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DOI
10.1016/j.jsv.2024.118749

Publication date
2025

Document Version
Final published version

Published in
Journal of Sound and Vibration

Journal of Sound and Vibration 595 (2025) 118749

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

The effect of small internal and dashpot damping on a trapped mode of a semi-infinite string

A.K. Abramian [a,*], S.A. Vakulenko [b], W.T. van Horssen [d], A. Jikhareva [c]

[a] *Institute for Problems in Mechanical Engineering Russian Academy of Sciences, V.O., Bolshoy pr. 61, St. Petersburg, 199178, Russia*
[b] *Saint Petersburg Electrotechnical University, 197022 Saint Petersburg, Russia*
[c] *Saint Petersburg University for Industrial Technology and Design, Russia*
[d] *Department of Applied Mathematical Analysis, Faculty EEMCS, Delft University of Technology, Mekelweg 4, 2628 CD Delft, Netherlands*

ARTICLE INFO

Keywords:
Internal damping
Trapped mode
Winkler foundation
Oscillator
String

ABSTRACT

The effect of small internal and dashpot damping on a trapped mode of a 1D-waveguide, that is, a semi-infinite string on a Winkler elastic foundation, has been investigated. At the edge of the string a mass–spring–damper system is attached. The string is assumed to have an internal damping. Four models for the internal damping are considered: air damping, Kelvin–Voigt damping, local Kelvin–Voigt damping, and damping related to time hysteresis. Depending on the internal damping and the parameters in the formulated problem, it will be shown that the amplitude of a trapped mode of the string can decrease or increase with time.

1. Introduction

One dimensional models in mechanics such as strings and beams frequently occur and are, for instance, used to describe the dynamics of suspended bridges or of overhead power transmission lines, stay cables, marine and underwater cables, and the contact cable of a catenary system [1–5]. The authors of [3] use a semi-infinite string which is resting on an elastic foundation as a model for an underwater cable. In [4] an infinite string on a distributed visco-elastic foundation subjected to a gravitational load and two point moving loads is used as “a model for a one-level catenary”. The steady-state solutions were found and analyzed. The dynamic behavior of a long marine cable (string) with a suspended object was considered in [5]. In [5] the dynamics of a string like system with a distributed tangential damping along the cable was investigated. The problem was solved by using a numerical method. Axial vibrations of a slender bar, embedded in an elastic medium can also be described mathematically by a string equation [6]. Another example of a physical object which is modeled as a string with a mass–spring inclusion, is a crystalline lattice [7]. The existence of inclusions in the above mentioned systems leads sometimes to a sharp capture of energy and to an increase in the vibration amplitudes near obstacles [7]. This effect depends sharply on the distribution of the natural frequencies of the string system in the complex frequency plane. Examples of vibration localization in periodic or nearly periodic engineering structures like bladed disks in turbomachines, large space antennas, etc. were considered in [8]. The author of [8] uses the string equation in the analysis of localizations in multi-bay trusses. In [9] the influence of structural damping on propagating waves in periodic and disordered nearly periodic systems has been considered. In case of absence of damping a localization factor is introduced which characterizes the spatial amplitude decay of propagating waves. It was shown that the rate of spatial decay may increase with damping for particular values of the system parameters. Mode localization in a translating string connected to a spring–mass–damper system was considered in [10]. The authors of [10] find that the string attached to a spring–mass–damper system leads to the localization of vibration

* Corresponding author.
E-mail addresses: andabr33@yahoo.co.uk (A.K. Abramian), vakulenfr@mail.ru (S.A. Vakulenko), W.T.vanHorssen@tudelft.nl (W.T. van Horssen), jikhareva.sg@mail.ru (A. Jikhareva).

https://doi.org/10.1016/j.jsv.2024.118749
Received 27 May 2024; Received in revised form 31 August 2024; Accepted 18 September 2024
Available online 26 September 2024

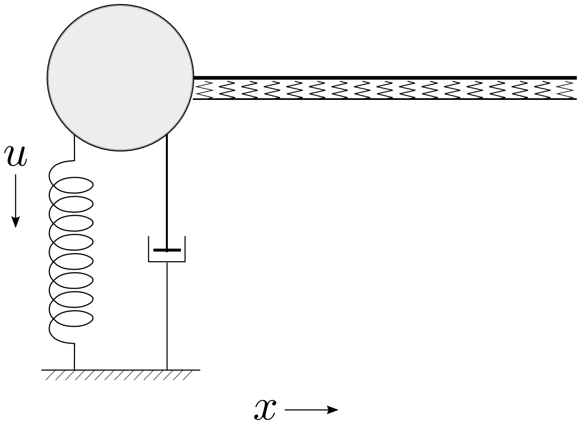

Fig. 1. String with an attached mass–spring–damper system at $x = 0$.

modes in the downstream or upstream region. The existence of localized (trapped) modes in an infinite or semi-infinite string on a Winkler foundation with an inclusion was investigated in [11–15]. This inclusion can be a mass, a spring, or a simple oscillator. When the string tension is constant, one natural frequency corresponds to a trapped mode of oscillation localized near the inclusion. The influence of a non-linearity on the localized mode behavior in such conservative systems was investigated in [16–18]. Other examples of trapped modes which can exist in mechanical elastic systems can be found in [19–23]. The influence of damping on non-linear waves like solitons, breathers etc. are beyond the scope of this paper. In this paper the effect of small internal and dashpot damping on a trapped mode in a 1D-waveguide will be investigated. The 1D-waveguide is a semi-infinite string on a Winkler elastic foundation, which at its edge is attached to a mass–spring–damper system. The rigidity of the spring may be positive or negative. Usually such systems which are attached to the edge of the string are used for damping of undesirable oscillations [5,24]. Four models for the internal damping are considered in this paper: air damping, Kelvin–Voigt damping, local Kelvin–Voigt damping, and damping related to a time hysteresis. In case the infinite system has no internal or dashpot damping it has a natural frequency and corresponding to this frequency a trapped mode [11–14]. For undamped, semi-infinite strings, which we consider in this paper, it will be shown that trapped modes exist and are located in the region of the edge of the string. The expression for the eigenfrequency which corresponds to the trapped mode of the undamped string will be given. Also by using energy estimates, the stability of the undamped system will be investigated. Due to damping in the system it is not clear whether trapped modes will still exist or not. For an undamped finite length string, which is coupled to a linear spring–dashpot system, it has been shown for a special combination of the system parameters that damping destroys the normal modes of vibration and traveling modes are formed [25]. In [26] the dynamics of an "oscillator onto whose mass one end of a string of infinite length is attached" was considered. It was found that in such a system the oscillator's motion will be damped. The author of [26] uses the obtained results to illustrate such physical concepts as "radiation damping, the impedance of a wave propagating medium, Browning motion and Langevin equation". On the basis of these, and on the afore-mentioned results, we can expect that in the systems which are considered in our paper, the trapped modes can vanish because of the presence of a spring–mass–damper or internal damping in the string. Nevertheless, in this paper with the help of an analytical perturbation method it is proved that for particular values of the system parameters the trapped modes still can exist. Moreover, the amplitudes of these localized modes can decrease or increase with time depending on the type of internal damping and spring rigidity. Whether there is damping or not, this growth of the amplitudes in the region of the system, where it is not expected, can be troublesome and should be taken into account in the process of cable design.

2. Statement of the problem

The system under investigation is shown in Fig. 1. The equation of motion for a differential element of the string which describes the oscillations of a semi-infinite string on a Winkler foundation is given as:

$$\rho u_{xx} - N u_{tt} - ku - \epsilon D[u] = 0, \tag{1}$$

where $u(x,t)$ is the displacement of the string in transverse direction, ρ is the mass density per unit length of the string, N is the tensile force in the string, k is the string elastic foundation coefficient (assumed to be constant), ϵ is the string damping coefficient, $t > 0$ is the time, $x \in (0, +\infty)$ is the longitudinal coordinate, $D[u]$ is a linear operator, which corresponds to a small damping effect. The operator $D[u]$ can have different forms:

(i) For local simple air damping one has

$$D[u] = \mu(x) u_t,$$

where $\mu(x) \geq 0$ is a function that defines the space localization;

(**ii**) for local Kelvin–Voigt damping (see [27])

$$D[u] = -(\mu(x)u_{xt})_x, \tag{2}$$

(**iii**) and for local time hysteresis damping

$$D[u] = \mu(x)\beta \int_0^t \exp(-\beta(t-\tau))u_\tau d\tau.$$

To obtain a non-local uniform in space damping we set $\mu(x) \equiv 1$. The following boundary conditions are taken into account for air damping and for time hysteresis damping:

$$u(x,t) \to 0, \quad x \to \infty \tag{3}$$

$$u_x(x,t)|_{x=0} = Mq_{tt} + \eta q_t + Gq. \tag{4}$$

For the Kelvin–Voigt damping the boundary conditions are as follows:

$$u(x,t) \to 0, \quad x \to \infty, \tag{5}$$

$$(u_x(x,t) + \epsilon\mu(x)u_{xt})|_{x=0} = Mq_{tt} + \eta q_t + Gq, \tag{6}$$

where M is the constant mass in the system, η is the small damping coefficient (referred to as a damping coefficient coupled to the string system), G is the constant stiffness which can be positive or negative, and $q = q(t) = u(0,t)$ is the displacement of the mass–spring–damper system at $x = 0$,. Notice that differential Eq. (1), and the boundary conditions can be transformed to a dimensionless form when we rescale the variables. For the rescaling, the following relations are used:

$$a = \frac{kL^2}{N}, \quad \delta = \frac{\epsilon L^2}{NT_0}, \quad c_0^2 = \frac{N}{\rho}, \quad T_0 = 2\pi\sqrt{\frac{M}{G}}, \quad L = c_0T_0, \quad x = L\bar{x},$$
$$u = L\bar{u}, \quad t = T_0\bar{t}, \quad b_1 = \frac{ML}{NT_0^2}, \quad b_2 = \frac{L\eta}{NT_0}, \quad b_3 = \frac{GL}{N},$$

where $\bar{x}$ is the dimensionless longitudinal coordinate, $\bar{t}$ is the dimensionless time. Note that $\delta = \frac{\epsilon}{NT_0}$ for Kelvin–Voigt damping, and $\delta = \frac{\epsilon L^2}{N}$ for a time hysteresis damping. To simplify notations, the overbar is omitted and we obtain the equation:

$$u_{xx} - u_{tt} - au - \delta D[u] = 0. \tag{7}$$

The following boundary conditions are taken into account for air damping and for time hysteresis damping:

$$u(x,t) \to 0, \quad x \to \infty, \tag{8}$$

$$u_x(x,t)|_{x=0} = b_1q_{tt} + b_2q_t + b_3q. \tag{9}$$

For the Kelvin–Voigt damping the boundary conditions are as follows:

$$u(x,t) \to 0, \quad x \to \infty, \tag{10}$$

$$(u_x(x,t) + \delta\mu(x)u_{xt})|_{x=0} = b_1q_{tt} + b_2q_t + b_3q, \tag{11}$$

We introduce a small dimensionless parameter δ, and suppose that $b_2 = \bar{C}\delta$, where the positive coefficient $\bar{C}$ defines the ratio between the damping coefficient of the (to the string) attached system and the internal damping of the string itself. This coefficient may be large, or small or of order 1. However, the condition $\bar{C} \ll \delta^{-1}$ always has to be satisfied.

3. Oscillatory localized solutions

Let us outline the main ideas of our mathematical approach. For the undamped system we use some energy estimates (see the next subsection), and then we substitute a specific exponential solution into the string equation. The energy estimate helps makes it possible to draw some conclusions about the system behavior. Similar estimates can be made for the cases of air and Kelvin–Voigt damping. For the case with time hysteresis damping such an estimate leads to a complicated expression which cannot be analyzed easily, simple, and for that reason is not presented in this paper. The direct substitution of an exponential solution into the governing equation gives a better possibility to analyze the system behavior. This solution is defined by a complex number, for which the imaginary part $i\Omega$ is an oscillation frequency, and its real part, depending on its sign, defines an increase or decrease rate of the solution in time. From the boundary conditions we obtain a nonlinear equation for Ω. For small damping coefficients this equation can be investigated by standard perturbation methods. Further, we validate the correctness of the obtained results by numerical methods.

3.1. Energy considerations

We consider first the case $\delta = 0$, and we derive energy expressions for (complex-valued) solutions. To do that we multiply both sides of (1) by u_t, and then integrate with respect to x over $(0, +\infty)$ assuming that all norms $\|u\|^2 = \int_0^{+\infty} |u|^2 dx$, $\|u_x\|^2 = \int_0^{+\infty} |u_x|^2 dx$

and $\|u_t\|^2 = \int_0^{+\infty} |u_t|^2 dx$ are bounded. Integrating by parts and taking into account the boundary conditions (8) and (9), one obtains the energy conservation law for the total energy consisting of the string energy term and the oscillator energy term:

$$E[u] = E_s[u] + E_{osc}[u] = const, \tag{12}$$

where

$$2E_s[u] = \|u_x\|^2 + a\|u\|^2 + \|u_t\|^2 = \int_0^{+\infty} (|u_x(x,t)|^2 + a|u_x(x,t)|^2 + |u_t(x,t)|^2) dx,$$

and

$$2E_{osc}[u] = b_1|u_t(0)|^2 + b_3|u(0)|^2.$$

If $b_3 \geq 0$ and $\delta = 0$ the energy conserved, according to Eq. (12). Then, both the oscillator energy E_{osc} and the string energy E_s are nonnegative, therefore, this energy estimate implies that solutions are stable: and so, the exponential instability is impossible.

If $b_3 < 0$ the situation is more intriguing. Then, it is possible that $E_{osc} < 0$, and exponentially growing (or decreasing) in t solutions can exist only if the total energy of these solutions is equal to zero. In fact, we shall see that solutions $u(x,t)$ exponentially depend on t.

Now let us consider a simple air damping with $\delta > 0$. For simple air damping we find (for $\delta \neq 0$):

$$dE[u]/dt = -\delta\|u_t\|^2. \tag{13}$$

Then, the energy decreases in time according to (13). This implies stability for $b_3 \geq 0$ as is explained above. It is possible to get the same type of estimate for Kelvin–Voigt damping in the form:

$$dE[u]/dt = -\delta \int_0^{+\infty} \mu(x) u_{xt}^2 dx. \tag{14}$$

For this case we can draw the same conclusion about stability for the system as for the case with air damping. In the next subsections we study, construct, and consider some special solutions of our formulated problems for $u(x,t)$.

3.2. Harmonic solutions

We are looking for (almost) harmonic oscillations, which might be slowly decreasing in time:

$$u(x,t) = U(x)\exp(-i\Omega t),$$

where U is still an unknown function in x, and $i = \sqrt{-1}$. The function $U(x)$ has to satisfy the following equation:

$$U_{xx} + (\Omega^2 - a)U + i\delta\Omega\bar{D}[U] = 0, \tag{15}$$

where the operator $\bar{D}[U]$ is equal to μU in the case of simple air damping, $\bar{D}[U] = -(\mu U_x)_x$ in the case of Kelvin–Voigt damping, and $\bar{D}[U] = -U(x)\mu(x)\frac{i\beta\Omega}{\beta - i\Omega}$ for time hysteresis damping.

3.3. The undamped case ($\delta = 0$, $b_2 = 0$)

Let us suppose that $\delta = 0$, and let $U_0(x)$ and Ω_0 be solutions of (15) and the corresponding non-perturbed frequency equation,respectively. We introduce $d^2 = a - \Omega_0^2$, and choose d such that $\mathrm{Re}\, d > 0$. Then $U_0(x) = C\exp(-dx)$ is a solution of Eq. (15) which is decreasing for $x \to +\infty$ (where C is a constant). To obtain the equation for Ω_0, we use the boundary condition (9), and as a result we have

$$b_1\Omega_0^2 - b_3 = d = \sqrt{a - \Omega_0^2}. \tag{16}$$

This implies

$$b_1^2\Omega_0^4 + (1 - 2b_1b_3)\Omega_0^2 + b_3^2 - a = 0. \tag{17}$$

From (17) it follows for physically relevant cases that $a > \Omega_0^2 > b_3/b_1$, and

$$\Omega_0^2 = \frac{(2b_1b_3 - 1) + \sqrt{1 + 4ab_1^2 - 4b_1b_3}}{2b_1^2}. \tag{18}$$

The properties of the solutions for Ω_0 with Re $d > 0$ depend on the parameters b_1, b_3, and a in a complicated way. To simplify the analysis, we choose the Winkler foundation coefficient a as a bifurcation parameter. Let us introduce $a_c = b_3/b_1$. It should be observed that a_c is the square of the (attached mass–spring) oscillator dimensionless frequency ω_{osc}, that is, $\omega_{osc}^2 = b_3/b_1$. Note that for harmonic localized solutions relation (12) gives

$$E[U_0(x)\exp(-i\Omega_0 t)] = E_s + E_{osc}, \tag{19}$$

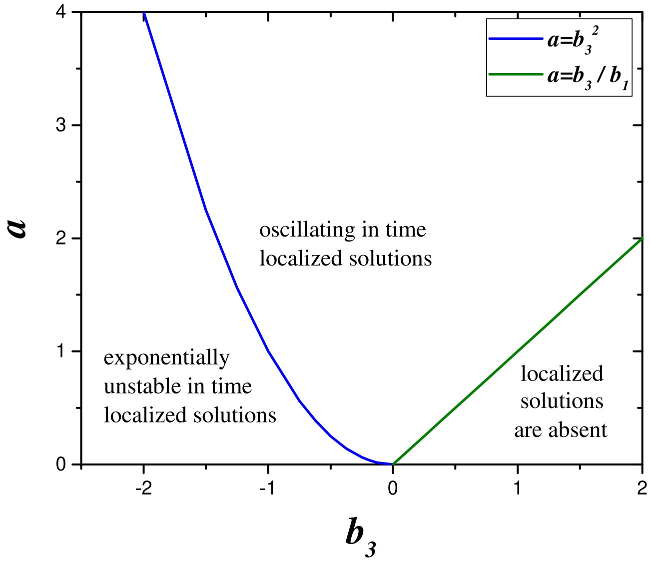

Fig. 2. This bifurcation diagram shows how the localized solutions depend on the two parameters a and b_3, while $b_1 = 1$ is fixed (note that we can take $b_1 = 1$ by changing the variables x, t; so, really we have two parameters). The whole parameter plane is divided into three regions, corresponding to the absence of solutions, stable oscillating solutions, and, in time exponentially increasing solutions.

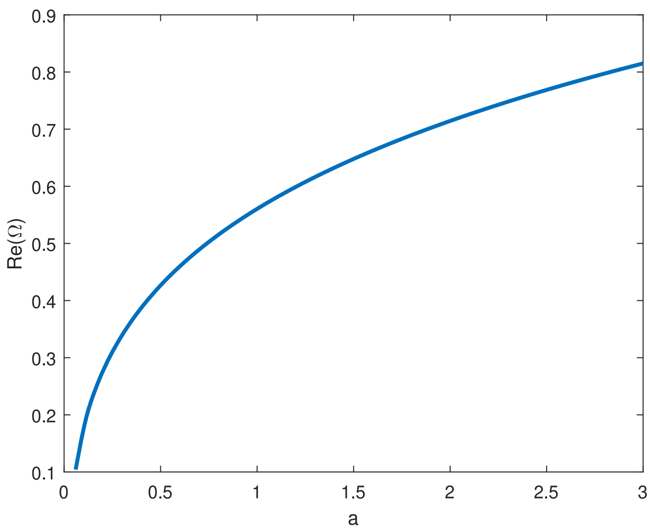

Fig. 3. This plot shows the real part of the oscillation frequency Ω_0 as a function of the Winkler foundation coefficient a. The imaginary part of Ω_0 is equal to zero. The parameter values are $b_1 = 2$, and $b_3 = 2$.

where

$$E_s = (2\,\mathrm{Re})^{-1}\left(|d|^2 + |\Omega_0|^2 + a\right),$$

$$E_{osc} = b_1|\Omega_0|^2 + b_3.$$

From (18), and some numerical simulations (see also Fig. 2) we obtain the following:

(I) For $b_3 > 0$ and $a > a_c$: there exist monotone decreasing localized in x solutions $U_0(x)$ with $\Im\Omega_0 = 0$;

(II) For $b_3 > 0$ and $0 < a < a_c$ localized in x solutions $U_0(x)$ are absent;

(III) For $b_3 < 0$ there is a bifurcation value $\tilde{a}_c = b_3^2$. For $a > \tilde{a}_c$, we have solutions $U_0(x)\exp(-i\Omega_0 t)$ with $\Re\Omega_0 = 0$, which exponentially decrease in t for $\mathrm{Im}\,\Omega_0 < 0$ and exponentially increase in t for $\mathrm{Im}\,\Omega_0 > 0$. For $0 < a < \tilde{a}_c$, one has solutions $U_0(x)\exp(-i\Omega_0 t)$ with $\Re\Omega_0 \neq 0$, which oscillate in t.

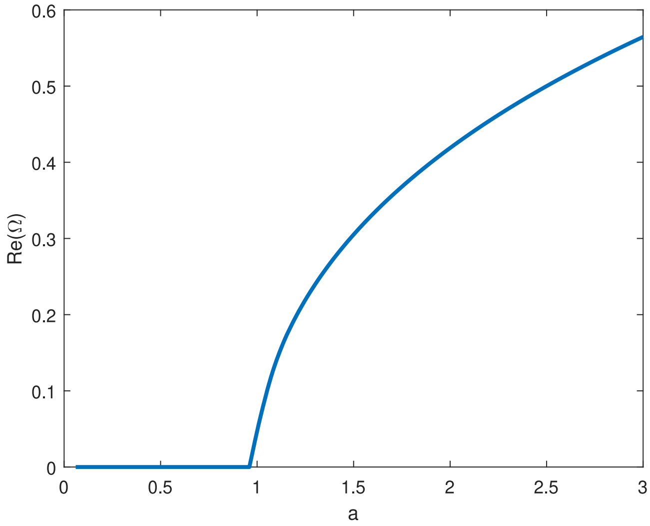

Fig. 4. This plot shows the real part of the oscillation frequency Ω_0 as a function of the Winkler foundation coefficient a. The parameter values are $b_1 = 2$, and $b_3 = -1$.

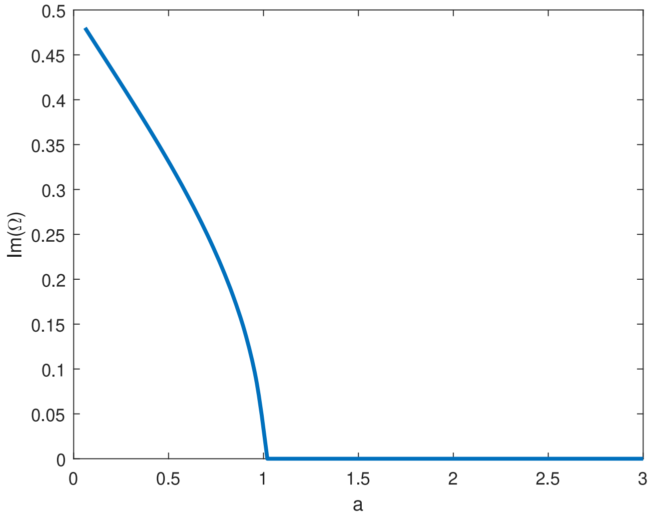

Fig. 5. This plot shows the imaginary part of the oscillation frequency Ω_0 as a function of the Winkler foundation coefficient a. The parameter values are $b_1 = 2$, and $b_3 = -1$.

These properties are illustrated in the bifurcation diagram Fig. 2 and in Figs. 3–5. The results (I, II) can be obtained as follows. Note that for $b_3 > 0$ relation (12) implies that Ω_0 is a real number: $\operatorname{Im}\Omega_0 = 0$. Therefore, $\Omega_0 \in (0, a)$ and Ω_0^2 can be found as an intersection of two curves (see comment in Fig. 6).

Consider the case (III). Note that the existence of the second bifurcation value $\tilde{a}_c = b_3^2$ follows from (18). In fact, at $a = \tilde{a}_c$ the root Ω_0^2 defined by Eq. (18) changes sign. For example, if $2b_1b_3 < 1$ the quantity Ω_0^2 is a negative real number for $0 < a < \tilde{a}_c$, thus $i\Omega_0$ is real, see also Fig. 6. Furthermore, according to (12) in the case $b_3 < 0$ exponentially decreasing or exponentially increasing solutions should have constant in time total energy E_{tot}. It is clear that E_{tot} for exponentially decreasing solutions is equal to 0 (because all terms that define this energy go to zero as $t \to +\infty$). Similarly for exponentially increasing solutions all these terms are proportional to $\exp(ct)$ for some constant $c > 0$, thus, the energy is conserved only under the condition that $E_{tot} = 0$ when the sum of all these terms are equal to zero. So, we obtain that all decreasing or increasing solutions have zero energy. It can be checked by (19) that for such solutions $E_s = -E_{osc}$, and thus these solutions really have zero energy. So, for $b_3 < 0$ we encounter an instability. In the next section we consider how damping influences this instability.

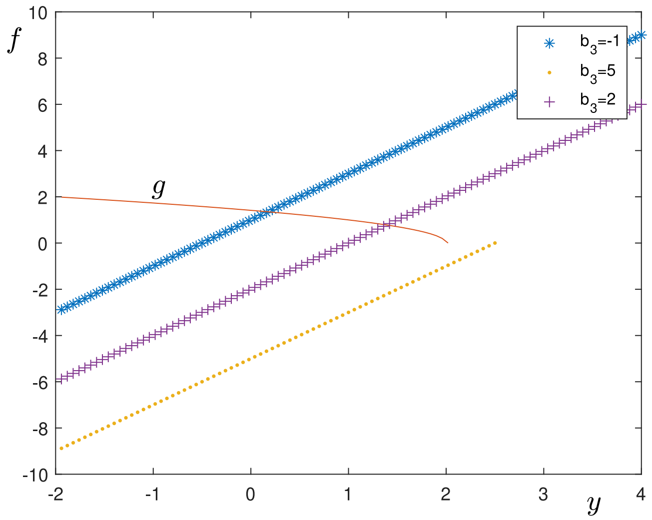

Fig. 6. The frequency Ω_0 (for real or purely imaginary values) can be obtained from $\Omega_0^2 = y$ and $f(y) = g(y)$, where $g(y) = \sqrt{a-y}$ and $f(y) = b_1 y - b_3$. This plot shows the intersections of the curves $f(y)$ and $g(y)$ for $a = 2, b_1 = 2$ and for different values of b_3. For $b_3 < 0$ the corresponding value of y is negative.

3.4. Perturbation theory for weak damping

For small $\delta > 0$ we apply the standard perturbation method, that is, the Rayleigh–Schrödinger perturbation theory, which is well known in quantum mechanics [28,29], but, was earlier developed by Lord Rayleigh for acoustics [30]. Note that this approach can be considered as a variant of the Lyapunov–Schmidt reduction method (see [31]), and we follow [31] (adapted to our problem). We consider here the case $\operatorname{Im}\Omega_0 = 0$. We are looking for solutions in the form

$$u(x,t) = \exp(-i\Omega t)w(x), \quad \Omega = \Omega_0 + \delta\tilde{\Omega}, \tag{20}$$

where $w(x)$ is a new unknown function, and $\tilde{\Omega}$ is a small perturbation of the frequency. It is natural to look for w in the form [31]

$$w(x) = cU_0(x) + \delta\tilde{U}(x), \tag{21}$$

where $\tilde{U}$ is a small perturbation and c is a unknown constant. As it was shown above, $U_0 = C_0\exp(-dx)$, and we choose C_0 such that $\|U_0\|^2 = \int_0^\infty U_0^2(x)dx = 1$. To obtain the minimum in L_2-norm of the perturbation we adjust it to c :

$$P(c) = \delta^2\int_0^{+\infty}\tilde{U}(x)^2 dx = \int_0^{+\infty}(w(x) - -cU_0(x))^2 dx. \tag{22}$$

Differentiating the function $P(c)$ with respect to c we get

$$c = \int_0^\infty wU_0 dx\Big(\int_0^\infty U_0^2(x)dx\Big)^{-1}, \tag{23}$$

and

$$\int_0^\infty U_0(x)\tilde{U}(x)dx = 0. \tag{24}$$

The last relations have a simple geometric interpretation. To obtain the minimum in norm, in the one-dimensional subspace of $L_2(0,\infty)$ containing those functions $cU_0(x)$ with different c, we choose the orthogonal projection of w on this subspace. By using these relations in the next subsections, we will study different damping mechanisms.

3.4.1. Air damping

Taking into account terms of order δ, we obtain for the simplest air damping the following boundary value problem for $\tilde{\Omega}$ and $\tilde{U}$:

$$\tilde{U}_{xx} - a\tilde{U} - \theta_0\tilde{U} + i\Omega_0\mu U_0 = \tilde{\theta}U_0, \tag{25}$$

$$\Big(\tilde{U}_x - (b_1\theta_0 + b_3)\tilde{U}(x)\Big)|_{x=0} = \Big(b_1\tilde{\theta}U_0(x) + \bar{C}(-i\Omega_0)U_0(x)\Big)|_{x=0}, \tag{26}$$

$$\tilde{U}(x) \to 0 \quad x \to +\infty, \tag{27}$$

where $\theta_0 = -\Omega_0^2$ and $\theta_0 + \delta\tilde{\theta} = -\Omega^2$. Note that $\tilde{\Omega} = -\tilde{\theta}/2\Omega_0$, and so $\tilde{\Omega}$ is involved in these equations via $\tilde{\theta}$. We multiply both sides of (25) with U_0, and further we integrate with respect to x from $x = 0$ to $x = \infty$. To simplify formulas, it is convenient to introduce $\gamma = -\tilde{\Omega}$. Then, by taking into account (24), by observing that $\|U_0\| = 1$, and by using (26) and (27), we obtain

$$\tilde{\theta} = i\Omega_0(1 + 2b_1 d)^{-1}\Big(2\bar{C}d + \int_0^\infty \mu(x)U_0^2 dx\Big). \tag{28}$$

Finally we find that

$$u = \exp(-i\Omega_0 t + \delta\gamma t)(U_0(x) + \delta\tilde{U}(x)) \tag{29}$$

with

$$\gamma = -(2(1 + 2b_1 d))^{-1}(2\bar{C}d + \int_0^\infty \mu(x)U_0^2 dx). \tag{30}$$

This shows the existence of solutions which slowly decrease in time. If the damping is uniform in space, that is, $\mu(x) = \mu = const$, we have

$$\gamma = -(2(1 + 2b_1 d))^{-1}(2\bar{C}d + \mu). \tag{31}$$

3.4.2. Local Kelvin–Voigt damping

For the problem with local Kelvin–Voigt damping one has

$$\tilde{U}_{xx} - a\tilde{U} - \theta_0\tilde{U} - i\Omega_0(\mu U_{0x})_x = \tilde{\theta}U_0, \tag{32}$$

$$\big(\tilde{U}_x - (b_1\theta_0 + b_3)\tilde{U}(x)\big)|_{x=0} = \Big(b_1\tilde{\theta}U_0(x) + \bar{C}(-i\Omega_0)U_0(x) - i\mu(0)\Omega_0 U_0\Big)|_{x=0}. \tag{33}$$

As before we multiply both sides of (32) with U_0 and integrate with respect to x from $x = 0$ to $x = \infty$. By using (27) and (33) we obtain:

$$\tilde{\theta} = -((1 + 2b_1 d))^{-1} i\Omega_0\Big(\int_0^\infty (\mu(x)U_{0x}^2 dx + 2\bar{C}d\Big) \tag{34}$$

and

$$\gamma = -(2(1 + 2b_1 d))^{-1}\Big(2d\bar{C} + \int_0^\infty \mu(x)U_{0x}^2 dx\Big). \tag{35}$$

For uniform in space Kelvin–Voigt damping we obtain similarly

$$\gamma = -(2(1 + 2b_1 d))^{-1}\big(\mu d^2 + 2d\bar{C}\big). \tag{36}$$

3.4.3. Hysteresis damping

Let us consider the case of localized in space time hysteresis damping. The operator $D[u]$ acts on non-perturbed solutions $U_0(x)\exp(-i\Omega_0 t$ in the following way:

$$D[U_0(x)e^{-i\Omega_0 t}] = I(x,t) = \beta\mu(x)\int_0^t \exp(-\beta(t-\tau))(U_0(x)\exp(-i\Omega_0\tau))_\tau d\tau.$$

We note that for large t and $\operatorname{Im}\Omega_0 = 0$ one has

$$I(x,t) \approx \frac{-i\Omega_0\beta\mu(x)U(x)}{\beta - i\Omega_0}\exp(-i\Omega_0 t)$$

up to terms which are exponentially small in t. We see that for large t the hysteresis damping works as a simple air damping multiplied with a complex coefficient $\frac{\beta}{\beta - i\Omega_0}$. Therefore, one has (for large t)

$$\gamma \approx -\frac{\beta}{2(1 + 2b_1 d)(\beta - i\Omega_0)}\Big(\int_0^\infty \mu(x)U_0^2 dx + 2d\bar{C}\Big). \tag{37}$$

We observe that hysteresis not only leads to a small amplitude decrease but also produces a small perturbation in the oscillation frequency. In Fig. 7 and in Fig. 8 the order ϵ corrections (that is, $\widetilde{\Omega}$) in $\Omega = \Omega_0 + \delta\widetilde{\Omega}$ are presented for the cases with Kelvin–Voigt damping and with time hysteresis damping, respectively. In Fig. 7 and in Fig. 8 it is easy to observe that instabilities do not occur.

4. Numerical results, validation and discussion

To verify the obtained analytical results, the problem is also studied numerically for comparison. As numerical method the "sponge layer" method [32,33] can be applied to model the boundary condition (10) at infinity. The name "sponge layer" is adopted to indicate that wave absorption in the present study is achieved by introducing damping terms in the governing equations. The method consists of adding an additional dissipative term $\nu(x)u_t$ in a small absorption zone adjacent to the boundary. The wave noticeably loses energy when passing through the absorption zone which reduces the reflection of the wave from the boundary of the computational domain. The size of the absorption zone and the function $\nu(x)$ were chosen for the least reflection of waves

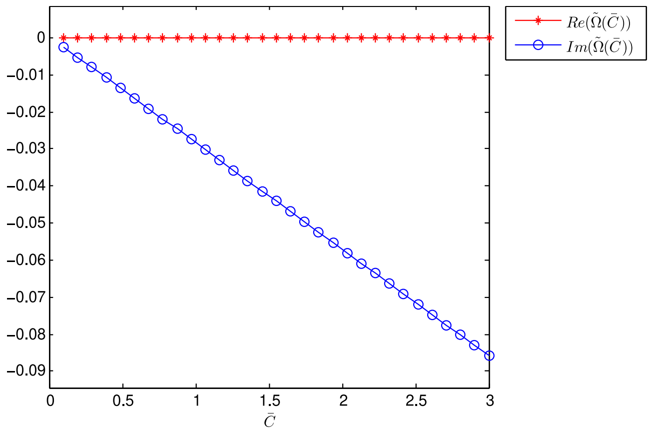

Fig. 7. This plot shows the results for the case with Kelvin–Voigt damping. The parameters are $a = 0.3$, $\bar{C}$ in the interval $[0, 3]$, and $b_3 = 0.01$.

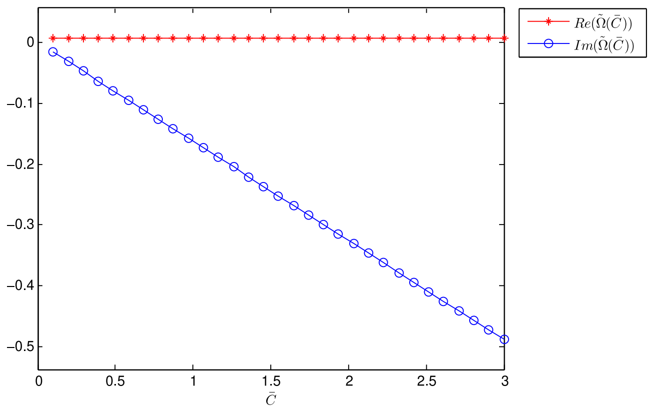

Fig. 8. This plot shows the results for the case with time hysteresis damping. The parameters are: $a_1 = 0.3$, $b_1 = 0.1$, $b_3 = 0.01$.

from the domain boundary. The optimal size of the "sponge layer" can be determined based on the analytical results for a trapped oscillation frequency. The "sponge layer" should contain 15 waves. The size of the "sponge layer" is 50 units of length which is equivalent to 15 maximum wavelengths for the considered parameters. The calculation domain is 200 units of length. The equation with the introduced "sponge layer" is taken in the form:

$$u_{xx} - u_{tt} - v(x)u_t - au - \delta D[u] = 0, \tag{38}$$

where $v(x) = (0.02x - 3)H(x - 150)$, $x = 0$ to 200 units of length, and the equation is approximated by a three-point implicit difference scheme. Eq. (38) is solved by using the matrix sweep method [34]. The uniform grid is chosen with a step size of 0.05 units for length, and a time step of 0.0001 units for time. The calculations for a string without damping are presented in Fig. 9, and show that a trapped mode can exist and it is localized at the edge of the system.

In Fig. 9a the mode shape is represented, and in Fig. 9b we can see the corresponding time history at $x = 0$. Also, from what we can observe in Fig. 9a and in Fig. 9b the analytical and numerical results are very close to each other for times t of order ϵ^{-1}. The curve presented in Fig. 10 shows the instability case when the amplitude of the string vibration is growing in time at the edge $x = 0$. The numerical results for the case when the string has air damping is presented in Fig. 11. As in the case without damping the differences between the analytical and numerical results are small. The curves for the Kelvin–Voigt and for the hysteresis damping are similar to the curves for air damping, and are for that reason not presented. The curve presented in Fig. 11a shows that near the edge of the string a localized mode still exists. The difference in behavior between the trapped mode of the string without damping and the localized mode with damping, is that in the last case the mode is slowly decaying in time as we can observe in Fig. 11b.

5. Conclusions

In this paper a semi-infinite string on a Winkler elastic foundation has been considered. The edge of the string is attached to a mass–spring–damper system. An analytical perturbation method is proposed in this paper which allows us to analyze the dynamics of the considered system. We can conclude that in this paper for an undamped system the localized solution was found, as well

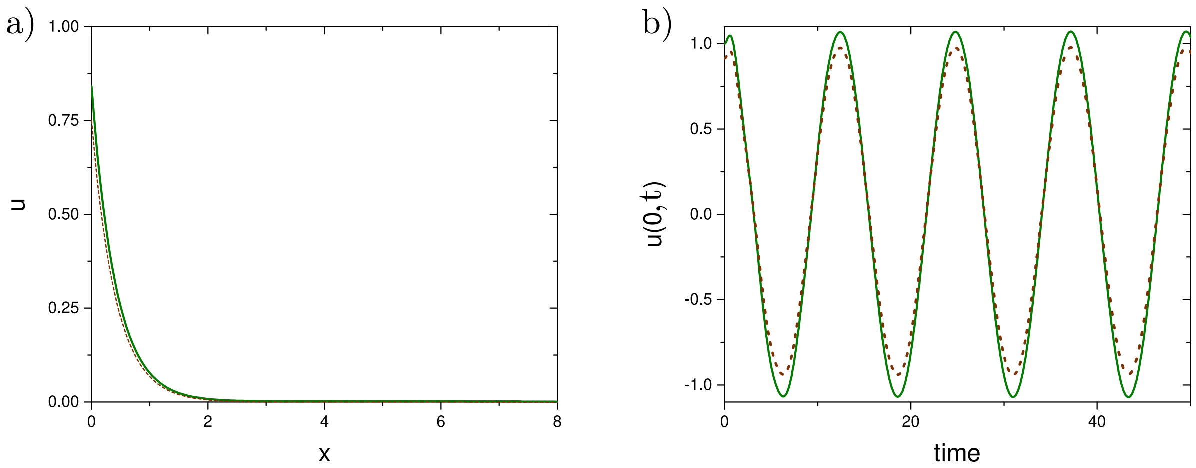

Fig. 9. This plot shows:(a) the shape, and (b) the time behavior at $x = 0$ of a localized mode for the case without damping. The solid line corresponds to the numerical results and the dotted line to the analytical results. The parameters are: $b_3 = -2$, $a = 6$, $b_1 = 1$.

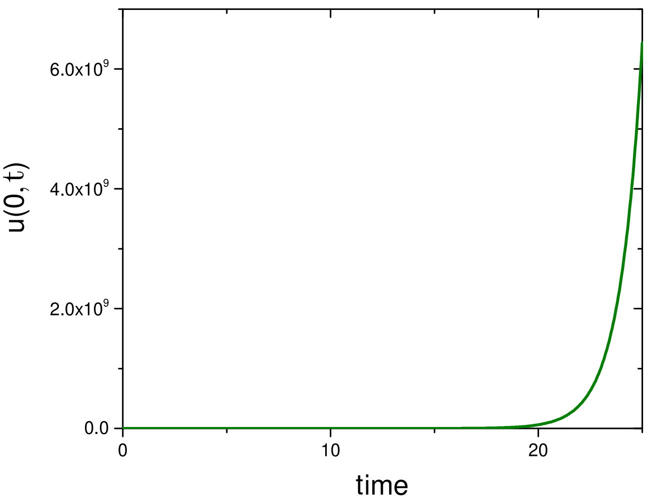

Fig. 10. This plot shows the increasing amplitude of a localized mode in time for the case without damping. The parameters are: $b_3 = -2$, $a = 0.3$, $b_1 = 0.1$.

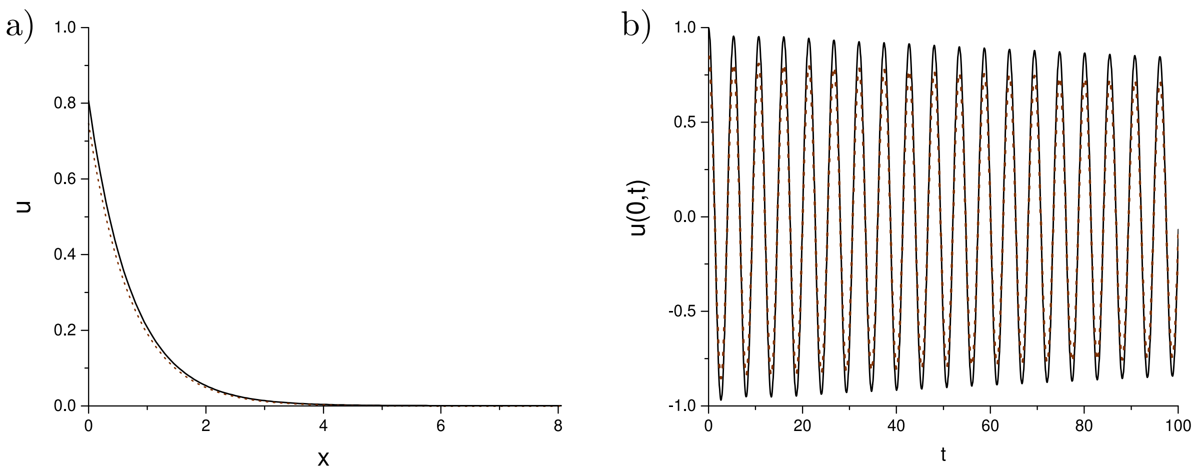

Fig. 11. This plot shows: (a) the shape, and (b) the time behavior of a localized mode for the case with damping. The solid line corresponds to the numerical results and the dotted line to the analytical results. The parameters are: $b_3 = -2$, $a = 6$, $b_1 = 1$, $\delta = 0.1$, $b_2 = 0.01$.

as the frequency, which corresponds to this trapped mode. With the help of an energy estimate the stability of the undamped system was investigated. A bifurcation diagram, which shows how the localized solution depends on the system parameters, has been given. For different kinds of damping mechanisms in the string (such as air damping, (local) Kelvin–Voigt damping, or a (local) time hysteresis damping) it has been shown that localized modes of vibrations can exist for particular values of the system parameters. The amplitudes of these localized modes of vibration can increase in time or can decrease in time depending on the negative or positive stiffness coefficient in the attached mass–spring–damping system, respectively. In case of absence of damping in the problem, and for a positive stiffness coefficient in the attached system at the edge of the string, it has been shown that depending on the coefficient of the Winkler foundation that localized in space solutions are absent, or that oscillating in time and localized in space solutions are present. The increase of amplitudes of localized modes can be interpreted as system instabilities. The nature of these instabilities are close to the types of instabilities considered in [35–37] for some physical systems. Compared to the studies in [35–37] we restrict our analysis to (in-)stabilities which can occur when localized modes exist. So, from a practical point of view, we can get undesirable growth of string amplitudes even in the presence of damping. This shows the importance of finding the parameters for which this occurs. Such instabilities can also be seen as the beginning of a fracture process in a crystalline lattice when the lattice is modeled as a string with inclusions as has been done in [7]. Also knowledge of the trapped mode location can be used to suppress the cable(string) oscillations by installing a damper at its edge, or to create an acoustic emitter at one particular frequency. Note that in this paper we only focus on trapped mode-solutions. Other types of solutions, like traveling waves, which may exist in the system, were not considered yet.

Funding

This research is supported by the MSHERF project 124040800009-8.

CRediT authorship contribution statement

A.K. Abramian: Project administration, Methodology, Formal analysis, Conceptualization. **S.A. Vakulenko:** Methodology, Formal analysis. **W.T. van Horssen:** Writing – review & editing, Validation, Conceptualization. **A. Jikhareva:** Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] G.J. Boertjens, W.T. van Horssen, An asymptotic theory for a weakly nonlinear beam equation with a quadratic perturbation, SIAM J. Appl. Math. 60 (2) (2000) 602–632, http://dx.doi.org/10.1137/S0036139998336766.

[2] Z. Yu, Y.L. Xu, Mitigation of three dimensional vibration of inclined sag cable using discrete oil dampers-i. formulation, J. Sound Vib. 214 (4) (1998) 659–673, http://dx.doi.org/10.1006/jsvi.1998.1609.

[3] L. Demeio, S. Lenci, Second order solutions for the dynamics of a semi-infinite cable on a lateral substrate, J. Sound Vib. 315 (3) (2008) 414–432, http://dx.doi.org/10.1016/j.jsv.2008.03.010.

[4] A.V. Metrikine, Steady state responce of an infinite string on a non-linear visco-elastic foundation to moving point loads, J. Sound Vib. 272 (3–5) (2004) 1033–1046, http://dx.doi.org/10.1016/j.jsv.2003.04.001.

[5] M.A. Vaz, X. Li, J. Liu, X. Ma, Analytical model for axial vibration of marine cables considering equivalent distributed viscous damping, Appl. Ocean Res. 113 (2021) 102733, http://dx.doi.org/10.1016/j.apor.2021.102733.

[6] M.S. Taima, T. El-Sayed, S.H. Farghaly, Longitudinal vibration analysis of a stepped nonlocal rod embedded in several elastic media, J. Vibr. Engn. Technol. 10 (2022) 1399–1412, http://dx.doi.org/10.1007/s42417-022-00454-7.

[7] E. Glushkov, N. Glushkova, J. Wauer, Wave propagation in an elastically supported string with point-wise defects: gap-band and pass-band effects, ZAMM Z. Angew. Math. Mech. 91 (1) (2011) 4–22, http://dx.doi.org/10.1002/zamm.201000039.

[8] O.O. Bendiksen, Localization phenomena in structural dynamics, Chaos, Solit. Fract. 11 (10) (2000) 1621–1660, http://dx.doi.org/10.1016/S0960-0779(00)00013-8.

[9] M.P. Castanier, C. Pierre, Individual and interactive mechanisms for localization and dissipation in a mono-coupled nearly-periodic structure, J. Sound Vib. 168 (3) (1993) 479–505, http://dx.doi.org/10.1006/jsvi.1993.1387.

[10] S.-Y. Lee, C.D. Mote Jr., Travelling wave dynamics in a translating string coupled to stationary constraints:energy transfer and mode localization, J. Sound Vib. 212 (1) (1998) 1–22, http://dx.doi.org/10.1006/jsvi.1997.1285.

[11] A.K. Abramian, V.L. Andreyev, D.A. Indejtchev, Resonance oscillations of infinite and finite elastic structures with inclusions, J. Acoust. Soc. Am. 95 (1994) 3007–3008, http://dx.doi.org/10.1121/1.408819.

[12] J. Kaplunov, S. Sorokin, A simple example of a trapped mode in an unbounded waveguide, J. Acoust. Soc. Am. 97 (6) (1995) 3898–3899, http://dx.doi.org/10.1121/1.412405.

[13] E.V. Shishkina, S.N. Gavrilov, Y.A. Mochalova, Passage through a resonance for a mechanical system, having time varying parameters and possessing a single trapped mode. the principal term of the resonant solution 481 (2) (2020) 115422, http://dx.doi.org/10.1016/j.jsv.2020.115422.

[14] S.N. Gavrilov, E.V. Shishkina, Y.A. Mochalova, The anti-localization of non-stationary linear waves and its relation to the localization, J. Sound Vib. 553 (1–2) (2023) 117673, http://dx.doi.org/10.1016/j.jsv.2023.117673.

[15] S.N. Gavrilov, E.V. Shishkina, Y.A. Mochalova, Non-stationary localized oscillations of an infinite string, with time-varying tension, lying on the Winkler foundation with a point elastic inhomogeneity, Nonlinear Dynam. 95 (2019) http://dx.doi.org/10.1007/s11071-018-04735-3.

[16] D.A. Indeitsev, E.V. Osipova, Localization of nonlinear waves in elastic bodies with inclusions, Acoust. Phys. 50 (2004) 420–426, http://dx.doi.org/10.1134/1.1776219.

[17] J. Kaplunov, E. Nolde, An example of a quasi-trapped mode in a weakly non-linear elastic waveguide, C. R. M.C. 336 (7) (2008) 553–558, http://dx.doi.org/10.1016/j.crme.2008.04.005.

[18] A.K. Abramian, S.A. Vakulenko, W.T. van Horssen, Dynamics of a weakly nonlinear string on an elastic foundation with a partly prescribed discrete spectrum, Nonlinear Dynam. 111 (2023) 5221–5235, http://dx.doi.org/10.1007/s11071-022-08142-7.

[19] R. Porter, Trapped waves in thin elastic plates, Wave Motion 45 (1–2) (2007) 3–15, http://dx.doi.org/10.1016/j.wavemoti.2007.04.001.

[20] V. Pagneux, Trapped modes and edge resonances in acoustics and elasticity, in: R. Craster, J. Kaplunov (Eds.), Dynamic Localization Phenomena in Elasticity, Acoustics and Electromagnetism, Springer, Berlin, 2013, pp. 181–223.

[21] R. Porter, D.V. Evans, Trapped modes due to narrow cracks in thin simply-supported elastic plates, Wave Motion 51 (3) (2014) 533–546, http://dx.doi.org/10.1016/j.wavemoti.2014.01.002.

[22] J. Kaplunov, G. Rogerson, P. Tovstik, Localized vibration in elastic structures with slowly varying thickness, Q. J. Mech. Appl. Math. 58 (4) (2005) 645–664, http://dx.doi.org/10.1093/qjmam/hbi028.

[23] A. Abramyan, D. Indeitsev, Trapping modes in a membrane with an inhomogeneity, Acoust. Phys. 44 (4) (1998) 371–376.

[24] P. Zhou, H. Liu, Modeling and control perfomance of negative stiffness damper for suppressing stay cable vibrations, Struct. Control Health Monit. 23 (4) (2016) 764–782, http://dx.doi.org/10.1002/stc.1809.

[25] A. Blanchard, O.V. Gendelman, D.M. McFarland, L.A. Bergman, A.F. Vakakis, Mode complexity in a harmonically forced string with a local spring-damper and transition from vibrations to waves, J. Sound Vib. 334 (6) (2015) 282–295, http://dx.doi.org/10.1016/j.jsv.2014.08.033.

[26] B. Yurke, Conservative model for the damped harmonic oscillator, Am. J. Phys. 52 (1984) 1099–1102, http://dx.doi.org/10.1119/1.13768.

[27] S. Chen, K. Liu, Z. Liu, Spectrum and stability for elastic systems with global or local Kelvin–Voigt damping, SIAM J. Appl. Math. 59 (2) (1998) 651–668, https://www.jstor.org/stable/118440.

[28] L.D. Landau, E.M. Lifschitz, Quantum Mechanics: Non-Relativistic Theory, Elsevier, third ed., 2013.

[29] E. Schrödinger, Quantisierung als eigenwertproblem [quantization as an eigenvalue problem], Annalen der Physik (in German) 80 (13) (1926) 437–490, http://dx.doi.org/10.1002/andp.19263851302, Bibcode:1926AnP.385.437S..

[30] J.W.S. Rayleigh, Theory of Sound. I, second ed., Macmillan, London, ISBN: 978-1-152-06023-4, 1894, pp. 115–118.

[31] L. Nirenberg, Topics in Nonlinear Functional Analysis, New York Univ., 1974, Lecture Notes.

[32] M. Israeli, S.A. Orszag, Approximation of radiation boundary conditions, J. Comput. Phys. 41 (1) (1981) 115–135.

[33] S.V. Tsynkov, Numerical solution of problems on unbounded domains, A Review, Appl. Numer. Math. 27 (4) (1998) 465–532.

[34] D. Potter, Computational Methods in Physics, Wiley, New York, 1973.

[35] O.N. Kirillov, F. Verhulst, Dissipation-induced instabilities and symmetry, Acta Mech. Sin. 27 (1) (2011) 2–6, http://dx.doi.org/10.1007/s10409-011-0409-0.

[36] M. Cavalkanti, V. Cavalkanti, L. Tebou, Stabilization of the wave equation with localized compensating frictional and and Kelvin–Voigt dissipating mechanisms, Electron. J. Differential Equations 83 (2017) 1–18, http://ejde.math.txstate.edu or http://ejde.math.unt.edu.

[37] Z. Liu, Q. Zhang, Stability of a string with local Kelvin–Voigt damping and nonsmooth coefficient at interface, SIAM J. Contr. Opt. 54 (4) (2016) 1859–1871, http://dx.doi.org/10.1137/15M1049385.