On Ocean Currents with Constant Vorticity
Explicit Solutions and an Application to the ACC

Geyer, Anna; Quirchmayr, Ronald

DOI
10.1111/sapm.70081

Publication date
2025

Document Version
Final published version

Published in
Studies in Applied Mathematics

Citation (APA)
Geyer, A., & Quirchmayr, R. (2025). On Ocean Currents with Constant Vorticity: Explicit Solutions and an Application to the ACC. *Studies in Applied Mathematics*, *155*(1), Article e70081. https://doi.org/10.1111/sapm.70081

Studies in Applied Mathematics

WILEY

STUDIES IN APPLIED MATHEMATICS

ORIGINAL ARTICLE OPEN ACCESS

On Ocean Currents with Constant Vorticity: Explicit Solutions and an Application to the ACC

Anna Geyer[1] | Ronald Quirchmayr[2]

[1]Delft Institute of Applied Mathematics, TU Delft, Delft, The Netherlands | [2]Department of Mathematics, Babeş-Bolyai University, Cluj-Napoca, Romania

Correspondence: Ronald Quirchmayr (ronald.quirchmayr@ubbcluj.ro)

Received: 11 April 2025 | **Revised:** 12 June 2025 | **Accepted:** 30 June 2025

Funding: This work was funded by the European Union – NextGenerationEU – and Romanian Government, under National Recovery and Resilience Plan for Romania, contract no. 760040/23.05.2023, cod PNRR-C9-I8-CF 185/22.11.2022, through the Romanian Ministry of Research, Innovation and Digitalization, within Component 9, Investment I8.

Keywords: Antarctic Circumpolar Current | constant vorticity | f-plane | geophysical flows

ABSTRACT

We study the three-dimensional, divergence-free, incompressible Euler equations in the f-plane approximation for off-equatorial oceanic flows of constant vorticity, where the fluid domain is bounded by a free surface and a flat bed. The major difference, compared to related earlier works, is that we refrain from any global restrictions on solutions with respect to the latitudinal coordinate y, which we justify by the f-plane approximation's locality. The resulting flows are necessarily steady (despite the time dependence of the governing equations), zonal, independent of the zonal coordinate x, and fully explicit; the corresponding free surface exhibits a nontrivial parabolic structure in y. We also provide an application to the Antarctic Circumpolar Current (ACC) for which we compare the sea surface height predicted by our constant vorticity model with satellite altimetry measurements available in the literature.

2020 Mathematics Subject Classification: 35Q31, 35Q86

1 | Introduction

The mathematical analysis of geophysical fluid dynamics is very intricate and complicated due to many factors, including the nonlinear character of the equations, the three-dimensional nature of the water flows, stratification of the fluid, the Coriolis effects due to the Earth's rotation around its axis, and the presence of swirling motions of the fluid. The latter is mathematically captured by the vorticity vector, which is defined as the curl of the velocity field and measures the strength of a fluid's local rotation. While irrotational flows (i.e., flows with zero vorticity) have very nice mathematical properties, such as the existence of a velocity potential enabling far-reaching applications of harmonic and complex analysis, realistic flows are typically nonuniform. The simplest nontrivial rotational flows are linear shear flows, that is, currents with constant vorticity. Compared to uniform flows, they are more complex, as they may have stagnation points or critical layers. Significant analytical progress has been made rather recently in the investigation of steady periodic gravity waves with constant vorticity; see [1, 2] and the references therein.

Nonzero constant vorticity flows have been shown to be in essence two-dimensional, for constant density flows, as well as for vertically stratified flows [3–9]. Even more specifically, constant vorticity flows were shown to give rise to only trivial bounded solutions in the equatorial β-plane [10]. The recent studies [11, 12] explored the existence of solutions in the f-plane approximation at arbitrary off-equatorial latitudes under the constant vorticity assumption and found that any bounded solution is almost trivial with a flat surface, vanishing vertical fluid velocity, and the horizontal velocity components independent of spatial variables, taking the form of temporal simple harmonic oscillators with the frequency determined by the Coriolis parameter f.

In the present study, we dispense with the assumption that solutions have to be bounded with respect to the latitudinal direction y. This can be justified, since the f-plane approximation is accurate and valid only in a narrow latitudinal strip, and therefore global boundedness restrictions on solutions with respect to y are unnecessary. Precisely this relaxation gives rise to nontrivial off-equatorial flows of constant nonzero vorticity satisfying the f-plane approximation of Euler's equations. These solutions are necessarily steady, zonal, independent of the zonal direction x, and fully explicit. The azimuthal velocity component is linear in both y and the vertical coordinate z; the other velocity components are zero; the pressure is quadratic in y and z; the free surface exhibits a parabolic shape. Finally, we apply our result to the Antarctic Circumpolar Current (ACC), for which we compare the sea surface height predicted by our constant vorticity model with satellite altimetry measurements available in the literature.

The paper is structured as follows: Section 2 introduces the geophysical setting and further assumptions. In Section 3, we state and prove our main result. Section 4 provides an application of this result to the ACC.

2 | The Geophysical Setting

We consider the f-plane approximation for the governing equations of geophysical fluid dynamics and denote by $(x, y, z) \in \mathbb{R}^3$ the position vector in the corresponding rotating Cartesian coordinate frame for the standard basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \in \mathbb{R}^3$ pointing eastward, northward, and upward, respectively. The fluid domain lies between a flat ocean bed at depth $z = -d < 0$ and a free surface η, which is an a priori unknown part of the solution (see Figure 1). We assume that the free surface is described by the graph of a continuously differentiable function $\eta : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, t) \mapsto \eta(x, y, t)$. At each instance of $t \in \mathbb{R}$, we denote the fluid domain by $D_\eta(t)$, that is,

$$D_\eta(t) := \{(x, y, z) \in \mathbb{R}^3 : -d < z < \eta(x, y, t)\},$$

and set

$$D_\eta := \{(x, y, z, t) \in \mathbb{R}^4 : (x, y, z) \in D_\eta(t) \text{ and } t \in \mathbb{R}\}.$$

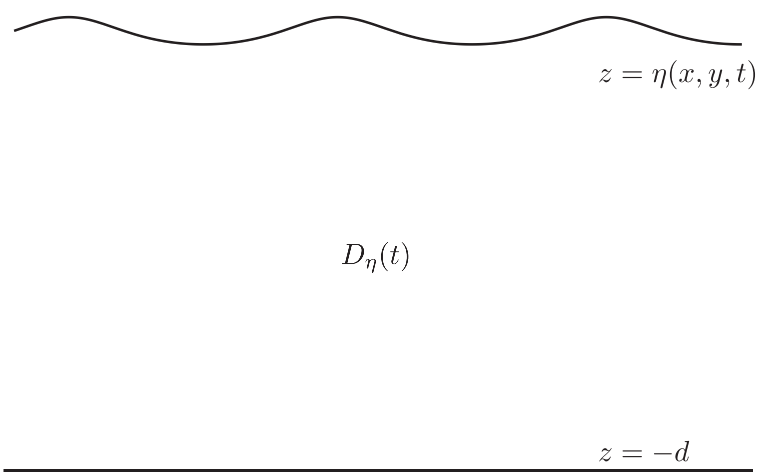

FIGURE 1 | A cross section of the (a priori unknown) fluid domain $D_\eta(t)$ bounded by the free surface η and the flat bed.

We denote by $\Omega \approx 7.29 \cdot 10^{-5}$ rad s^{-1} the angular velocity of the Earth and set

$$f := 2\Omega \sin\theta, \quad \hat{f} := 2\Omega\cos\theta,$$

where $\theta \in (-\pi/2, \pi/2)$ is a fixed angle of latitude (the Equator is located at $\theta = 0$ and θ is positive on the Northern Hemisphere). Furthermore, $(u, v, w) : D_\eta \to \mathbb{R}^3$ denotes the velocity field, where u, v, and w point in the directions x, y, and z, respectively, and $P : D_\eta \to \mathbb{R}$ is the pressure. Our set of governing equations, namely, the f-plane approximation for Euler's equations, reads

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + \hat{f}w - fv &= -\rho^{-1}P_x, \\ v_t + uv_x + vv_y + wv_z + fu &= -\rho^{-1}P_y, \\ w_t + uw_x + vw_y + ww_z - \hat{f}u &= -\rho^{-1}P_z - g. \end{aligned} \tag{2.1}$$

The density $\rho > 0$ is assumed to be fixed and g denotes the gravitational constant. In addition, we require conservation of mass, that is,

$$u_x + v_y + w_z = 0. \tag{2.2}$$

We impose the usual dynamic and kinematic boundary conditions at the free surface and the flat ocean bed, that is,

$$\begin{cases} P = P_{\text{atm}} \\ w = \eta_t + u\eta_x + v\eta_y \end{cases} \quad \text{on} \quad z = \eta(x, y, t), \tag{2.3}$$

where $P_{\text{atm}} > 0$ denotes a given constant atmospheric pressure, and

$$w = 0 \quad \text{on} \quad z = -d. \tag{2.4}$$

The relative vorticity in D_η (with respect to the rotating frame of reference) is given by

$$\omega = (\omega_1, \omega_2, \omega_3) = (w_y - v_z, u_z - w_x, v_x - u_y). \tag{2.5}$$

Assumptions. Our main result (Theorem 3.1) relies on several requirements, which are listed below in Assumption 2.2. They are similar to those employed in [11], except for one crucial difference: in our present study, we do not assume boundedness of solutions with respect to the y-coordinate. It is precisely this relaxation which permits the existence of nontrivial flows of constant nonzero vorticity governed by the f-plane approximation of Euler's equation off the equator. Our motivation for this relaxation stems from the fact that the f-plane approximation is accurate in a very narrow latitudinal strip about the underlying fixed latitude θ. Geophysicists typically require deviations from $y = 0$ to lie in a range of about ±50 km, which is of an order 100 less than the zonal dimension of large (zonal) ocean currents. Due to the strong locality of this approximation, we refrain from imposing any kind of global restriction for solutions with respect to y. Thus, (u, v, w), P, and η can become unbounded or undefined as $y \to \pm\infty$; the graph of η might potentially intersect the line $\{z = -d\}$ at very large latitudinal distances. We emphasize at this point that we are not interested here in studying properties of solutions near regions where they become potentially undefined. Instead, we argue that regions where such mathematical singularities might occur do certainly lie far beyond the f-plane approximation's scope of applicability and thus unlikely bear geophysical significance.

Notation 2.1. In view of the above discussion, we impose a suitable restriction on the fluid domain in y-direction to rule out ill-posedness. For this purpose, we set

$$Y_{\max} := \frac{(g - \hat{f}u_0)^2}{2g|f\omega_2|}, \quad Y := \frac{Y_{\max}}{1000}, \tag{2.6}$$

and restrict the y-range to the interval $(-Y, Y)$. Here, u_0 is associated with the azimuthal velocity component (cf. Theorem 3.1). We denote the resulting restricted fluid domain by

$$D_{\eta,Y} := \{(x,y,z,t) \in \mathbb{R}^4 : |y| < Y,\ (x,y,z) \in D_\eta(t) \text{ and } t \in \mathbb{R}\}.$$

We emphasize that we do not impose additional lateral boundary conditions (they would be physically meaningless); we simply restrict (2.1)–(2.4) to $D_{\eta,Y}$. The formula for $Y_{\max}$ comes from (3.1): it equals the absolute value of the zero of the discriminant. We will see that Y is large enough to fully cover realistic geophysical applications such as the one given in Section 4.

Assumption 2.2. We assume that $Y_{\max} > 0$ and impose the following requirements.

i. The vorticity vector $\boldsymbol{\omega} \in \mathbb{R}^3$ is constant, nonzero and

$$\omega_2' := \omega_2 + \hat{f} \neq 0, \tag{2.7a}$$
$$\omega_3' := \omega_3 + f \neq 0. \tag{2.7b}$$

ii. The latitudinal angle θ is nonzero: $\theta \in (-\pi/2, \pi/2) \setminus \{0\}$.

iii. The free surface $\eta : \mathbb{R} \times (-Y, Y) \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies the following sea-level normalization:

$$\eta(0,0,0) := 0. \tag{2.8}$$

iv. The fluid velocity vector

$$(u, v, w) : D_{\eta,Y} \to \mathbb{R}^3$$

is twice continuously differentiable; furthermore it is bounded with respect to $x, t \in \mathbb{R}$ for every fixed $y \in (-Y, Y)$.

v. The pressure

$$P : D_{\eta,Y} \to \mathbb{R}$$

is continuously differentiable; moreover it is bounded with respect to $x, t \in \mathbb{R}$ for every fixed $y \in (-Y, Y)$.

Remark 2.3. We are only interested in classical solutions of (2.1)–(2.4), which explains the smoothness requirements in 2.2(iii)–(v). Note that conservation of mass (2.2) in combination with the assumption of constant vorticity and smoothness entails harmonicity and thereby real analyticity of u, v, w in the spatial variables within the connected domain $D_{\eta,Y}$ (we will prove that η satisfies (3.1), which ensures connectedness for real-world scenarios). That is,

$$\Delta u = \Delta v = \Delta w = 0$$

throughout $D_{\eta,Y}$, where $\Delta := \nabla \cdot \nabla$ with $\nabla := (\partial_x, \partial_y, \partial_z)$.

3 | Main Result

In this section, we present and prove our main result, which is stated as follows.

Theorem 3.1. *The set of solutions fulfilling the free boundary value problem (2.1)–(2.4) under Assumption 2.2 is characterized as follows:*

i. The vorticity satisfies $\boldsymbol{\omega} = (0, \omega_2, \tilde{f}\omega_2)$ for some $\omega_2 \in \mathbb{R} \setminus \{0\}$ with $\tilde{f} := f/\hat{f} = \tan\theta$.

ii. Throughout $D_{\eta,Y}$, $v = w \equiv 0$ and u depends only on (y, z); it is given by

$$u(y,z) = u_0 + \omega_2(z - \tilde{f}y)$$

for some $u_0 \in \mathbb{R}$. In particular, the flow is steady and zonal.

iii. *Throughout $D_{\eta,Y}$, the pressure P depends only on (y,z) and satisfies*

$$P(y,z) = \rho\Big[\hat f\Big(\frac{\omega_2}{2}((z-\tilde f y)^2) + u_0(z-\tilde f y)\Big) - gz\Big] + P_{\mathrm{atm}}.$$

iv. *The free surface η depends only on $y \in (-Y,Y)$ and is given by*

$$\eta(y) = \frac{g + \hat f(\tilde f \omega_2 y - u_0) - \sqrt{\left(g - \hat f u_0\right)^2 + 2\omega_2 g f y}}{\hat f \omega_2}. \tag{3.1}$$

Proof. One can directly check that u, v, w, P, and η given by Theorem 3.1(i)–(iv) are indeed a solution of the free boundary value problem (2.1)–(2.4), which fulfills Assumption 2.2.

Let, on the other hand, u, v, w, P, and η be a solution of (2.1)–(2.4) satisfying Assumption 2.2. We first prove Theorem 3.1(i). To this end, we consider the vorticity equation, which is obtained by taking the curl of (2.1). In view of mass conservation (2.2) and the assumptions that density ρ and $\boldsymbol{\omega}$ are constant, the vorticity equation written out componentwise simplifies to

$$\omega_1 u_x + \omega_2' u_y + \omega_3' u_z = 0, \tag{3.2a}$$

$$\omega_1 v_x + \omega_2' v_y + \omega_3' v_z = 0, \tag{3.2b}$$

$$\omega_1 w_x + \omega_2' w_y + \omega_3' w_z = 0. \tag{3.2c}$$

By combining (2.4), (2.7b), (2.8), (3.2c), and continuity of the free surface, we infer the existence of a small open subset in $D_{\eta,Y}$ lying between the ocean bed and the surface, where w vanishes. Thanks to harmonicity of w in $D_{\eta,Y}$ (cf. Remark 2.3), we infer that

$$w = 0 \qquad \text{in} \quad D_{\eta,Y}. \tag{3.3}$$

Therefore (cf. (2.5)), we infer that

$$\begin{cases} \omega_1 = -v_z \\ \omega_2 = u_z \end{cases} \qquad \text{in} \quad D_{\eta,Y},$$

which yields that

$$\begin{cases} u(x,y,z,t) = \bar u(x,y,t) + \omega_2 z \\ v(x,y,z,t) = \bar v(x,y,t) - \omega_1 z \end{cases} \qquad \text{in} \quad D_{\eta,Y} \tag{3.4}$$

for some functions $\bar u, \bar v : \mathbb{R}\times(-Y,Y)\times\mathbb{R} \to \mathbb{R}$. From (2.2), (3.3), and (3.4) we deduce that

$$\bar u_x + \bar v_y = 0 \qquad \text{in} \quad \mathbb{R}\times(-Y,Y)\times\mathbb{R}$$

giving rise to a function $\psi : \mathbb{R}\times(-Y,Y)\times\mathbb{R} \to \mathbb{R}$, $(x,y,t) \mapsto \psi(x,y,t)$, satisfying

$$\psi_y = \bar u, \quad \psi_x = -\bar v. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.2a), (3.2b), and (2.5), shows that

$$\begin{aligned} \omega_1\psi_{xy} + \omega_2'\psi_{yy} + \omega_3'\omega_2 &= 0, \\ \omega_1\psi_{xx} + \omega_2'\psi_{xy} + \omega_3'\omega_1 &= 0, \\ -(\psi_{xx} + \psi_{yy}) &= \omega_3. \end{aligned} \tag{3.6}$$

By solving (3.6) for ψ_{xx}, ψ_{xy}, and ψ_{yy}, we obtain that

$$\psi_{xx} = \frac{\omega_2'(f\omega_2 - \hat{f}\omega_3) - \omega_1^2\omega_3'}{\omega_1^2 + {\omega_2'}^2} =: A, \tag{3.7a}$$

$$\psi_{xy} = -\frac{\omega_1(\omega_2\omega_3' + f\omega_2')}{\omega_1^2 + {\omega_2'}^2} =: B, \tag{3.7b}$$

$$\psi_{yy} = \frac{f\omega_1^2 - \omega_2\omega_2'\omega_3'}{\omega_1^2 + {\omega_2'}^2} =: C. \tag{3.7c}$$

These fractions are well-defined by (2.7a), that is, $\omega_1^2 + {\omega_2'}^2 > 0$. Next, we integrate (3.7) to infer that

$$\psi(x, y, t) = \frac{A}{2}x^2 + Bxy + \frac{C}{2}y^2 + a(t)x + b(t)y + c(t), \qquad (x, y, t) \in \mathbb{R} \times (-Y, Y) \times \mathbb{R}$$

for some functions $a, b, c : \mathbb{R} \to \mathbb{R}$. Thus, for all $(x, y, t) \in \mathbb{R} \times (-Y, Y) \times \mathbb{R}$,

$$\begin{aligned} \bar{u}(x, y, t) &= \psi_y = Cy + Bx + b(t), \\ \bar{v}(x, y, t) &= -\psi_x = -(Ax + By + a(t)). \end{aligned} \tag{3.8}$$

Since we require $u(\cdot, y, z, t) : \mathbb{R} \to \mathbb{R}$ and $v(\cdot, y, z, t) : \mathbb{R} \to \mathbb{R}$ to be bounded for fixed (y, z, t), $\bar{u}(\cdot, y, t) : \mathbb{R} \to \mathbb{R}$ and $\bar{v}(\cdot, y, t) : \mathbb{R} \to \mathbb{R}$ have to be bounded as well, which implies that

$$A = B = 0. \tag{3.9}$$

To show that

$$\omega_1 = 0, \tag{3.10}$$

we employ a contradiction argument. Let us assume that $\omega_1 \neq 0$. Then it follows from (3.7b), (3.9), and (2.7b) that

$$\omega_2\omega_3' = -f\omega_2' = -\omega_3'\omega_2' + \omega_3\omega_2'. \tag{3.11}$$

From (3.7a), (3.9), and (2.7), it follows that

$$\omega_2'(\omega_3'\omega_2 - \omega_2'\omega_3) = \omega_1^2\omega_3'. \tag{3.12}$$

Next, we substitute (3.11) into (3.12), divide the outcome by $\omega_3' \neq 0$ (cf. (2.7b)), and use the assumption that $\omega_1 \neq 0$ to infer that

$$-(\omega_2')^2 = \omega_1^2 > 0,$$

which is obviously a contradictory statement. Therefore, (3.10) must hold true. Furthermore, we employ (2.7a), (3.7a), (3.9), and (3.10) to conclude that

$$f\omega_2 = \hat{f}\omega_3. \tag{3.13}$$

This proves Theorem 3.1(i), since $\boldsymbol{\omega}$ is nonzero by assumption.

As an immediate consequence, we infer from (3.13) that $\omega_3'\omega_2 = \omega_2'\omega_3$, which in combination with (3.7c) and (3.10) implies that

$$C = -\omega_3 = -\tilde{f}\omega_2.$$

In view of (3.4), (3.8), (3.10) and by setting $u_0(t) := b(t)$, $v_0(t) := -a(t)$, $t \in \mathbb{R}$, we obtain

$$\begin{cases} u(y, z, t) = u_0(t) + \omega_2(z - \tilde{f}y) \\ v(t) = v_0(t) \end{cases} \qquad \text{in} \quad D_{\eta,Y}.$$

Particularly, u is independent of x and v depends only on t. Together with (3.3), this reduces the governing Equations (2.1) to

$$P_x = \rho c_1(t),$$
$$P_y = \rho\big(c_2(t) - f\omega_2(z - \tilde{f}y)\big),$$
$$P_z = \rho\big(c_3(t) + \hat{f}\omega_2(z - \tilde{f}y)\big),$$

where

$$c_1(t) := -u_0'(t) + \omega_3' v_0(t),$$
$$c_2(t) := -v_0'(t) - f u_0(t),$$
$$c_3(t) := \hat{f} u_0(t) - g.$$

Thus, straightforward integration yields

$$P(x, y, z, t) = \rho\left[\frac{\hat{f}\omega_2}{2}(z - \tilde{f}y)^2 + c_1(t)x + c_2(t)y + c_3(t)z\right] + P_{\text{atm}}. \tag{3.14}$$

Note that the dynamic boundary condition in (2.3) combined with (2.8) determines the constant of integration to be P_{atm}. To ensure boundedness of P in the x-coordinate (cf. Assumption 2.2(v)), it is required that

$$c_1(t) = 0 \quad \text{for all} \quad t \in \mathbb{R}, \tag{3.15}$$

hence P is independent of x. Thanks to (3.14) in combination with the dynamic boundary condition in (2.3) and (2.8)—the latter determines the physically meaningful branch of the root—we obtain the following explicit formula for the surface η:

$$\eta(y, t) = \frac{g - \hat{f}u_0(t) + f\omega_2 y - \sqrt{\Delta(y,t)}}{\omega_2 \hat{f}}, \tag{3.16}$$

where

$$\Delta(y, t) := \big(g - \hat{f}u_0(t)\big)^2 + 2\omega_2\big(gf + \hat{f}v_0'(t)\big)y.$$

In particular, η is independent of x, which—in combination with (3.3)—simplifies the kinematic boundary condition in (2.3) to

$$\eta_t(y, t) = -v_0(t)\,\eta_y(y, t), \qquad (y, t) \in (-Y, Y) \times \mathbb{R}. \tag{3.17}$$

In order to show that $v_0 \equiv 0$ and u_0 is constant, we calculate η_t and η_y using (3.16):

$$\eta_t(y, t) = -\frac{1}{\omega_2 \hat{f}}\left[\hat{f}u_0'(t) + \frac{\Delta_t(y,t)}{2\sqrt{\Delta(y,t)}}\right],$$
$$\eta_y(y, t) = \frac{1}{\omega_2 \hat{f}}\left[f\omega_2 - \frac{\Delta_y(t)}{2\sqrt{\Delta(y,t)}}\right],$$

where

$$\Delta_t(y, t) = 2\hat{f}\big(\big(\hat{f}u_0(t) - g\big)u_0'(t) + \omega_2 v_0''(t)y\big),$$
$$\Delta_y(t) = 2\omega_2\hat{f}\big(\tilde{f}g + v_0'(t)\big).$$

We observe that the y-dependencies of η_t and η_y are of a different form if $v_0''(t) \neq 0$ for some $t \in \mathbb{R}$, as then Δ_t exhibits a linear y-term, while Δ_y is independent of y. Therefore, (3.17) cannot be satisfied for all $y \in (-Y, Y)$ unless $v_0 \equiv 0$. The case $v_0'' \equiv 0$ also enforces $v_0 \equiv 0$, because the y-dependent terms in (3.17) can again be isolated on the right-hand side of the equation, with the left-hand side depending only on t. We conclude that (3.17) is valid only if $v_0 \equiv 0$, in which case u_0 is constant by (3.15). This proves Theorem 3.1(ii). Plugging the steady velocity field into (3.14) and (3.16) validates Theorem 3.1(iii) and (iv). □

Remark 3.2 (The zero vorticity case). We find that the uniform flow $(u, v, w) = (u_0, 0, 0)$ in combination with the pressure

$$P(y,z) = \rho\left[\hat{f}\left(u_0(z - \tilde{f}y)\right) - gz\right] + P_{\text{atm}}$$

satisfies the governing equations and boundary conditions (2.1)–(2.4), where the surface η is a linear function of y given by

$$\eta(y) = \frac{f u_0}{\hat{f}u_0 - g}\, y. \tag{3.18}$$

Remark 3.3 (Equatorial f-plane). Let us briefly discuss the case $\theta = 0$, that is, $f = 0$. The proof of Theorem 3.1 shows that in this case, Assumption (2.7b) would lead to contradictions. This explains Assumption 2.2(ii). However, one can directly check that the flow given by

$$u(z) := u_0 + \omega_2 z, \quad v = w \equiv 0, \quad P(z) := \rho\left[2\Omega\left(\frac{\omega_2}{2}z^2 + u_0 z\right) - gz\right] + P_{\text{atm}}, \quad \eta \equiv 0$$

is a steady constant vorticity solution of (2.1)–(2.4) in the equatorial f-plane. But this is not the only constant vorticity solution. It has been shown in [13] that the Coriolis terms can be eliminated in the equatorial f-plane approximation by means of a transformation, provided that steady two-dimensional flows are considered. Thus, the large class of steady constant vorticity flows gives rise to corresponding solutions in the equatorial f-plane. The analytical study of equatorial waves and currents has been the subject of numerous investigations in recent years (see, e.g., [14–19] and the references therein).

4 | Application to the ACC

This section provides an application of Theorem 3.1 to a scenario which is relevant for the ACC. We are particularly interested in the theoretical prediction of the sea surface height (both qualitatively and quantitatively); a comparison with satellite altimetry measurements from the literature will be made.

The ACC is the world's largest and strongest ocean current. It completely encircles Antarctica in the eastward direction due to the absence of continental barriers and thereby connects the Pacific, Atlantic, and Indian Oceans. Figure 2 depicts its location and mean current speed. It is driven by strong westerly winds and has a mean mass transport of (possibly more than) 130 Sverdrups (Sv; $1\ \text{Sv} = 10^6\ \text{m}^3\,\text{s}^{-1}$) (see, e.g., [20, 21]). There is a rapidly growing number of studies on various different mathematical aspects of the ACC, which are based on analytical approaches for partial differential equations; we refer to [22–28] and the references therein for some recent developments.

To define a steady velocity field according to Theorem 3.1, we rely on the following observations. ACC's mean surface velocity ranges between 20 and 50 $\text{cm}\,\text{s}^{-1}$ [29]. In contrast to earlier studies, which suggest almost zero bottom velocity (see, e.g., [30]), modern observations found relatively high and stable near-bottom mean speeds of around 10 $\text{cm}\,\text{s}^{-1}$ (cf. [31, 32]).

We apply Theorem 3.1 with the following set of parameters:

- The angle of latitude θ is fixed at 49° South, that is, $\theta = -\frac{49\pi}{180}$, which on average locates the center of the Subantarctic Front (SAF)—one of the three fronts in the ACC.
- The depth $d = 5000$ m is fixed.
- At $y = 0$, the azimuthal surface and bottom velocities are assumed to be 25 $\text{cm}\,\text{s}^{-1}$ and 10 $\text{cm}\,\text{s}^{-1}$, respectively. Theorem 3.1 applied to this situation yields $u(y,z) = u_0 + \omega_2(z - \tilde{f}y)$ with $u_0 = 1/4\ \text{m}\,\text{s}^{-1}$ and $\omega_2 = \frac{u_0 - 10^{-1}}{d}\ \text{s}^{-1}$ (and $v = w \equiv 0$).
- The constant atmospheric pressure equals $P_{\text{atm}} = 101\,000\ \text{kg}\,\text{m}^{-1}\,\text{s}^{-2}$.
- The constant sea water density is $\rho = 1030\ \text{kg}\,\text{m}^{-3}$.

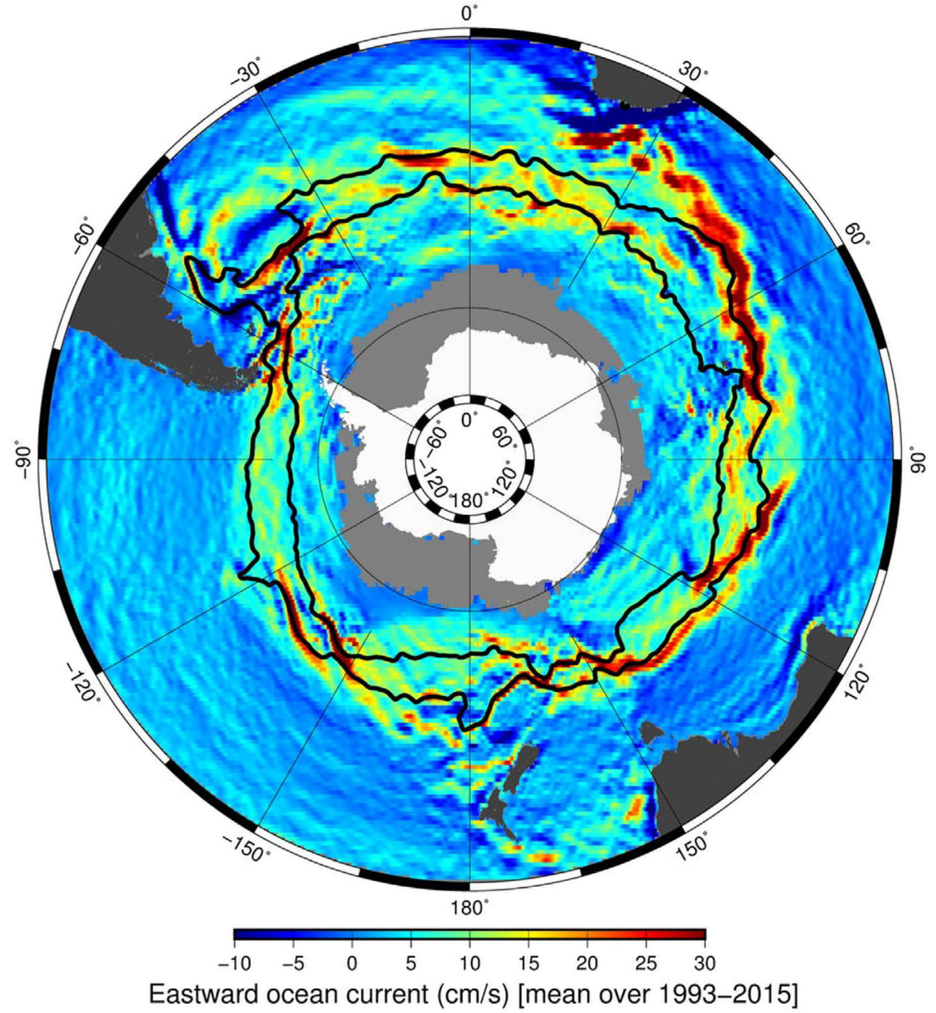

FIGURE 2 | Mean eastward current speed of the ACC; black lines represent the Subantarctic Front and the Polar Front. Image credit: Hellen Phillips (Senior Research Fellow, Institute for Marine and Antarctic Studies, University of Tasmania), Benoit Legresy (CSIRO), and Nathan Bindoff (Professor of Physical Oceanography, Institute for Marine and Antarctic Studies, University of Tasmania).

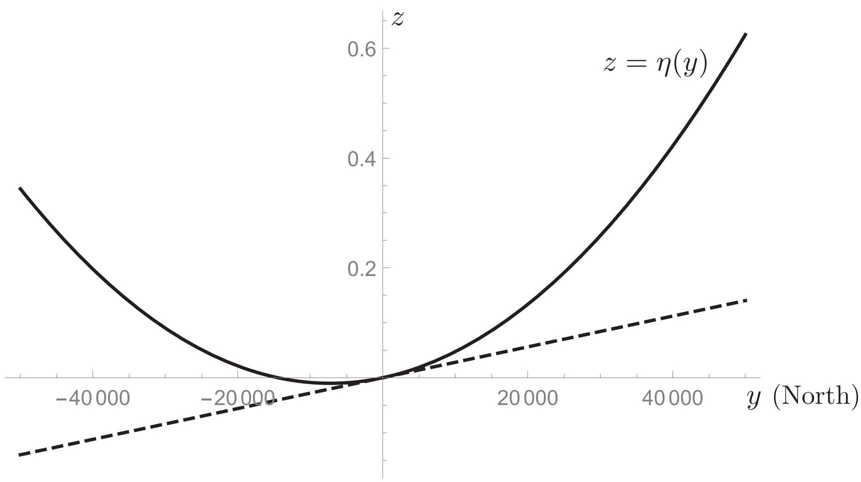

FIGURE 3 | A plot of the nonlinear sea surface profile η given by (3.1) (solid line); the unit is meter (m). The dashed line, which is the tangent at $y = 0$, shows the corresponding uniform flow scenario ($\omega_2 = 0$), in which η is given by (3.18).

- The gravitational acceleration equals $g = 9.8\,\mathrm{m\,s^{-2}}$.
- The Earth's angular velocity is $\Omega = 7.29 \cdot 10^{-5}\,\mathrm{rad\,s^{-1}}$.

First, we calculate $Y_{\max} \approx 1.48 \times 10^6$ km, hence $Y \approx 1480$ km. In view of the f-plane approximation's range of accuracy (a roughly 100 km wide strip about the fixed angle of latitude), we see that (2.6) is not restrictive at all. A plot of the explicit parabolic sea surface height function η given by (3.1) is shown in Figure 3 (solid line). It shows a variation of roughly 0.6 m along a latitudinal range of 100 km.

Satellite altimetry measurements, see, for example, [33, Figure 6], suggest that the sea surface height is monotonically increasing at a rate of approximately 0.5 m per 1° latitude (roughly 100 km) in the center of the SAF (at −49°). The graph of η depicted in Figure 3 shows local surface height variations of similar size and form.

Acknowledgments

This work was supported by the project "Nonlinear Studies of Stratified Oceanic and Atmospheric Flows" funded by the European Union – NextGenerationEU – and Romanian Government, under National Recovery and Resilience Plan for Romania, contract no. 760040/23.05.2023, cod PNRR-C9-I8-CF 185/22.11.2022, through the Romanian Ministry of Research, Innovation and Digitalization, within Component 9, Investment I8.

Open access publishing facilitated by Anelis Plus (the official name of Asociatia Universitatilor, a Institutelor de Cercetare - Dezvoltare si a Bibliotecilor Centrale Universitare din Romania), as part of the Wiley - Anelis Plus agreement.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

1. A. Constantin and E. Vărvărucă, "Steady Periodic Water Waves With Constant Vorticity: Regularity and Local Bifurcation," *Archive for Rational Mechanics and Analysis* 199 (2011): 33–67.

2. A. Constantin, E. Vărvărucă, and W. Strauss, "Global Bifurcation of Steady Gravity Water Waves With Critical Layers," *Acta Mathematica* 217, no. 2 (2016): 195–262.

3. A. Constantin, "Two-Dimensionality of Gravity Water Flows of Constant Nonzero Vorticity Beneath a Surface Wave Train," *European Journal of Mechanics - B/Fluids* 30 (2011): 12–16.

4. E. Wahlén, "Non-Existence of Three-Dimensional Travelling Water Waves With Constant Non-Zero Vorticity," *Journal of Fluid Mechanics* 746 (2014): R2.

5. C. I. Martin, "Two Dimensionality of Gravity Water Flows Governed by the Equatorial f-Plane Approximation," *Annali di Matematica Pura ed Applicata* 196 (2017): 2253–2260.

6. C. I. Martin, "Non-Existence of Time-Dependent Three-Dimensional Gravity Water Flows With Constant Non-Zero Vorticity," *Physics of Fluids* 30, no. 10 (2018): 107102.

7. C. I. Martin, "Liouville-Type Results for the Time-Dependent Three-Dimensional (Inviscid and Viscous) Water Wave Problem With an Interface," *Journal of Differential Equations* 362 (2023): 88–105.

8. A. Geyer and C. I. Martin, "Three-Dimensional Time-Dependent Water Flows With Constant Non-Vanishing Vorticity and Depth Dependent Density," *Physica D* 472 (2025): 134523.

9. R. M. Chen, L. Fan, S. Walsh, and M. H. Wheeler, "Rigidity of Three-Dimensional Internal Waves With Constant Vorticity," *Journal of Mathematical Fluid Mechanics* 25 (2023): 71.

10. C. I. Martin, "On Constant Vorticity Water Flows in the β-Plane Approximation," *Journal of Fluid Mechanics* 865 (2019): 762–774.

11. C. I. Martin, "Constant Vorticity Water Flows With Full Coriolis Term," *Nonlinearity* 32 (2019): 2327–2336.

12. R. Quirchmayr, "Stratified Ocean Currents With Constant Vorticity," *Journal of Mathematical Fluid Mechanics* 27 (2025): 7.

13. M. H. Wheeler, "On Stratified Water Waves With Critical Layers and Coriolis Forces," *Discrete and Continuous Dynamical Systems* 39, no. 8 (2019): 4747–4770.

14. A. Constantin and R. I. Ivanov, "Equatorial Wave-Current Interactions," *Communications in Mathematical Physic* 370 (2019): 1–48.

15. A. Constantin and R. S. Johnson, "The Dynamics of Waves Interacting With the Equatorial Undercurrent," *Geophysical & Astrophysical Fluid Dynamics* 109, no. 4 (2015): 311–358.

16. J. Chu, Q. Ding, and J. Escher, "Variational Formulation of Rotational Steady Water Waves in Two-Layer Flows," *Journal of Mathematical Fluid Mechanics* 23 (2021): 91.

17. A. Geyer and R. Quirchmayr, "Shallow Water Models for Stratified Equatorial Flows," *Discrete and Continuous Dynamical Systems* 39 (2019): 4533–4545.

18. D. Henry, "Exact Equatorial Water Waves in the f-Plane," *Nonlinear Analysis: Real World Applications* 28 (2016): 284–289.

19. D. Ionescu-Kruse and C. I. Martin, "Periodic Equatorial Water Flows From a Hamiltonian Perspective," *Journal of Differential Equations* 262 (2017): 4451–4474.

20. V. O. Ivchenko, K. J. Richards, and D. P. Stevens, "The Dynamics of the Antarctic Circumpolar Current," *Journal of Physical Oceanography* 26 (1996): 753–774.

21. S. R. Rintoul, C. W. Hughes, and D. Olbers, "The Antarctic Circumpolar Current System," in *International Geophysics*, ed. G. Siedler, J. Church, and J. Gould, Vol. 77 (Academic Press, 2001).

22. A. Constantin and R. S. Johnson, "An Exact, Steady, Purely Azimuthal Flow as a Model for the Antarctic Circumpolar Current," *Journal of Physical Oceanography* 46, no. 12 (2016): 3585–3594.

23. J. Chu, K. Marynets, and Z. Wang, "Existence and Approximate Solutions of a Nonlinear Model for the Antarctic Circumpolar Current," *Differential and Integral Equations* 36, no. 7/8 (2023): 537–558.

24. S. V. Haziot, "Study of an Elliptic Partial Differential Equation Modelling the Antarctic Circumpolar Current," *Discrete and Continuous Dynamical Systems A.* 39, no. 8 (2019): 4415–4427.

25. H.-C. Hsu and C. I. Martin, "On the Existence of Solutions and the Pressure Function Related to the Antarctic Circumpolar Current," *Nonlinear Analysis: Theory, Methods & Applications* 155 (2017): 285–293.

26. K. Marynets, "The Antarctic Circumpolar Current as a Shallow-Water Asymptotic Solution of Euler's Equation in Spherical Coordinates," *Deep-Sea Research Part II: Topical Studies in Oceanography* 160 (2019): 58–62.

27. C. I. Martin and R. Quirchmayr, "Explicit and Exact Solutions Concerning the Antarctic Circumpolar Current With Variable Density in Spherical Coordinates," *Journal of Mathematical Physics* 60 (2019): 101505.

28. C. I. Martin and R. Quirchmayr, "Exact Solutions and Internal Waves for the Antarctic Circumpolar Current in Spherical Coordinates," *Studies in Applied Mathematics* 148, no. 3 (2022): 1021–1039.

29. M. H. Derkani, A. Alberello, F. Nelli, et al., "Wind, Waves, and Surface Currents in the Southern Ocean: Observations From the Antarctic Circumnavigation Expedition," *Earth System Science Data* 13, no. 3 (2021): 1189–1209.

30. D. Olbers and M. Wenzel, "Determining Diffusivities From Hydrographic Data by Inverse Methods With Applications to the Circumpolar Current," in *Oceanic Circulation Models: Combining Data and Dynamics*, ed. D. L. T. Anderson and J. Willebrand, NATO ASI Series, Vol. 284 (Springer, 1989).

31. K. A. Donohue, K. L. Tracey, D. R. Watts, M. P. Chidichimo, and T. K. Chereskin, "Mean Antarctic Circumpolar Current Transport Measured in Drake Passage," *Geophysical Research Letters* 43 (2016): 11760–11767.

32. S. Wu, L. Lembke-Jene, F. Lamy, et al., "Orbital- and Millennial-Scale Antarctic Circumpolar Current Variability in Drake Passage Over the Past 140,000 Years," *Nature Communications* 12 (2021): 3948.

33. S. T. Gille, "Mean Sea Surface Height of the Antarctic Circumpolar Current from Geosat Data: Method and Application," *Journal of Geophysical Research* 99, no. C9 (1994): 18255–18273.