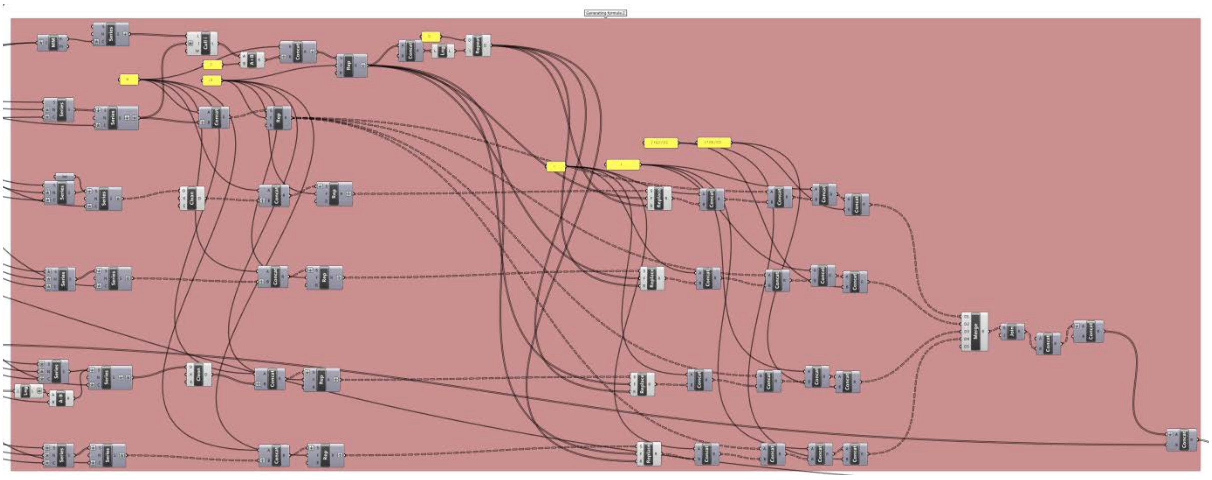


Figure 6.8 Grasshopper: generating equations to calculate the height of a node

The equations mentioned in paragraph 2.4 are generated in this part. First it searches for the free nodes and then it searches for the four nodes around the free nodes. When this is known, the force density method equations can be generated. After that the forces on the nodes should be subtracted. This has to be exported to Excel. The formulas are designed in such a way that Excel immediately can start calculating, see figure 6.9.

Figure 6.9 Grasshopper: generated equations for calculating the height of the nodes

6.2.6 Excel

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Height	Equation_z	Length_y	Force_y	EC_y	Length_x	Force_	EC_x					
2	0	0,000	1,50	0,714	0,765	1,50	0,476	0,340	0,01			Ref perim	10
3	0	0,000	1,50	0,714	0,765	1,50	0,476	0,340					6
4	0	0,000	1,50	0,714	0,765	1,50	0,476	0,340					10
5	0	0,000	1,50	0,714	0,765	1,50	0,476	0,340				Perimeter	10,00
6	5,948412	0,000	1,50	0,714	0,765	1,50	0,476	0,340				EC_total	21,43
7	7,058001	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
8	5,160904	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
9	0	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
10	0	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
11	8,297706	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
12	10,17046	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
13	7,422679	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
14	0	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
15	0	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
16	8,630302	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
17	10,76327	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
18	7,901055	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
19	0	0,000	1,50	0,714	0,765	1,50	0,476	0,340					
20	0	0,000	1,50	0,714	0,765			0,000					
21	7,519453	0,000	1,50	0,714	0,765			0,000					
22	9,436206	0,000			0,000			0,000					
23	7,013658	0,000			0,000			0,000					
24	0	0,000			0,000			0,000					
25	0	0,000			0,000			0,000					
26	4,893738	0,000			0,000			0,000					
27	6,102009	0,000			0,000			0,000					
28	4,636999	0,000			0,000			0,000					
29	0	0,000			0,000			0,000					
30	0	0,000			0,000			0,000					
31	0	0,000			0,000			0,000					

Calculate Z

Figure 6.10 Excel: the calculation sheet

Figure 6.10 shows the calculation sheet in Excel. The values from column B to M are the exported values from Grasshopper, except for the values in cell D2 and G2. The values in those two cells are calculated using the dimensions of the primal grid and the amount of divisions in the primal grid. In paragraph 3.2 the relation between the dimensions of the primal grid and the reciprocal figure were explained according to this formula:

$$l : h = \frac{F_y}{n_y} : \frac{F_x}{n_x}$$

With this formula and the given perimeter of the reciprocal figure, Excel can calculate the forces Force_y and Force_x. Those forces are copied downward for the amount of members in that direction. With the force and the length, the complementary energy can be calculated per member (EC_y and EC_x). The sum of those values is the total complementary energy.

In column B the equations are presented to calculate the height of the free nodes, see figure 6.9 for the equations as exported from Grasshopper.

In Excel a script is created in Visual Basic for Applications (see Appendix 3). This script contains the Solver function of Excel. This script is started by clicking on the button 'Calculate Z' in the Excel sheet.

The Solver function uses the Generalized Reduced Gradient and changes the values in column A (with the height of the nodes) in such a way that the results of the equations from figure 6.9 equal to zero.

A big disadvantage for now is the limitations in the Solver function. This function accepts no more than 100 constraints. To calculate the height of the nodes, there is a maximum of 100 nodes. If there was no maximum, or a higher maximum, it was possible to calculate a much higher accurate structure.

6.2.7 Import height free nodes

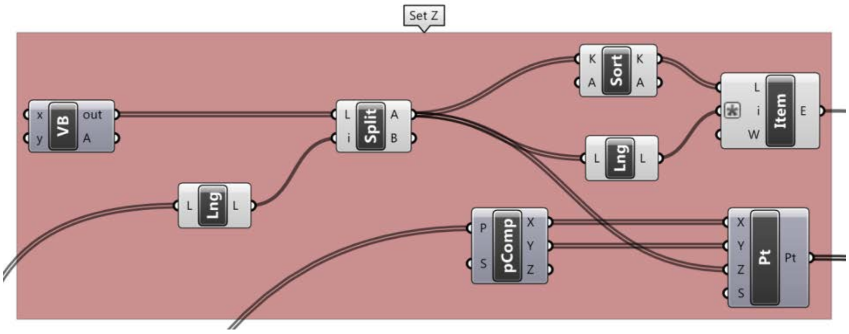


Figure 6.11 Grasshopper: import the height of the free nodes

In this part there is piece of Visual Basic code (see Appendix 4) that imports the height which are calculated in Excel, as described in 6.1.6. Those heights are combined with the x and y coordinates of the primal grid, such that the coordinates of the nodes are known.

6.2.8 Visualization of the load

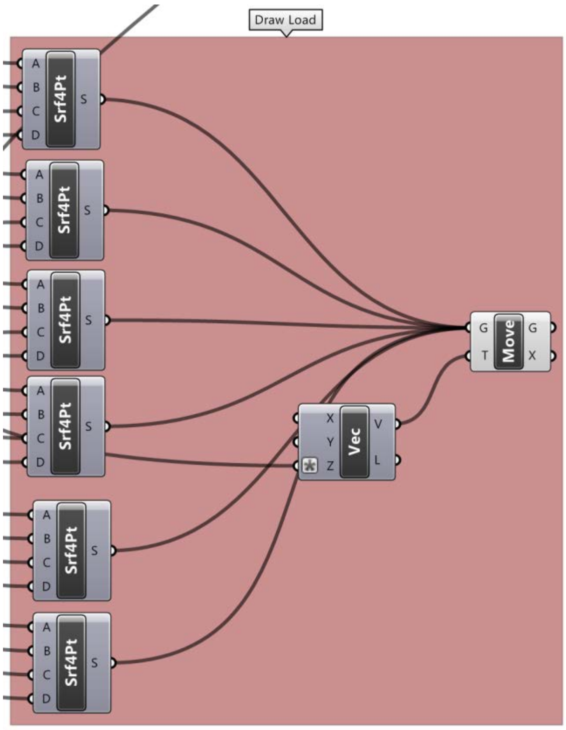


Figure 6.12 Grasshopper: visualization of the load

In this part the load, as set in 6.1.1, will be visualized in Rhino, see figure 6.12 and 6.15.

6.2.9 Visualization of the reciprocal grid

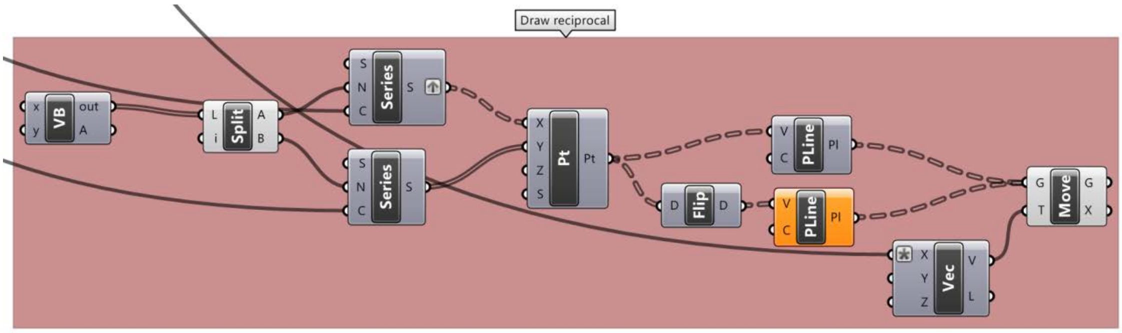


Figure 6.13 Grasshopper: visualization of the reciprocal grid

This part of the tool imports the optimum forces in the x and y direction for the least complementary energy from Excel, by using a Visual Basic script (see Appendix 5). With those values the reciprocal figure is generated and drawn in Rhino, see figure 6.13 and 6.15.

6.2.10 Visualization of the final structure

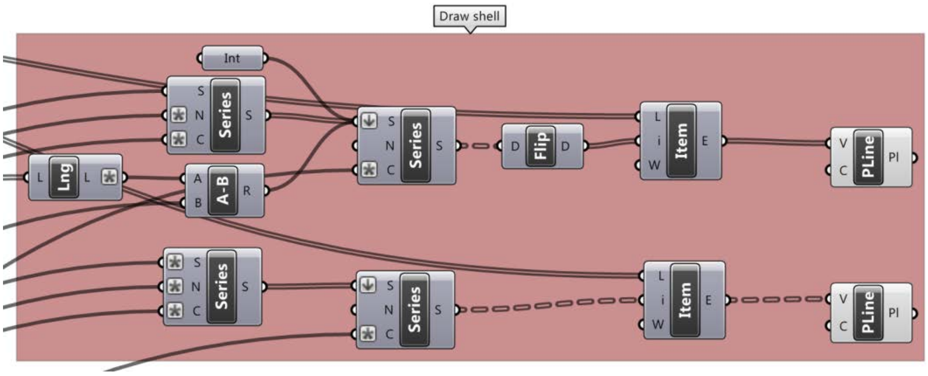


Figure 6.14 Grasshopper: visualization of the final structure

With this part of the tool the final structure is drawn, see figure 6.14 and 6.15.

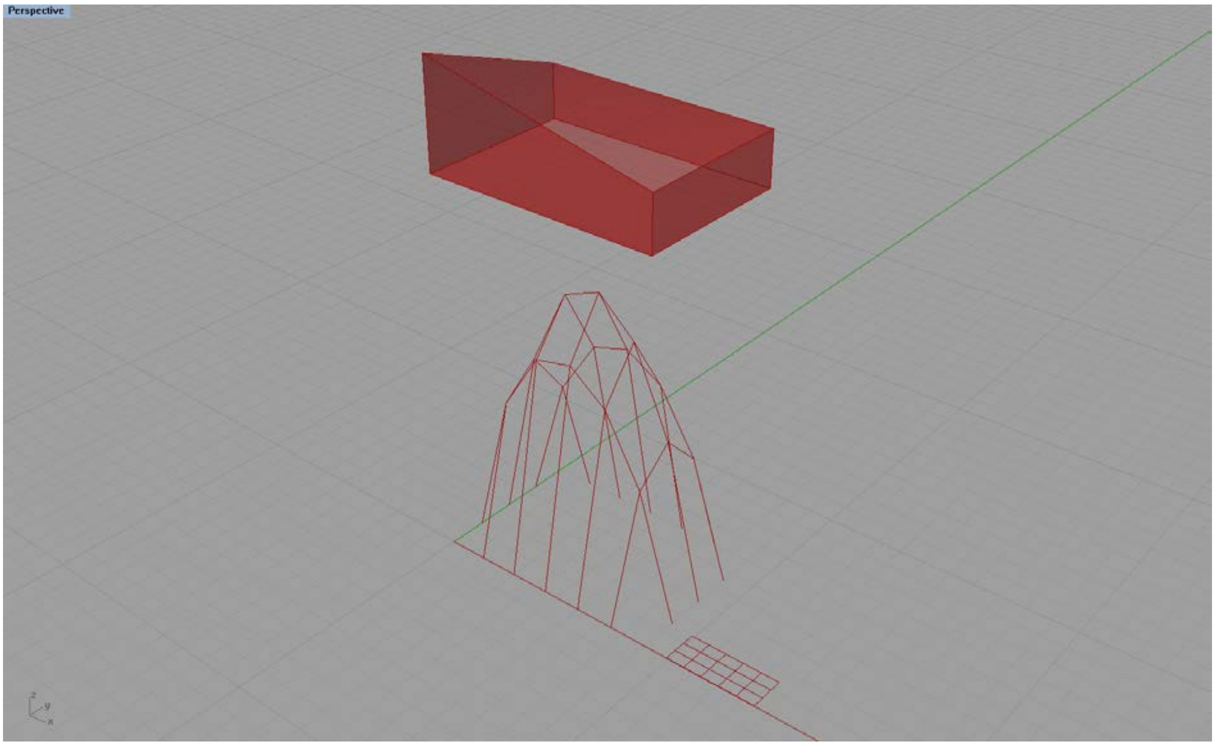


Figure 6.15 Rhino: example of a result of a calculation

6.3 Conclusion

The steps as explained before can be summarized in the process diagram as presented in figure 6.16.

It is easy to calculate different structures.

As mentioned in paragraph 6.2.6, Excel has a maximum of constraint in the Solver function. Therefore the accuracy of the model is not that high as expected. A solution for that problem is maybe another solver function which has not a maximum or a higher maximum.

Another solution is to use a matrix and an inverse matrix. The disadvantage of a matrix is that every calculation needs another matrix. That makes this solution more complex.

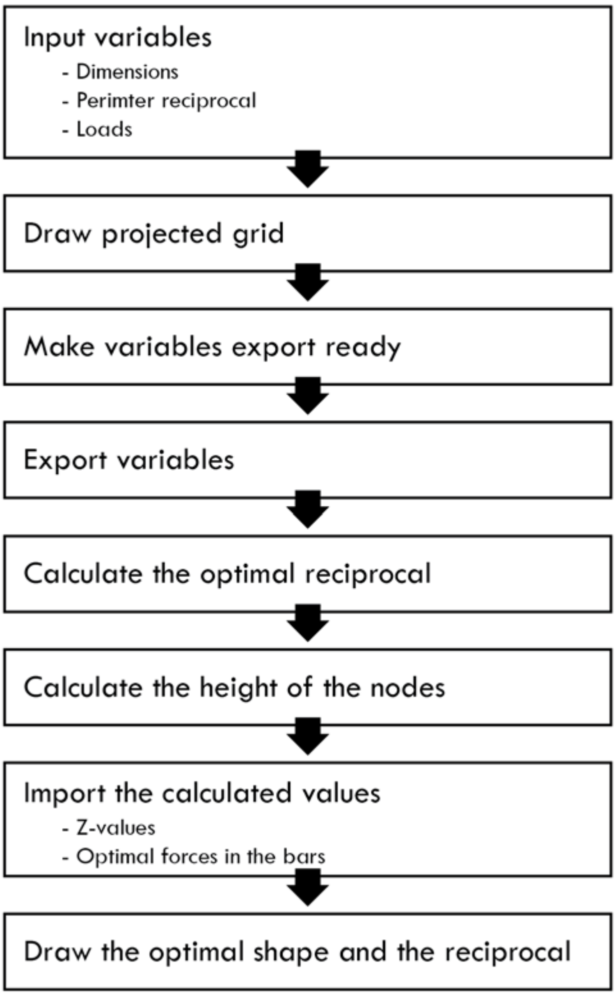


Figure 6.16 Process diagram

7 Results

In figure 7.1 are two results generated by the developed tool. Both solutions have the reciprocal figure and thus the same perimeter, see figure 7.2. The only difference is the magnitude of the point loads. In the left example all the loads are the same, in the right example the magnitude of the load at the right side is bigger than at the left side.

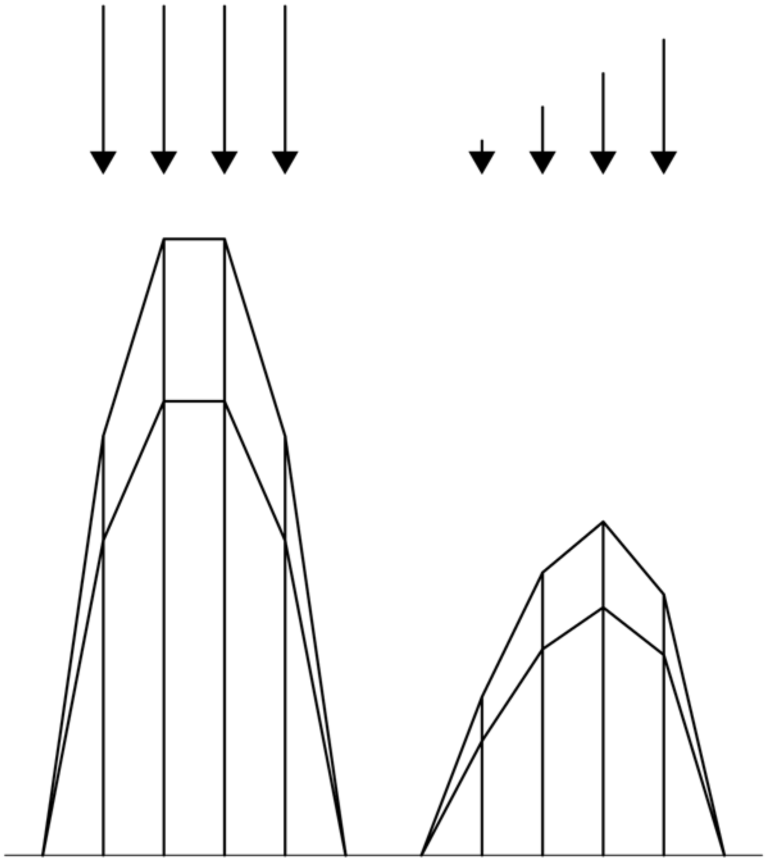


Figure 7.1 Two example results of the tool; left with equal loads, right with non equal loads

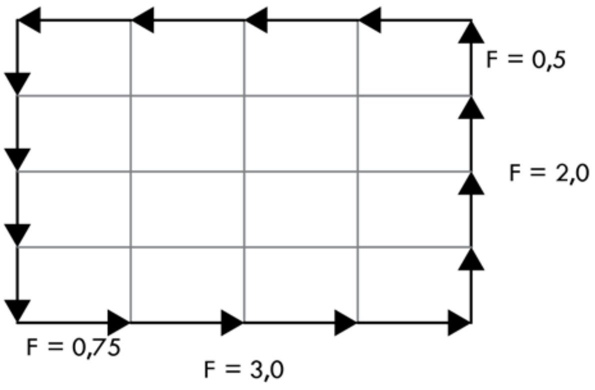


Figure 7.2 Reciprocal figure by the example of figure 7.1

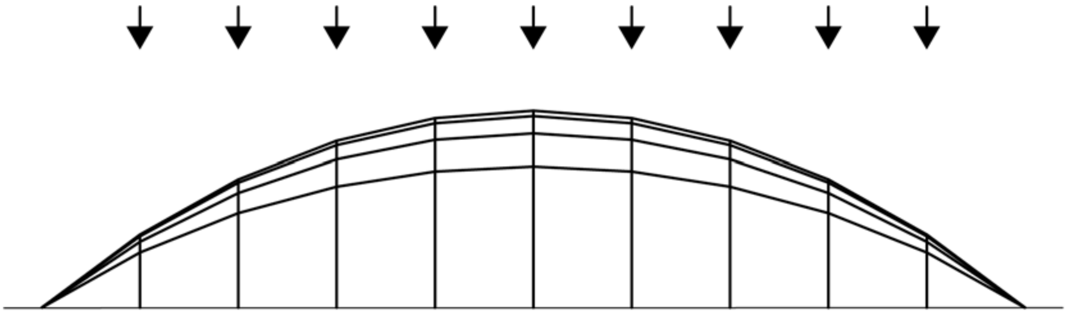


Figure 7.3 Example result with a high step size

In figure 7.3 there is a shell structure with the highest possible amount of calculation steps in the tool. It is a relatively fluent structure, but when it is possible to use more steps the result is a more smooth shape.

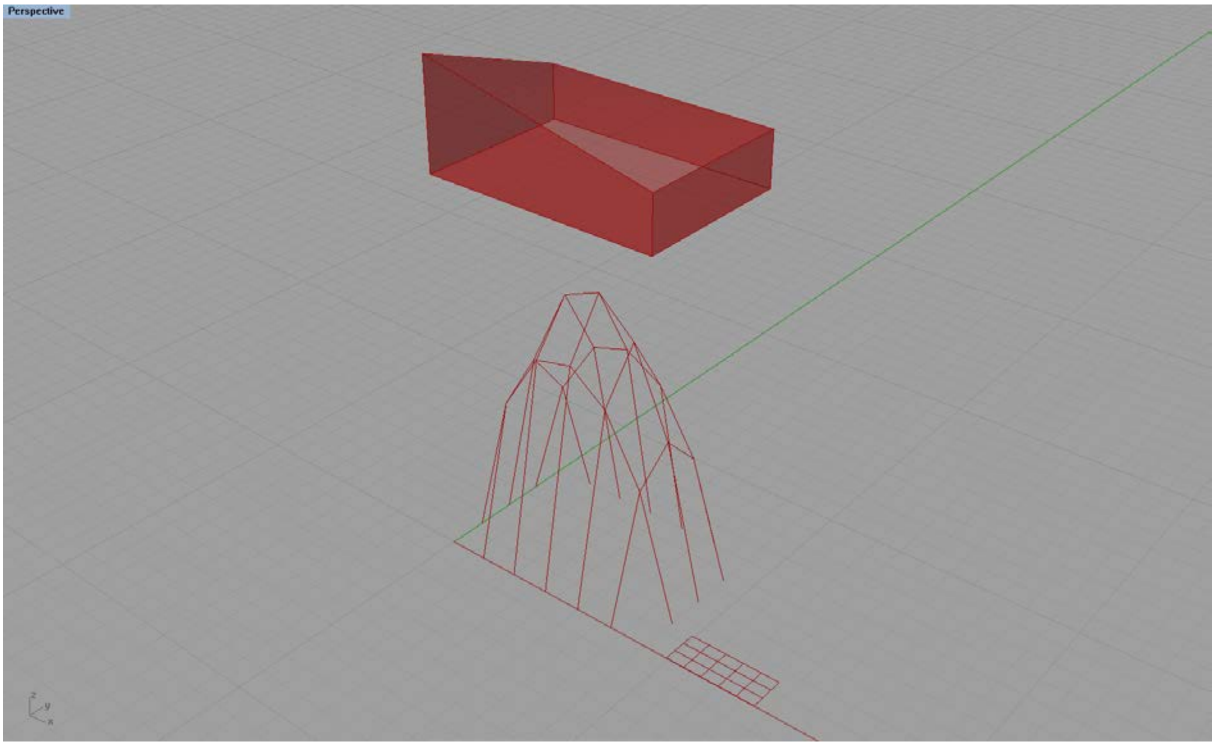


Figure 7.4 Result of a calculation

Figure 7.4 contains the results of a calculation. The inputs are the values as presented in figure 6.3. What we can see in this figure is at the top the graphical representation of the volume of the load. Below the load is the final optimized structure. Right of the structure is the associated reciprocal grid. The magnitude of the forces in the reciprocal figure can be found in the Excel calculating sheet, see figure 6.10.

Part IV
Conclusions

8 Conclusion

This chapter gives a short conclusion of the research and some recommendations to be elaborated further on this subject.

8.1 Conclusion

The research as done in the second part of this thesis provides some techniques to calculate on a faster way the optimal shell structure.

With the found relation between the dimensions of the primal grid and the dimensions of the reciprocal figure it is possible to describe on a fast way the right reciprocal figure.

The tool, designed in Grasshopper, functions as expected; it is simple, relatively fast and with the description in chapter 6 it is good to understand what every part does in the tool.

By programming the Solver function in Excel the limitation of this function came up. Afterwards it was better to look first what the limitations of the programs are, so one can anticipate in that during the development.

It can be considered that the results are only valid for the orthogonal grids. When there is a non-orthogonal grid the results will probably different.

The difference in results from an orthogonal versus a non-orthogonal grid however was not part of this thesis but can be elaborated in future.

8.2 Recommendations

The tool as described in the third part is good enough to make calculations to determine the shape of a shell by different loads, but there are some recommendations:

- As mentioned in paragraph 6.2.6, Excel has a maximum of constraint in the Solver function. Therefore the accuracy of the model is not that high as expected. The maximum of accuracy with the tool is showed in figure 8.1.

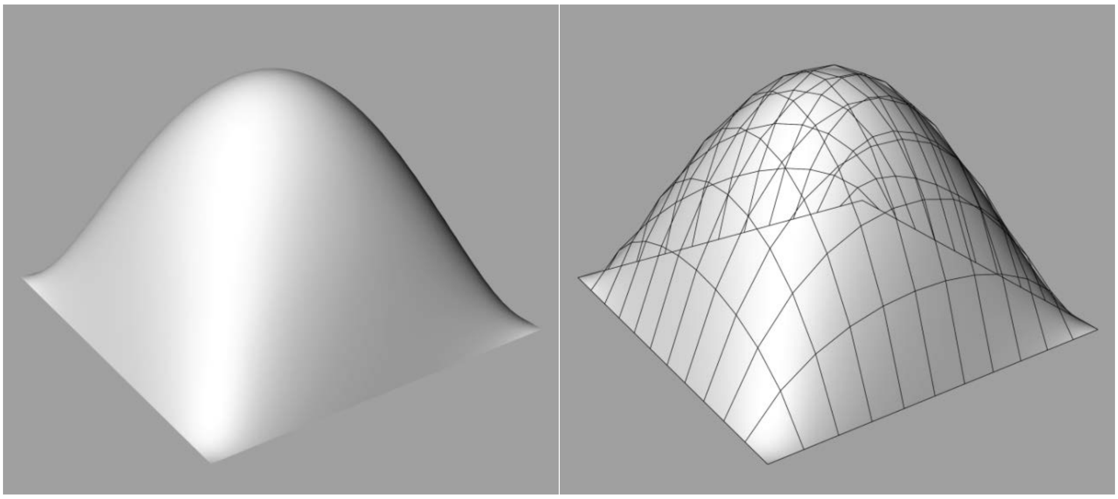
  

  Figure 8.1 Render of a shell, with at the right a calculated structure as layer on it

  When it was possible to use more calculation steps, the members of the structure become smaller and the model more accurate.
  To eliminate this problem, probably the use of a matrix and an inverse matrix in Excel is the way to go.

- To use the tool for more practical load cases, there must be the possibility to make loads with another direction then only in the z direction, and a combination of different load cases. These aspects have not been investigated because of the time.

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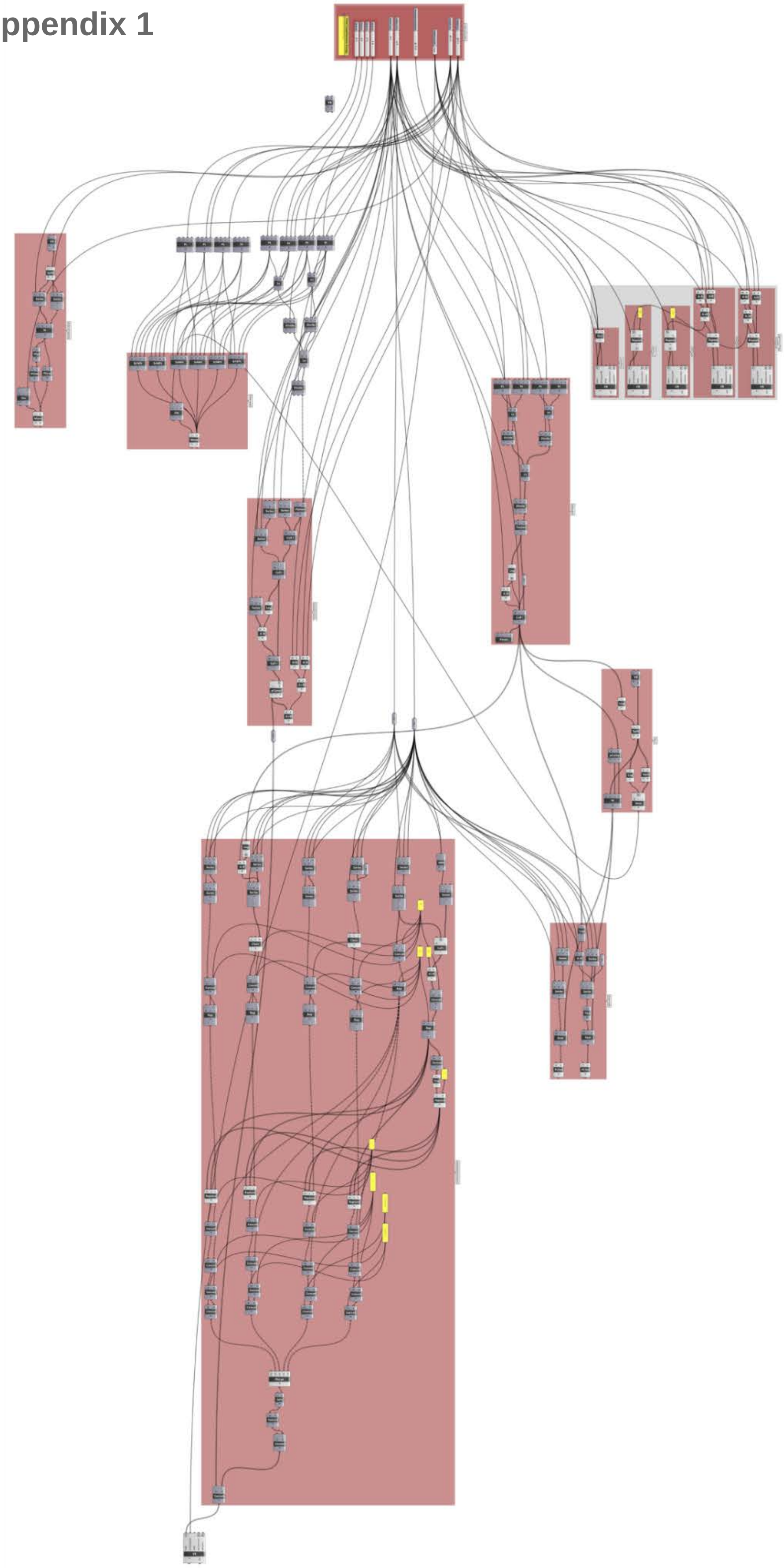
Appendix 1



Appendix 2

Visual Basic script in Grasshopper to export variables to Excel.

```
If sendToExcel Then

      Dim columnList As New Collection
      columnList.Add(data)

' override language
      Dim oldCI As System.Globalization.CultureInfo = sys-
      tem.Threading.Thread.CurrentThread.CurrentCulture
      System.Threading.Thread.CurrentThread.CurrentCulture = New
      System.Globalization.CultureInfo("en-US")

'Grab a running instance of Excel
      Dim xlapp As Object
      xlapp = Sys-
      tem.Runtime.InteropServices.Marshal.GetActiveObject("Excel.App
      lication")


' Get open document
      Dim wb As Object
      wb = xlapp.worksheets(1)

      Dim i As int32
      i = 0
      'For i = 0 To columnName.Count() - 1
      'Retrieve column input from columnList Collection
      Dim columnNum As List (Of Object)
      columnNum = columnList(i + 1)

      If Not columnNum.Count() = 0 Then
      'number of rows for current column
      Dim count As int32
      count = columnNum.Count()
      'Place information for a column
            Dim j As Int32
            For j = 0 To count - 1
            Dim cellPosition As Object
            cellPosition = wb.Cells(j + rowStart + 1, columnStart +
            i)
            cellPosition.value = columnNum(j)

      Next
      End If

      ' Give the user control of Excel
      xlapp.UserControl = True

End If
```

Appendix 3

Visual Basic for Application Macro in Excel to activate the Solver function.

```
Sub Solver_Z()

     ActiveWorkbook.RefreshAll

     Range("A2:A200").Select
     Selection.ClearContents

     SolverReset
     SolverAdd CellRef:="$B$3:$B$102", Relation:=2, Formu-
     laText:="0"
     SolverOk SetCell:="$B$2", MaxMinVal:=3, ValueOf:=0, By-
     Change:="$A$2:$A$150", _
     Engine:=1, EngineDesc:="GRG Nonlinear"
     SolverSolve True

End Sub
```

Appendix 4

Visual Basic script in Grasshopper to import the z coordinates to draw the resulting structure.

```
Dim xlApp As Object

Dim oldCI As System.Globalization.CultureInfo = sys-
tem.Threading.Thread.CurrentThread.CurrentCulture
System.Threading.Thread.CurrentThread.CurrentCulture = New Sys-
tem.Globalization.CultureInfo("en-US")

'Grab a running instance of Excel
     xlApp = Sys-
     tem.Runtime.InteropServices.Marshal.GetActiveObject("Excel.App
     lication")


'call objects
     Dim wb As Object = xlApp.ActiveWorkbook
     Dim sheet As Object
     Dim val As Double
     Dim i As Integer

     sheet = wb.Worksheets(1)

     For i = 2 To 200
     val = sheet.cells(i, 1).value
     print(val)
     Next
```

Appendix 5

Visual Basic script in Grasshopper to import the force size to draw the reciprocal figure.

```
Dim xlApp As Object

Dim oldCI As System.Globalization.CultureInfo = sys-
tem.Threading.Thread.CurrentThread.CurrentCulture
System.Threading.Thread.CurrentThread.CurrentCulture = New Sys-
tem.Globalization.CultureInfo("en-US")

'Grab a running instance of Excel
      xlApp = Sys-
      tem.Runtime.InteropServices.Marshal.GetActiveObject("Excel.App
      lication")


'call objects
      Dim wb As Object = xlApp.ActiveWorkbook
      Dim sheet As Object
      Dim val As Double
      Dim val2 As Double

      sheet = wb.Worksheets(1)

      val = sheet.cells(2, 4).value
      val2 = sheet.cells(2, 7).value
      print(val)
      print(val2)
```