Rapport No. 174A

LABORATORIUM VOOR SCHEEPSBOUWKUNDE

TECHNISCHE HOGESCHOOL DELFT

COMPUTATION OF THE HYDRODYNAMIC COEFFICIENTS OF OSCILLATING CYLINDERS

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november 1969

Contents

page

Introduction 3

1. Formulation of the problem 5

2. Integral equation for the velocity potential; Green's functions; Source potential 9

3. Determination of the velocity potential for a heaving circular cylinder according to Ursell; added mass and damping 20

4. Heaving of a cylinder with an arbitrary cross-section

   4.1. Curvilinear coordinates and conformal transformations 27
   4.2. Calculation of the added mass and damping 37
   4.3. Added mass and damping of a cylinder with a "Lewis form" 44

5. Swaying and rolling

   5.1. Potential of a dipole in the origin; asymmetric multipole potentials 48
   5.2. Added mass and damping for swaying; added moment of inertia and damping for rolling produced by the swaying motion 51
   5.3. Added moment of inertia and damping for rolling; added mass and damping for swaying produced by the rolling motion 61

References.

Computation of the hydrodynamic coefficients of oscillating cylinders

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Preface.

This report is a translation from Dutch of an earlier report [1], which has been written by the author in order to provide formulas for added mass and damping, which are used in computer programs for ship motions, devised by members of the Shipbuilding Laboratory in Delft.

The intention of this report is to be a manual for those,who want to acquaint themselves with the hydrodynamic backgrounds of the methods, which are used to determine the hydrodynamic properties of ships according to the strip method.

The reader is supposed to be familiar with the fundamentals of the hydrodynamics and the infinitesimal surface wave theory. For a study of these theories the reader is referred to [2], [3], [4] and [5].

Introduction

The last few years much attention has been paid to the theoretical approximation of the hydrodynamic coefficients of a ship in which great advance has been made by the availability of computers. Since in general a three-dimensional method leads to calculations which are too complicated, the problem is considered two-dimensional by application of the so-called strip method. In this case the ship is divided up into a number of sections and of each section, which is supposed to have a constant profile, the hydrodynamic properties are determined, assuming that the disturbances in the fluid due to the motions of the sections only propagate in the direction perpendicular to its axes. Therefore application of the above mentioned method requires information about the hydrodynamic properties of infinitely-long cylinders (or finite cylinders contained between vertical walls at right angles to the axis) with cross-sections, which are equal to those of the considered sections of the ship.

Ursell [6], [7] made the first contribution to the solution of this problem. He considered the problem of a circular cylinder, which oscillates harmonically with small amplitude, while the mean position of the axis coincides with the mean surface of the fluid. Ursell starts from the following assumptions:

1. the fluid is inviscid, incompressible and irrotational.
2. the oscillation is of such a nature that linearization is allowed.

From 1. follows that the velocity potential ϕ satisfies the equation of Laplace $\Delta\phi = 0$, while according to 2. the accessory boundary conditions are linear. Consequently it follows from 1. and 2. that above mentioned problem can be formulated as a linear potential problem.

Ursell found a solution by superimposing suitably chosen functions such that each separate function satisfies the equation of Laplace and the linearized free-surface condition, while a combination of these functions satisfies the remaining boundary conditions.

Tasai [8], [9] generalized Ursell's method for more general cross-sections, the so-called Lewis-forms, which are characterized by three parameters. Tasai applied a conformal transformation with which the Lewis-form is mapped onto a semi-circle. Because of the restricted number of parameters, the transformation formula's can be determined in an analytical way.

Porter [10] derived expressions for the hydrodynamic coefficients of cylinders, which cannot be approximated with a Lewis-form in a satisfactory way and for which more complicated transformation formulas are required. Moreover he verified some results experimentally. A method however to find the transformation formulas mapping an arbitrary cross-section of a ship onto a semi-circle is not given by him.

On the Shipbuilding Laboratory in Delft Smith [11] devised a computer-program of the iterative process of Fil'chakova [12], with which the transformation formula can be determined for every arbitrary cross-section, which maps this cross-section onto a semi-circle. After this the hydrodynamic coefficients of this section can rather easily be determined. It is noteworthy that strictly speaking this method can be applied only if the cross-section intersects the fluid surface perpendicularly.

The English edition of this report has been supplemented with another transformation method (see section 4.1.2.), which appears to be very useful.

1. Formulation of the Problem

In a fluid of infinite depth a cylinder is considered which is oscillating one-dimensionally and harmonically with frequency σ while the mean position of its axis is assumed to lie in the free surface of the undisturbed fluid (Fig. 1.1.). As possible ways of oscillation we shall consider here heaving, swaying and rolling.

The x-axis is horizontal and coinciding with the free surface of the fluid and perpendicular to the axis of the cylinder and the y-axis is vertical, positive downwards and going through the mean position of the axis of the cylinder.

Fig. 1.1.

Further we assume the amplitude of the oscillation being small with respect to the diameter of the cylinder and the length of the waves, generated by the oscillation, so that we may relate the value of all physical quantities to the centre-position of the cylinder in the linearized approximation. Taking the cylinder very long with respect to the breadth or enclosing the cylinder at both ends between two infinitely long walls perpendicular to the axis of the cylinder, we can neglect the velocity components parallel to the axis of the cylinder and consequently the motion is two-dimensional.

The determination of the motions of the fluid under influence of the harmonic oscillation of the cylinder can be reduced to the solution of a boundary-value problem from the linear potential-theory. Consequently the velocity potential $\Phi(x, y, t)$ is also a harmonic function of the time.

Therefore using complex notation we may write the potential in the following form:

$$\Phi\ (x,\ y,\ t) = -i\phi(x,\ y)\ e^{i\sigma t} \qquad (1,1)$$

From this the actual potential is obtained by taking the real part of the right-hand side. In the future for calculations in which the time-dependence of the variables is not mentioned, we shall always work with the time-independent part $\phi(x, y)$.

The velocity potential should satisfy the equation of Laplace everywhere in the fluid:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y} = 0 \qquad (1.2)$$

Because of (1,1) we may write (1.2) as:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (1.3)$$

If $\eta = \eta\ (x,t)$ is the wave-height in consequence of the oscillation of the cylinder, for the linearized case ([5], ch 2, (2.1.14)) for waves which have small amplitudes in proportion to their length the following relation holds:

$$\frac{\partial\eta}{\partial t} = -\frac{\partial\Phi}{\partial y} \quad (y = 0) \quad \text{(kinematic surface condition)} \qquad (1.4)$$

We see that in the linearized form this relation is referred to the mean surface : $y = 0$.

Condition (1.4) is based on the hypothesis that any fluid particle once being on a boundary surface, will remain on it ([5], §1.4).

A second condition which Φ has to satisfy at the free surface, follows from Bernouilli's law:

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right\} + \frac{p}{\rho} + g\eta = C\ (t) \qquad (1.5)$$

Since the pressure at the free surface is constant and an addition of constant or time-dependent terms to Φ has no influence upon the velocity distribution $\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$ in the fluid, we can reduce (1.5) to:

$$\frac{\partial\Phi}{\partial t} + g\eta = 0 \quad (y=0) \quad \text{(dynamic surface condition)} \qquad (1.6)$$

where we only retained the linear terms.

From the conditions (1.4) and (1.6) η can be eliminated. Differentiating (1.6) with respect to t and substituting successively for $\frac{\partial \eta}{\partial t}$ the righthand side of (1.4) and for Φ the expression (1.1), we finally obtain:

$$K\,\phi + \frac{\partial \phi}{\partial y} = 0 \quad (y=0) \quad \text{(linearized free surface condition)} \qquad (1.7)$$

in which:

$$K = \frac{\sigma^2}{g}.$$

On basis of the earlier mentioned hypothesis with respect to fluid particles on a boundary surface, we can derive, that the normal velocity component of the cylinder at the hull due to the forced oscillation, is equal to the corresponding velocity component of the fluid particles on the cylinder, so:

$$\frac{\partial \Phi}{\partial n} = -iU_n\,(x,y)e^{i\sigma t}$$

or:

$$\frac{\partial \phi}{\partial n} = U_n(x,y) \quad \text{(boundary condition on the cylinder)} \qquad (1.8)$$

In this relation n refers to the normal outward direction to the surface of the cylinder (Fig. 1.1). It should be noticed that, on account of the linearizing of the problem, relation (1.8) is referred here to the mean position of the cylinder again.

For physical reasons it is easy to see that the disturbances in the fluid, as a result of the oscillation of the cylinder, decrease with increasing depth so that:

$$\lim_{y \to \infty} \text{grad}\,\phi = 0 \qquad (1.9)$$

Since the forced oscillation is harmonic, waves are excited at the fluid surface, which are composed of a standing-wave, rapidly decreasing in amplitude with the distance from the cylinder, and a regular progressive wave, which travels to infinity on both sides of the cylinder. The lastnamed wave effects a radiation of energy, withdrawn from the motion of the cylinder, whereby the fluid has a damping influence on the motion of the cylinder. Thus:

$$\Phi \to C_1 e^{-Ky}\, e^{i(-Kx + \sigma t)} \quad \text{for } x \to +\infty$$

$$\Phi \to C_2 e^{-Ky}\, e^{i(Kx + \sigma t)} \quad \text{for } x \to -\infty$$

or:

2. Integral-equation for the velocity potential; Green's functions, Source potential

Using Green's theorem we derive in this chapter an integral equation for the velocity potential for the case of a vertical oscillating cylinder. In addition much attention will be paid to the Green's function and its physical meaning.

We assume that the cylinder is carrying out a vertical harmonic oscillating motion. Consequently according to (1.11) the velocity potential is a symmetric function:

$$\phi(x,y) = \phi(-x,y) \tag{2.1}$$

Fig. 2.1.

We apply Green's theorem, which in the two-dimensional case has the following form:

$$\iint_{V} (\phi\Delta\psi - \psi\Delta\phi)\,dV = \oint_{S} \left(\phi \frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) ds \tag{2.2}$$

V is an area enclosed by a contour S while <u>n</u> represents the outward normal to the contour S.

For the function ϕ we choose the velocity potential in consequence of the heaving motion of the cylinder. The function ψ is chosen in such a way that after substitution of ψ into (2.2) the requirements with respect to the uniqueness and existence of the solution for the resulting integral equation are fulfilled. The function ψ should then have a source singularity in a point (a,b) in the area $y > 0$, outside the cylinder. For ψ we choose a function of the form([13] ch. VI-3):

$$\psi\ (x,y;\ a,b) = \log\sqrt{(x-a)^2+(y-b)^2} + \psi_r(x,y;\ a,b) \tag{2.3}$$

The first term on the righthand side is the potential of a source in the point (a,b). The function ψ_r is regular in the area $y>0$, outside the cylinder. We now choose the contour S in such a way that ϕ and ψ are regular in the area V enclosed by S. For these reasons S is composed of the lines II and V along the free surface, the line IV along the contour of the cylinder, a small circle III with radius δ enclosing the point (a,b) and a large circle I with radius r (fig. 2.1). ϕ and ψ satisfy the Laplace equation within this area so that the lefthand side of (2,2) becomes zero and thus

$$\oint\limits_{I+II+III+IV+V} (\phi\ \frac{\partial\psi}{\partial n} - \psi\ \frac{\partial\phi}{\partial n})\ ds = 0 \tag{2.4}$$

We shall first determine the limit value of the integral along the small circle III, for the case $\delta \to 0$. It is remarked that for small values of δ the expression (2.3) on the small circle III may be written as:

$$\psi = \log\ \delta + \psi_r \tag{2.5}$$

Further we may write on the small circle:

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial\delta} \quad \text{and} \quad ds = \delta d\theta,$$

so that:

$$\int\limits_{III} = -\int_{\pi}^{-\pi} \left\{ -\phi\ \frac{\partial}{\partial\delta}\ (\log\ \delta + \psi_r) + (\log\ \delta + \psi_r)\ \frac{\partial\phi}{\partial\delta} \right\}\ \delta d\theta$$

or:

$$\int\limits_{III} = \int_{-\pi}^{+\pi} \left\{ -\phi(\frac{1}{\delta} + \frac{\partial\psi_r}{\partial\delta}) + (\log\ \delta + \psi_r)\ \frac{\partial\phi}{\partial\delta} \right\}\ \delta d\theta$$

As ψ_r and $\frac{\partial\psi_r}{\partial\delta}$ are limited we obtain for $\delta \to 0$:

$$\int\limits_{III} = -\ 2\pi\phi\ (a,b) \tag{2.6}$$

Next we consider the integral along the lines II and IV.

We now choose ψ_r in such a way that the line integral along the free surface becomes zero:

$$\int_{II+V} \left\{ \phi \frac{\partial \psi}{\partial y} - \psi \frac{\partial \phi}{\partial y} \right\} dx = 0 \qquad (y=0)$$

This condition is satisfied if the integrand of the above expression vanishes:

$$\phi \, \psi_y - \psi \, \phi_y = 0 \qquad (y=0)$$

With the aid of (1.7) we see that on account of the symmetry with respect to ϕ and ψ of above expression, this relation is valid if:

$$K \, \psi + \psi_y = 0 \qquad (y=0) \tag{2.7}$$

We further investigate which requirements ψ has to satisfy in order that the line integral on the large circle vanishes for the limit $r \to \infty$:

$$\lim_{r \to \infty} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left\{ \phi \frac{\partial \psi}{\partial r} - \psi \frac{\partial \phi}{\partial r} \right\} r \, d\theta = 0 \tag{2.8}$$

For $r \to \infty$ the potential ϕ has to satisfy the condition (1.10) on the large circle. As the relation (2.8) is symmetrical with respect to ϕ and ψ these functions have to be equal on the large circle; so the function ψ represents for $|x| \to \infty$ a regular progressive wave:

$$\begin{aligned} \psi &\to C_1 e^{-Ky - iKx} \qquad \text{for } x \to +\infty \\ \psi &\to C_1 e^{-Ky + iKx} \qquad \text{for } x \to -\infty \end{aligned} \tag{2.9}$$

Hence, if the function ψ, as given by (2.3.), satisfies the relations (2.7.) and (2.9.) then (2.4.)results in the following integral equation for $\phi(x,y)$

$$\phi(a,b) = \frac{1}{2\pi} \int_B \left\{ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right\} ds \tag{2.10}$$

In this expression B represents the contour of the cylinder on which the normal velocity $\frac{\partial \phi}{\partial n}$ is given. In the integrand of (2.10) the function ϕ is integrated over the contour of the cylinder. When we take now the point (a,b) on the contour of the cylinder, we obtain, after application of the same procedure (in which the small circle III changes into a semi-circle on B), the integral equation:

$$\phi(a,b) = \frac{1}{\pi} \int_B \left\{ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right\} ds \tag{2.11}$$

In this expression both $\phi(a,b)$ and ϕ in the integrand refer to values of ϕ on the cylinder B. With the aid of (2.11) we first determine now the value of ϕ on the contour B and after that with the aid of (2.10) we are able to determine the value of ϕ in an arbitrary point of the fluid. The function ψ, being chosen in such a way that the line integrals along I, II and V vanish while moreover the requirements with respect to the existence and the uniqueness of the solution of (2.10) are satisfied, is called a Green's function.

Our last task is the determination of the function $\psi_r(x,y;\ a,b)$ in (2.3). We substitute (2.3) into (2.7):

$$K\psi_r + \psi_{r_y} = -\frac{K}{2}\log\left\{(x-a)^2+b^2\right\} + \frac{b}{(x-a)^2+b^2} \qquad (y=0)$$

We search for a solution of this differential equation in the form:

$$\psi_r(x,y;\ a,b) = -\frac{1}{2}\log\left\{(x-a)^2+(y+b)^2\right\} + \psi^1(x,y;\ a,b)$$

The first term on the righthand side represents a sink in the imagepoint $(a, -b)$ of (a,b) with respect to the free surface. Then the function ψ^1 has to satisfy the relation:

$$K\psi^1 + \psi^1_y = \frac{2b}{(x-a)^2+b^2} \qquad (y=0) \qquad (2.12)$$

Apart from a factor 2 the righthand side of (2.12) is exactly the Laplace transformation of $\cos p(x-a)$, hence (2.12) can be written as:

$$K\psi^1 + \psi^1_y = 2\int_0^\infty e^{-pb}\cos p(x-a)\,dp \qquad (y=0) \qquad (2.13)$$

Consider the integral:

$$\int_0^\infty e^{-pb}\cos p(x-a)\,dp \qquad (2.14)$$

We consider p as a complex variable: $p = \alpha + i\beta$. As the integral has only a singularity for $p = -\infty$, we may change the path of integration, which leads along the real axis in the above mentioned case, into an arbitrary line L between the origin (0,0) and $(\infty,0)$ (Fig. 2.2).

Fig. 2.2.

One of the elementary properties of Green's functions is their symmetry with respect to the points (x,y) and (a,b),([14] Chp. IX-3), which means that the function remains the same if we interchange (x,y) and (a,b). Consequently we substitute into (2.13) for ψ^1 the expression:

$$\psi^1(x,y;\ a,b) = 2 \int_L P(p,K)e^{-p(y+b)} \cos p(x-a)dp \qquad (2.15)$$

The function P(p,K) has to be determined in such a way that (2.13) is satisfied. Substitution of (2.15) into (2.13) yields:

$$-2 \int_L p.P(p,K)e^{-pb} \cos p(x-a)dp + 2 \int_L K\ P(p,K)e^{-pb} \cos p.$$

$$.(x-a)dp = 2 \int_L e^{-pb} \cos p(x-a)dp \qquad (y=0)$$

which may be written as:

$$\int_L \left\{ (K-p)P(p,K) - 1 \right\} e^{-pb} \cos p(x-a)dp = 0$$

we find:

$$P(p.K) = \frac{1}{K-p} \qquad (2.16)$$

and:

$$\psi^1(x,y;\ a,b) = 2 \int_L \frac{e^{-p(y+b)}}{K-p} \cos p(x-a)dp \qquad (2.17)$$

The integrand of (2.17.) has a pole of the first order in $p = K$ ($K = \frac{\sigma^2}{g}$, real). We may choose the contour L in two ways now: over the singularity $p = k$, e.g. L, or underneath it, e.g. L_1 (fig. 2.2.). As the residue of the integrand is not equal to zero, the values of the integral for these two contours will be different. So the function ψ^1 and as a consequence also the function ψ are not uniquely determined. It appears that this uniqueness is caused by condition (2.9.): ψ has to represent a regular progressive wave at infinite distance from the cylinder.

We now proceed to study the behaviour of ψ successively for the contours L and L_1. As we remain on the same side of the pole we are allowed to change the contour L_1 into M_1 and L into M_2 (Fig. 2.3.).

Fig. 2.3.

For $|x| \to \infty$ the function : $\log \sqrt{\frac{(x-a)^2+(y-b)^2}{(x-a)^2+(y+b)^2}}$ vanishes. Consequently only the behaviour of ψ^1 for $|x| \to \infty$ on the contours M_1 and M_2 remains to be studied. We shall show now that ψ^1 represents regular progressive waves for the contour M_2 when $|x|$ increases to infinity. Making the transformations $x' = x - a$ and $y' = y + b$ and skipping after that again the indices we obtain for (2.17.) for the contour M_2:

$$\psi^1{}_{M_2} = \int_{M_2} \frac{e^{-py'}\left\{e^{ipx'} + e^{-ipx'}\right\}}{K - p} dp = \int_{M_2} \frac{e^{-p(y-ix)}}{K - p} dp + \int_{M_2} \frac{e^{-p(y+ix)}}{K - p} dp \quad (2.18.)$$

The first integral gives:

$$\int_{M_2} \frac{e^{-p(y-ix)}}{K-p} dp = \underbrace{\int_0^{K-\varepsilon} \frac{e^{-\alpha(y-ix)}}{K-\alpha} d\alpha}_{I_1} + \underbrace{\int_{K-\varepsilon}^{K+\varepsilon} \frac{e^{-(\alpha+i\beta)(y-ix)}}{K-p} dp}_{I_2} + \underbrace{\int_{K+\varepsilon}^{\infty} \frac{e^{-\alpha(y-ix)}}{K-\alpha} d\alpha}_{I_3}$$

We denote the first, second and third integral of the right-hand side respectively by I_1, I_2 and I_3. Partial integrating of I_1 gives:

$$I_1 = -\frac{e^{-\alpha(y-ix)}}{(K-\alpha)(y-ix)}\Bigg|_{\alpha=0}^{\alpha=K-\varepsilon} + \int_0^{K-\varepsilon} \frac{e^{-\alpha(y-ix)}}{(K-\alpha)^2(y-ix)}\,d\alpha$$

It is easy to see that $I_1 \to 0$ for $|x| \to \infty$. In the same way it can be proved that $I_3 \to 0$ for $|x| \to \infty$. Further because of $\beta>0$ the integral I_2 vanishes for $|x| \to \infty$. It likewise can be proved that for the second integral on the right-hand side of (2.18) the integrations along the real axis over $(0, K-\varepsilon)$ and $(K+\varepsilon,\infty)$ become zero, whereas on account of the theorem of residues the integration along the semicircle $(K-\varepsilon,K+\varepsilon)$ yields:

$$\int_{K-\varepsilon\ (\smile)}^{K+\varepsilon} \frac{e^{-p(y+ix)}}{K-p}\,dp = \int_{K-\varepsilon\ (\frown)}^{K+\varepsilon} \frac{e^{-p(y+ix)}}{K-p}\,dp + 2\pi i\ (\text{residue } p = K).$$

After substitution of $p = \alpha + i\beta$ into the left-hand side we see that (since β assumes only negative values) this integral vanishes for $|x| \to \infty$. For the residue in $p = K$ we find:

$$\lim_{p \to K} (p-K)\,\frac{e^{-p(y+ix)}}{K-p} = -\,e^{-K(y+ix)}$$

so:

$$\int_{K-\varepsilon\ (\frown)}^{K+\varepsilon} \frac{e^{-p(y+ix)}}{K-p}\,dp = 2\pi\, i\, e^{-K(y+ix)}$$

and:

$$\lim_{|x| \to \infty} \int_{M_2} \frac{e^{-py}\cos p x}{K-p}\,dp = 2\pi\, i\, e^{-K(y+ix)} \qquad (2.19)$$

Consequently ψ^1 and for this reason also ψ represents for $|x| \to \infty$ a regular progressive wave if the path of integration L in expression (2.17) has the shape of M_2.

In the same way it can be proved that ψ gives a regular incoming wave for $|x| \to \infty$ if L has the shape of M_1.

Therefore in order to make ψ satisfy condition (2.9), the integration in (2.17) is carried out along the path M_2. The Green's function ψ which has to be substituted into the integral equations (2.10) and (2.11) has now an unique representation (We divide by a factor 2):

$$\psi(x,y;\ a,b) = \frac{1}{2}\log\sqrt{\frac{(x-a)^2+(y-b)^2}{(x-a)^2+(y+b)^2}} + \int_{\longrightarrow \overset{\frown}{K} \longrightarrow} \frac{e^{-p(y+b)}}{K-p}\cos p(x-a)\,dp \qquad (2.20)$$

So summing up above results, we constructed a function ψ with the following properties:

(i) $\Delta\psi=0$

(ii) the linearized free-surface condition (2.7)

(iii) for $|x| \to \infty$, ψ represents a regular progressive wave (2.9) (2.21)

In chapter 3 we shall solve above mentioned boundary-value problem by taking for ψ a linear combination of potential functions. One of these components is the potential function of a source in the origin (a=b=o), which satisfies in addition the conditions (2.21.). For this reason we shall consider here the potential function of such a source more precisely. Setting a=b=o in (2.20.) and calling the potential now ϕ, we obtain:

$$\phi = \int_{\underset{K}{\frown}} \frac{e^{-py}}{K-p} \cos px \, dp \qquad (2.22)$$

In literature (2.22) is mostly given in another form, which can be derived form (2.22) by applying theorems of the complex function theory. We shall give this derivation here:

If we split up the integral (2.22) into two integrals I_1, and I_2, it gives:

$$\int_{\underset{K}{\frown}} \frac{e^{-py} \cos px}{K-p} dp = \frac{1}{2}\left\{ \underbrace{\int_{\underset{K}{\frown}} \frac{e^{-py+ipx}}{K-p} dp}_{I_1} + \underbrace{\int_{\underset{K}{\frown}} \frac{e^{-py-ipx}}{K-p} dp}_{I_2} \right\} \qquad (2.23)$$

It appears that the calculation gives different results for $x>0$ and $x<0$. We shall consider here the case $x>0$. For $x<0$ the reasoning proceeds in an analogous way.

Fig. 2.4.

For the calculation of I_1 we close the contour M with the arc of a circle C_B and the positive imaginary axis I_B so that application of Cauchy's theorem leads to:

$$\int_{\frown} \frac{e^{-py+ipx}}{K-p}\,dp + \lim_{R\to\infty}\int_0^{\frac{\pi}{2}} \frac{e^{-Re^{i\theta}y + iRe^{i\theta}x}.Re^{i\theta}}{K-Re^{i\theta}}\,i d\theta + \int_\infty^0 \frac{e^{-i\beta y - \beta x}}{K-i\beta}\,i d\beta = 0$$

This is easily reduced to:

$$\int_{\frown} \frac{e^{-py+ipx}}{K-p}\,dp + \lim_{R\to\infty}\int_0^{\frac{\pi}{2}} \frac{e^{-(Ry\cos\theta+Rx\sin\theta) - i(Ry\sin\theta - Rx\cos\theta)}}{K - R\cos\theta - iR\sin\theta}\,Re^{i\theta} i d\theta +$$

$$\int_\infty^0 \frac{e^{-i\beta y - \beta x}}{K-i\beta}\,i d\beta = 0 \qquad (2.24.)$$

The first integral in this expression is a principal value integral.

We notice that on C_B: $\cos\theta > 0$, $\sin\theta > 0$ and $x>0$, $y>0$. So the third integral vanishes for $R \to \infty$. Consequently (2.24.) may be deduced to:

$$\int_{\frown} \frac{e^{-py+ipx}}{K-p}\,dp + \int_\infty^0 \frac{e^{-i\beta y - \beta x}}{K-i\beta}\,i\,d\beta = 0 \qquad (2.25.)$$

For the calculation of I_2 we close the contour with the arc of a circle C_o and the negative imaginary axis. As in this case the path of integration encloses the pole $p=K$ with residue

$$\lim_{p\to K} (p-K)\,\frac{e^{-py-ipx}}{K-p} = -e^{-Ky-iKx}$$

We find by applying the theorem of residues:

$$\int_{\frown} \frac{e^{-py-ipx}}{K-p}\,dp + \lim_{R\to\infty}\int_0^{-\frac{\pi}{2}} \frac{e^{-Re^{i\theta}y - iRe^{i\theta}x}.Re^{i\theta}}{K-Re^{i\theta}}\,i d\theta +$$

$$+ \int_{-\infty}^0 \frac{e^{-i\beta y + \beta x}}{K-i\beta}\,i d\beta = -2\pi i(-e^{-Ky-iKx}) = 2\pi i\, e^{-ky-ikx}.$$

In a similar way as in (2.24.) the second integral in this relation vanishes so that after evaluation of the first integral the above mentioned form results into:

$$\int_{\frown} \frac{e^{-py+ipx}}{K-p}\,dp + \int_{-\infty}^{0} \frac{e^{-i\beta y+\beta x}}{K-i\beta}\,i\,d\beta = 2\pi i\, e^{-Ky-iKx} \qquad (2.26)$$

From (2.23), (2,25) and (2.26) follows:

$$\int_{\frown} \frac{e^{-py}\cos px}{K-p}\,dp =$$

$$= \frac{1}{2}\left\{ \int_{0}^{\infty} \frac{e^{-i\beta y-\beta x}}{K-i\beta}\,i\,d\beta + \int_{0}^{\infty} \frac{e^{i\beta y-\beta x}}{K+i\beta}\,(-i)\,d\beta \right\} + \pi i\, e^{-Ky-iKx} \qquad (2.27)$$

(2.27) may further be reduced to:

$$\int_{\frown} \frac{e^{-py}\cos px}{K-p}\,dp =$$

$$= \int_{0}^{\infty} \frac{e^{-\beta x}(K\sin\beta y - \beta\cos\beta y)}{K^2+\beta^2}\,d\beta + \pi i\, e^{-Ky-iKx} \qquad (x>0)$$

In the same way it can be proved that for $x<0$ the integral on the right-hand side of above expression has to be replaced by:

$$\int_{0}^{\infty} e^{\beta x}\,\frac{(K\sin\beta y - \beta\cos\beta y)}{K^2+\beta^2}\,d\beta + \pi i\, e^{-Ky+iKx} \qquad (x<0)$$

Consequently the potential ϕ of a source in the origin satisfying the free-surface condition and the radiation condition is given by:

$$\phi = \int_{0}^{\infty} e^{-\beta|x|}\,\frac{(K\sin\beta y - \beta\cos\beta\, y)}{K^2+\beta^2}\,d\beta + \pi i\, e^{-Ky-iK|x|} \qquad (2.28)$$

In this report we will use the following definition for the source potential:

$$\Phi = \frac{gb}{\pi\sigma}\,\mathrm{Re}\left\{-i\phi\, e^{i\sigma t}\right\} = \frac{gb}{\pi\sigma}\left\{\phi_c \cos\sigma t + \phi_s \sin\sigma t\right\},$$

where b is the wave height at an infinite distance from the cylinder.

Comparing this expression with (2.28) we find for the non-dimensional quantities ϕ_s and ϕ_c:

$$\phi_c = \pi e^{-Ky}\cos Kx$$

$$\phi_s = \pi e^{-Ky}\sin K|x| - \int_{0}^{\infty} \frac{e^{-\beta|x|}}{K^2+\beta^2}\,(\beta\cos\beta y - K\sin\beta y)\,d\beta \qquad (2.29)$$

Remark

It appears that the integral in (2.29) converges very slowly when it is calculated in a numerical way. Porter [10 page 148] has given the following power series expansion, which affords good results:

$$\int_0^\infty \frac{e^{-\beta x}}{K^2 + \beta^2} (K\sin\beta y - \beta\cos\beta y)\, d\beta = e^{-Ky}\left\{Q \cos Kx + S \sin Kx\right\} - \pi e^{-Ky}\sin Kx$$

where

$$Q = \gamma + \ln\left[K(x^2 + y^2)^{\frac{1}{2}}\right] + \sum_{n=1}^{\infty} \frac{K^n(x^2 + y^2)^{n/2}}{n\,!\,n} \cos n\chi,$$

$$S = \chi + \sum_{n=1}^{\infty} \frac{K^n(x^2 + y^2)^{n/2}}{n\,!\,n} \sin n\chi,$$

$\gamma = 0{,}5772156649\ldots\ldots$: Euler's constant, and

$$\chi = \text{arc tan}\,\frac{x}{y}$$

3. Determination of the velocity potential for a heaving circular cylinder, according to Ursell; added mass and damping

In the previous chapter we saw that the potential can be determined by solving the integral equation (2.11). In general however it appears that this leads to rather complicated numerical calculations. Ursell [6] has developed a method of solution, which consists of superposition of potential functions, which all satisfy the equation of Laplace and the free surface condition. The solution is composed of the source potential (2.28) and a linear combination of multipole potentials which are represented by:

$$\phi_{2m} = a^{2m}\left\{\frac{\cos 2m\,\theta}{r^{2m}} + \frac{K}{2m-1}\,\frac{\cos(2m-1)\theta}{r^{2m-1}}\right\} \qquad m = 1,2,3,\ldots.. \qquad (3.1)$$

where (r,θ) are the polar coordinates:

$$x = r\sin\theta, \qquad y = r\cos\theta \qquad (3.2)$$

while a represents the radius of the circular cylinder (Fig. 3.1).

Fig. 3.1.

Since the cylinder is carrying out a vertical oscillation, the corresponding velocity potential is a symmetric function with respect to the y-axis (see (1.11)). Consequently it suffices to restrict our future considerations to the range $0\leq\theta\leq\frac{\pi}{2}$. The time dependent potential Φ is expressed by:

$$\Phi = \frac{gb}{\pi\sigma}\left[\phi_c\,(Kr;\theta)\cos\sigma t + \phi_s\,(Kr;\theta)\sin\sigma t + \right.$$

$$+\cos\sigma t\sum_{m=1}^{\infty} p_{2m}(Ka)a^{2m}\left\{\frac{\cos 2m\,\theta}{r^{2m}} + \frac{K}{2m-1}\,\frac{\cos(2m-1)\theta}{r^{2m-1}}\right\} + \qquad (3.3)$$

$$\left. + \sin\sigma t\sum_{m=1}^{\infty} q_{2m}(Ka)a^{2m}\left\{\frac{\cos 2m\,\theta}{r^{2m}} + \frac{K}{2m-1}\,\frac{\cos(2m-1)\theta}{r^{2m-1}}\right\}\right]$$

according to (2.29) and (3.1), while:

$$\phi_c\ (Kr,\Theta) = \pi e^{-Kr\cos\Theta}\cos(Kr\sin\Theta)$$

$$\phi_s\ (Kr,\Theta) = -\int_0^\infty \frac{e^{-\beta r\ \sin\Theta}}{K^2+\beta^2}\left\{\beta\cos\ (\beta r\cos\Theta)-K\sin(\beta r\cos\Theta)\right\}\ d\beta \qquad (3.4)$$

$$+\ \pi e^{-Kr\cos\Theta}\sin(Kr\sin\Theta).$$

It is easy to verify that the multipole potentials ϕ_{2m} satisfy the free-surface condition. In chapter 2 we showed that the source potential $\frac{gb}{\pi\sigma}\left\{\phi_c\ \cos\ \sigma t + \phi_s\ \sin\ \sigma t\right\}$ is determined in such a way that the free-surface condition is satisfied. Furthermore we showed there that this potential represents for $|\ x\ | \to \infty$ a regular progressive wave. As the multipole-potentials vanish for $r \to \infty$ the total potential Φ, represented by (3.3), satisfies the radiation condition (1.11). It still remains to determine the coefficients p_{2m} and q_{2m} in such a way that the boundary condition is satisfied. For the case of a circular cylinder the boundary condition (1.8) at the cylinder is reduced to:

$$\frac{dy}{dt}\cos\ \Theta = \frac{\partial\phi}{\partial r} \qquad (3.5)$$

The Cauchy-Riemann conditions which relate the velocity potential Φ and the conjugate stream function Ψ have in polar coordinates the form:

$$\frac{\partial\Phi}{\partial r} = -\frac{1}{r}\ \frac{\partial\Psi}{\partial\Theta}$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial\Theta} = \frac{\partial\Psi}{\partial r} \qquad (3.6)$$

Substituting Φ in (3.6) yields us for the stream function:

$$\Psi = \frac{gb}{\pi\sigma}\Bigg[\ \psi_c(Kr,\Theta)\cos\ \sigma t + \psi_s(Kr,\Theta)\sin\ \sigma t\ +$$

$$+\ \cos\ \sigma t\sum_1^\infty p_{2m}(Ka)a^{2m}\left\{\frac{\sin\ 2m\ \Theta}{r^{2m}} + \frac{K}{2m-1}\ \frac{\sin\ (2m-1)\Theta}{r^{2m-1}}\right\} + \qquad (3.7)$$

$$+\ \sin\ \sigma t\sum_1^\infty q_{2m}(Ka)a^{2m}\left\{\frac{\sin\ 2m\ \Theta}{r^{2m}} + \frac{K}{2m-1}\ \frac{\sin\ (2m-1)\Theta}{r^{2m-1}}\right\}\Bigg]$$

where:

$$\psi_c(Kr;\Theta) = \pi e^{-Kr\cos\Theta}\sin(Kr\sin\Theta)$$

$$\psi_s(Kr;\Theta) = \int_0^\infty \frac{e^{-\beta r\ \sin\Theta}}{K^2+\beta^2}\ \left\{\beta\sin\ (\beta r\cos\Theta) + K\cos\ (\beta r\cos\Theta)\right\}\ d\beta\ - \qquad (3.8)$$

$$- \pi e^{-Kr\cos\Theta}\cos(Kr\sin\Theta).$$

Using (3.6) the boundary condition at the cylinder (3.5) may be written as:

$$-\frac{1}{a}\frac{\partial\Psi}{\partial\Theta} = \frac{dy}{dt}\cos\Theta \qquad (r=a)$$

Integrating with respect to Θ we obtain:

$$\Psi = -a\frac{dy}{dt}\sin\Theta + c\,(t) \qquad (r=a)$$

By substituting $\Theta = 0$ the integration constant $c(t)$ is found to be zero. We assume that the ordinate of the axis of the cylinder is given by:

$$y = l\cos(\sigma t + \varepsilon) \tag{3.9}$$

Consequently the streamfunction at the cylinder has to satisfy:

$$\Psi = l\sigma a\sin(\sigma t + \varepsilon)\sin\Theta \qquad (r=a) \tag{3.10}$$

From (3.7) and (3.10) we find:

$$\psi_c(Ka;\Theta)\cos\sigma t + \psi_s(Ka;\Theta)\sin\sigma t +$$

$$\cos\sigma t\sum_1^\infty p_{2m}(Ka)\left\{\sin 2m\Theta + \frac{Ka}{2m-1}\sin(2m-1)\Theta\right\} + \tag{3.11}$$

$$\sin\sigma t\sum_1^\infty q_{2m}(Ka)\left\{\sin 2m\Theta + \frac{Ka}{2m-1}\sin(2m-1)\Theta\right\} = \frac{la\pi K}{b}\sin(\sigma t+\varepsilon).\sin\Theta$$

As (3.11) holds for the range $0\leqslant\Theta\leqslant\frac{\pi}{2}$ we find by substituting $\Theta = \frac{\pi}{2}$ in this expression:

$$\psi_c(Ka;\tfrac{\pi}{2})\cos\sigma t + \psi_s(Ka;\tfrac{\pi}{2})\sin\sigma t + \cos\sigma t\sum_1^\infty p_{2m}(Ka)\,\frac{Ka(-1)^{m-1}}{2m-1} +$$

$$+ \sin\sigma t\sum_1^\infty q_{2m}(Ka)\,\frac{Ka(-1)^{m-1}}{2m-1} = \frac{la\pi K}{b}\sin(\sigma t + \varepsilon) \tag{3.12}$$

With this relation we eliminate the factor $\frac{la\pi K}{b}\sin(\sigma t + \varepsilon)$ from (3.11). It is easily seen that in the resulting form the coefficients p_{2m} and q_{2m} have to satisfy the relations:

$$\psi_c(Ka;\Theta)-\psi_c(Ka;\tfrac{\pi}{2})\sin\Theta = \sum_1^\infty p_{2m}(Ka)f_{2m}(Ka;\Theta)$$

$$\psi_s(Ka;\Theta)-\psi_s(Ka;\tfrac{\pi}{2})\sin\Theta = \sum_1^\infty q_{2m}(Ka)f_{2m}(Ka;\Theta) \tag{3.13}$$

where:

$$f_{2m}(Ka;\Theta) = -\left[\sin 2m\Theta + \frac{Ka}{2m-1}\left\{\sin(2m-1)\Theta - \sin\Theta\sin\frac{1}{2}(2m-1)\pi\right\}\right]$$

In (3.13) the left-hand side is expanded in a series of functions $f_{2m}(Ka;\Theta)$, where m = 1,2,3,.... In practical applications we take of course only a finite number N of terms, where N determines the accuracy of one approximation. The coefficients p_{2m} and q_{2m} for example can now be determined with the least-squares approximation method.

From Bernoulli's law (1.5) we derive that the hydrodynamic pressure in a point of the liquid in linearized form is given by:

$$p = -\rho\frac{\partial\Phi}{\partial t} \tag{3.14}$$

Consequently the hydrodynamic force per unit-length on the cylinder is:

$$P = \rho\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left[a\frac{\partial\Phi}{\partial t}\right]_{r=a}\cos\Theta d\Theta = \frac{2\rho abg}{\pi}(M_o\cos\sigma t - N_o\sin\sigma t) \tag{3.15}$$

where:

$$M_o = \int_0^{\frac{\pi}{2}}\phi_s(Ka;\Theta)\cos\Theta d\Theta + \sum_1^{\infty}\frac{(-1)^{m-1}q_{2m}(Ka)}{4m^2-1} + \frac{1}{4}\pi\, Kaq_2(Ka)$$

$$N_o = \int_0^{\frac{\pi}{2}}\phi_c(Ka;\Theta)\cos\Theta d\Theta + \sum_1^{\infty}\frac{(-1)^{m-1}p_{2m}(Ka)}{4m^2-1} + \frac{1}{4}\pi\, Kap_2(Ka).$$

A long cylinder which is completely submerged in an ideal infinite fluid experiences a hydrodynamic force $-M\ddot{r}$ per unit-length which is equal to the product of the relative acceleration $\ddot{r}$ and the displaced volume of fluid M per unit cylinder length.

The situation remains the same if we remove the fluid and add to the cylinder per unit-length a mass M. For this reason M is called the added mass of the cylinder. If the cylinder is moving in a fluid with a free surface then the force is no longer in phase with the acceleration. We resolve this force into a component in phase with the acceleration, which does not dissipate any energy, and a component in phase with the velocity, which has the same character as a frictional force and which is responsible for the dissipation of energy in the form of outward-going waves. The acceleration component of the hydrodynamic force is determined by the added mass and the velocity component by the damping of the cylinder.

We shall calculate now these two quantities. From (3.9) it follows for the velocity of the cylinder:

$$\frac{dy}{dt} = - l\sigma \sin\,(\sigma t + \varepsilon),$$

combining this with (3.12) we find:

$$\frac{dy}{dt} = - \frac{\sigma b}{\pi a K} \left\{ A\cos\,\sigma t + B\sin\,\sigma t \right\} \qquad (3.16)$$

where:

$$A(Ka) = \psi_c(Ka;\tfrac{\pi}{2}) + \sum_1^{\infty} \frac{(-1)^{m-1} Ka}{2m-1}\, p_{2m}(Ka)$$

$$(3.17)$$

$$B(ka) = \psi_s(Ka;\tfrac{\pi}{2}) + \sum_1^{\infty} \frac{(-1)^{m-1} Ka}{2m-1}\, q_{2m}(Ka)$$

The acceleration of the cylinder is given by:

$$\frac{d^2y}{dt^2} = \frac{b\sigma^2}{\pi a K} \left\{ A\sin\sigma t - B\cos\sigma t \right\} \qquad (3.18)$$

The force in phase with the acceleration follows from (3.15) and (3.18):

$$- \frac{2\rho a\ bg}{\pi} \frac{M_oB + N_oA}{A^2 + B^2} \left\{ A\ \sin\sigma t - B\ \cos\sigma t \right\} \qquad (3.19)$$

The force in phase with the velocity is found from (3.15) and (3.16):

$$\frac{2\rho a\ bg}{\pi} \frac{M_oA - N_oB}{A^2 + B^2} \left\{ A\ \cos\sigma t + B\ \sin\sigma t \right\} \qquad (3.20)$$

Fig. 3.2.

The validity of the formulas (3.19) and (3.20) is easily seen from the vector-diagram in Fig. 3.2.

We assume that the hydrodynamic force, the velocity and the acceleration are respectively given by:

$$\underline{P} = p_1\cos\sigma t + p_2\sin\sigma t$$

$$\underline{v} = v_1\cos\sigma t + v_2\sin\sigma t$$

$$\underline{a} = a_1\cos\sigma t + a_2\sin\sigma t.$$

The velocity and the acceleration have a phase difference of 90 degrees. The component of $\underline{P}$ in phase with the acceleration is expressed by:

$$p_a = \frac{(\underline{P}.\underline{a})}{|\underline{a}|} = \frac{(p_1a_1 + p_2a_2)}{\sqrt{a_1^2 + a_2^2}}$$

In vector notation this component is represented by:

$$p_a \cdot \frac{\underline{a}}{|\underline{a}|} = \frac{p_1a_1 + p_2a_2}{a_1^2 + a_2^2} \, (a_1\cos\sigma t + a_2\sin\sigma t)$$

It is easy to check now that in this way expression (3.19) is derived from (3.15) and (3.18). The derivation of the expression for the force component in phase with the velocity is similar.

The added mass of the cylinder per unit length is defined as the negative value of the ratio between (3.19) and (3.18):

$$m'(Ka) = 2\rho a^2 \frac{M_oB + N_oA}{A^2 + B^2} \qquad (3.21)$$

The dimensionless expression $\frac{M_oB + N_oA}{A^2 + B^2}$ is defined as the added mass coefficient.

The damping coefficient of the cylinder per unit length is defined as the negative value of the ratio between (3.20) and (3.16):

$$N'(Ka) = 2\rho a^2\sigma \frac{M_oA - N_oB}{A^2 + B^2} \qquad (3.22)$$

From (3.7) and (3.17) it follows that on the cylinder for $\theta = \frac{\pi}{2}$ the stream function can be written as:

$$\Psi = \frac{gb}{\pi\sigma}\left\{A\cos\sigma t + B\sin\sigma t\right\} \qquad (r=a;\ \theta=\tfrac{\pi}{2})$$

Comparing this with the expression which results from (3.10) when we set $\theta = \frac{\pi}{2}$, it follows that the ratio: $\frac{\text{wave amplitude at infinity}}{\text{amplitude of the forced oscillation}}$ is equal to:

$$\frac{b}{l} = \frac{\pi Ka}{\sqrt{A^2 + B^2}} \tag{3.23}$$

Finally we observe that the work done by the cylinder in one cycle, must be equal to the energy radiated by the regular progressive wave during the same time, which is twice the energy of one wavelength of the regular progressive wave:

$$\int_0^{\frac{2\pi}{\sigma}} P \frac{dy}{dt} \,.\, dt = \rho b^2 g^2 \frac{\pi}{\sigma^2}$$

Substituting the expressions (3.9) and (3.15) we find the relation:

$$M_o A - N_o B = \frac{\pi^2}{2} \tag{3.24}$$

Consequently the damping coefficient can be simplified to:

$$N'(Ka) = \frac{\rho a^2 \sigma \pi^2}{A^2 + B^2} \tag{3.25}$$

4. Heaving of a cylinder with an arbitrary cross-section

In this chapter we will discuss in which way Ursell's method can be modified for the calculation of the added mass and damping of an arbitrary cylinder. The essential point in this process is the mapping of a semi-circle onto the cross section S of the cylinder (Fig. 4.1.1) by means of a conformal transformation, i.e. we determine such a system of curvilinear coordinates that one of the coordinate-lines coincides with the contour of the cross-section.

4.1. Curvilinear coordinates and conformal transformations

Fig. 4.1.1.

We take the origin of a rectangular coordinate system at the mean position of the axis of the cross-section in the free-surface of the fluid. The x-axis is taken horizontally and the y-axis vertically in downward direction. This plane is often called the physical plane and is denoted here by the z-plane. The plane of the semi-circle or reference plane is here called the ζ-plane. In the ζ-plane we assume a polar-coordinate system (r,θ) with origin in the centre of the circle. With a conformal mapping of the z-plane onto the ζ-plane every point (r,θ) in the ζ-plane corresponds with a point (x,y) of the z-plane. Consequently there exist relations between the variables x,y and r,θ of the form:

$$x = f(r,\theta)$$
$$y = g(r,\theta) \qquad (4.1.1)$$

The corresponding inverse relations are written as:

$$r = \bar{f}\,(x,y)$$
$$\theta = \bar{g}\,(x,y) \qquad (4.1.2.)$$

We now require that the conformal transformation maps the cross-section S onto the semi-circle. If the circle has a radius r = a then S is given by:

$$x = f(a,\theta) \qquad\qquad (4.1.3)$$
$$y = g(a,\theta)$$

Now we also conceive as coordinates in the z-plane the variables r and θ. So along the contour S only the variable θ changes in value while r = a remains constant. The conformal transformation brings about a coordinate transformation (4.1.2) of rectangular coordinates (x,y) into curvilinear coordinates (r,θ) in such a way that one of the coordinate-lines (in this case r = a) coincides with the cross-section. The coordinate-lines r = constant and θ = constant represent two sets of curves in the z-plane which are mapped in the ζ-plane as the lines r = constant and θ = constant, which represent there circles with the origin as centre and straight lines through the origin (Fig. 4.1.1). We know from the theory of conformal transformations that right angles at the intersection points of lines correspond with right angles at the intersection points of the transforms of these lines. Consequently the coordinate lines r = constant and θ = constant intersect each other also perpendicularly in the z-plane. The important consequence of this is that differentiation along the cross-section with line-coordinate s corresponds with differentiation to θ:

$$\frac{\partial}{\partial s} = \frac{1}{r}\frac{\partial}{\partial \theta}$$

and differentiation along the normal n with differentiation to r:

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$$

In the future we shall see that the place-dependence of many physical quantities like for example the streamfunction ψ and the potential ∅ is expressed by curvilinear coordinates r and θ.
Using a conformal transformation, we determine the relations (4.1.1.), which satisfy the conditions (4.1.3.) on the contour S.
We map the region outside the unit circle $|\zeta| > 1$, represented in polar coordinates by $\zeta = re^{i\varphi}$ onto the region outside the closed curve S of the complex z-plane, where z = x + iy. The region outside S is supposed to be simply connected (Fig. 4.1.2.).

We determine a transformation in the form of a series with a finite number of terms:

$$z = \sum_{n=-1}^{m-2} C_n \zeta^{-n} = \sum_{n=-1}^{m-2} (A_n + i\, B_n) r^{-n} (\cos n\varphi - i \sin n\varphi) \qquad (4.1.4.)$$

Equating the real and imaginary parts of this equation, we obtain:

$$x = \sum_{n=-1}^{m-2} r^{-n} (A_n \cos n\varphi + B_n \sin n\varphi)$$
$$y = \sum_{n=-1}^{m-2} r^{-n} (-A_n \cos n\varphi + B_n \sin n\varphi) \qquad (4.1.5.)$$

As we are only interested in cylinders which are symmetrical with respect to the x- and y-axis (in fact we only consider the region y>0 of the cross-section; consequently we can imagine the cross-section to be symmetrical with respect to the x-axis), (4.1.5.) can be reduced to:

$$x = \sum_{n=-1}^{m-2} r^{-2n-1} A_{2n+1} \cos(2n+1)\varphi$$
$$y = \sum_{n=-1}^{m-2} r^{-2n-1} A_{2n+1} \sin(2n+1)\varphi \qquad (4.1.6.)$$

We notice that in this case the rectangular coordinate-axes ξ and η of the ζ-plane are transformed into the coordinate-axes x and y of the z-plane and since the circle intersects the horizontal axis perpendicularly, the cross-section of this cylinder intersects the x-axis perpendicularly. Consequently this transformation is restricted to cross-sections which intersect the x-axis perpendicularly.

Comparing Fig. 4.1.1. and Fig. 4.1.2., we see that $\varphi = \frac{\pi}{2} - \Theta$. Substituting this for φ into (4.1.6.) we obtain:

$$x = A_{-1} r \sin\Theta + \sum_{n=0}^{m-2} (-1)^n \frac{A_{2n+1} \sin(2n+1)\Theta}{r^{2n+1}}$$
$$y = A_{-1} r \cos\Theta + \sum_{n=0}^{m-2} (-1)^n \frac{A_{2n+1} \cos(2n+1)\Theta}{r^{2n+1}} \qquad (4.1.7.)$$

These are just the transformation formulas used by Porter [10]. Porter makes use of a slightly different notation. For reasons of simplicity we will adopt here this notation; (4.1.7.) then becomes:

$$x = a\left\{ r \sin\Theta + \sum_{n=0}^{N} (-1)^n \frac{a_{2n+1}}{r^{2n+1}} \sin(2n+1)\Theta \right\}$$

$$y = a\left\{ r \cos\Theta + \sum_{n=0}^{N} (-1)^{n+1} \frac{a_{2n+1}}{r^{2n+1}} \cos(2n+1)\Theta \right\} \qquad (4.1.8.)$$

Finally we remark that for the case of a circular cylinder the coefficients $a_1, a_2, a_3, \ldots$ are all zero. The coefficient a represents then the ratio of the radii of the circles in the z-plane and the ζ-plane. For that reason a is called the scale factor of the transformation.

Fig. 4.1.2.

4.1.1. Transformation method of Fil'chakova [11].

This method is closely related to Melentiev's method, described by Kantorovich and Krylov [15]. The method is derived for cylinders with an arbitrary shape. Consequently we have to start from the relation (4.1.4.). For the determination of the coefficients C_n we choose 2 m points at the unit circle r = 1, which are dividing the circle in equal parts, thus the polar angle of every point always differs $\Delta\varphi = \frac{\pi}{m}$ with those of his two adjacent points. Next we divide these points into two systems of points: an even system $\varphi_{2k} = \frac{2k\pi}{m}$ and an odd system $\varphi_{2k-1} = \frac{(2k-1)\pi}{m}$, k = 1,2, ...,m.

The images $z_{2k} = x_{2k} + i\, y_{2k}$ respectively $z_{2k-1} = x_{2k-1} + i\, y_{2k-1}$ (k= 1,2...m) of these points on the cross-section S are called nodal points. The coordinates of these points are given by:

$$x_{2k} = \sum_{n=-1}^{m-2} (A_n \cos n\varphi_{2k} + B_n \sin n\varphi_{2k})$$

$$y_{2k} = \sum_{n=-1}^{m-2} (-A_n \sin n\varphi_{2k} + B_n \cos n\varphi_{2k}) \qquad (4.1.9a.)$$

$$\left.\begin{aligned} x_{2k-1} &= \sum_{n=-1}^{m-2} (A_n \cos n\varphi_{2k-1} + B_n \sin n\varphi_{2k-1}) \\ y_{2k-1} &= \sum_{n=-1}^{m-2} (A_n \cos n\varphi_{2k} + B_n \sin n\varphi_{2k}) \end{aligned}\right\} \qquad (4.1.9b.)$$

So we have an even sytem of nodal points (x_{2k}, y_{2k}) and an odd system (x_{2k-1}, y_{2k-1}), $k = 1, 2, \ldots, m$, which are the image-points of the points $(1, \varphi_{2k})$ respectively $(1, \varphi_{2k-1})$ at the unit circle.

Making use of the properties of orthogonality for trigonometric functions of discrete equally spaced arguments (here: $\frac{2\pi}{m}$), we can invert (4.1.9) in a simple way with respect to the coefficients A_n and B_n. For the even system these relations of orthogonality are:

$$\sum_{k=1}^{m} \sin j\varphi_{2k} \sin n\varphi_{2k} = \sum_{k=1}^{m} \cos j\varphi_{2k} \cos n\varphi_{2k} = \begin{cases} 0, & j \neq n \\ \frac{m}{2}, & j = n \end{cases} \qquad (4.1.10.)$$

$$\sum_{k=1}^{m} \sin j\varphi_{2k} \cos n\varphi_{2k} = 0$$

We multiply first the equations (4.1.9) by $\cos j\varphi_{2k}$ and $\sin j\varphi_{2k}$ respectively and after that we take the sum with respect to k. Combining this with the relations (4.1.10) this affords:

$$\sum_{k=1}^{m} (x_{2k} \cos j\varphi_{2k} - y_{2k} \sin j\varphi_{2k}) = \sum_{n=-1}^{m-2} A_n \Big(\sum_{k=1}^{m} \cos j\varphi_{2k} \cos n\varphi_{2k} +$$

$$+ \sum_{k=1}^{m} \sin j\varphi_{2k} \sin n\varphi_{2k} \Big) + \sum_{n=-1}^{m-2} B_n \Big(\sum_{k=1}^{m} \cos j\varphi_{2k} \sin n\varphi_{2k} -$$

$$- \sum_{k=1}^{m} \sin j\varphi_{2k} \cos n\varphi_{2k} \Big) = m\, A_j$$

In an analogous manner we multiply (4.1.9a) respectively by $\sin j\varphi_{2k}$ and $\cos j\varphi_{2k}$, which results in the following relations:

$$A_j^{(+m)} = \frac{1}{m}\sum_{k=1}^{m} (x_{2k}\cos j\varphi_{2k} - y_{2k}\sin j\varphi_{2k})$$

$$B_j^{(+m)} = \frac{1}{m}\sum_{k=1}^{m} (x_{2k}\sin j\varphi_{2k} + y_{2k}\cos j\varphi_{2k}) \qquad (4.1.11)$$

$$j = -1, 0, 1, \ldots, m-2$$

The index (+m) has been added to A_j and B_j in order to indicate that these coefficients are determined on the basis of the even nodal points.

Starting from (4.1.9b) we can determine in an analogous way a relation between A_j and B_j and the odd nodal points. In this case the index (-m) is added to A_j and B_j:

$$A_j^{(-m)} = \frac{1}{m}\sum_{k=1}^{m} (x_{2k-1}\cos j\varphi_{2k-1} - y_{2k-1}\sin j\varphi_{2k-1})$$

$$B_j^{(-m)} = \frac{1}{m}\sum_{k=1}^{m} (x_{2k-1}\sin j\varphi_{2k-1} + y_{2k-1}\cos j\varphi_{2k-1}) \qquad (4.1.12)$$

$$j = -1, 0, 1, \ldots, m-2$$

If the nodal points are known we can determine the coefficients by means of the expressions (4.1.11) and (4.1.12) where $A_j = A_j^{(+m)} = A_j^{(-m)}$ and $B_j = B_j^{(+m)} = B_j^{(-m)}$. However the locations of the nodal points are unknown.

We shall devise now an iteration process, based on the property that $A_j^{(+m)} = A_j^{(-m)}$ and $B_j^{(+m)} = B_j^{(-m)}$. So on account of (4.1.7) and (4.1.8) there exists a relation between the even and odd nodal points. One possibility to determine this relation is to eliminate $A_n^{(-m)}$ and $B_n^{(-m)}$ from (4.1.9a) and (4.1.12)

$$x_{2\nu} = \sum_{n=-1}^{m-2} (A_n^{(-m)}\cos n\varphi_{2\nu} + B_n^{(-m)}\sin n\varphi_{2\nu}) =$$

$$= \frac{1}{m}\left\{\sum_{k=1}^{m} x_{2k-1}\sum_{n=-1}^{m-2} \cos n(\varphi_{2k-1}-\varphi_{2\nu}) - y_{2k-1}\sum_{m=-1}^{m-2} \sin n(\varphi_{2k-1}-\varphi_{2\nu})\right\}$$

$$y_{2\nu} = \sum_{n=-1}^{m-2} (-A_n^{(-m)}\sin n\varphi_{2\nu} + B_n^{(-m)}\cos n\varphi_{2\nu}) =$$

$$= \frac{1}{m}\left\{\sum_{k=1}^{m} x_{2k-1}\sum_{n=-1}^{m-2} \sin n(\varphi_{2k-1}-\varphi_{2\nu}) + y_{2k-1}\sum_{n=-1}^{m-2} \cos n(\varphi_{2k-1}-\varphi_{2\nu})\right\}$$

In the same way we obtain by eliminating $A_n^{(+m)}$ and $B_n^{(+m)}$ from (4.1.9b) by means of (4.1.11):

$$x_{2\nu-1} = \frac{1}{m}\left\{\sum_{k=1}^{m} x_{2k} \sum_{n=-1}^{m-2} \cos n(\varphi_{2k}-\varphi_{2\nu-1}) - y_{2k} \sum_{n=-1}^{m-2} \sin n(\varphi_{2k}-\varphi_{2\nu-1})\right\}$$

$$y_{2\nu-1} = \frac{1}{m}\left\{\sum_{k=1}^{m} x_{2k} \sum_{n=-1}^{m-2} \sin n(\varphi_{2k}-\varphi_{2\nu-1}) + y_{2k} \sum_{n=-1}^{m-2} \cos n(\varphi_{2k}-\varphi_{2\nu-1})\right\}$$

We define the following new quantities:

$$\gamma^{I(m)}_{2k-1,2\nu} = \frac{1}{m}\sum_{n=-1}^{m-2} \sin n(\varphi_{2k-1}-\varphi_{2\nu}); \quad \gamma^{I(m)}_{2k,\ 2\nu-1} = \frac{1}{m}\sum_{n=-1}^{m-2} \sin n(\varphi_{2k}-\varphi_{2\nu-1});$$

$$\gamma^{II(m)}_{2k-1,2\nu} = \frac{1}{m}\sum_{n=-1}^{m-2} \cos n(\varphi_{2k-1}-\varphi_{2\nu}); \quad \gamma^{II(m)}_{2k,\ 2\nu-1} = \frac{1}{m}\sum_{n=-1}^{m-2} \cos n(\varphi_{2k}-\varphi_{2\nu-1}); \tag{4.1.13}$$

Then we obtain the following recurrence formulas of the iteration process:

$$x^{(n)}_{2\nu-1} = \sum_{k=1}^{m} x^{(n)}_{2k}\, \gamma^{II(m)}_{2k,2\nu-1} - y^{(n)}_{2k}\, \gamma^{I(m)}_{2k,2\nu-1}$$

$$y^{(n)}_{2\nu-1} = \sum_{k=1}^{m} x^{(n)}_{2k}\, \gamma^{I(m)}_{2k,2\nu-1} + y^{(n)}_{2k}\, \gamma^{II(m)}_{2k,2\nu-1} \tag{4.1.14}$$

$$x^{(n+1)}_{2\nu} = \sum_{k=1}^{m} x^{(n)}_{2k-1}\gamma^{II(m)}_{2k-1,2\nu} - y^{(n)}_{2k-1}\gamma^{I(m)}_{2k-1,2\nu}$$

$$y^{(n+1)}_{2\nu} = \sum_{k=1}^{m} x^{(n)}_{2k-1}\gamma^{I(m)}_{2k-1,2\nu} + y^{(n)}_{2k-1}\gamma^{II(m)}_{2k-1,2\nu} \tag{4.1.15}$$

The iteration process is carried out in the following manner:

For some m = 4,8,16... we select on the basis of a graphical consideration an estimation for the zeroth approximation to the m even points ($x^{(0)}_{2k}$, $y^{(0)}_{2k}$), k = 1,2,....m. Kantorovich and Krylov describe in chapter V §7 [15] various methods to obtain a suitable estimation for the locations of the nodal points.

By means of (4.1.14) we calculate the accessory odd points which in general will not lie on S. After that we carry these points to the contour for example along the line which connects this point with the origin and thus obtain the zeroth approximation for the odd nodal points $(x^{(0)}_{2k-1}, y^{(0)}_{2k-1})$. With these points we calculate with (4.1.11) the accessory even points, carry them to the contour and in this way get the first approximation for the even nodal points $(x^{(1)}_{2k}, y^{(1)}_{2k})$, etc.
We repeat this process until a subsequent approximation coincides with sufficient accuracy with the previous one. In order to increase the accuracy of the transformation we have to take a larger value for m, for example 2m, taking the even and odd nodal points of the previous iteration as an estimation for the m even nodal points of the new iteration, after which we repeat the iteration process as we described above. In this way we can determine the locations of the nodal points more accurately.
So the transformation-equations:

$$x = \sum_{n=-1}^{m-2} (A_n \cos n\varphi + B_n \sin n\varphi)$$

$$y = \sum_{n=-1}^{m-2} (-A_n \cos n\varphi + B_n \sin\varphi) \qquad (4.1.16)$$

are completely determined now.

4.1.2. Alternative method for the determination of the transformation coefficients.

We shall now consider another method which has been developed by "Rescona Engineering" in Amstelveen (Holland) in cooperation with W.E. Smith. From the analytical point of view this method is much more simple than the one discussed in the preceding section. Applications of this method proved that it is very useful.
This method is expounded here for cylinders, which are symmetrical with respect to the x- and y-axis. This is a case which is usually encountered in naval architecture.

We start form the equations (4.1. q) where we put r = 1:

$$x = a\left\{\sin\Theta + \sum_{n=0}^{N} (-1)^n a_{2n+1} \sin(2n+1)\Theta\right\}$$
$$y = a\left\{\cos\Theta + \sum_{n=0}^{N} (-1)^{n+1} a_{2n+1} \cos(2n+1)\Theta\right\} \qquad (4.1.17)$$

These equations describe the relation between the variable Θ on the unit-circle in the ζ-plane and the variables x and y along the given contour in the z-plane.

We substitute in (4.1.17) the expressions:

$$\sin(2n+1)\Theta = \sum_{r=0}^{n} (-1)^r \frac{2n+1}{(2r+1)!}\left[\prod_{k=1}^{r}\left\{(2n+1)^2 - (2k-1)^2\right\}\right]\sin^{2r+1}\Theta$$

$$\cos(2n+1)\Theta = \frac{1}{2}(2\cos\Theta)^{2n+1} +$$
$$+ \frac{1}{2}\sum_{r=1}^{n}(-1)^r \frac{2n+1}{r!}\left[\prod_{k=r+1}^{2r-1}(2n-k+1)\right](2\cos\Theta)^{2n-2r+1}$$

This yields:

$$\frac{x}{a} = \sin\Theta + \sum_{k=0}^{N} b_k \sin^{2k+1}\Theta$$
$$\frac{y}{a} = \cos\Theta + \sum_{k=0}^{N} c_k \cos^{2k+1}\Theta \qquad (4.1.18)$$

where the coefficients b_k and c_k are linear combinations of $a_1, a_3, \ldots, a_{2N+1}$.

We now choose on the right half of the cross-section in the z-plane a sequence of m points, such that the first point coincides with the point (x=0,y=T) while the last point coincides with ($x=B_o$,y=0). T and B_o represent respectively the draft and the half beam of the cross-section (Fig. 4.1.1). The points of this sequence, which we represent by (x_i, y_i), where i = 1,2,....,m, originate from points on the unit circle in the ζ-plane represented by $(1,\Theta_i)$. Consequently with the points (0,T) and $(B_o,0)$ in the z-plane correspond the points (1,0) and $(1,\frac{\pi}{2})$ in the ζ-plane. Substituting these values for (x_i, y_i) and Θ_i in (4.1.18), we obtain the following system of equations:

$$\frac{x_i}{a} - \sin \Theta_i - \sum_{k=0}^{N} b_k \sin^{2k+1}\Theta_i = 0$$

$$i = 2,3,\ldots,m$$

$$\frac{y_i}{a} - \cos \Theta_i - \sum_{k=0}^{N} c_k \cos^{2k+1}\Theta_i = 0 \qquad (4.1.19)$$

$$i = 1,2,\ldots,m-1$$

In this way we obtain a system of 2m-2 equations with the following N+m unknown variables:

$$a,\ a_1,\ a_3,\ \ldots\ldots,\ a_{2N+1}$$

and

$$\Theta_2,\ \Theta_3,\ \ldots\ldots,\ \Theta_{m-1}$$

In order to make the number of equations equal to the number of unknown variables the relation

$$m = N + 2 \qquad (4.1.20)$$

between the number of points m, chosen along the cross-section, and the number of terms N, we want to consider in the series expansion (4.1.17), has to be satisfied.

The set of equations (4.1.19) are solved by the Newton Raphson Method [16].

Representing the left-hand terms of (4.1.19) by:

$F_i = F_i\ (x_i, \Theta_i, a, a_1, a_3, \ldots\ldots, a_{2N+1})$ and

$G_i = G_i\ (y_i, \Theta_i, a, a_1, a_3, \ldots\ldots, a_{2N+1})$ respectively,

we obtain the following set of iteration equations:

$$\Delta a^j \frac{\partial F_i^j}{\partial a} + \sum_{k=0}^{N} \Delta a_{2k+1}^j \frac{\partial F_i^j}{\partial a_{2k+1}} + \Delta\Theta_i^j \frac{\partial F_i^j}{\partial \Theta_i} = - F_i^j$$

$$i = 2,3,\ldots,m \qquad (4.1.21)$$

$$\Delta a^j \frac{\partial G_i^j}{\partial a} + \sum_{k=0}^{N} \Delta a_{2k+1}^j \frac{\partial G_i^j}{\partial a_{2k+1}} + \Delta\Theta_i^j \frac{\partial G_i^j}{\partial \Theta_i} = - G_i^j$$

$$i = 1,\ 2,\ \ldots,m-1$$

In these equations $\Delta a^j \equiv a^{j+1} - a^j$, $\Delta a_{2k+1}^j \equiv a_{2k+1}^{j+1} - a_{2k+1}^j$ and $\Delta\Theta_i^j \equiv \Theta_i^{j+1} - \Theta_i^j$ represent the corrections which are to be added to the j-th iterates a^j, a_{2k+1}^j and Θ_i^j in order to obtain the (j+1)-th iterates of these variables.

The numerical values of the constants $\frac{\partial F_i^j}{\partial a}$, $\frac{\partial F_i^j}{\partial a_{2k+1}}$, $\frac{\partial F_i^j}{\partial \theta_i}$, F_i^j and $\frac{\partial G_i^j}{\partial a}$, $\frac{\partial G_i^j}{\partial a_{2k+1}}$, $\frac{\partial G_i^j}{\partial \theta_i}$, G_i^j are obtained by substituting the j-th iterates in the corresponding functions for the variables $\theta_i, a, a_1, a_3, \ldots, a_{2N+1}$.

4.2. Calculation of added mass and damping [10]

Completely similar to the method Ursell used for circular cylinders, the velocity potential for the case of an arbitrary cross-section is also composed of a source potential and a linear combination of multipole potentials. For the source potential we take again expression (2.28), which satisfies the surface condition and the radiation condition. To satisfy the boundary condition on the cylinder we superimpose a suitably chosen linear combination of multipole potentials, whereby each multipole potential satisfies the surface condition and vanishes for $|x| \to \infty$. The multipole potential is defined by:

$$\varphi_{2m} = \frac{\cos 2m\theta}{r^{2m}} + ka\left\{\frac{\cos(2m-1)\theta}{(2m-1)r^{2m-1}} + \sum_{n=0}^{N} (-1)^n \frac{(2n+1)a_{2n+1}\cos(2m+2n+1)\theta}{(2m+2n+1)r^{2m+2n+1}}\right\}$$

$$m = 1,2,3,\ldots \qquad (4.2.1)$$

We remark that φ_{2m} vanishes vor $r \to \infty$. In order to prove that φ_{2m} satisfies the free-surface condition, we transform this relation first by means of (4.1.8) from rectangular coordinates (x,y) into curvilinear coordinates (r,θ).

As $\frac{\partial x}{\partial \theta} = 0$ for $\theta = \pm\frac{\pi}{2}$ (the θ-lines intersect the x-axis perpendicularly), the free surface condition.

$$K\cdot \phi + \phi_y = 0 \qquad (y=0)$$

can be written as:

$$K\phi + \frac{1}{\left(\frac{\partial y}{\partial \theta}\right)}\cdot\frac{\partial \phi}{\partial \theta} = 0 \qquad \left(\theta = \pm\frac{\pi}{2}\right) \qquad (4.2.2)$$

From (4.1.8) it follows:

$$\frac{dy}{d\theta} = -a\left\{r \sin\theta + \sum_{n=0}^{N} \frac{(-1)^{n+1}(2n+1)a_{2n+1}}{r^{2n+1}} \sin(2n+1)\theta\right\}$$

whence for $\theta = \pm\frac{\pi}{2}$:

$$\frac{dy}{d\theta} = \mp a\left\{r - \sum_{n=0}^{N} (2n+1)a_{2n+1}\cdot\frac{1}{r^{2n+1}}\right\}$$

In curvilinear coordinates the free-surface condition obtains the form:

$$Ka\left\{r-\sum_{n=0}^{N}(2n+1)\frac{a_{2n+1}}{r^{2n+1}}\right\}\phi \mp \frac{\partial\phi}{\partial\theta}=0 \qquad \theta=\pm\frac{\pi}{2} \tag{4.2.3}$$

Substitution of (4.2.1) into (4.2.3) shows that φ_{2m} satisfies the free-surface condition. So the velocity potential which is the solution of the boundary value problem can be written as:

$$\Phi=\frac{gb}{\pi\sigma}\left\{(\phi_c+\sum_{m=1}^{\infty}p_{2m}\varphi_{2m})\cos\sigma t+(\phi_s+\sum_{m=1}^{\infty}q_{2m}\varphi_{2m})\sin\sigma t\right\} \tag{4.2.4}$$

where ϕ_c and ϕ_s are defined in accordance with (2.29):

$$\phi_c=\pi e^{-Ky}\cos Kx$$

$$\phi_s=\pi e^{-Ky}\sin Kx-\int_0^{\infty}\frac{e^{-\beta x}(\beta\cos\beta y-K\sin\beta y)}{\beta^2+K^2}\,d\beta \tag{4.2.5}$$

By means of the Cauchy-Riemann relations (3.6) we calculate the conjugate stream-function:

$$\Psi=\frac{gb}{\pi\sigma}\left\{(\psi_c+\sum_{m=1}^{\infty}p_{2m}\psi_{2m})\cos\sigma t+(\psi_s+\sum_{m=1}^{\infty}q_{2m}\psi_{2m})\sin\sigma t\right\} \tag{4.2.6}$$

where:

$$\psi_c=\pi e^{-Ky}\sin Kx$$

$$\psi_s=-\pi e^{-Ky}\cos Kx+\int_0^{\infty}\frac{e^{-\beta x}(\beta\sin\beta y+K\cos\beta y)}{\beta^2+K^2}\,d\beta$$

$$\psi_{2m}=\frac{\sin 2m\theta}{r^{2m}}+Ka\left\{\frac{\sin(2m-1)\theta}{(2m-1)r^{2m-1}}+\sum_{n=0}^{N}(-1)^n\frac{(2n+1)a_{2n+1}}{2m+2n+1}\cdot\frac{\sin(2m+2n+1)\theta}{r^{2m+2n+1}}\right\} \tag{4.2.7}$$

It remains to determine the coefficients p_{2m} and q_{2m} in such a way that the boundary condition on the cylinder is satisfied. The reasoning proceeds analogous to the corresponding calculation in chapter 3. The boundary condition on the cylinder has the form:

$$\frac{\partial\Phi}{\partial n}=\frac{dy}{dt}\cos\alpha \tag{4.2.8}$$

α is the angle between the positive normal on the cross-section and the positive y-axis (Fig. 4.2.1).

Fig. 4.2.1

It is further easily seen that at the cylinder surface:

$$\cos\alpha = \frac{\partial x}{\partial s} = \frac{\partial y}{\partial n}$$

$$\sin\alpha = -\frac{\partial y}{\partial s} \qquad (4.2.9)$$

$$\frac{\partial \Phi}{\partial n} = -\frac{\partial \Psi}{\partial s} = -\frac{\partial \Psi}{r\partial \Theta}$$

So the boundary condition at the cylinder (r=1) can also be written as:

$$-\frac{\partial \Psi}{\partial s} = \frac{dy}{dt}\frac{\partial x}{\partial s}$$

After integration this leads to:

$$\Psi\,(r=1,\Theta) = -\frac{dy}{dt}\,x(r=1,\Theta) + c(t) \qquad (4.2.10)$$

Substitution of $\Theta=0$ ($x=0$) yields for the integration constant: $c(t)=0$, so:

$$\Psi\,(r=1,\Theta) = -\frac{dy}{dt}\,x(r=1,\Theta) \qquad (4.2.11)$$

Substitution of $\Theta = \frac{\pi}{2}$ gives:

$$\Psi\,(1,\tfrac{\pi}{2}) = -\frac{dy}{dt}\,B_o \qquad (4.2.12)$$

$B_o = x(1,\frac{\pi}{2})$ is the half beam of the cross-section (Fig. 4.1.1). Eliminate $\frac{dy}{dt}$ from (4.2.11) and (4.2.12) to obtain:

$$\Psi\,(1,\Theta) = \frac{x(1,\Theta)}{B_o}\,\Psi(1,\tfrac{\pi}{2}) \qquad (4.2.13)$$

Substituting (4.2.6) in this expression and equating successively the coefficients of $\cos\sigma t$ and $\sin\sigma t$, we see that the coefficients p_{2m} and q_{2m} have to satisfy the relations:

$$\psi_c(1,\theta) + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}(1,\theta) = \frac{x(1,\theta)}{B_0}\left\{\psi_c(1,\tfrac{\pi}{2}) + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}(1,\tfrac{\pi}{2})\right\}$$

$$\psi_s(1,\theta) + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}(1,\theta) = \frac{x(1,\theta)}{B_0}\left\{\psi_s(1,\tfrac{\pi}{2}) + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}(1,\tfrac{\pi}{2})\right\}$$

or:

$$\psi_c(1,\theta) - \frac{x(1,\theta)}{B_0}\,\psi_c(1,\tfrac{\pi}{2}) = \sum_{m=1}^{\infty} p_{2m}f_{2m}(1,\theta)$$

$$\psi_s(1,\theta) - \frac{x(1,\theta)}{B_0}\,\psi_s(1,\tfrac{\pi}{2}) = \sum_{m=1}^{\infty} q_{2m}f_{2m}(1,\theta) \qquad (4.2.14)$$

where:

$$f_{2m}(1,\theta) = \frac{x(1,\theta)}{B_0}\,\psi_{2m}(1,\tfrac{\pi}{2}) - \psi_{2m}(1,\theta) \quad ^{*)}$$

The equations (4.2.14) have the same structure as (3.13). The coefficients p_{2m} and q_{2m} may be calculated in a corresponding way.

The velocity potential Φ at the contour of the cylinder (r=1) is written as:

$$\Phi(1,\theta) = \frac{gb}{\pi\sigma}\,(M \sin \sigma t + N \cos \sigma t) \qquad (4.2.15)$$

where:

$$M(1,\theta) = \phi_s(1,\theta) + \sum_{m=1}^{\infty} q_{2m}\varphi_{2m}(1,\theta)$$

$$N(1,\theta) = \phi_c(1,\theta) + \sum_{m=1}^{\infty} p_{2m}\varphi_{2m}(1,\theta)$$

We calculate the pressure along the contour according to (3.14):

$$p(1,\theta) = -\frac{\rho g b}{\pi}\,(M \cos \sigma t - N \sin \sigma t) \qquad (4.2.16)$$

We define:

$$\frac{dy}{dt} = \frac{gb}{\pi\sigma B_0}\,(-A \cos \sigma t - B \sin \sigma t) \qquad (4.2.17)$$

hence:

$$\frac{d^2y}{dt^2} = \frac{gb}{\pi B_0}\,(A \sin \sigma t - B \cos \sigma t) \qquad (4.2.18)$$

*) See the remark at the end of this section.

where according to (4.2.12):

$$A = \psi_c(1,\tfrac{\pi}{2}) + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}(1,\tfrac{\pi}{2})$$

$$B = \psi_s(1,\tfrac{\pi}{2}) + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}(1,\tfrac{\pi}{2}) \tag{4.2.19}$$

We can resolve the pressure into a component in phase with the velocity and a component in phase with the acceleration. This is done in a similar way as in chapter 3.

$$p(1,\theta) = \frac{\rho g b}{\pi}\frac{MB+NA}{A^2+B^2}(A\sin\sigma t - B\cos\sigma t) - \frac{\rho g b}{\pi}\frac{MB-NA}{A^2+B^2}(A\cos\sigma t + B\sin\sigma t)$$

or:

$$p(1,\theta) = \rho B_0 \frac{MB+NA}{A^2+B^2}\ddot{y} + \rho B_0 \frac{MA-NB}{A^2+B^2}\sigma\dot{y} \tag{4.2.20}$$

The total vertical force on the cylinder per unit length becomes:

$$F = -2 \int_{S(0<\theta<\frac{\pi}{2})} p(1,\theta)\cos\alpha ds \tag{4.2.21}$$

From (4.2.9) and (4.1.8) it follows that at the contour of the cylinder:

$$\cos\alpha ds = \frac{\partial x}{\partial s} ds = dx =$$

$$= a\left\{\cos\theta + \sum_{n=0}^{N}(-1)^n(2n+1)a_{2n+1}\cos(2n+1)\theta\right\}d\theta = a\,W(\theta)d\theta \tag{4.2.22}$$

where the function between parentheses is denoted by $W(\theta)$

Substitution of $\theta = \frac{\pi}{2}$ and $r = 1$ into the first equation of (4.1.8) produces:

$$B_0 = a\left\{1 + \sum_{n=0}^{N} a_{2n+1}\right\}$$

Introducing the constant G, defined by $G = \{1 + \sum_{n=0}^{N} a_{2n+1}\}$, this reduced to:

$$a = \frac{B_0}{G} \tag{4.2.23}$$

consequently (4.2.21) becomes:

$$F = -2 B_0 \int_0^{\frac{\pi}{2}} p(1,\theta)\frac{W(\theta)}{G}d\theta.$$

Substitution of (4.2.16) into this equation leads to:

$$F = -\frac{2\rho g b B_o}{\pi}\,(N_o \sin\sigma t - M_o \cos\sigma t) \tag{4.2.24}$$

where:

$$M_o = \int_0^{\frac{\pi}{2}} M(1,\theta)\,\frac{W(\theta)}{G}\,d\theta$$

$$N_o = \int_0^{\frac{\pi}{2}} N(1,\theta)\,\frac{W(\theta)}{G}\,d\theta \tag{4.2.25}$$

We resolve the vertical force into a component in phase with the acceleration and a component in phase with the velocity:

$$F = -\frac{2\rho g b B_o}{\pi}\,\frac{M_o B + N_o A}{A^2+B^2}\,(A \sin\sigma t - B \cos\sigma t) -$$

$$\frac{2\rho g b B_o}{\pi}\,\frac{M_o A - N_o B}{A^2+B^2}\,(-A \cos\sigma t - B \sin\sigma t).$$

With the aid of (4.2.17) and (4.2.8) the above mentioned expression becomes;

$$F = -2\rho B_o^2\,\frac{M_o B + N_o A}{A^2+B^2}\,\ddot{y} - 2\rho B_o^2\,\frac{M_o A - N_o B}{A+B}\,\sigma\dot{y} \tag{4.2.26}$$

Defining the added mass m'' and the damping N' by representing the force according to $F = -m''\ddot{y} - N'\dot{y}$, we find:

$$m'' = 2\rho B_o^2\,\frac{M_o B + N_o A}{A^2+B^2} \tag{4.2.27}$$

So in non-dimensional form we obtain for the coefficient of added mass:

$$\frac{M_o B + N_o A}{A^2+B^2} \tag{4.2.28}$$

The damping of the cylinder per unit length becomes:

$$N' = 2\rho B_o^2\sigma\,\frac{M_o A - N_o B}{A^2+B^2} \tag{4.2.29}$$

By equating the dissipated energy to the work done by the cylinder, we obtain a relation which is identical to (3.24):

$$M_o A - N_o B = \frac{\pi^2}{2} \tag{4.2.30}$$

So (4.2.29) can be written as:

$$N' = \frac{\rho B_o^2\sigma\pi^2}{A^2+B^2} \tag{4.2.31}$$

Analogous to (3.23) we find for the ratio of the wave amplitude at infinite distance from the cylinder and the amplitude of the forced oscillation:

$$\frac{K\, B_o\, \pi}{\sqrt{A^2 + B^2}} \tag{4.2.32}$$

Remark: Instead of eliminating $\frac{dy}{dt}$ from (4.2.11) and (4.2.12) which finally leads to the set of equations (4.2.14) for p_{2m} and q_{2m}, we can also substitute expression (4.2.17) for $\frac{dy}{dt}$ into (4.2.11) and after that equate the coefficients of cos σt and sin σt. Successively we then obtain:

$$\Psi\ (1,\Theta) = \frac{gb}{\pi\sigma B_o}\ (A\ \cos\sigma t + B\ \sin\sigma t).x(1,\Theta) \tag{4.2.33}$$

this becomes:

$$\psi_c(1,\theta) + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}(1,\theta) = \frac{gb}{\pi\sigma B_o}\ x\ (1,\theta).A$$

$$\psi_s(1,\theta) + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}(1,\theta) = \frac{gb}{\pi\sigma B_o}\ x\ (1,\theta).B$$

or:

$$\psi_c(1,\theta) = \sum_{m=0}^{\infty} p_{2m}f_{2m}$$

$$\psi_s(1,\theta) = \sum_{m=0}^{\infty} q_{2m}f_{2m}$$

where:

$$f_o = \frac{gb}{\pi\sigma B_o}\ x\ (1,\theta)$$

$$f_{2m} = -\ \psi_{2m}(1,\theta)\quad ,\quad m\neq 0$$

and:

$$p_o = A, \qquad q_o = B.$$

The calculation of A and B according to (4.2.19) is dropped now.

4.3. Added mass and damping of a cylinder with a "Lewis-form"

We will consider a special case now of the theory discussed in 4.2, where we take in the transformation formulas (4.1. 8) for N the value 2. In this case the tranformed shape is often a reasonable approximation of a cross-section of a ship. This kind of sections have frequently been used by Lewis and Grim in their calculations and are known as "Lewis-forms".

Fig. 4.3.1.

Tasai made an extensive study of this case in [8]. For the sake of simplicity we shall adopt the notation which he uses: instead of the polar coordinate r the variable α is introduced which is related with r by: $r = e^{\alpha}$. So the unit circle in the ζ-plane corresponds with $\alpha=0$ and in the z-plane the coordinate line $\alpha=0$ coincides with the contour of the section. In Tasai's notation the transformation formulas (4.1. 8) become:

$$\frac{x}{M} = e^{\alpha}\sin\theta + a_1 e^{-\alpha}\sin\theta - a_3 e^{-3\alpha}\sin 3\theta$$
$$\frac{y}{M} = e^{\alpha}\cos\theta - a_1 e^{-\alpha}\cos\theta + a_3 e^{-3\alpha}\cos 3\theta \qquad (4.3.1)$$

At the contour of the section where $\alpha=0$, the following relations hold:

$$\frac{x_o}{M} = (1+a_1)\sin\theta - a_3\sin 3\theta$$
$$\frac{y_o}{M} = (1-a_1)\cos\theta + a_3\cos 3\theta \qquad (4.3.2)$$

According to (4.2.23) for the scale factor of the transformation which is denoted here by M we find:

$$M = \frac{B_o}{1+a_1+a_3}$$

As parameters for the cross-section we take:

$$H_o = \frac{B_o}{T} \quad \text{and} \quad \sigma = \frac{S}{2B_o T} \tag{4.3.3}$$

S is the area of the section, T the draught and B_o the half beam. On account of the restricted number of terms in the transformation equations it is possible to find in an analytic way an explicit expression for the coefficients a_1 and a_3. In this way we avoid the iteration process which may require much computer time; we find:

$$a_1 = \frac{H_o-1}{H_o+1}(a_3+1); \qquad a_3 = \frac{-c_2+\sqrt{c_2^2-4c_1c_3}}{2c_1} \tag{4.3.4}$$

where:

$$c_1 = (3+\frac{2\sigma}{\pi}) + (1-\frac{2\sigma}{\pi})(\frac{H_o-1}{H_o+1})^2,$$

$$c_2 = 2\left\{(1-\frac{2\sigma}{\pi})(\frac{H_o-1}{H_o+1})^2 + \frac{2\sigma}{\pi}\right\},$$

$$c_3 = (1-\frac{2\sigma}{\pi})\left\{(\frac{H_o-1}{H_o+1})^2 - 1\right\}.$$

From (4.2.1) it follows that in this case the multipole-potentials φ_{2m} obtain the following shape:

$$\varphi_{2m} = e^{-2m\alpha}\cos 2m\theta + \frac{\xi_o}{1+a_1+a_3}\left\{\frac{e^{-(2m-1)\alpha}}{2m-1}\cos(2m-1)\theta + a_1\frac{e^{-(2m+1)\alpha}}{2m+1}\right.$$

$$\left. .\cos(2m+1)\theta - \frac{3a_3}{2m+3}e^{-(2m+3)\alpha}\cos(2m+3)\theta\right\} \tag{4.3.5}$$

$$m = 1,2,3....$$

where:

$$\xi_o = K.B_o = \frac{\sigma^2}{g}B_o.$$

For the conjugate streamfunction we obtain from (4.2.7):

$$\psi_{2m} = e^{-2m\alpha}\sin 2m\theta + \frac{\xi_o}{1+a_1+a_3}\left\{\frac{e^{-(2m-1)\alpha}}{2m-1}\sin(2m-1)\theta + a_1\frac{e^{-(2m+1)\alpha}}{2m+1}\cdot\right.$$

$$\left.\cdot\sin(2m+1)\theta - \frac{3a_3}{2m+1}e^{-(2m+3)\alpha}\sin(2m+3)\theta\right\} \qquad (4.3.6)$$

$m = 1,2,3,\ldots.$

So the total potential respectively streamfunction obtains the shapes (4.2.4) and (4.2.6) where we substitute for φ_{2m} and ψ_{2m} the expressions (4.3.5) and (4.3.6). For the calculation of the coefficients p_{2m} and q_{2m} we proceed in a similar way as for the general case in section 4.2:

The boundary condition at the contour of the cylinder results in condition (4.2.11) which has to be satisfied by the streamfunction ψ :

$$\Psi(\alpha=0,\theta) = -\frac{dy}{dt}x(\alpha=0,\theta) \qquad (4.3.7)$$

Now we eliminate $\frac{dy}{dt}$ by means of (4.2.12) and after that we equate successively the coefficients of cos σt and sin σt which results in the two expressions (4.2.14), which here obtain the following shape:

$$\psi_c(\alpha=0,\theta\ ;\ \xi_o,a_1,a_3) - \frac{\sin\theta+a_1\sin\theta-a_3\sin3\theta}{1+a_1+a_3}\psi_c(\alpha=0,\theta=\tfrac{\pi}{2};\xi_o,a_1,a_3) =$$

$$=\sum_{m=1}^{\infty} p_{2m}f_{2m}.$$

$$\psi_s(\alpha=0,\theta\ ;\ \xi_o,a_1,a_3) - \frac{\sin\theta+a_1\sin\theta-a_3\sin3\theta}{1+a_1+a_3}\psi_s(\alpha=0,\theta=\tfrac{\pi}{2};\xi_o,a_1,a_3) =$$

$$=\sum_{m=1}^{\infty} q_{2m}f_{2m},$$

where:

$$f_{2m} = -\left[\sin 2m\theta + \frac{\xi_o}{1+a_1+a_3}\left\{\frac{\sin(2m-1)\theta}{2m-1} + a_1\frac{\sin(2m+1)\theta}{2m+1} - \right.\right.$$

$$\left.\left. - \frac{3a_3\sin(2m+3)\theta}{2m+3}\right\} + \frac{\xi_o(-1)^m}{(1+a_1+a_3)^2}\left\{\frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3}\right\}\right.\cdot$$

$$\cdot(\sin\theta + a_1\sin\theta - a_3\sin3\theta) \qquad (4.3.9)$$

The added mass and damping are calculated according to (4.2.27) respectively (4.2.29).

In accordance with (4.2.25) and (4.2.10) the quantities M_o, N_o, A and B obtain the form:

$$M_o = \int_o^{\frac{\pi}{2}} \phi_s(0,\theta;a_1,a_3,\xi_o) \frac{\cos\theta + a_1\cos\theta - 3a_3\cos 3\theta}{1+a_1+a_3} d\theta + \frac{1}{1+a_1+a_3} \cdot$$

$$\left[\sum_{m=1}^{\infty} (-1)^{m-1} q_{2m} \left(\frac{1+a_1}{4m^2-1} + \frac{9a_3}{4m^2-9}\right) + \frac{\pi\xi_o}{4(1+a_1+a_3)} \{(1+a_1-a_1a_3)q_2 - a_3q_4\}\right].$$

$$N_o = \int_o^{\frac{\pi}{2}} \phi_c(0,\theta;a_1,a_3,\xi_o) \frac{\cos\theta + a_1\cos\theta - 3a_3\cos 3\theta}{1+a_1+a_3} d\theta + \frac{1}{1+a_1+a_3} \cdot$$

$$\left[\sum_{m=1}^{\infty} (-1)^{m-1} p_{2m} \left(\frac{1+a_1}{4m^2-1} + \frac{9a_3}{4m^2-9}\right) + \frac{\pi\xi_o}{4(1+a_1+a_3)} \{(1+a_1-a_1a_3)p_2 - a_3p_4\}\right].$$

$$A = \psi_c(0,\tfrac{\pi}{2};a_1,a_3,\xi_o) + \sum_{m=1}^{\infty} p_{2m}(\xi_o)(-1)^{m-1} \frac{\xi_o}{1+a_1+a_3} \left\{\frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3}\right\}$$

$$B = \psi_s(0,\tfrac{\pi}{2},a_1,a_3,\xi_o) + \sum_{m=1}^{\infty} q_{2m}(\xi_o)(-1)^{m-1} \frac{\xi_o}{1+a_1+a_3} \left\{\frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3}\right\}$$

5. Swaying and rolling.

In this chapter we calculate the hydrodynamic coefficients for a cylinder which is carrying out a forced harmonic swaying or rolling motion. In addition we shall pay attention to the coupled motion between rolling and swaying. According to (1.11)the velocity potential which is a solution of this boundary value problem has to be an asymmetric function. Consequently the source potential (2.28) and the multipole potentials which are used for solving the symmetric problem are no longer usable here. Consequently the first thing we have to do in this chapter is to derive the potentials which will replace the source potential and the multipole potentials respectively. After that we proceed to the calculation of the hydrodynamic coefficients.

5.1. Potential of a dipole in the origin; asymmetric multipole potentials.

For physical reasons it is easy to see that a dipole produces a flow field which is asymmetric with respect to a line through the dipole perpendicular to the direction of its axis.

Fig. 5.1.1.

We know that the potential in a point (x,y) due to the presence of a dipole in a point (a,b) with moment M is given by the formula [13]:

$$\phi_{dip} = \frac{-M(\underline{r}_1 \cdot \underline{n}_1)}{|\underline{r}|} = \frac{-M(\underline{r} \cdot \underline{n}_1)}{|\underline{r}|^2} = \frac{-M(x-a)}{(x-a)^2+(y-b)^2} \qquad (5.1.1.)$$

$\underline{n}_1$ is the unit vector in the direction of the axis, $\underline{r}$ the vector which connects (a,b) with (x,y).

We assume a dipole of strength $\frac{1}{2K}$ in the point (a,b) and a dipole of equal strength in the point (a,-b). Both dipoles have their axis in the negative x-direction. The potentials of these dipoles are represented by:

$$\phi_{dip}(a,b) = \frac{1}{2K} \frac{(x-a)}{(x-a)^2+(y-b)^2}$$

$$\phi_{dip}(a,-b) = \frac{1}{2K} \frac{(x-a)}{(x-a)^2+(y+b)^2} \qquad (5.1.2.)$$

We now consider the potential:

$$\phi = \phi_{dip}(a,b) + \phi_{dip}(a,-b) + \phi_r \qquad (5.1.3.)$$

ϕ_r is a regular function, which we determine in such a way that ϕ satisfies the free-surface condition $K\phi+\phi_y = 0$. Substituting ϕ in this relation, we obtain for ϕ_r the condition:

$$K\phi_r + \phi_{r_y} = -\frac{(x-a)}{(x-a)^2+b^2} = -\int_0^\infty e^{-pb}\sin(x-a)p \, dp \quad (b>0) \qquad (5.1.4.)$$

since

$$\frac{x-a}{(x-a)^2+b^2}$$

is the Laplace transform of sin(x-a)p.

In an analogous manner as for the source potential in chapter 2 we find for ϕ_r the expression:

$$\phi_r = -\int_{M_2} \frac{e^{-p(y+b)}\sin p(x-a)}{K-p} \, dp \qquad (5.1.5.)$$

Here also the uniqueness of the regular function ϕ_r is furnished by the condition that for the limit case $|x| \to \infty$ ϕ_r has to represent the regular outgoing progressive wave (1.10). For this reason the contour $\underset{\frown}{\overset{1}{\longrightarrow}}$ is excluded (Fig. 2.3.). So finally we obtain for the potential of a dipole in the origin (a=b=0) the expression:

$$\phi = \frac{x}{K(x^2 + y^2)} - \int_{M_2} \frac{e^{-py}\sin px}{K-p}\, dp \qquad (5.1.6.)$$

which is constructed in such a way that it satisfies the free surface condition, represents for $|x| \to \infty$ a regular outgoing progressive wave (1.10) and is an asymmetric function.

In an analogous manner as ϕ in (2.22) we can reduce the second term of (5.1.6.) to:

$$\phi_r = \mp \int_0^{\infty} \frac{e^{\mp \beta x}\{K\cos\beta y + \beta\sin\beta y\}}{K^2 + \beta^2}\, d\beta \pm \pi e^{-Ky \mp iKx} \quad x \gtrless 0 \qquad (5.1.7.)$$

In this chapter we will use for the dipole potential Φ_{dip}, which satisfies both the free-surface condition and the radiation condition, the expression defined by:

$$\Phi_{dip} = \frac{gb}{\pi\sigma} \mathrm{Re}\left\{-i\phi e^{i\sigma t}\right\} = \frac{gb}{\pi\sigma}\left\{\phi_c \cos\sigma t + \phi_s \sin\sigma t\right\} \qquad (5.1.8.)$$

Comparing this with (5.1.6.) and (5.1.7.), we obtain:

$$\phi_c = -\pi e^{-Ky}\sin kx$$

$$\phi_s = \pm\pi e^{-Ky}\cos Kx \mp \int_0^{\infty} \frac{K\cos\beta y + \beta\sin\beta y}{\beta^2 + K^2} e^{\mp \beta x}\, d\beta + \frac{x}{K(x^2+y^2)} \qquad x \gtrless 0$$

The conjugate streamfunction ψ_{dip} of Φ_{dip} is given by:

$$\psi_{dip} = \frac{gb}{\pi\sigma}\{\psi_c \cos\sigma t + \psi_s \sin\sigma t\}$$

where:

$$\psi_c = \pi e^{-Ky}\cos Kx$$

$$\psi_s = \pi e^{-Ky}\sin K|x| - \int_0^\infty e^{-\beta|x|}\frac{\beta\cos\beta y - K\sin\beta y}{K^2+\beta^2}\,d\beta - \frac{y}{K(x^2+y^2)} \qquad (5.1.9.)$$

Finally we define the asymmetric multipole potentials

$$\varphi_{2m} = \frac{\sin(2m+1)\theta}{r^{2m+1}} + Ka\left\{\frac{\sin 2m\theta}{2mr^{2m}} + \sum_{n=0}^{N}\frac{(-1)^n a_{2n+1}(2n+1)\sin(2n+2m+2)\theta}{(2m+2n+2)r^{2m+2n+2}}\right\} \qquad (5.1.10)$$

m= 1,2,3,....

Substituting (5.1.10) in (4.2.3.) we see that φ_{2m} satisfies the free surface condition. The conjugate streamfunction ψ_{2m} of φ_{2m} is:

$$\psi_{2m} = \frac{\cos(2m+1)\theta}{r^{2m}} - Ka\left\{\frac{\cos 2m\theta}{2mr^{2m}} + \sum_{n=0}^{N}\frac{(-1)^n a_{2n+1}(2n+1)\cos(2m+2n+2)\theta}{(2m+2n+2)r^{2m+2n+2}}\right\} \qquad (5.1.11.)$$

m=1,2,3,...

Completely analogous to the problem of the heaving cylinder we determine a solution of the boundary value problem, where the cylinder is making a swaying- or rolling motion.

The velocity potential is composed of the dipole potential (5.1.8.) and a linear combination of multipole potentials (5.1.10.):

$$\Phi = \frac{gb}{\pi\sigma}\left[\left\{\phi_c + \sum_{m=1}^{\infty} p_{2m}\varphi_{2m}\right\}\cos\sigma t + \left\{\phi_s + \sum_{m=1}^{\infty} q_{2m}\varphi_{2m}\right\}\sin\sigma t\right] \qquad (5.1.12.)$$

For the conjugate streamfunction we find:

$$\Psi = \frac{gb}{\pi\sigma}\left[\left\{\psi_c + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}\right\}\cos\sigma t + \left\{\psi_s + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}\right\}\sin\sigma t\right] \qquad (5.1.13.)$$

5.2. Added mass and damping for swaying; added moment of inertia and damping for rolling produced by the swaying motion.

The boundary condition at the cylinder becomes in this case (Fig. 4.2.1.):

$$\frac{\partial\Phi}{\partial n} = \frac{dx}{dt}\sin\alpha \qquad (5.2.1.)$$

With (4.2.9.) this can be written as:

$$\frac{\partial\Psi}{\partial s} = \frac{dx}{dt}\,\frac{dy}{ds} \qquad (5.2.2.)$$

After integration this leads to:

$$\Psi(r=1,\theta) = \frac{dx}{dt}\,y\,(r=1,\theta) + C(t) \qquad (5.2.3.)$$

Substitution of $\theta = \frac{\pi}{2}$ gives: $C(t) = \Psi\,(1,\frac{\pi}{2})$.

Consequently:

$$\Psi(1,\theta) - \Psi(1,\frac{\pi}{2}) = \frac{dx}{dt}\;y(1,\theta) \qquad (5.2.4.)$$

Analogous to the procedure we followed in reducing (4.2.11.) to (4.2.13.) we can now eliminate $\frac{dx}{dt}$. The other method which we discussed in the remark at the end of section 4.2. is to substitute into the above mentioned form the expression $\frac{dx}{dt} = - x_a\sigma\,\sin(\sigma t + \gamma)$ (assuming that x is given by $x = x_a\cos(\sigma t + \gamma)$), so that an expression is obtained similar to (4.2.33.).
Here we apply the first method.
Substitution of $\theta = 0$ into (5.2.4.) gives:

$$\Psi(1,0) - \Psi(1,\frac{\pi}{2}) = \frac{dx}{dt}\,T \qquad (5.2.5.)$$

Eliminating $\frac{dx}{dt}$ from (5.2.4.) and (5.2.5.) leads to:

$$\frac{1}{y(1,\theta)}\left\{\Psi(1,\theta) - \Psi(1,\frac{\pi}{2})\right\} = \frac{1}{T}\left\{\Psi(1,0) - \Psi(1,\frac{\pi}{2})\right\}$$

Next, we substitute (5.1.13.) in this expression and equate successively the coefficients of $\cos\sigma t$ and $\sin\sigma t$.
This produces:

$$\left\{\psi_c(1,\theta) - \psi_c(1,\tfrac{\pi}{2})\right\} - \frac{y(1,\theta)}{T}\left\{\psi_c(1,0) - \psi_c(1,\tfrac{\pi}{2})\right\} = \sum_{m=1}^{\infty} p_{2m} f_{2m}$$

$$\left\{\psi_s(1,\theta) - \psi_s(1,\tfrac{\pi}{2})\right\} - \frac{y(1,\theta)}{T}\left\{\psi_s(1,0) - \psi_s(1,\tfrac{\pi}{2})\right\} = \sum_{m=1}^{\infty} q_{2m} f_{2m} \qquad (5.2.6.)$$

where:

$$f_{2m} = \frac{y(1,\theta)}{T}\left\{\psi_{2m}(1,0) - \psi_{2m}(1,\tfrac{\pi}{2})\right\} - \left\{\psi_{2m}(1,\theta) - \psi_{2m}(1,\tfrac{\pi}{2})\right\}$$

(5.2.6.) represents a set of equations from which p_{2m} and q_{2m} can be calculated in the usual way.
Analogous to (4.2.17.) we define:

$$\frac{dx}{dt} = \frac{gb}{\pi\sigma T}\left\{- A\cos\sigma t - B\sin\sigma t\right\} \qquad (5.2.7.)$$

so that:

$$\frac{d^2x}{dt^2} = \frac{gb}{\pi T}\left\{A\sin\sigma t - B\cos\sigma t\right\}$$

where according to (5.1.13.) and (5.2.5.):

$$A = \psi_c(1,\tfrac{\pi}{2}) - \psi_c(1,0) + \sum_{m=1}^{\infty} p_{2m}\left\{\psi_{2m}(1,\tfrac{\pi}{2}) - \psi_{2m}(1,0)\right\}$$

$$B = \psi_s(1,\tfrac{\pi}{2}) - \psi_s(1,0) + \sum_{m=1}^{\infty} q_{2m}\left\{\psi_{2m}(1,\tfrac{\pi}{2}) - \psi_{2m}(1,0)\right\} \qquad (5.2.8.)$$

Entirely equivalent to (4.2.15.) we define for the potential along the cylinder:

$$\Phi(1,\theta) = \frac{gb}{\pi\sigma}(M\sin\sigma t + N\cos\sigma t) \qquad (5.2.9.)$$

Then we find for the pressure along the cylinder:

$$p(1,\theta) = -\frac{\rho g b}{\pi}(M\cos\sigma t - N\sin\sigma t) \qquad (5.2.10.)$$

where on account of (5.1.12.);

$$M(1,\theta) = \emptyset_s(1,\theta) + \sum_{m=1}^{\infty} q_{2m}\varphi_{2m}(1,\theta)$$

$$N(1,\theta) = \emptyset_c(1,\theta) + \sum_{m=1}^{\infty} p_{2m}\varphi_{2m}(1,\theta)$$

Analogous to (4.2.20.) we find from (5.2.7.) and (5.2.10.);

$$p(1,\theta) = \rho T\frac{MB+NA}{A^2+B^2}\ddot{x} + \rho T\frac{MA-NB}{A^2+B^2}\sigma\dot{x} \qquad (5.2.11.)$$

The total horizontal hydrodynamic force becomes:

$$F_s = -\int_{S(0<\theta<\frac{\pi}{2})}\left\{p(1,\theta) - p(1,-\theta)\right\}\sin\alpha ds \qquad (5.2.12.)$$

From (4.2.9a), (4.1. 8) and (4.2.23a) it follows:

$$\sin\alpha ds = -\frac{\partial y}{\partial s}ds = -dy = -a\left\{-\sin\theta + \sum_{n=0}^{N}(-1)^n a_{2n+1}(2n+1)\sin(2n+1)\theta\right\}d\theta =$$

$$= \frac{B_o}{G} V(\theta)\,d\theta \qquad (5.2.13.)$$

where the function between parentheses is denoted by $V(\theta)$.
As the pressure is asymmetric in θ (5.2.12.) becomes:

$$F_s = -2B_o \int_0^{\frac{\pi}{2}} p(1,\theta)\frac{V(\theta)}{G}d\theta \qquad (5.2.14.)$$

Substitution of (5.2.11) gives:

$$F_s = -2\rho T\,B_o\frac{M_oB+N_oA}{A^2+B^2}\ddot{x} - 2\rho T\,B_o\frac{M_oA-N_oB}{A^2+B^2}\sigma\dot{x}$$

where:

$$N_o = \int_0^{\frac{\pi}{2}} N(1,\theta)\,\frac{V(\theta)}{G}\,d\theta$$

$$M_o = \int_0^{\frac{\pi}{2}} M(1,\theta)\,\frac{V(\theta)}{G}\,d\theta$$

When we define the relation between the swaying force and the added mass M_s and damping N_s for the swaying motion by

$$F_s = - M_s\ddot{x} - N_s\dot{x}$$

we find for the added mass per unit length:

$$M_s = 2\,\rho T\,B_0\,\frac{M_o\hat{\ } B+N_o A}{A^2 + B^2} \qquad (5.2.15.)$$

and for the damping per unit length:

$$N_s = 2\,\rho T\,B_0\,\frac{M_o A-N_o B}{A^2 + B^2}\,\sigma \qquad (5.2.16.)$$

From fig. 4.2.1. we see that on account of the asymmetry of $p(1,\theta)$ in θ the moment on the cylinder produced by the swaying motion (clockwise is positive) is expressed by:

$$M_{RS} = 2\int_{S(0<\theta<\frac{\pi}{2})} \left\{p\sin\alpha . y - p\cos\alpha . x\right\} ds$$

Combining this with (4.2.9.) yields:

$$M_{RS} = -2\int_0^{\frac{\pi}{2}} p\left(x\,\frac{\partial x}{\partial \theta} + y\,\frac{\partial y}{\partial \theta}\right)d\theta \qquad (5.2.17.)$$

Substitution of (5.2.10) gives:

$$M_{RS} = \frac{B_0{}^2 2\rho g b}{\pi}\left\{ -X_R \sin\sigma t + Y_R \cos\sigma t\right\} \qquad (5.2.18.)$$

where:

$$X_R = \frac{1}{B_0^2} \int_0^{\frac{\pi}{2}} N(1,\theta)(x\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta})d\theta$$

$$Y_R = \frac{1}{B_0^2} \int_0^{\frac{\pi}{2}} M(1,\theta)(x\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta})d\theta$$

We resolve the moment into a component in phase with the acceleration and a component in phase with the velocity according to:

$$M_{RS} = I_{RS}\left(-\frac{d^2x}{dt^2}\right) + N_{RS}\left(-\frac{dx}{dt}\right) \qquad (5.2.19.)$$

N_{RS} and I_{RS} represent the damping and added moment of inertia for the rolling motion produced by swaying.

From (5.2.7.) and (5.2.18.) it follows:

$$M_{RS} = \frac{-2\rho g b B_0^2}{\pi} \cdot \frac{BY_R+AX_R}{A^2+B^2}(A\sin\sigma t - B\cos\sigma t) - \frac{2\rho g b B_0^2}{\pi} \cdot \frac{AY_R-BX_R}{A^2+B^2}(-A\cos\sigma t - B\sin\sigma t)$$

$$= -2\rho\, TB_0^2\, \frac{BY_R+AX_R}{A^2+B^2}\,\ddot{x} - 2\,\rho\sigma TB_0^2\,\frac{AY_R-BX_R}{A^2+B^2}\,\dot{x}$$

hence:

$$I_{RS} = 2\rho TB_0^2\,\frac{BY_R+AX_R}{A^2+B^2}$$

$$N_{RS} = 2\rho\sigma TB_0^2\,\frac{AY_R-BX_R}{A^2+B^2} \qquad (5.2.20.)$$

For the ratio of the wave amplitude at infinite distance from the cylinder and the amplitude of the forced oscillation we obtain:

$$\frac{b}{x_a} = \frac{\pi\, KT}{\sqrt{A^2+B^2}}$$

Between the coefficients M_o, N_o, A and B, the relation $M_oA-N_oB = \frac{\pi^2}{2}\frac{T}{B_o}$ is valid.

We derive now expressions [9] for the above mentioned quantities M_s, N_s and I_{RS}, N_{RS} for the special case of a Lewis-form.
The multipole potential obtains the form:

$$\varphi_{2m}(\alpha,\theta) = \left[e^{-(2m+1)\alpha} \sin(2m+1)\theta + \frac{\xi_o}{1+a_1+a_3} \left\{ \frac{e^{-2m\alpha}}{2m} \sin 2m\theta + \right.\right.$$

$$\left.\left. + \frac{a_1 e^{-(2m+2)\alpha}}{2m+2} \sin(2m+2)\theta + \frac{3a_3}{2m+4} e^{-(2m+4)\alpha} \sin(2m+4)\theta \right\}\right] \qquad (5.2.21.)$$

$m = 1,2,\ldots.$

The streamfunction becomes:

$$\psi_{2m}(\alpha,\theta) = \left[e^{-(2m+1)\alpha} \cos(2m+1)\theta - \frac{\xi_o}{1+a_1+a_3} \left\{ \frac{e^{-2m\alpha}}{2m} \cos 2m\theta + \right.\right.$$

$$\left.\left. + \frac{a_1 e^{-(2m+2)\alpha}}{2m+2} \cos(2m+2)\theta - \frac{3a_3}{2m+4} e^{-(2m+4)\alpha} \cos(2m+4)\theta \right\}\right] \qquad (5.2.22.)$$

$m = 1,2,\ldots$

while (5.2.4.) has the form:

$$\left[\psi_c(\alpha=0,\theta) - \psi_c(\alpha=0,\tfrac{\pi}{2}) \right] \cos\sigma t + \left[\psi_s(0,\theta) - \psi_s(0,\tfrac{\pi}{2}) \right] \sin\sigma t +$$

$$\cos\sigma t \sum_{m=1}^{\infty} p_{2m} \left[-\cos(2m+1)\theta - \frac{\xi_o}{1+a_1+a_3} \left\{ \frac{\cos 2m\theta}{2m} + \frac{a_1\cos(2m+2)\theta}{2m+2} - \frac{3a_3\cos(2m+4)\theta}{2m+4} \right\} \right.$$

$$\left. - \frac{\xi_o(-1)^{m+1}}{1+a_1+a_3} \left(\frac{1}{2m} \; \frac{a_1}{2m+2} - \frac{3a_3}{2m+4} \right) \right] + \sin\sigma t \sum_{m=1}^{\infty} q_{2m} \left[-\cos(2m+1)\theta - \right.$$

$$- \frac{\xi_o}{1+a_1+a_3} \left\{ \frac{\cos 2m\theta}{2m} + \frac{a_1\cos(2m+2)\theta}{2m+2} - \frac{3a_3\cos(2m+4)\theta}{2m+4} \right\} - \frac{\xi_o(-1)^{m+1}}{1+a_1+a_3} \cdot$$

$$\left. \cdot \left(\frac{1}{2m} - \frac{a_1}{2m+2} - \frac{3a_3}{2m+4} \right) \right] = \left(\frac{\sigma\pi}{g\eta} \right) \frac{dx}{dt} M \left\{ (1-a_1)\cos\theta + a_3\cos 3\theta \right\} \qquad (5.2.23.)$$

We now eliminate $\frac{dx}{dt}$ in the usual way.
However here, in accordance with the method followed by Tasai in his publication (see remark at the end of §4.2.), we shall substitute an expression for $\frac{dx}{dt}$ with the structure of (5.2.7.).
We define:

$$x = x_a \cos(\sigma t + \gamma)$$

then:

$$\frac{dx}{dt} = - x_a \, \sigma \sin(\sigma t + \gamma) \qquad (5.2.24.)$$

The righthand side of (5.2.23.) can now be written as:

$$\left(\frac{\sigma\pi}{gb}\right) \frac{dx}{dt} M \left\{ (1-a_1)\cos\theta + a_3\cos 3\theta \right\} = h(\theta) \, (p_o \cos\sigma t + q_o \sin\sigma t) \qquad (5.2.25.)$$

where:

$$h(\theta) = \frac{\left\{(1-a_1)\cos\theta + a_3\cos 3\theta\right\}}{1+a_1+a_3} \qquad (5.2.26.)$$

$$p_o = - \frac{\pi x_a}{b} \xi_o \sin\gamma \quad ; \quad q_o = - \frac{\pi x_a}{b} \xi_o \cos\gamma$$

After equating in (5.2.25.) successively the coefficients of $\cos\sigma t$ and $\sin\sigma t$, we obtain a system of linear equations for the coefficients p_{2m} and q_{2m}:

$$\psi_c(0,\theta) - \psi_c(0, \tfrac{\pi}{2}) = \sum_{m=0}^{\infty} f_{2m}(\theta) \, p_{2m}$$

$$\psi_s(0,\theta) - \psi_s(0, \tfrac{\pi}{2}) = \sum_{m=0}^{\infty} f_{2m}(\theta) \, q_{2m}$$

where:

$$f_o(\theta) = h(\theta)$$

$$f_{2m}(\theta) = \cos(2m+1)\theta + \frac{\xi_o}{1+a_1+a_3}\left\{\frac{\cos 2m\theta}{2m} + \frac{a_1\cos(2m+2)\theta}{2m+2} - \frac{3a_3\cos(2m+4)\theta}{2m+4}\right\} + \frac{\xi_o(-1)^{m+1}}{1+a_1+a_3}\left\{\frac{1}{2m} - \frac{a_1}{2m+2} - \frac{3a_3}{2m+4}\right\} \qquad (5.2.28.)$$

$m = 1,2.....$

The values for the coefficients p_{2m} and q_{2m} can now be determined in the usual way.

From (5.2.25.) it follows:

$$\frac{dx}{dt} = \frac{gb}{\pi\sigma B_0}\left\{p_o \cos\sigma t + q_o \sin\sigma t\right\}$$

Hence: (5.2.29.)

$$\frac{d^2x}{dt^2} = \frac{gb}{\pi B_0}\left\{q_o \cos\sigma t - p_o \sin\sigma t\right\}$$

According to (5.2.14.) we obtain for the hydrodynamic force in the x-direction:

$$F = 2\,\rho B_0\left(\frac{gb}{\pi}\right)\left\{-N_o \sin\sigma t + M_o \cos\sigma t\right\} \qquad (5.2.30)$$

where:

$$N_o = -\int_0^{\frac{\pi}{2}} \phi(0,\theta)\,\frac{(1-a_1)\sin\theta + 3a_3\sin 3\theta}{1+a_1+a_3}\,d\theta - \frac{3a_3}{1+a_1+a_3}\cdot\frac{\pi}{4}\,p_2 - \sum_{m=1}^{\infty} p_{2m}\,\frac{\xi_o(-1)^{m-1}}{(1+a_1+a_3)^2}\left[\left\{\frac{1}{4m^2-1} - \frac{a_1}{(2m+2)^2-1} - \frac{3a_3}{(2m+4)^2-1}\right\}(1-a_1) + 3a_3\left\{\frac{-1}{4m^2-9} + \frac{a_1}{(2m+2)^2-9} + \frac{3a_3}{(2m+4)^2-9}\right\}\right]$$

M_o is obtained by replacing in the above mentioned form $\emptyset_c$ by $\emptyset_s$ and p_{2m} by q_{2m}. From (5.2.29.) and (5.2.30.) we can derive added mass and damping in the usual way:

$$M_S = 2\,\rho B_0{}^2 \;\frac{N_o p_o + M_o q_o}{p_o{}^2 + q_o{}^2}$$

(5.2.31.)

$$N_S = 2\rho\sigma E_0{}^2 \;\frac{M_o p_o - N_o q_o}{p_o{}^2 + q_o{}^2}$$

Substitution of (5.2.10.) and (4.3.2.) into (5.2.17.), where in the expression for $M(1,\theta)$ and $N(1,\theta)$ we have to substitute for φ_{2m} the expression (5.2.21.), yields the rolling moment produced by the swaying motion:

$$M_{RS} = \frac{4\rho B_0{}^2 gb}{\pi}\left\{ X_R \sin\sigma t - Y_R \cos\sigma t \right\} \tag{5.2.32.}$$

where:

$$X_R = \int_0^{\frac{\pi}{2}} \frac{\emptyset_c(0,\theta)}{(1+a_1+a_3)^2}\left\{a_1(1+a_3)\sin 2\theta - 2a_3\sin 4\theta\right\}d\theta +$$

$$+ \frac{\pi\xi_o(a_1p_2 - a_3p_4)}{8(1+a_1+a_3)^2} + \sum_{m=1}^{\infty} \frac{p_{2m}(-1)^{m+1}}{(1+a_1+a_3)^2}\left\{\frac{2a_1(1+a_3)}{(2m+1)^2-4} + \frac{8a_3}{(2m+1)^2-16}\right\}$$

Y_R is obtained by replacing in above expression $\emptyset_c$ by $\emptyset_s$ and p_{2m} by q_{2m}. The added moment of inertia and the damping moment for the rolling motion produced by swaying now follow from(5.2.29.), (5.2.32.) and (5.2.19.):

$$I_{RS} = 4\, B_o{}^3 \frac{p_o X_R + q_o Y_R}{p_o^2 + q_o^2}$$

$$N_{RS} = 4\, B_o{}^3 \frac{p_o Y_R - q_o X_R}{p_o^2 + q_o^2} \qquad (5.2.33.)$$

5.3. Added moment of inertia and damping for rolling; added mass and damping for swaying produced by the rolling motion.

When the cylinder is carrying out an harmonic rolling motion about the origin, represented by $\vartheta = \vartheta_a \cos(\sigma t + \gamma)$, then the boundary condition at the cylinder has the form (Fig. 5.3.1.):

$$\frac{\partial \Phi}{\partial n} = R\frac{d\vartheta}{dt} \sin\varphi = R.\ \frac{d\vartheta}{dt} \cdot \frac{dR}{ds} \qquad (5.3.1.)$$

Fig. 5.3.1.

As can be seen from Fig. 5.3.1. φ is the angle between the tangent on the contour and the velocity along the surface, ϑ the rolling angle (positive in clockwise direction) and R is the distance between the origin and a point on the surface.

Combining (5.3.1.) with (4.2.9.) yields:

$$-\frac{\partial\Psi}{\partial s} = \frac{d\vartheta}{dt}\ \frac{d}{ds}\left\{\tfrac{1}{2}(x^2(1,\theta) + y^2(1,\theta))\right\} \qquad (5.3.2.)$$

After integration this is reduced to:

$$\Psi(1,\theta) = -\tfrac{1}{2}\left(\frac{d\vartheta}{dt}\right)\left\{x^2(1,\theta)+ y^2(1,\theta)\right\} + C(t) \qquad (5.3.3.)$$

Substitution of $\theta = \frac{\pi}{2}$ gives:

$$C(t) = \Psi(1,\tfrac{\pi}{2}) + \tfrac{1}{2}\left(\frac{d\vartheta}{dt}\right)B_o{}^2$$

So we obtain for the streamfunction the expression:

$$\Psi(1,\theta) - \Psi(1,\tfrac{\pi}{2}) = -\tfrac{1}{2}\,\frac{d\vartheta}{dt}\left\{x^2(1,\theta) + y^2(1,\theta) - B_o{}^2\right\} \qquad (5.3.4.)$$

For a change we shall not eliminate $\frac{d\vartheta}{dt}$ but we substitute for this quantity $\frac{d\vartheta}{dt} = -\vartheta_a\sigma\sin(\sigma t + \gamma)$ (see remark at the end of section 4.2.). Consequently:

$$\Psi(1,\theta) - \Psi(1,\tfrac{\pi}{2}) = \tfrac{1}{2}\vartheta_a\sigma\sin(\sigma t + \gamma)\left\{x^2(1,\theta) + y^2(1,\theta) - B_o{}^2\right\}$$

or:

$$\frac{\pi\sigma}{gb}\left\{\Psi(1,\theta) - \Psi(1,\tfrac{\pi}{2})\right\} = \frac{\pi\sigma}{gb}\ \tfrac{1}{2}\vartheta_a\sigma\left\{x^2(1,\theta)+y^2(1,\theta) - B_o{}^2\right\} . \sin(\sigma t + \gamma) \qquad (5.3.5.)$$

The righthandside of the above mentioned form is written as:

$$g(\theta)\ (p_o\cos\sigma t + q_o\sin\sigma t)$$

where:

$$g(\theta) = \frac{x^2(1,\theta)+y^2(1,\theta)-B_o{}^2}{B_o{}^2}$$

$$p_o = \frac{\pi\vartheta_a K B_o{}^2}{2b}\ \sin\gamma \qquad (5.3.6.)$$

$$q_o = \frac{\pi\vartheta_a K B_o{}^2}{2b}\ \cos\gamma$$

Substituting (5.1.13.) into (5.3.5.) and equating successively the coefficients of $\cos\sigma t$ and $\sin\sigma t$, we obtain a set of linear equations for p_{2m} and q_{2m}.

$$\psi_c(1,\theta) - \psi_c(1,\tfrac{\pi}{2}) = \sum_{m=0}^{\infty} p_{2m} f_{2m}(\theta)$$

$$\psi_s(1,\theta) - \psi_s(1,\tfrac{\pi}{2}) = \sum_{m=0}^{\infty} q_{2m} f_{2m}(\theta) \qquad (5.3.7.)$$

where:

$$f_o = g(\theta) = \frac{x^2(1,\theta)+y^2(1,\theta)-B_o^2}{B_o^2}$$

$$f_{2m} = \psi_{2m}(1,\tfrac{\pi}{2}) - \psi_{2m}(1,\theta)$$

$m \neq 0$

From (5.3.6.) it follows:

$$\sqrt{p_o^2 + q_o^2} = \frac{\pi \vartheta_a K B_o^2}{2b}$$

So for the ratio: $\frac{\text{wave amplitude at infinity}}{\text{oscillation amplitude of the cylinder}}$ is found:

$$\frac{b}{\vartheta_a} = \frac{\pi K B_o^2}{2\sqrt{p_o^2 + q_o^2}} \qquad (5.3.8.)$$

The hydrodynamic moment becomes:

$$M_R = -2 \int_o^{\frac{\pi}{2}} p\left(x\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta}\right) d\theta$$

which can be reduced to:

$$M_R = \frac{-2\rho g b B_o^2}{\pi} \left\{ X_R \sin\sigma t - Y_R \cos\sigma t \right\} \qquad (5.3.9.)$$

where X_R and Y_R are defined in the same way as in 5.2. for the swaying motion. By means of (5.3.6.) we can write $\frac{d\vartheta}{dt} = -\vartheta_a \sigma \sin(\sigma t + \gamma)$ as:

$$\frac{d\vartheta}{dt} = \vartheta_a \sigma \left(\sin\sigma t \cos\gamma + \cos\sigma t \sin\gamma\right)$$

$$= \frac{2bg}{\pi\sigma B_o^2}\left(-q_o \sin\sigma t - p_o \cos\sigma t\right) \qquad (5.3.10.)$$

The acceleration becomes:

$$\frac{d^2\vartheta}{dt^2} = \frac{2bg}{\pi B_o^2}\left(-q_o \cos\sigma t + p_o \sin\sigma t\right)$$

We now resolve M_R into a component in phase with the velocity and a component in phase with the acceleration. From (5.3.9.) and (5.3.10.) it follows:

$$M_R = -\frac{2\rho g b B_o^2}{\pi} \frac{Y_R q_o + p_o X_R}{p_o^2 + q_o^2}\left(-q_o \cos\sigma t + p_o \sin\sigma t\right) - \frac{2\rho g b B_o^2}{\pi} \frac{-q_o X_R + p_o Y_R}{p_o^2 + q_o^2}\left(-q_o \sin\sigma t - p_o \cos\sigma t\right) \qquad (5.3.11.)$$

By means of (5.3.10.) we may write (5.3.11.) as:

$$M_R = -\rho B_o^4 \frac{Y_R q_o + p_o X_R}{p_o^2 + q_o^2}\ddot{\vartheta} - \rho\sigma B_o^4 \frac{p_o Y_R - q_o X_R}{p_o^2 + q_o^2}\dot{\vartheta} \qquad (5.3.12.)$$

By defining the hydrodynamic moment by $M_R = -I_R\ddot{\vartheta} - N_R\dot{\vartheta}$ we find for the added moment of inertia:

$$I_R = \rho B_o^4 \frac{p_o X_R + Y_R q_o}{p_o^2 + q_o^2}$$

and for the damping:

$$N_R = \rho\sigma B_o^4 \frac{p_o Y_R - q_o X_R}{p_o^2 + q_o^2}$$

where X_R and Y_R are defined analogous to the similar constants in (5.2.18.). However, for the coefficients p_{2m} and q_{2m}, which are found in the expressions for M and N, we substitute the values, which satisfy the set of equations (5.3.7.).

The swaying force produced by the rolling motion is calculated with:

$$F_{SR} = -2 \int_0^{\frac{\pi}{2}} p \sin\alpha ds \qquad (5.3.14.)$$

Substituting (5.2.13.) we obtain:

$$F_{SR} = -2B_0 \int_0^{\frac{\pi}{2}} \frac{p(1,\theta)V(\theta)}{G} d\theta \qquad (5.3.15.)$$

Next, we substitute for the pressure p expression (5.2.10.) where for the coefficients p_{2m} and q_{2m} in the expressions for M and N the values are substituted, which satisfy the set of equations (5.3.7.).

$$F_{SR} = \frac{2B_0\rho g b}{\pi}(M_o \cos\sigma t - N_o \sin\sigma t) \qquad (5.3.16.)$$

where:

$$M_o = \int_0^{\frac{\pi}{2}} M(1,\theta) \frac{V(\theta)}{G} d\theta \text{ and } N_o = \int_0^{\frac{\pi}{2}} N(1,\theta) \frac{V(\theta)}{G} d\theta$$

By means of (5.3.10.) the expression (5.3.16.) is written as:

$$F_{SR} = -\rho B_0^3 \frac{M_o q_o + N_o p_o}{p_o^2 + q_o^2} \ddot{\vartheta} - \rho B_0^3 \sigma \frac{M_o p_o - N_o q_o}{p_o^2 + q_o^2} \dot{\vartheta} \qquad (5.3.17.)$$

The swaying force produced by the rolling motion is defined by:

$$F_{SR} = M_{SR}\left(-\frac{d^2\vartheta}{dt^2}\right) + N_{SR}\left(-\frac{d\vartheta}{dt}\right) \qquad (5.3.18.)$$

Comparing (5.3.17.) and (5.3.18.) we obtain the added mass and damping for swaying produced by the rolling motion:

$$M_{SR} = \rho B_0^3 \frac{M_o q_o + N_o p_o}{p_o^2 + q_o^2}$$

$$(5.3.19.)$$

$$N_{SR} = \rho B_0^3 \sigma \frac{M_o p_o - N_o q_o}{p_o^2 + q_o^2}$$

Finally by equating the radiated energy with the work done by the cylinder, we obtain:

$$p_o Y_R - q_o X_R = \frac{\pi^2}{8} \qquad (5,3,20.)$$

Tasai has carried out these calculations for a Lewis-form [9].
His results can be derived from the above mentioned formulas by substituting for $\emptyset_{2m}$ and ψ_{2m} respectively (5.2.21.) and (5.2.22.).

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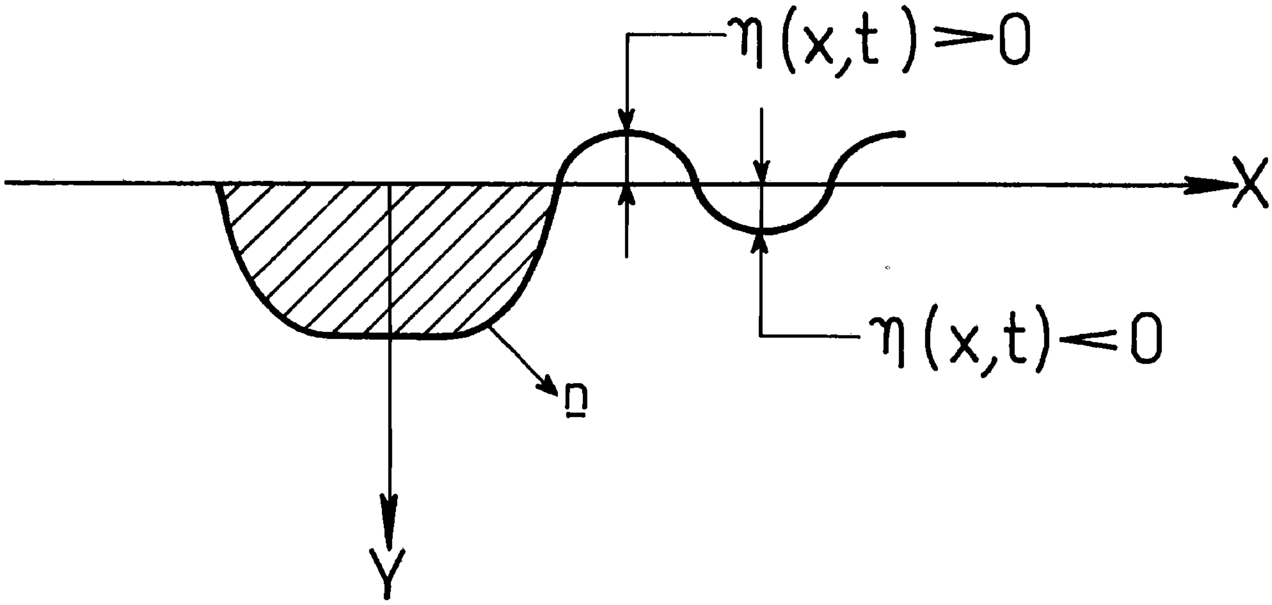

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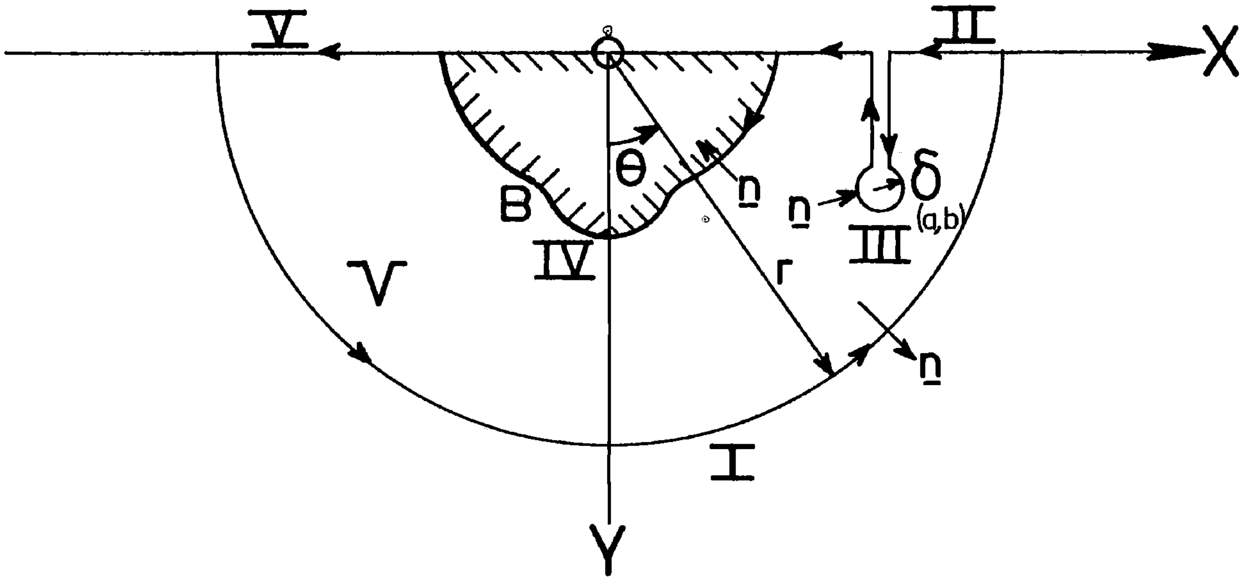

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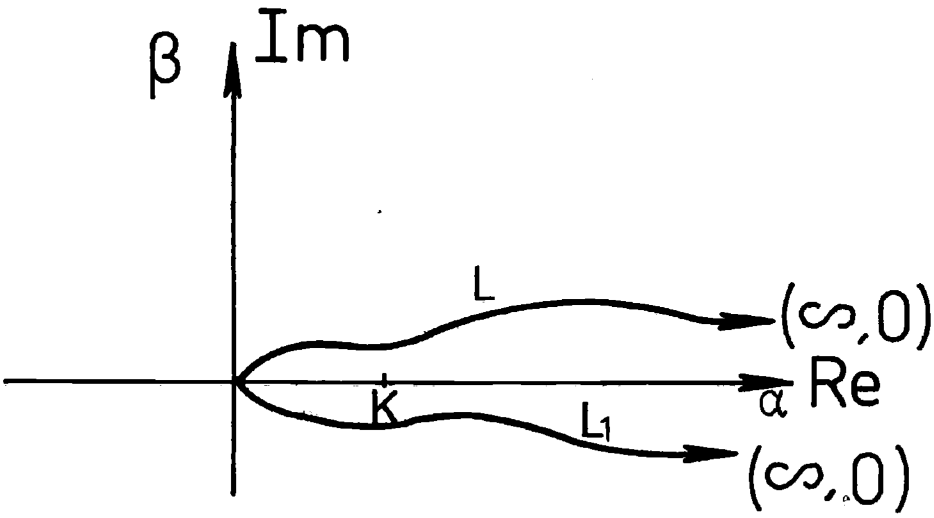

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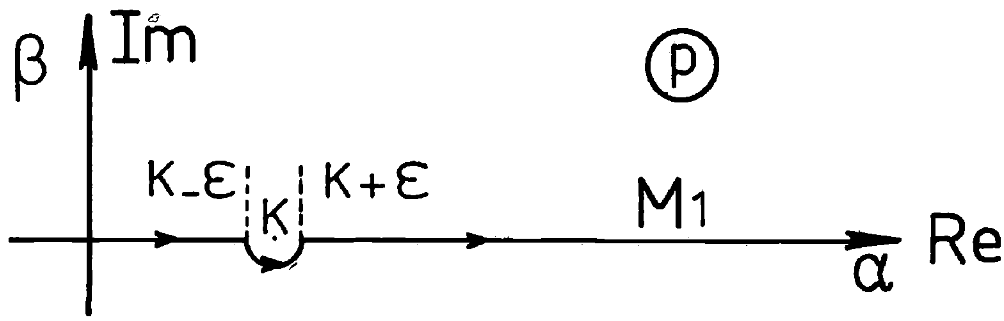

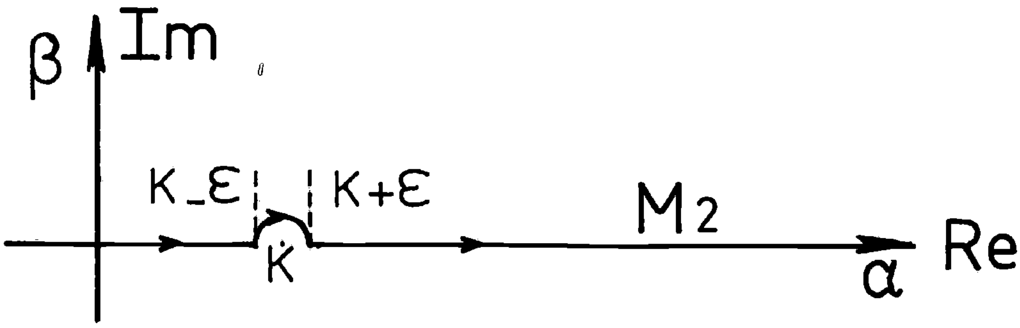

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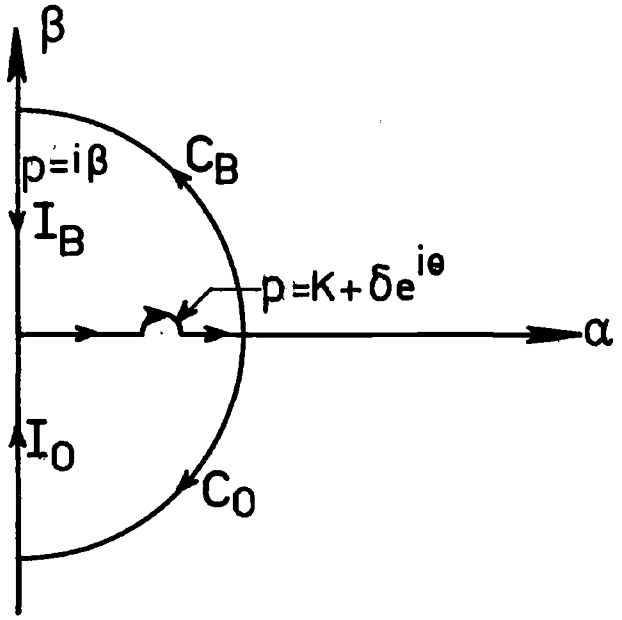

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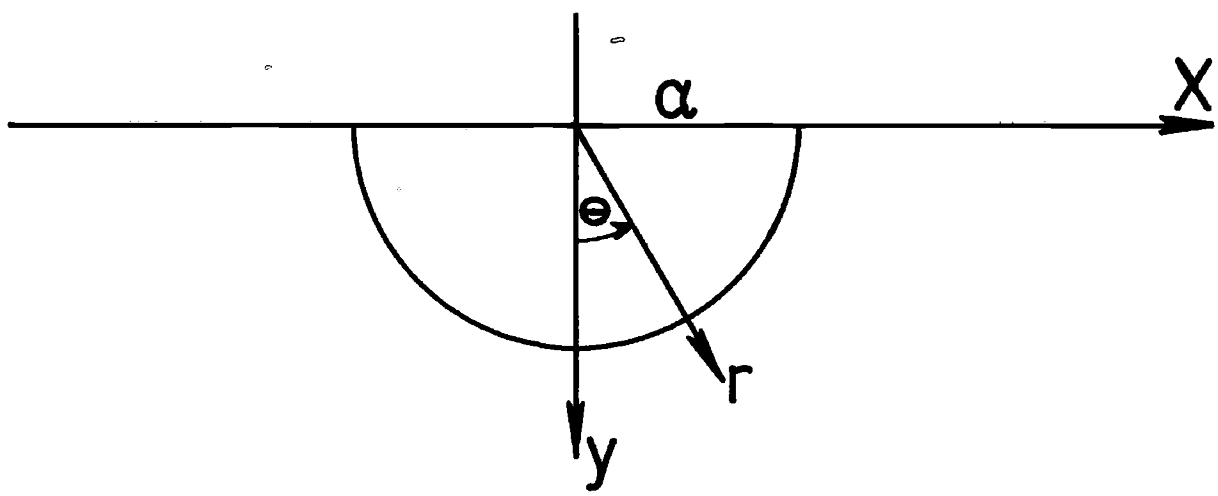

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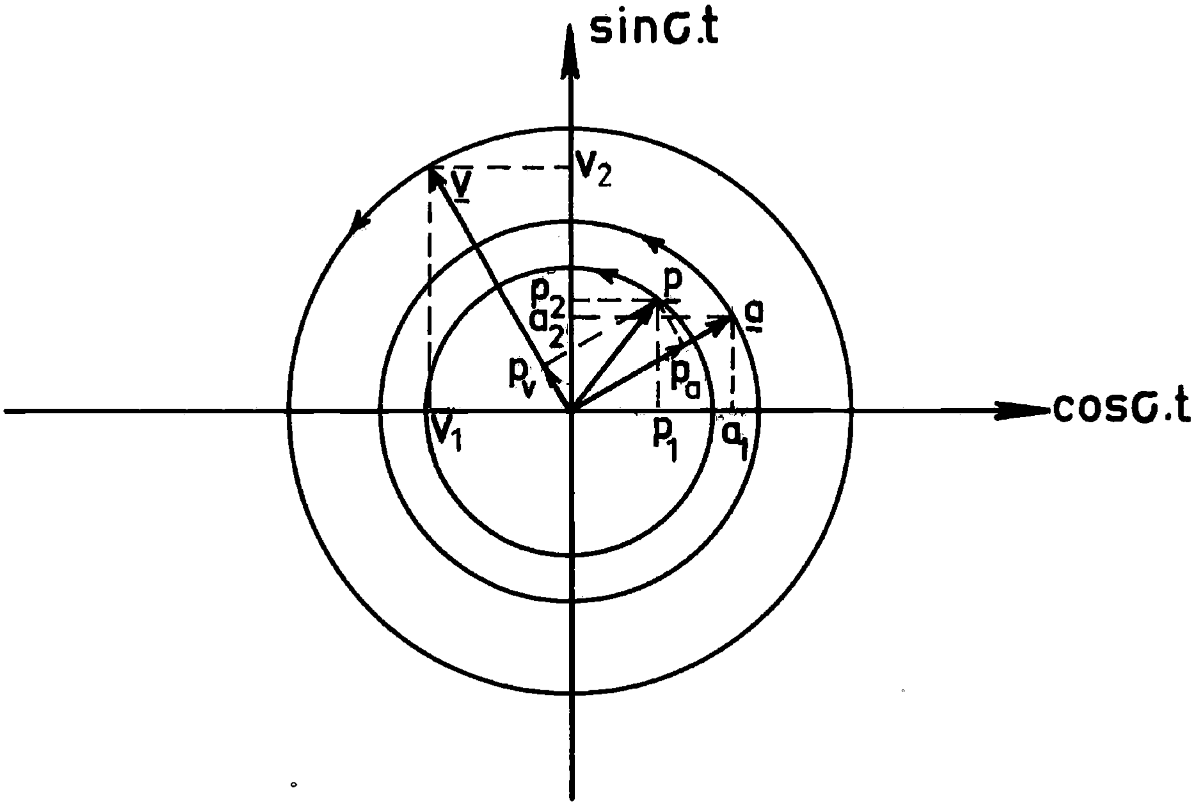
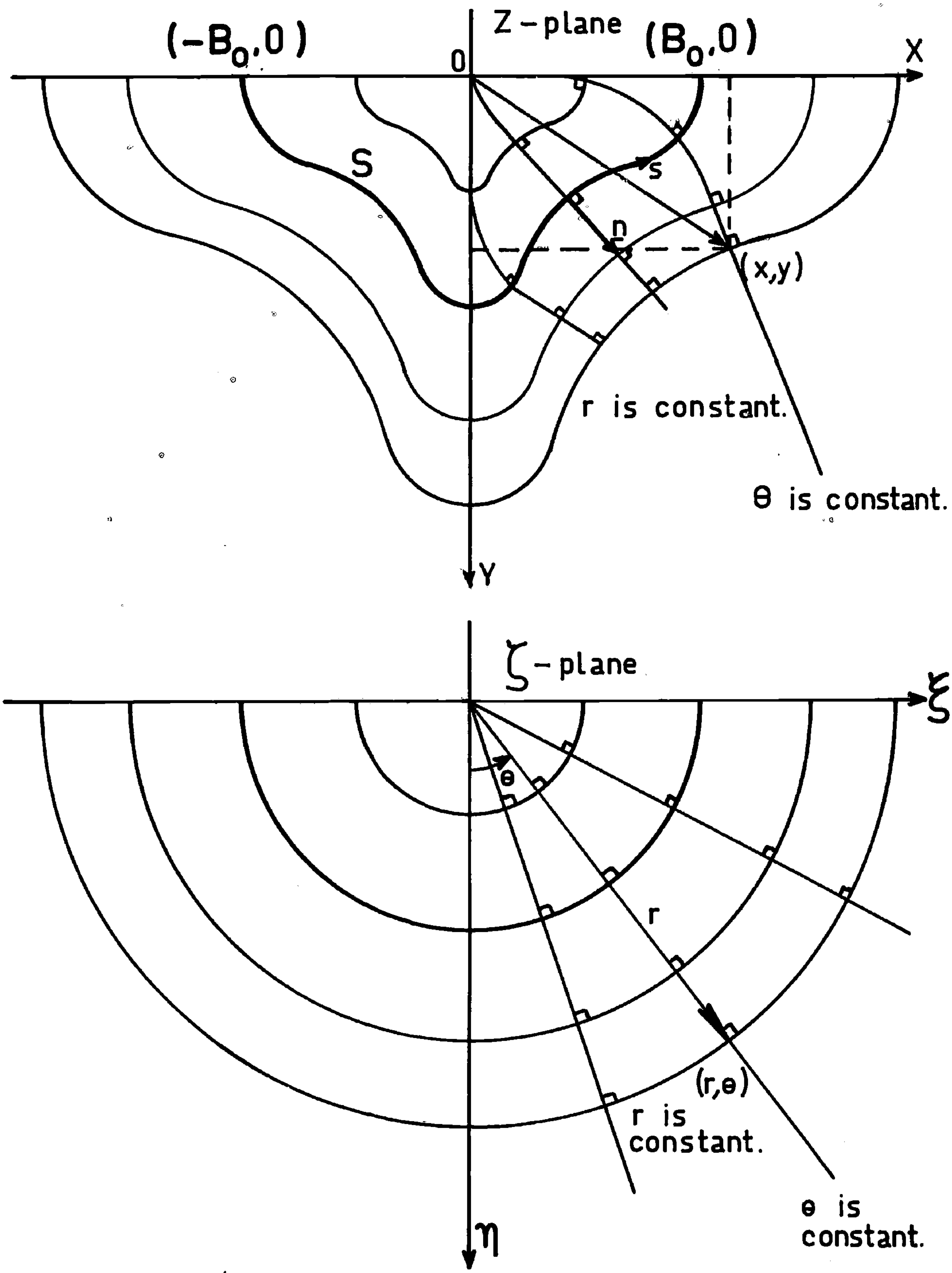


fig 3.2


Z-plane
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(B0,0)
0
X
S
s
n
(x,y)
r is constant.
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r is constant.
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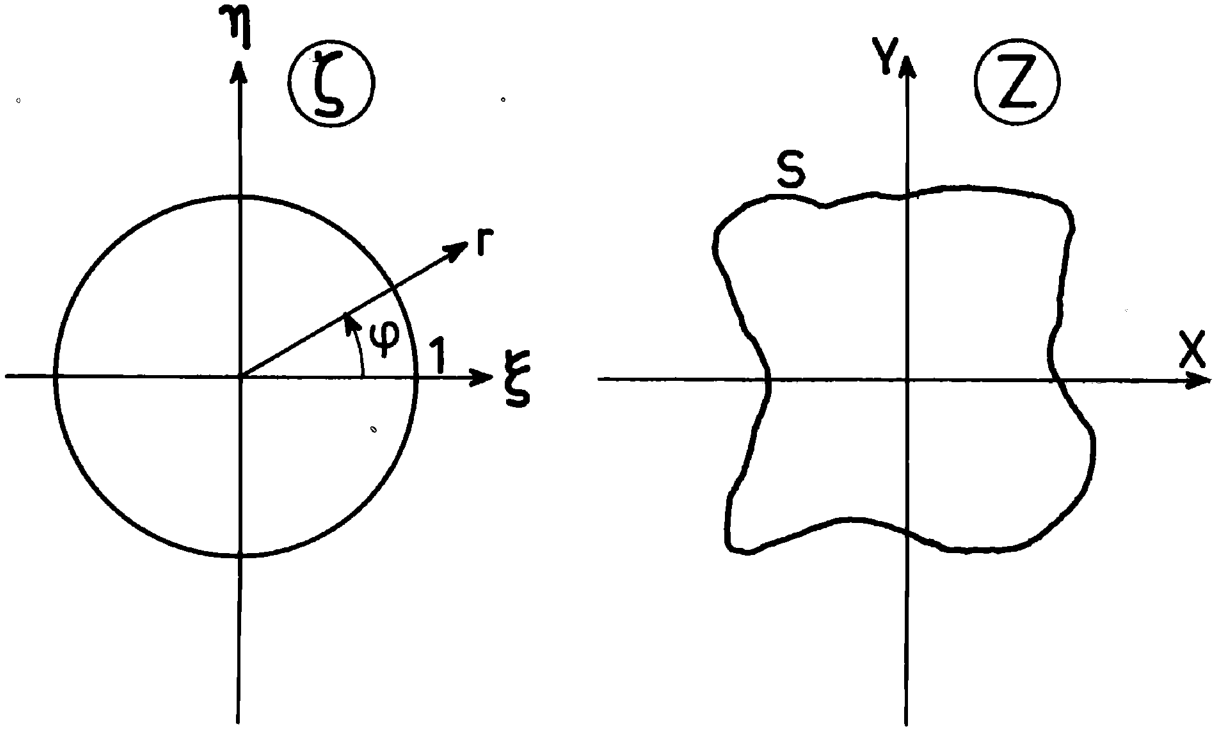
η

fig 4.1.1.



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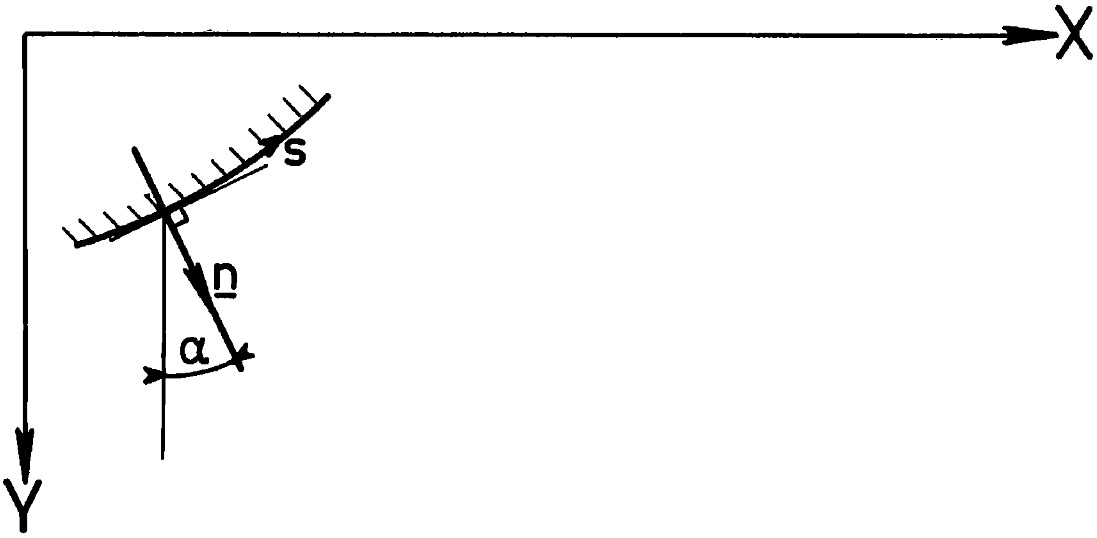

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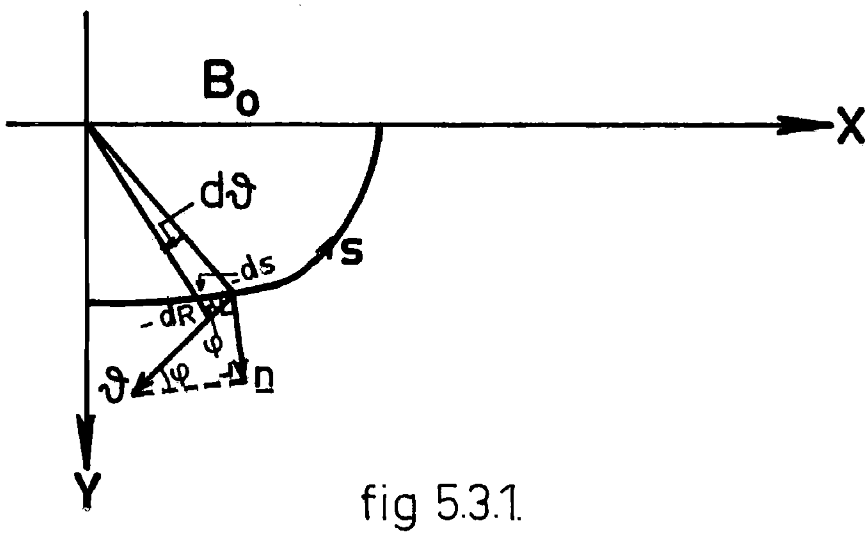

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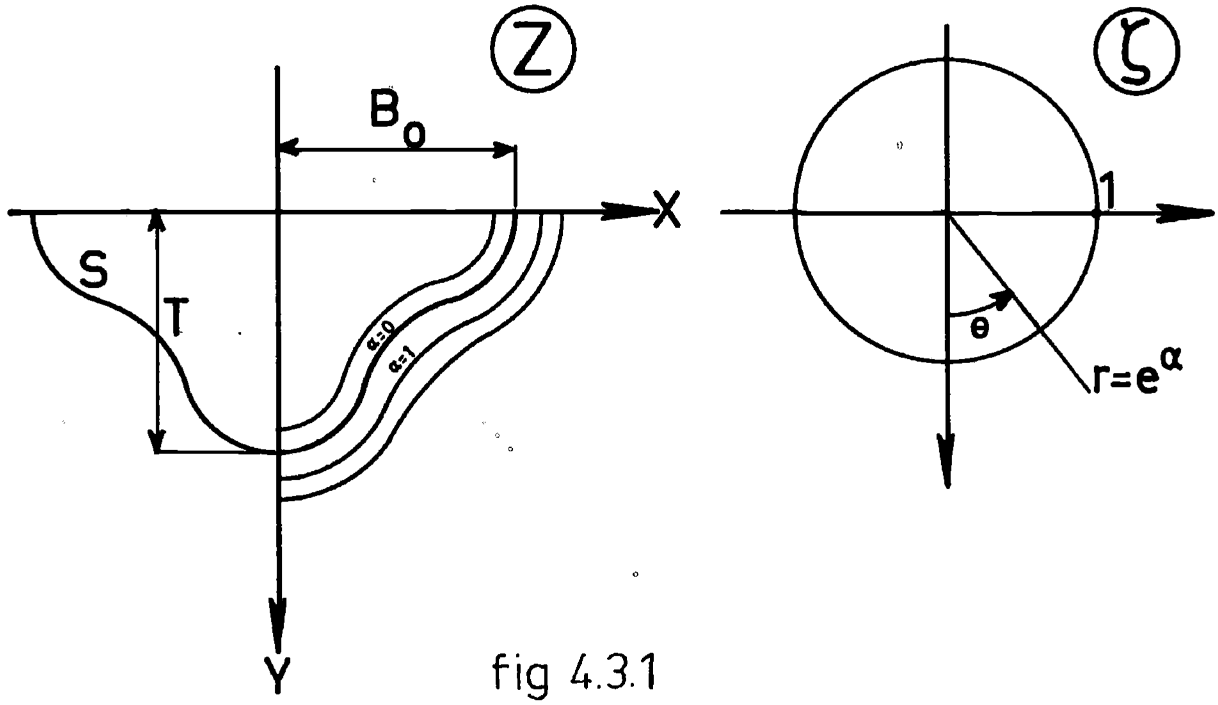

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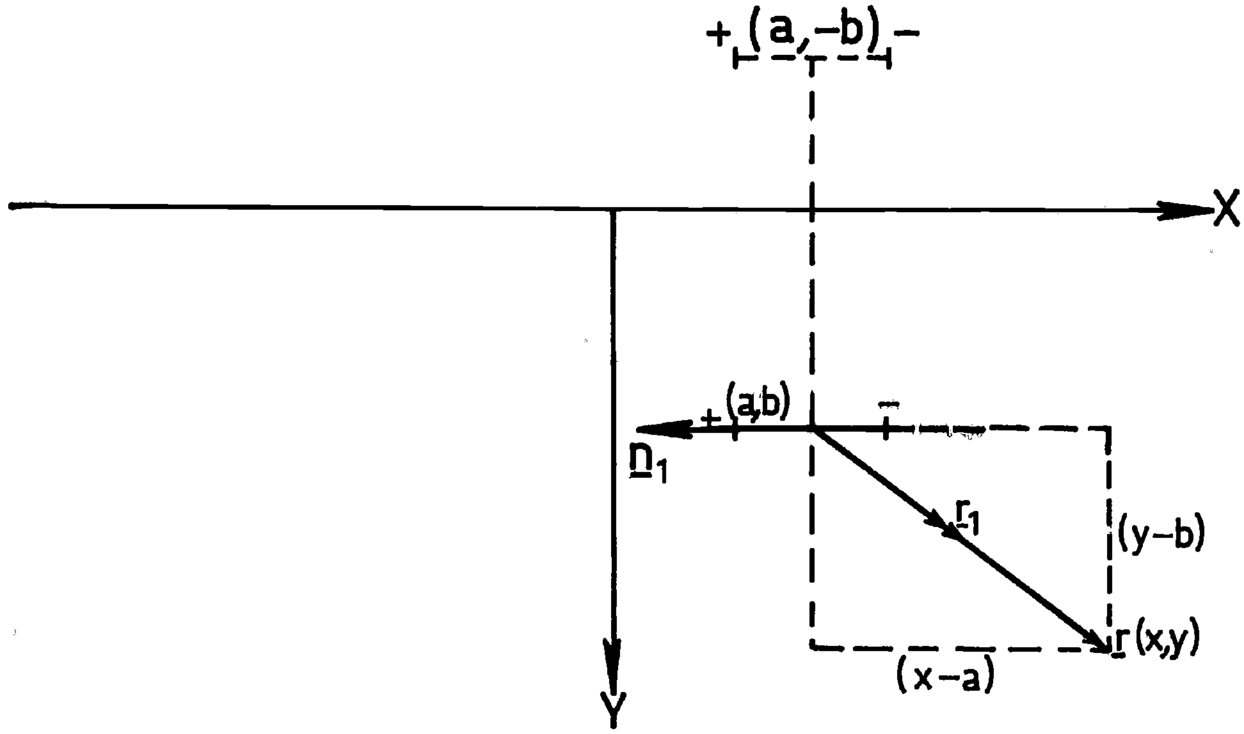

fig 5.1.1.

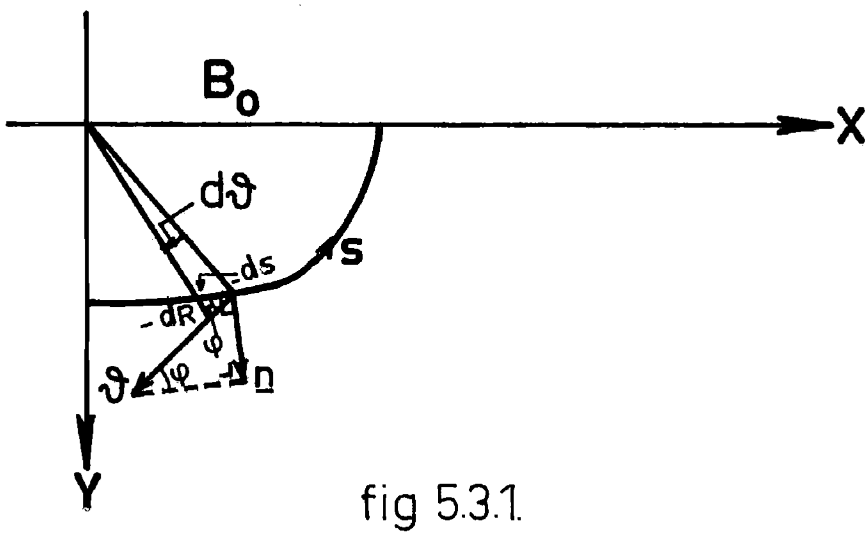

fig 5.3.1.

Rep. 174-A.

REPORT No. 145 S
(SH/–)

June 1973

NEDERLANDS SCHEEPSSTUDIECENTRUM TNO
NETHERLANDS SHIP RESEARCH CENTRE TNO
SHIPBUILDING DEPARTMENT LEEGHWATERSTRAAT 5, DELFT

★

COMPUTATION OF THE HYDRODYNAMIC COEFFICIENTS OF OSCILLATING CYLINDERS

(BEREKENING VAN DE HYDRODYNAMISCHE COEFFICIENTEN VAN OSCILLERENDE CILINDERS)

by

DR. IR. B. DE JONG

ALLEEN VOOR REPRODUKTIE

Issued by the Council

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DR. IR. B. DE JONG
(Shipbuilding Laboratory, Delft University of Technology,
now at Twente University of Technology)

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VOORWOORD

Dit rapport is een vertaling van een reeds eerder door de auteur in de nederlandse taal geschreven rapport [1] dat tot doel had formules voor de toegevoegde massa en de demping te geven, welke gebruikt konden worden in computerprogramma's voor het berekenen van scheepsbewegingen, zoals die ontwikkeld werden door medewerkers van het Laboratorium voor Scheepsbouwkunde van de Technische Hogeschool te Delft. De vertaling kwam tot stand met de medewerking van de heer W. Beukelman van het Laboratorium.

Het rapport is bedoeld als handleiding voor diegenen die zich op de hoogte willen stellen van de hydrodynamische achtergronden van de methoden die gehanteerd worden om volgens de strip methode hydrodynamische eigenschappen van schepen te bepalen.

Verondersteld wordt dat de lezer bekend is met de grondslagen van de hydrodynamica en de theorie van de infinitesimale oppervlaktegolven. Voor het bestuderen van deze theorieën wordt de lezer verwezen naar de referenties [2], [3], [4] en [5].

HET NEDERLANDS SCHEEPSSTUDIECENTRUM TNO

PREFACE

This report is a translation from Dutch of an earlier report [1], written by the author in order to provide formulas for added mass and damping, which are used in computer programs for the calculation of ship motions, devised by members of the Shipbuilding Laboratory of the Delft University of Technology. The translation was prepared with the assistance of Mr. W. Beukelman of the Laboratory.

The intention of the report is to be a manual for those who want to acquaint themselves with the hydrodynamic backgrounds of the methods used to determine the hydrodynamic properties of ships according to the strip method.

The reader is supposed to be familiar with the fundamentals of hydrodynamics and the infinitesimal surface wave theory. For a study of these theories the reader is referred to references [2], [3], [4] and [5].

THE NETHERLANDS SHIP RESEARCH CENTRE TNO

CONTENTS

		page
	Summary	7
1	**Introduction**	7
2	**Formulation of the problem**	8
3	**Integral equation for the velocity potential; Green's functions; Source potential**	9
4	**Determination of the velocity potential for a heaving circular cylinder according to Ursell; added mass and damping**	14
5	**Heaving of a cylinder with an arbitrary cross section**	18
	5.1 Curvilinear coordinates and conformal transformations	18
	5.1.1 Transformation method of Fil'chakova	20
	5.1.2 Alternative method for the determination of the transformation coefficients	21
	5.2 Calculation of the added mass and damping	23
	5.3 Added mass and damping of a cylinder with a "Lewis-form"	26
6	**Swaying and rolling**	28
	6.1 Potential of a dipole in the origin; skew-symmetric multipole potentials	28
	6.2 Added mass and damping for swaying; added moment of inertia and damping for rolling produced by the swaying motion	30
	6.3 Added moment of inertia and damping for rolling; added mass and damping for swaying produced by the rolling motion	33
	References	35

LIST OF THE PRINCIPAL SYMBOLS

a	Radius of semi-circle (eq. 4.1)
$a_1, a_3, a_5 \ldots a_{2n+1}, \ldots$	Transformation coefficients
$\underline{a}$	Acceleration of motion
b	Wave height (amplitude) at infinite distance (eq. 4.3)
g	Acceleration of gravity
m'	Added mass of circular cylinder per unit length
m''	Added mass of arbitrarily shaped cylinder per unit length
n	Normal to surface of cylinder (Fig. 5.1.1)
p	Pressure at point of cylinder surface also: complex variable $p = \alpha + i\beta$
p_0	Coefficient (eqs. 6.2.26, 6.3.6)
q_0	Coefficient (eqs. 6.2.26, 6.3.6)
r	Radius (section 4) also: polar coordinate (radius) (sections 5 and 6)
s	Line coordinate (Fig. 5.1.1)
t	Time
x, y	Coordinates
A, B	Coefficients (eqs. 4.17, 5.2.19, 6.2.8)
B_0	Half breadth of cylinder
F	Total vertical force per unit length on cylinder
I_R	Added moment of inertia per unit length
I_{RS}	Added moment of inertia per unit length for the rolling motion produced by swaying
K	Wave number ω^2/g
M_0	Coefficient (eqs. 4.15, 5.2.25)
M_1, M_2	Path of integration (Fig. 3.3)
M_S	Added mass per unit length for swaying
M_{SR}	Added mass per unit length for the swaying motion produced by rolling
N_0	Coefficient (eqs. 4.15, 5.2.25)
N'	Damping coefficient of cylinder per unit length (eq. 4.22)
N_R	Damping per unit length for rolling
N_{RS}	Damping per unit length for the rolling motion produced by swaying
N_S	Damping per unit length for swaying
N_{SR}	Damping per unit length for the swaying motion produced by rolling
T	Draught
U	Velocity of fluid particles on cylinder
U_n	Normal component of this velocity
α	Angle (between normal to contour and y-axis) also: variable in $r = e^{\alpha}$ (section 5.3)
η	height of wave caused by oscillation
ϑ	Rolling angle
θ	Polar coordinate (argument)
ξ	Coefficient (eq. 5.3.5)
ϱ	Specific mass of fluid
ϕ	Time independent part of Φ
φ	Polar coordinate (argument)
φ_{2m}	Multipole potential
ψ	Time independent part of Ψ
ψ_{2m}	Conjugate stream function of φ_{2m}
Δ	Two dimensional Laplace operator
Φ	Velocity potential
Ψ	Stream function conjugate to Φ
ω	Circular frequency

COMPUTATION OF THE HYDRODYNAMIC COEFFICIENTS OF OSCILLATING CYLINDERS *

by

Dr. Ir. B. DE JONG

Summary

After an explanation of the basic principles of the strip theory, the formulation, as a linear potential problem for an infinitely long cylinder is shown. Using Green's theorem, an integral equation for the velocity potential is derived for the case of a vertically oscillating cylinder.
To solve this equation for a circular cylinder, Ursell's method of superimposing potential functions, all satisfying the equation of Laplace and the free surface condition, is used.
The concepts of added mass and damping are introduced and defined and their calculation is shown for the circular cylindrical shape. For the computation of added mass and damping for the vertical (heaving) motion of arbitrarily shaped cylinders, Ursell's method is modified, applying conformal transformation. For this purpose two transformation methods are described. Also Tasai's modification of Ursell's method, applying Lewis-forms is reproduced. In the last chapter expressions are derived for the hydrodynamic coefficients, viz. added mass, added moment of inertia and damping, of a cylinder carrying out a forced harmonic swaying or rolling motion, also including the coupling between these two motions. Contrary to the case of heaving, the skew-symmetric approach has to be applied here.

1 Introduction

The last few years much attention has been paid to the theoretical approximation of the hydrodynamic coefficients of a ship in which great advance has been made by the availability of computers. Since in general a three-dimensional method leads to calculations which are too complicated, the problem is considered as a two-dimensional one by the application of the so-called strip method. In this case the ship is divided into a number of sections and of each section, which is supposed to have a constant profile, the hydrodynamic properties are determined, assuming that the disturbances in the fluid due to the motions of the sections only propagate into the direction perpendicular to its axes. Therefore, application of the above-mentioned method requires information about the hydrodynamic properties of infinitely-long cylinders, (or finite cylinders contained between vertical walls at right angles to the axis), with cross sections which are equal to those of the considered sections of the ship.

Ursell [6], [7] made the first contribution to the solution of this problem. He considered the problem of a circular cylinder which oscillates harmonically with small amplitude, while the mean position of the axis coincides with the mean surface of the fluid. Ursell starts from the following assumptions:

1. the fluid is inviscid, incompressible and irrotational.
2. the oscillation is of such a nature that linearization is allowed.

From 1. it follows that the velocity potential Φ satisfies the equation of Laplace $\Delta\Phi = 0$, while according to 2. the accessory boundary conditions are linear. Consequently, it follows from 1. and 2. that the above-mentioned problem can be formulated as a linear potential problem.

Ursell found a solution by superimposing suitably chosen functions such that each separate function satisfies the equation of Laplace and the linearized free-surface condition, while a combination of these functions satisfies the remaining boundary conditions.

Tasai [8], [9] generalized Ursell's method for more general cross sections, the so-called Lewis forms, which are characterized by three parameters. Tasai applied a conformal transformation with which the Lewis form is mapped onto a semi-circle. Because of the restricted number of parameters, the transformation formulas can be determined in an analytical way.

Porter [10] derived expressions for the hydrodynamic coefficients of cylinders which cannot be approximated with Lewis forms in a satisfactory way and for which more complicated transformation formulas are required. Moreover, he verified some results experimentally. A method, however, to find the transformation formulas, mapping an arbitrary cross section of a ship onto a semi-circle, is not given by him.

In the Shipbuilding Laboratory at Delft, Smith [11] devised a computer program of the iterative process of Fil'chakova, [12], by which the transformation formula can be determined for every arbitrary cross section which maps this cross section onto a semi-circle. After this the hydrodynamic coefficients of this section can be determined rather easily. It is noteworthy that strictly speaking this method can be applied only if the cross section intersects the fluid surface perpendicularly.

* Report no. 174A of the Shipbuilding Laboratory, Delft University of Technology.

The English edition of this report has been supplemented with another transformation method, (see section 5.1.2), which appears to be very useful.

2 Formulation of the problem

In a fluid of infinite depth a cylinder is considered which is oscillating one-dimensionally and harmonically with frequency ω while the mean position of its axis is assumed to lie in the free surface of the undisturbed fluid (Fig. 2.1). As possible ways of oscillation we shall consider heaving, swaying and rolling here.
The x-axis is horizontal, coinciding with the free surface of the fluid and perpendicular to the axis of the cylinder and the y-axis is vertical, positive in downward direction and going through the mean position of the axis of the cylinder.

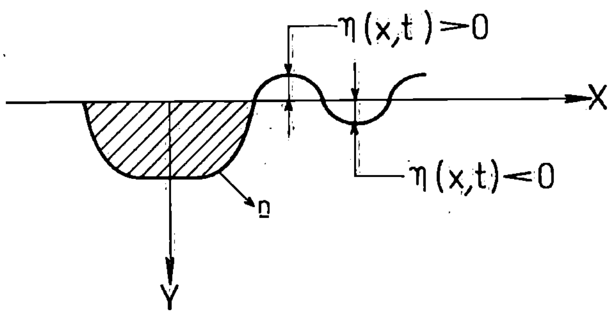

Fig. 2.1

Further, we assume the amplitude of the oscillation being small with respect to the diameter of the cylinder and the length of the waves, generated by the oscillation, so that, in the linearized approximation, the values of all physical quantities can be referred to the mean position of the cylinders. Taking the cylinder very long with respect to the breadth or enclosing the cylinder at both ends between two infinitely long walls perpendicular to the axis of the cylinder, we can neglect the velocity components parallel to the axis of the cylinder and consequently the motion is two-dimensional.

The determination of the motions of the fluid under influence of the harmonic oscillation of the cylinder can be reduced to the solution of a boundary-value problem from the linear potential theory. Consequently, the velocity potential $\Phi(x, y, t)$ is also a harmonic function of the time.

Therefore, using complex notation, we may write the potential in the following form

$$\Phi(x, y, t) = -i\phi(x, y)e^{i\omega t} \tag{2.1}$$

From this the actual potential is obtained by taking the real part of the right-hand side expression. In future calculations where the time dependence of the variables is not involved, we shall always work with the time-independent part $\phi(x, y)$.

The velocity potential has to satisfy the equation of Laplace everywhere in the fluid:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{2.2}$$

Because of equation (2.1), we may write equation (2.2) as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{2.3}$$

If $\eta = \eta(x, t)$ is the wave height in consequence of the oscillation of the cylinder, then for the linearized case ([5], Ch. 2, equation (2.1.14)) for waves which have small amplitudes in proportion to their length the following relation holds

$$\frac{\partial\eta}{\partial t} = -\frac{\partial\Phi}{\partial y}, \quad (y = 0) \quad \text{(kinematic surface condition)} \tag{2.4}$$

We observe that in the linearized form this relation is referred to the mean surface of the fluid: $y = 0$.

Condition (2.4) is based on the hypothesis that any fluid particle, once being on a boundary surface, will remain on it, ([5], section 1.4).

A second condition which Φ has to satisfy at the free surface follows from Bernouilli's law:

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right\} + \frac{p}{\varrho} - g\eta = C(t) \tag{2.5}$$

Since the pressure at the free surface is constant and an addition of constant or time-dependent terms to Φ has no influence upon the velocity distribution $(\partial\phi/\partial x, \partial\phi/\partial y)$ in the fluid, we can reduce (2.5) to

$$\frac{\partial\Phi}{\partial t} + g\eta = 0, \quad (y = 0) \tag{2.6}$$

(dynamic surface condition)

where we only retained the linear terms.

From the conditions (2.4) and (2.6) η can be eliminated. Differentiating (2.6) with respect to t and substituting successively for $\partial\eta/\partial t$ the righthand side of (2.4) and for Φ the expression (2.1), we finally obtain

$$K\phi + \frac{\partial\phi}{\partial y} = 0, \quad (y = 0) \tag{2.7}$$

(linearized free surface condition)

in which

$$K = \frac{\omega^2}{g} \quad \text{represents the wave number}$$

On basis of the earlier mentioned hypothesis with respect to fluid particles on a boundary surface, we can derive that the normal velocity component at the hull of the cylinder, due to the forced oscillation, is equal to the corresponding velocity component of the fluid particles on the cylinder, so

$$\frac{\partial \Phi}{\partial n} = -iU_n(x, y)e^{i\omega t}$$

or

$$\frac{\partial \phi}{\partial n} = U_n(x, y) \tag{2.8}$$

(boundary condition on the cylinder)

In this relation $\underline{n}$ refers to the normal outward direction to the surface of the cylinder, (figure 2.1). It should be noticed that, on account of linearizing the problem, relation (2.8) is here also referred to the mean position of the cylinder.

For physical reasons it is easy to see that the disturbances in the fluid, as a result of the oscillation of the cylinder, decrease with increasing depth so that

$$\lim_{y \to \infty} \text{grad}\, \phi = 0 \tag{2.9}$$

Since the forced oscillation is harmonic, waves are excited at the fluid surface, which are composed of a standing wave, rapidly decreasing in amplitude with the distance from the cylinder and a regular progressive wave, which travels to infinity on both sides of the cylinder. The last-named wave effects a radiation of energy, withdrawn from the motion of the cylinder, in which the fluid has a damping influence on the motion of the cylinder. Thus

$$\Phi \to C_1 \; \bar{e}^{Ky} e^{i(-Kx+\omega t)} \quad \text{as} \quad x \to +\infty$$

$$\Phi \to C_2 e^{-Ky} e^{i(Kx+\omega t)} \quad \text{as} \quad x \to -\infty$$

or

$$\phi \to C_1 e^{-Ky-iKx} \quad \text{as} \quad x \to +\infty$$

$$\phi \to C_2 e^{-Ky+iKx} \quad \text{as} \quad x \to -\infty \tag{2.10}$$

(For an analytical derivation of this condition see section 6.7 of [5]).

Resuming we are faced with the problem now of determining a potential ϕ which is a solution of the Laplace equation $\Delta\phi = 0$ everywhere in the fluid and which in addition satisfies the following boundary conditions

(i) the linearized free-surface condition (2.7)
(ii) the boundary condition on the cylinder (2.8)
(iii) if $y \to \infty$, every disturbance vanishes in the fluid, (2.9)
(iv) the radiation condition (2.10)
(v) for heaving the potential has to be symmetrical:

$$\phi(x, y) = \phi(-x, y)$$

and for rolling and swaying skew-symmetric:

$$\phi(x, y) = -\phi(-x, y) \tag{2.11}$$

For physical reasons condition (v) is easy to see. We further remark that conditon (iv) implicates condition (iii) on account of (2.10).

Up to now we have not defined the form of the function $U_n(x, y)$, the normal velocity component of the forced oscillation of the cylinder. It is clear that $U_n(x, y)$ both depends on the mode of oscillation of the cylinder, thus heaving, swaying or rolling and on the shape of its cross section.

In the next chapters we shall solve the above-mentioned problem for the three modes of oscillation, mentioned above, while the shape of the cylinder may be taken arbitrarily.

3 Integral equation for the velocity potential; Green's functions; Source potential

Using Green's theorem, we derive an integral equation for the velocity potential for the case of a vertically oscillating cylinder, in this chapter. In addition, much attention will be paid to the Green's function and its physical meaning.

We assume that the cylinder is carrying out a vertical harmonic oscillating motion. Consequently, according to (2.11), the velocity potential is a symmetric function

$$\phi(x, y) = \phi(-x, y) \tag{3.1}$$

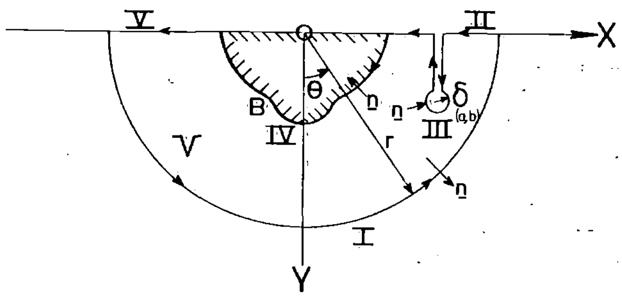

Fig. 3.1

We apply Green's theorem, which in the two-dimensional case has the following form

$$\iint_V (\phi\Delta\psi - \psi\Delta\phi)\,\mathrm{d}V = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}s \tag{3.2}$$

V is an area enclosed by a contour S while n represents the outward normal to the contour S.

For the function ϕ we choose the velocity potential in consequence of the heaving motion of the cylinder. The function ψ is chosen in such a way that after substitution of ψ into (3.2) the requirements with respect to the uniqueness and existence of the solution for the resulting integral equation are fulfilled.

It appears that the function ψ must have a source singularity in a point (a, b) in the region $y > 0$, outside the cylinder. For ψ we choose a function of the form, ([13] Ch. VI-3):

$$\psi(x, y; a, b) = \log\sqrt{(x-a)^2+(y-b)^2} + \psi_r(x, y; a, b) \tag{3.3}$$

The first term on the righthand side is the potential of a source in the point (a, b). The function ψ_r is regular in the region $y > 0$, outside the cylinder. We now choose the contour S in such a way that ϕ and ψ are regular in the region V enclosed by S. For these reasons S is composed of the lines II and V along the free surface, the line IV along the contour of the cylinder, a small circle III with radius δ enclosing the point (a, b) and a large circle I with radius r (Fig. 3.1). ϕ and ψ satisfy the Laplace equation within this region so, that the lefthand side of (3.2) becomes zero and thus.

$$\oint_{\mathrm{I+II+III+IV+V}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}s = 0 \tag{3.4}$$

We shall first determine the limit value of the integral along the small circle III, for the case $\delta \to 0$. It is observed that for small values of δ the expression (3.3) on the small circle III may be written as:

$$\psi = \log\delta + \psi_r \tag{3.5}$$

Further, we may write on the small circle:

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial\delta} \quad \text{and} \quad \mathrm{d}s = \delta\,\mathrm{d}\theta,$$

so that

$$\int_{\mathrm{III}} = -\int_{\pi}^{-\pi} \left\{-\phi\frac{\partial}{\partial\delta}(\log\delta+\psi_r) + (\log\delta+\psi_r)\frac{\partial\phi}{\partial\delta}\right\}\delta\,\mathrm{d}\theta$$

or

$$\int_{\mathrm{III}} = \int_{-\pi}^{+\pi} \left\{-\phi\left(\frac{1}{\delta} + \frac{\partial\psi_r}{\partial\delta}\right) + (\log\delta+\psi_r)\frac{\partial\phi}{\partial\delta}\right\}\delta\,\mathrm{d}\theta$$

As ψ_r and $\partial\psi_r/\partial\delta$ are bounded we obtain for $\delta \to 0$

$$\int_{\mathrm{III}} = -2\pi\phi(a, b) \tag{3.6}$$

Next, we consider the integral along the lines II and V. We choose ψ_r in such a way that the line integral along the free surface equals zero:

$$\int_{\mathrm{II+V}} \left\{\phi\frac{\partial\psi}{\partial y} - \psi\frac{\partial\phi}{\partial y}\right\}\mathrm{d}x = 0 \qquad (y = 0)$$

This condition is satisfied if the integrand of the above expression vanishes:

$$\phi\psi_y - \psi\phi_y = 0 \qquad (y = 0)$$

With the aid of (2.7) we see that on account of the symmetry with respect to ϕ and ψ of above-mentioned expression, this relation is valid if

$$K\psi + \psi_y = 0 \qquad (y = 0) \tag{3.7}$$

We further investigate what requirements ψ has to satisfy in order that the line integral on the large circle vanishes for the limit $r \to \infty$:

$$\lim_{r\to\infty} \int_{-\pi/2}^{+\pi/2} \left\{\phi\frac{\partial\psi}{\partial r} - \psi\frac{\partial\phi}{\partial r}\right\} r\,\mathrm{d}\theta = 0 \tag{3.8}$$

When $r \to \infty$ the potential ϕ has to satisfy the condition (2.10) on the large circle. As the relation (3.8) is symmetric with respect to ϕ and ψ these functions have to be equal on the large circle; so the function ψ has to represent a regular progressive wave if $|x| \to \infty$:

$$\psi \to C_1 e^{-Ky-iKx} \quad \text{as} \quad x \to +\infty$$
$$\psi \to C_1 e^{-Ky+iKx} \quad \text{as} \quad x \to -\infty \tag{3.9}$$

Hence, if the function ψ, as given by (3.3), satisfies the relations (3.7) and (3.9) then (3.4) results in the following integral equation for $\phi(x, y)$

$$\phi(a, b) = \frac{1}{2\pi}\int_B \left\{\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right\}\mathrm{d}s \tag{3.10}$$

In this expression B represents the contour of the cylinder on which the normal velocity $\partial\phi/\partial n$ is given. In the integrand of (3.10) the function ϕ is integrated over the contour of the cylinder. When we take now the point (a, b) on the contour of the cylinder, we obtain, after application of the same procedure (in which the small circle III changes into a semi-circle on B), the integral equation

$$\phi(a,b) = \frac{1}{\pi}\int_B \left\{\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right\} ds \qquad (3.11)$$

In this expression both $\phi(a, b)$ and ϕ in the integrand refer to values of ϕ on the cylinder B. With the aid of equation (3.11) we determine first the value of ϕ on the contour B after which by using (3.10) we are able to determine the value of ϕ in an arbitrary point of the fluid. The function ψ, being chosen in such a way that the line integrals along I, II and V vanish while, moreover, the requirements with respect to the existence and the uniqueness of the solution of (3.10) are satisfied, is called a Green's function.

It remains to determine the function $\psi_r(x, y; a, b)$ in (3.3). We substitute (3.3) into (3.7):

$$K\psi_r + \psi_{r_y} = -\frac{K}{2}\log\{(x-a)^2+b^2\} + \frac{b}{(x-a)^2+b^2} \qquad (y=0)$$

We search for a solution of this differential equation in the form

$$\psi_r(x, y; a, b) = -\tfrac{1}{2}\log\{(x-a)^2+(y+b)^2\} +$$
$$+\psi^1(x, y; a, b)$$

The first term on the righthand side represents a sink in the imagepoint $(a, -b)$ of (a, b) with respect to the free surface. Then the function ψ^1 has to satisfy the relation:

$$K\psi^1 + \psi^1_y = \frac{2b}{(x-a)^2+b^2} \qquad (y=0) \qquad (3.12)$$

Apart from a factor 2 the righthand side of (3.12) is exactly the Laplace transformation of $\cos p(x-a)$, hence (3.12) can be written as

$$K\psi^1 + \psi^1_y = 2\int_0^\infty e^{-pb}\cos p(x-a)\,dp \quad (y=0) \qquad (3.13)$$

Consider the integral

$$\int_0^\infty e^{-pb}\cos p(x-a)\,dp \qquad (3.14)$$

We consider p as a complex variable: $p = \alpha + i\beta$. As the integral has only a singularity for $p = -\infty$, we may change the path of integration, which is along the real axis in the above-mentioned case, into an arbitrary line L between the origin $(0, 0)$ and the point $(\infty, 0)$ at infinity (Fig. 3.2).

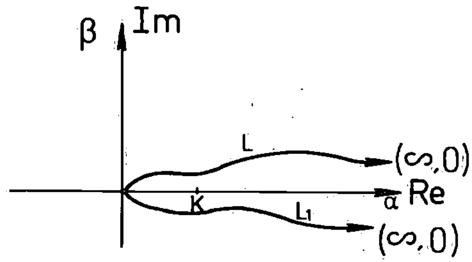

Fig. 3.2

One of the elementary properties of Green's functions is their symmetry with respect to the points (x, y) and (a, b), ([14] Ch. IX-3), which means that the function remains the same if we interchange (x, y) and (a, b). Consequently we substitute into (3.13) for ψ^1 the expression

$$\psi^1(x, y; a, b) = 2\int_L P(p, K)e^{-p(y+b)}\cos p(x-a)\,dp \qquad (3.15)$$

The function $P(p, K)$ has to be determined in such a way that (3.13) is satisfied. Substitution of (3.15) into (3.13) yields

$$-2\int_L pP(p,K)e^{-pb}\cos p(x-a)\,dp +$$
$$+2\int_L KP(p,K)e^{-pb}\cos p\cdot(x-a)\,dp =$$
$$= 2\int_L e^{-pb}\cos p(x-a)\,dp \qquad (y=0)$$

which may be written as

$$\int_L \{(K-p)P(p,K)-1\}\,e^{-pb}\cos p(x-a)\,dp = 0$$

We find

$$P(p, K) = \frac{1}{K-p} \qquad (3.16)$$

and

$$\psi^1(x, y; a, b) = 2\int_L \frac{e^{-p(y+b)}}{K-p}\cos p(x-a)\,dp \qquad (3.17)$$

The integrand of (3.17) has a pole of the first order in $p = K$, $(K = \omega^2/g$ is real). We may choose the contour L in two ways: either over the singularity $p = K$, e.g. L, or underneath it, e.g. L_1 (Fig. 3.2). As the residue of the integrand is not equal to zero, the values of the integral for these two contours will be different. So the function ψ^1 and as a consequence also the function ψ are not uniquely determined. It appears that this uniqueness is achieved by condition (3.9): ψ has to represent a regular progressive wave at infinite distance from the cylinder.

We now proceed to study the behaviour of ψ successively for the contours L and L_1. As we remain on the same side of the pole we are allowed to change the contour L_1 into M_1 and L into M_2, (Fig. 3.3).

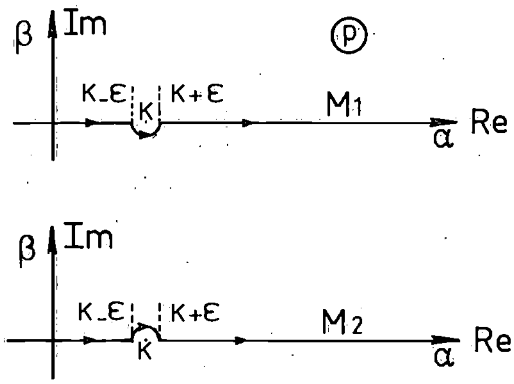

Fig. 3.3

When $|x| \to \infty$ the function

$$\log \sqrt{\frac{(x-a)^2+(y-b)^2}{(x-a)^2+(y+b)^2}}$$

vanishes. Consequently, it suffices to study only the behaviour of ψ^1 on the contours M_1 and M_2 if $|x| \to \infty$. We shall show that ψ^1 represents regular progressive waves for the contour M_2 when $|x|$ increases to infinity. Making the transformations $x' = x-a$ and $y' = y+b$ and skipping the indices after that again, we obtain for (3.17) for the contour M_2

$$\psi^1_{M_2} = \int_{M_2} \frac{e^{-py'}\{e^{ipx'}+e^{-ipx'}\}}{K-p}\,dp =$$

$$= \int_{M_2} \frac{e^{-p(y-ix)}}{K-p}\,dp + \int_{M_2} \frac{e^{-p(y+ix)}}{K-p}\,dp \tag{3.18}$$

The first integral gives

$$\int_{M_2} \frac{e^{-p(y-ix)}}{K-p}\,dp = \underbrace{\int_0^{K-\varepsilon} \frac{e^{-\alpha(y-ix)}}{K-\alpha}\,d\alpha}_{I_1} +$$

$$+ \underbrace{\int_{K-\varepsilon}^{K+\varepsilon} \frac{e^{-(\alpha+i\beta)(y-ix)}}{K-p}\,dp}_{I_2} + \underbrace{\int_{K+\varepsilon}^{\infty} \frac{e^{-\alpha(y-ix)}}{K-\alpha}\,d\alpha}_{I_3}$$

We denote the first, second and third integral of the right-hand side respectively by I_1, I_2 and I_3. Integrating I_1 by parts, we obtain

$$I_1 = -\frac{e^{-\alpha(y-ix)}}{(K-\alpha)(y-ix)}\Bigg|_{\alpha=0}^{\alpha=K-\varepsilon} + \int_0^{K-\varepsilon} \frac{e^{-\alpha(y-ix)}}{(K-\alpha)^2(y-ix)}\,d\alpha$$

It is easy to see that $I_1 \to 0$ as $|x| \to \infty$. In the same way it can be proved that $I_3 \to 0$ as $|x| \to \infty$. Further, since $\beta > 0$ the integral I_2 vanishes as $|x| \to \infty$. It likewise can be proved that for the second integral on the right-hand side of (3.18) the integrations along the real axis over $(0, K-\varepsilon)$ and $(K+\varepsilon, \infty)$ become zero, whereas on account of the theorem of residues the integration along the semicircle $(K-\varepsilon, K+\varepsilon)$ yields

$$\int_{K-\varepsilon}^{K+\varepsilon} \frac{e^{-p(y+ix)}}{K-p}\,dp = \int_{K-\varepsilon}^{K+\varepsilon} \frac{e^{-p(y+ix)}}{K-p}\,dp +$$

$$+2\pi i(\text{residue } p = K).$$

After substitution of $p = \alpha + i\beta$ into the left-hand side we see that this integral vanishes when $|x| \to \infty$ since β assumes only negative values. For the residue in $p = K$ we find

$$\lim_{p \to K} (p-K)\frac{e^{-p(y+ix)}}{K-p} = -e^{-K(y+ix)}$$

so

$$\int_{K-\varepsilon}^{K+\varepsilon} \frac{e^{-p(y+ix)}}{K-p}\,dp = 2\pi i e^{-K(y+ix)}$$

and

$$\lim_{|x| \to \infty} \int_{M_2} \frac{e^{-py}\cos px}{K-p}\,dp = 2\pi i e^{-K(y+ix)} \tag{3.19}$$

Consequently, if $|x| \to \infty$, ψ^1 and for this reason also ψ represents a regular outgoing wave if the path of integration L in expression (3.17) has the shape of M_2. In the same way it can be proved that ψ gives a regular incoming wave when $|x| \to \infty$ if L has the shape of M_1. Therefore, in order to make ψ satisfy condition (3.9), the integration in (3.17) is carried out along the path M_2. The Green's function ψ which has to be substituted into the integral equations (3.10) and (3.11) has now a unique representation. Dividing by a factor 2, we have

$$\psi(x, y; a, b) = \tfrac{1}{2}\log \sqrt{\frac{(x-a)^2+(y-b)^2}{(x-a)^2+(y+b)^2}} +$$

$$+ \int_{M_2} \frac{e^{-p(y+b)}}{K-p}\cos p(x-a)\,dp \tag{3.20}$$

So summing up above results, we constructed a function ψ with the following properties

(i) $\Delta\psi = 0$
(ii) the linearized free-surface condition (3.7)
(iii) if $|x| \to \infty$, ψ represents a regular progressive wave (3.9) (3.21)

In chapter 4 we shall solve above-mentioned boundary value problem by taking for ψ a linear combination of potential functions. One of these components is the potential function of a source in the origin ($a = b = 0$), which satisfies, in addition, the conditions (3.21). For this reason we shall consider here the potential function of such a source more precisely. Setting $a = b = 0$ in (3.20) and calling the potential now ϕ, we obtain

$$\phi = \int_{M_2} \frac{e^{-py}}{K-p} \cos px \, \mathrm{d}p \qquad (3.22)$$

In literature (3.22) is mostly given in another form, which can be derived from (3.22) by applying theorems of the complex function theory. We shall give this derivation here:

If we split up the integral (3.22) into two integrals I_1 and I_2, this yields

$$\int_{M_2} \frac{e^{-py} \cos px}{K-p} \mathrm{d}p = \tfrac{1}{2} \Bigg\{ \underbrace{\int_{M_2} \frac{e^{-py+ipx}}{K-p} \mathrm{d}p}_{I_1} + \underbrace{\int_{M_2} \frac{e^{-py-ipx}}{K-p} \mathrm{d}p}_{I_2} \Bigg\} \qquad (3.23)$$

It appears that the calculation gives different results for $x > 0$ and $x < 0$. We shall consider here the case $x > 0$. For $x < 0$ the reasoning proceeds in an analogous way.

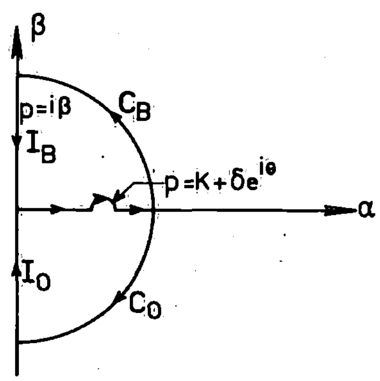

Fig. 3.4

For the calculation of I_1 we close the contour M_2 with the arc of a circle C_B and the positive imaginary axis I_B. Application of Cauchy's theorem leads to

$$\int_{M_2} \frac{e^{-py+ipx}}{K-p} \mathrm{d}p + \lim_{R\to\infty} \int_0^{\pi/2} \frac{e^{-Re^{i\theta}y+iRe^{i\theta}x} \cdot Re^{i\theta}}{K-Re^{i\theta}} i \, \mathrm{d}\theta + \int_\infty^0 \frac{e^{-i\beta y-\beta x}}{K-i\beta} i \, \mathrm{d}\beta = 0$$

This is easily reduced to

$$\int_{M_2} \frac{e^{-py+ipx}}{K-p} \mathrm{d}p + \lim_{R\to\infty} \int_0^{\pi/2} \frac{e^{-(Ry\cos\theta+Rx\sin\theta)-i(Ry\sin\theta-Rx\cos\theta)}}{K-R\cos\theta-iR\sin\theta} Re^{i\theta} i \, \mathrm{d}\theta + \int_\infty^0 \frac{e^{-i\beta y-\beta x}}{K-i\beta} i \, \mathrm{d}\beta = 0 \qquad (3.24)$$

We notice that on C_B: $\cos\theta > 0$, $\sin\theta > 0$ and $x > 0$, $y > 0$. So the third integral vanishes when $R \to \infty$. Consequently, (3.24) may be reduced to

$$\int_{M_2} \frac{e^{-py+ipx}}{K-p} \mathrm{d}p + \int_\infty^0 \frac{e^{-i\beta y-\beta x}}{K-i\beta} i \, \mathrm{d}\beta = 0 \qquad (3.25)$$

For the calculation of I_2 we close the contour with the arc of a circle C_0 and the negative imaginary axis. As in this case the path of integration encloses the pole $p = K$ with residue

$$\lim_{p\to K} (p-K) \frac{e^{-py-ipx}}{K-p} = -e^{-Ky-iKx}$$

we find by applying the theorem of residues

$$\int_{M_2} \frac{e^{-py-ipx}}{K-p} \mathrm{d}p + \lim_{R\to\infty} \int_0^{-\pi/2} \frac{e^{-Re^{i\theta}y-iRe^{i\theta}x} \cdot Re^{i\theta}}{K-Re^{i\theta}} i \, \mathrm{d}\theta + \int_{-\infty}^0 \frac{e^{-i\beta y+\beta x}}{K-i\beta} i \, \mathrm{d}\beta = -2\pi i(-e^{-Ky-iKx}) = 2\pi i e^{-Ky-iKx}$$

Analogous to the corresponding integral in (3.24) the second integral in this relation vanishes also. After evaluation of the first integral the above mentioned form results into

$$\int_{M_2} \frac{e^{-py+ipx}}{K-p} \mathrm{d}p + \int_{-\infty}^0 \frac{e^{-i\beta y+\beta x}}{K-i\beta} i \, \mathrm{d}\beta = 2\pi i e^{-Ky-iKx} \qquad (3.26)$$

From (3.23), (3.25) and (3.26) it follows

$$\int_{M_2} \frac{e^{-py}\cos px}{K-p}\,\mathrm{d}p =$$

$$= \tfrac{1}{2}\left\{\int_0^\infty \frac{e^{-i\beta y-\beta x}}{K-i\beta}\, i\,\mathrm{d}\beta + \int_0^\infty \frac{e^{i\beta y-\beta x}}{K+i\beta}(-i)\,\mathrm{d}\beta\right\} + \pi i e^{-Ky-iKx} \qquad (3.27)$$

(3.27) may further be reduced to

$$\int_{M_2} \frac{e^{-py}\cos px}{K-p}\,\mathrm{d}p =$$

$$= \int_0^\infty \frac{e^{-\beta x}(K\sin\beta y-\beta\cos\beta y)}{K^2+\beta^2}\,\mathrm{d}\beta + \pi i e^{-Ky-iKx} \qquad (x>0)$$

In the same way it can be proved that for $x<0$ the integral on the righthand side of above expression has to be replaced by

$$\int_0^\infty e^{\beta x}\frac{(K\sin\beta y-\beta\cos\beta y)}{K^2+\beta^2}\,\mathrm{d}\beta + \pi i e^{-Ky+iKx} \qquad (x<0)$$

Consequently, the potential ϕ of a source in the origin satisfying the free-surface condition and the radiation condition is given by

$$\phi = \int_0^\infty e^{-\beta|x|}\frac{(K\sin\beta y-\beta\cos\beta y)}{K^2+\beta^2}\,\mathrm{d}\beta + \pi i e^{-Ky-iK|x|} \qquad (3.28)$$

In this report we will use the following definition for the source potential

$$\Phi = \frac{gb}{\pi\omega}Re\{-i\phi e^{i\omega t}\} = \frac{gb}{\pi\omega}\{\phi_c\cos\omega t+\phi_s\sin\omega t\}$$

where b is the wave height at an infinite distance from the cylinder.

Comparing this expression with (3.28) we find for the non-dimensional quantities ϕ_s and ϕ_c

$$\phi_c = \pi e^{-Ky}\cos Kx$$

$$\phi_s = \pi e^{-Ky}\sin K|x| - \int_0^\infty \frac{e^{-\beta|x|}}{K^2+\beta^2}(\beta\cos\beta y-K\sin\beta y)\,\mathrm{d}\beta \qquad (3.29)$$

Remark

It appears that the integral in (3.29) converges very slowly when it is calculated in a numerical way. Porter [10 page 148] has given the following power series expansion, which affords good results

$$\int_0^\infty \frac{e^{-\beta x}}{K^2+\beta^2}(K\sin\beta y-\beta\cos\beta y)\,\mathrm{d}\beta =$$

$$= e^{-Ky}\{Q\cos Kx+S\sin Kx\} - \pi e^{-Ky}\sin Kx$$

where

$$Q = \gamma+\ln[K(x^2+y^2)^{\frac{1}{2}}] + \sum_{n=1}^\infty \frac{K^n(x^2+y^2)^{n/2}}{n!\,n}\cos n\chi,$$

$$S = \chi + \sum_{n=1}^\infty \frac{K^n(x^2+y^2)^{n/2}}{n!\,n}\sin n\chi,$$

γ = 0,5772156649......: Euler's constant, and

$$\chi = \arctan\frac{x}{y}$$

4 Determination of the velocity potential for a heaving circular cylinder according to Ursell; added mass and damping

In the previous chapter we saw that the potential can be determined by solving the integral equation (3.11). In general, however, it appears that this leads to rather complicated numerical calculations.

Ursell [6] has developed a method of solution which consists of superposition of potential functions which all satisfy the equation of Laplace and the free surface condition. The solution is composed of the source potential (3.28) and a linear combination of multipole potentials which are represented by

$$\varphi_{2m} = a^{2m}\left\{\frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}\right\}$$

$$m = 1,2,3,\ldots \qquad (4.1)$$

where (r, θ) are the polar coordinates

$$x = r\sin\theta, \qquad y = r\cos\theta \qquad (4.2)$$

while a represents the radius of the circular cylinder (figure 4.1).

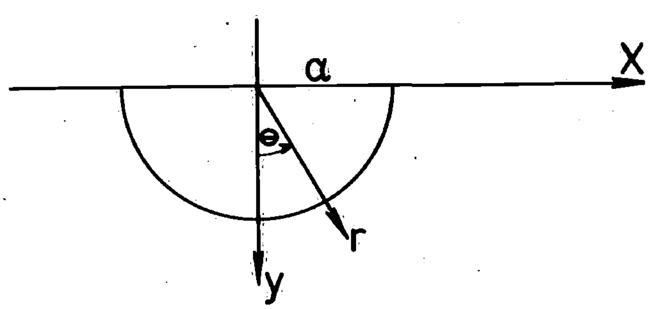

Fig. 4.1

Since the cylinder carries out a vertical oscillation, the corresponding velocity potential is a symmetric function with respect to the y-axis, (see (2.11)). Consequently, it suffices to restrict our future considerations to the range $0 \leqslant \theta \leqslant \pi/2$. According to (3.29) and (4.1) the time-dependent potential Φ is expressed by:

$$\Phi = \frac{gb}{\pi\omega}\Bigg[\phi_c(Kr;\theta)\cos\omega t + \phi_s(Kr;\theta)\sin\omega t + \\ + \cos\omega t \sum_{m=1}^{\infty} p_{2m}(Ka)a^{2m}\left\{\frac{\cos 2m\theta}{r^{2m}} + \\ + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}\right\} + \\ + \sin\omega t \sum_{m=1}^{\infty} q_{2m}(Ka)a^{2m}\left\{\frac{\cos 2m\theta}{r^{2m}} + \\ + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}\right\}\Bigg] \tag{4.3}$$

in which

$$\phi_c(Kr,\theta) = \pi e^{-Kr\cos\theta}\cos(Kr\sin\theta)$$

$$\phi_s(Kr,\theta) = -\int_0^\infty \frac{e^{-\beta r\sin\theta}}{K^2+\beta^2} \\ \{\beta\cos(\beta r\cos\theta) - K\sin(\beta r\cos\theta)\}\,d\beta + \\ + \pi e^{-Kr\cos\theta}\sin(Kr\sin\theta) \tag{4.4}$$

It is easy to verify that the multipole potentials φ_{2m} satisfy the free-surface condition. In chapter 3 we showed that the source potential $gb/\pi\omega\,\{\phi_c\cos\omega t + \phi_s\sin\omega t\}$ is determined in such a way that the free-surface condition is satisfied. Furthermore, we showed there that this potential represents a regular progressive wave when $|x| \to \infty$. As the multipole-potentials vanish if $r \to \infty$, the total potential Φ, represented by (4.3), satisfies the radiation condition (2.11). It still remains to determine the coefficients p_{2m} and q_{2m} in such a way that the boundary condition on the cylinder surface is satisfied. For the case of a circular cylinder the boundary condition (2.8) on the cylinder is reduced to

$$\frac{dy}{dt}\cos\theta = \frac{\partial\phi}{\partial r} \tag{4.5}$$

The Cauchy-Riemann conditions which relate the velocity potential Φ and the conjugate stream function Ψ have in polar coordinates the form

$$\frac{\partial\Phi}{\partial r} = -\frac{1}{r}\frac{\partial\Psi}{\partial\theta}$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{\partial\Psi}{\partial r} \tag{4.6}$$

Substituting Φ in (4.6) and performing the integration yields for the stream function

$$\Psi = \frac{gb}{\pi\omega}\Bigg[\psi_c(Kr,\theta)\cos\omega t + \psi_s(Kr,\theta)\sin\omega t + \\ + \cos\omega t \sum_1^{\infty} p_{2m}(Ka)a^{2m}\left\{\frac{\sin 2m\theta}{r^{2m}} + \\ + \frac{K}{2m-1}\frac{\sin(2m-1)\theta}{r^{2m-1}}\right\} + \\ + \sin\omega t \sum_1^{\infty} q_{2m}(Ka)a^{2m}\left\{\frac{\sin 2m\theta}{r^{2m}} + \\ + \frac{K}{2m-1}\frac{\sin(2m-1)\theta}{r^{2m-1}}\right\}\Bigg] \tag{4.7}$$

where

$$\psi_c(Kr;\theta) = \pi e^{-Kr\cos\theta}\sin(Kr\sin\theta)$$

$$\psi_s(Kr;\theta) = \int_0^\infty \frac{e^{-\beta r\sin\theta}}{K^2+\beta^2}\{\beta\sin(\beta r\cos\theta) + \\ + K\cos(\beta r\cos\theta)\}\,d\beta - \pi e^{-Kr\cos\theta}\cos(Kr\sin\theta) \tag{4.8}$$

Using (4.6), the boundary condition on the cylinder (4.5) may be written as

$$-\frac{1}{a}\frac{\partial\Psi}{\partial\theta} = \frac{dy}{dt}\cos\theta \qquad (r = a)$$

Integrating with respect to θ, we obtain

$$\Psi = -a\frac{dy}{dt}\sin\theta + c(t) \qquad (r = a)$$

By substituting $\theta = 0$ the integration constant $c(t)$ is found to be zero.

We assume that the ordinate of the axis of the cylinder is given by

$$y = l\cos(\omega t + \varepsilon) \tag{4.9}$$

Consequently the streamfunction on the cylinder has to satisfy:

$$\Psi = l\omega a\sin(\omega t + \varepsilon)\cdot\sin\theta \qquad (r = a) \tag{4.10}$$

From (4.7) and (4.10) we find

$$\psi_c(Ka;\theta)\cos\omega t+\psi_s(Ka;\theta)\sin\omega t+$$
$$+\cos\omega t\sum_1^\infty p_{2m}(Ka)\left\{\sin 2m\theta+\frac{Ka}{2m-1}\sin(2m-1)\theta\right\}+$$
$$+\sin\omega t\sum_1^\infty q_{2m}(Ka)\left\{\sin 2m\theta+\frac{Ka}{2m-1}\sin(2m-1)\theta\right\}=$$
$$=\frac{la\pi K}{b}\sin(\omega t+\varepsilon)\cdot\sin\theta \tag{4.11}$$

As (4.11) holds for the range $0\leqslant\theta\leqslant\pi/2$, we find by substituting $\theta=\pi/2$ in this expression

$$\psi_c\left(Ka;\frac{\pi}{2}\right)\cos\omega t+\psi_s\left(Ka;\frac{\pi}{2}\right)\sin\omega t+$$
$$+\cos\omega t\sum_1^\infty p_{2m}(Ka)\frac{Ka(-1)^{m-1}}{2m-1}+$$
$$+\sin\omega t\sum_1^\infty q_{2m}(Ka)\frac{Ka(-1)^{m-1}}{2m-1}=\frac{la\pi K}{b}\sin(\omega t+\varepsilon) \tag{4.12}$$

With this relation we eliminate the factor $la\pi K/b\sin(\omega t+\varepsilon)$ from (4.11). By equating in the resulting form the coefficients of $\cos\omega t$ and $\sin\omega t$, it is easy to see that the coefficients p_{2m} and q_{2m} have to satisfy the relations

$$\psi_c(Ka;\theta)-\psi_c\left(Ka;\frac{\pi}{2}\right)\sin\theta=\sum_1^\infty p_{2m}(Ka)f_{2m}(Ka;\theta)$$
$$\psi_s(Ka;\theta)-\psi_s\left(Ka;\frac{\pi}{2}\right)\sin\theta=\sum_1^\infty q_{2m}(Ka)f_{2m}(Ka;\theta) \tag{4.13}$$

where

$$f_{2m}(Ka;\theta)=-\left[\sin 2m\theta+\frac{Ka}{2m-1}\{\sin(2m-1)\theta-\sin\theta\sin\tfrac{1}{2}(2m-1)\pi\}\right]$$

In (4.13) the left-hand side represents an expansion in a series of functions $f_{2m}(Ka;\theta)$, where $m=1,2,3,\ldots$

Of course, in practical applications only a finite number of terms, N, is taken, the accuracy of the approximation being improved by increasing the value of N.

The coefficients p_{2m} and q_{2m} can now be determined with, for example, the least-squares approximation method.

From Bernoulli's law (2.5) we derive that the hydrodynamic pressure in a point of the liquid in linearized form is given by

$$p=-\varrho\frac{\partial\Phi}{\partial t} \tag{4.14}$$

Consequently the hydrodynamic force per unit-length on the cylinder is

$$P=\varrho\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}a\frac{\partial\Phi}{\partial t}\Big|_{r=a}\cos\theta\,d\theta=$$
$$=\frac{2\varrho abg}{\pi}(M_0\cos\omega t-N_0\sin\omega t) \tag{4.15}$$

where

$$M_0=\int_0^{\pi/2}\phi_s(Ka;\theta)\cos\theta\,d\theta+\sum_1^\infty\frac{(-1)^{m-1}q_{2m}(Ka)}{4m^2-1}+$$
$$+\tfrac{1}{4}\pi Kaq_2(Ka)$$
$$N_0=\int_0^{\pi/2}\phi_c(Ka;\theta)\cos\theta\,d\theta+\sum_1^\infty\frac{(-1)^{m-1}p_{2m}(Ka)}{4m^2-1}+$$
$$+\tfrac{1}{4}\pi Kap_2(Ka)$$

It is a well-known fact that a long cylinder which is completely submerged in an ideal infinite fluid experiences a hydrodynamic force $M\ddot{r}$ per unit-length which is equal to the product of the relative acceleration $\ddot{r}$ and the displaced volume of fluid M per unit cylinder length. The situation remains the same if we remove the fluid and add per unit-length a mass M to the cylinder. For this reason M is called the added mass of the cylinder. If the cylinder moves in a fluid with a free surface then the force is no longer in phase with the acceleration. We dissolve this force into a component in phase with the acceleration, which does not dissipate any energy, and a component in phase with the velocity, which has the same character as a frictional force and which is responsible for the dissipation of energy in the form of outward-going waves. The acceleration component of the hydrodynamic force is determined by the added mass and by the velocity component by the damping of the cylinder.

We shall calculate these two quantities now.

From (4.9) we find for the velocity of the cylinder

$$\frac{dy}{dt}=-l\omega\sin(\omega t+\varepsilon)$$

Combining this with (4.12), we find

$$\frac{dy}{dt}=-\frac{\omega b}{\pi aK}\{A\cos\omega t+B\sin\omega t\} \tag{4.16}$$

where

$$A(Ka)=\psi_c\left(Ka;\frac{\pi}{2}\right)+\sum_1^\infty\frac{(-1)^{m-1}Ka}{2m-1}p_{2m}(Ka)$$
$$B(Ka)=\psi_s\left(Ka;\frac{\pi}{2}\right)+\sum_1^\infty\frac{(-1)^{m-1}Ka}{2m-1}q_{2m}(Ka) \tag{4.17}$$

The acceleration of the cylinder is given by

$$\frac{d^2y}{dt^2}=\frac{b\omega^2}{\pi a K}\{A\sin\omega t-B\cos\omega t\} \tag{4.18}$$

The force in phase with the acceleration is found from (4.15) and (4.18):

$$-\frac{2\varrho abg}{\pi}\,\frac{M_0B+N_0A}{A^2+B^2}\{A\sin\omega t-B\cos\omega t\} \tag{4.19}$$

The force in phase with the velocity is found from (4.15) and (4.16):

$$\frac{2\varrho abg}{\pi}\,\frac{M_0A-N_0B}{A^2+B^2}\{A\cos\omega\;+B\sin\omega t\} \tag{4.20}$$

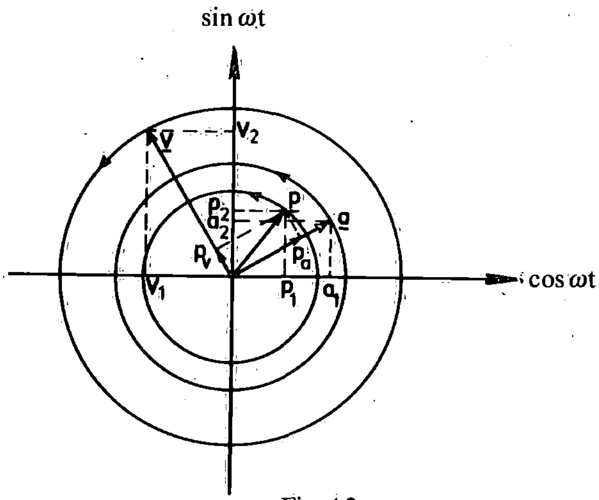

Fig. 4.2

The validity of the formulas (4.19) and (4.20) is easy to see from the vector-diagram in figure 4.2.

We assume that the hydrodynamic force, the velocity and the acceleration are respectively given by

$$\underline{P}=p_1\cos\omega t+p_2\sin\omega t$$

$$\underline{v}\;=v_1\cos\omega t+v_2\sin\omega t$$

$$\underline{a}\;=a_1\cos\omega t+a_2\sin\omega t$$

The velocity and the acceleration have a phase difference of 90 degrees. The component of $\underline{P}$ in phase with the acceleration is expressed by

$$p_a=\frac{(\underline{P}\cdot\underline{a})}{|\underline{a}|}=\frac{(p_1a_1+p_2a_2)}{\sqrt{a_1^2+a_2^2}}$$

In vector notation this component is represented by

$$p_a\cdot\frac{\underline{a}}{|\underline{a}|}=\frac{p_1a_1+p_2a_2}{a_1^2+a_2^2}(a_1\cos\omega t+a_2\sin\omega t)$$

It is easy to check now that in this way expression (4.19) has been derived from (4.15) and (4.18). The derivation of the expression for the force component in phase with the velocity is similar.

The added mass of the cylinder per unit length is defined as the negative value of the ratio between (4.19) and (4.18):

$$m'(Ka)=2\varrho a^2\frac{M_0B+N_0A}{A^2+B^2} \tag{4.21}$$

The dimensionless expression $(M_0B+N_0A)/(A^2+B^2)$ is defined as the added mass coefficient.

The damping coefficient of the cylinder per unit length is defined as the negative value of the ratio between (4.20) and (4.16):

$$N'(Ka)=2\varrho a^2\omega\frac{M_0A-N_0B}{A^2+B^2} \tag{4.22}$$

From (4.7) and (4.17) it follows that for $\theta=\pi/2$ the stream function on the cylinder can be written as

$$\Psi=\frac{gb}{\pi\omega}\{A\cos\omega t+B\sin\omega t\}\qquad\left(r=a;\theta=\frac{\pi}{2}\right)$$

Comparing this with the expression which results from (4.10) when we set $\theta=\pi/2$, it follows that the ratio

$$\frac{\text{wave amplitude at infinity}}{\text{amplitude of the forced oscillation}}$$

is equal to

$$\frac{b}{l}=\frac{\pi Ka}{\sqrt{A^2+B^2}} \tag{4.23}$$

Finally we observe that the work done by the cylinder in one cycle must be equal to the energy radiated by the regular progressive wave during the same time, which is twice the energy of one wavelength of the regular progressive wave:

$$\int_0^{2\pi/\omega}P\frac{dy}{dt}\cdot dt=\varrho b^2g^2\frac{\pi}{\omega^2}$$

Substituting the expressions (4.9) and (4.15) we find the relation

$$M_0A-N_0B=\frac{\pi^2}{2} \tag{4.24}$$

Consequently the damping coefficient can be simplified to

$$N'(Ka)=\frac{\varrho a^2\omega\pi^2}{A^2+B^2} \tag{4.25}$$

5 Heaving of a cylinder with an arbitrary cross section

In this chapter we will discuss in which way Ursell's method can be modified for the calculation of the added mass and damping of an arbitrary cylinder. The essential point in this process is the mapping of a semi-circle onto the cross section S of the cylinder (figure 5.1.1) by means of a conformal transformation, i.e., we determine such a system of curvilinear coordinates that one of the coordinate-lines coincides with the contour of the cross section.

5.1 *Curvilinear coordinates and conformal transformations*

We take the origin of a rectangular coordinate system at the mean position of the axis of the cross section in the free surface of the fluid. The x-axis is taken horizontally and the y-axis vertically in downward direction. This plane is often called the physical plane and is denoted here by the z-plane. The plane of the semi-circle or reference plane is here called the ζ-plane. In the ζ-plane we assume a polar coordinate system (r, θ) having its origin in the centre of the circle. With a conformal mapping of the z-plane onto the ζ-plane every point (r, θ) in the ζ-plane corresponds with a point (x, y) of the z-plane. Consequently, relations exist between the variables x, y and r, θ of the form

$$x = f(r, \theta)$$
$$y = g(r, \theta) \qquad (5.1.1)$$

The corresponding inverse relations are written as

$$r = \bar{f}(x, y)$$
$$\theta = \bar{g}(x, y) \qquad (5.1.2)$$

We now require that the conformal transformation maps the cross section S onto the semi-circle. If the circle has a radius $r = a$ then S is given by

$$x = f(a, \theta)$$
$$y = g(a, \theta) \qquad (5.1.3)$$

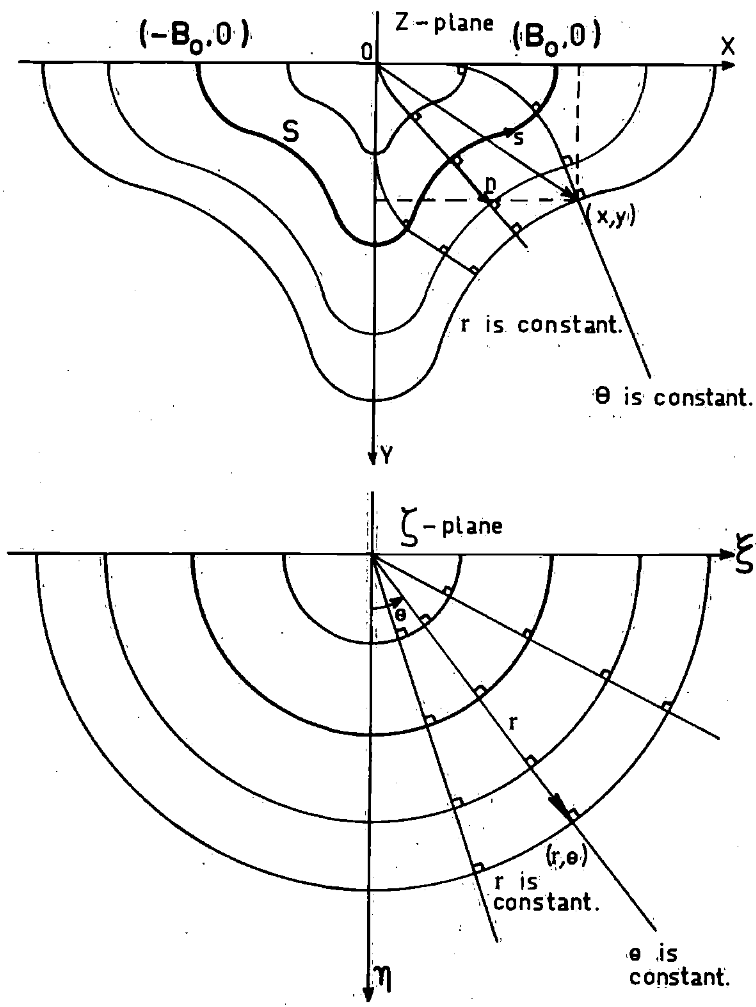

Fig. 5.1.1

Now we also conceive the variables r and θ as coordinates in the z-plane. So along the contour S only the variable θ changes in value while $r=a$ remains constant.

The conformal transformation brings about a coordinate transformation (5.1.2) of rectangular coordinates (x, y) into curvilinear coordinates (r, θ) in such a way that one of the coordinate lines (in this case $r=a$) coincides with the cross section. The coordinate lines $r=$ constant and $\theta=$ constant represent two sets of curves in the z-plane which are mapped in the ζ-plane as the lines $r=$ constant and $\theta=$ constant, which represent circles there with the origin as centre and straight lines through the origin (figure 5.1.1). We know from the theory of conformal transformations that right angles at the intersection points of lines correspond with right angles at the intersection points of the transforms of these lines. Consequently the coordinate lines $r=$ constant and $\theta=$ constant intersect each other also perpendicularly in the z-plane. The important consequence of this is that differentiation along the cross section with line-coordinate s corresponds with differentiation with respect to θ

$$\frac{\partial}{\partial s}=\frac{1}{r}\frac{\partial}{\partial \theta}$$

and differentiation along the normal $\underline{n}$ with differentiation with respect to r:

$$\frac{\partial}{\partial n}=\frac{\partial}{\partial r}$$

In future we shall see that the place dependence of many physical quantities just as, e.g., the streamfunction ψ and the potential φ is expressed by curvilinear coordinates r and θ.

Using a conformal transformation, we determine the relations (5.1.1), which satisfy the conditions (5.1.3) on the contour S. We map the region outside the unit circle $|\zeta|>1$, represented in polar coordinates by $\zeta = re^{i\varphi}$ onto the region outside the closed curve S of the complex z-plane, where $z=x+iy$. The region outside S is supposed to be simply connected, (figure 5.1.2).

We determine a transformation in the form of a series with a finite number of terms

$$z=\sum_{n=-1}^{m-2} C_n\zeta^{-n}=\sum_{n=-1}^{m-2}(A_n+iB_n)r^{-n}(\cos n\varphi - i\sin n\varphi) \tag{5.1.4}$$

Equating the real and imaginary parts of this equation, we obtain

$$x=\sum_{n=-1}^{m-2} r^{-n}(A_n\cos n\varphi+B_n\sin n\varphi)$$

$$y=\sum_{n=-1}^{m-2} r^{-n}(-A_n\sin n\varphi+B_n\cos n\varphi) \tag{5.1.5}$$

As we are only interested in cylinders which are symmetric with respect to the x- and y-axis (in fact we only consider the region $y>0$ of the cross-section; consequently we can imagine the cross section to be symmetric with respect to the x-axis), (5.1.5) can be reduced to

$$x=\sum_{n=-1}^{m-2} r^{-2n-1}A_{2n+1}\cos(2n+1)\varphi$$

$$y=\sum_{n=-1}^{m-2} r^{-2n-1}A_{2n+1}\sin(2n+1)\varphi \tag{5.1.6}$$

We notice that in this case the rectangular coordinate axes ξ and η of the ζ-plane are transformed into the coordinate axes x and y of the z-plane and since the circle intersects the horizontal axis perpendicularly, the cross section of this cylinder intersects the x-axis perpendicularly. Consequently, this transformation is restricted to cross sections which intersect the x-axis perpendicularly.

Comparing figure 5.1.1 and figure 5.1.2, we see that $\varphi=\pi/2-\theta$. Substituting this for φ into (5.1.6) we obtain

$$x=A_{-1}r\sin\theta+\sum_{n=0}^{m-2}(-1)^n\frac{A_{2n+1}\sin(2n+1)\theta}{r^{2n+1}} \tag{5.1.7}$$

$$y=A_{-1}r\cos\theta+\sum_{n=0}^{m-2}(-1)^n\frac{A_{2n+1}\cos(2n+1)\theta}{r^{2n+1}}$$

These are just the transformation formulas used by Porter [10]. Porter makes use of a slightly different notation. For reasons of simplicity we will adopt here this notation, (5.1.7) then becomes

$$x=a\left\{r\sin\theta+\sum_{n=0}^{N}(-1)^n\frac{a_{2n+1}}{r^{2n+1}}\sin(2n+1)\theta\right\}$$

$$y=a\left\{r\cos\theta+\sum_{n=0}^{N}(-1)^{n+1}\frac{a_{2n+1}}{r^{2n+1}}\cos(2n+1)\theta\right\} \tag{5.1.8}$$

Finally, we remark that for the case of a circular cylinder the coefficients a_1, a_2, a_3, ... are all zero. Then the coefficient a represents the ratio of the radii of the circles in the z-plane and the ζ-plane. For that reason a is called the scale factor of the transformation.

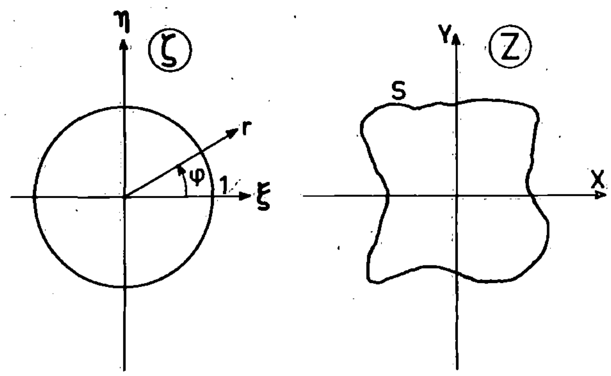

Fig. 5.1.2

5.1.1 Transformation method of Fil'chakova [12]

This method is closely related to Melentiev's method, described by Kantorovich and Krylov [15]. The method is derived for cylinders with an arbitrary shape. Consequently, we have to start from the relation (5.1.4). For the determination of the coefficients C_n we choose $2m$ points at the unit circle $r=1$ which divide the circle in equal parts, thus the polar angle of every point always differs $\Delta\varphi = \pi/m$ with those of his two adjacent points. Next, we divide these points into two systems of points: an even system $\varphi_{2k} = 2k\pi/m$ and an odd system

$$\varphi_{2k-1} = \frac{(2k-1)\pi}{m}, \quad k = 1, 2, \ldots, m.$$

The images $z_{2k} = x_{2k} + iy_{2k}$ respectively $z_{2k-1} = x_{2k-1} + iy_{2k-1}$ $(k = 1, 2, \ldots, m)$ of these points on the cross-section S are called nodal points. The coordinates of these points are given by

$$x_{2k} = \sum_{n=-1}^{m-2} (A_n \cos n\varphi_{2k} + B_n \sin n\varphi_{2k})$$

$$y_{2k} = \sum_{n=-1}^{m-2} (-A_n \sin n\varphi_{2k} + B_n \cos n\varphi_{2k}) \qquad (5.1.9a)$$

$$x_{2k-1} = \sum_{n=-1}^{m-2} (A_n \cos n\varphi_{2k-1} + B_n \sin n\varphi_{2k-1})$$

$$y_{2k-1} = \sum_{n=-1}^{m-2} (-A_n \sin n\varphi_{2k-1} + B_n \cos n\varphi_{2k-1}) \qquad (5.1.9b)$$

So we have an even system of nodal points (x_{2k}, y_{2k}) and an odd system (x_{2k-1}, y_{2k-1}), $k = 1, 2, \ldots, m$, which are the image-points of the points $(1, \varphi_{2k})$ respectively $(1, \varphi_{2k-1})$ at the unit circle. Making use of the properties of orthogonality for trigonometric functions of discrete equally spaced arguments (here: $2\pi/m$), we can invert (5.1.9) in a simple way with respect to the coefficients A_n and B_n. For the even system these relations of orthogonality are

$$\sum_{k=1}^{m} \sin j\varphi_{2k} \sin n\varphi_{2k} = \sum_{k=1}^{m} \cos j\varphi_{2k} \cos n\varphi_{2k} = \begin{cases} 0, & j \neq n \\ m/2, & j = n \end{cases}$$

$$\sum_{k=1}^{m} \sin j\varphi_{2k} \cos n\varphi_{2k} = 0 \qquad (5.1.10)$$

We multiply first the equations (5.1.9a) by $\cos j\varphi_{2k}$ and $\sin j\varphi_{2k}$, respectively, and after that we take the sum with respect to k. Combining this with the relations (5.1.10) affords

$$\sum_{k=1}^{m} (x_{2k} \cos j\varphi_{2k} - y_{2k} \sin j\varphi_{2k}) =$$

$$= \sum_{n=-1}^{m-2} A_n \left(\sum_{k=1}^{m} \cos j\varphi_{2k} \cos n\varphi_{2k} + \right.$$

$$\left. + \sum_{k=1}^{m} \sin j\varphi_{2k} \sin n\varphi_{2k} \right) + \sum_{n=-1}^{m-2} B_n \left(\sum_{k=1}^{m} \cos j\varphi_{2k} \sin n\varphi_{2k} - \right.$$

$$\left. - \sum_{k=1}^{m} \sin j\varphi_{2k} \cos n\varphi_{2k} \right) = mA_j$$

In an analogous manner we multiply (5.1.9a) respectively by $\sin j\varphi_{2k}$ and $\cos j\varphi_{2k}$.

The following results are obtained

$$A_j^{(+m)} = \frac{1}{m} \sum_{k=1}^{m} (x_{2k} \cos j\varphi_{2k} - y_{2k} \sin j\varphi_{2k})$$

$$B_j^{(+m)} = \frac{1}{m} \sum_{k=1}^{m} (x_{2k} \sin j\varphi_{2k} + y_{2k} \cos j\varphi_{2k}) \qquad (5.1.11)$$

$$j = -1, 0, 1, \ldots, m-2$$

The index $(+m)$ has been added to A_j and B_j in order to indicate that these coefficients are determined on the basis of the even nodal points.

Starting from (5.1.9b) we can determine in an analogous way a relation between A_j and B_j and the odd nodal points. In this case the index $(-m)$ is added to A_j and B_j:

$$A_j^{(-m)} = \frac{1}{m} \sum_{k=1}^{m} (x_{2k-1} \cos j\varphi_{2k-1} - y_{2k-1} \sin j\varphi_{2k-1})$$

$$B_j^{(-m)} = \frac{1}{m} \sum_{k=1}^{m} (x_{2k-1} \sin j\varphi_{2k-1} + y_{2k-1} \cos j\varphi_{2k-1})$$

$$j = -1, 0, 1, \ldots, m-2 \qquad (5.1.12)$$

If the nodal points are known we can determine the coefficients by means of the expressions (5.1.11) and (5.1.12) where $A_j = A_j^{(+m)} = A_j^{(-m)}$ and $B_j = B_j^{(+m)} = B_j^{(-m)}$. However, the locations of the nodal points are unknown.

We shall devise now an iteration process, based on the property that $A_j^{(+m)}=A_j^{(-m)}$ and $B_j^{(+m)}=B_j^{(-m)}$. So on account of (5.1.7) and (5.1.8) a relation exists between the even and odd nodal points. One possibility to determine this relation is to eliminate $A_n^{(-m)}$ and $B_n^{(-m)}$ from (5.1.9a) and (5.1.12):

$$x_{2\nu}=\sum_{n=-1}^{m-2}(A_n^{(-m)}\cos n\varphi_{2\nu}+B_n^{(-m)}\sin n\varphi_{2\nu})=$$
$$=\frac{1}{m}\left\{\sum_{k=1}^{m}x_{2k-1}\sum_{n=-1}^{m-2}\cos n(\varphi_{2k-1}-\varphi_{2\nu})-\right.$$
$$\left.-y_{2k-1}\sum_{m=-1}^{m-2}\sin n(\varphi_{2k-1}-\varphi_{2\nu})\right\}$$

$$y_{2\nu}=\sum_{n=-1}^{m-2}(-A_n^{(-m)}\sin n\varphi_{2\nu}+B_n^{(-m)}\cos n\varphi_{2\nu})=$$
$$=\frac{1}{m}\left\{\sum_{k=1}^{m}x_{2k-1}\sum_{n=-1}^{m-2}\sin n(\varphi_{2k-1}-\varphi_{2\nu})-\right.$$
$$\left.-y_{2k-1}\sum_{n=-1}^{m-2}\cos n(\varphi_{2k-1}-\varphi_{2\nu})\right\}$$

In the same way we obtain by eliminating $A_n^{(+m)}$ and $B_n^{(+m)}$ from (5.1.9b) by means of (5.1.11)

$$x_{2\nu-1}=\frac{1}{m}\left\{\sum_{k=1}^{m}x_{2k}\sum_{n=-1}^{m-2}\cos n(\varphi_{2k}-\varphi_{2\nu-1})-\right.$$
$$\left.-y_{2k}\sum_{n=-1}^{m-2}\sin n(\varphi_{2k}-\varphi_{2\nu-1})\right\}$$

$$y_{2\nu-1}=\frac{1}{m}\left\{\sum_{k=1}^{m}x_{2k}\sum_{n=-1}^{m-2}\sin n(\varphi_{2k}-\varphi_{2\nu-1})+\right.$$
$$\left.+y_{2k}\sum_{n=-1}^{m-2}\cos n(\varphi_{2k}-\varphi_{2\nu-1})\right\}$$

We define the following new quantities

$$\gamma^{I(m)}_{2k-1,2\nu}=\frac{1}{m}\sum_{n=-1}^{m-2}\sin n(\varphi_{2k-1}-\varphi_{2\nu});$$
$$\gamma^{I(m)}_{2k,2\nu-1}=\frac{1}{m}\sum_{n=-1}^{m-2}\sin n(\varphi_{2k}-\varphi_{2\nu-1});$$
$$\gamma^{II(m)}_{2k-1,2\nu}=\frac{1}{m}\sum_{n=-1}^{m-2}\cos n(\varphi_{2k-1}-\varphi_{2\nu});$$
$$\gamma^{II(m)}_{2k,2\nu-1}=\frac{1}{m}\sum_{n=-1}^{m-2}\cos n(\varphi_{2k}-\varphi_{2\nu-1}) \qquad (5.1.13)$$

Then we obtain the following recurrence formulas of the iteration process:

$$x^{(n)}_{2\nu-1}=\sum_{k=1}^{m}x^{(n)}_{2k}\gamma^{II(m)}_{2k,2\nu-1}-y^{(n)}_{2k}\gamma^{I(m)}_{2k,2\nu-1}$$
$$y^{(n)}_{2\nu-1}=\sum_{k=1}^{m}x^{(n)}_{2k}\gamma^{I(m)}_{2k,2\nu-1}+y^{(n)}_{2k}\gamma^{II(m)}_{2k,2\nu-1} \qquad (5.1.14)$$

$$x^{(n+1)}_{2\nu}=\sum_{k=1}^{m}x^{(n)}_{2k-1}\gamma^{II(m)}_{2k-1,2\nu}-y^{(n)}_{2k-1}\gamma^{I(m)}_{2k-1,2\nu}$$
$$y^{(n+1)}_{2\nu}=\sum_{k=1}^{m}x^{(n)}_{2k-1}\gamma^{I(m)}_{2k-1,2\nu}+y^{(n)}_{2k-1}\gamma^{II(m)}_{2k-1,2\nu} \qquad (5.1.15)$$

The iteration process is carried out in the following manner: For some $m=4, 8, 16, \ldots$ we select on the basis of a graphical consideration an estimation for the zeroth approximation for the m even points $(x^{(0)}_{2k}, y^{(0)}_{2k})$, $k=1, 2, \ldots, m$. (Kantorovich and Krylov describe in Chapter V, §7, [15], various methods to obtain a suitable estimation for the locations of the nodal points.) By means of (5.1.14) we calculate the accessory odd points which in general will not lie on S. After that, we carry these points to the contour, e.g., along the line which connects this point with the origin and thus obtain the zeroth approximation for the odd nodal points $(x^{(0)}_{2k-1}, y^{(0)}_{2k-1})$. With these points we calculate with (5.1.15) the accessory even points, carry them to the contour and in this way obtain the first approximation for the even nodal points $(x^{(1)}_{2k}, y^{(1)}_{2k})$, etc. We repeat this process until a subsequent approximation coincides with sufficient accuracy with the previous one. In order to increase the accuracy of the transformation we have to take a larger value for m, for example $2m$, taking the even and odd nodal points of the previous iteration as an estimation for the m even nodal points of the new iteration, after which we repeat the iteration process as we described above. In this way we can determine the locations of the nodal points more accurately. So the transformation equations

$$x=\sum_{n=-1}^{m-2}(A_n\cos n\varphi+B_n\sin n\varphi)$$
$$y=\sum_{n=-1}^{m-2}(-A_n\sin n\varphi+B_n\cos\varphi) \qquad (5.1.16)$$

are completely determined now.

5.1.2 Alternative method for the determination of the transformation coefficients

We shall now consider another method which has been developed in 1966 by "Rescona Engineering" in Amstelveen (Holland) in cooperation with W. E. Smith. From the analytical point of view this method is much simpler than the one discussed in the preceding section. Applications of this method proved that it is very useful.

This method is expounded here for cylinders which are symmetrical with respect to the x- and y-axis. This is a case which is usually encountered in naval architecture.

We start from the equations (5.1.8) where we put $r=1$:

$$x = a\left\{\sin\theta + \sum_{n=0}^{N}(-1)^n a_{2n+1}\sin(2n+1)\theta\right\}$$
$$y = a\left\{\cos\theta + \sum_{n=0}^{N}(-1)^{n+1} a_{2n+1}\cos(2n+1)\theta\right\} \tag{5.1.17}$$

These equations describe the relation between the variable θ on the unit-circle in the ζ-plane and the variables x and y along the given contour in the z-plane.

We substitute in (5.1.17) the expressions

$$\sin(2n+1)\theta = \sum_{r=0}^{n}(-1)^r\frac{2n+1}{(2r+1)!}\left[\prod_{k=1}^{r}\{(2n+1)^2-(2k-1)^2\}\right]\sin^{2r+1}\theta$$

$$\cos(2n+1)\theta = \tfrac{1}{2}(2\cos\theta)^{2n+1} + \tfrac{1}{2}\sum_{r=1}^{n}(-1)^r\frac{2n+1}{r!}\left[\prod_{k=r+1}^{2r-1}(2n-k+1)\right](2\cos\theta)^{2n-2r+1}$$

This yields

$$\frac{x}{a} = \sin\theta + \sum_{k=0}^{N} b_k\sin^{2k+1}\theta$$
$$\frac{y}{a} = \cos\theta + \sum_{k=0}^{N} c_k\cos^{2k+1}\theta \tag{5.1.18}$$

where the coefficients b_k and c_k are linear combinations of $a_1, a_3, \ldots, a_{2N+1}$:

$$b_k = \sum_{n=k}^{N}(-1)^{n+k}a_{2n+1}\frac{2n+1}{(2k+1)!}\left[\prod_{s=1}^{K}\{(2n+1)^2-(2s-1)^2\}\right]$$

$$c_k = \sum_{n=k+1}^{N}(-1)^{2n-k+1}a_{2n+1}\frac{2n+1}{(n-k)!}2^{2k}\left[\prod_{s=n-k+1}^{2n-2k-1}(2n-s+1)\right] + (-1)^{k+1}a_{2k+1}2^{2k}$$

On the right half of the cross section in the z-plane we now choose a sequence of m points, such that the first point coincides with the point $(x=0, y=T)$ while the last point coincides with $(x=B_0, y=0)$. T and B_0 represent respectively the draft and the half beam of the cross section (figure 5.1.1). The points of this sequence, which we represent by (x_i, y_i), where $i=1, 2, \ldots, m$, originate from points on the unit circle in the ζ-plane represented by $(1, \theta_i)$. Consequently with the points $(0, T)$ and $(B_0, 0)$ in the z-plane correspond the points $(1, 0)$ and $(1, \pi/2)$ in the ζ-plane. Substituting these values for (x_i, y_i) and θ_i in (5.1.18), we obtain the following system of equations

$$\frac{x_i}{a} - \sin\theta_i - \sum_{k=0}^{N} b_k\sin^{2k+1}\theta_i = 0 \qquad i=2,3,\ldots,m$$
$$\frac{y_i}{a} - \cos\theta_i - \sum_{k=0}^{N} c_k\cos^{2k+1}\theta_i = 0 \qquad i=1,2,\ldots,m-1 \tag{5.1.19}$$

In this way we obtain a system of $2m-2$ equations with the following $N+m$ unknown variables

$$a, a_1, a_3, \ldots, a_{2N+1}$$

and

$$\theta_2, \theta_3, \ldots, \theta_{m-1}$$

In order to make the number of equations equal to the number of unknown variables the relation

$$m = N+2 \tag{5.1.20}$$

between the number of points m, chosen along the cross section, and the number of terms N, we want to consider in the series expansion (5.1.17), has to be satisfied.

The set of equations (5.1.19) are solved by the Newton Raphson Method [16].

Representing the left-hand terms of (5.1.19) by

$$F_i = F_i(x_i, \theta_i, a, a_1, a_3, \ldots, a_{2N+1})$$

and

$$G_i = G_i(y_i, \theta_i, a, a_1, a_3, \ldots, a_{2N+1}),$$

respectively, we obtain the following set of iteration equations

$$\Delta a^j\frac{\partial F_i^j}{\partial a} + \sum_{k=0}^{N}\Delta a_{2k+1}^j\frac{\partial F_i^j}{\partial a_{2k+1}} + \Delta\theta_i^j\frac{\partial F_i^j}{\partial\theta_i} = -F_i^j$$
$$i=2,3,\ldots,m$$

$$\Delta a^j \frac{\partial G_i^j}{\partial a} + \sum_{k=0}^{N} \Delta a_{2k+1}^j \frac{\partial G_i^j}{\partial a_{2k+1}} + \Delta\theta_i^j \frac{\partial G_i^j}{\partial \theta_i} = -G_i^j$$

$$i = 1, 2, \ldots, m-1 \qquad (5.1.21)$$

In these equations $\Delta a^j \equiv a^{j+1} - a^j, \Delta a_{2k+1}^j \equiv a_{2k+1}^{j+1} - a_{2k+1}^j$ and $\Delta\theta_i^j \equiv \theta_i^{j+1} - \theta_i^j$ represent the corrections which are to be added to the j-th iterates a^j, a_{2k+1}^j and θ_i^j in order to obtain the $(j+1)$-th iterates of these variables.

The numerical values of the constants

$$\frac{\partial F_i^j}{\partial a}, \frac{\partial F_i^j}{\partial a_{2k+1}}, \frac{\partial F_i^j}{\partial \theta_i}, F_i^j \quad \text{and} \quad \frac{\partial G_i^j}{\partial a}, \frac{\partial G_i^j}{\partial a_{2k+1}}, \frac{\partial G_i^j}{\partial \theta_i}, G_i^j$$

are obtained by substituting the j-th iterates in the corresponding functions for the variables θ_i, a, a_1, a_3, ..., a_{2N+1}.

It appears that the sections of a conventionally framed ship can be approximated in a reasonable manner with a Lewis-form. Consequently, it is adequate to use for the zeroth iterates of the coefficients a, a_1 and a_3 the Lewis-values which are given in section 5.3. It is observed that the scale factor is there denoted by M. The remaining coefficients $a_5, a_7, \ldots$ are all set equal to zero.

However, for a ship with unconventional sectional shapes, e.g. bulb sections, the Lewis-approximation is not satisfactory anymore. Consequently, we may expect that the Lewis coefficients are rather bad starting values for the iteration process. A suitable procedure for such a ship is to start the transformation process with the midship sections. These sections are in general rather well approximated by Lewis-forms and, therefore, the Lewis-coefficients are suitable starting values. Next, the transformation is carried out for the neighbouring forebody section, while the coefficients of the midship section are used as zeroth iterate for the coefficients of this section, etc.

After having determined the coefficients for the forebody sections, we proceed in a simular manner for the afterbody sections.

Finally, with regard to the starting values of the variables θ_i, equidistant values appear to be adequate in most cases.

5.2 *Calculation of added mass and damping* [10]

Completely similar to the method Ursell used for circular cylinders, the velocity potential for the case of an arbitrary cross section is also composed of a source potential and a linear combination of multipole potentials. For the source potential we take expression (3.28) again, which satisfies the surface condition and the radiation condition. To satisfy the boundary condition on the cylinder we superimpose a suitably chosen linear combination of multipole potentials. Each multipole potential satisfies the surface condition and vanishes when $|x| \to \infty$. The multipole potential is defined by

$$\varphi_{2m} = \frac{\cos 2m\theta}{r^{2m}} + Ka \left\{ \frac{\cos(2m-1)\theta}{(2m-1)r^{2m-1}} + \sum_{n=0}^{N} (-1)^n \frac{(2n+1)a_{2n+1} \cos(2m+2n+1)\theta}{(2m+2n+1)r^{2m+2n+1}} \right\}$$

$$m = 1, 2, 3, \ldots \qquad (5.2.1)$$

We remark that φ_{2m} vanishes for $r \to \infty$.

In order to prove that φ_{2m} satisfies the free-surface condition, we transform this relation first by means of (5.1.8) from rectangular coordinates (x, y) into curvilinear coordinates (r, θ). As $\partial x/\partial\theta = 0$ for $\theta = \pm\pi/2$, (the θ-lines intersect the x-axis perpendicularly), the free surface condition

$$K\phi + \phi_y = 0 \qquad (y = 0)$$

can be written as

$$K\phi + \frac{1}{\left(\frac{\partial y}{\partial \theta}\right)} \cdot \frac{\partial \phi}{\partial \theta} = 0 \qquad \left(\theta = \pm\frac{\pi}{2}\right) \qquad (5.2.2)$$

From (5.1.8) it follows

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -a \left\{ r \sin\theta + \sum_{n=0}^{N} \frac{(-1)^{n+1}(2n+1)a_{2n+1}}{r^{2n+1}} \sin(2n+1)\theta \right\}$$

whence for $\theta = \pm\pi/2$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \mp a \left\{ r - \sum_{n=0}^{N} (2n+1)a_{2n+1} \frac{1}{r^{2n+1}} \right\}$$

In curvilinear coordinates the free-surface condition obtains the form

$$Ka \left\{ r - \sum_{n=0}^{N} (2n+1) \frac{a_{2n+1}}{r^{2n+1}} \right\} \phi \mp \frac{\partial \phi}{\partial \theta} = 0 \qquad \theta = \pm\frac{\pi}{2}$$

$$(5.2.3)$$

Substitution of (5.2.1) into (5.2.3) shows that φ_{2m} satisfies the free-surface condition. Thus the velocity potential which is the solution of the boundary value problem can be written as

$$\Phi = \frac{gb}{\pi\omega}\left\{\left(\phi_c + \sum_{m=1}^{\infty} p_{2m}\varphi_{2m}\right)\cos\omega t + \right.$$

$$\left. + \left(\phi_s + \sum_{m=1}^{\infty} q_{2m}\varphi_{2m}\right)\sin\omega t\right\} \qquad (5.2.4)$$

where ϕ_c and ϕ_s are defined in accordance with (3.29):

$$\phi_c = \pi e^{-Ky}\cos Kx$$

$$\phi_s = \pi e^{-Ky}\sin Kx - \int_0^{\infty} \frac{e^{-\beta x}(\beta\cos\beta y - K\sin\beta y)}{\beta^2 + K^2}\,\mathrm{d}\beta \qquad (5.2.5)$$

By means of the Cauchy-Riemann relations (4.6) we calculate the conjugate stream-function

$$\Psi = \frac{gb}{\pi\omega}\left\{\left(\psi_c + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}\right)\cos\omega t + \right.$$

$$\left. + \left(\psi_s + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}\right)\sin\omega t\right\} \qquad (5.2.6)$$

where

$$\psi_c = \pi e^{-Ky}\sin Kx$$

$$\psi_s = -\pi e^{-Ky}\cos Kx + \int_0^{\infty} \frac{e^{-\beta x}(\beta\sin\beta y + K\cos\beta y)}{\beta^2 + K^2}\,\mathrm{d}\beta$$

$$\psi_{2m} = \frac{\sin 2m\theta}{r^{2m}} + Ka\left\{\frac{\sin(2m-1)\theta}{(2m-1)r^{2m-1}} + \right.$$

$$\left. + \sum_{n=0}^{N}(-1)^n \frac{(2n+1)a_{2n+1}}{2m+2n+1}\cdot\frac{\sin(2m+2n+1)\theta}{r^{2m+2n+1}}\right\} \qquad (5.2.7)$$

It remains to determine the coefficients p_{2m} and q_{2m} in such a way that the boundary condition on the cylinder is satisfied. The reasoning proceeds analogous to the corresponding calculation in chapter 4. The boundary condition on the cylinder has the form

$$\frac{\partial\Phi}{\partial n} = \frac{\mathrm{d}y}{\mathrm{d}t}\cos\alpha \qquad (5.2.8)$$

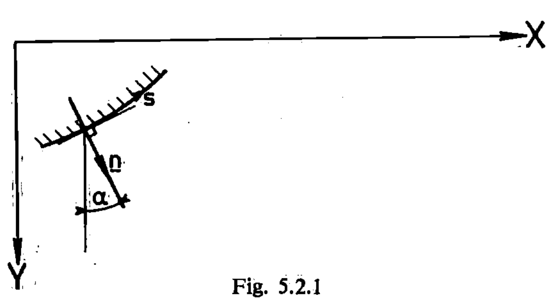

Fig. 5.2.1

α is the angle between the positive normal on the cross section and the positive y-axis, (figure 5.2.1).

It is further easy to see that at the cylinder surface

$$\cos\alpha = \frac{\partial x}{\partial s} = \frac{\partial y}{\partial n}$$

$$\sin\alpha = -\frac{\partial y}{\partial s} \qquad (5.2.9)$$

$$\frac{\partial\Phi}{\partial n} = -\frac{\partial\Psi}{\partial s} = -\frac{\partial\Psi}{r\partial\theta}$$

So the boundary condition at the cylinder, $(r = 1)$, can also be written as

$$-\frac{\partial\Psi}{\partial s} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial x}{\partial s}$$

After integration this leads to

$$\Psi(r = 1, \theta) = -\frac{\mathrm{d}y}{\mathrm{d}t}x(r = 1, \theta) + c(t) \qquad (5.2.10)$$

Substitution of $\theta = 0$, $(x = 0)$, shows that the integration constant $c(t)$ has zero value, so

$$\Psi(r = 1, \theta) = -\frac{\mathrm{d}y}{\mathrm{d}t}x(r = 1, \theta) \qquad (5.2.11)$$

Substitution of $\theta = \pi/2$ gives

$$\Psi\left(1, \frac{\pi}{2}\right) = -\frac{\mathrm{d}y}{\mathrm{d}t}B_0 \qquad (5.2.12)$$

$B_0 = x(1, \pi/2)$ is the half beam of the cross section (figure 5.1.1). Eliminate $\mathrm{d}y/\mathrm{d}t$ from (5.2.11) and (5.2.12) to obtain

$$\Psi(1, \theta) = \frac{x(1, \theta)}{B_0}\Psi\left(1, \frac{\pi}{2}\right) \qquad (5.2.13)$$

Substituting (5.2.6) in this expression and equating successively the coefficients of $\cos\omega t$ and $\sin\omega t$, we see that the coefficients p_{2m} and q_{2m} have to satisfy the relations

$$\psi_c(1, \theta) + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}(1, \theta) =$$

$$\frac{x(1, \theta)}{B_0}\left\{\psi_c\left(1, \frac{\pi}{2}\right) + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}\left(1, \frac{\pi}{2}\right)\right\},$$

$$\psi_s(1, \theta) + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}(1, \theta) =$$

$$= \frac{x(1, \theta)}{B_0}\left\{\psi_s\left(1, \frac{\pi}{2}\right) + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}\left(1, \frac{\pi}{2}\right)\right\}$$

or:

$$\psi_c(1,\theta) - \frac{x(1,\theta)}{B_0}\psi_c\left(1,\frac{\pi}{2}\right) = \sum_{m=1}^{\infty} p_{2m} f_{2m}(1,\theta)$$

$$\psi_s(1,\theta) - \frac{x(1,\theta)}{B_0}\psi_s\left(1,\frac{\pi}{2}\right) = \sum_{m=1}^{\infty} q_{2m} f_{2m}(1,\theta) \quad (5.2.14)$$

where:

$$f_{2m}(1,\theta) = \frac{x(1,\theta)}{B_0}\psi_{2m}\left(1,\frac{\pi}{2}\right) - \psi_{2m}(1,\theta)$$

(See the remark at the end of this section)

The equations (5.2.14) have the same structure as (4.13). The coefficients p_{2m} and q_{2m} may be calculated in a corresponding way.

The velocity potential Φ on the contour of the cylinder, ($r=1$), is written as

$$\Phi(1,\theta) = \frac{gb}{\pi\omega}(M\sin\omega t + N\cos\omega t) \quad (5.2.15)$$

where

$$M(1,\theta) = \phi_s(1,\theta) + \sum_{m=1}^{\infty} q_{2m}\varphi_{2m}(1,\theta)$$

$$N(1,\theta) = \phi_c(1,\theta) + \sum_{m=1}^{\infty} p_{2m}\varphi_{2m}(1,\theta)$$

We calculate the pressure along the contour according to (4.14)

$$p(1,\theta) = -\frac{\varrho g b}{\pi}(M\cos\omega t - N\sin\omega t) \quad (5.2.16)$$

We define

$$\frac{dy}{dt} = \frac{gb}{\pi\omega B_0}(-A\cos\omega t - B\sin\omega t) \quad (5.2.17)$$

hence

$$\frac{d^2y}{dt^2} = \frac{gb}{\pi B_0}(A\sin\omega t - B\cos\omega t) \quad (5.2.18)$$

where according to (5.2.12)

$$A = \psi_c\left(1,\frac{\pi}{2}\right) + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}\left(1,\frac{\pi}{2}\right)$$

$$B = \psi_s\left(1,\frac{\pi}{2}\right) + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}\left(1,\frac{\pi}{2}\right) \quad (5.2.19)$$

We can resolve the pressure into a component in phase with the velocity and a component in phase with the acceleration. This is done in a similar way as in chapter 4.

$$p(1,\theta) = \frac{\varrho g b}{\pi}\frac{MB+NA}{A^2+B^2}(A\sin\omega t - B\cos\omega t) - \frac{\varrho g b}{\pi}\frac{MB-NA}{A^2+B^2}(A\cos\omega t + B\sin\omega t)$$

or

$$p(1,\theta) = \varrho B_0\frac{MB+NA}{A^2+B^2}\ddot{y} + \varrho B_0\frac{MA+NB}{A^2+B^2}\omega\dot{y} \quad (5.2.20)$$

The total vertical force on the cylinder per unit length becomes

$$F = -2\int_{S(0<\theta<\pi/2)} p(1,\theta)\cos\alpha\, ds \quad (5.2.21)$$

From (5.2.9) and (5.1.8) it follows that at the contour of the cylinder

$$\cos\alpha\, ds = \frac{\partial x}{ds}ds = dx =$$

$$= a\left\{\cos\theta + \sum_{n=0}^{N}(-1)^n(2n+1)a_{2n+1}\cos(2n+1)\theta\right\}d\theta =$$

$$aW(\theta)d\theta \quad (5.2.22)$$

where the function between parentheses is denoted by $W(\theta)$. Substitution of $\theta=\pi/2$ and $r=1$ into the first equation of (5.1.8) produces

$$B_0 = a\left\{1 + \sum_{n=0}^{N} a_{2n+1}\right\}$$

Introducing the constant G, defined by

$$G = \left\{1 + \sum_{n=0}^{N} a_{2n+1}\right\}$$

this reduces to

$$a = \frac{B_0}{G} \quad (5.2.23)$$

Consequently (5.2.21) becomes

$$F = -2B_0\int_0^{\pi/2} p(1,\theta)\frac{W(\theta)}{G}d\theta$$

Substitution of (5.2.16) into this equation leads to

$$F = -\frac{2\varrho g b B_0}{\pi}(N_0\sin\omega t - M_0\cos\omega t) \quad (5.2.24)$$

where

$$M_0 = \int_0^{\pi/2} M(1,\theta)\frac{W(\theta)}{G}\,d\theta$$

$$N_0 = \int_0^{\pi/2} N(1,\theta)\frac{W(\theta)}{G}\,d\theta \tag{5.2.25}$$

We resolve the vertical force into a component in phase with the acceleration and a component in phase with the velocity

$$F = -\frac{2\varrho g b B_0}{\pi}\frac{M_0B+N_0A}{A^2+B^2}(A\sin\omega t - B\cos\omega t) - \frac{2\varrho g b B_0}{\pi}\frac{M_0A-N_0B}{A^2+B^2}(-A\cos\omega t - B\sin\omega t)$$

With the aid of (5.2.17) and (5.2.18) the above-mentioned expression becomes

$$F = -2\varrho B_0^2\frac{M_0B+N_0A}{A^2+B^2}\ddot{y} - 2\varrho B_0^2\frac{M_0A-N_0B}{A^2+B^2}\omega\dot{y} \tag{5.2.26}$$

Defining the added mass m'' and the damping N' by representing the force according to $F = -m''\ddot{y} - N'\dot{y}$, we find

$$m'' = 2\varrho B_0^2\frac{M_0B+N_0A}{A^2+B^2} \tag{5.2.27}$$

So in non-dimensional form we obtain for the coefficient of added mass

$$\frac{M_0B+N_0A}{A^2+B^2} \tag{5.2.28}$$

The damping of the cylinder per unit length becomes

$$N' = 2\varrho B_0^2\omega\frac{M_0A-N_0B}{A^2+B^2} \tag{5.2.29}$$

By equating the dissipated energy to the work done by the cylinder, we obtain a relation which is identical to (4.24)

$$M_0A - N_0B = \frac{\pi^2}{2} \tag{5.2.30}$$

So (5.2.29) can be written as

$$N' = \frac{\varrho B_0^2\omega\pi^2}{A^2+B^2} \tag{5.2.31}$$

Analogous to (4.23) we find for the ratio between the wave amplitude at infinite distance from the cylinder and the amplitude of the forced oscillation

$$\frac{KB_0\pi}{\sqrt{A^2+B^2}} \tag{5.2.32}$$

Remark

Instead of eliminating dy/dt from (5.2.11) and (5.2.12) which finally leads to the set of equations (5.2.14) for p_{2m} and q_{2m}, we can also substitute expression (5.2.17) for dy/dt into (5.2.11) and after that equate the coefficients of $\cos\omega t$ and $\sin\omega t$. Successively, we then obtain

$$\Psi(1,\theta) = \frac{gb}{\pi\omega B_0}(A\cos\omega t + B\sin\omega t)\cdot x(1,\theta) \tag{5.2.33}$$

this becomes

$$\psi_c(1,\theta) + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}(1,\theta) = \frac{gb}{\pi\omega B_0}x(1,\theta)\cdot A$$

$$\psi_s(1,\theta) + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}(1,\theta) = \frac{gb}{\pi\omega B_0}x(1,\theta)\cdot B$$

or

$$\psi_c(1,\theta) = \sum_{m=0}^{\infty} p_{2m}f_{2m}$$

$$\psi_s(1,\theta) = \sum_{m=0}^{\infty} q_{2m}f_{2m}$$

where

$$f_0 = \frac{gb}{\pi\omega B_0}x(1,\theta)$$

$$f_{2m} = -\psi_{2m}(1,\theta), \qquad m \neq 0$$

and

$$p_0 = A, \qquad q_0 = B$$

The calculation of A and B according to (5.2.19) is dropped now.

5.3 *Added mass and damping of a cylinder with a "Lewis-form"*

We will consider a special case now of the theory discussed in section 5.2, where we take in the transformation formulas (5.1.8) for N the value 2. In this case the transformed shape is often a reasonable approximation of a cross section of a ship. This kind of sections has frequently been used by Lewis and Grim in their calculations and are known as "Lewis-forms".

Tasai made an extensive study of this case in [8]. For the sake of simplicity we shall adopt the notation which he uses: instead of the polar coordinate r the variable α is introduced which is related with r by: $r = e^{\alpha}$. So the unit circle in the ζ-plane corresponds with $\alpha = 0$ and in the z-plane the coordinate line $\alpha = 0$

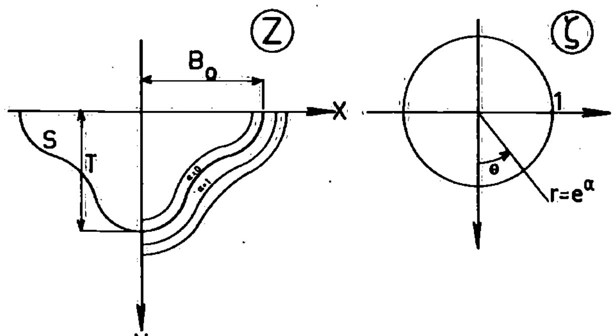

Fig. 5.3.1

coincides with the contour of the section. In Tasai's notation the transformation formulas (5.1.8) become

$$\frac{x}{M} = e^{\alpha}\sin\theta + a_1 e^{-\alpha}\sin\theta - a_3 e^{-3\alpha}\sin 3\theta$$

$$\frac{y}{M} = e^{\alpha}\cos\theta - a_1 e^{-\alpha}\cos\theta + a_3 e^{-3\alpha}\cos 3\theta \qquad (5.3.1)$$

On the contour of the section where $\alpha = 0$, the following relations hold

$$\frac{x_0}{M} = (1+a_1)\sin\theta - a_3\sin 3\theta$$

$$\frac{y_0}{M} = (1-a_1)\cos\theta + a_3\cos 3\theta \qquad (5.3.2)$$

According to (5.2.23), we find for the scale factor of the transformation which is denoted here by M

$$M = \frac{B_0}{1+a_1+a_3}$$

As parameters for the cross section we take

$$H_0 = \frac{B_0}{T} \quad \text{and} \quad \sigma = \frac{S}{2B_0T} \qquad (5.3.3)$$

S is the area of the section, T the draught and B_0 the half beam. On account of the restricted number of terms in the transformation equations it is possible to find an explicit expression for the coefficients a_1 and a_3 in an analytic way. In this way we avoid the iteration process which may require much computer time; we find

$$a_1 = \frac{H_0-1}{H_0+1}(a_3+1)$$

$$a_3 = \frac{-c_1+3+\sqrt{9-2c_1}}{c_1} \qquad (5.3.4)$$

where

$$c_1 = \left(3+\frac{4\sigma}{\pi}\right)+\left(1-\frac{4\sigma}{\pi}\right)\left(\frac{H_0-1}{H_0+1}\right)^2$$

From (5.2.1) it follows that in this case the multipole-potentials φ_{2m} obtain the following shape

$$\varphi_{2m} = e^{-2m\alpha}\cos 2m\theta + \frac{\xi_0}{1+a_1+a_3}\left\{\frac{e^{-(2m-1)\alpha}}{2m-1}\cos(2m-1)\theta + a_1\frac{e^{-(2m+1)\alpha}}{2m+1}\cdot\cos(2m+1)\theta - \frac{3a_3}{2m+3}e^{-(2m+3)\alpha}\cos(2m+3)\theta\right\}$$

$$m = 1, 2, 3 \ldots \qquad (5.3.5)$$

where

$$\xi_0 = K\cdot B_0 = \frac{\omega^2}{g}B_0$$

For the conjugate streamfunction we obtain from (5.2.7)

$$\psi_{2m} = e^{-2m\alpha}\sin 2m\theta + \frac{\xi_0}{1+a_1+a_3}\left\{\frac{e^{-(2m-1)\alpha}}{2m-1}\sin(2m-1)\theta + a_1\frac{e^{-(2m+1)\alpha}}{2m+1}\sin(2m+1)\theta - \frac{3a_3}{2m+1}e^{-(2m+3)\alpha}\sin(2m+3)\theta\right\} \quad m = 1, 2, 3, \ldots \qquad (5.3.6)$$

So the total potential or streamfunction respectively obtains the shapes (5.2.4) and (5.2.6) where we substitute for φ_{2m} and ψ_{2m} the expressions (5.3.5) and (5.3.6). For the calculation of the coefficients p_{2m} and q_{2m} we proceed in a similar way as for the general case in section 5.2.

We saw that the boundary condition on the contour of the cylinder results in condition (5.2.11) which has to be satisfied by the streamfunction Ψ:

$$\Psi(\alpha = 0, \theta) = -\frac{dy}{dt}x(\alpha = 0, \theta) \qquad (5.3.7)$$

Next, we eliminate dy/dt by means of (5.2.12) and after that we equate successively the coefficients of $\cos\omega t$ and $\sin\omega t$ which results in the two expressions (5.2.14), which obtain the following shape here

$$\psi_c(\alpha = 0, \theta; \xi_0, a_1, a_3) - \frac{\sin\theta + a_1\sin\theta - a_3\sin 3\theta}{1+a_1+a_3}\cdot\psi_c\left(\alpha = 0, \theta = \frac{\pi}{2}; \xi_0, a_1, a_3\right) = \sum_{m=1}^{\infty} p_{2m}f_{2m}$$

$$\psi_s(\alpha=0,\theta;\xi_0,a_1,a_3) - \frac{\sin\theta+a_1\sin\theta-a_3\sin 3\theta}{1+a_1+a_3}\cdot$$

$$\cdot\psi_s\left(\alpha=0,\theta=\frac{\pi}{2};\xi_0,a_1,a_3\right)=\sum_{m=1}^{\infty} q_{2m}f_{2m} \qquad (5.3.8)$$

where

$$f_{2m}=-\left[\sin 2m\theta+\frac{\xi_0}{1+a_1+a_3}\left\{\frac{\sin(2m-1)\theta}{2m-1}+\right.\right.$$

$$\left.+a_1\frac{\sin(2m+1)\theta}{2m+1}-\frac{3a_3\sin(2m+3)\theta}{2m+3}\right\}+$$

$$+\frac{\xi_0(-1)^m}{(1+a_1+a_3)^2}\left\{\frac{1}{2m-1}-\frac{a_1}{2m+1}-\frac{3a_3}{2m+3}\right\}\cdot$$

$$\left.\cdot(\sin\theta+a_1\sin\theta-a_3\sin 3\theta)\right] \qquad (5.3.9)$$

The added mass and damping are calculated according to (5.2.27) and (5.2.29) respectively.

In accordance with (5.2.25) and (5.2.10) the quantities M_0, N_0, A and B obtain the form

$$M_0=$$

$$=\int_0^{\pi/2}\phi_s(0,\theta;a_1,a_3,\xi_0)\frac{\cos\theta+a_1\cos\theta-3a_3\cos 3\theta}{1+a_1+a_3}\,d\theta+$$

$$\frac{1}{1+a_1+a_3}\left[\sum_{m=1}^{\infty}(-1)^{m-1}q_{2m}\left(\frac{1+a_1}{4m^2-1}+\frac{9a_3}{4m^2-9}\right)+\right.$$

$$\left.+\frac{\pi\xi_0}{4(1+a_1+a_3)}\{(1+a_1-a_1a_3)q_2-a_3q_4\}\right]$$

$$N_0=$$

$$=\int_0^{\pi/2}\phi_c(0,\theta;a_1,a_3,\xi_0)\frac{\cos\theta+a_1\cos\theta-3a_3\cos 3\theta}{1+a_1+a_3}\,d\theta+$$

$$+\frac{1}{1+a_1+a_3}\left[\sum_{m=1}^{\infty}(-1)^{m-1}p_{2m}\left(\frac{1+a_1}{4m^2-1}+\frac{9a_3}{4m^2-9}\right)+\right.$$

$$\left.+\frac{\pi\xi_0}{4(1+a_1+a_3)}\{(1+a_1-a_1a_3)p_2-a_3p_4\}\right]$$

$$A=\psi_c\left(0,\frac{\pi}{2};a_1,a_3,\xi_0\right)+\sum_{m=1}^{\infty}p_{2m}(\xi_0)(-1)^{m-1}$$

$$\frac{\xi_0}{1+a_1+a_3}\left\{\frac{1}{2m-1}-\frac{a_1}{2m+1}-\frac{3a_3}{2m+3}\right\}$$

$$B=\psi_s\left(0,\frac{\pi}{2},a_1,a_3,\xi_0\right)+\sum_{m=1}^{\infty}q_{2m}(\xi_0)(-1)^{m-1}$$

$$\frac{\xi_0}{1+a_1+a_3}\left\{\frac{1}{2m-1}-\frac{a_1}{2m+1}-\frac{3a_3}{2m+3}\right\}$$

6 Swaying and rolling

In this chapter we determine expressions for the hydrodynamic coefficients for a cylinder which carries out a forced harmonic swaying or rolling motion. In addition we shall pay attention to the coupled motion between rolling and swaying.

According to (2.11) the velocity potential which is a solution of this boundary value problem has to be a skew-symmetric function. Consequently, the source potential (3.28) and the multipole potentials which are used for solving the symmetric problem are no longer usable here. Consequently, the first thing we have to do in this chapter is to derive the potentials which will replace the source potential and the multipole potentials respectively. After that we proceed to the calculation of the hydrodynamic coefficients.

6.1 *Potential of a dipole in the origin, skew-symmetric multipole potentials*

For physical reasons it is easy to see that a dipole produces a flow field which is skew-symmetric with respect to a line through the dipole perpendicular to the direction of its axis.

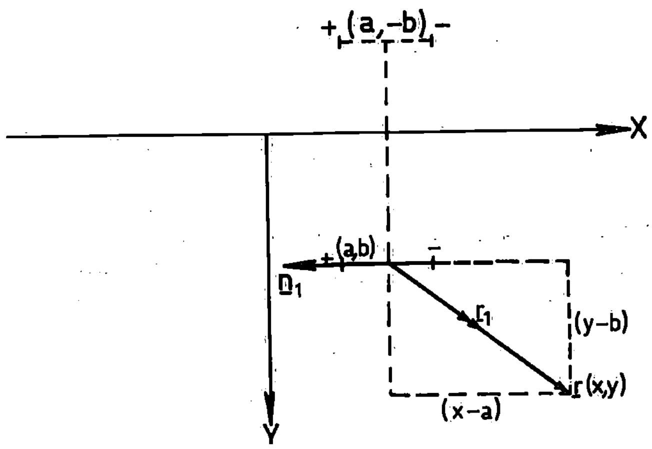

Fig. 6.1.1

We know that the potential in a point (x, y) due to the presence of a dipole in a point (a, b) with moment M is given by the following formula [13]

$$\phi_{\text{dip}}=\frac{-M(\underline{r}_1\cdot\underline{n}_1)}{|\underline{r}|}=\frac{-M(\underline{r}\cdot\underline{n}_1)}{|\underline{r}|^2}=\frac{-M(x-a)}{(x-a)^2+(y-b)^2} \qquad (6.1.1)$$

$\underline{n}_1$ is the unit vector in the direction of the axis, $\underline{r}$ the vector which connects (a, b) with (x, y) (see figure 6.1.1).

We assume a dipole of strength $\frac{1}{2}K$ in the point (a, b) and a dipole of equal strength in the point

$(a, -b)$. Both dipoles have their axis in the negative x-direction. The potentials of these dipoles are represented by

$$\phi_{dip}(a, b) = \frac{1}{2K} \cdot \frac{(x-a)}{(x-a)^2+(y-b)^2}$$

$$\phi_{dip}(a, -b) = \frac{1}{2K} \cdot \frac{(x-a)}{(x-a)^2+(y+b)^2} \qquad (6.1.2)$$

We now consider the potential

$$\phi = \phi_{dip}(a, b) + \phi_{dip}(a, -b) + \phi_r \qquad (6.1.3)$$

ϕ_r is a regular function, which we determine in such a way that ϕ satisfies the free-surface condition $K\phi + \phi_y = 0$. Substituting ϕ in this relation, we obtain for ϕ_r the condition

$$K\phi_r + \phi_{r_y} = -\frac{(x-a)}{(x-a)^2+b^2} = -\int_0^\infty e^{-pb} \sin(x-a)p \, dp \qquad (b > 0) \quad (6.1.4)$$

since

$$\frac{x-a}{(x-a)^2+b^2}$$

is the Laplace transform of $\sin(x-a)p$.

In an analogous manner as for the source potential in chapter 3 we find for ϕ_r the expression

$$\phi_r = -\int_{M_2} \frac{e^{-p(y+b)} \sin p(x-a)}{K-p} dp \qquad (6.1.5)$$

Here also the uniqueness of the regular function ϕ_r is furnished by the condition that for the limit case $|x| \to \infty$ ϕ_r has to represent the regular outgoing progressive wave (2.10). For this reason the contour M_1 is excluded, (figure 3.3). So, finally, for the potential of a dipole in the origin ($a = b = 0$) we obtain the expression

$$\phi = \frac{x}{K(x^2+y^2)} - \int_{M_2} \frac{e^{-py} \sin px}{K-p} dp \qquad (6.1.6)$$

which has been constructed in such a way that it satisfies the free surface condition, represents the regular outgoing progressive wave (2.10) if $|x| \to \infty$ and is a skew-symmetric function.

In an analogous manner as ϕ in (3.22) we can reduce the second term of (6.1.6) to

$$\phi_r = \mp \int_0^\infty \frac{e^{\mp\beta x}\{K\cos\beta y + \beta \sin \beta y\}}{K^2+\beta^2} d\beta \pm \pi e^{-Ky \mp iKx} \qquad x \gtrless 0 \quad (6.1.7)$$

In this chapter we will use for the dipole potential, Φ_{dip}, which satisfies both the free-surface condition and the radiation condition, the expression defined by

$$\Phi_{dip} = \frac{gb}{\pi\omega} Re\{-i\phi e^{i\omega t}\} = \frac{gb}{\pi\omega}\{\phi_c \cos\omega t + \phi_s \sin\omega t\} \qquad (6.1.8)$$

Comparing this with (6.1.6) and (6.1.7), we obtain

$$\phi_c = -\pi e^{-Ky} \sin Kx$$

$$\phi_s = \pm \pi e^{-Ky} \cos Kx \mp \int_0^\infty \frac{K\cos\beta y + \beta \sin\beta y}{\beta^2+K^2} e^{\mp\beta x} d\beta + \frac{x}{K(x^2+y^2)} \qquad x \gtrless 0$$

The conjugate streamfunction Ψ_{dip} of Φ_{dip} is given by

$$\Psi_{dip} = \frac{gb}{\pi\omega}\{\psi_c \cos\omega t + \psi_s \sin\omega t\}$$

where

$$\psi_c = \pi e^{-Ky} \cos Kx$$

$$\psi_s = \pi e^{-Ky} \sin K|x| - \int_0^\infty e^{-\beta|x|} \frac{\beta\cos\beta y - K\sin\beta y}{K^2+\beta^2} d\beta - \frac{y}{K(x^2+y^2)} \qquad (6.1.9)$$

Finally, we define the skew-symmetric multipole potentials by

$$\varphi_{2m} = \frac{\sin(2m+1)\theta}{r^{2m+1}} + Ka\left\{\frac{\sin 2m\theta}{2mr^{2m}} + \sum_{n=0}^{N} \frac{(-1)^n a_{2n+1}(2n+1)\sin(2n+2m+2)\theta}{(2m+2n+2)r^{2m+2n+2}}\right\}$$
$$m = 1, 2, 3, \ldots \qquad (6.1.10)$$

Substituting (6.1.10) in (5.2.3), we see that φ_{2m} satisfies the free surface condition. The conjugate streamfunction ψ_{2m} of φ_{2m} is

$$\psi_{2m} = -\frac{\cos(2m+1)\theta}{r^{2m+1}} - Ka\left\{\frac{\cos 2m\theta}{2mr^{2m}} + \sum_{n=0}^{N} \frac{(-1)^n a_{2n+1}(2n+1)\cos(2m+2n+2)\theta}{(2m+2n+2)r^{2m+2n+2}}\right\}$$
$$m = 1, 2, 3, \ldots \qquad (6.1.11)$$

Completely analogous to the problem of the heaving cylinder we determine a solution of the boundary value problem, where the cylinder is making a swaying or rolling motion. The velocity potential is composed of the dipole potential (6.1.8) and a linear combination of multipole potentials (6.1.10):

$$\Phi = \frac{gb}{\pi\omega}\left[\left\{\phi_c + \sum_{m=1}^{\infty} p_{2m}\varphi_{2m}\right\}\cos\omega t + \right.$$
$$\left. + \left\{\phi_s + \sum_{m=1}^{\infty} q_{2m}\varphi_{2m}\right\}\sin\omega t\right] \tag{6.1.12}$$

For the conjugate streamfunction we find

$$\Psi = \frac{gb}{\pi\omega}\left[\left\{\psi_c + \sum_{m=1}^{\infty} p_{2m}\psi_{2m}\right\}\cos\omega t + \right.$$
$$\left. + \left\{\psi_s + \sum_{m=1}^{\infty} q_{2m}\psi_{2m}\right\}\sin\omega t\right] \tag{6.1.13}$$

6.2 *Added mass and damping for swaying; added moment of inertia and damping for rolling produced by the swaying motion*

The boundary condition at the cylinder becomes in this case (see figure 5.2.1)

$$\frac{\partial\Phi}{\partial n} = \frac{dx}{dt}\sin\alpha \tag{6.2.1}$$

With (5.2.9) this can be written as

$$\frac{\partial\Psi}{\partial s} = \frac{dx}{dt}\frac{dy}{ds} \tag{6.2.2}$$

After integration this leads to

$$\Psi(r=1,\theta) = \frac{dx}{dt}y(r=1,\theta) + C(t) \tag{6.2.3}$$

Substitution of $\theta = \pi/2$ gives: $C(t) = \psi(1, \pi/2)$.
Consequently

$$\Psi(1,\theta) - \Psi\left(1,\frac{\pi}{2}\right) = \frac{dx}{dt}y(1,\theta) \tag{6.2.4}$$

Analogous to the procedure we followed in reducing (5.2.11) to (5.2.13) we can eliminate dx/dt. The other method which we discussed in the remark at the end of section 5.2 is to substitute into the above-mentioned form the expression $dx/dt = -x_a\omega\sin(\omega t+\gamma)$, (assuming x to be given by $x = x_a\cos(\omega t+\gamma)$), so that an expression is obtained similar to (5.2.33). We apply the first method here.

Substitution of $\theta = 0$ into (6.2.4) gives

$$\Psi(1,0) - \Psi\left(1,\frac{\pi}{2}\right) = \frac{dx}{dt}T \tag{6.2.5}$$

Eliminating dx/dt from (6.2.4) and (6.2.5) leads to

$$\frac{1}{y(1,\theta)}\left\{\Psi(1,\theta) - \Psi\left(1,\frac{\pi}{2}\right)\right\} = \frac{1}{T}\left\{\Psi(1,0) - \Psi\left(1,\frac{\pi}{2}\right)\right\}$$

Next, we substitute (6.1.13) in this expression and equate the coefficients of $\cos\omega t$ and $\sin\omega t$ successively. This produces

$$\left\{\psi_c(1,\theta) - \psi_c\left(1,\frac{\pi}{2}\right)\right\} - \frac{y(1,\theta)}{T}\left\{\psi_c(1,0) - \psi_c\left(1,\frac{\pi}{2}\right)\right\} = $$
$$= \sum_{m=1}^{\infty} p_{2m}f_{2m}$$

$$\left\{\psi_s(1,\theta) - \psi_s\left(1,\frac{\pi}{2}\right)\right\} - \frac{y(1,\theta)}{T}\left\{\psi_s(1,0) - \psi_s\left(1,\frac{\pi}{2}\right)\right\} = $$
$$= \sum_{m=1}^{\infty} q_{2m}f_{2m} \tag{6.2.6}$$

where

$$f_{2m} = \frac{y(1,\theta)}{T}\left\{\psi_{2m}(1,0) - \psi_{2m}\left(1,\frac{\pi}{2}\right)\right\} - $$
$$\left\{\psi_{2m}(1,\theta) - \psi_{2m}\left(1,\frac{\pi}{2}\right)\right\}$$

(6.2.6) represents a set of equations from which p_{2m} and q_{2m} can be calculated in the usual way.

Analogous to (5.2.17) we define

$$\frac{dx}{dt} = \frac{gb}{\pi\omega T}\{-A\cos\omega t - B\sin\omega t\} \tag{6.2.7}$$

hence

$$\frac{d^2x}{dt^2} = \frac{gb}{\pi T}\{A\sin\omega t - B\cos\omega t\}$$

where according to (6.1.13) and (6.2.5)

$$A = \psi_c\left(1,\frac{\pi}{2}\right) - \psi_c(1,0) + $$
$$+ \sum_{m=1}^{\infty} p_{2m}\left\{\psi_{2m}\left(1,\frac{\pi}{2}\right) - \psi_{2m}(1,0)\right\}$$

$$B = \psi_s\left(1,\frac{\pi}{2}\right) - \psi_s(1,0) + $$
$$+ \sum_{m=1}^{\infty} q_{2m}\left\{\psi_{2m}\left(1,\frac{\pi}{2}\right) - \psi_{2m}(1,0)\right\} \tag{6.2.8}$$

Entirely equivalent to (5.2.15) we define for the potential along the cylinder

$$\Phi(1,\theta) = \frac{gb}{\pi\omega}(M\sin\omega t + N\cos\omega t) \tag{6.2.9}$$

Then we find for the pressure along the cylinder

$$p(1,\theta) = -\frac{\varrho g b}{\pi}(M\cos\omega t - N\sin\omega t) \tag{6.2.10}$$

where on account of (6.1.12)

$$M(1,\theta) = \phi_s(1,\theta) + \sum_{m=1}^{\infty} q_{2m}\varphi_{2m}(1,\theta)$$

$$N(1,\theta) = \phi_c(1,\theta) + \sum_{m=1}^{\infty} p_{2m}\varphi_{2m}(1,\theta)$$

Analogous to (5.2.20) we find from (6.2.7) and (6.2.10)

$$p(1,\theta) = \varrho T\frac{MB+NA}{A^2+B^2}\ddot{x} + \varrho T\frac{MA-NB}{A^2+B^2}\omega\dot{x} \tag{6.2.11}$$

The total horizontal hydrodynamic force becomes

$$F_s = -\int_{S(0<\theta<\pi/2)} \{p(1,\theta) - p(1,-\theta)\}\sin\alpha\,\mathrm{d}s \tag{6.2.12}$$

From (5.2.9), (5.1.8) and (5.2.23) it follows

$$\sin\alpha\,\mathrm{d}s = -\frac{\partial y}{\partial s}\mathrm{d}s = -\mathrm{d}y = -a\Big\{-\sin\theta +$$

$$+\sum_{n=0}^{N}(-1)^n a_{2n+1}(2n+1)\sin(2n+1)\theta\Big\}\mathrm{d}\theta = \frac{B_0}{G}V(\theta)\,\mathrm{d}\theta \tag{6.2.13}$$

where the function between parentheses is denoted by $V(\theta)$.

As the pressure is skew-symmetric in θ, (6.2.12) becomes

$$F_s = -2B_0\int_0^{\pi/2} p(1,\theta)\frac{V(\theta)}{G}\mathrm{d}\theta \tag{6.2.14}$$

Substitution of (6.2.11) gives

$$F_s = -2\varrho T B_0\frac{M_0B+N\ A}{A^2+B^2}\ddot{x} - 2\varrho T B_0\frac{M_0A-N_0B}{A^2+B^2}\omega\dot{x}$$

where

$$N_0 = \int_0^{\pi/2} N(1,\theta)\frac{V(\theta)}{G}\mathrm{d}\theta$$

$$M_0 = \int_0^{\pi/2} M(1,\theta)\frac{V(\theta)}{G}\mathrm{d}\theta$$

When we define the relation between the swaying force and the added mass M_s and damping N_s for the swaying motion by

$$F_s = -M_s\ddot{x} - N_s\dot{x}$$

we find for the added mass per unit length

$$M_s = 2\varrho T B_0\frac{M_0B+N_0A}{A^2+B^2} \tag{6.2.15}$$

and for the damping per unit length

$$N_s = 2\varrho T B_0\frac{M_0A-N_0B}{A^2+B^2}\omega \tag{6.2.16}$$

From figure 5.2.1 we see that on account of the skew-symmetry of $p(1,\theta)$ in θ the moment on the cylinder produced by the swaying motion, (clockwise is positive), is expressed by

$$M_{RS} = 2\int_{S(0<\theta<\pi/2)} \{p\sin\alpha\cdot y - p\cos\alpha\cdot x\}\,\mathrm{d}s$$

Combining this with (5.2.9) yields

$$M_{RS} = -2\int_0^{\pi/2} p\left(x\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta}\right)\mathrm{d}\theta \tag{6.2.17}$$

Substitution of (6.2.10) gives

$$M_{RS} = \frac{B_0^2 2\varrho g b}{\pi}\{-X_R\sin\omega t + Y_R\cos\omega t\} \tag{6.2.18}$$

where

$$X_R = \frac{1}{B_0^2}\int_0^{\pi/2} N(1,\theta)\left(x\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta}\right)\mathrm{d}\theta$$

$$Y_R = \frac{1}{B_0^2}\int_0^{\pi/2} M(1,\theta)\left(x\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta}\right)\mathrm{d}\theta$$

We resolve the moment into a component in phase with the acceleration and a component in phase with the velocity according to

$$M_{RS} = I_{RS}\left(-\frac{d^2x}{\mathrm{d}t^2}\right) + N_{RS}\left(-\frac{\mathrm{d}x}{\mathrm{d}t}\right) \tag{6.2.19}$$

N_{RS} and I_{RS} represent the damping and added moment of inertia for the rolling motion produced by swaying.

From (6.2.7) and (6.2.18) it follows

$$M_{RS} = \frac{-2\varrho g b B_0^2}{\pi}\frac{BY_R+AX_R}{A^2+B^2}(A\sin\omega t - B\cos\omega t) -$$

$$\frac{2\varrho g b B_0^2}{\pi}\frac{AY_R - BX_R}{A^2+B^2}(-A\cos\omega t - B\sin\omega t) =$$

$$= -2\varrho T B_0^2\frac{BY_R+AX_R}{A^2+B^2}\ddot{x} - 2\varrho\omega T B_0^2\frac{AY_R-BX_R}{A^2+B^2}\dot{x}$$

hence

$$I_{RS} = 2\varrho T B_0^2\frac{BY_R+AX_R}{A^2+B^2}$$

$$N_{RS} = 2\varrho\omega T B_0^2\frac{AY_R-BX_R}{A^2+B^2} \qquad (6.2.20)$$

For the ratio between the wave amplitude at infinite distance from the cylinder and the amplitude of the forced oscillation we obtain

$$\frac{b}{x_a} = \frac{\pi K T}{\sqrt{A^2+B^2}}$$

Between the coefficients M_0, N_0, A and B, the relation

$$M_0 A - N_0 B = \frac{\pi^2}{2}\frac{T}{B_0}$$

is valid.

We derive now expressions [9] for the above mentioned quantities M_s, N_s and I_{RS}, N_{RS} for the special case of a Lewis-form.

The multipole potential obtains the form

$$\varphi_{2m}(\alpha,\theta) = \left[e^{-(2m+1)\alpha}\sin(2m+1)\theta + \frac{\xi_0}{1+a_1+a_3}\left\{\frac{e^{-2m\alpha}}{2m}\sin 2m\theta + \frac{a_1 e^{-(2m+2)\alpha}}{2m+2}\sin(2m+2)\theta + \frac{3a_3}{2m+4}e^{-(2m+4)\alpha}\sin(2m+4)\theta\right\}\right] \qquad (6.2.21)$$

$m = 1, 2, \ldots$

The streamfunction becomes

$$\psi_{2m}(\alpha,\theta) = \left[e^{-(2m+1)\alpha}\cos(2m+1)\theta - \frac{\xi_0}{1+a_1+a_3}\left\{\frac{e^{-2m\alpha}}{2m}\cos 2m\theta + \frac{a_1 e^{-(2m+2)\alpha}}{2m+2}\cos(2m+2)\theta - \frac{3a_3}{2m+4}e^{-(2m+4)\alpha}\cos(2m+4)\theta\right\}\right] \qquad (6.2.22)$$

$m = 1, 2, \ldots$

while (6.2.4) has the form

$$\left[\psi_c(\alpha=0,\theta) - \psi_c\left(\alpha=0,\frac{\pi}{2}\right)\right]\cos\omega t + \left[\psi_s(0,\theta) - \psi_s\left(0,\frac{\pi}{2}\right)\right]\sin\omega t + \cos\omega t\sum_{m=1}^{\infty} p_{2m}\left[-\cos(2m+1)\theta - \frac{\xi_0}{1+a_1+a_3}\left\{\frac{\cos 2m\theta}{2m} + \frac{a_1\cos(2m+2)\theta}{2m+2} - \frac{3a_3\cos(2m+4)\theta}{2m+4}\right\} - \frac{\xi_0(-1)^{m+1}}{1+a_1+a_3}\left(\frac{1}{2m} - \frac{a_1}{2m+2} - \frac{3a_3}{2m+4}\right)\right] + \sin\omega t\sum_{m=1}^{\infty} q_{2m}\left[-\cos(2m+1)\theta - \frac{\xi_0}{1+a_1+a_3}\left\{\frac{\cos 2m\theta}{2m} + \frac{a_1\cos(2m+2)\theta}{2m+2} - \frac{3a_3\cos(2m+4)\theta}{2m+4}\right\} - \frac{\xi_0(-1)^{m+1}}{1+a_1+a_3}\left(\frac{1}{2m} - \frac{a_1}{2m+2} - \frac{3a_3}{2m+4}\right)\right] = \left(\frac{\omega\pi}{g\eta}\right)\frac{dx}{dt}M\{(1-a_1)\cos\theta + a_3\cos 3\theta\} \qquad (6.2.23)$$

We now eliminate dx/dt in the usual way.

However, in accordance with the method followed by Tasai in his publication, (see remark at the end of section 5.2), we shall here substitute an expression for dx/dt with the structure of (6.2.7).

We define

$$x = x_a\cos(\omega t + \gamma)$$

then

$$\frac{dx}{dt} = -x_a\omega\sin(\omega t+\gamma) \qquad (6.2.24)$$

The righthand side of (6.2.23) can now be written as

$$\left(\frac{\omega\pi}{gb}\right)\frac{dx}{dt}M\{(1-a_1)\cos\theta + a_3\cos 3\theta\} =$$

$$= h(\theta)(p_0\cos\omega t + q_0\sin\omega t) \qquad (6.2.25)$$

where

$$h(\theta) = \frac{\{(1-a_1)\cos\theta + a_3\cos 3\theta\}}{1+a_1+a_3} \qquad (6.2.26)$$

$$p_0 = -\frac{\pi x_a}{b}\xi_0\sin\gamma; \quad q_0 = -\frac{\pi x_a}{b}\xi_0\cos\gamma$$

After equating in (6.2.25) the coefficients of $\cos\omega t$ and $\sin\omega t$ successively, we obtain a system of linear equations for the coefficients p_{2m} and q_{2m}

$$\psi_c(0,\theta)-\psi_c\left(0,\frac{\pi}{2}\right)=\sum_{m=0}^{\infty} f_{2m}(\theta)p_{2m}$$

$$\psi_s(0,\theta)-\psi_s\left(0,\frac{\pi}{2}\right)=\sum_{m=0}^{\infty} f_{2m}(\theta)q_{2m}$$

where

$$f_0(\theta)=h(\theta)$$

$$f_{2m}(\theta)=\cos(2m+1)\theta+\frac{\xi_0}{1+a_1+a_3}\left\{\frac{\cos 2m\theta}{2m}+\right.$$

$$\left.+\frac{a_1\cos(2m+2)\theta}{2m+2}-\frac{3a_3\cos(2m+4)\theta}{2m+4}\right\}+$$

$$+\frac{\xi_0(-1)^{m+1}}{1+a_1+a_3}\left\{\frac{1}{2m}-\frac{a_1}{2m+2}-\frac{3a_3}{2m+4}\right\}.$$

$$m=1,2,\ldots \tag{6.2.28}$$

The values for the coefficients p_{2m} and q_{2m} can now be determined in the usual way.

From (6.2.25) it follows

$$\frac{dx}{dt}=\frac{gb}{\pi\omega B_0}\{p_0\cos\omega t+q_0\sin\omega t\}$$

Hence

$$\frac{d^2x}{dt^2}=\frac{gb}{\pi B_0}\{q_0\cos\omega t-p_0\sin\omega t\} \tag{6.2.29}$$

According to (6.2.14) we obtain for the hydrodynamic force in the x-direction

$$F=2\varrho B_0\left(\frac{gb}{\pi}\right)\{-N_0\sin\omega t+M_0\cos\omega t\} \tag{6.2.30}$$

where

$$N_0=-\int_0^{\pi/2}\phi(0,\theta)\frac{(1-a_1)\sin\theta+3a_3\sin 3\theta}{1+a_1+a_3}d\theta-$$

$$\frac{3a_3}{1+a_1+a_3}\cdot\frac{\pi}{4}p_2-\sum_{m=1}^{\infty}p_{2m}\frac{\xi_0(-1)^{m-1}}{(1+a_1+a_3)^2}$$

$$\left[\left\{\frac{1}{4m^2-1}-\frac{a_1}{(2m+2)^2-1}-\frac{3a_3}{(2m+4)^2-1}\right\}(1-a_1)+\right.$$

$$\left.+3a_3\left\{\frac{-1}{4m^2-9}+\frac{a_1}{(2m+2)^2-9}+\frac{3a_3}{(2m+4)^2-9}\right\}\right]$$

M_0 is obtained by replacing ϕ_c by ϕ_s and p_{2m} by q_{2m} in the above-mentioned form. From (6.2.29) and (6.2.30) we can derive the added mass and damping in the usual way:

$$M_S=2\varrho B_0^2\frac{N_0p_0+M_0q_0}{p_0^2+q_0^2}$$

$$N_S=2\varrho\omega B_0^2\frac{M_0p_0-N_0q_0}{p_0^2+q_0^2} \tag{6.2.31}$$

Substitution of (6.2.10) and (5.3.2) into (6.2.17), where in the expression for $M(1,\theta)$ and $N(1,\theta)$ we have to substitute for φ_{2m} the expression (6.2.21), yields the rolling moment produced by the swaying motion:

$$M_{RS}=\frac{4\varrho B_0^2 gb}{\pi}\{X_R\sin\omega t-Y_R\cos\omega t\} \tag{6.2.32}$$

where

$$X_R=\int_0^{\pi/2}\frac{\phi_c(0,\theta)}{(1+a_1+a_3)^2}\{a_1(1+a_3)\sin 2\theta-2a_3\sin 4\theta\}d\theta+$$

$$+\frac{\pi\xi_0(a_1p_2-a_3p_4)}{8(1+a_1+a_3)^2}+\sum_{m=1}^{\infty}\frac{p_{2m}(-1)^{m+1}}{(1+a_1+a_3)^2}$$

$$\left\{\frac{2a_1(1+a_3)}{(2m+1)^2-4}+\frac{8a_3}{(2m+1)^2-16}\right\}$$

Y_R is obtained by replacing in above expression ϕ_c by ϕ_s and p_{2m} by q_{2m}. The added moment of inertia and the damping moment for the rolling motion produced by swaying are now derived from (6.2.29), (6.2.32) and (6.2.19):

$$I_{RS}=4B_0^3\frac{p_0X_R+q_0Y_R}{p_0^2+q_0^2}$$

$$N_{RS}=4B_0^3\frac{p_0Y_R-q_0X_R}{p_0^2+q_0^2} \tag{6.2.33}$$

6.3 *Added moment of inertia and damping for rolling, added mass and damping for swaying produced by the rolling motion*

When the cylinder carries out a harmonic rolling motion about the origin, represented by $\vartheta=\vartheta_a\cos(\omega t+\gamma)$, then the boundary condition on the cylinder has the form (see figure 6.3.1)

$$\frac{\partial\Phi}{\partial n}=R\frac{d\vartheta}{dt}\sin\varphi=R\frac{d\vartheta}{dt}\frac{dR}{ds} \tag{6.3.1}$$

As can be seen from figure 6.3.1, φ is the angle between the tangent on the contour and the velocity along the surface, ϑ the rolling angle, (positive in clockwise direction), and R is the distance between the origin and a point on the surface. Combining (6.3.1) with (5.2.9) yields

$$-\frac{\partial\Psi}{\partial s}=\frac{d\vartheta}{dt}\frac{d}{ds}\{\tfrac{1}{2}(x^2(1,\theta)+y^2(1,\theta))\} \tag{6.3.2}$$

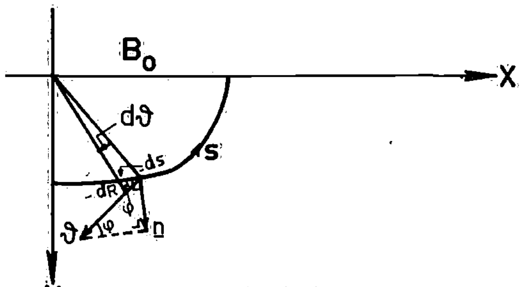

Fig. 6.3.1

After integration this is reduced to

$$\Psi(1,\theta) = -\tfrac{1}{2}\left(\frac{d\vartheta}{dt}\right)\{x^2(1,\theta)+y^2(1,\theta)\}+C(t) \qquad (6.3.3)$$

Substitution of $\theta = \pi/2$ gives

$$C(t) = \Psi\left(1,\frac{\pi}{2}\right)+\tfrac{1}{2}\left(\frac{d\vartheta}{dt}\right)B_0^2$$

So we obtain for the streamfunction the expression

$$\Psi(1,\theta)-\Psi\left(1,\frac{\pi}{2}\right) = -\tfrac{1}{2}\frac{d\vartheta}{dt}\{x^2(1,\theta)+y^2(1,\theta)-B_0^2\} \qquad (6.3.4)$$

For a change we shall not eliminate $d\vartheta/dt$ but we substitute for this quantity $d\vartheta/dt = -\vartheta_a\omega\sin(\omega t+\gamma)$ (see remark at the end of section 5.2). Consequently

$$\Psi(1,\theta)-\Psi\left(1,\frac{\pi}{2}\right) = \tfrac{1}{2}\vartheta_a\omega\sin(\omega t+\gamma)\{x^2(1,\theta)+$$
$$+y^2(1,\theta)-B_0^2\}$$

or

$$\frac{\pi\omega}{gb}\left\{\Psi(1,\theta)-\Psi\left(1,\frac{\pi}{2}\right)\right\} = \frac{\pi\omega}{gb}\tfrac{1}{2}\vartheta_a\omega\{x^2(1,\theta)+$$
$$+y^2(1,\theta)-B_0^2\}\sin(\omega t+\gamma). \qquad (6.3.5)$$

The righthandside of the above mentioned form is written as

$$g(\theta)(p_0\cos\omega t+q_0\sin\omega t)$$

where

$$g(\theta) = \frac{x^2(1,\theta)+y^2(1,\theta)-B_0^2}{B_0^2}$$

$$p_0 = \frac{\pi\vartheta_a K B_0^2}{2b}\sin\gamma$$

$$q_0 = \frac{\pi\vartheta_a K B_0^2}{2b}\cos\gamma \qquad (6.3.6)$$

Substituting (6.1.13) into (6.3.5) and equating successively the coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain a set of linear equations for p_{2m} and q_{2m}:

$$\psi_c(1,\theta)-\psi_c\left(1,\frac{\pi}{2}\right) = \sum_{m=0}^{\infty} p_{2m}f_{2m}(\theta)$$

$$\psi_s(1,\theta)-\psi_s\left(1,\frac{\pi}{2}\right) = \sum_{m=0}^{\infty} q_{2m}f_{2m}(\theta) \qquad (6.3.7)$$

where

$$f_0 = g(\theta) = \frac{x^2(1,\theta)+y^2(1,\theta)-B_0^2}{B_0^2}$$

$$f_{2m} = \psi_{2m}\left(1,\frac{\pi}{2}\right)-\psi_{2m}(1,\theta) \qquad m\neq 0$$

From (6.3.6) it follows:

$$\sqrt{p_0^2+q_0^2} = \frac{\pi\vartheta_a K B_0^2}{2b}$$

So for the ratio

$$\frac{\text{wave amplitude at infinity}}{\text{oscillation amplitude of the cylinder}}$$

is found

$$\frac{b}{\vartheta_a} = \frac{\pi K B_0^2}{2\sqrt{p_0^2+q_0^2}} \qquad (6.3.8)$$

The hydrodynamic moment becomes

$$M_R = -2\int_0^{\pi/2} p\left(x\frac{\partial x}{\partial\theta}+y\frac{\partial y}{\partial\theta}\right)d\theta$$

which can be reduced to

$$M_R = \frac{-2\varrho g b B_0^2}{\pi}\{X_R\sin\omega t - Y_R\cos\omega t\} \qquad (6.3.9)$$

where X_R and Y_R are defined in the same way as in 6.2 for the swaying motion. By means of (6.3.6) we can write $d\vartheta/dt = -\vartheta_a\omega\sin(\omega t+\gamma)$ as

$$\frac{d\vartheta}{dt} = \vartheta_a\omega(\sin\omega t\cos\gamma+\cos\omega t\sin\gamma) =$$
$$= \frac{2gb}{\pi\omega B_0^2}(-q_0\sin\omega t - p_0\cos\omega t) \qquad (6.3.10)$$

The acceleration becomes

$$\frac{d^2\vartheta}{dt^2} = \frac{2bg}{\pi B_0^2}(-q_0\cos\omega t+p_0\sin\omega t)$$

We now resolve M_R into a component in phase with the velocity and a component in phase with the acceleration. From (6.3.9) and (6.3.10) it follows

$$M_R = -\frac{2\varrho bB_0^2}{\pi}\frac{Y_R q_0 + p_0 X_R}{p_0^2+q_0^2}(-q_0\cos\omega t + p_0\sin\omega t) - \frac{2\varrho bB_0^2}{\pi}\frac{-q_0 X_R + p_0 Y_R}{p_0^2+q_0^2}(-q_0\sin\omega t - p_0\cos\omega t) \quad (6.3.11)$$

By means of (6.3.10) we may write (6.3.11) as

$$M_R = -\varrho B_0^4\frac{Y_R q_0 + p_0 X_R}{p_0^2+q_0^2}\ddot{\vartheta} - \varrho\omega B_0^4\frac{p_0 Y_R - q_0 X_R}{p_0^2+q_0^2}\dot{\vartheta} \quad (6.3.12)$$

By defining the hydrodynamic moment by $M_R = -I_R\ddot{\vartheta} - N_R\dot{\vartheta}$ we find for the added moment of inertia

$$I_R = \varrho B_0^4\frac{p_0 X_R + Y_R q_0}{p_0^2+q_0^2}$$

and for the damping

$$N_R = \varrho\omega B_0^4\frac{p_0 Y_R - q_0 X_R}{p_0^2+q_0^2}$$

where X_R and Y_R are defined analogous to the similar constants in (6.2.18). However, for the coefficients p_{2m} and q_{2m}, which are found in the expressions for M and N, we substitute the values, which satisfy the set of equations (6.3.7).

The swaying force produced by the rolling motion is given by

$$F_{SR} = -2\int_0^{\pi/2} p\sin\alpha\,\mathrm{d}s \quad (6.3.14)$$

Substituting (6.2.13) we obtain

$$F_{SR} = -2B_0\int_0^{\pi/2}\frac{p(1,\theta)V(\theta)}{G}\mathrm{d}\theta \quad (6.3.15)$$

Next, we substitute for the pressure p expression (6.2.10) where for the coefficients p_{2m} and q_{2m} in the expressions for M and N the values are substituted, which satisfy the set of equations (6.3.7):

$$F_{SR} = \frac{2B_0\varrho gb}{\pi}(M_0\cos\omega t - N_0\sin\omega t) \quad (6.3.16)$$

where

$$M_0 = \int_0^{\pi/2} M(1,\theta)\frac{V(\theta)}{G}\mathrm{d}\theta \text{ and}$$

$$N_0 = \int_0^{\pi/2} N(1,\theta)\frac{V(\theta)}{G}\mathrm{d}\theta$$

By means of (6.3.10) the expression (6.3.16) is written as

$$F_{SR} = -\varrho B_0^3\frac{M_0 q_0 + N_0 p_0}{p_0^2+q_0^2}\ddot{\vartheta} - \varrho B_0^3\omega\frac{M_0 p_0 - N_0 q_0}{p_0^2+q_0^2}\dot{\vartheta} \quad (6.3.17)$$

The swaying force produced by the rolling motion is defined by

$$F_{SR} = M_{SR}\left(\frac{d^2\vartheta}{dt^2}\right) + N_{SR}\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}t}\right) \quad (6.3.18)$$

Comparing (6.3.17) and (6.3.18) we obtain the added mass and damping for swaying produced by the rolling motion:

$$M_{SR} = \varrho B_0^3\frac{M_0 q_0 + N_0 p_0}{p_0^2+q_0^2}$$

$$N_{SR} = \varrho B_0^3\omega\frac{M_0 p_0 - N_0 q_0}{p_0^2+q_0^2} \quad (6.3.19)$$

Finally, by equating the radiated energy with the work done by the cylinder, we obtain

$$p_0 Y_R - q_0 X_R = \frac{\pi^2}{8} \quad (6.3.20)$$

Tasai has carried out these calculations for a Lewis-form [9]. His results can be derived from the above-mentioned formulas by substituting for φ_{2m} and ψ_{2m} respectively (6.2.21) and (6.2.22).

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