COMPUTATION OF THE INNER-OUTER FACTORIZATION FOR TIME-VARYING SYSTEMS

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An inner-outer factorization theorem for linear time-varying systems is obtained via an extension of the classical Beurling-Lax theorem to the time-varying context. This provides characteristic features of the inner factor, which can be used to compute realizations of the inner and outer factors from a realization of the given transfer operator. The resulting algorithm is unidirectional in time. The outer factor can also be obtained by an expression involving a Riccati recursive equation.

1. Introduction

Recently, there has been some interest in computing inner-outer factorizations of time-varying systems, as a basic step in robust control applications such as the design of feedback controllers and sensitivity minimization [1, 2]. For *time-invariant* single-input single-output systems, the inner-outer factorization is a factorization of an analytical (causal) transfer function $T(z)$ into the product of an inner and an outer system: $T(z) = V(z)T_0(z)$. The inner part $V(z)$ is analytical (*i.e.*, has its poles outside the unit disc) and has modulus 1 on the unit circle, whereas the outer part $T_0(z)$ is analytical and may have zeros only outside the unit disc. For example, (with $|\alpha|, |\beta| < 1$)

$$z\frac{z-\alpha^*}{1-\beta z} \;=\; z\frac{z-\alpha^*}{1-\alpha z}\cdot\frac{1-\alpha z}{1-\beta z}\,.$$

The resulting outer factor is such that its inverse is again a stable system, provided there are no zeros on the unit circle. For multi-input multi-output systems, the definition of the outer factor is more abstract (see *e.g.*, Halmos [3]) and takes the form of a range condition: $T_0(z)$ is outer if $T_0(z)H_m^2 = H_n^2$, where H_m^2 is the Hardy space of analytical m-dimensional vector-valued functions. A generalization of this definition applies in the time-varying context.

The existence of inner-outer factorizations in any context is more or less fundamental to analytical Hilbert spaces. Abstract mathematical formulations of it which also apply to the time-varying setting can be found in [4, 5]. In this paper, we connect the abstract theory to a computational scheme acting on state space realizations. One of the aspects of time-varying systems is that the state dimension can vary, and as a result, the number of 'zeros' in the inner and outer factors can vary, too. The theory in this paper handles such variations automatically. Full details can be found in [6].

An application of the inner-outer factorization is the computation of inverse systems: if T is a causal and invertible system, then its inverse is not necessarily causal: the inversion might have introduced an anti-causal part. This effect is known as a dichotomy; it is in general not a trivial task to determine the causal and anti-causal parts of T^{-1}. With the inner-outer factorization, however, the inverse of the outer factor is again causal, whereas the inverse of the inner factor is fully anti-causal, and determines which part of the inverse outer factor is made anti-causal. This application of the inner-outer factorization plays a crucial role in *e.g.*, the computation of optimal feedback controllers [1].

2. Notation

We adopt the notation of [7, 8, 9, 6], so that the description of it in this paper will be terse. For $i = \cdots,-1,0,1,\cdots$, let $\mathcal{M}_i$ be a separable Hilbert space. We will usually take $\mathcal{M}_i = \mathbb{C}^{M_i}$ with M_i a finite number, so that $\mathcal{M}_i$ is a finite vector space, but we will have to allow $\mathcal{M} = \ell_2$ in section 5. The space $\mathcal{M} = \cdots\times\mathcal{M}_0\times\mathcal{M}_1\times\cdots$ is the space of (non-uniform) sequences $u = [\cdots,u_0,u_1,\cdots]$ with entries in $\mathcal{M}_i$. Such sequences will represent the signals in our systems. If all $\mathcal{M}_i$ are finite, then we call $\mathcal{M}$ *locally finite.* Some (or most) of the dimensions may be zero, and in this way finite non-uniform vectors are also included in the formalism. The space $\ell_2^{\mathcal{M}}$ is the space of sequences in $\mathcal{M}$ with finite 2-norm. The space $\mathcal{X}(\mathcal{M},\mathcal{N})$ is the space of bounded operators $T : y = uT$ acting from $\ell_2^{\mathcal{M}}$ into $\ell_2^{\mathcal{N}}$. An operator in such a space has an (infinite) matrix representation where the ij-th entry is an operator $\mathcal{M}_i \to \mathcal{N}_j$ (an $M_i\times N_j$ matrix). The space $\mathcal{U}\subset\mathcal{X}$ consists of bounded operators T which are upper: $T_{ij} = 0$ $(i > j)$. Likewise, we define $\mathcal{L}\subset\mathcal{X}$ to be the space of lower operators and $\mathcal{D} = \mathcal{U}\cap\mathcal{L}$ to be the space of diagonals.

In $\mathcal{X}$, the causal bilateral shift-operator Z is defined via $[\cdots,\boxed{u_0},u_1,\cdots]Z = [\cdots,\boxed{u_{-1}},u_0,\cdots]$ (the square identifies the position of the 0-th entry). If $u\in\mathcal{M}$, then $uZ\in\mathcal{M}^{(1)}$, where $\mathcal{M}^{(1)}$ is equal to the space sequence $\mathcal{M}$, shifted over one position. The k-th diagonal shift of an operator A into the South-East direction is denoted as $A^{(k)} = Z^{*k}AZ^k$.

We shall need Hilbert-Schmidt spaces $\mathcal{X}_2,\mathcal{U}_2$, $\mathcal{L}_2$, $\mathcal{D}_2$, consisting of those elements of $\mathcal{X},\mathcal{U}$, $\mathcal{L}$, $\mathcal{D}$ for which the norms of the entries are square summable. These spaces are Hilbert spaces for the usual Hilbert-Schmidt inner product. They can be considered as input or output spaces for our system operators: if T is a bounded operator $\ell_2^{\mathcal{M}}\to\ell_2^{\mathcal{N}}$, then it may be extended as a bounded operator $\mathcal{X}_2\to\mathcal{X}_2$ by stacking sequences in ℓ_2 to form elements of $\mathcal{X}_2$. This leads for example to the expression $y = uT$, where $u\in\mathcal{X}_2(\mathbb{C}^{\mathbb{Z}},\mathcal{M})$ and $y\in\mathcal{X}_2(\mathbb{C}^{\mathbb{Z}},\mathcal{N})$ [8]. We will use the shorthand $\mathcal{X}_2^{\mathcal{M}}$ for $\mathcal{X}_2(\mathbb{C}^{\mathbb{Z}},\mathcal{M})$, but continue to write $\mathcal{X}_2$ if the precise form of $\mathcal{M}$ is not of interest.

We define $\mathbf{P}$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{U}_2$, $\mathbf{P}_0$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{D}_2$, and $\mathbf{P}_{\mathcal{L}_2Z^{-1}}$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{L}_2Z^{-1}$. If $X\in\mathcal{X}_2$, then its k-th diagonal is defined in terms of $\mathbf{P}_0$ by $X_{[k]} = \mathbf{P}_0(Z^{-k}X)$, and $X = \sum Z^kX_{[k]}$. The domain of the projector $\mathbf{P}_0$ can be extended to operators in $\mathcal{X}$.

A subspace $\mathcal{H}$ of $\mathcal{X}_2^{\mathcal{M}}$ is called (left) D-invariant if $D\mathcal{H}\subset\mathcal{H}$ for all $D\in\mathcal{D}$, and shift-invariant if $Z\mathcal{H}\subset\mathcal{H}$. A D-invariant subspace $\mathcal{H}$ falls apart into 'slices' (rows) $\mathcal{H}_k$ such that $\mathcal{H} = \cdots\times\mathcal{H}_0\times\mathcal{H}_1\times\cdots$, where each $\mathcal{H}_k$ is a subspace in $\ell_2^{\mathcal{M}}$ (the k-th row of $\mathcal{X}_2^{\mathcal{M}}$) [7, 9, 6]. Let d_k be the dimension of $\mathcal{H}_k$, then we call $d = [d_k]_{-\infty}^{\infty}$ the sequence of dimensions of $\mathcal{H}$: $d = \text{s-dim}\,\mathcal{H}$. If all d_k are finite, then $\mathcal{H}$ is said to be *locally finite.* Let $\mathcal{B}_k = \mathbb{C}^{d_k}$ ($\mathcal{B}_k = \ell_2$ if $d_k = \infty$), and let $\mathcal{B} = \cdots\times\mathcal{B}_k\times\cdots$. Each $\mathcal{H}_k$ has a basis representation $\mathbf{F}_k$ such that $\mathcal{H}_k = \mathcal{B}_k\mathbf{F}_k$, where the rows of $\mathbf{F}_k$ are the individual basis vectors. Likewise, a D-invariant subspace $\mathcal{H}$ has a basis representation $\mathbf{F}$ such that $\mathcal{H} = \mathcal{D}_2\mathbf{F}$ [7], where the k-th (block)-row of $\mathbf{F}$ is $\mathbf{F}_k$. The diagonal operator $\Lambda_{\mathbf{F}} := \mathbf{P}_0(\mathbf{F}\mathbf{F}^*) = \text{diag}[\mathbf{F}_k\mathbf{F}_k^*]_{-\infty}^{\infty}$ plays the role of Gram operator. If $\Lambda_{\mathbf{F}}$ is uniformly positive (*i.e.,* boundedly invertible), the basis representation is called *strong,* and the projection operator onto $\mathcal{H}$ is in this case given by $\mathbf{P}_{\mathcal{H}}(\,\cdot\,) = \mathbf{P}_0(\,\cdot\,\mathbf{F}^*)\Lambda_{\mathbf{F}}^{-1}\mathbf{F}$. If $\Lambda_{\mathbf{F}} = I$, then $\mathbf{F}$ is called an orthonormal basis representation.

3. Time-varying systems

An operator $T\in\mathcal{U}(\mathcal{M},\mathcal{N})$ is called a causal transfer operator: it maps sequences $u\in\ell_2^{\mathcal{M}}$ to sequences $y = uT\in\ell_2^{\mathcal{N}}$ in a causal way. When T is viewed as an operator from $\mathcal{X}_2^{\mathcal{M}}$ to $\mathcal{X}_2^{\mathcal{N}}$, then because $\mathcal{X}_2 = \mathcal{L}_2Z^{-1}\oplus\mathcal{U}_2$, its action on $\mathcal{L}_2Z^{-1}$ can be decomposed into two operators H_T and K_T:

$$\cdot T\big|_{\mathcal{L}_2Z^{-1}} = \cdot K_T + \cdot H_T : \qquad \cdot H_T = \mathbf{P}(\,\cdot T\big|_{\mathcal{L}_2Z^{-1}})\,; \quad \cdot K_T = \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\,\cdot T\big|_{\mathcal{L}_2Z^{-1}})$$

H_T is called the Hankel operator of T. The range and kernel of H_T and H_T^* are D-invariant subspaces with important system-theoretic properties [7]:

$$\begin{array}{lllll} \mathcal{K}(T) & = & \ker(\cdot H_T) & = & \{U \in \mathcal{L}_2 Z^{-1} : \mathbf{P}(UT) = 0\} \\ \mathcal{H}(T) & = & \mathrm{ran}(\cdot H_T^*) & = & \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 T^*) \\ \mathcal{H}_0(T) & = & \mathrm{ran}(\cdot H_T) & = & \mathbf{P}(\mathcal{L}_2 Z^{-1} T) \\ \mathcal{K}_0(T) & = & \ker(\cdot H_T^*) & = & \{Y \in \mathcal{U}_2 : \mathbf{P}(YT^*) = 0\}. \end{array}$$

These subspaces provide decompositions of $\mathcal{L}_2 Z^{-1}$ and $\mathcal{U}_2$ as

$$\begin{array}{lllll} \overline{\mathcal{H}} & \oplus & \mathcal{K} & = & \mathcal{L}_2 Z^{-1} \\ \overline{\mathcal{H}}_0 & \oplus & \mathcal{K}_0 & = & \mathcal{U}_2\,, \end{array}$$

(the overbar denotes closure). $\mathcal{H}(T)$ is called the (natural) input state space, and $\mathcal{H}_0(T)$ the (natural) output state space of T. If these subspaces are locally finite, then they have the same s-dimension, and T is said to be *locally finite.* In this case, one can obtain minimal realizations of the type

$$UT = Y \qquad \Leftrightarrow \qquad \begin{array}{lll} X_{[i+1]}^{(-1)} & = & X_{[i]}A + U_{[i]}B \\ Y_{[i]} & = & X_{[i]}C + U_{[i]}D \end{array} \qquad \mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \tag{1}$$

where

$$\begin{array}{llllll} A & \in & \mathcal{D}(\mathcal{B}, \mathcal{B}^{(-1)})\,, & C & \in & \mathcal{D}(\mathcal{B}, \mathcal{N})\,, \\ B & \in & \mathcal{D}(\mathcal{M}, \mathcal{B}^{(-1)})\,, & D & \in & \mathcal{D}(\mathcal{M}, \mathcal{N})\,. \end{array} \tag{2}$$

The space sequence $\mathcal{B}$ is called the system order of the realization. Let A_k be the k-th entry of the diagonal operator A, and likewise for B_k, C_k, D_k. If $\mathcal{B}_k = \mathbb{C}^{d_k}$, then A_k is a $d_k \times d_{k+1}$ matrix, B_k : $M_k \times d_{k+1}$, $C_k : d_k \times N_k$, $D_k : M_k \times N_k$, and all dimensions are time-varying. The realization equations (1) are equivalent to

$$\begin{array}{lll} x_{i,k+1} & = & x_{i,k}A_k + u_{i,k}B_k \\ y_{i,k} & = & x_{i,k}C_k + u_{i,k}D_k\,. \end{array}$$

$\mathbf{T}$ is a realization of T if its entries T_{ij} or diagonals $T_{[i]}$ are given by

$$T_{ij} = \begin{cases} 0\,, & i > j \\ D_i\,, & i = j \\ B_i A_{i+1} \cdots A_{j-1} C_j\,, & i < j\,. \end{cases} \qquad \Leftrightarrow \qquad T_{[i]} = \begin{cases} 0, & i < 0 \\ D, & i = 0 \\ B^{(i)} A^{(i-1)} \cdots A^{(1)} C, & i > 0 \end{cases} \tag{3}$$

Let ℓ_A denote the spectral radius of the operator AZ. If $\ell_A < 1$ (the realization is said to be strictly stable), then $(I - AZ)$ is invertible, and $T = D + BZ(I - AZ)^{-1}C$. In this case, the operators $\mathbf{F}$ and $\mathbf{F}_0$ defined by

$$\begin{array}{lll} \mathbf{F} & := & \left(BZ + BZAZ + BZ\,(AZ)^2 + \cdots \right)^* \\ \mathbf{F}_0 & := & C + AZC + (AZ)^2 C + \cdots \end{array} \tag{4}$$

are bounded operators in $\mathcal{L}Z^{-1}$ and $\mathcal{U}$, respectively, and given by $\mathbf{F} = \left(BZ(I - AZ)^{-1} \right)$ and $\mathbf{F}_0 = (I - AZ)^{-1}C$, respectively. In case $\ell_A \le 1$, then $\mathbf{F}$ and $\mathbf{F}_0$ are bounded operators on $\mathcal{D}_2$, and can be defined via (4) on a dense subset of $\mathcal{X}_2$. The realization is called (uniformly) controllable if the controllability Gramian $\Lambda_{\mathbf{F}} := \mathbf{P}_0(\mathbf{F}\mathbf{F}^*)$ is (uniformly) positive, (uniformly) observable if the observability Gramian $\Lambda_{\mathbf{F}_0} = \mathbf{P}_0(\mathbf{F}_0 \mathbf{F}_0^*)$ is (uniformly) positive, and minimal if it is both controllable and observable. Equivalently, a realization is controllable if $\cdot\mathbf{F}\big|_{\mathcal{D}_2}$ is one-to-one (injective), and observable if $\cdot\mathbf{F}_0\big|_{\mathcal{D}_2}$ is one-to-one. For minimal realizations, $\mathbf{F}$ and $\mathbf{F}_0$ are basis representations of $\overline{\mathcal{H}}(T)$ and $\overline{\mathcal{H}}_0(T)$, respectively:

$$\overline{\mathcal{H}}(T) = \overline{\mathcal{D}_2^{\mathcal{B}} \mathbf{F}}\,, \qquad \overline{\mathcal{H}}_0(T) = \overline{\mathcal{D}_2^{\mathcal{B}} \mathbf{F}_0}\,.$$

More in general, for a controllable realization, $\mathcal{H}_0(T) \subset \mathcal{D}_2\mathbf{F}_0$, and for an observable realization, $\mathcal{H}(T) \subset \mathcal{D}_2\mathbf{F}$. We mention the following properties, which are valid for $\ell_A \le 1$ [7, 6]:

$$\begin{cases} \mathbf{F}_0 &= C + AZ\mathbf{F}_0 \\ T &= D + BZ\mathbf{F}_0 \end{cases} \qquad \begin{cases} Z\mathbf{F} &= A^*\mathbf{F} + B^* \\ T^* &= D^* + C^*\mathbf{F} \end{cases} \qquad \begin{cases} \Lambda_{\mathbf{F}_0} = I &\Rightarrow \quad AA^* + CC^* = I \\ \Lambda_{\mathbf{F}} = I &\Rightarrow \quad A^*A + B^*B = I \end{cases} \tag{5}$$

4. State space properties of inner systems

A system V is an isometry if $VV^* = I$, a co-isometry if $V^*V = I$, and unitary if both $VV^* = I$ and $V^*V = I$. A system is inner if it is unitary and upper. A realization $\mathbf{V}$ is called unitary (or lossless) if $\mathbf{V}\mathbf{V}^* = I$ and $\mathbf{V}^*\mathbf{V} = I$, where

$$\mathbf{V} = \begin{bmatrix} A_V & C_V \\ B_V & D_V \end{bmatrix}. \tag{6}$$

Proposition 1. *Let $V \in \mathcal{U}$. Then*

$$VV^* = I \quad \Rightarrow \quad \mathcal{K}_0(V) = \mathcal{U}_2 V \oplus \ker(\,\cdot V^*|_{\mathcal{U}_2}).$$

If $VV^* = I$ and $\ker(\,\cdot V^*|_{\mathcal{U}_2}) = \emptyset$ then V is inner. Dually,

$$V^*V = I \quad \Rightarrow \quad \mathcal{K}(V) = \mathcal{L}_2 Z^{-1} V^* \oplus \ker(\,\cdot V|_{\mathcal{L}_2 Z^{-1}}).$$

If $V^*V = I$ and $\ker(\,\cdot V|_{\mathcal{L}_2 Z^{-1}}) = \emptyset$, then V is inner.

PROOF Let $VV^* = I$. Because V is an isometry, the subspaces $\mathcal{X}_2 V = \operatorname{ran}(V)$, $\mathcal{L}_2 Z^{-1} V$ and $\mathcal{U}_2 V$ are closed, and $\mathcal{X}_2 V = \mathcal{L}_2 Z^{-1} V \oplus \mathcal{U}_2 V$. $\mathcal{U}_2 V \subset \mathcal{K}_0$, because $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}([\mathcal{U}_2 V]V^*) = 0$. The remaining subspace $\mathcal{K}_0 \ominus \mathcal{U}_2 V$ consists of elements

$$\begin{aligned} \mathcal{K}_0 \ominus \mathcal{U}_2 V &= \{X \in \mathcal{U}_2 : \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(XV^*) = 0 \wedge \mathbf{P}(XV^*) = 0\} \\ &= \{X \in \mathcal{U}_2 : XV^* = 0\} \\ &= \ker(\,\cdot V^*|_{\mathcal{U}_2}). \end{aligned}$$

Hence $\mathcal{K}_0 = \mathcal{U}_2 V \oplus \ker(\,\cdot V^*|_{\mathcal{U}_2})$.

If $\ker(\,\cdot V^*|_{\mathcal{U}_2}) = \emptyset$, then $X \in \mathcal{U}_2,\ XV^* = 0 \Rightarrow X = 0$. This implies

$$X \in Z^{-n}\mathcal{U}_2,\ XV^* = 0 \quad \Rightarrow \quad X = 0 \qquad (\text{all } n \in \mathbb{Z}),$$

since $(Z^n X)V^* = 0 \Leftrightarrow XV^* = 0$. Letting $n \to \infty$ yields $\ker(\,\cdot V^*) = \emptyset$, so that V has a left inverse, which must be equal to the right inverse V^*. Hence $V^*V = I$ and V is inner. Dual results hold in case $V^*V = I$. □

Unitary realizations and inner systems are closely connected: one can show that a locally finite inner system has a unitary realization, and the converse is true at least when the realization is strictly stable. More precisely, we have the following theorem.

Theorem 2. *Let $\mathbf{V}$ given by equation (6) be a state realization of a bounded transfer operator $V \in \mathcal{U}(\mathcal{M},\mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are locally finite spaces of sequences. Let $\Lambda_{\mathbf{F}}$ and $\Lambda_{\mathbf{F}_0}$ be the controllability and the observability Gramians of the given realization. If $\ell_{A_V} < 1$, then*

$$\begin{aligned} \mathbf{V}^*\mathbf{V} = I \quad &\Rightarrow \quad V^*V = I,\quad \Lambda_{\mathbf{F}} = I, \\ \mathbf{V}\mathbf{V}^* = I \quad &\Rightarrow \quad VV^* = I,\quad \Lambda_{\mathbf{F}_0} = I. \end{aligned} \tag{7}$$

If $\ell_{A_V} \le 1$, then

$$\begin{aligned} \mathbf{V}^*\mathbf{V} = I,\quad \Lambda_{\mathbf{F}} = I \quad &\Rightarrow \quad V^*V = I, \\ \mathbf{V}\mathbf{V}^* = I,\quad \Lambda_{\mathbf{F}_0} = I \quad &\Rightarrow \quad VV^* = I. \end{aligned}$$

PROOF A proof for the case $\ell_{A_V} < 1$ appears in [9]. The proof for the generalization to $\ell_A = 1$ is omitted in this paper. $\square$

5. Beurling-Lax theorem

Theorem 3. *All DZ-invariant subspaces $\mathcal{K}_0$ in $\mathcal{U}_2^{\mathcal{N}}$ have the form $\mathcal{K}_0 = \mathcal{U}_2^{\mathcal{M}} V$, where $V \in \mathcal{U}(\mathcal{M},\mathcal{N})$ is an isometry ($VV^* = I$).*

PROOF Let $\mathcal{R}_0 = \mathcal{K}_0 \ominus Z\mathcal{K}_0$. This is a D-invariant subspace in $\mathcal{U}_2^{\mathcal{N}}$. We can assume it is non-empty, for else $\mathcal{K}_0 = Z\mathcal{K}_0 = Z^n\mathcal{K}_0$ for all $n \geq 0$, and since $X \in \mathcal{U}_2 \Rightarrow \lim_{n\to\infty} \mathbf{P}(Z^{-n}X) = 0$, this implies that $\mathcal{K}_0 = 0$, and there is nothing to prove. Likewise, define $\mathcal{R}_n = Z^n\mathcal{K}_0 \ominus Z^{n+1}\mathcal{K}_0$. Then $\mathcal{R}_n = Z^n\mathcal{R}_0$, and $\mathcal{K}_0 = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots$.

Suppose s-dim $\mathcal{R}_0 = M$, and define the sequence of Hilbert spaces $\mathcal{M}$ to have entries $\mathcal{M}_k = \mathbb{C}^{M_k}$ ($\mathcal{M}_k = \ell_2$ if $M_k = \infty$). Then there exist isometries $V_k : \mathcal{M}_k \to (\mathcal{R}_0)_k$ such that $(\mathcal{R}_0)_k = \mathcal{M}_k V_k$. Let V be the operator whose k-th block-row is equal to V_k. V is an orthonormal basis representation of $\mathcal{R}_0$, as in section 2, such that

$$\mathcal{R}_0 = \mathcal{D}_2^{\mathcal{M}} V\,, \qquad \mathbf{P}_0(VV^*) = I\,.$$

Then $\mathcal{R}_n = \mathcal{D}_2 Z^n V$. Because $\mathcal{R}_i \perp \mathcal{R}_j$ $(i \neq j)$, it follows that $D_1 Z^n V \perp D_2 V$ $(n \geq 1)$ for all $D_{1,2} \in \mathcal{D}_2$, *i.e.,*

$$\begin{aligned} \mathbf{P}_0(Z^n VV^*) &= 0 \\ \mathbf{P}_0(VV^* Z^{-n}) &= 0 \end{aligned}$$

so that $VV^* = I$: V is an isometry. The orthogonal collection $\{\mathcal{D}_2 Z^n V\} \in \mathcal{K}_0$ $(n \geq 0)$, and together spans the space $\mathcal{U}_2 V$. Hence $\mathcal{K}_0 = \{\mathcal{D}_2 Z^n V\} = \mathcal{U}_2 V$. The factor V is unique up to a left diagonal unitary factor. $\square$

The above proof is along the lines of the proof of Helson [3, §VI.3] for the time-invariant Hardy space setting. This proof was in turn based on the work of Beurling for the scalar (SISO) case and Lax for the extension to vector valued functions. A remaining question is to give conditions under which V is actually unitary. For time-invariant systems, this condition is that $\mathcal{K}_0$ is “full range” [3]. Systems T for which $\mathcal{K}_0(T)$ is full range have been called “roomy” in [10]. Systems of finite degree are roomy: if $\mathcal{H}_0(T)$ is finite dimensional, then its complement $\mathcal{K}_0(T)$ is automatically full range. For time-varying systems, only less definite results can be obtained, which we omit in this paper.

Note that, in the above theorem, $\mathcal{M}$ can have components $\mathcal{M}_k$ which are infinite dimensional, even if $\mathcal{N}$ is locally finite, depending on $\mathcal{K}_0$. In our application of the theorem in the next section, however, $\mathcal{K}_0$ is such that $\mathcal{M}$ will be locally finite automatically, starting from locally finite spaces.

Corollary 4. *If $V \in \mathcal{U}(\mathcal{M},\mathcal{N})$ is an isometry, then there exists an isometry $U \in \mathcal{U}(\mathcal{M}_U,\mathcal{N})$ such that* $\ker(\,\cdot V^*|_{\mathcal{U}_2^{\mathcal{N}}}) = \mathcal{U}_2^{\mathcal{M}_U} U$. *The operator*

$$W = \begin{bmatrix} U \\ V \end{bmatrix}$$

is inner, with $\mathcal{H}_0(W) = \overline{\mathcal{H}}_0(V)$.

PROOF If V is an isometry, then (proposition 1)

$$\mathcal{U}_2^{\mathcal{N}} = \overline{\mathcal{H}}_0(V) \oplus \ker(\,\cdot V^*|_{\mathcal{U}_2^{\mathcal{N}}}) \oplus \mathcal{U}_2^{\mathcal{M}} V\,, \tag{8}$$

where $\mathcal{K}_0' := \ker(\,\cdot V^*|_{\mathcal{U}_2})$ is shift-invariant, so that according to theorem 3 there exists an isometry $U \in \mathcal{U}(\mathcal{M}_U,\mathcal{N})$ such that $\mathcal{K}_0' = \mathcal{U}_2^{\mathcal{M}_U} U$. In view of proposition 1, W is inner if $WW^* = I$ and

$\ker(\cdot W^*|_{\mathcal{U}_2}) = \emptyset$. $WW^* = I$ requires $UV^* = 0$, which is true because $\mathcal{U}_2 V \perp \mathcal{U}_2 U$. Hence $\mathcal{U}_2 W = \mathcal{U}_2 U \oplus \mathcal{U}_2 V$, and since $\overline{\mathcal{H}}_0(W) \supset \overline{\mathcal{H}}_0(V)$, we must have (from equation (8)) that $\overline{\mathcal{H}}_0(W) = \overline{\mathcal{H}}_0(V)$ and $\ker(\cdot W^*|_{\mathcal{U}_2}) = \emptyset$. Hence W is inner, and $\mathcal{H}_0(W)$ is closed. $\square$

6. Inner-outer factorization

We will say that an operator $T_0 \in \mathcal{U}$ is (left) *outer* if

$$\overline{\mathcal{U}_2 T_0} \;=\; \mathcal{U}_2\,. \tag{9}$$

Other definitions are possible (see *e.g.,* Arveson [4]); the above definition is such that $\overline{\mathrm{ran}}(\cdot T_0) = \overline{\mathcal{X}_2 T_0} = \mathcal{X}_2$, so that $\ker(\cdot T_0^*) = \emptyset$ and T_0 has an algebraic left inverse which is upper (it can be unbounded if $\mathcal{X}_2 T_0$ is not closed). A factorization of an operator T into $T = T_0 V$, where T_0 is outer and V is inner (or an isometry: $VV^* = I$) is called an outer-inner factorization. This factorization can be obtained from theorem 3 by defining $\mathcal{K}_0$ to be $\mathcal{K}_0 = \overline{\mathcal{U}_2 T}$. The closures here and in (9) are necessary in cases where $\mathcal{U}_2 T$ is not a closed subspace. This happens when there are ‘zeros on the unit circle’, for example when $T = I - Z$. The existence of inner-outer factorizations is established in the following theorem. A more general proof (in the context of nest algebras which specializes to the current setting) is given by Arveson [4].

Theorem 5. *Let $T \in \mathcal{U}(\mathcal{M},\mathcal{N})$. Then T has a factorization*

$$T \;=\; T_0 V$$

where $V \in \mathcal{U}(\mathcal{M}_V,\mathcal{N})$ is an isometry ($VV^* = I$), $T_0 \in \mathcal{U}(\mathcal{M},\mathcal{M}_V)$ is outer, and $\mathcal{M}_V \subset \mathcal{M}$ (entrywise).

PROOF Define $\mathcal{K}_0 = \overline{\mathcal{U}_2 T}$. Then $\mathcal{K}_0$ is a D-invariant subspace which is shift-invariant: $Z\mathcal{K}_0 \subset \mathcal{K}_0$. According to theorem 3, there exists a space sequence $\mathcal{M}_V$ and an isometric operator $V \in \mathcal{U}(\mathcal{M}_V,\mathcal{N})$ such that $\overline{\mathcal{U}_2^{\mathcal{M}} T} = \mathcal{U}_2^{\mathcal{M}_V} V$. By construction, $\overline{\mathcal{U}_2^{\mathcal{M}} T} = \mathcal{D}_2^{\mathcal{M}_V} V \oplus Z\overline{\mathcal{U}_2^{\mathcal{M}} T}$ with $\mathcal{M}_V$ of minimal dimensions. Because also $\overline{\mathcal{U}_2^{\mathcal{M}} T} = \overline{[\mathcal{D}_2^{\mathcal{M}} \oplus Z\mathcal{U}_2^{\mathcal{M}}] T}$, but $\mathcal{D}_2^{\mathcal{M}} T$ is not necessarily orthogonal to $Z\mathcal{U}^{\mathcal{M}} T$, it follows that $\mathcal{M}_V \subset \mathcal{M}$. In particular, the entries of $\mathcal{M}_V$ are finite vector spaces.

Define $T_0 = TV^*$. Then $\overline{\mathcal{U}_2 T_0} = \overline{\mathcal{U}_2 TV^*} = \overline{\overline{\mathcal{U}_2 T} V^*} = \overline{\mathcal{U}_2 VV^*} = \mathcal{U}_2$, so that T_0 is outer. It remains to prove that $T = T_0 V$, *i.e.,* $T = TV^*V$. This is immediate if V is inner. If V is not inner, then corollary 4 ensures the existence of an isometry U such that

$$\mathcal{U}_2 \;=\; \overline{\mathcal{H}}_0(V) \;\oplus\; \mathcal{U}_2 U \;\oplus\; \mathcal{U}_2 V\,,$$

where $\mathcal{K}_0' := \mathcal{U}_2 U = \ker(\cdot V^*|_{\mathcal{U}_2})$, and $W = \begin{bmatrix} U \\ V \end{bmatrix}$ is inner. Then $U^*U + V^*V = I$, $VU^* = 0$, and $T = TV^*V \;\Leftrightarrow\; T(I - V^*V) = 0 \;\Leftrightarrow\; TU^*U = 0$. But $\mathcal{U}_2 TU^* \subset \mathcal{U}_2 VU^* = \emptyset$, which implies $TU^* = 0$. Hence $T = T_0 V$. $\square$

One can show that, in theorem 5, V is inner if and only if $\ker(\cdot T^*) = \emptyset$. If V is not inner, then the extension W of V in theorem 5 is inner, and such that $\overline{\mathcal{H}}_0(V) = \mathcal{H}_0(W)$, but the resulting factor $T_0 = TW^*$ based on W is not precisely outer according to the definition in (9):

$$\overline{\mathcal{U}_2 T_0} \;=\; \overline{\mathcal{U}_2 T} W^* \;=\; \mathcal{U}_2 V W^* \;=\; \mathcal{U}_2 [0 \quad I]$$

so that this T_0 reaches only a subset of $\mathcal{U}_2$ and maps the rest to 0.

The inner-outer factorization is based on the identification of a subspace $\mathcal{K}_0 = \overline{\mathcal{U}_2 T}$ as $\mathcal{K}_0 = \mathcal{U}_2 V$. The complement in $\mathcal{U}_2$ of this space is $\overline{\mathcal{H}}_0(V) \oplus \mathcal{K}_0'$ and is characterized by the elements $X \in \mathcal{U}_2$ satisfying $\mathbf{P}_0(\mathcal{U}_2 TX^*) = 0$, that is, $XT^* \perp \mathcal{U}_2$. Hence

$$\overline{\mathcal{H}}_0(V) \;\oplus\; \mathcal{K}_0' \;=\; \{X \in \mathcal{U}_2 : XT^* \in \mathcal{L}_2 Z^{-1}\} \;=\; \{X \in \mathcal{U}_2 : \mathbf{P}(XT^*) = 0\}\,.$$

In this expression, $\mathcal{K}_0' = \mathcal{U}_2 U = \ker(\,\cdot V^*\big|_{\mathcal{U}_2})$ according to its definition. We now show that also $\mathcal{K}_0' = \ker(\,\cdot T^*\big|_{\mathcal{U}_2}) = \{X \in \mathcal{U}_2 : XT^* = 0\}$. Indeed, if $X \in \mathcal{K}_0'$, then $X = X_1 U$ for some $X_1 \in \mathcal{U}_2$, and because $UT^* = 0$, it follows that $XT^* = 0$. Conversely, if $XT^* = 0$, then $XV^*T_0^* = 0$, and because $\ker(\,\cdot T_0^*) = \emptyset$, it follows that $XV^* = 0$ so that $X \in \mathcal{K}_0'$. Hence $\mathcal{K}_0' = \ker(\,\cdot T^*\big|_{\mathcal{U}_2})$.

7. Computation of the inner-outer factorization $T = VT_0$

Let $T \in \mathcal{U}(\mathcal{M},\mathcal{N})$, with $\mathcal{M},\mathcal{N}$ locally finite spaces of sequences. In this section, we work with a dual factorization of T: $T = VT_0$ (for different V and T_0), where T_0 is 'right outer': $\overline{\mathcal{L}_2 Z^{-1} T_0^*} = \mathcal{L}_2 Z^{-1}$ (or $\overline{T_0 \mathcal{U}_2} = \mathcal{U}_2$), and where the left inner (isometric) factor V satisfies $V^*V = I$ and is obtained by identifying the subspace $\mathcal{K}(V) = \mathcal{L}_2 Z^{-1} V^*$ with $\overline{\mathcal{L}_2 Z^{-1} T^*}$. For this factorization,

$$\overline{\mathcal{H}}(V) \oplus \mathcal{K}' \;=\; \{U \in \mathcal{L}_2 Z^{-1} : UT \in \mathcal{U}_2\}, \qquad \mathcal{K}' \;=\; \ker(\,\cdot T\big|_{\mathcal{L}_2 Z^{-1}})\,.$$

We have defined in section 3 the decomposition of T, restricted to $\mathcal{L}_2 Z^{-1}$, as

$$\cdot T\big|_{\mathcal{L}_2 Z^{-1}} = \,\cdot K_T + \,\cdot H_T\,, \qquad \cdot K_T = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,\cdot T\big|_{\mathcal{L}_2 Z^{-1}})\,.$$

It is thus seen that $\overline{\mathcal{H}}(V)$ is the *largest* subspace in $\mathcal{L}_2 Z^{-1}$ for which $\overline{\mathcal{H}}(V) K_T = 0$ and which is orthogonal to $\mathcal{K}'$. This property provides a way to compute the inner-outer factorization.

Let $\mathbf{Q}$ be an orthonormal basis representation of $\overline{\mathcal{H}}(V)$: $\overline{\mathcal{H}}(V) = \mathcal{D}_2 \mathbf{Q}$, and let $\mathbf{F}_0$ be a basis representation of $\overline{\mathcal{H}}_0(T)$, or more generally, a subspace in $\mathcal{U}_2$ containing $\overline{\mathcal{H}}_0(T)$. The fact that $\overline{\mathcal{H}}(V) K_T = \emptyset$ translates to the condition $\mathbf{Q}T \in \mathcal{U}$. Because $\overline{\mathcal{H}}(V) T \subset \overline{\mathcal{H}}_0(T)$, we must have that $\mathbf{Q}T = Y\mathbf{F}_0$ for some bounded diagonal operator Y, which plays an instrumental role in the derivation of a state realization for V. It remains to implement the condition $\overline{\mathcal{H}}(V) \perp \mathcal{K}'$. Suppose that $\mathbf{Q}$ has a component in $\mathcal{K}'$, so that $D\mathbf{Q} \in \mathcal{K}'$, for some $D \in \mathcal{D}_2$. Then, since $\mathcal{K}' = \ker(\,\cdot T\big|_{\mathcal{L}_2 Z^{-1}})$,

$$D\mathbf{Q} \in \mathcal{K}' \quad \Leftrightarrow \quad D\mathbf{Q}T = DY\mathbf{F}_0 = 0 \quad \Leftrightarrow \quad D \in \ker(\,\cdot Y)\,.$$

Hence $\overline{\mathcal{H}}(V) = \mathcal{D}_2 \mathbf{Q}$ can be described as the largest subspace $\mathcal{D}_2 \mathbf{Q}$ in $\mathcal{L}_2 Z^{-1}$ for which $\mathbf{Q}T = Y\mathbf{F}_0$ with $\ker(\,\cdot Y) = \emptyset$.

If $\mathcal{B}$ is the state sequence space of T, and $\mathcal{B}_V$ is the state sequence space of V, then $Y \in \mathcal{D}(\mathcal{B}_V, \mathcal{B})$. The condition $\ker(\,\cdot Y) = \emptyset$ implies that $\mathcal{B}_V \subset \mathcal{B}$ (pointwise), so that the state dimension of V is at each point in time less than or equal to the state dimension of T at that point.

Proposition 6. *Let $T \in \mathcal{U}$ be a locally finite transfer operator, let $\mathbf{T} = \{A,B,C,D\}$ be an observable realization of T, and assume $\ell_A < 1$. Let V be a left inner (isometric) factor of T so that $T_0 = V^*T$ is right outer. Then the pair (A_V, B_V) that corresponds to an orthonormal basis representation $\mathbf{Q}$ of $\overline{\mathcal{H}}(V)$ satisfies*

$$\begin{array}{lrcl} (i) & A_V^* Y A + B_V^* B &=& Y^{(-1)} \\ (ii) & A_V^* Y C + B_V^* D &=& 0 \\ (iii) & A_V^* A_V + B_V^* B_V &=& I \\ (iv) & \ker(\cdot Y) &=& \emptyset\,. \end{array}$$

for some bounded $Y \in \mathcal{D}$, and conversely, all solutions (A_V, B_V) of these equations give basis representations of $\overline{\mathcal{H}}(V)$.

Proof Let $\mathbf{F}_0 = (I - AZ)^{-1} C$. Because $\mathcal{H}_0(T) \subset \mathcal{D}_2 \mathbf{F}_0$, we have $\mathbf{P}(\mathbf{Q}T) = Y\mathbf{F}_0$ for some bounded $Y \in \mathcal{D}$, and we will show that Y is given by a solution to equation (i). Indeed, let Y be defined by $\mathbf{P}(\mathbf{Q}T) = Y\mathbf{F}_0$. We will apply the relations $Z\mathbf{Q} = A_V^* \mathbf{Q} \,+\, B_V^*$; $\mathbf{F}_0 = C + AZ\mathbf{F}_0$, $T = D + BZ\mathbf{F}_0$ (*cf.*

equation (5)). Firstly, $\mathbf{P}(Z^{-1}Y\mathbf{F}_0) = Y^{(1)}\mathbf{P}(Z^{-1}\mathbf{F}_0) = (YA)^{(1)}\mathbf{F}_0$. On the other hand,

$$\begin{aligned} A_V^{*(1)}\mathbf{P}(Z^{-1}\mathbf{Q}T) &= \mathbf{P}(Z^{-1}[A_V^*\mathbf{Q}]T) \\ &= \mathbf{P}(Z^{-1}[Z\mathbf{Q}-B_V^*]T) \\ &= \mathbf{P}(\mathbf{Q}T) - B_V^{*(1)}B^{(1)}\mathbf{F}_0 \\ &= Y\mathbf{F}_0 - B_V^{*(1)}B^{(1)}\mathbf{F}_0 . \end{aligned}$$

Hence, because observability means that $\cdot\mathbf{F}_0\big|_{\mathcal{D}_2}$ is one-to-one,

$$\begin{aligned} & \mathbf{P}(Z^{-1}Y\mathbf{F}_0) = \mathbf{P}(Z^{-1}\mathbf{Q}T) \\ \Rightarrow\quad & (A_V^*YA)^{(1)}\mathbf{F}_0 + (B_V^*B)^{(1)}\mathbf{F}_0 = Y\mathbf{F}_0 \\ \Leftrightarrow\quad & A_V^*YA + B_V^*B = Y^{(-1)} . \end{aligned}$$

Conversely, since $\ell_A < 1$ implies that any solution Y of (i) must be unique, it follows that this solution will satisfy $\mathbf{P}(\mathbf{Q}T) = Y\mathbf{F}_0$.

Let Y be given by $\mathbf{P}(\mathbf{Q}T) = Y\mathbf{F}_0$. To derive the equivalence of (ii) with the condition $\mathbf{Q}T = \mathcal{U}$, we will use the fact that $\mathbf{Q}T \in \mathcal{U} \Leftrightarrow \mathbf{P}_0(Z^n\mathbf{Q}T) = 0$ for all $n > 0$.

$$\begin{aligned} n=1: \qquad \mathbf{P}_0(Z\mathbf{Q}T) &= \mathbf{P}_0([A_V^*\mathbf{Q}+B_V^*]T) \\ &= A_V^*\mathbf{P}_0(\mathbf{Q}T)+B_V^*D \\ &= A_V^*YC+B_V^*D \end{aligned}$$

Hence $\mathbf{P}_0(Z\mathbf{Q}T) = 0 \Leftrightarrow A_V^*YC+B_V^*D = 0$. For $n > 1$, assume $\mathbf{P}_0(Z^{n-1}\mathbf{Q}T) = 0$. Then

$$\begin{aligned} \mathbf{P}_0(Z^n\mathbf{Q}T) &= \mathbf{P}_0(Z^{n-1}[Z\mathbf{Q}T]) \\ &= \mathbf{P}_0(Z^{n-1}[A_V^*\mathbf{Q}]T) + \mathbf{P}_0(Z^{n-1}B_V^*T) \\ &= A_V^{*(n-1)}\mathbf{P}_0(Z^{n-1}\mathbf{Q}T) + B_V^{*(n-1)}\mathbf{P}_0(Z^{n-1}T) \\ &= 0 + 0 . \end{aligned}$$

Hence (ii) is both necessary and sufficient for the condition $\mathbf{Q}T \in \mathcal{U}$ to be satisfied. The fact that we took $\mathbf{Q}$ to be an orthonormal basis representation implies condition (iii), and condition (iv) has already been derived. □

It is possible to construct solutions (A_V, B_V) for the four equations in proposition 6, and from these solutions a realization $\mathbf{V}$ for the inner (isometric) factor V of T follows. Taking the k-th entry of each diagonal in (i)–(iv) gives the recursive equations

$$\left\{ \begin{array}{lll} (i) & A_{V,k}^*Y_kA_k + B_{V,k}^*B_k & = \; Y_{k+1} \\ (ii) & A_{V,k}^*Y_kC_k + B_{V,k}^*D_k & = \; 0 \\ (iii) & A_{V,k}^*A_{V,k} + B_{V,k}^*B_{V,k} & = \; I \\ (iv) & Y_{k+1} \text{ full row-rank.} & \end{array} \right.$$

A_V and B_V can be computed from these equations starting at some point in time, once an initial value for Y is known (this is discussed below). The recursion for Y_{k+1} is convergent because $\ell_A < 1$. Assuming Y_k known, the computation of Y_{k+1}, $A_{V,k}$ and $B_{V,k}$ requires four steps:

$$\begin{array}{llcll} (a) & \begin{bmatrix} A'_{V,k} \\ B'_{V,k} \end{bmatrix} & = & \begin{bmatrix} Y_kC_k \\ D_k \end{bmatrix}^{\perp} & [\text{for } (ii)] \\ (b) & Y'_{k+1} & = & [A'^*_{V,k} \;\; B'^*_{V,k}] \begin{bmatrix} Y_kA_k \\ B_k \end{bmatrix} & [\text{for } (i)] \\ (c) & \begin{bmatrix} Y_{k+1} \\ 0 \end{bmatrix} & = & \begin{bmatrix} Q_{1,k} \\ Q_{2,k} \end{bmatrix} Y'_{k+1} & [\text{QR-factorization of } Y'_{k+1} \text{ for } (iv)] \\ (d) & \begin{bmatrix} A_{V,k} \\ B_{V,k} \end{bmatrix} & = & \begin{bmatrix} A'_{V,k} \\ B'_{V,k} \end{bmatrix} Q^*_{1,k} , & \end{array}$$

where $[\cdot]^\perp$ denotes the linear algebra operation of taking a minimal orthonormal basis of the full orthogonal complement of the column space of its argument (the basis vectors form the columns of the result). Steps (a) and (b) determine Y'_{k+1}, which can be too large: its kernel needs not be empty. In step (c), the kernel is determined as the span of the rows of $Q_{2,k}$ and subsequently removed, which yields Y_{k+1} and $A_{V,k}, B_{V,k}$.

With A_V and B_V known, we can proceed in two directions. It was noted in the previous section that it will not always be possible to obtain an inner factor V: if $\ker(\cdot T|_{\mathcal{L}_2Z^{-1}}) \neq \emptyset$, then V will be isometric. V can be extended to an inner operator $W = [U \quad V]$, where U is the isometry satisfying $\mathcal{L}_2 Z^{-1} U^* = \ker(\cdot T|_{\mathcal{L}_2Z^{-1}})$. The resulting W is too large in the sense that $U^*T = 0$, but since $\mathcal{H}(W) = \overline{\mathcal{H}}(V)$, a realization $\mathbf{W}$ is readily obtained from A_V, B_V by requiring that $\mathbf{W}$ is unitary (theorem 2):

$$\mathbf{W} = \begin{bmatrix} A_V & C_W \\ B_V & D_W \end{bmatrix} \text{ unitary } \quad \Rightarrow \quad \begin{bmatrix} C_{W,k} \\ D_{W,k} \end{bmatrix} = \begin{bmatrix} A_{V,k} \\ B_{V,k} \end{bmatrix}^\perp$$

A realization for U is obtained from the condition $U^*T = 0$, where U^*T evaluates as

$$\begin{aligned} U^*T &= [D_U^* + C_U^* \mathbf{Q}]T \\ &= D_U^*[D + BZ\mathbf{F}_0] + C_U^* Y \mathbf{F}_0 \\ &= [D_U^* D + C_U^* YC] + [D_U^* B + C_U^* YA] Z \mathbf{F}_0 \end{aligned} \tag{10}$$

Hence $U^*T = 0$ requires both $C_U^*YA + D_U^*B = 0$ and $C_U^*YC + D_U^*D = 0$, and in view of the above steps (a)–(d), it follows that

$$(e) \qquad \begin{bmatrix} C_{U,k} \\ D_{U,k} \end{bmatrix} = \begin{bmatrix} A'_{V,k} \\ B'_{V,k} \end{bmatrix} Q_{2,k}^*, \qquad \begin{bmatrix} C_{V,k} \\ D_{V,k} \end{bmatrix} = \begin{bmatrix} A'_{V,k} \\ B'_{V,k} \end{bmatrix}^\perp.$$

With $\mathbf{V}$ known, a realization for the outer factor T_0 is obtained by evaluating $T_0 = V^*T$ in terms of state space quantities. This yields, much as in equation (10),

$$\begin{aligned} T_0 &= [C_V^*YC + D_V^*D] + [C_V^*YA + D_V^*B]\,\mathbf{F}_0 \\ \mathbf{T}_0 &= \begin{bmatrix} A & C \\ C_V^*YA + D_V^*B & C_V^*YC + D_V^*D \end{bmatrix}. \end{aligned} \tag{11}$$

An algorithm to compute $\mathbf{V}$ and $\mathbf{T}_0$ from a realization of T for finite $n \times n$ (block)-matrices is given as algorithm 1. The body of the algorithm consists of the steps (a)–(e) that have been explained above. One issue that remains to be discussed concerns the initialization of Y. In an algorithm for finite matrices, we can take $Y_1 = [\cdot]$ because the input space $\mathcal{M}$ for T (and hence V) has empty dimensions before time instant 1, so that a minimal realization for V has zero states before time instant 1. For the more general class of systems which are time-invariant before, say, point 1 in time, an initial value for Y is determined in the following way. Y_1 now has to satisfy an equation rather than a recursion: $Y_1 = Y_0 = A_{V,0}^* Y_0 A_0 + B_{V,0}^* B_0$, where, as before,

$$\begin{aligned} A_{V,0}^* A_{V,0} + B_{V,0}^* B_{V,0} &= I, \\ A_{V,0}^* Y_0 C_0 + B_{V,0}^* D_0 &= 0. \end{aligned}$$

We will show that the solution of these equations is the same as the classical solution of the inner-outer factorization, and is determined by the zeros of the time-invariant part of T that are in the unit disc. For convenience of notation, define $y = Y_0$, $a = A_0$, $b = B_0$, $c = C_0$, $d = D_0$, $\alpha = A_{V,0}$, $\beta = B_{V,0}$. We will also assume that d (and hence T) is invertible, and that its zeros are distinct. Then

$$\begin{aligned} y &= \alpha^* ya + \beta^* b \\ 0 &= \alpha^* yc + \beta^* d \\ I &= \alpha^*\alpha + \beta^*\beta \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} \beta^* &= -\alpha^* ycd^{-1} \\ y &= \alpha^* y(a - cd^{-1}b) \\ I &= \alpha^*\alpha + \beta^*\beta. \end{aligned} \tag{12}$$

In: $\{\mathbf{T}_k\}$ (an observable realization of T)
Out: $\{\mathbf{V}_k\}, \{(\mathbf{T}_0)_k\}$ (realizations of the isometric and outer factors)

$Y_1 = [\,\cdot\,]$

for $k = 1, \cdots, n$

$$\left[\begin{array}{lrcl}
(a) & \begin{bmatrix} A'_{V,k} \\ B'_{V,k} \end{bmatrix} & = & \begin{bmatrix} Y_k C_k \\ D_k \end{bmatrix}^{\perp} \\
(b) & Y'_{k+1} & = & [A'^*_{V,k} \quad B'^*_{V,k}] \begin{bmatrix} Y_k A_k \\ B_k \end{bmatrix} \\
(c) & \begin{bmatrix} Y_{k+1} \\ 0 \end{bmatrix} & = & \begin{bmatrix} Q_{1,k} \\ Q_{2,k} \end{bmatrix} Y'_{k+1} \qquad \text{[QR-factorization of } Y'_{k+1}] \\
(d) & \begin{bmatrix} A_{V,k} \\ B_{V,k} \end{bmatrix} & = & \begin{bmatrix} A'_{V,k} \\ B'_{V,k} \end{bmatrix} Q^*_{1,k} \\
(e) & \begin{bmatrix} C_{V,k} \\ D_{V,k} \end{bmatrix} & = & \begin{bmatrix} A'_{V,k} \\ B'_{V,k} \end{bmatrix}^{\perp} \\
 & \mathbf{V}_k & = & \begin{bmatrix} A_{V,k} & C_{V,k} \\ B_{V,k} & D_{V,k} \end{bmatrix} \\
 & (\mathbf{T}_0)_k & = & \begin{bmatrix} A_k & C_k \\ C^*_{V,k} Y_k A_k + D^*_{V,k} B_k & C^*_{V,k} Y_k C_k + D^*_{V,k} D_k \end{bmatrix}
\end{array}\right.$$

end

Algorithm 1. Inner-outer factorization algorithm for $T = VT_0$ (finite matrix case).

Bring in eigenvalue decompositions of α and $(a - cd^{-1}b)$: $\alpha = r\phi r^{-1}$; $a - cd^{-1}b \;=\; s\psi s^{-1}$. Then

$$(r^* y s) \;=\; \phi^*(r^* y s)\psi\,.$$

Because both ϕ and ψ are diagonal matrices, the above expression shows that $(r^* ys)$ must be a rectangular diagonal matrix (or a permutation thereof), and hence the diagonal entries of ϕ are equal to a subset of the diagonal entries of ψ^{-*}. In view of the requirement $\alpha^*\alpha = I - \beta^*\beta$, ϕ can contain only the entries of ψ^{-*} that are smaller than 1. Because V must be of the highest possible system order while y must have full row rank, ϕ is precisely equal to those entries. It remains to note that the entries of $\psi^{-1} = \text{eig}(a - cd^{-1}b)^{-1}$ are equal to the zeros of T. This is because $T^{-1} = d^{-1} + d^{-1}bz\left[I - (a - cd^{-1}b)z\right]^{-1} cd^{-1}$ has poles equal to $\text{eig}(a - cd^{-1}b)^{-1}$. With the poles of the inner system thus determined, it is a straightforward matter (involving a Lyapunov equation) to compute α, β, and y from (12).

8. Closed-form expression for the outer factor realization

In the time-invariant setting, it is well known that the outer factor T_0 of T can be written in closed form in terms of the original state matrices $\{A, B, C, D\}$ of T and only one unknown intermediate quantity, which is the solution of a Riccati equation with $\{A, B, C, D\}$ as parameters. One way to obtain the Riccati equation is by performing a spectral factorization of the squared relation $T^*T = T_0^*T_0$. Riccati equations can be solved recursively; efficient solution methods for the recursive version are the so-called square-root algorithms, in which extra intermediate quantities are introduced to avoid the computation of inverses and square roots. The algorithm to compute the realization for T_0 given in

(11) can be viewed as such a square-root algorithm: besides Y, it contains the intermediate quantities A_V and B_V. We will show in this section how the corresponding Riccati recursion can be derived.

Theorem 7. *Let $T \in \mathcal{U}$ be a locally finite transfer operator, let $\mathbf{T} = \{A,B,C,D\}$ be an observable realization of T, and assume $\ell_A < 1$. Then a realization of the outer factor T_0 of T so that $T_0 = V^*T$ is given by*

$$\mathbf{T}_0 \;=\; \begin{bmatrix} I & \\ & R^* \end{bmatrix} \begin{bmatrix} A & C \\ C^*MA + D^*B & C^*MC + D^*D \end{bmatrix}$$

where $M \geq 0$ is the solution of maximal rank of

$$M^{(-1)} \;=\; A^*MA + B^*B - \left[A^*MC + B^*D\right] (D^*D + C^*MC)^\dagger \left[D^*B + C^*MA\right] \tag{13}$$

and R is a minimal full range factor ($\ker(\,\cdot R^*) = \emptyset$*) of*

$$RR^* \;=\; (D^*D + C^*MC)^\dagger\,,$$

provided the pseudo-inverse is bounded (see discussion below).

PROOF Let $\mathbf{T}_0$ be given by equation (11), so that C_V and D_V are given, according to steps (a) and (e), as

$$\begin{bmatrix} C_V \\ D_V \end{bmatrix} = \begin{bmatrix} YC \\ D \end{bmatrix}^{\perp\perp} = \begin{bmatrix} YC \\ D \end{bmatrix} R\,. \tag{14}$$

$R \in \mathcal{D}(\mathcal{N}, \mathcal{N}_V)$ is a diagonal whose 'tall' matrix entries R_k make the columns of $\left[\begin{smallmatrix} Y_kC_k \\ D_k \end{smallmatrix}\right]$ isometric, removing columns that are linearly dependent:

$$R^*\,(D^*D + C^*MC)\,R \;=\; I_{\mathcal{N}_V}\,, \qquad \text{where} \quad M := Y^*Y\,.$$

Let $X = D^*D + C^*MC$, then $R^*XR = I$ implies $RR^* = X^\dagger$, where $(\cdot)^\dagger$ denotes the operator pseudo-inverse [11]. According to step (c), $(Y^*Y)^{(-1)} = (Y'^*Y')^{(-1)}$, so that we obtain from step (b)

$$\begin{aligned}
(Y^*Y)^{(-1)} \;&=\; [A^*Y^* \quad B^*] \begin{bmatrix} A'_V \\ B'_V \end{bmatrix} [A'^*_V \quad B'^*_V] \begin{bmatrix} YA \\ B \end{bmatrix} \\
&=\; [A^*Y^* \quad B^*] \left(I - \begin{bmatrix} YC \\ D \end{bmatrix} RR^* [C^*Y^* \quad D^*] \right) \begin{bmatrix} YA \\ B \end{bmatrix} \\
&=\; A^*Y^*(I - YCRR^*C^*Y^*)\,YA \;+\; B^*(I - DRR^*D^*)\,B\, - \\
&\quad\; - A^*Y^*(YCRR^*D^*)\,B \,-\, B^*(DRR^*C^*Y^*)\,YA\,,
\end{aligned}$$

and with $M = Y^*Y$, M satisfies the equation

$$M^{(-1)} \;=\; A^*MA + B^*B \,-\, \left[A^*MC + B^*D\right] RR^* \left[D^*B + C^*MA\right]\,.$$

This equation has more solutions M. As $Y \in \mathcal{D}(\mathcal{B}_V, \mathcal{B})$ has $\mathcal{B}_V$ of maximal possible dimensions such that $\ker(\,\cdot Y) = \emptyset$, the solution M of the Riccati equation must be positive and of maximal rank to yield an outer factor T_0. (Note that if D^*D is invertible, then $M = 0$ is always a solution, and yields $T_0 = T$.) $\square$

The above Riccati equation bears a close resemblance to the Riccati equation that was obtained in the solution of the time-varying lossless embedding problem [12]. Indeed, it is well-known that the spectral factorization problem and the lossless embedding problem are connected: a Cayley transformation on $\mathbf{P}(T^*T)$ yields a contractive scattering operator which can be embedded to an inner operator, from which the outer factor can be extracted. Initial conditions for M can be obtained as $M_{k_0} = [\,\cdot\,]$

when T starts with zero states at some point k_0 in time, or from a solution of the Riccati equation if T is time-invariant before k_0. Again, the solution requires eigenvalue decompositions, and must satisfy the side conditions that $M \geq 0$ and has maximal rank. Initial conditions for the spectral factorization problem are investigated in [13].

In the above proof, we required the boundedness of the pseudo-inverse of $(D^*D + C^*MC)$ in case this operator is not uniformly positive (this is no issue when D^*D is uniformly positive). We will show that if $\mathrm{ran}(\cdot T)$ is closed, then the pseudo-inverse is also bounded. This condition is a generalization of the time-invariant "no zeros of T are on the unit circle". If $\mathrm{ran}(\cdot T)$ is not closed, then $\mathcal{L}_2 Z^{-1} T_0$ is dense in $\mathcal{L}_2 Z^{-1}$, but not closed. In this case, T_0 has a one-sided inverse which is unbounded. Similar issues played a role in the solution of the embedding problem [12], where it was shown that, even if R was unbounded, the products $R^*(D^*B + C^*MA)$ and $R^*(C^*MC + D^*D)$ remained bounded because of range inclusions that are automatically satisfied. The same happens here.

Proposition 8. *In theorem 7,* $(D^*D + C^*MC)^\dagger$ *is bounded if* $\mathrm{ran}(\cdot T)$ *is closed.*

Whether the range is closed or not, M is bounded, as are the products $R^*(D^*D + C^*MC)$ and $R^*(C^*MA + D^*B)$.

PROOF If $\mathrm{ran}(\cdot T_0)$ is closed, then T_0 has a one-sided inverse which is again upper. It follows that in this case $\mathrm{ran}(\cdot D_{T_0})$ is closed, so that $D_{T_0}^* D_{T_0} = XRR^*X = XX^\dagger X = X = D^*D + C^*MC$ has closed range and a bounded pseudo-inverse. Because $T_0 = V^*T$, $\mathrm{ran}(\cdot T_0)$ can be closed only if $\mathrm{ran}(\cdot T)$ is closed. If $\mathrm{ran}(\cdot T)$ is closed, then $(\mathcal{X}_2 V) T_0$ is closed. But from $V^*V = I$ it follows that $\mathcal{X}_2 V = \mathcal{X}_2$, so that in this case $\mathrm{ran}(\cdot T_0)$ is closed, too.

Because the realization of T is observable, it was argued in proposition 6 that Y (and hence M) is bounded. From the first equality in (14) we see that $\begin{bmatrix} C_V \\ D_V \end{bmatrix}$ is obtained by taking an orthonormal basis in the closure of the range of $\begin{bmatrix} YC \\ D \end{bmatrix}$. R is unbounded if the latter range is not closed. Nonetheless, $\begin{bmatrix} C_V \\ D_V \end{bmatrix}$ is well-defined and isometric, and $D_{T_0} = R(D^*D + C^*MC) = \begin{bmatrix} C_V \\ D_V \end{bmatrix}^* \begin{bmatrix} YC \\ D \end{bmatrix}$ is bounded. In the same way, it is shown that $C_{T_0} = R^*(C^*MA + D^*B) = \begin{bmatrix} C_V \\ D_V \end{bmatrix}^* \begin{bmatrix} YA \\ B \end{bmatrix}$ is bounded. $\square$

As is well known, M_{k+1} in the Riccati recursion can be computed more efficiently using square-root algorithms (see *e.g.*, Morf [14] for a list of pre-1975 references). In such algorithms, the square-root Y of M is computed, rather than M itself. The square-root algorithm that corresponds to the above equations is: find $\mathbf{W}_k$, unitary, such that the following product has zeros in the indicated positions:

$$\mathbf{W}_k^* \begin{bmatrix} Y_k & \\ & I \end{bmatrix} \begin{bmatrix} A_k & C_k \\ B_k & D_k \end{bmatrix} = \left[\begin{array}{cc} Y_{k+1} & 0 \\ 0 & 0 \\ \hline * & R_k^\dagger \end{array} \right].$$

Given Y_k, $\mathbf{W}_k$ can be obtained by a simple QR-factorization. Using the fact that $\mathbf{W}_k$ is unitary, multiplying the above equation with its transpose shows that two of the three non-zero block-entries of the right-hand side follow as Y_{k+1} and $R_k^\dagger$. In fact, $\mathbf{W}_k$ turns out to be precisely equal to the realization of the inner factor W as determined in the previous section. Initial values of Y can be obtained as discussed earlier.

9. Inner-outer factorization examples

We finish this paper with some example results of the inner-outer factorization algorithm on finite (4×4) matrices. In the finite matrix case, interesting things can occur only when T is singular or when the dimensions of T are not uniform.

1. The algorithm, applied to

$$T = \begin{bmatrix} \underline{0} & 1 & 4 & 6 \\ 0 & \underline{0} & 2 & 5 \\ 0 & 0 & \underline{0} & 3 \\ 0 & 0 & 0 & \underline{0} \end{bmatrix}$$

(the underlined entries form the 0-th diagonal) yields an almost trivial isometric factor V or inner factor W (the dots correspond to columns or rows with vanishing dimensions):

$$V = \begin{bmatrix} \underline{\cdot} & 1 & 0 & 0 \\ \cdot & \underline{0} & 1 & 0 \\ \cdot & 0 & \underline{0} & 1 \\ \cdot & 0 & 0 & \underline{0} \end{bmatrix} \qquad W = \begin{bmatrix} \underline{\cdot} & 1 & 0 & 0 & 0 \\ \cdot & \underline{0} & 1 & 0 & 0 \\ \cdot & 0 & \underline{0} & 1 & 0 \\ \cdot & 0 & 0 & \underline{0} & \underline{1} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_W & = & [1\ 1\ 1\ 1] \\ \#\mathcal{N}_W & = & [0\ 1\ 1\ 2] \\ \#\mathcal{B}_W & = & [0\ 1\ 1\ 1] \end{array}$$

It is seen that V is not inner, because T is singular. W is the inner extension of V. The only effect of W is a redefinition of time intervals: W acts as a shift operator. $T_0 = W^*T$ is

$$W^*T = \begin{bmatrix} \underline{\cdot} & \cdot & \cdot & \cdot \\ 0 & \underline{1} & 4 & 6 \\ 0 & 0 & \underline{2} & 5 \\ 0 & 0 & 0 & \underline{3} \\ 0 & 0 & 0 & \underline{0} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_{T_0} & = & [0\ 1\ 1\ 2] \\ \#\mathcal{N}_{T_0} & = & [1\ 1\ 1\ 1] \end{array}$$

The multiplication by W^* has shifted the rows of T downwards. This is possible: the result T_0 is still upper. V^*T is equal to W^*T with its last row removed.

2. Take

$$T = \begin{bmatrix} \underline{0} & 1 & 4 & 6 \\ 0 & \underline{1} & 2 & 5 \\ 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M} & = & [1\ 1\ 1\ 1] \\ \#\mathcal{N} & = & [1\ 1\ 1\ 1] \\ \#\mathcal{B} & = & [0\ 1\ 2\ 1] \end{array}$$

Hence T is again singular, but now a simple shift will not suffice. The algorithm computes W as

$$W = \begin{bmatrix} \underline{\cdot} & -0.707 & 0.577 & 0.367 & 0.180 \\ \cdot & \underline{-0.707} & -0.577 & -0.367 & -0.180 \\ \cdot & 0 & \underline{0.577} & -0.733 & -0.359 \\ \cdot & 0 & 0 & \underline{-0.440} & \underline{0.898} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_W & = & [1\ 1\ 1\ 1] \\ \#\mathcal{N}_W & = & [0\ 1\ 1\ 2] \\ \#\mathcal{B}_W & = & [0\ 1\ 1\ 1] \end{array}$$

$$T_0 = W^*T = \begin{bmatrix} \underline{\cdot} & \cdot & \cdot & \cdot \\ 0 & \underline{-1.414} & -4.243 & -7.778 \\ 0 & 0 & \underline{1.732} & 2.309 \\ 0 & 0 & 0 & \underline{-2.273} \\ 0 & 0 & 0 & \underline{0} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_{T_0} & = & [0\ 1\ 1\ 2] \\ \#\mathcal{N}_{T_0} & = & [1\ 1\ 1\ 1] \end{array}$$

V is equal to W with its last column removed, so that $T_0 = V^*T$ is equal to the above T_0 with its last row removed.

3. In the previous examples, we considered systems T with a constant number of inputs and outputs (equal to 1), for which $V \neq I$ only if T is singular. However, a non-identical V can also

occur if the number of inputs and outputs of T are varying in time. Thus consider

$$T = \begin{bmatrix} \underline{1.000} & 0.500 & 0.250 & 0.125 \\ \underline{1.000} & 0.300 & 0.100 & 0.027 \\ 0 & \underline{1.000} & 0.500 & 0.250 \\ 0 & 0 & \underline{1.000} & 0.300 \\ \cdot & \cdot & \cdot & \underline{\cdot} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M} & = & [2\ 1\ 1\ 0] \\ \#\mathcal{N} & = & [1\ 1\ 1\ 1] \\ \#\mathcal{B} & = & [0\ 1\ 2\ 1] \end{array}$$

$$V = \begin{bmatrix} \underline{-0.707} & 0.099 & 0.025 & -0.699 \\ \underline{-0.707} & -0.099 & -0.025 & 0.699 \\ 0 & \underline{0.990} & -0.005 & 0.139 \\ 0 & 0 & \underline{0.999} & 0.035 \\ \cdot & \cdot & \cdot & \underline{\cdot} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_V & = & [2\ 1\ 1\ 0] \\ \#\mathcal{N}_V & = & [1\ 1\ 1\ 1] \\ \#\mathcal{B}_V & = & [0\ 1\ 1\ 1] \end{array}$$

In this case, V is itself inner. The outer factor T_0 follows as

$$T_0 = V^* T = \begin{bmatrix} \underline{-1.414} & -0.565 & -0.247 & -0.107 \\ 0 & \underline{1.010} & 0.509 & 0.257 \\ 0 & 0 & \underline{1.001} & 0.301 \\ 0 & 0 & 0 & \underline{-0.023} \end{bmatrix} \qquad \begin{array}{lcl} \#\mathcal{M}_{T_0} & = & [1\ 1\ 1\ 1] \\ \#\mathcal{N}_{T_0} & = & [1\ 1\ 1\ 1] \end{array}$$

An interesting observation from these examples is that the inner-outer factorization of finite matrices T is equal to the QR-factorization of T when it is considered as an ordinary matrix without block entries.

10. Concluding remarks

We have derived, in section 7, a simple algorithm to compute realizations of the (left) inner and outer factors of a realization of a given system T. The computations are unidirectional: starting from an initial value of a quantity Y_k, state matrices are computed recursively from that point on. The initial value can be obtained straightforwardly in cases where the state dimension of T vanishes before some point in time, or where T is time-invariant before a point in time. From the algorithm, it can be observed that the number of states in the inner factor (the number of ‘zeros inside the unit disc’) is at each point k always less than the number of states of T, and cannot change at point k if D_k is square and invertible at that point, unless the number of states of T decreases at that point. It can increase if D_k is singular, or if the number of inputs increases at that point, and can decrease if the state dimension of T decreases, or if the number of inputs decreases. For finite matrices, the inner-outer factorization reduces to a QR-factorization.

The outer factor can be computed as a by-product of the same algorithm, or alternatively via a Riccati-type recursive equation.

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