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CRANFIELD

THE FREE STREAMLINE METHOD APPLIED TO THE FLOW AT THE REAR OF A DUCT

by

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-by-

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SUMMARY

The free streamline technique is extended to the problem of two-dimensional jet flow from the rear of a nacelle. Complex potentials for the jet flow and the flow in the free stream are found and from these the equation of the wake streamline and the velocity and pressure distributions are calculated. Some consideration is also given to the corresponding axi-symmetric problem.

MEP

CONTENTS

SYMBOLS

a	half width of jet at infinity
c	half width of nozzle
p	pressure
q	fluid speed ratio (Q/V)
Q	fluid speed
V_1, V_2	skin velocity outside and inside the free streamline respectively
α	angle between duct wall and x-axis
θ	direction of fluid velocity
ρ	density
ϕ	two dimensional velocity potential
Φ	axi-symmetric velocity potential
ψ	two dimensional stream function
Ψ	Stokes stream function in axi-symmetric motion
ω	complex potential $(\phi + i\psi)$
Ω	Kirchhoff's variable; $\log_e \left(-V \frac{dz}{d\omega}\right)$.

1. Introduction

The rear of a symmetrical two-dimensional duct can be idealised by a pair of converging thin straight plates (fig. 1). This paper is concerned with the problem of a jet issuing from such an idealised nacelle into a free stream together with the corresponding axi-symmetric problem. Both the jet and stream are considered to be inviscid and incompressible fluids.

Solutions to these problems are sought using free streamline techniques. The detailed analysis is given in an appendix; reference to the equations in the appendix are given in the form (A.2).

2. The free-streamline method applied to the flow at the rear of a two-dimensional duct

2.1. Steady irrotational discontinuous streamline motions

For two-dimensional irrotational motion in a z-plane ($z = x + iy$) the complex potential ω is given by

$$\omega = \phi + i\psi \qquad \text{.............(1)}$$

and if we consider a complex ω-plane then (1) implies a transformation between the z and ω planes. Now, if Q is the fluid speed at any point z and θ the direction of the fluid velocity at that point

$$\omega'(z) = -Qe^{-i\theta}$$

and in steady motion

$$\frac{p_1}{\rho} + \tfrac{1}{2}Q_1^2 = \frac{p_o}{\rho} + \tfrac{1}{2}V^2$$

where V is the fluid speed at some reference point (say at infinity) where the pressure is p_o.

2.1.1. Kirchhoff's method

The two-dimensional jet issuing from a nozzle leads to a problem in which the direction of the fluid is fixed along certain boundaries whereas along others the pressure is prescribed. Along the latter boundaries the speed is constant and equal to the 'skin velocity'. These boundaries are called free streamlines. To deal with this and similar problems it is usual to employ the method originally due to Kirchhoff[1] in which a third complex variable Ω is introduced and defined by

$$\Omega = \log_e\left(-V\frac{dz}{d\omega}\right) \qquad \text{............(2)}$$

or

$$\Omega = \log\left(\frac{V}{Q}\right) + i\theta \qquad \text{............(3)}$$

2.1.2. The corresponding transformations

If the space occupied by the fluid in the z-plane is mapped on to a complex Ω-plane it will be seen that a boundary for which the fluid speed Q has a constant value is represented by a portion of the axis of imaginaries, whilst corresponding to any straight boundary for which is constant in the z-plane we have a line parallel to the real axis in the Ω-plane. Thus, the diagrams in the ω- and Ω-planes consist, in general, of polygons. These polygonal figures are then mapped on to a complex t-plane so that corresponding points coincide. This mapping is usually effected by means of the Schwarz-Christoffel transformation, the differential equation of which is

$$\frac{dz}{dt} = \frac{C}{(t-t_1)^{a_1/\pi}(t-t_2)^{a_2/\pi}\cdots(t-t_n)^{a_n/\pi}} \qquad \ldots\ldots(4)$$

where $t = t_r$ (real) corresponds to the corner A_r of the polygon where the direction, keeping the interior of the polygon on the left, suddenly changes by a_r and C is a constant.

2.1.3. The intrinsic equation of the free streamline

From (3)

$$-V\frac{dz}{d\omega} = e^{\Omega}$$

and, along a free streamline, ψ is constant and the skin velocity is constant. Thus

$$\Omega = i\theta$$

and also

$$dz = ds\, e^{i\theta}$$

where ds is an element of arc of the free streamline.

Therefore

$$- V ds\, e^{i\theta} = e^{i\theta} d\omega = e^{i\theta} d\phi$$

or

$$ds = - \frac{d\phi}{V}$$

Hence the intrinsic equation of the free streamline is

$$s = \text{const} - \frac{\phi(\theta)}{V}$$

or if the stream function is chosen so that its value is zero on the free streamline

$$s = \text{const} - \frac{\omega(\theta)}{V} \qquad \text{.............(5)}$$

where ω is expressed as a function of θ by means of the $\omega - t$ and $\Omega - t$ transformations.

2.2. The solution of the problem of the idealised jet flow from a duct

This solution is presented in detail in the appendix. In the following discussion the suffices, $_1$ and $_2$ refer to the stream and jet respectively.

The jet is taken as issuing from the funnel-shaped nozzle, given by the equation

$$y = \pm (c - x \tan \alpha)$$

where α is the angle between the duct wall and the x-axis, into a stream of speed V_1 in the positive x direction at infinity downstream. The speed of the jet is V_2 at $x = \infty$, where its width is 2a (fig. 2a). The width of the nozzle opening is 2c. For $x < 0$ the jet and stream are separated by a dividing streamline which is a continuation of the duct wall, and across which the pressure must be continuous. It is shown (A.5,6) that this is satisfied only if the velocities are constant, but not necessarily equal on each side of the dividing streamline. The dividing streamline is thus a free streamline as defined previously (sec. 2.1.1).

2.2.1. The complex potentials

The Ω, ω, and t-plane mappings for this problem are shown in fig. 2b and using the Schwarz-Christoffel transformation (4) the appropriate mapping functions are found to be, for the domain (1) occupied by the stream (A.9,10,17)

$$\omega_1 = \frac{V_1 a}{\pi} \log_e\left(\frac{1-t_1}{2}\right) \;;\; t = \cosh \frac{\pi \Omega_1}{a} \qquad \text{.........(6)}$$

and for the jet domain (2), (A.11,13)

$$\omega_2 = \frac{V_2 a}{\pi} \log_e\left(\frac{1-t_2}{2}\right) \;;\; t_2 = \cosh \frac{\pi \Omega_2}{a} \qquad \text{........(7)}$$

where 2a is the jet width at infinity downstream.
The equations (6) and (7) give the complex potentials for both jet and stream respectively in terms of the speed ratio V/Q and the direction of flow θ.

2.2.2. The equation of the jet boundary

Substituting into (5) for the complex potential the intrinsic equation of the free streamline, which is the jet boundary, is found to be

$$s = -\frac{a}{\pi} \log_e\left(\sin^2 \frac{\pi\theta}{2\alpha}\right) \qquad \text{............(8)}$$

where s is measured from the lip of the nozzle where $\theta = -\alpha$.
Since along the free streamline

$$dz = ds\, e^{i\theta}$$

the equation of the free streamline can be written (A.18) in the form

$$z = ic_o - \frac{a}{\alpha} \int_{-\alpha}^{\theta} e^{i\theta} \cot \frac{\pi\theta}{2\alpha}\, d\theta \qquad \text{...........(9)}$$

for $-\alpha \leq \theta \leq 0$. (See fig. 3)

Also it is shown that the relation between the jet width at infinity (a) and at the origin (c) is

$$a = \frac{c}{1+I} \qquad \text{.............(10)}$$

$$\text{where} \quad I = \frac{2}{\pi} \int_0^{\pi/2} \sin \frac{2\alpha\phi}{\pi} \cot \phi \, d\phi$$

This completes the formal solution of the problem.

2.2.3. Application of the formal solution

Since the complex potentials (6) and (7) are not functions of z explicitly the actual determination of the velocity at any point in the field of flow can be difficult. As an example the flow in the stream (region 1) is calculated near the duct wall. Equation (A.20) expresses the distance up the duct wall from the exit lip C $(0, c_o)$ as an integral of the velocity ratio q_1 $(= Q_1/V_1)$ in the form

$$s = \frac{2a}{\alpha} \int_1^{q_1} \tanh \left(\frac{\pi}{\alpha} \log_e q_1 \right) dq_1 \qquad \text{.............(11)}$$

This equation has been solved numerically for the cases when the inclinations of the duct walls to the x-axis are $\alpha = \frac{\pi}{40}, \frac{\pi}{20}, \frac{\pi}{10}$. The results are plotted in figs. 4 and 4a. Further, simple calculations give the corresponding pressure distributions, which are shown in fig. 5.

2.3. Limitations of the method

It is immediately clear that this method can only apply at the rear of a duct and downstream of the jet exit. Also we can only use the method when the duct walls are straight. In practical cases however the duct angle α will be usually small, and thus the method can be applied to solve the idealised problem of the flow over a considerable portion of the tail of a body from which a jet issues.

The free-streamline technique does require that the fixed boundaries be thin to fulfil the condition that the slope of the free streamline and the slope of the boundary should be continuous at the lip. Also the method cannot solve the problem where the outside and inside walls of the duct are not parallel. Here again the condition of continuity of slope at the jet exit would be violated.

It would be interesting to consider the problem of the duct with a finite trailing edge angle, as mentioned above, more fully. It may be possible to represent the mixing region between the jet and stream by the space bounded by two free streamlines, one leaving the trailing edge parallel to the outer wall and the other parallel to the inner wall of the duct. This space is in some ways analgous to a deadwater region, and it should be possible to find values of the stream function consistent with this hypothesis.

3. Free streamlines in axi-symmetric flow

The existence of constant pressure free surfaces in steady axi-symmetric flow with prescribed fixed boundaries has been proved by Garabedian, Lewy and Schiffer[2]. Theorems proving the uniqueness of these flows have been given by Gilbarg[3], who has shown in particular that there can be only one axi-symmetric flow from an orifice with prescribed flux.

3.1. The stream function

The steady axi-symmetric irrotational motion of an incompressible fluid can be described by a Stokes' stream function Ψ in a meridian plane. If the x-axis is the axis of symmetry and r represents the radial displacement from this axis, Ψ satisfies the partial differential equation

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial\Psi}{\partial x}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right) = 0$$

which, in terms of the complex variables $z = x + ir$ and $z^* = x - ir$ taken as independent variables, becomes

$$\frac{\partial^2\Psi}{\partial z\partial z^*} + \frac{1}{2(z-z^*)}\left\{\frac{\partial\Psi}{\partial z} - \frac{\partial\Psi}{\partial z^*}\right\} = 0 \qquad \text{............(12)}$$

We require the stream function for the flow which has a given surface of revolution as its constant pressure free surface. This surface will intersect the meridian plane considered in a curve C (which we call the free streamline) upon which the stream function must satisfy the conditions

i) The stream function has a constant value which can be taken zero without loss of generality
i.e. on C,

$$\Psi = 0 \qquad \text{............(13)}$$

ii) the pressure is constant along C

i.e. $$\frac{1}{r}\frac{\partial\Psi}{\partial n} = 1 \qquad \text{............(14)}$$

Thus we have a Cauchy initial value problem for the differential equation (12) with initial data given on C by (13) and (14). The solution for Ψ has been given by Darboux[4], in terms of the Riemann function $R(z,z^*,t,t^*)$ as

$$\Psi(z,z^*) = \frac{1}{2i}\int_{\bar{z}^*}^{z} R(z,z^*;t;\bar{t})\,\frac{\partial\Psi}{\partial n_t}\,|dt| \qquad \text{..........(15)}$$

where the integration is carried out along C for $t^* = \bar{t}$ and n_t is the normal to C at the point t.

$$R(z,z^*;t,t^*) = \frac{\sqrt{(z-t^*)(t-z^*)}}{t-t^*}\, F\left(\frac{(z-t)(z^*-t^*)}{(z-t^*)(z^*-t)}\right)$$

where $F(w) = \sum_{m=0}^{\infty} \frac{\left[1.3.5...(2m-3)\right]^2}{2^{2m}(m!)^2}\, w^m$ is the hypergeometric series $F(-\frac{1}{2}, -\frac{1}{2}, 1, w)$ satisfying the equation

$$w(1-w)F''(w) + F'(w) - \frac{F(w)}{4} = 0$$

Garabedian[5] shows that, if the equation of the curve C is

$$\bar{z} = g(z)$$

then (15) reduces to

$$\Psi(z,\bar{z}) = \text{Re}\left\{\frac{1}{2i}\int_{z_o}^{z}\sqrt{(z-g(t))(\bar{z}-t)g'(t)}\; F\left(\frac{(z-t)(\bar{z}-g(t))}{(z-g(t))(\bar{z}-t)}\right)dt\right. \qquad \text{............(16)}$$

which is valid for any z_o on C. The equation (16) gives the stream function at any point in the flow in terms of the equation of the free streamline.

3.2. The free streamline

We now have to apply this stream function to the problem of jet flow. Garabedian considers a special problem where the axis of symmetry is a tangent to the curve C at some point, so that the value of the stream function is zero on the axis of symmetry, which is a streamline, as well as on the free streamline C. This assumption is obviously not valid in the case of idealised jet flow, since there is flow in the jet at infinity downstream, and therefore, the two streamlines cannot intersect or touch. In fact on the axis of symmetry the value of the stream function is $V_2a^2/2$ where V_2 is the jet speed at infinity, where the jet radius is a.

Now on the x-axis for which $z = \bar{z}$

$$F\left(\frac{(z-t)(\bar{z}-g(t))}{(z-g(t))(\bar{z}-t)}\right) = F(1) = \frac{4}{\pi}$$

and hence (16) becomes

$$\frac{V_2 a^2}{2} = -\frac{2}{\pi}\,\mathrm{Im}\left\{\int_{z_o}^{z}\left[(x-g(t))(x-t)g'(t)\right]^{\frac{1}{2}}\,dt\right\} \quad \ldots\ldots(17)$$

where z_o is the point (∞, ia) and the direction of the free streamline at the trailing edge of the duct $(x = 0)$ must be the same as the direction of the duct wall, i.e.

$$\tan\alpha = \frac{d}{dx}\left[\frac{g(z)-\bar{g}(\bar{z})}{2i}\right] \quad \ldots\ldots\ldots\ldots\ldots(18)$$

To find the equation of the free streamline we have to solve (17) with the condition that the free streamline is a continuation of the duct trailing edge (18), and also show that the free streamline so found satisfies both the jet flow and the flow in the free stream.

Garabedian suggests that a known form for C can be carried over from the two-dimensional case. This is not justified in his paper and would appear to be in error since the velocity potential Φ and the stream function Ψ in axi-symmetric motion do not both satisfy the same equation. For this same reason Kirchhoff's method (2.1.1) cannot be applied to the axi-symmetric problem since a complex potential ω, defined by $\Phi + i\Psi$ and which satisfies the equations of motion, does not exist. Also the Kirchhoff velocity parameter Ω cannot be defined.

From these considerations it is seen that a solution of the axi-symmetric jet flow problem at the rear of a nacelle cannot follow the lines of the corresponding two-dimensional problem. The only known complete solution is that for the axi-symmetric equivalent of a Borda mouth-piece found by Southwell and Vaisey[6] using relaxation methods. Having solved (17) for the equation of the dividing streamline the stream function (16) is then known, and from it the velocity field in either jet or stream can be calculated.

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APPENDIX

The 'Free Streamline' method applied to the flow at the rear of a two-dimensional duct

1. Consider a jet of incompressible fluid moving irrotationally in two dimensions through a funnel-shaped duct DC given by the equation

$$y = c - x \tan \alpha \quad \text{for} \quad x \leqslant 0 \quad \text{and} \quad c \quad \text{positive} \qquad (1)$$

and

$$y = x \tan \alpha - c \quad \text{for} \quad x \leqslant 0 \qquad \text{............(2)}$$

and issuing into a surrounding stream of velocity V_1 in the positive x-direction at $x = +\infty$. The issuing jet asymptotes to a stream of speed V_2 and half width a at $x = +\infty$. Due to the symmetry of the configuration, only the flow in the upper half of the z-plane $(z = x + iy)$ will be considered (see fig. 2a). The jet and stream are separated by a dividing, or wake, streamline CB.

The total heads H_1, and H_2 of stream and jet respectively are constant, and thus, if Q_1 and Q_2 are fluid velocities in their respective domains

$$(p_1+\tfrac{1}{2}\rho Q_1^2) - (p_2+\tfrac{1}{2}\rho Q_2^2) = H_1 - H_2 = \text{constant} \qquad \text{........(3)}$$

Now the pressure p is continuous across the free streamline, i.e. $p_1 = p_2$. Thus from (3), along CB

$$Q_1^2 - Q_2^2 = \text{const.} \qquad \text{............(4)}$$

Assuming constant vorticity along CB

$$Q_1 - Q_2 = \text{const.} \qquad \text{............(5)}$$

and hence from (4)

$$Q_1 + Q_2 = \text{const.} \qquad \text{............(6)}$$

Therefore, from (5) and (6), Q_1 and Q_2 are constants along CB and equal to V_1 and V_2 respectively by consideration of their values at $x = +\infty$.

2. The mappings

The upper half z-plane is mapped on the whole of the t-plane so that the stream occupies the upper half, and the jet the lower half, of the t-plane. The corresponding figures in the ω-plane ($\omega = \phi + i\psi$) and the Ω-plane ($\Omega = \log_e \frac{V}{Q} + i\theta$) are mapped upon the t-plane. ϕ and ψ are respectively the velocity potential and the stream function. Ω can be written

$$\Omega = \log_e \left(- V \frac{dz}{d\omega} \right) \; ; \; \frac{d\omega}{dz} = - Qe^{-i\theta} \qquad \ldots\ldots\ldots\ldots(7)$$

where V is the free streamline skin velocity, i.e. V_1 in the stream and V_2 in the jet. These mappings are shown in fig. 2b.

At $x = +\infty$

$$\omega'(z) = - V$$

$$\text{or} \quad \omega(z) = - V(z-ia)$$

so that $\psi = V(a-y)$ at ∞, taking $\psi = 0$ along the dividing streamline which asymptotes to $y = a$. Along AOB, $y = 0$, so this is the streamline $\psi = V_2 a$.

Mapping the semi-infinite strip BCD in the Ω-plane corresponding to the stream, upon the upper half t-plane we have, from the Schwarz-Christoffel theorem, using suffix 1 for region 1, i.e. the stream,

$$\frac{d\Omega_1}{dt_1} = \frac{C}{(t^2-1)^{\frac{1}{2}}}$$

or

$$t_1 = \cosh \left(\frac{\Omega_1}{C} + D \right) \qquad \ldots\ldots\ldots\ldots(8)$$

where C and D are constants. The points B, C and D are taken to correspond to $t = +1, -1, \infty$ respectively.

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Since $t = 1$ at B where $\Omega = 0$, $D = 0$
and $t = -1$ at C where $\Omega = -ia$ $C = \frac{a}{\pi}$

Equation (8) then becomes

$$t_1 = \cosh \frac{\pi \Omega_1}{a} \qquad \text{.............(9)}$$

Similarly, mapping the lower half ω-plane ($\psi \leqslant 0$) upon the upper half t-plane we have

$$\frac{d\omega_1}{dt_1} = \frac{A}{1-t_1}$$

where A is a constant, or

$$\omega_1 = - A \log_e \frac{(1-t_1)}{2} \qquad \text{.............(10)}$$

if we make the point C $(t = -1)$ correspond to the origin in the ω-plane.

Now consider the mapping of the jet (region 2). The transformation between the Ω and t-planes yields, as in equation (9)

$$t_2 = \cosh \frac{\pi \Omega_2}{a} \qquad \text{.............(11)}$$

where, in this case, the real part of Ω_2 is $\log \frac{V_2}{Q_2}$.

Mapping the infinite strip BCAO of the ω-plane on to the lower half t-plane we have

$$\frac{d\omega_2}{dt} = \frac{B}{1-t}$$

or

$$\omega_2 = - B \log_2 \left(\frac{1-t}{2}\right) \qquad \text{.............(12)}$$

On passing through B $(t = 1)$ arg $(1-t)$ changes by π, or ψ_2 changes by $B\pi$ on passing through $t = 1$.

Hence

$$B\pi = - V_2 a$$

and, from (12)

$$\omega_2 = \frac{V_2 a}{\pi} \log_e \left(\frac{1-t_2}{2}\right) \qquad \ldots\ldots\ldots\ldots(13)$$

Thus the complex potentials for each region is known (equations (10) and (13)) in terms of $\frac{V}{Q}$ and θ, except for a constant A which is determined in the next section.

3. <u>The equation of the dividing streamline</u>

The intrinsic equation for a free streamline is, in general,

$$s = \text{const} - \frac{\phi(\theta)}{V}$$

or, if the free streamline is $\psi = 0$

$$s = \text{const} - \frac{\omega(\theta)}{V} \qquad \ldots\ldots\ldots\ldots(14)$$

Thus, from (11) and (13), along CB where $\Omega = i\theta$

$$s = \text{const} - \frac{1}{V_2} \cdot \frac{aV_2}{\pi} \log_e \left(\frac{1 - \cos\frac{\pi\theta}{a}}{2}\right)$$

or

$$s = -\frac{a}{\pi} \log_e \left(\sin^2 \frac{\pi\theta}{2a}\right) \qquad \ldots\ldots\ldots\ldots(15)$$

if we take $s = 0$ at C where $\theta = -a$.

Also, from (9), (10) and (14), the intrinsic equation again measured from C is

$$s = \frac{A}{V_1} \log_e \left(\sin^2 \frac{\pi\theta}{2a}\right) \qquad \ldots\ldots\ldots\ldots(16)$$

and since (15) and (16) must represent the same curve

$$A = -\frac{V_1 a}{\pi} \qquad \ldots\ldots\ldots\ldots(17)$$

The equation of the dividing streamline can be obtained from the relation

$$dz = ds e^{i\theta}$$

from which, using (15)

$$dz = -\frac{a}{\pi}\,\frac{2\sin\frac{\pi\theta}{2\alpha}\cos\frac{\pi\theta}{2\alpha}}{\sin^2\frac{\pi\theta}{2\alpha}}\cdot\frac{\pi}{2\alpha}\,e^{i\theta}\,d\theta$$

$$= -\frac{a}{\alpha}\cot\frac{\pi\theta}{2\alpha}\,e^{i\theta}\,d\theta$$

Thus, since $z = ic$ when $x = 0$, $\theta = -\alpha$, the equation of the dividing streamline becomes

$$z = ic - \frac{a}{\alpha}\int_{-\alpha}^{\theta} e^{i\theta}\cot\frac{\pi\theta}{2\alpha}\,d\theta \qquad \text{.............(18)}$$

for $-\alpha \leqslant \theta \leqslant 0$.

In particular equation (18) gives a relation between the jet width at the duct exit and at infinity. The imaginary part of (18) is

$$y = c - \frac{a}{\alpha}\int_{-\alpha}^{\theta} \sin\theta\cot\frac{\pi\theta}{2\alpha}\,d\theta$$

and, at infinity, $y = a$ and $\theta = 0$

thus $$a = c - \frac{a}{\alpha}\int_{-\alpha}^{0} \sin\theta\cot\frac{\pi\theta}{2\alpha}\,d\theta$$

or $$a = \frac{c}{1+I} \qquad \text{.............(19)}$$

where $$I = \frac{2}{\pi}\int_{0}^{\pi/2} \sin\frac{2\alpha\phi}{\pi}\cot\phi\,d\phi$$

putting $\phi = -\frac{\pi\theta}{2\alpha}$.

4. The flow in the stream near the solid boundary

As an example, consider the flow on the free-stream side of the solid boundary. Here the complex potential is

$$\omega_1 = \frac{aV_1}{\pi} \log_e \left(\frac{1-t_1}{2}\right); \quad t_1 = \cosh \frac{\pi \Omega_1}{2\alpha}$$

Also

$$\frac{d\omega_1}{d\,z} = -\, Q_1 e^{-i\theta}$$

$$= -\, Q_1 e^{i\alpha} \quad \text{on CD}$$

Thus $$d\omega_1 = -\, Q_1 dz\, e^{i\alpha} = -\, Q_1\, ds$$

or $$ds = -\frac{1}{Q_1}\, d\omega_1$$

$$= -\frac{aV_1}{\pi Q_1}\, d\left\{\log_e \frac{\left[1 - \cosh \frac{\Omega\pi}{2\,\alpha}\right]}{2}\right\}$$

Now, in the stream, the real part of Ω, i.e. $\log_e \frac{V_1}{Q_1}$ is negative, thus

$$s = \frac{2a}{\alpha} \int_1^q \tanh \left(\frac{\pi}{\alpha} \log_e q\right) dq \qquad \ldots\ldots\ldots\ldots(20)$$

for $1 \leq q \leq \infty$, where $q = \frac{Q_1}{V_1}$

It will be seen that $s = 0$ when $Q_1 = V_1$ i.e. at C, and that s tends to infinity with Q_1. A graph of Q_1 against s for various values of α is given in fig. 4. In the limiting case $\alpha = 0$, equation (20) shows that there is no variation of Q_1 as s varies.

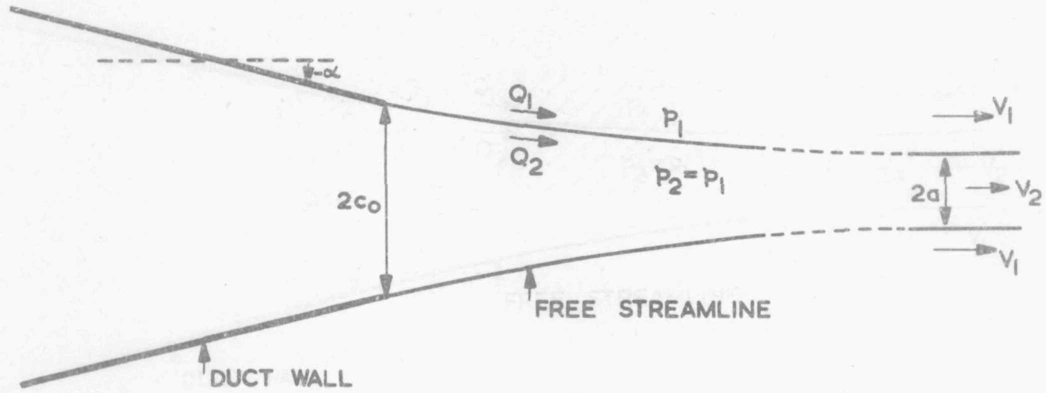

FIG. 1. IDEALISED DUCT WITH JET.

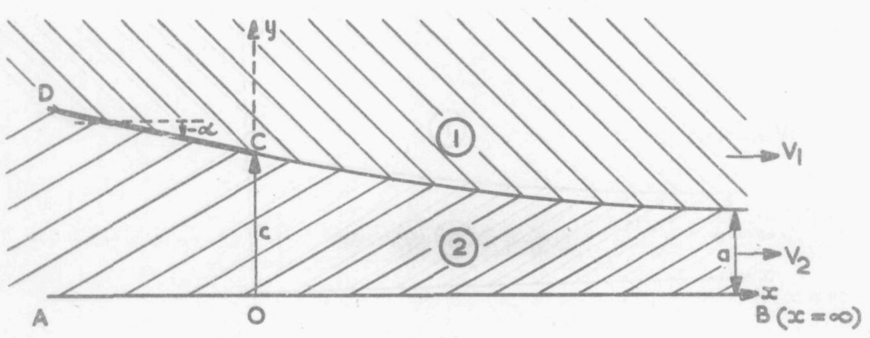

FIG. 2a. FREE STREAMLINE METHOD; Z—PLANE DIAGRAM.

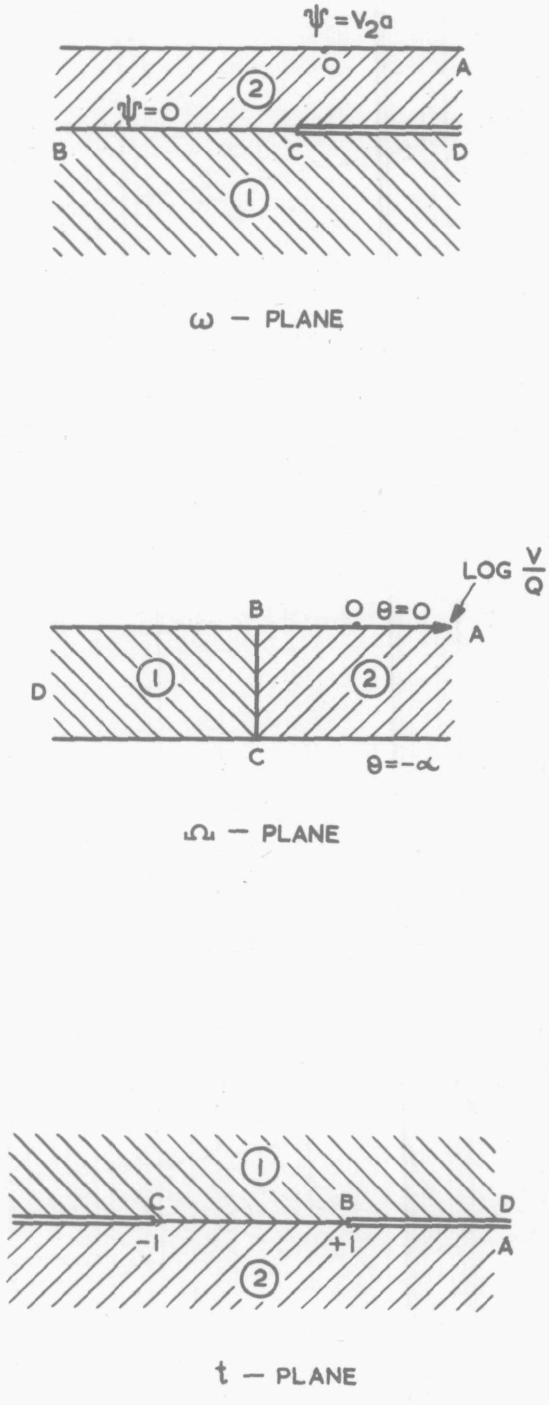

FIG. 2b. FREE STREAMLINE METHOD; ω, Ω & t-PLANE MAPPINGS.

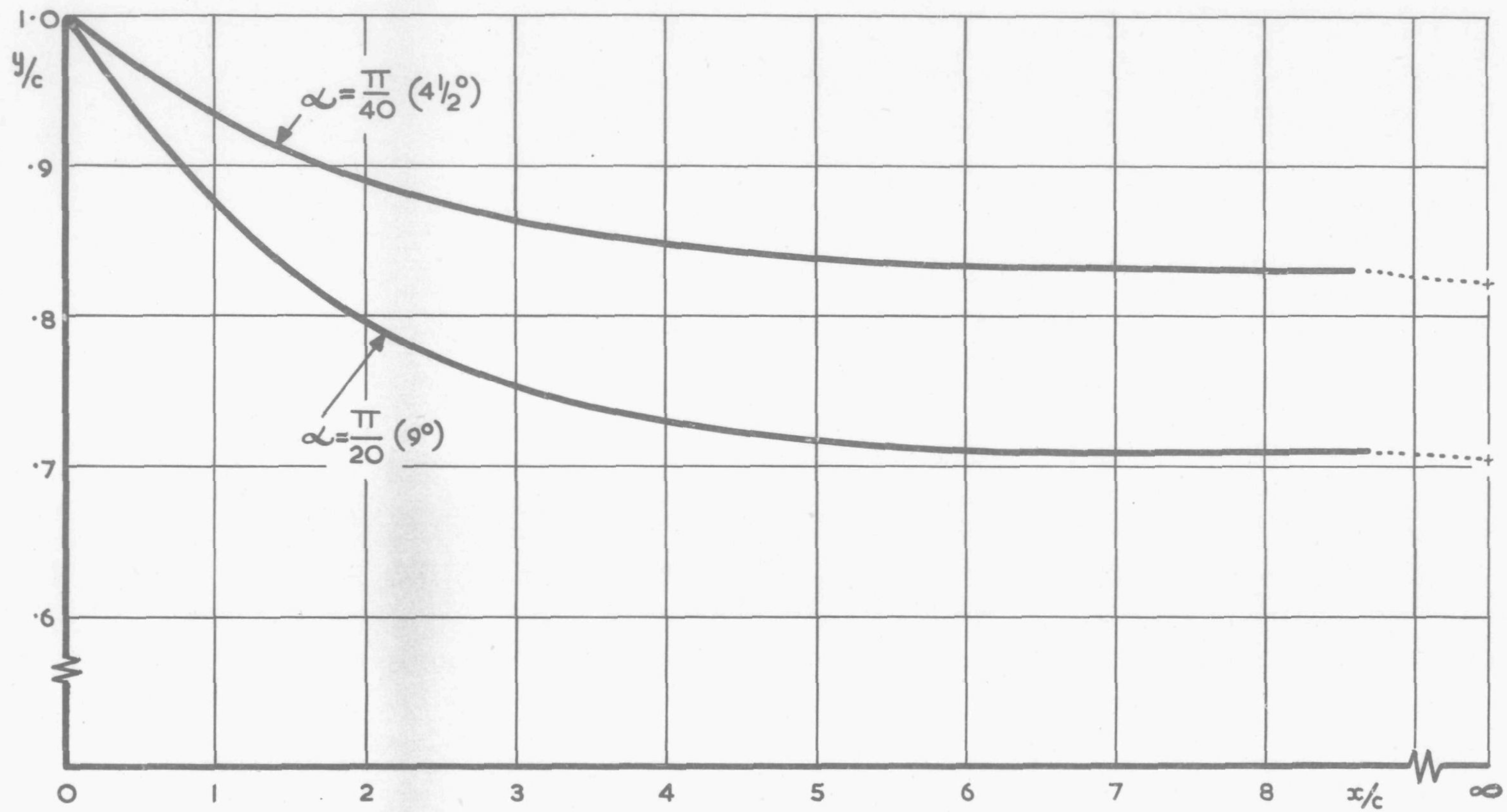

FIG. 3. THE FREE STREAMLINE POSITION (y/c) IN RELATION TO DISTANCE (x/c) DOWNSTREAM OF EXIT.

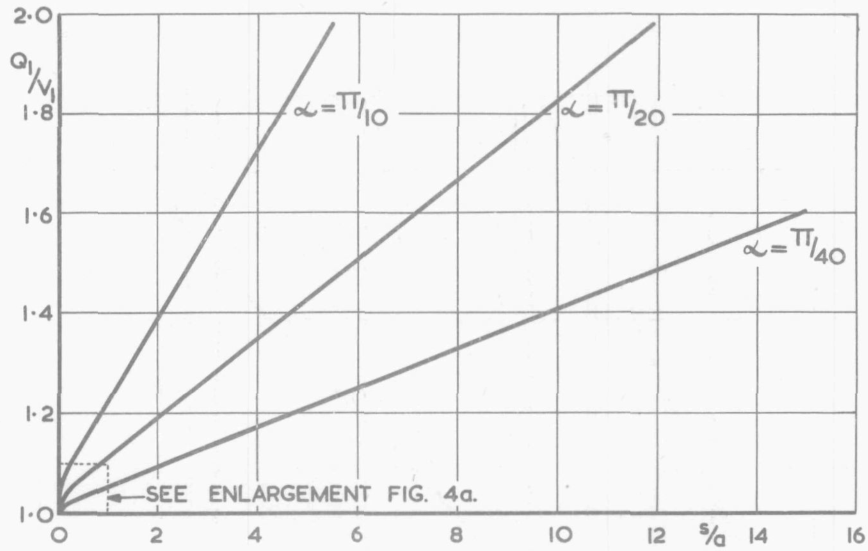

FIG. 4. VELOCITY VARIATION WITH DISTANCE FROM NOZZLE EXIT ON THE OUTSIDE OF THE IDEALISED TWO DIMENSIONAL DUCT.

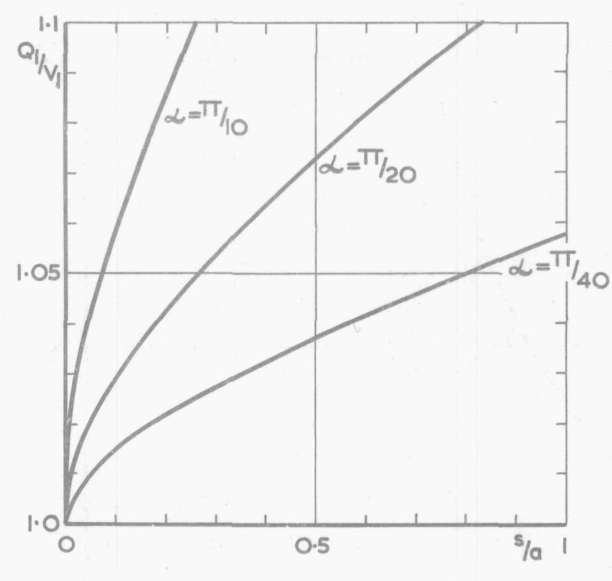

FIG. 4a. ENLARGEMENT OF VELOCITY VARIATION NEAR THE END OF THE IDEALISED DUCT [SEE FIG. 4.]

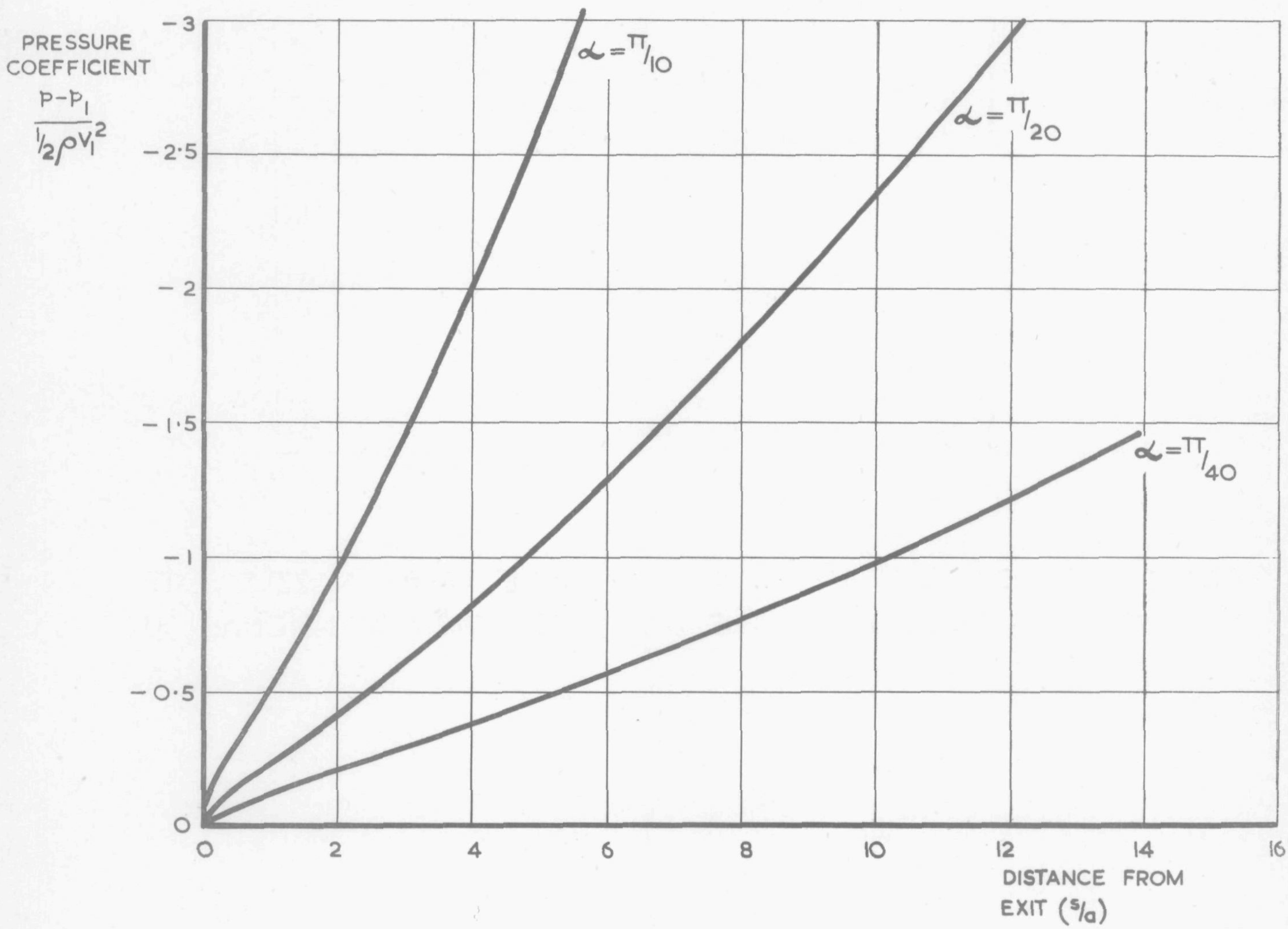

FIG. 5. PRESSURE DISTRIBUTION ON OUTSIDE OF IDEALISED TWO DIMENSIONAL DUCT. (a = HALF JET WIDTH AT ∞)