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Goede ultrafilters en verzadigde modellen

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"Good ultrafilters and saturated models"

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Summary

This thesis focusses on good ultrafilters. There are two main theorems which are proven. The first is the theorem which states that good, countably incomplete ultrafilters exist. The second theorem proves that good ultrafilters make ultraproducts saturated.

In the first section we start by defining ultrafilters. We also discuss an alternative definition and prove that ultrafilters exist. Subsequently we define countably incomplete ultrafilters and again give an alternative definition. Finally good ultrafilters are defined and an example is given by showing that every ultrafilter on a set of cardinality $\aleph_0$ is good.

This brings us to the next section where it is shown that good, countably incomplete ultrafilters exist. To prove the existence of good ultrafilters we use the notion of an independent set of functions. We construct sequences of filters $\{\mathcal{F}_\eta\}_{\eta<2^\alpha}$ and independent sets $\{\mathcal{S}_\eta\}_{\eta<2^\alpha}$ which satisfy a list of conditions. Then the filter $\mathcal{U} = \bigcup_{\eta<2^\alpha} \mathcal{F}_\eta$ is a good, countably incomplete ultrafilter.

Having proven that good, countably incomplete ultrafilters actually exist, we next use them to show that they make ultraproducts saturated. Before giving the proof of the theorem, the third section contains a short introduction in model theory. The definitions of a language and its terms and formulas are given. We then discuss when a sentence is satisfied in order to come to a definition for a model.

The next part gives the definition of an ultraproduct and states the fundamental theorem of ultraproducts. This theorem is used in the proofs of the two theorems in the last part of this thesis.

Finally we come to the part where the theorem stating that good, countably incomplete filters make ultraproducts saturated is given. At first a weaker version of this theorem is proven. The structure of this proof is similar to the one proving good ultrafilters make ultraproducts saturated. Then we finish with the proof of the theorem.

Contents

Introduction

Ultrafilters are special subsets of the power set of a set. They are mostly known for their use in model theory. An example of an application of ultrafilters is in the construction of the hyperreals. These are obtained when making an ultraproduct of the real numbers. The idea here is that the universe of the real numbers is extended to a universe which contains sequences of real numbers instead of real numbers. Each real number is identified with the corresponding constant sequence. The model that is then obtained is the model of the non-standard analysis. In this model many theorems of the standard analysis have a much simpler proof because of the existence infinitely large and small numbers.

Next to that, the non-standard analysis has a very interesting property, namely that it is $\aleph_1$-saturated. Roughly speaking this means that if a statement holds for all finite subsets of a countable set, then it holds for the whole set. For example if we look at the statement $0 \leq x \leq \frac{1}{n}$, then for each finite subset of the natural numbers we can find an x which satisfies the statement for all n in that subset. It then follows that $0 \leq x \leq \frac{1}{n}$ is true for every natural number, and thus there exists an infinitely small element x in the model of the non-standard analysis.

The degree of saturation of a model tells us something about the number of statements which we can realize. The $\aleph_1$-saturation of the non-standard analysis can be achieved by using countably incomplete ultrafilters. The question which then arises is how to achieve higher order saturation for models.

In order to do this we will need a stronger filter than a countably incomplete ultrafilter, a *good* ultrafilter. One of the main theorems of this thesis proves that this kind of ultrafilters actually exists. After an introduction in model theory we show that good, countably incomplete filters make ultraproducts α-saturated.

1. Ultrafilters

This first section gives an introduction into the theory of ultrafilters. We will give the definition of an ultrafilter and show that ultrafilters exist. Then we can move on to the definition of a good ultrafilter, whose existence will be proven in section 2.

For a set I, $\mathcal{P}(I)$ is the set of all subsets of I and $\mathcal{S}_\omega(I)$ is the set of all *finite* subsets of I. If I is a set of cardinality α, then $|\mathcal{P}(I)| = 2^\alpha$ and, if α is infinite, $|\mathcal{S}_\omega(I)| = \alpha$.

Definition 1.1. A *filter* $\mathcal{F}$ over a non-empty set I is a subset of $\mathcal{P}(I)$ such that:

(F1) $\emptyset \notin \mathcal{F}$;

(F2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;

(F3) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$ then $B \in \mathcal{F}$.

The following properties are a few simple consequences of this definition. Firstly, with a simple induction argument it follows from (F2) that a filter is closed under finite intersections. From (F3) we see that $I \in \mathcal{F}$, since for any $A \in \mathcal{F}$ we have $A \subseteq I \subseteq I$. Moreover, note that $\mathcal{P}(I)$ is not a filter but does satisfy (F2) and (F3). Finally, the following lemma follows easily from the definition and will be useful in many proofs.

Lemma 1.2. *Let $\mathcal{F}$ be a filter over a set I and $A \subseteq I$. Then $I \setminus A \notin \mathcal{F}$ iff for every $F \in \mathcal{F}$ the intersection $F \cap A$ is non-empty.*

Proof. Let $I \setminus A \notin \mathcal{F}$ and suppose that $F \cap A$ is empty for some $F \in \mathcal{F}$. Then $F \subseteq I \setminus A$ and thus, by (F3), $I \setminus A \in \mathcal{F}$. This gives a contradiction, so $F \cap A$ must be non-empty.

On the other hand let $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$. Suppose $I \setminus A \in \mathcal{F}$. Then, by (F2), $F \cap I \setminus A \in \mathcal{F}$ for any $F \in \mathcal{F}$. But $(F \cap I \setminus A) \cap A = \emptyset$ which contradicts our assumption. Hence, $I \setminus A \notin \mathcal{F}$. □

We say that any subset $\mathcal{A} \subset \mathcal{P}(I)$ has the *finite intersection property* iff every finite intersection of elements of $\mathcal{A}$ is non-empty. Note that any filter $\mathcal{F}$ has the finite intersection property since it is closed under finite intersections and the empty set is not an element of $\mathcal{F}$.

For $\mathcal{A} \subset \mathcal{P}(I)$ having the finite intersection property, the filter $\langle \mathcal{A} \rangle$ is the intersection of all filters containing $\mathcal{A}$, i.e.

$$\langle \mathcal{A} \rangle = \bigcap \{\mathcal{F} : \mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{P}(I) \ \& \ \mathcal{F} \text{ is a filter on } I\}.$$

We call $\langle \mathcal{A} \rangle$ the filter generated by $\mathcal{A}$.

Definition 1.3. Let $\mathcal{U}$ be a filter over a set I, then $\mathcal{U}$ is an *ultrafilter* iff for every $A \subseteq I$ either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Example. Any filter $\mathcal{U}$ on a set I generated by a singleton $\{x\}$, $x \in I$, is an ultrafilter. Namely, let A be any subset of I. Suppose $x \in A$, then $A \in \mathcal{U}$. On the other hand, if $x \notin A$, then $x \in I \setminus A$ and therefore $I \setminus A \in \mathcal{U}$.

An ultrafilter can also be defined as a maximal filter over I. In other words, if $\mathcal{U}$ is an ultrafilter over I there is no filter $\mathcal{F}$ over I such that $\mathcal{U} \subsetneq \mathcal{F}$. This is proven in the following proposition.

Proposition 1.4. *A filter $\mathcal{U}$ is an ultrafilter over I iff it is a maximal filter over I.*

Proof. Let $\mathcal{U}$ be a maximal filter over I and A be any subset of I. If $A \in \mathcal{U}$ and $I \setminus A \in \mathcal{U}$, then $A \cap (I \setminus A) = \emptyset \in \mathcal{U}$. This contradicts the first property of a filter, so we can not have both $A \in \mathcal{U}$ and $I \setminus A \in \mathcal{U}$. We now show that if $I \setminus A \notin \mathcal{U}$ then $A \in \mathcal{U}$. This is sufficient since if $A = I \setminus B \notin \mathcal{U}$ it would follow that $I \setminus A = B \in \mathcal{U}$.
Suppose $I \setminus A \notin \mathcal{U}$. Consider the set $\mathcal{U} \cup \{A\}$. Since $\mathcal{U}$ is a filter it has the finite intersection property. Furthermore, by lemma 1.2, $U \cap A \neq \emptyset$ for all $U \in \mathcal{U}$ since $I \setminus A \notin \mathcal{U}$. It follows that $\mathcal{U} \cup \{A\}$ also has the finite intersection property. Now let $\mathcal{V}$ be the filter generated by $\mathcal{U} \cup \{A\}$, so $\mathcal{V} = \langle \mathcal{U} \cup \{A\} \rangle$. Then $\mathcal{U} \subseteq \mathcal{V}$, since $\mathcal{U} \subseteq \mathcal{U} \cup \{A\}$ and thus $\mathcal{U} \subseteq \mathcal{F}$ for every filter $\mathcal{F}$ which contains $\mathcal{U} \cup \{A\}$. Because $\mathcal{U}$ is a maximal filter it follows that $\mathcal{U} = \mathcal{V}$. Therefore $A \in \mathcal{U}$.

On the other hand let $\mathcal{U}$ be an ultrafilter and suppose that $\mathcal{U}$ is not a maximal filter. Let $\mathcal{V}$ be a filter such that $\mathcal{U} \subsetneq \mathcal{V}$. Take $A \in \mathcal{V} \setminus \mathcal{U}$, then since $A \notin \mathcal{U}$ we must have $I \setminus A \in \mathcal{U}$. But, since $\mathcal{U} \subsetneq \mathcal{V}$, we now find that $\emptyset = A \cap (I \setminus A) \in \mathcal{V}$. This is in contradiction with the first property of a filter, so $\mathcal{U}$ must be a maximal filter. □

The existence of ultrafilters is proven in the Ultrafilter Theorem. This theorem states that any $\mathcal{A} \subseteq \mathcal{P}(I)$ which has the finite intersection property can be extended to an ultrafilter on I. The proof uses the axiom of choice in the form of Zorn's lemma (see Appendix A).

Theorem 1.5 (Ultrafilter Theorem)**.** *Let I be a set and $\mathcal{A} \in \mathcal{P}(I)$. If $\mathcal{A}$ has the finite intersection property, then there is an ultrafilter $\mathcal{U}$ over I such that $\mathcal{A} \subseteq \mathcal{U}$.*

Proof. Let $\mathcal{B}$ be the set of all filters containing $\mathcal{A}$, i.e.

$$\mathcal{B} = \{\mathcal{F} : \mathcal{A} \subseteq \mathcal{F}, \mathcal{F} \text{ is a filter }\},$$

$\mathcal{B}$ is not empty since $\langle \mathcal{A} \rangle \in \mathcal{B}$. We will use Zorn's lemma (A.1) to show that $\mathcal{B}$ has a maximal element.

Let $\mathcal{C}$ be any chain in $\mathcal{B}$. We have to show that $\bigcup \mathcal{C}$ is an element of $\mathcal{B}$, this means that $\bigcup \mathcal{C}$ is filter containing $\mathcal{A}$. Since every $C \in \mathcal{C}$ contains $\mathcal{A}$ it follows that $\mathcal{A} \subseteq \bigcup \mathcal{C}$. Furthermore, we have:

(F1) For every $C \in \mathcal{C}$, $\emptyset \notin C$, so $\emptyset \notin \bigcup \mathcal{C}$.

(F2) Let $A, B \in \bigcup \mathcal{C}$, then $A \in C_1$ and $B \in C_2$ for some $C_1, C_2 \in \mathcal{C}$. Since $\mathcal{C}$ is a chain we have $C_1 \subseteq C_2$ without loss of generality. It follows that $A, B \in C_2$ and thus, since C_2 is a filter, $A \cap B \in C_2 \subseteq \bigcup \mathcal{C}$.

(F3) Let $A \in \bigcup \mathcal{C}$ and $B \supseteq A$. Then $A \in C$ for some filter $C \in \mathcal{C}$, so $B \in C \subseteq \bigcup \mathcal{C}$.

It follows from (F1)-(F3) that $\bigcup \mathcal{C}$ is a filter. Hence, $\bigcup \mathcal{C} \in \mathcal{B}$.

We have now found a chain $\mathcal{C}$ in $\mathcal{B}$ such that $\bigcup \mathcal{C} \in \mathcal{B}$, therefore we can apply Zorn's lemma. It follows that $\mathcal{B}$ has a maximal element, say $\mathcal{U}$. Since $\mathcal{U}$ is an element of $\mathcal{B}$, $\mathcal{A} \subseteq \mathcal{U}$ and by the maximality of $\mathcal{U}$, if $\mathcal{F}$ is any filter containing $\mathcal{U}$, then $\mathcal{F} = \mathcal{U}$. Therefore, by proposition 1.4, $\mathcal{U}$ is an ultrafilter containing $\mathcal{A}$. □

Definition 1.6. An ultrafilter $\mathcal{U}$ is *countably incomplete* iff there is a countable $\mathcal{A} \subset \mathcal{U}$ such that $\bigcap \mathcal{A} = \emptyset$.

An equivalent way to define a countably incomplete ultrafilter is given by the following proposition.

Proposition 1.7. *An ultrafilter $\mathcal{U}$ over I is countably incomplete iff there is a countable decreasing chain*

$$I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

of elements $I_n \in \mathcal{U}$ such that $\bigcap_n I_n = \emptyset$.

Proof. Suppose $\mathcal{U}$ is a countably incomplete ultrafilter over I. Let $\mathcal{A}$ be a countable subset of $\mathcal{U}$ such that $\bigcap \mathcal{A} = \emptyset$. Let $\{X_1, \ldots, X_n, \ldots\}$ be an enumeration of the elements of $\mathcal{A}$ and define

$$J_n = \bigcap_{i=1}^{n} X_i.$$

Since $\bigcap_n X_n = \emptyset$, we know that $\bigcap_n J_n = \emptyset$. Every J_n is also an element of $\mathcal{U}$ since $\mathcal{U}$ is closed under finite intersections. Furthermore, $J_n \supseteq J_{n+1}$ for every $n \in \mathbb{N}$. It follows that for every n there is an $m > n$ such that $J_n \supsetneq J_m$. Otherwise we would find $\bigcap_n J_n = J_m$ for some m. Now we can construct a decreasing chain of sets $J_n \in \mathcal{U}$ such that $J_n \supsetneq J_{n+1}$ for every $n \in \mathbb{N}$. Then the chain

$$I \supset J_0 \supset J_1 \supset J_2 \supset \cdots$$

is a decreasing chain of elements of $\mathcal{U}$ such that its intersection is the empty set.

Conversely let $I = I_0 \supset I_1 \supset \cdots$ be a decreasing chain of elements of $\mathcal{U}$ such that $\bigcap_n I_n = \emptyset$. Then the set $\mathcal{I} = \{I_0, I_1, I_2, \ldots\}$ is a countable subset of $\mathcal{U}$ with $\bigcap \mathcal{I} = \emptyset$. Therefore, $\mathcal{U}$ is countably incomplete. □

The focus of this thesis lies on a special kind of ultrafilter called a *good ultrafilter*. Before we give the definition of a good ultrafilter we need some notations about functions.

Consider a set I and two functions p and q from $\mathcal{S}_\omega(I)$ into $\mathcal{P}(I)$.

- p is *multiplicative* iff for every $X, Y \in \mathcal{S}_\omega(I)$, $p(X \cup Y) = p(X) \cap p(Y)$.
- p is *monotone* iff for every $X \subseteq Y \in \mathcal{S}_\omega(I)$, $p(X) \supseteq p(Y)$.
- $q \leq p$ iff for all $X \in S_\omega(I)$, $q(X) \subseteq p(X)$.

Definition 1.8. An ultrafilter $\mathcal{U}$ on I is *good* iff for every monotone $p : \mathcal{S}_\omega(I) \to \mathcal{U}$ there is a multiplicative $q : \mathcal{S}_\omega(I) \to \mathcal{U}$ such that $q \leq p$.

Note that if I has cardinality α and $\mathcal{U}$ is a good ultrafilter over I, then for every $\beta < \alpha$ and every monotone $p : \mathcal{S}_\omega(\beta) \to \mathcal{U}$ there is a multiplicative $q : \mathcal{S}_\omega(\beta) \to \mathcal{U}$. To see this let $p : \mathcal{S}_\omega(\beta) \to \mathcal{U}$ be a monotone function. Take $f : \mathcal{S}_\omega(I) \to \mathcal{U}$ such that $f(A) = p(A \cap \beta)$. Then, since $\mathcal{U}$ is good, there is a multiplicative $g : \mathcal{S}_\omega(I) \to \mathcal{U}$ such that $g \leq f$. If we take $q : \mathcal{S}_\omega(\beta) \to \mathcal{U}$ to be the restriction of g to $\mathcal{S}_\omega(\beta)$ we find that q is multiplicative and $q \leq p$.

Example. Any ultrafilter on a set I of cardinality $\aleph_0$ is good. It is sufficient to show that an ultrafilter on $\mathbb{N}$ is good. So let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let $p : \mathcal{S}_\omega(\mathbb{N}) \to \mathcal{U}$ be a monotone function. Let $X_n = \{i \in \mathbb{N} : i \leq n\}$ for any $n \in \mathbb{N}$. For any finite set $F \subseteq \mathbb{N}$ define

$$n(F) = \bigcap_{n \in \mathbb{N}} \{X_n : F \subseteq X_n\} \in \mathcal{S}_\omega(\mathbb{N}).$$

We now show that there is a multiplicative function $q : \mathcal{S}_\omega(\mathbb{N}) \to \mathcal{U}$ such that $q \leq p$.

Define $q : \mathcal{S}_\omega(\mathbb{N}) \to \mathcal{U}$ by $q(F) = p(n(F))$. Then since $F \subseteq n(F)$ and p is monotone we find

$$q(F) = p(n(F)) \subseteq p(F), \quad \text{for all } F \in \mathcal{S}_\omega(\mathbb{N}),$$

so $q \leq p$. Finally, to show that q is multiplicative, take $F, G \in \mathcal{S}_\omega(\mathbb{N})$. Then we have $n(F \cup G) = n(F) \cup n(G)$. Without loss of generality we may assume that $n(F) \cup n(G) = n(F)$, hence $p(n(F) \cup n(G)) = p(n(F))$. Since $n(G) \subseteq n(F) \cup n(G)$ and p is monotone, $p(n(F) \cup n(G) \subseteq p(n(G))$. Hence we find that $p(n(F) \cup n(G)) = p(n(F)) \cap p(n(G))$. This gives us

$$q(F \cup G) = p(n(F \cup G)) = p(n(F)) \cap p(n(G)) = q(F) \cap q(G).$$

Therefore, q is multiplicative and it follows that $\mathcal{U}$ is a good ultrafilter.

2. Existence of good ultrafilters

In the previous section we have seen that ultrafilters on sets exists. We will now prove the existence of good ultrafilters. The proof we give is from Kunen [3].

Theorem 2.1. *Let I be a set of cardinality $\alpha \geq \aleph_0$. Then there exists a good, countably incomplete ultrafilter over I.*

Before we give the proof of this theorem we need some more definitions and theorems. First of all the proof from Kunen uses the notion of an independent set of functions.

Definition 2.2. Let $\mathcal{F}$ be a filter over a set I and $\mathcal{S} \subseteq I^I$. $\mathcal{S}$ is *independent from $\mathcal{F}$* iff for any distinct $f_1, \ldots, f_n \in \mathcal{S}$ and $i_1, \ldots, i_n \in I$,

$$F \cap \{j \in I : f_k(j) = i_k \text{ for all } 1 \leq k \leq n\} \neq \emptyset, \text{ for all } F \in \mathcal{F}.$$

$\mathcal{S}$ is *independent* iff $\mathcal{S}$ is independent from $\{I\}$, i.e. $\{j \in I : f_k(j) = i_k \text{ for all } 1 \leq k \leq n\}$ is not empty.

We see that if $\mathcal{S}$ is independent from $\mathcal{F}$ then, by lemma 1.2, $I \backslash \{j : f_k(j) = i_k, 1 \leq k \leq n\} \notin \mathcal{F}$. To prove theorem 2.1 we will construct a sequence of filters and independent sets of functions such that every filter and every set satisfies a list of conditions. These independent sets of functions will be constructed such that each set has cardinality 2^α. The following theorem proves that this is possible.

Theorem 2.3. *If $|I| = \alpha \geq \aleph_0$, then there is an independent $\mathcal{S} \subseteq I^I$ such that $|\mathcal{S}| = 2^\alpha$.*

Proof. Consider the set $\{\langle S, r\rangle : S \in \mathcal{S}_\omega(I) \text{ and } r \in I^{\mathcal{P}(S)}\}$. We know that $|\mathcal{S}_\omega(I)| = \alpha$ and $|I^{\mathcal{P}(S)}| = \alpha^{2^n} = \alpha$. Hence, we can enumerate the set as $\langle S_i, r_i\rangle_{i \in I}$. For a set $A \subseteq I$ let $f_A \in I^I$ be the function defined by $f_A(i) = r_i(A \cap S_i)$. Then $\mathcal{S} = \{f_A : A \subseteq I\}$ is a set which satisfies our conditions. Firstly, since $|\mathcal{P}(I)| = 2^\alpha$, we see that

$$|\mathcal{S}| = |\{f_A : A \subseteq I\}| = |\{f_A : A \in \mathcal{P}(I)\}| = |\mathcal{P}(I)| = 2^\alpha.$$

Moreover, let $f_{A_1}, \ldots, f_{A_n}$ be distinct members of $\mathcal{S}$, where $A_1, \ldots, A_n$ are distinct sets, and $i_1, \ldots, i_n \in I$. Take $S \in \mathcal{S}_\omega(I)$ such that $S \cap A_k \neq S \cap A_l$ if $k \neq l$. Then we can construct a function $r : \mathcal{P}(S) \to I$ such that $r(A_k \cap S) = i_k$. Now let $j \in I$ such that $\langle S, r\rangle = \langle S_j, r_j\rangle$. Then we find

$$f_{A_k}(j) = r_j(A_k \cap S_j) = r(A_k \cap S) = i_k.$$

It follows that $\{j : f_{A_k}(j) = i_k, 1 \leq k \leq n\}$ is non-empty. Thus, $\mathcal{S}$ is an independent set of cardinality 2^α. □

Finally, to make sure that our construction in the proof of theorem 2.1 is correct, we will need the following two lemmas.

Lemma 2.4. *Let I be a set, $\mathcal{S} \subseteq I^I$ a set independent from the filter $\mathcal{F}$ over I and $A \subseteq I$. Then there are $\mathcal{S}' \subseteq \mathcal{S}$ and $\mathcal{F}' \supseteq \mathcal{F}$ such that*

(i) *$\mathcal{S}'$ is independent from $\mathcal{F}'$,*

(ii) *$\mathcal{S} \setminus \mathcal{S}'$ is finite, and*

(iii) *A or $I \setminus A$ is an element of $\mathcal{F}'$.*

Proof. We distinguish two cases in our proof.
Case 1. Suppose $\mathcal{S}$ is independent from $\langle \mathcal{F} \cup \{A\} \rangle$. Take $\mathcal{S}' = \mathcal{S}$ and $\mathcal{F}' = \langle \mathcal{F} \cup \{A\} \rangle$. Then $\mathcal{S}'$ is independent from $\mathcal{F}'$ by assumption, $\mathcal{S} \setminus \mathcal{S}' = \emptyset$ is finite, and $A \in \mathcal{F}'$ by definition of $\mathcal{F}'$.
Case 2. $\mathcal{S}$ is *not* independent from $\langle \mathcal{F} \cup \{A\} \rangle$. Let $f_1, \ldots, f_n$ be distinct members of $\mathcal{S}$ and $i_1, \ldots, i_n \in I$ such that

$$G \cap A \cap B = \emptyset, \text{ for some } G \in \mathcal{F}$$

where $B = \{j \in I : f_k(j) = i_k, 1 \leq k \leq n\}$. By independence we know that $F \cap B \neq \emptyset$ for all $F \in \mathcal{F}$. So the set $F \cup \{B\}$ has the finite intersection property and thus we can define the filter $\mathcal{F}' = \langle \mathcal{F} \cup \{B\} \rangle$. Since $G \cap B \subseteq I \setminus A$ it follows that $I \setminus A \in \mathcal{F}'$, thus condition (iii) holds.

Now define $\mathcal{S}' = \mathcal{S} \setminus \{f_1, \ldots, f_n\}$, then $\mathcal{S} \setminus \mathcal{S}' = \{f_1, \ldots, f_n\}$ is finite and condition (ii) is satisfied.

Let $g_1, \ldots, g_m$ be any distinct functions in $\mathcal{S}'$, $\iota_1, \ldots, \iota_n \in I$ and $C = \{j \in I : g_k(j) = \iota_k, 1 \leq k \leq m\}$. Then for every $F \in \mathcal{F}$ we have $F \cap (B \cap C) \neq \emptyset$ since $\mathcal{S}$ is independent from $\mathcal{F}$ and $\mathcal{S}' \subseteq \mathcal{S}$. For any $F' \in \mathcal{F}'$ we have that $F' \supseteq F \cap B$ for some $F \in \mathcal{F}$, hence the intersection $F' \cap C$ is non-empty. Thus we find that $\mathcal{S}'$ is independent from $\mathcal{F}'$ satisfying condition (i). □

Lemma 2.5. *Let I be a set, $\mathcal{S} \subseteq I^I$ a set independent from the filter $\mathcal{F}$ over I and $p : \mathcal{S}_\omega(I) \to \mathcal{F}$ a monotone function. Then there are $\mathcal{S}' \subseteq \mathcal{S}$, $\mathcal{F}' \supseteq \mathcal{F}$ and a multiplicative $q : \mathcal{S}_\omega(I) \to \mathcal{F}'$ such that*

(i) $\mathcal{S}'$ is independent from $\mathcal{F}'$,

(ii) $\mathcal{S} \setminus \mathcal{S}'$ is finite, and

(iii) $q \leq p$.

Proof. Fix $g \in \mathcal{S}$. Let $\mathcal{S}' = \mathcal{S} \setminus \{g\}$, then $\mathcal{S} \setminus \mathcal{S}' = \{g\}$ is finite. For each $T \in \mathcal{S}_\omega(I)$ define $q_T : \mathcal{S}_\omega(I) \to \mathcal{F}$ by

$$q_T(S) = \begin{cases} \emptyset & S \not\subseteq T \\ p(T) & S \subseteq T \end{cases}.$$

Let $\{T_i\}_{i \in I}$ enumerate $\mathcal{S}_\omega(I)$ and define $q : \mathcal{S}_\omega(I) \to \mathcal{F}'$ by

$$q(S) = \bigcup_{i \in I} q_{T_i}(S) \cap g^{-1}(\{i\}),$$

where $\mathcal{F}' = \langle \mathcal{F} \cup \text{range } q \rangle$. Then q is multiplicative and $q \leq p$. Before proving this, we show that $\mathcal{F} \cup \text{range } q$ has the finite intersection property and thus that $\mathcal{F}'$ is well-defined.

Let $S \in \mathcal{S}_\omega(I)$ and $i \in I$ such that $S \subseteq T_i$. Then, since $p(T_i) \in \mathcal{F}$ and $\mathcal{S}$ is independent from $\mathcal{F}$, $F \cap p(T_i) \cap g^{-1}(\{i\}) \neq \emptyset$ for all $F \in \mathcal{F}$. It follows that $F \cap q(S)$ is non-empty. By the multiplicativity of q we now find for any $S_1, \ldots, S_k \in \mathcal{S}_\omega(I)$ and $F \in \mathcal{F}$,

$$F \cap q(S_1) \cap \ldots \cap q(S_k) = F \cap q(S_1 \cup \ldots \cup S_k) \neq \emptyset,$$

since $S_1 \cup \ldots \cup S_k \in \mathcal{S}_\omega(I)$. Hence it follows that $F \cup \text{range } q$ has the finite intersection property.

Now to show that q is indeed multiplicative let $X, Y \in \mathcal{S}_\omega(I)$, then

$$\begin{aligned} q(X \cup Y) &= \bigcup_{i \in I} q_{T_i}(X \cup Y) \cap g^{-1}(\{i\}) \\ &= \bigcup_{i \in I : X \cup Y \subseteq T_i} p(T_i) \cap g^{-1}(\{i\}) \\ &= \left[\bigcup_{i \in I : X \subseteq T_i} p(T_i) \cap g^{-1}(\{i\}) \right] \cap \left[\bigcup_{i \in I : Y \subseteq T_i} p(T_i) \cap g^{-1}(\{i\}) \right] \\ &= \left[\bigcup_{i \in I} q_{T_i}(X) \cap g^{-1}(\{i\}) \right] \cap \left[\bigcup_{i \in I} q_{T_i}(Y) \cap g^{-1}(\{i\}) \right] \\ &= q(X) \cap q(Y). \end{aligned}$$

Furthermore, let $S \in \mathcal{S}_\omega(I)$. Notice that, since p is a monotone function, $p(T) \subseteq p(S)$ for all $T \supseteq S$. It follows that

$$\begin{aligned} q(S) &= \bigcup_{i \in I} \{ q_{T_i}(S) \cap g^{-1}(\{i\}) \} \\ &= \bigcup_{i \in I : S \subseteq T_i} \{ p(T_i) \cap g^{-1}(\{i\}) \} \\ &\subseteq \bigcup_{i \in I : S \subseteq T_i} p(T_i) \\ &\subseteq p(S), \end{aligned}$$

and thus $q \leq p$.

Finally we show that $\mathcal{S}'$ is independent from $\mathcal{F}'$. Let $f_1, \ldots f_n \in \mathcal{S}$, $i_1, \ldots, i_n \in I$ and $A = \{j \in I : f_k(j) = i_k, 1 \leq k \leq n\}$. Let $S \in \mathcal{S}_\omega(I)$, then

$$\begin{aligned} F \cap q(S) \cap A &= F \cap \left[\bigcup_{i \in I} q_{T_i}(S) \cap g^{-1}(\{i\}) \right] \cap A \\ &= F \cap \bigcup_{i \in I : S \subseteq T_i} p(T_i) \cap g^{-1}(\{i\}) \cap A \\ &= \bigcup_{i \in I : S \subseteq T_i} F \cap p(T_i) \cap \{j \in I : g(j) = i, f_k(j) = i_k, 1 \leq k \leq n\} \\ &\neq \emptyset, \end{aligned}$$

where the last step follows from the independence of $\mathcal{S}$ from $\mathcal{F}$. Hence we find that $F' \cap A \neq \emptyset$ for any $F' \in \mathcal{F}'$, since $F \cap q(S) \subseteq F'$ for some $F \in \mathcal{F}$ and $S \in \mathcal{S}_\omega(I)$. □

This gives us all the necessary definitions and theorems to prove that good, countably incomplete ultrafilters exist.

Proof of 2.1. Let $\{A_\eta\}_{\eta < 2^\alpha}$ enumerate $\mathcal{P}(I)$. Let $\{p_\eta\}_{\eta < 2^\alpha}$ enumerate all monotone functions from $\mathcal{S}_\omega(I)$ into $\mathcal{P}(I)$, such that each monotone $p : \mathcal{S}_\omega(I) \to \mathcal{P}(I)$ is listed 2^α times. We will construct sequences $\{\mathcal{F}_\eta\}_{\eta < 2^\alpha}$ and $\{\mathcal{S}_\eta\}_{\eta < 2^\alpha}$, such that for all $\eta < 2^\alpha$ the following hold:

(1) $\mathcal{F}_\eta$ is a filter over I and $\mathcal{S}_\eta \subseteq I^I$ is independent from $\mathcal{F}_\eta$;

(2) For $\xi < \eta < 2^\alpha$, $\mathcal{F}_\xi \subseteq \mathcal{F}_\eta$ and $\mathcal{S}_\xi \supseteq \mathcal{S}_\eta$;

(3) $|\mathcal{S}_\eta| = 2^\alpha$;

(4) If η is a limit ordinal, $\mathcal{F}_\eta = \bigcup_{\xi<\eta} \mathcal{F}_\xi$ and $\mathcal{S}_\eta = \bigcap_{\xi<\eta} \mathcal{S}_\xi$;

(5) $\mathcal{F}_0$ is generated by sets $\{B_n\}_{n<\aleph_0}$ such that $\bigcap_{n<\aleph_0} B_n = \emptyset$;

(6) $\mathcal{S}_\eta \setminus \mathcal{S}_{\eta+1}$ is finite;

(7) Either A_η or $I \setminus A_\eta$ is an element of $\mathcal{F}_{\eta+1}$;

(8) If $p_\eta : \mathcal{S}_\omega(I) \to \mathcal{F}_\eta$, then there is a multiplicative $q : \mathcal{S}_\omega(I) \to \mathcal{F}_{\eta+1}$ such that $q \leq p_\eta$.

The first four conditions will take care of themselves. Condition (7) will ensure that $\mathcal{U} = \bigcup_{\eta<2^\alpha} \mathcal{F}_\eta$ is an ultrafilter since either A_η or $I \setminus A_\eta$ will be an element of $\mathcal{U}$ for all $\eta < 2^\alpha$.

By (5), $\mathcal{U}$ will have a countable subset $\mathcal{B} = \{B_n : n < \aleph_0\}$ such that $\bigcap \mathcal{B} = \emptyset$, hence $\mathcal{U}$ will be countably incomplete. To make this condition hold, take $\mathcal{S}_0 \cup \{f\}$ independent of power 2^α. Define for $n < \aleph_0$ the set $B_n = \{i \in I : n < f(i) < \aleph_0\}$ and let $\mathcal{F}_0 = \langle \{B_n : n < \aleph_0\} \rangle$. Then $\bigcap_{n<\aleph_0} B_n = \emptyset$.

Now let $p : \mathcal{S}_\omega(I) \to \mathcal{U} = \bigcup_{\eta<2^\alpha} \mathcal{F}_\eta$ be a monotone function. Define range $p = \{p_\xi : \xi < \alpha\}$. Then $p_\xi \in \mathcal{F}_{\eta_\xi}$ for all $\xi < \alpha$. Furthermore $\eta_\xi < 2^\alpha$ for all $\xi < \alpha$, hence by König (appendix A),

$$\sum_{\xi<\alpha} \eta_\xi < \prod_{\xi<\alpha} 2^\alpha = (2^\alpha)^\alpha = 2^\alpha.$$

So we find that there is an $\eta < 2^\alpha$ such that $\eta_\xi < \eta$ for all $\xi < \alpha$. This means that $p_\xi \in \mathcal{F}_\eta$ for all $\xi < \alpha$, so $p : \mathcal{S}_\omega(I) \to \mathcal{F}_\eta$. By applying (8) for some $\eta' > \eta$ such that $p_{\eta'} = p$, we find that there is a multiplicative $q : \mathcal{S}_\omega(I) \to \mathcal{F}_{\eta'+1}$. Thus it follows that $\mathcal{U}$ will be a good ultrafilter.

By applying lemma 2.4 and 2.5 at each stage $\eta < 2^\alpha$ we see that the last three conditions hold. This concludes our proof. $\square$

3. Saturated ultraproducts

Good ultrafilters are of great interest in model theory. This is because they make ultraproducts saturated, which will be the main conclusion of this section. We will start with an introduction to model theory, where we discuss all the necessary definitions.

3.1. Model theory

In model theory we look at the relation between a formal language and the interpretations of this language. In this interpretation a sentence can be given a truth value, *true* or *false*. A model can then be defined on the basis of these truth values.

Definition 3.1. A *language* $\mathcal{L} = \{g_1, g_2, \ldots, P_1, P_2, \ldots, c_1, c_2, \ldots\}$ is a collection of (n-placed) *function symbols*, (m-placed) *relation symbols* and *constant symbols*.

If a language $\mathcal{L}'$ contains all of the symbols of $\mathcal{L}$ and some additional symbols, we say that $\mathcal{L}'$ is an *expansion* of $\mathcal{L}$ and write $\mathcal{L} \subset \mathcal{L}'$. We define the *power* of $\mathcal{L}$, denoted by $||\mathcal{L}||$, as $||\mathcal{L}|| = \omega \cup |\mathcal{L}|$. An *interpretation* $\mathbb{A}$ of $\mathcal{L}$ consists of a non-empty set A, called the domain, and an interpretation function. This interpretation function maps each function symbol g to an n-placed function $f : A^n \to A$ on A, each relation symbol P to an n-placed relation $R \subseteq A^n$ on A and each constant symbol c to a constant $a \in A$.

Example 3.2. Let $\mathcal{L} = \{g, P, c\}$ be a language. Then the following could be an interpretation of $\mathcal{L}$:

- The domain A is the set of positive integers;
- $x + y$ is the interpretation of $g(x, y)$;
- $x \leq y$ is the interpretation of $P(x, y)$;
- c is interpreted as 1.

In a language $\mathcal{L}$ there are different kind of strings of symbols. One of these are the *terms* of $\mathcal{L}$. A term is defined as follows:

(i) A variable or a constant symbol is a term;

(ii) For a function symbol g and terms $t_1, \ldots, t_n$, $g(t_1, \ldots, t_n)$ is a term;

(iii) t is a term iff it can be shown to be a term on the basis of the conditions above.

The next string of symbols is an *atomic formula* of $\mathcal{L}$. To define atomic formulas we need the identity symbol $\equiv$, this symbol denotes a binary relation. For any two terms t_1 and t_2 of $\mathcal{L}$, $t_1 \equiv t_2$ is an atomic formula, and $P(t_1, \ldots, t_m)$ is an atomic formula, where P is a relation symbol of $\mathcal{L}$ and $t_1, \ldots, t_m$ are terms of $\mathcal{L}$. Lastly $\mathcal{L}$ has *formulas*, which are defined as follows:

(i) Atomic formulas are formulas;

(ii) For any formulas φ and ψ of $\mathcal{L}$, $(\varphi \wedge \psi)$ and $(\neg\varphi)$ are formulas;

(iii) For a variable v and a formula φ, $(\forall v)\varphi$ is a formula;

(iv) A string of symbols is a formula iff it can be shown to be a formula on the basis of the conditions above.

A variable in a formula φ is said to be *free* iff it is not quantified in φ. For example, in the formula $(\forall x)\varphi(x, y)$, the variable x is not free but y is. We will write $t(v_0, \ldots, v_n)$, respectively $\varphi(v_0, \ldots, v_n)$, for a term, respectively a formula, whose free variables form a subset of $\{v_0, \ldots, v_n\}$. If a formula contains no free variables, then it is called a *sentence*. Each sentence in a language $\mathcal{L}$ is either true or false in an interpretation of $\mathcal{L}$. A formula with free variables, on the other hand, may be satisfied by some elements in the domain A and not satisfied by others.

Let $\varphi(v_0, \ldots, v_n)$ be a formula of a language $\mathcal{L}$ and $\mathbb{A}$ an interpretation of $\mathcal{L}$. If $\varphi(a_0, \ldots, a_n)$ is true for some collection of elements $a_0, \ldots, a_n \in A$, then we say that *φ is satisfied in $\mathbb{A}$ by $a_0, \ldots, a_n$*, and we write

$$\mathbb{A} \models \varphi(a_0, \ldots, a_n).$$

We say that $\mathbb{A}$ *satisfies*, or *realizes*, some formula ψ iff ψ is satisfied in $\mathbb{A}$ by some sequence $a_0, \ldots, a_n$. Notation: $\mathbb{A} \models \psi$. If $\Sigma(v_0, \ldots, v_n)$ is a set of formulas in the variables $v_0, \ldots, v_n$, then we say that $\mathbb{A}$ satisfies Σ iff every $\sigma \in \Sigma$ is satisfied in $\mathbb{A}$ by some sequence $a_0, \ldots, a_n$.

Example 3.3. Let $\mathcal{L}$ be the language with interpretation as in example 3.2. The formula $(\forall y)P(x, y)$ has one free variable, this variable is x, and thus it is not a sentence. The element $1 \in A$ satisfies this formula, since $1 \leq y$ for every $y \in A$.

Now look at the formula $(\exists x)(\forall y)P(x, y)$. This formula does not have any free variables and is thus a sentence. This sentence states that the positive integers have a smallest element. As we have seen above this element exists, so the sentence is true in our interpretation of $\mathcal{L}$.

Definition 3.4. A *theory* T of a language $\mathcal{L}$ is a collection of sentences of $\mathcal{L}$. We call these sentences the *axioms* of T.

From the axioms of a theory T we can derive all other sentences which hold in T. This leads us to the definition of a model.

Definition 3.5. A *model* $\mathbb{A}$ of a theory T is an interpretation of $\mathcal{L}$ for which all axioms of T are true.

The *power* of a model $\mathbb{A}$ is the cardinal $|A|$. If $X \subset A$, then $(\mathbb{A}, x)_{x \in X}$ is a model in the expanded language $\mathcal{L} \cup X$ where each $x \in X$ is interpreted as a constant symbol.

A set of formulas $\Sigma = \Sigma(v_0, \ldots, v_n)$ is *consistent* with a theory T iff there is a model of T which realizes every $\sigma(v_0, \ldots, v_n) \in \Sigma$. If $\Sigma(v_0, \ldots, v_n)$ is a maximal consistent set of formulas, we call $\Sigma(v_0, \ldots, v_n)$ a *type*. For example, if $\mathbb{A}$ is a model and $\Sigma(v_0, \ldots, v_n)$ is the set of all formulas satisfied by $a_0, \ldots, a_n \in A$, then Σ is a type.

Proposition 3.6 gives a useful method to show if a set of formulas is consistent. The proof of this proposition uses the compactness theorem which states that a set of sentences Σ is satisfiable iff every finite subset of Σ is satisfiable. The theorem and the proof can be found in Chang & Keisler [1].

Proposition 3.6. *Let T be a theory and let $\Sigma = \Sigma(v_0, \ldots, v_n)$ be a set of formulas. Then Σ is consistent with T iff every finite subset of Σ is realized in some model of T.*

Proof. Suppose Σ is consistent with T. Then by definition T has a model $\mathbb{A}$ which satisfies Σ, so

$$\mathbb{A} \models \sigma, \quad \text{for all } \sigma \in \Sigma.$$

It follows that for any finite subset $\{\sigma_1, \ldots, \sigma_n\} \subset \Sigma$, $\mathbb{A}$ satisfies every σ_i, $i = 1, \ldots, n$, and thus

$$\mathbb{A} \models \{\sigma_1, \ldots, \sigma_n\}.$$

On the other hand, suppose that every finite subset of Σ is realized in some model of T. Then by the compactness theorem there is a model which satisfies every $\sigma \in \Sigma$. So it follows that Σ is consistent with T. □

3.2. Ultraproducts

Now that we have discussed the necessary introduction on model theory we can define the ultraproducts. We will first discuss the construction of an ultraproduct on sets and then define the ultraproduct for models.

For an ultrafilter $\mathcal{U}$ over a set I and $\{A_i\}_{i\in I}$ a collection of non-empty sets let C be the Cartesian product over these sets, that is

$$C = \prod_{i\in I} A_i = \left\{ f : I \to \bigcup_{i\in I} A_i \,|\, f(i) \in A_i \right\}.$$

We say that two functions $f, g \in C$ are *$\mathcal{U}$-equivalent*, $f =_{\mathcal{U}} g$, iff

$$\{i \in I : f(i) = g(i)\} \in \mathcal{U}.$$

The relation $=_{\mathcal{U}}$ is an equivalence relation and we denote the equivalence class of a function f as $f_{\mathcal{U}} = \{g \in C : f =_{\mathcal{U}} g\}$.

Definition 3.7. The set $\prod_{\mathcal{U}} A_i$ of all equivalence classes of $=_{\mathcal{U}}$ is the *ultraproduct of A_i modulo $\mathcal{U}$*, i.e.

$$\prod_{\mathcal{U}} A_i = \{f_{\mathcal{U}} : f \in \prod_{i\in I} A_i\}.$$

Now, to define the ultraproduct of models let $\mathbb{A}_i$ be a model for a language $\mathcal{L}$ for every $i \in I$. The *ultraproduct* $\prod_{\mathcal{U}} \mathbb{A}_i$ is the model for $\mathcal{L}$ with universe $\prod_{\mathcal{U}} A_i$ where each symbol in $\mathcal{L}$ is interpreted in the following way:

(i) For a constant symbol c of $\mathcal{L}$ let a_i be the interpretation of c in $\mathbb{A}_i$ for each $i \in I$. Then the interpretation of c in $\prod_{\mathcal{U}} \mathbb{A}_i$ is the element $b \in \prod_{\mathcal{U}} A_i$ such that $b = \langle a_i : i \in I\rangle_{\mathcal{U}}$.

(ii) For a relation symbol P of $\mathcal{L}$ let R_i be the interpretation of P in $\mathbb{A}_i$ for every $i \in I$. Then the interpretation of P in $\prod_{\mathcal{U}} \mathbb{A}_i$ is the relation S such that

$$S(f^1_{\mathcal{U}}, \dots, f^n_{\mathcal{U}}) \quad \text{iff} \quad \{i \in I : R_i(f^1(i), \dots, f^n(i))\} \in \mathcal{U}$$

(iii) For a function symbol G of $\mathcal{L}$ let F_i be the interpretation of G in $\mathbb{A}_i$ for every $i \in I$. Then the interpretation of G in $\prod_{\mathcal{U}} \mathbb{A}_i$ is the function H given by

$$H(f^1_{\mathcal{U}}, \dots, f^n_{\mathcal{U}}) = \langle F_i(f^1(i), \dots, f^n(i)) : i \in I\rangle_{\mathcal{U}}.$$

If $\mathbb{A}_i = \mathbb{A}$ for all $i \in I$and a certain model $\mathbb{A}$, then we call $\prod_{\mathcal{U}} \mathbb{A}$ the *ultrapower* of $\mathbb{A}$.

We now give an important theorem about ultraproducts. This theorem will be useful in the proofs of the next section.

Theorem 3.8 (The Fundamental Theorem of Ultraproducts)**.** *Let I be a set and let $\mathbb{B}$ be the ultraproduct $\prod_{\mathcal{U}} \mathbb{A}_i$. Then:*

(i) For any term $t(x_1, \dots, x_n)$ of $\mathcal{L}$ and elements $f^1_{\mathcal{U}}, \dots, f^n_{\mathcal{U}} \in B$, we have

$$t_{\mathbb{B}}\left(f^1_{\mathcal{U}}, \dots, f^n_{\mathcal{U}}\right) = \left\langle t_{\mathbb{A}_i}\left(f^1(i), \dots, f^n(i)\right) : i \in I\right\rangle_{\mathcal{U}}.$$

(ii) For any formula $\varphi(x_1, \dots, x_n)$ of $\mathcal{L}$ and $f^1_{\mathcal{U}}, \dots f^n_{\mathcal{U}} \in B$, we have

$$\mathbb{B} \models \varphi\left(f^1_{\mathcal{U}}, \dots, f^n_{\mathcal{U}}\right) \quad \textit{iff} \quad \left\{i \in I : \mathbb{A}_i \models \varphi\left(f^1(i), \dots, f^n(i)\right)\right\} \in \mathcal{U}.$$

The proof of this theorem can be found in Chang & Keisler [1].

3.3. Saturated models

We are now able to show that good ultrafilters make ultraproducts saturated. An α-saturated model is a model which realizes a maximum number of types. This makes it possible to realize a large amount of formulas with only one element.

Definition 3.9. Let α be a cardinal and $\mathbb{A}$ a model. Then $\mathbb{A}$ is *α-saturated* iff for any $X \subseteq A$, $|X| < \alpha$, $(\mathbb{A}, x)_{x\in X}$ realizes every type $\Sigma(v)$ of the language $\mathcal{L} \cup X$ which is consistent with the theory of $(\mathbb{A}, x)_{x\in X}$.

Note that by 3.6 we know that if $(\mathbb{A}, x)_{x\in X}$ realizes every subset of $\Sigma(v)$ then $\Sigma(v)$ is consistent with the theory of $(\mathbb{A}, x)_{x\in X}$. So if we want to show that a model A is α-saturated it suffices to prove that if $(\mathbb{A}, x)_{x\in X}$ realizes every finite subset of every set of formulas $\Sigma(v)$, then $(\mathbb{A}, x)_{x\in X}$ realizes $\Sigma(v)$.

Before coming to the main result of this section we first prove a similar weaker theorem. The structure of this proof will be similar to the proof of our main theorem.

Theorem 3.10. *Let $\mathcal{L}$ be a countable language, and let $\mathcal{U}$ be a countably incomplete ultrafilter over a set I. Then the ultraproduct $\prod_{\mathcal{U}} \mathbb{A}_i$ is $\aleph_1$-saturated for every family $\mathbb{A}_i$, $i \in I$, of models for $\mathcal{L}$.*

Proof. Let X be any countable subset of $\prod_{\mathcal{U}} A_i$ and $\Sigma(v)$ be any set of formulas of $\mathcal{L} \cup X$. We must show that if every finite subset of $\Sigma(v)$ is satisfiable in $(\prod_{\mathcal{U}} \mathbb{A}_i, x)_{x\in X}$, then $\Sigma(v)$ is satisfiable in $(\prod_{\mathcal{U}} \mathbb{A}_i, x)_{x\in X}$.

Let $\{a_n\}_{n<\aleph_0}$ enumerate X. Note that if $a_n = \langle a_n(i) : i \in I\rangle_{\mathcal{U}}$, then

$$\left(\prod_{\mathcal{U}} \mathbb{A}_i, a_n\right)_{n<\aleph_0} = \prod_{\mathcal{U}} \left((\mathbb{A}_i, a_n(i))_{n<\aleph_0}\right).$$

Now, since $\mathcal{L}$ is an arbitrary countable language and $\mathcal{L} \cup X$ is also countable, it suffices to prove:

(i) For every set $\Sigma(v)$ of formulas of $\mathcal{L}$, if each finite subset of $\Sigma(v)$ is satisfiable in $\prod_{\mathcal{U}} \mathbb{A}_i$, then $\Sigma(v)$ is satisfiable in $\prod_{\mathcal{U}} \mathbb{A}_i$.

Suppose $\prod_{\mathcal{U}} \mathbb{A}_i$ realizes every finite subset of a set of formulas $\Sigma(v)$ of $\mathcal{L}$. The language $\mathcal{L}$ is countable, so $\Sigma(v)$ is also countable and we can write $\Sigma(v) = \{\sigma_1(v), \sigma_2(v), \ldots\}$. Since $\mathcal{U}$ is countably incomplete there is a descending chain $I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of elements of $\mathcal{U}$ such that $\bigcap_{n<\aleph_0} I_n = \emptyset$. Let $X_0 = I$, $n < \aleph_0$ and define

$$X_n = I_n \cap \{i \in I : \mathbb{A}_i \models (\exists x)(\sigma_1(x) \wedge \cdots \wedge \sigma_n(x))\}.$$

We know that $\prod_{\mathcal{U}} \mathbb{A}_i$ realizes every finite subset of $\Sigma(v)$, hence by the fundamental theorem 3.8 $\{i \in I : \mathbb{A}_i \models (\exists x)(\sigma(x) \wedge \ldots \wedge \sigma_n(x))\} \in \mathcal{U}$ for any $n < \aleph_0$. Therefore, $X_n \in \mathcal{U}$ for all $n < \aleph_0$. Moreover, $\bigcap_{n<\aleph_0} X_n = \emptyset$ and $X_n \supseteq X_{n+1}$ for all $n < \aleph_0$. It follows that for each $i \in I$ there is a greatest $n_i < \aleph_0$ such that $i \in X_{n_i}$.

Now we choose a function $f \in \prod_{i\in I} A_i$ as follows:

(a) if $n_i = 0$ let $f(i)$ be an arbitrary element of A_i,

(b) if $n_i > 0$, choose $f(i)$ so that $\mathbb{A}_i \models \sigma_1 \wedge \ldots \wedge \sigma_{n_i}[f(i)]$.

Let $n > 0$. Then $n \leq n_i$ for all $i \in X_n$. So it follows that $\mathbb{A}_i \models \sigma_n[f(i)]$ for every $i \in X_n$ and thus $X_n \subseteq \{i \in I : \mathbb{A}_i \models \sigma_n[f(i)]\}$. This implies that $\{i \in I : \mathbb{A}_i \models \sigma_n[f(i)]\} \in \mathcal{U}$ and thus, by the fundamental theorem of ultraproducts (3.8), $\prod_{\mathcal{U}} \mathbb{A}_i \models \sigma_n[f_{\mathcal{U}}]$ for every $n > 0$. Hence, $f_{\mathcal{U}}$ satisfies $\Sigma(v)$ in $\prod_{\mathcal{U}} \mathbb{A}_i$. This proves (i). $\square$

Example. As an example for an $\aleph_1$-saturated model we look at the non-standard model of the natural numbers. Let $\mathcal{N}$ be the model of the natural numbers with domain $\mathbb{N}$ and $\mathcal{U}$ a countably incomplete ultrafilter over $\mathbb{N}$. Then the ultrapower $\prod_{\mathcal{U}} \mathcal{N}$ is $\aleph_1$-saturated.

Now look at the set of formulas $\Sigma(v)$ given by $\Sigma(v) = \{v > n : n \in \mathbb{N}\}$. Each finite subset of $\Sigma(v)$ can be realized since we can always find an $m \in \mathbb{N}$ such that $m > n$ for a given n. By the $\aleph_1$-saturation it follows that $\Sigma(v)$ is satisfiable in $\prod_{\mathcal{U}} \mathcal{N}$. This means that there is an element in $\prod_{\mathcal{U}} \mathcal{N}$ which is greater than every natural number.

The theorem which states that good ultrafilters make an ultraproduct saturated is a generalization of theorem 3.10.

Theorem 3.11. *Let $\mathcal{U}$ be a good, countably incomplete ultrafilter over a set I of cardinality $\alpha \geq \aleph_0$. Suppose $||\mathcal{L}|| \leq \alpha$. Then for any family $\mathbb{A}_i$, $i \in I$, of models for $\mathcal{L}$, the ultraproduct $\prod_{\mathcal{U}} \mathbb{A}_i$ is α^+-saturated.*

Proof. We see that in the same way as in the proof of theorem 3.10 it is sufficient to prove:

(i) For every set $\Sigma(v)$ of formulas of $\mathcal{L}$, if each finite subset of $\Sigma(v)$ is satisfiable in $\prod_{\mathcal{U}} \mathbb{A}_i$, then $\Sigma(v)$ is satisfiable in $\prod_{\mathcal{U}} \mathbb{A}_i$.

In our proof we will first construct a monotone function p. Then we will use that $\mathcal{U}$ is a good ultrafilter to find a multiplicative function q. To conclude our proof we will find an element $h_{\mathcal{U}}$ using our function q which satisfies $\Sigma(v)$ in $\prod_{\mathcal{U}} \mathbb{A}_i$.

Let $\Sigma(v)$ be a set of formulas of $\mathcal{L}$ and suppose that $\prod_{\mathcal{U}} \mathbb{A}_i$ realizes every finite subset of $\Sigma(v)$. Since $\mathcal{U}$ is countably incomplete there is a descending chain $I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of elements of $\mathcal{U}$ such that $\bigcap_{n<\aleph_0} I_n = \emptyset$. Define $p : \mathcal{S}_\omega(\Sigma) \to \mathcal{U}$ as follows:

$$p(\sigma) = I_{|\sigma|} \cap \{i \in I : \mathbb{A}_i \models (\exists x) \bigwedge \sigma\}, \tag{1}$$

where $p(\emptyset) = I_0$. Each $\sigma \in \mathcal{S}_\omega(\Sigma)$ is finite and thus satisfiable in $\prod_{\mathcal{U}} \mathbb{A}_i$, so $\prod_{\mathcal{U}} \mathbb{A}_i \models (\exists x) \bigwedge \sigma$. By the fundamental theorem of ultraproducts it follows that $\{i \in I : \mathbb{A}_i \models (\exists x) \bigwedge \sigma\} \in \mathcal{U}$. Therefore, $p(\sigma) \in \mathcal{U}$ and p is well-defined.

The function p is monotone. To show this, let $\sigma \subseteq \tau \in \mathcal{S}_\omega(\Sigma)$. Since $|\sigma| \leq |\tau|$ we have $I_{|\sigma|} \supseteq I_{|\tau|}$. Moreover, if $\mathbb{A}_i$ realizes τ for any $i \in I$ then $\mathbb{A}_i$ also realizes σ. Thus

$$\{i \in I : \mathbb{A}_i \models (\exists x) \bigwedge \tau\} \subseteq \{i \in I : \mathbb{A}_i \models (\exists x) \bigwedge \sigma\}.$$

So we find $p(\sigma) \supseteq p(\tau)$.

Because $||\mathcal{L}|| \leq \alpha$ we have that $|\Sigma| \leq \alpha$ and thus $|\mathcal{S}_\omega(\Sigma)| \leq \alpha$. So, since $\mathcal{U}$ is a good ultrafilter, we can now find a multiplicative $q : \mathcal{S}_\omega(\Sigma) \to \mathcal{U}$ such that $q \leq p$. Define, for each $i \in I$,

$$\sigma_i = \{\theta \in \Sigma : i \in q(\{\theta\})\}. \tag{2}$$

Notice that σ_i is finite for all $i \in I$. To see this, note that if $|\sigma_i| \geq n$, then $i \in I_n$. Because, if σ_i has at least n distinct elements $\theta_1, \ldots, \theta_n$, then $i \in q(\{\theta_s\})$ for all $s \leq n$. Using the multiplicity of q and $q \leq p$, we find

$$i \in q(\{\theta_s\}) \cap \ldots \cap q(\{\theta_s\}) = q(\{\theta_1, \ldots, \theta_n\}) \subseteq p(\{\theta_1, \ldots, \theta_n\}) \subseteq I_n.$$

If σ_i would not be finite then, $|\sigma_i| \geq n$ for all n and thus $i \in I_n$ for all n. But since $\bigcap_{n<\aleph_0} I_n = \emptyset$ this i does not exists. It follows that σ_i is finite.

We now construct $h_{\mathcal{U}} \in \prod_{i \in I} A_i$ satisfying $\Sigma(v)$ by specifying $h_i \in A_i$ for all $i \in I$. Let $i \in I$. We will take h_i such that it satisfies all $\theta \in \sigma_i$ in $\mathbb{A}_i$.

Now, by (1), (2) and multiplicity of q, for every $i \in I$,

$$i \in \bigcap \{q(\{\theta\}) : \theta \in \sigma_i\} = q\left(\bigcup \{\theta : \theta \in \sigma_i\}\right) = q(\sigma_i) \subseteq p(\sigma_i),$$

so $i \in p(\sigma_i)$. By (1) for every $i \in I$ there is an element $h_i \in A_i$ such that $\mathbb{A}_i \models \bigwedge \sigma_i(h_i)$. Hence, we have found our element $h_i \in A_i$ such that $\mathbb{A}_i \models \theta(h_i)$ for any $\theta \in \sigma_i$ and $i \in I$.

Finally, if $\{i \in I : \mathbb{A}_i \models \theta(h_i)\} \in \mathcal{U}$, then by the fundamental theorem of ultraproducts $\prod_{\mathcal{U}} \mathbb{A}_i \models \theta(h_{\mathcal{U}})$. For every $\theta \in \Sigma$ we have that $q(\{\theta\}) \subseteq \{i \in I : \mathbb{A}_i \models \theta(h_i)\}$ and thus, since $q(\{\theta\}) \in \mathcal{U}$, $\{i \in I : \mathbb{A}_i \models \theta(h_i)\} \in \mathcal{U}$. It follows that $\prod_{\mathcal{U}} \mathbb{A}_i \models \theta(h_{\mathcal{U}})$ for all $\theta \in \Sigma$. Therefore, $h_{\mathcal{U}}$ satisfies Σ in $\prod_{\mathcal{U}} \mathbb{A}_i$. $\square$

A. Set theory

The subjects studied in this thesis require a certain knowledge of set theory. We especially need to know about orderings on sets, ordinals and cardinals. This appendix contains a short introduction into these subjects. For a more extensive documentation on set theory we refer to [2]. The lemmas and theorems in this section will be left unproven.

A *relation* R is a set of ordered pairs $\langle x, y\rangle$. We write xRy instead of $\langle x, y\rangle \in R$. A relation can have the following properties over a set X:

(i) *reflexive*: xRx for all $x \in X$;

(ii) *transitive*: if xRy and yRz, then xRz for all $x, y, z \in X$;

(iii) *antisymmetric*: if xRy and yRx, then $x = y$ for all $x, y \in X$;

(iv) *connected*: for all $x, y \in X$ either xRy or yRx.

A *partial ordering* of a set X is a relation R over X which is reflexive, transitive and antisymmetric. A connected partial ordering of X is called a *simple ordering* of X. Any set X of sets is partially ordered by the inclusion relation $\subseteq$. If X is simply ordered by $\subseteq$, then we call X a *chain*.

Lemma A.1 (Zorn's Lemma)**.** *Let X be a non-empty set of sets and Y a chain in X. If $\bigcup Y \in X$, then X has a maximal element.*

The lemma of Zorn is equivalent to the axiom of choice. Two other theorems which are equivalent to the axiom of choice are the well ordering principle and the enumeration principle.

We say that a simple ordering on a set X is a *well ordering* iff every non-empty subset Y of X has a smallest element, that is an $y \in Y$ such that yRz for all $z \in Y$. A *strict* well ordering is a well ordering that is irreflexive, i.e. $\neg(xRx)$ for all $x \in X$.

Theorem A.2 (Well Ordering Principle)**.** *Every set can be well ordered.*

Before we can state the enumeration principle we need to define what an ordinal is. An *ordinal* is a set α such that α is strictly well ordered by the relation $\in$ and $\bigcup \alpha \subseteq \alpha$. The first three ordinals are: $0 = \emptyset, 1 = \{\emptyset\}$ and $2 = \{\emptyset, \{\emptyset\}\}$. An ordinal β is a *successor* if there is an α such that $\beta = \alpha \cup \{\alpha\}$. If α is not the successor of any ordinal, then α is a limit ordinal. The first limit ordinal after 0 is the ordinal ω, this ordinal represents the natural numbers.

An enumeration of a set X is a function whose domain is an ordinal α and whose range is the set X. We use the notation $\{x_\beta\}_{\beta<\alpha}$ to denote an enumeration of the set X.

Theorem A.3 (Enumeration Principle)**.** *Every set can be enumerated.*

For a set X, $|X|$ is the smallest ordinal α such that there is a bijection between X and α. An ordinal α is called a *cardinal* iff $\alpha = |\alpha|$. The least cardinal greater than α is called the *successor* of α and is denoted as α^+. We write $\aleph_0$ for the cardinal of the set of natural numbers $\mathbb{N}$. The successor of $\aleph_0$ is $\aleph_1$.

If $\{X_i : i \in I\}$ is a collection of sets with cardinalities α_i then we can define the sum of cardinals as follows

$$\sum_{i\in I} \alpha_i = \sum_{i\in I} |X_i| = \left|\bigcup_{i\in I} X_i \times \{i\}\right|.$$

The product is defined as

$$\prod_{i\in I}\alpha_i = \left|\prod_{i\in I} X_i\right|,$$

where $\prod_{i\in I} X_i$ denotes the Cartesian product of the sets X_i, that is the set of all functions f with domain I for which $f(i) \in X_i$.

For a set X of cardinality α and a set Y of cardinality β the set X^Y denotes the set of all functions from Y into X. The power of the α with exponent β is defined as $\alpha^\beta = |X^Y|$. If α is infinite and $n > 0$, then $\alpha^n = \alpha$ and $(2^\alpha)^\alpha = 2^\alpha$.

Theorem A.4 (König's Theorem)**.** *Let I be a set and let $\{\alpha_i : i \in I\}$ and $\{\beta_i : i \in I\}$ be sets of cardinals. If $\alpha_i < \beta_i$ for all $i \in I$, then*

$$\sum_{i\in I}\alpha_i < \prod_{i\in I}\beta_i.$$

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