Optimal Deposit Rate Modelling and Risk Assessment for Non-Maturing Deposits

an application to aggregate data of the Dutch banking sector

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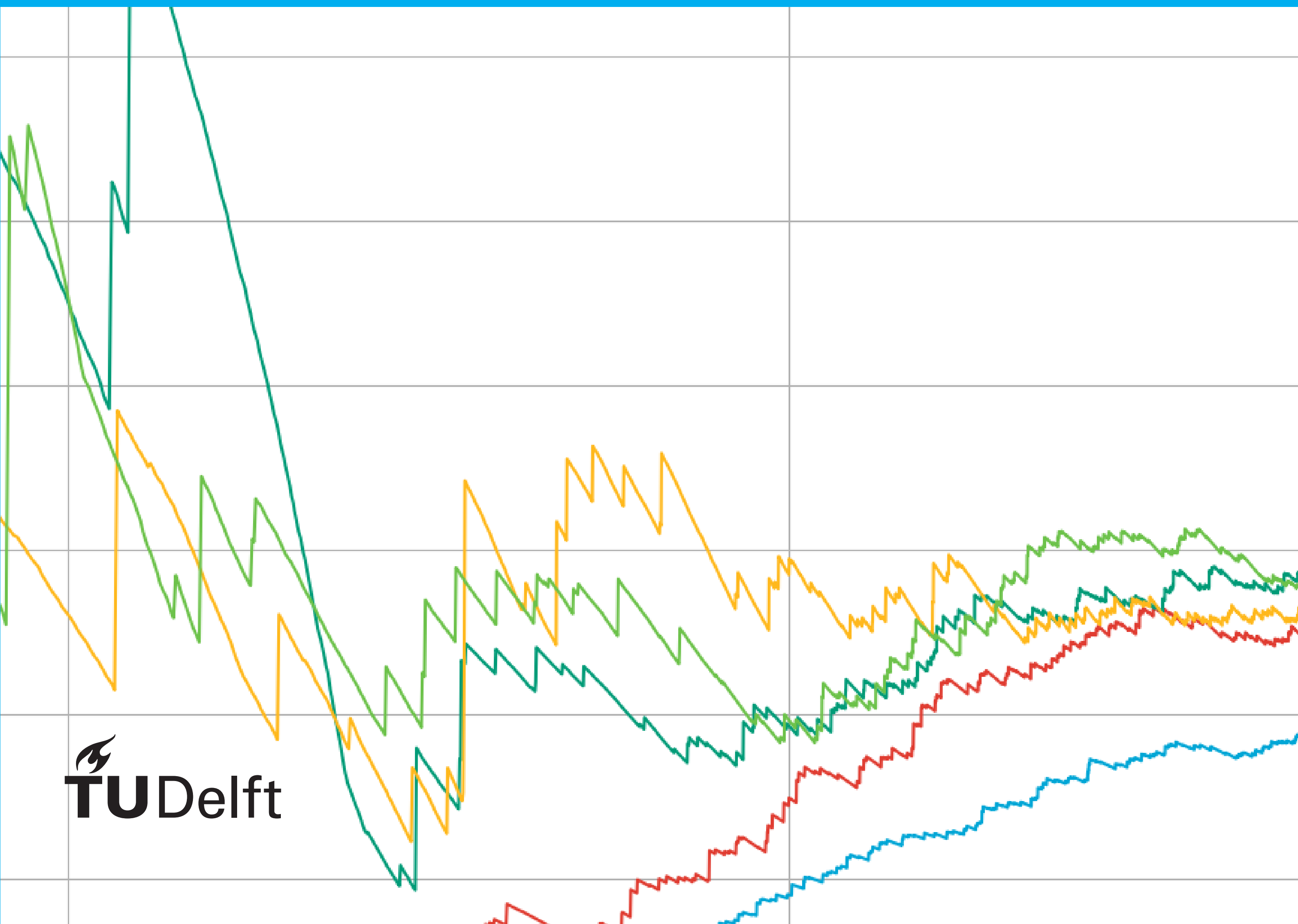

TU Delft

Optimal Deposit Rate Modelling and Risk Assessment for Non-Maturing Deposits

an application to aggregate data of the Dutch banking sector

by

P.W. Oppelaar

to obtain the degree of Master of Science
in Applied Mathematics (Financial Engineering)
at the Delft University of Technology,
faculty Electrical Engineering, Mathematics and Computer Science
to be defended publicly on June 26th, 2024 at 15:00 PM.

Student number:	4958225	
Project duration:	October 2, 2023 – June 26, 2024	
Thesis committee:	Prof. dr. ir. C. Vuik,	TU Delft, Responsible Professor
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An electronic version of this thesis is available at http://repository.tudelft.nl/.

Abstract

This thesis concerns the modelling, risk analysis and deposit rate optimization for Non-Maturing Deposits (NMDs), applied on aggregate data for the Dutch banking sector. The final NMD model consists of three parts, where there is a clear separation between the model for the term structure, the deposit rate and the deposit volume. The final model is calibrated on public data from 2004 to 2023.

For the term structure, we calibrate a shifted CIR model using a genetic algorithm on historical bond rates that were obtained via a Svensson model. The deposit volume is modelled via the change in the logarithm of the volume, which is regressed using Ordinary Least Squares on market and deposit rate parameters. The model is made autoregressive to remove serial correlation in the residuals.

Additionally, we describe and conduct a deposit rate model optimization for several classes of deposit rate models. We define two properties, the profit and the shortage, and maximize over a weighted sum of the two. For the optimization, we use a Robbins-Monro type model for its ability to deal with expectations, without the requirement of approximating the expectation at every step.

With the calibrated model and optimized deposit rate models, we introduce a way of risk analysis, by looking at the quantiles of the deposit volume, conditioned on the minimum of the market rate being in the lowest quantiles. For our model, this risk analysis indicates that a low interest rate environment is not problematic for the final, maximum and minimum volumes over a period of at least 10 years. On the contrary, the quantiles of the volume are higher when conditioned on rates being very low, indicating that the worst scenario for the deposit volume is not a scenario with low interest rates.

For simple models, we find that the procedure converges to an optimum value within an acceptable timeframe; for more complicated models, we find that the convergence is either slow, or that the these parameters do not influence the criterion that is optimized, compared to, for instance, the random component in the volume evolution.

When the volume is considered as an input for the deposit rate model, an optimal model is obtained that outperforms models only considering the market rate significantly on the two considered properties. This suggests that taking the volume into account is beneficial for maintaining a strong NMD portfolio.

Our methods are applied to public, aggregate data. As a result, we are unable to observe the effects of consumers switching banks within the Dutch system. We suspect that our approach works better when single bank data is available to the experimenter. All the methods that were described in this thesis are applicable to such data, yielding a possibly interesting way of conducting optimization, as well as risk analysis within a bank.

This thesis adds to the literature by introducing a way of risk analysis that is incorporated in the model itself. For this analysis quantiles of the distribution of the dependent variable are calculated for both unconditioned and conditioned scenarios. Concretely for this thesis, quantiles of the deposit volume, conditioned on the market rate minimum over the whole period being in the lowest quantile, are calculated.

Secondly, the thesis adds to the literature by introducing a method for optimizing the model for setting the deposit rate. Optimal models obtained via this approach can be used to challenge current ways that banks set their deposit rate.

Acknowledgements

Before you lies my MSc thesis titled *Optimal Deposit Rate Modelling and Risk Assessment for Non-Maturing Deposits: an application to aggregate data of the Dutch banking sector*. I have written this thesis to finalize my MSc in Applied Mathematics with a specialization in Financial Engineering. This thesis was written in the Research group Statistics, in collaboration with Triple A - Risk Finance B.V., where I had the opportunity to work on my thesis full-time from October 2023 until April 2024.

During my thesis, I enjoyed the support from the thesis committee, consisting of Kees Vuik from the Numerical Analysis department, Fabian Mies from the Statistics department and Niels van der Kleij from Triple A- Risk Finance B.V., for which I am thankful.

I am very thankful that I was given the opportunity to write my thesis within the Banking department at Triple A - Risk Finance. The support from my colleagues there, and especially the support, ideas and challenges from Niels van der Kleij and Romy Mieras have been of great help in the construction of this thesis. Their expertise was valuable for choosing a subject for this thesis and introducing me to the field of banking, which was relatively new for me at the start of this thesis.

Additionally, the frequent meetings and enthusiastic supervision of Fabian Mies ensured that I was never stuck in the experimentation and could quickly pitch ideas for more areas to explore in this work.

Finally, I would like to thank my friends and family for their continued support during the writing of this thesis, and whole of my studies at TU Delft.

I wish you a pleasant reading.

P.W. Oppelaar
Delft, June 2024

Abbreviations

- **ACM** (The Netherlands) Authority for Consumers and Markets
- **AR** Autoregressive
- **BCBS** Basel Committee on Banking Supervision
- **bp** basis point (one hundredth of a percentage point)
- **CDF** Cumulative Density Function
- **CIR** Cox-Ingersoll-Ross (Model)
- **CSRBB** Credit Spread Risk in the Banking Book
- **DF** Deposit Facility
- **DNB** De Nederlandsche Bank (Dutch Central Bank)
- **EBA** European Banking Authority
- **ECB** European Central Bank
- **EONIA** Euro Overnight Index Average
- **ESTR** Euro Short Term Rate
- **EURIBOR** Euro Interbank Offer Rate
- **G2++** Two-Additive-Factor Gaussian (Model)
- **GA** Genetic Algorithm
- **GAM** General Additive Model
- **GDTSM** Gaussian Deterministic Term Structure Model
- **HJM** Heath-Jarrow-Morton (Framework)
- **i.i.d.** Independent and Identically Distributed
- **IRRBB** Interest Rate Risk in the Banking Book
- **MFI** Monetary Financial Institution
- **MLF** Marginal Lending Facility
- **MMDA** Money Market Deposit Account
- **MRO** Main Refinancing Option
- **NIG** Normal Inverse Gaussian (Distribution)
- **NII** Net Interest Income
- **OLS** Ordinary Least Squares
- **OU** Ornstein-Uhlenbeck (Process)
- **TSL** Term Structure of Liquidity
- **VaR** Value at Risk
- **ZLB** Zero Lower Bound

Summary

Introduction

Non-Maturing Deposits (NMDs) make up a large part of the funding of a bank. They are defined as products without a specified maturity. This means that the consumer (the depositor) is free to withdraw their funds at any time without notice, while the bank (the issuer) is free to change the rewarded interest rate at any moment, also without notice. Checking accounts and variable savings accounts are two important classes of NMDs.

The freedom of depositors to withdraw their deposits at any time poses a risk to the bank. When a large part of a bank's deposits is withdrawn in a short timeframe, a so called 'bank run' may occur.

Changing market rates (short rates/bond rates) pose a second risk to the bank. Even though, theoretically, the bank receives the short rate and awards the deposit rate (which they set themselves), a large part of a banks funds is locked up in longer term investments, such as mortgages and longer term loans. This maturity mismatch can result in a situation where rising interest rates eliminate a bank's profit margin on NMDs.

While these pose significant risks to the stability of a bank, the literature on NMDs is relatively limited. The most influential papers on this subject are by Blöchlinger (2015), Jarrow and Van Deventer (1998), Kalkbrener and Willing (2004), and Selvaggio (1996), discussing the valuation and risk management of these products. Specifically for the Dutch market, there is a paper by de Jong and Wielhouwer (2003).

To further indicate the relevance of this topic, the Netherlands Authority for Consumers and Markets (ACM) recently published a concept report, indicating that, due to limited competition between banks in the Dutch banking landscape, depositors benefit insufficiently from higher European Central Bank (ECB) rates (ACM, 2024). We should see this in the experiments of this thesis as a low optimal deposit rate.

In this thesis, a model for NMDs is developed and calibrated to aggregate data from the Dutch market. This form of data gives rise to some difficulties in the model calibration, as it is limited to monthly frequency and groups the data of all banks into a single time series. Resulting in possible deposit volume movements between banks being unobserved by the model.

The deposit rate model is viewed as an input in this thesis, and several types of models are optimized to yield the most profit for a bank, while maintaining a sufficiently high deposit volume. The obtained optimal models are compared in different interest rate scenarios, and for one of the models, a novel way of risk analysis, making use of the extreme quantiles in market rate distribution is carried out.

This thesis adds to the literature by introducing this optimization over the deposit rate model, which can be used to challenge current ways of determining the deposit rate. Additionally, the way the risk analysis is carried out is innovative and finally, we add to the literature by calibrating an NMD model to aggregate data.

Methodology

An NMD model essentially consists of three sub-models, namely:

1. the term structure of interest rates,
2. the deposit rate, and
3. the deposit volume.

For the first and third, we describe the calibration method. After this, we describe the method for the optimization of the deposit rate and the method for the risk analysis. Finally, we describe the comparison of the different optimal deposit rate models.

Term structure model calibration: For the term structure of interest rates, we calibrate a Cox-Ingersoll-Ross (CIR) (Cox et al., 1985) model, using the approach set out by De Rossi (2010). Due to the inherent Zero Lower Bound in the CIR model, rates are shifted upward by 100bp (one percentage point), and shifted back down by the same amount when using the model.

The method of De Rossi (2010) uses Maximum Likelihood to estimate the coefficients of the CIR model. In this specific case, the model is given by

$$dr = a(\theta - r) + \sigma\sqrt{r}dW,$$

where this denotes the model in the real world measure. The parameter λ, for conversion to the risk-neutral measure, is also found in the calibration.

The calibration is done using a genetic algorithm, combined with a particle filtering approach for the calculation of the likelihood of a particular parameter set.

For the term structure, we are able to calibrate a continuous time model, because the term structure data is available with much higher frequency than the data on either the deposit rate, or the deposit volume (daily versus monthly frequency).

Deposit volume model calibration: For the calibration of the deposit volume model, an Ordinary Least Squares (OLS) approach was used. The evolution of the logarithm of the deposit volume was regressed on short rate and deposit rate parameters in a model of the form:

$$d\log V(t_i) = \boldsymbol{\alpha}'\boldsymbol{\rho}(t_i) + \gamma\epsilon_{t_i},$$

where $\boldsymbol{\alpha}$ is a vector of coefficients that is calibrated, $\boldsymbol{\rho}(t_i)$ is a vector of a constant, market rates and deposit rates (and quantities derived from them, possibly lagged), γ is a positive constant and ϵ_{t_i} are samples of a zero mean, unit variance error process.

The inclusion of variables in the vector $\boldsymbol{\rho}$ is determined via a mixed selection approach, here, we systematically include and exclude variables based on their p-values, determined by fitting them to the residuals of the then considered model.

After this procedure, we check if the residuals demonstrate serial correlation by conducting a Durbin-Watson test (Durbin & Watson, 1950). In our case, we find serial correlation. This correlation is removed via a Cochrane-Orcutt procedure, which effectively turns the model into an autoregressive model of order one ($AR(1)$) (Cochrane & Orcutt, 1949).

$$d\log V(t_i) - b\, d\log V(t_{i-1}) = \boldsymbol{\alpha}'[\boldsymbol{\rho}(t_i) - b\boldsymbol{\rho}(t_{i-1})] + \gamma\epsilon,$$

where b is the serial correlation coefficient from the previous model. Note that $\boldsymbol{\alpha}$, γ and ϵ are different from the previous model.

Deposit rate model optimization: For the optimization of the deposit rate model, we define two objectives, what we call the profit and the shortage, and optimize for a combination of the two. The optimization is carried out using a Robbins-Monro algorithm (Robbins & Monro, 1951). This iterative scheme is a form of stochastic approximation. We use this, because of its ability to efficiently deal with optimizing the objective when it is an expectation.

The algorithm starts with an initial guess $\boldsymbol{\theta}_0$ and iterates via the formula (importantly, this is a different $\boldsymbol{\theta}$) than in the CIR-model.

$$\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n - a_n g(\boldsymbol{\theta}_n),$$

where $(a_n)_n \geq 0$ is a positive, decreasing sequence and $g(\boldsymbol{\theta}_n)$ is the derivative with respect to $\boldsymbol{\theta}$ at $\boldsymbol{\theta}_n$ of $F(\boldsymbol{\theta}, \chi)$, which is the objective function for a random realization of the random variable χ. The overall objective is thus the expectation of F over χ. In our case, the χ corresponds to the random parts of the term structure model and the deposit volume model.

We optimize three types of deposit rate model: one with constant absolute margin, one with constant relative margin and the final one with a dependence on the volume. All are constrained to have a zero lower bound. These models are given by

$$i(t) = \max\{0, r(t) - \mu\},$$

$$i(t) = \max\{0, \alpha r(t)\}$$

Table 1: Quantiles of the minimum of the deposit volume over 10 years, for different conditionings on the minimum of the market rate. We see that these increase when conditioned on the market rate being low.

Deposit Volume Quantile	5%	1%	0.1%
Unconditioned	472 EUR B	445 EUR B	410 EUR B
r_{min} in bottom 5%	489 EUR B	461 EUR B	426 EUR B
r_{min} in bottom 1%	493 EUR B	467 EUR B	432 EUR B
r_{min} in bottom 0.1%	498 EUR B	474 EUR B	437 EUR B

and

$$i(t) = \max\left\{\left(\theta_1 + \theta_2 \frac{V(t) - V^*(t)}{V^*(t)}\right) r(t), 0\right\}$$

respectively. The parameters for which the optimal values are found are μ, α, θ_1 and θ_2.

Main Results

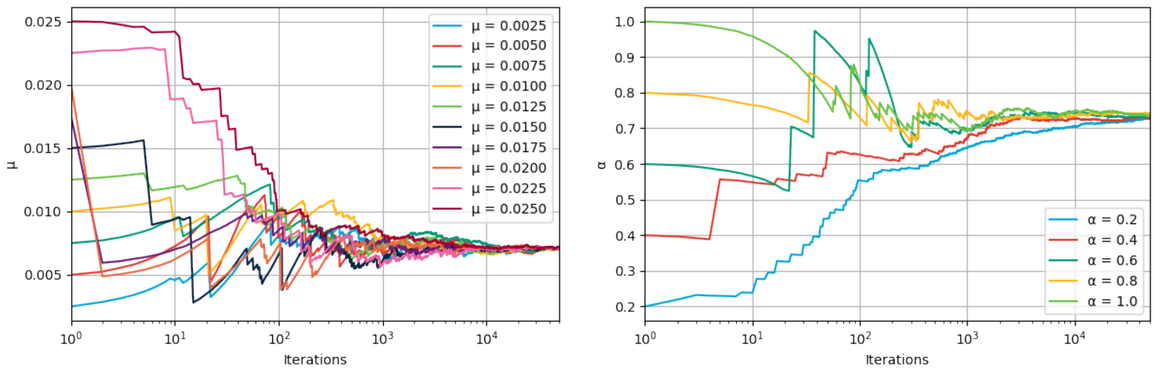
The main results of this thesis are the deposit rate optimization, the risk analysis and the deposit rate model comparison. We optimize for a combination of the profit and the shortage, where we define the target volume (or minimum target volume) V^* as 80% of the initial volume, and increase this with the market rate. The result for the optimization, when we give both criteria equal weight, is an optimal constant, absolute margin around $\mu = 0.8\%$; for a constant relative margin model, the found optimum lies around $\alpha = 0.73$; for the volume dependent model, convergence is less clear, but since the initial guess of $(\theta_1, \theta_) = (2.5, -45)$ does not change much during the optimization procedure, this seems like a good candidate for an optimal volume dependent model. These results can be found in Figures 1a, 1b and 2 respectively. Of course, changing the weight of the criteria will change these optimal values. Optimizing for only one of the two yields more trivial results, with only optimizing for profit suggesting a minimal deposit rate and only optimizing for minimum shortage suggesting a maximal deposit rate. Of these, the first points towards a relative unwillingness of consumers to withdraw their deposits and invest outside the Dutch NMD market.

The main results for the risk analysis of the deposit volume are in Figure 3. Here, we plot the minimum volume over a period of 10 years, as forecasted using 10,000,000 simulations of the calibrated NMD model. The minimum volume is presented for its significance in 'bank run' scenarios. Figure 3a contains the results for all the simulations, where Figure 3b contains the results for the subset of scenarios with the minimum rate in the lowest 0.1%-quantile. From these results, mainly the quantiles and also the 'fatness' of the left tail of the distribution, we see that very low interest rate scenarios are not necessary adverse for the size of the NMD portfolio. A possible explanation for this is that the zero lower bound on the deposit rate decreases the spread between market and deposit rate when the market rate is very low, thus making saving relatively more attractive than in a high interest rate scenario. The quantiles for other conditionings on the minimum of the interest rate can be found in Table 1.

Finally, the results for the comparison of the different optimal deposit rate models can be found in Figure 4. Here it can be seen that the volume dependent model outperforms both the constant absolute margin and relative margin models for most interest rate scenarios, both in expectation and for the lowest considered quantile. This can be seen from the objective values for the volume dependent model lying above those for the other two models. This result is expected for the expectation, since that this the quantity that is optimized for, and the volume dependent model has an extra parameter to use to its advantage. That the volume dependent model also outperforms the other two models in the lowest quantile is more surprising, but is likely the result of the volume dependent model being able to somewhat limit the outflow of deposit volume when it comes close to the target volume, thus limiting the shortage.

Conclusion

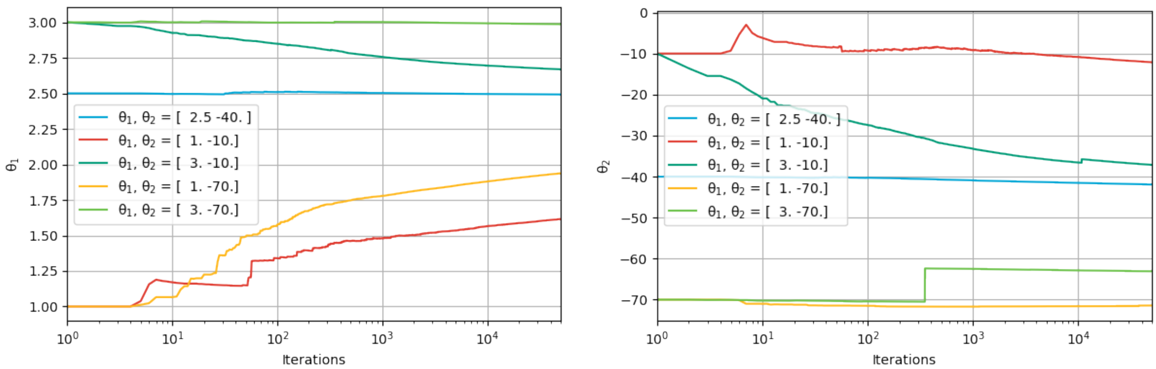
The results for the optimization of the deposit rate model, show us that we can find such an optimal model, at least when considering classes of simple deposit rate models. This provides a tool to find an optimal strategy for a bank.



(a) Robbins-Monro optimization for the parameter μ in the constant, absolute margin model. The convergence is plotted for different starting values μ_0.

(b) Robbins-Monro optimization for the parameter α in the constant, relative margin model. The convergence is plotted for different starting values α_0

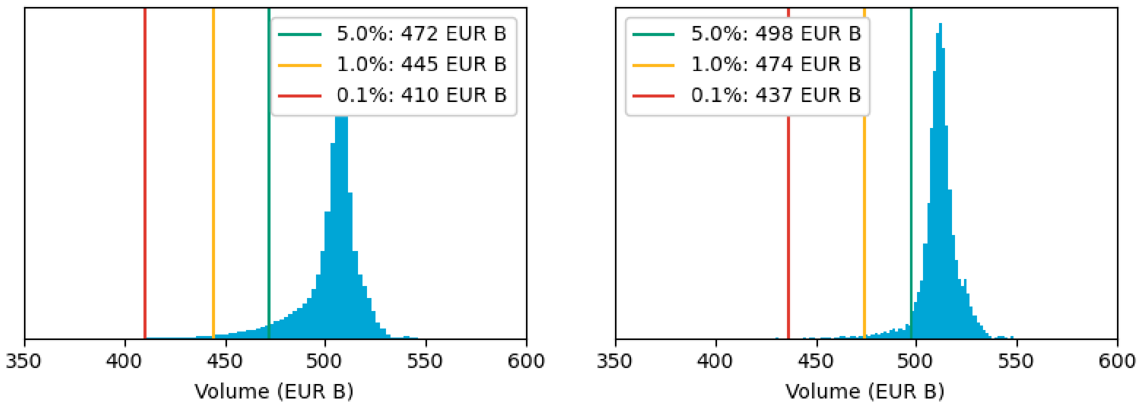
Figure 1: Robbins-Monro optimization results for two types of deposit rate models, for different starting values of the optimized parameter. In both plots, the number of iterations on the x-axis is logarithmic for better readability.



(a) θ_1

(b) θ_2

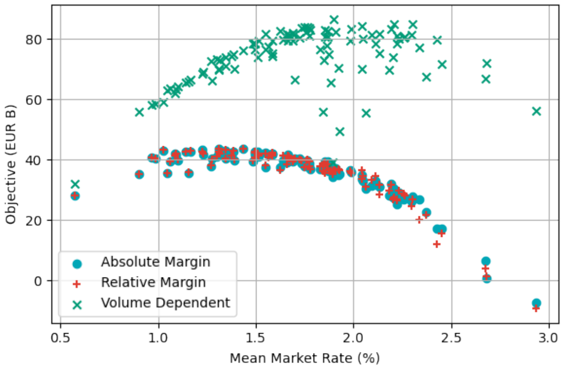
Figure 2: Robbins-Monro optimization results for the parameters θ_1 and θ_2 in the volume dependent model, for different starting values. In both plots, the number of iterations on the x-axis is logarithmic for better readability.



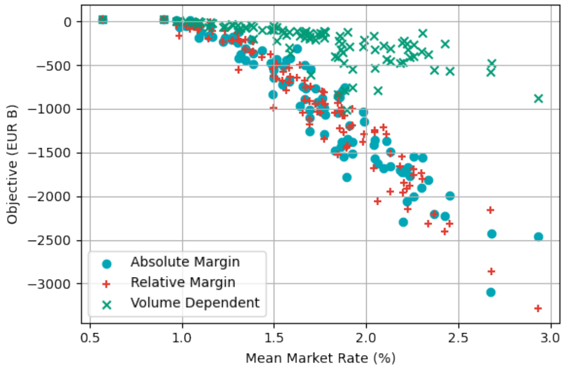
(a) Distribution minimum volume, unconditioned.

(b) Distribution of the minimum volume, conditioned on the minimum rate being in the lowest 0.1%-quantile.

Figure 3: Minimum deposit volume distributions for a ten-year period. Plots of the simulation results for the minimum volume over ten years, unconditioned and conditioned on the minimum of the rates being in the lowest 0.1%-quantile. All plots have lines at the 0.1%-, 1%- and 5%-quantiles as vertical lines.



(a) Expected value



(b) 0.1%-quantile

Figure 4: Comparison of the three optimal deposit rate models on the expected value and 0.1%-quantile of the objective for different interest rate scenarios.

From our results on the risk analysis, we see that very low market rate scenarios do not have an adverse effect on the total NMD volume. Rather, the calibrated deposit volume model suggest that the worst interest rate scenarios are when the margin between the market and deposit rate is maximal.

Finally, the results indicate that a volume dependent model significantly outperforms models that do not depend on the volume, and therefore volume should be used as an input in deposit rate models.

Another interesting point of this analysis, is that the results suggest that for optimal bank profit, a bank should keep its deposit rate at zero. The resulting decrease in volume appears to be not large enough to counteract the increase in profit from the increased margin. We suspect that this is partly because of the use of aggregate data, an important factor that is not captured in this data, is that a customer can switch banks within the Dutch system to obtain a higher rate. It does however show that the barrier for a depositor to withdraw its funds from NMDs completely is relatively high, and only becomes a significant effect when the difference between market and deposit rate is rather large. This conclusion corresponds with the findings of the ACM (2024).

Since the framework we have developed could also be used with single bank data, we suggest this as a topic for further research. The individual volumes might be more sensitive to rate parameters than the total volume, because of the interbank switching not being captured in the latter.

Other extensions could be to fit the deposit volume model on some non-rate parameters as well. This was tried for the time as proxy of macroeconomic effects, where no correlation was found.

The final extension we propose, is extending the model to include more than one bank, and model their interactions. This would, however, require a substantial amount of data, which is highly likely to be confidential in nature.

Contents

1 Introduction

Non-maturing deposits (NMDs), or variable savings, form an important part of a banks funding. In the Netherlands, around three quarters of household savings are in accounts with no agreed maturity.[1] These accounts not having an agreed maturity means that depositors can withdraw their deposit at any time, and deposit at any time as well. In turn, the bank can change the interest rate the depositors earn on their deposits, the deposit rate, at will. These dynamics imply that NMDs come with some risks to the bank. Modelling the dynamics of these products to assess these risks is thus important. Additionally, the bank makes a profit on the difference between deposit and market rates. Therefore, modelling this is also of interest to banks.

The risks, involved in NMDs, arise directly from the dynamics. In the literature, two types of risk are identified. Liquidity, or sometimes called option, risk is the risk that arises from the ability of the depositors to withdraw funding. When the bank does not sufficiently account for this risk, a so called 'bank run', where a large group of depositors withdraws their funds in a short time, could result in the downfall of a bank.[2]

The second risk comes from the non-constant market rate. In general (in the case of negative market rates, this was not the case), a bank can earn money by investing the deposited funds in the shortest maturity zero coupon bond, receiving the market rate, and rewarding the depositor with a lower deposit rate. This arbitrage can remain, because only banks can issue NMDs and the number of banks is limited (Jarrow & Van Deventer, 1998). Interest rate risk comes in when the market rate fluctuates. For instance, a falling interest rate eliminates the profit margin for the bank. The bank is then forced to change the deposit rate as well.

Relatively little attention has been given to the valuation and risk analysis of NMDs in literature, which might be due to the confidentiality of data on NMD volumes at individual banks. Nevertheless, there is vast literature on the evolution of term structures and the pass-through of market rates to deposit rates. Recently, developments have been made in term structure models that cope well around zero. In literature on pass-through rates, behaviour around zero received considerable attention during the last few years as well.

Additionally, we will describe and carry out an optimization procedure for the deposit rate. This is a natural next step, since this is the one input that a bank can use to influence the profit on their NMD portfolio, as well as the size of their portfolio. The optimization is carried out using a Robbins-Monro algorithm (Robbins & Monro, 1951), possibly on a multivariate function (Blum, 1954). We define several classes of possible deposit rate models, and optimize over each of these. The final optimum is then obtained as the best optimum over the classes. We define a flexible criterion for the optimization, and the procedure is flexible regarding possible deposit rate model forms. We thus end up with a framework for deposit rate optimization that is widely useable.

A concept report, published by the Netherlands Authority for Consumers and Markets (ACM), concludes that the deposit rate on the Dutch market remains low, compared to the ECB rate, due to the market being oligopolistic (ACM, 2024). A large portion of the total deposit volume is controlled by a

[1]See data from DNB (Table 5.2.6)

[2]See, for example, the collapse of Silicon Valley Bank in 2023.

limited number of banks, while the influence and market share of smaller banks remain limited. The ACM concludes that this is at least partly due to consumers not being inclined to switch banks quickly. This means that the result for the optimal deposit rate will likely be on the lower side.

This thesis uses data that is not confidential, which means that the available data on the deposit volume is limited. The most granular deposit volume data available, is with monthly frequency from De Nederlandsche Bank (DNB). Although the deposit rate on bank level is publicly available with high frequency, the weighed average of this rate is again only available with monthly frequency from DNB.[3] This data is more useful, since it includes information on the volume the deposit rate is awarded on. Data on the term structure, data from the interbank market[4] and data from the ECB[5] are available at a much higher frequency. The models in this thesis will be developed in such a way that they are also applicable to single bank data.

We contribute to the literature in the following ways. Firstly, we provide a framework for the analysis and valuation of an NMD portfolio, specifically on the Dutch market, where we add to a preceding paper by de Jong and Wielhouwer (2003). Secondly, the risk analysis procedure, where we look at quantiles of the distribution of the volume that follows from our calibrated model, has not been done in the literature on NMDs, the most similar risk analysis method is the method described by Marena et al. (2023). Additionally, we add to the literature by describing a deposit rate model optimization procedure, which optimizes over various deposit rate model types found in literature. Finally, we only use public data for our analysis, meaning that the used methods can be applied by anyone, and are not restricted to those with access to confidential data. The use of confidential data might, however, improve the model performance.

The following section, Section 2, gives a detailed overview of the literature on NMDs and relevant to NMDs. Section 3 gives an overview of the data, that was used in this thesis. The subsequent section (Section 4) sets out the methods that were used to obtain the results found in Section 5. The results are discussed in the same section. The thesis ends with a conclusion in Section 6.

[3]See data from DNB (Table 5.2.7.1: 'MFI households deposits and loans, interest rates, adjusted for breaks (Month)', row: 'Overnight deposits - Total')
[4]Available via the ECB
[5]See data from the ECB

2

Literature

This section gives an overview of the existing literature on NMDs and related subjects. Additionally, our contribution to the literature is stated by identifying an open question that this thesis aims to answer. First, an overview of the literature regarding the valuation of NMDs is given. This is literature that contains models for the term structure, deposit rate and deposit volume. Secondly, the literature on individual parts of the model is presented. The section starts with models for European Central Bank (ECB) policy rate and term structure models. After this, it moves on to models for deposit rates and deposit volumes. The section ends with an overview of regulatory guidelines and literature on interest rate risk.

2.1 NMD Modelling

NMD valuation involves the modelling of the evolution of the cash flows associated with the NMD process. These cash flows consist of the interest payments made by the bank to the depositors, inflow and outflow of deposits and interest payments the bank can receive on their deposits by lending it on the interbank market or to central banks like the ECB. Determining this requires modelling the three different components that determine this cash flow. These are:

1. the short rate (or whole term structure),
2. the deposit rate (which is the rate a bank pays their depositors), and
3. the deposit volume.

Modelling the deposit volume is vital in the analysis of NMDs, as this is the source of a bank's profit. Furthermore, rapid decreases in the volume can result in liquidity problems, because a part of the volume is locked up in longer term investments. When a bank accounts insufficiently for these decreases, or does not adequately prevent them, a so-called bank run can result in the collapse of a bank (Silicon Valley Bank and Credit Suisse are recent examples). The minimum of this quantity over the course of the considered horizon gives an indication to banks what part of the deposits needs to be available on short notice to return to their depositors, and what part of the volume can be put into longer term investments (which might have higher expected returns). Some studies focus on modelling the minimum of this quantity (i.e. Kalkbrener & Willing, 2004).

A fundamental paper in the NMD field is by Jarrow and Van Deventer (1998). They derive a valuation method for NMDs (and corresponding assets, i.e. credit card loans). They do this by introducing a market separation between banks and non-banks, where the former are the only ones that can issue demand deposits. This results in an arbitrage opportunity for (a limited number of) banks, when the deposit rate is lower than the rate on the interbank market. This is usually the case, although not when the rate approaches zero. In their analysis, Jarrow and Van Deventer (1998) use a one-factor Hull and White (1994) (HW) model for the short rate. The deposit rate is modelled as the short rate minus a constant, and subsequently (in discrete time) as a function of the previous deposit rate, the current short rate and the change in short rate. The deposit volume is modelled first as linear in the market rate, and subsequently as a function of the previous deposit volume, time (explicitly) the short rate and the change in short rate. Furthermore, the logarithm of the deposit volume is modelled, this ensures positivity of the deposit volume. Continuous analogues for these processes are also derived.

Selvaggio (1996), while using a similar approach to model the market and deposit rates, uses a different approach for modelling the deposit volume. He defines an equation for a target volume, that depends on a vector of macroeconomic variables. The real volume then approaches this target volume with a predefined speed. Frachot (2001) employs a similar approach, but uses a behavioural model.

Kalkbrener and Willing (2004) use a three-factor model for the valuation of NMDs, similar to Jarrow and Van Deventer (1998). They, however, opt for a more involved model for the market rate, and use the framework of Heath et al. (1992) (HJM). They use the same model for the deposit rate as Jarrow and Van Deventer (1998). For the deposit volume, Kalkbrener and Willing (2004) implement both a normal and log-normal model. The normal model consists of a deterministic part, linear in time, and an Ornstein-Uhlenbeck (OU) process. They allow the Brownian motion in this process to be correlated to the ones in the market rate model. There is no explicit coupling to the market rate or deposit rate in their model. The log-normal model they propose is the exponential of the proposed normal model. They use these models to derive what they call the ”Term Structure of Liquidity” or TSL. Where TSL(t, p) is the exact amount available during the whole period $[0, t]$, with probability p. They also use their model to construct a replicating bond portfolio for the NMD deposits. This replicating portfolio can then be used to hedge the risks involved with issuing NMDs.

Blöchlinger (2015, 2021) has published several studies on the subject of NMD valuation and risk management. Blöchlinger (2015) proposes ways to hedge option/liquidity risks involved in NMD products. Blöchlinger (2021) expands on this by introducing analytical solutions via a Gaussian approximation. This speeds up the computation of the valuation of the NMDs. Blöchlinger (2015) criticizes the replicating portfolio approach of Kalkbrener and Willing (2004), and argues that it does not account for the volatility risks involved. A replicating portfolio should therefore include volatility sensitive instruments, as opposed to solely straight bonds. Blöchlinger (2015) uses a discrete, autoregressive model for the short rate (with time dependent coefficients). The deposit rate is modelled similarly, but includes an upper and lower bound. It can also only take certain values (round numbers). The volume modelling allows for adjustments based on the costs of holding a deposit and the market rate (changes). Blöchlinger (2021) models the short rate discretely as an autoregressive process. For the deposit rate and volume, Blöchlinger (2021) extends the models of Jarrow and Van Deventer (1998) to be able to deal with bank specific optimal rates and corresponding volume reactions.

Nyström (2008) develops a model for NMDs, where the focus is more on the modelling of the deposit volume. There, the customer has the option to divide their savings over a number of savings accounts with different policy functions for the deposit rate, as well as a transaction account. Through these policy functions, more involved deposit rate protocols can be incorporated in the model, such as a rate that is adjusted according to the total volume a customer has in their account. For the deposit volume, and, particularly, the division of a depositors funds over the transaction and savings accounts, Nyström (2008) derives a behavioural model for the ”average customer”. The valuation of the deposits is done in the same way as by Jarrow and Van Deventer (1998). While this approach makes sense when modelling the NMD process for a single bank make, where policy functions for the deposit rate are deterministic, this framework is less suitable when working with aggregate data, since such policy functions are then not available.

Marena et al. (2023) propose a model that is entirely of OU type. They use a Vasicek (1977) model for the market rate (see next section) for its compatibility with negative interest rates. The ZLB is imposed on the deposit rate and deposit volume by modelling them indirectly via their logarithms. They compare the performance of different Lévy distributions in the models and find that not the classical Gaussian (associated with Brownian motion), but a Normal-Inverse-Gaussian (NIG) (Barndorff-Nielsen, 1995) process provides a better fit to the observed data. They later use the flexibility of this distribution to perform a risk analysis, where they derive the Value at Risk (VaR) of the deposit volume in a ’bank run’ scenario.

2.2 ECB Rate

An important player in determining the market rate, is the central bank. In general, short term market rates follow the rates that are set out by these entities, since banks can deposit and lend from the central bank at these rates. Modelling these rates, instead of the market rate, using a more traditional term structure model, might result in a more intuitive way to add risks into the model.

The central bank in the Euro area, and therefore also the Netherlands, is the ECB. The ECB, alongside providing bank supervision, provides deposit and lending facilities to banks, and sets the interest rates on these facilities with the goal of maintaining economic stability. The ECB sets three rates on three different facilities:

- the Main Refinancing Option (MRO), which provides loans with maturity of one week;
- the Marginal Lending Facility (MLF), which provides overnight loans; and
- the Deposit Facility (DF), which provides overnight deposits.

The Governing Council, which is responsible for setting the rates on these facilities, sets these or decides to keep them the same every six weeks. Any changes take immediate effect. Rates are generally changed in steps of 25bp (basis points, $1\text{bp} = 0.01\%$, where this denotes percentage points). Exceptions are that sometimes a jump of 50bp occurs and that jumps are smaller when the rates are close to zero. Importantly, the following inequality for these rates always holds:

$$r_{\text{MDF}} < r_{\text{MRO}} < r_{\text{MLF}}. \tag{2.1}$$

With setting these rates, the ECB influences the rates on the interbank market, which are the Euro Interbank Offer Rates (EURIBOR) with maturities ranging from one week up to one year and the Euro Overnight Index Average (EONIA) / Euro Short Term Rate (€STR) with overnight maturity. On this interbank market, banks offer loans and deposits to each other. Short term changes in the ECB rates are usually expected, either inferred from macroeconomic trends or from communications from the Governing Council (Cour-Thimann & Jung, 2021). On longer time scales, they may however be treated as random.

Some models are available for the behaviour of the ECB rates. Carfora et al. (2014) fit a model to the rates, identifying periods of expansion and regression, where the rates increase during expansion and decrease during recession. This model is of the following form:

$$\begin{aligned} r(t) &= r(0) + \sum_{j=1}^{N_t} a_j b_j, \\ r(0) &= r_0. \end{aligned} \tag{2.2}$$

Equation 2.2 corresponds to Eq. 4 in Carfora et al. (2014). In this equation, r is (one of the) ECB rate(s), r_0 is the initial rate, N_t is the number of ECB interventions up to time t, $a_j \in \{0, 25\text{bp}, 50\text{bp}\}$ is the intervention step, and $b_j \in \{-1, +1\}$ determines the direction of the intervention. In this, N_t follows a Poisson process.

Slight alterations to this model yield the following model:

$$\begin{aligned} r(t) &= r(0) + \sum_{j=1}^{M_t} k_j l_j, \\ r(0) &= r_0. \end{aligned} \tag{2.3}$$

In Equation 2.3, M_t denotes the number of six-week periods before t, $k_j \in \{0, 25\text{bp}, 50\text{bp}\}$ and $l_j \in \{-1, 0, +1\}$. The difference is in the amount of interventions in a certain time frame, which is deterministic with Equation 2.3 and stochastic with Equation 2.2, and the inclusion of an uncertain or constant state in Equation 2.3 ($l_j = 0$).

The ZLB is not explicitly accounted for in this model. While the interest rate on the DF has been negative in the past, the rates on the MRO and MLF have remained non-negative.[6] When modelling the MRO, a ZLB can be enforced by forcing the rate to zero whenever it becomes negative after an adjustment.

[6] See data from the ECB

2.3 Term Structure

The term structure is, at a given time, the collection prices for zero coupon bonds for all maturities. A zero coupon bond is a (theoretical) product that pays one unit of currency at maturity. It does not involve any payments in the period between the current time and the maturity (such payments are called coupons). This term structure determines risk-free rates over all periods of time, and tells something about the way the short rate (instantaneous rate of return) is expected to evolve. In the literature, several models are described that generate the term structure.

2.3.1 One-Factor Models

The simplest models that generate the term structure, are one-factor models. These models contain only one factor, namely the short rate and derive the term structure through no-arbitrage arguments and taking expectations of the short rate. Since the short rate is a continuous rate, models for this rate are continuous models. Models for the short rate take the form of Lévy driven stochastic differential equations, and are given by:

$$dr(t) = \alpha(r(t), t)dt + \sigma(r(t), t)dL(t). \tag{2.4}$$

In Equation 2.4, r denotes the short rate, $\alpha(r(t), t)$ is a function describing the drift of $r(t)$ (possibly mean reverting), $\sigma(r(t), t)$ is a function describing the variance of the process and $L(t)$ is a zero drift Lévy process (commonly Brownian motion and denoted by $W(t)$).

The following describes some important variants of Equation 2.4 and discuss their advantages and limitations. For a more detailed description of these models, as well as derivations of forward rates and bond prices, see for instance Brigo and Mercurio (2007) and papers where the models are introduced.

Vasicek (1977) model

Vasicek (1977) was the first to introduce a term structure model. In this model, both the drift and variance are time-independent. The process is mean reverting to some long term mean θ. The Lévy process, that drives the model, is Brownian motion. The model is thus given by:

$$dr(t) = a(\theta - r(t))dt + \sigma dW(t). \tag{2.5}$$

In Equation 2.5, a describes the speed of mean reversion, σ^2 is the variance (for one unit of time) and $W(t)$ denotes Brownian motion. This model results in a short rate that is normally distributed for all future times. This means that the short rate can become negative in the Vasicek (1977) model.

Dothan (1978) Model

Dothan (1978) proposed a model that does not suffer from the possibility of negative rates, but follows a geometric Brownian motion. This, combined with the introduction of a constant ”market price of risk” (see Brigo & Mercurio, 2007, p. 51-53), results in the following model (Brigo & Mercurio, 2007):

$$dr(t) = ar(t)dt + \sigma r(t)dW(t). \tag{2.6}$$

Equation 2.6 is limited in mean reversion, it can only revert to zero for negative a. Positivity is however ensured due to the $r(t)$ factor in front of the Brownian motion. In this model, the short rate for any given future time is distributed according to a log-normal distribution. Due to the limitation on the mean reversion of this model, is sees little usage in practice.

CIR (Cox et al., 1985) Model

The model introduced by Cox et al. (1985) (Cox-Ingersoll-Ross or CIR model) addresses the main concern of the Vasicek model, namely its possibility to generate negative interest rates, by introducing a different function for the variance of the process. In their model, they incorporate the short rate in the variance, resulting in the square root of the short rate in the stochastic differential equation (SDE). This gives in the following model:

$$dr(t) = a(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t). \tag{2.7}$$

The only difference between Equation 2.7 and Equation 2.5 is the inclusion of the square root of the short rate in the former. This, however, ensures that the short rate can never become negative, as the Brownian motion term vanishes near zero, making the drift dominant. The resulting distribution for the short rate, at any given time, is now a non-central chi-squared distribution.

Adaptations

In subsequent literature, these models are adapted to mitigate some of their drawbacks. Mainly with regard to fitting to real world data and behaviour around the zero lower bound (ZLB). Hull and White (1990), for instance, extend the Vasicek model to include time varying coefficients. This enables the model to be perfectly fitted to the current term structure. Where in (Hull & White, 1990), all coefficients are allowed to be time dependent, this is restricted somewhat in (Hull & White, 1993), where only the parameter θ is allowed to be a function of time. This does not restrict the fitting of the model to the current term structure of interest rates. This model is widely used in practice, despite it having no restriction on negative interest rates.

Hull and White (1990) proposed allowing θ to be a function for the CIR model as well, which has proven less successful (Brigo & Mercurio, 2007). This is because in those models, positivity is no longer ensured (which is problematic due to the square root of the short rate in the process). The function that replaces θ in the model is not available analytically.

Other, more recent, extensions of the CIR model are available as well. Starting with the CIR++ model (Brigo & Mercurio, 2007). This model is obtained by applying a deterministic shift to a general CIR process. Explicitly:

$$\begin{aligned} dx(t) &= k(\theta - x(t))dt + \sigma\sqrt{x(t)}dW(t), \quad x(0) = x_0, \\ r(t) &= x(t) + \phi(t). \end{aligned} \tag{2.8}$$

In Equation 2.8, k, θ, σ and x_0 are all positive constants. The function $\phi(t)$ describes the deterministic shift as a function of time. This model is able to deal with negative interest rates, in sharp contrast with the 'regular' CIR model. This model can also be fitted exactly to the current term structure, and remains tractable.

More recently, Orlando and Bufalo (2021) have developed a further adaptation of the CIR model, which they call the CIR# model. In addition to negative rates, this model can incorporate cluster volatility, rate jumps and can capture time changes in the volatility of interest rates. This model is, however, significantly more involved, and involves partitioning of the data.

2.3.2 Multifactor Models

The main drawback of using a one-factor model for the short rate, is that this results in perfect correlation between rates for all maturities (Brigo & Mercurio, 2007). In practice, these rates are not perfectly correlated and especially when considering shocks to the short rate, having these effects propagate equally to rates for short and long maturities might be undesirable.

Multifactor models do not suffer from this problem. Introducing more and more factors makes it possible to fit more complex correlation structures, at the cost of increased computational complexity. Only a two-factor Gaussian model is presented here. Extension to more than two factors is straightforward. Brigo and Mercurio (2007) introduce the following two-factor model:

$$\begin{aligned} r(t) &= x(t) + y(t) + \phi(t), \quad r(0) = r_0, \\ dx(t) &= -axdt + \sigma dW_1(t), \quad x(0) = 0, \\ dy(t) &= -bydt + \eta dW_2(t), \quad y(0) = 0, \\ dW_1(t)dW_2(t) &= \rho dt. \end{aligned} \tag{2.9}$$

In Equation 2.9, a, b, σ and η are positive constants, ρ is a constant between -1 and 1 (the correlation between the Brownian Motions $W_1(t)$ and $W_2(t)$), and r_0 is the initial rate. The function $\phi(t)$ can be used to fit the model to the current term structure and $W_1(t)$ and $W_2(t)$ are (correlated) Brownian motions. Brigo and Mercurio (2007) name this model the two-additive-factor Gaussian Model (G2++). This model is equivalent to that of Hull and White (1994).

Multifactor extensions of the CIR model also exist (Chen & Scott, 2003; Longstaff & Schwartz, 1992), but they lose their tractability when choosing $\rho \neq 0$ (Brigo & Mercurio, 2007).

2.3.3 Other Term Structure Models

The models, discussed above, are all models for the short rate. This is, however, not the only way to tackle the modelling of the term structure. Heath et al. (1992) develop a framework for the term structure that models the evolution of the forward rate curve, instead of the short rate. This eliminates what Heath et al. (1992) call the ”inversion of the term structure”, which is required in all the above models. This inversion is computationally expensive, and might lead to a model that admits arbitrage (Cox et al., 1985; Heath et al., 1992). A disadvantage of this model is that the corresponding short rate is no longer Markov, but is path dependent. The advantage of this model is that it can be fitted to all forward rate curves. This is not the case for, for instance, the CIR (Cox et al., 1985) model.

2.4 Deposit Rate

The deposit rate is the rate of return a depositor receives on their deposit. This rate is set by the bank and can be changed at will by the bank. The dynamics of setting this rate, and especially its connection to the money market rate, have received considerable attention in literature. A discussion of the different aspects that could be included in a deposit rate model can be found in Paraschiv (2011). This is presented here as well, with some additions concerning behaviour at the ZLB.

2.4.1 Deposit Rate Characteristics

Firstly, some characteristics are set out that could be incorporated in a deposit rate model. The subsequent section presents a model and discusses how these characteristics can be incorporated. These characteristics are adapted from Paraschiv (2011).

Incomplete Pass-Through

Andries and Billon (2016) presents an overview of the empirical literature on the retail interest rate pass-through, specifically for the Euro area. The pass-through describes how much of an increase/decrease in the market rate, is reflected in a bank rate (deposit or loan) after a certain time period. A long term pass-through of at most 40% is found by de Bondt (2005) for overnight deposits and short-term deposits with maturity up to three months. Nehls (2006) analyses the pass-through in the German market and finds differences between banks of different sizes. Sørensen and Werner (2006) analyse the pass-through for the euro area and find that this pass-through is not the same for all countries.

Jump Process

Although deposit rates can be changed by banks at any time, and without the penalty of a cost to the depositor. It is observed that banks change their deposit rate in jumps, and keep it at ’round’ values (mostly intervals of 25bp) (Kahn et al., 1999). Dueker (2000) and Mester and Saunders (1995) find this for the prime rate (another interest rate set by banks). This is explained by postulating that there is a cost involved for a bank to change its deposit rate, which would for example be in the form of an administrative cost (Paraschiv, 2011).

Slow Adjustment

Some delay is observed in the pass-through of changes in market rates to deposit rates. Mester and Saunders (1995) observe this for the prime rate. This is sometimes credited to the notion that banks only change their rates when they believe the market rate change is longer term, and not a short term spike. This also connects to the idea that changing the deposit rate is of non-zero cost to the bank, so that a bank is somewhat reluctant to change their rate. Another explanation would be that information is not immediately available to customers, of which banks could take advantage (Rosen, 2002).

Asymmetric Adjustment

The adjustment of the interest rate by banks is not necessarily symmetric (Hannan & Berger, 1991; Neumark & Sharpe, 1992). When market rates decrease, this narrows the margin of the bank on deposits, while an increase in market rate widens the banks margin. This would make banks more inclined to change the deposit rate in the former case, than in the latter case. Sander and Kleimeier (2002) find asymmetric pass-through in their analysis of countries in the European Monetary Union (EMU), but for other countries than the Netherlands. There, they find symmetric adjustments.

Imposing the ZLB
Similar to market rates, deposit rates were thought to have a ZLB, where it was postulated that depositors would exchange their deposits for cash in order to avoid paying the negative interest rate (Black, 1995). While market rates have been negative in the Euro area between 2014 and 2022,[6] deposit rates remained above zero regardless.[7] Only for large (institutional) clients, or for large volume accounts, banks have successfully charged negative interest rates (although still higher than the deposit facility rate).[8] This was explained by Altavilla et al. (2022) as being due to a non-zero cost for the clients, associated with holding their funds in cash, rather than at a bank. A recent study by Ulate (2021) proposes models for the deposit rate, that cope well with the ZLB.

Variable Spread
At any moment, the rate a bank charges on new loans is higher than the rate it pays its new depositors. The difference between these rates is the spread, a source of profit for banks. This (absolute) spread is not constant (not even for the ECB),[6] but tends to be wider for higher rates and narrower for lower rates. Rosen (2002) observe this for the US market. This is again connected to the notion that banks try to optimize their profits and are able to do this due to imperfect competition.

2.4.2 Deposit Rate Models
Now, it is specified how the preceding characteristics can be (and have been) included in models. This is done for both discrete and continuous time models.

Continuous Time Deposit Rate Models
The simplest forms of deposit rate models are models where the deposit rate is just an affine function of the market rate. Such an approach is, for instance, taken by Jarrow and Van Deventer (1998) (at first). Such a model is given by:

$$i(t) = r(t) - \mu. \tag{2.10}$$

In Equation 2.10, $i(t)$ is the deposit rate at time t, $r(t)$ the short rate at time t, and μ a bank margin. The profit for a bank is then captured in the variable μ. This model does not account for a ZLB on the deposit rate, as it will drop below zero whenever the market rate is below μ. A more general model, that can take into account more of the effects above, is given by (in the form of an SDE):

$$\begin{aligned} di(t) &= \alpha(\boldsymbol{\beta} \cdot \boldsymbol{r}(t) - \mu(\boldsymbol{r}(t)) - i(t))dt, \\ i(0) &= i_0. \end{aligned} \tag{2.11}$$

In Equation 2.11, α is included as the adjustment speed of the rate. This α can be a function of the difference between the deposit rate and market rate minus a margin, to make the model asymmetric.

Different aforementioned characteristics could be incorporated in Equation 2.11 as follows. Different market rates can be included, for example: a short term rate and longer term rate, in the vector $\boldsymbol{r}(t)$ with weights given by $\boldsymbol{\beta}$. $\mu(\boldsymbol{r}(t))$ is now a function of the market rates as well, this can be used to account for an incomplete pass-through, to impose a ZLB and to introduce a variable spread. To account for the deposit rate being a jump process between distinct values, possible values for the deposit rate can be defined and put in a set I. Then the rounded deposit rate is given by

$$i_{\text{true}}(t) = \max\{\iota \leq i(t) | \iota \in I\}. \tag{2.12}$$

The choice of the set I, in combination with the process for $i(t)$ should be made in such a way that this maximum always exists.

Alternatively, one could model the deposit rate as an OU process, reverting to the value $\boldsymbol{\beta} \cdot \boldsymbol{r}(t) - \mu(\boldsymbol{r}(t))$. This could be advantageous when the bank behaviour is not entirely certain, or when modelling the volume weighted average of the deposit rate, where the behavioural characteristics of depositors could be of significant influence. This is less common in the literature, but is the approach taken by Marena et al. (2023) for instance.

[7] See data from DNB (Table 5.2.7.1: 'MFI households deposits and loans, interest rates, adjusted for breaks (Month)', row: 'Overnight deposits - Total')
[8] See data from DNB (Table 5.2.7.3: 'MFI non-financial corporations deposits and loans, interest rates, adjusted for breaks (Month)', row: 'Overnight deposits - Total')

Discrete Time Deposit Rate Models

When data is less granular, it could make more sense to model the deposit rate as a discrete process. One could also argue that the deposit rate process should be modelled as a process of Poisson type, due to the jump characteristic of the rate. This is, however, not common in literature. Discrete processes for the deposit rate are given by the discrete counterparts of Equation 2.10 and Equation 2.11 and the OU process. The analogue for Equation 2.10 is exactly the same, while the analogues for the other processes are given by their autoregressive (AR) variants. These are given by, for a sequence of times $(t_n)_{n\geq 0}$, by: for Equation 2.11

$$\begin{aligned} i(t_n) &= i(t_{n-1}) + \alpha(\boldsymbol{\beta} \cdot \boldsymbol{r}(t_n) - \mu(\boldsymbol{r}(t_n)) - i(t_{n-1})), \\ i(0) &= i_0; \end{aligned} \tag{2.13}$$

and a general AR process with adjustment $\boldsymbol{\beta} \cdot \boldsymbol{r}(t_n) - \mu(\boldsymbol{r}(t_n)) - i(t_{n-1})$ and a zero mean error for the OU process.

2.5 Deposit Volume

Outside of studies concerning NMDs completely, there is not much literature concerning deposit volumes (and some of the literature on NMDs assumes constant volumes or a relatively simple model). Nevertheless, some features of the deposit volume that could be incorporated in a model are discussed here. A detailed discussion on the literature on deposit volume models can again be found in Paraschiv (2011).

2.5.1 Deposit Volume Characteristics

Analogue to the previous section, some of the characteristics that could be incorporated in a deposit volume model are set out first. The next subsection covers possible models and how these characteristics are then incorporated.

Explanatory Variables

Obvious candidates for explanatory variables are the deposit rate and the market rate, as these rates determine the return on the deposit for the depositors. Additionally, since the deposit rate is a variable that can be controlled by the bank, having this variable in the model could give rise to strategies for setting the deposit rates (provided that it has, indeed, a significant influence). Most authors include the market and deposit rate (or their difference) in their model for the deposit volume (most often in a linear fashion). Other authors however follow an approach similar to Jarrow and Van Deventer (1998), where they include only the market short rate in the model, but omit the deposit rate. Models also differ in which market rates they include in the model. There is some consensus on the inclusion of longer term interest rates in modelling the deposit volume (Blöchlinger, 2015; de Jong & Wielhouwer, 2003; Paraschiv, 2011).

Other explanatory variables could be some macroeconomic factors, these factors are not often considered in the literature (Paraschiv, 2011). O'Brien (2001) does however take the average income as an input for the deposit volume model, as does Nyström (2008). However, "Income is not a significant explanatory variable for MMDA (Money Market Deposit Account) deposit balances demands." (O'Brien, 2001, p. 9). As an alternative to macroeconomic variables, a time trend is often included (i.e. Hutchison & Pennacchi, 1996; Jarrow & Van Deventer, 1998; Paraschiv & Schürle, 2010). Including this time trend avoids having to forecast macroeconomic variables. Another way of dealing with the influence of macroeconomic variables is by detrending the time series for the deposits. This is, for instance, done by de Jong and Wielhouwer (2003) for the Dutch market. They also introduce a long term mean deposit volume, which is possible when the volume is detrended (otherwise, one would expect this to rise with, for instance, inflation).

Seasonality

Some amount of seasonality is expected in the deposit volume, especially when considering data from one country in particular. A person receives their salary on a certain date, and then gradually spends (part of) this money during the course of the month. Since, in general, salaries are awarded towards the end of the month, this effect is expected to be visible in aggregate data as well. Furthermore, on a larger timescale, seasonality is also expected, since bonuses and holiday allowances are generally awarded at roughly the same time for the majority of depositors. During the summer, an outflow of funds would be expected, since the majority of depositors would go on holiday during this period. This seasonality is often incorporated by introducing indicator variables for specific months (Bardenhewer, 2006; Castagna & Scaravaggi, 2017; Selvaggio, 1996). However, seasonal effects are often omitted entirely, especially when only valuation of NMDs is considered.

2.5.2 Deposit Volume Models

Here, it is set out how to include the properties, discussed above, in models. This is done for both continuous time and discrete time models. The choice for either model is mostly dependent on the available data. For deposit volumes, it is common to model the dynamics of the logarithm of the volume, instead of the volume directly. This enforces a ZLB on the deposit volume.

Continuous Time Deposit Volume Models

In the literature, volume models without an explicit stochastic process are often used (i.e. Jarrow & Van Deventer, 1998). Some of the above-mentioned explanatory variables are included, while others are approximated using, for instance, a general time trend. A model with explicit volatility is presented here. This model is able to capture erratic behaviour of the deposit volume. The market and deposit rates are included explicitly, while a time trend is incorporated as a proxy for general macroeconomic variables. A periodic function, that can be used to model seasonal effects, is included as well. The model is governed by the following SDE:

$$
\begin{aligned}
d\log V(t) &= (\boldsymbol{\gamma}\cdot\boldsymbol{r}(t)+\delta i(t)+\zeta t+\xi(t))dt+dL(t),\\
V(0) &= V_0.
\end{aligned}
\tag{2.14}
$$

In Equation 2.14, $V(t)$ is the deposit volume at time t, $\boldsymbol{\gamma}\cdot\boldsymbol{r}(t)$ is a weighted vector of market rates. For instance, both the short rate and a long term rate, say the 1 year EURIBOR could be included. δ is a constant. $i(t)$ is the deposit rate at time t. ζ is a constant, which covers the time trend. $\xi(t)$ is a periodic function for potential seasonal characteristics. $L(t)$ is a driftless Lévy process. And V_0 the deposit volume at $t=0$. This model is very general, but it does include all the above characteristics. Jarrow and Van Deventer (1998) let the evolution of the volume depend on the change in market rate as well. A similar effect to including the change in short rate can be accomplished via including longer term rates in the vector $\boldsymbol{r}(t)$, as longer term rates tend to convey the expectation of the evolution of the short rate.

Discrete Time Deposit Volume Models

A discrete version of Equation 2.14 might be more appropriate, when the available data is less granular. The discrete version takes the form of an AR model, and is, for the sequence of times $(t_n)_{n\geq 0}$ with $t_{n+1}-t_n=\Delta t$ for all $n\geq 0$, as follows:

$$
\begin{aligned}
\log V(t_n) &= \log V(t_{n-1})+\boldsymbol{\gamma}\cdot\boldsymbol{r}(t_n)+\delta_1 i(t_{n-1})+\delta_2 i(t_n)+\zeta t_n+\xi(t_n)+\epsilon(t_n),\\
V(0) &= V_0.
\end{aligned}
\tag{2.15}
$$

In Equation 2.15, the variables are similar to those in Equation 2.14, but do not necessarily have the same value. An extra term $\delta_1 i(t_{n-1})$ is included to account for the interest a depositor receives over the previous time period. Note that the time step between the time points is constant. The Lévy process has been replaced by a zero mean error term.

2.6 Risk Assessment

In this section, an outline the standards and regulations that banks use to assess and report their interest rate risk is given. These standards and regulations are set out by central bank authorities that supervise banks and other financial institutions. Additionally, how these risks are evaluated in the literature are discussed.

2.6.1 Supervision and Guidelines

Banks in the Netherlands are under supervision of either DNB or the ECB. Large banks are under direct supervision of the ECB, while smaller financial entities are supervised indirectly through DNB.

The European Banking Authority (EBA) has published a set of guidelines for managing Interest Rate Risk arising form Non-Trading Book Activities (Interest Rate Risk in the Banking Book or IRRBB) (EBA, 2022c). Different stress tests are specified in this documentation (EBA, 2022c, sec. 4, art. 99). The guidelines include qualitative descriptions of the types of stress-tests that a bank should carry out. The EBA (2022a, 2022b) also provided drafts for regulation, to be approved by the European Commission, on explicit stress tests to be carried out by banks. These are adaptations of the standards, set out by the Basel Committee on Banking Supervision (BCBS, 2016, section IV.3). The proposed framework is as follows.

For the calculation of IRRBB, the whole volume of NMDs is first separated into retail and wholesale, where retail deposits are deposits from individuals or small businesses. Wholesale deposits are deposits from larger businesses and institutions. Then, the NMDs are further separated into core and non-core deposits. Core deposits are deposits that are regarded as stable over a relatively long time period (i.e. these deposits are not likely to be withdrawn). Finally, the core deposits are slotted into different time buckets according to their expected maturity (specified by the BCBS (2016, section IV, Table 1) and adopted by the EBA (2022a, Annex I)). The BCBS (2016, section IV, Table 2) provides caps for the percentage that is allowed to be considered as core, as well as what the maximum average core maturity is allowed to be. The EBA (2022a, article 7) adopts these values as well.

To perform the risk analysis, the EBA (2022b) has defined shock scenarios. In these scenarios, the interest rates on the buckets are shocked up or down by a set amount (EBA, 2022b, Annex I). This amount is scaled per bucket according to the specific shock scenario, of which there are 6 (EBA, 2022b, Article 1.1). In some scenarios, both short rates and longer term rates are affected equally, while, in others, they are affected either disproportionately (short rates more affected than longer rates) or inversely (one up, one down). Explicit mathematical expressions for the rates per bucket after the shock are also given by the EBA (2022b, Article 3).

2.6.2 Risk Analysis in Literature

Aside from the guidelines, set out by the BCBS and EBA, there are some risk analyses for NMDs in the literature. Jarrow and Van Deventer (1998) develop a framework which enables hedging of an NMD portfolio using ”standard techniques from the interest rate derivatives literature.” de Jong and Wielhouwer (2003) discuss hedging of the portfolio with long and short term investments. Kalkbrener and Willing (2004) looks at the minimum of the deposit volume over a time period and with this determines the liquidity risk. For the calculation of interest rate risk, Kalkbrener and Willing (2004) construct a replicating bond portfolio, where the bonds have fixed maturities. They do this by matching the delta (partial derivative of the value with respect to the short rate) profile of the NMDs. For the pricing of the NMDs, they follow an approach similar to Jarrow and Van Deventer (1998), but they replace the volume process by the minimum of the volume process up to that time. These approaches are criticized by Blöchlinger (2015), since they do not include hedges for the volatility risks that arise from the option risks that are implicit in NMDs.

2.7 Contribution

This thesis will contribute to the literature in the following ways. Firstly, it adds to the literature by creating and calibrating a model for NMDs on the Dutch market. De Jong and Wielhouwer (2003) did this before, but their paper is from before the global financial crisis and the extended period of market rates being at the ZLB, and a different model is used. Although specified on data concerning the Netherlands, the model will be constructed in such a way, that it is applicable to data from other regions or countries after calibration. Testing this could be the subject of subsequent research.

The second way this thesis contributes to the literature is through the way the risk analysis will be performed. This risk analysis will entirely be based on extreme quantiles of the distributions of the market rate and the eventual deposit volume. In this way, the risk analysis is completely inside the model. This has the benefit of being rather flexible and universal regarding which model this technique is used on. A somewhat similar analysis is done by Kalkbrener and Willing (2004). In this thesis, we however look at extreme scenarios by conditioning on edge-case market rate scenarios.

Finally, we add to the literature by, as opposed to many studies in this field, regarding the deposit rate model as an input that can be controlled, which is the case for a bank. The optimization of several deposit rate models and their comparison is, to our knowledge, not found in previous literature.

In practice, the risk analysis procedure can be used in addition to the risk analysis that is imposed by the regulator, and can serve to quantify risks that are in the model, without a prior shock scenario. The deposit rate model optimization, and the optimal models it produces can be used to challenge existing ways of setting the deposit rate, especially if there is a large discrepancy between the two.

3 Data

In this section, the data that is available for the empirical part of this thesis is discussed. Characteristics of this data are explored and shortcomings are identified. Solving these shortcomings could be the basis for subsequent experiments. The outline of this section follows the same structure as Section 2. The section starts with data on the term structure, moves on to data on the deposit rate and ends with data on the deposit volume. The section is concluded with a summary of the limitations that arise from the limited data availability.

3.1 Term Structure Data

For the calibration of the term structure model, data on historical yield curves in the Euro area is available from the ECB data portal. There, parameters for a Svensson (1994) model are published with daily frequency. These parameters specify a yield curve with yields for all maturities. These parameters are for AAA-rated bonds.

3.1.1 Svensson (1994) model

The Svensson (1994) model is an extension of the Nelson-Siegel model (Nelson & Siegel, 1987), and provides an analytical expression for the forward rates on zero-coupon bonds for all maturities at time $t = 0$. Note that the forward rate is defined as the constant, continuously compounding rate over the maturity of the zero-coupon bond. I.e.

$$P(T) = \exp(-f(T)T), \tag{3.1}$$

where $P(T)$ denotes the price of a zero coupon bond, maturing at time T, with a payout of one unit of currency. $f(T)$ is the corresponding forward rate. The Svensson (1994) model describes these rates analytically, using a function of 6 variables, which are denoted by $\boldsymbol{b} = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$. In this, the τ are in units of time, where $\tau_1 < \tau_2$ can be identified with a short and a longer maturity component; the β are in units of percent per year (or just yield per year). The equation for the forward rate in the Svensson (1994) model is given by

$$f(T; \boldsymbol{b}) = \beta_0 + \beta_1 \exp\left(-\frac{T}{\tau_1}\right) + \beta_2 \frac{T}{\tau_1} \exp\left(-\frac{T}{\tau_1}\right) + \beta_3 \frac{T}{\tau_2} \exp\left(-\frac{T}{\tau_2}\right). \tag{3.2}$$

The short rate r can be obtained by setting $T = 0$, which would yield $r = \beta_0 + \beta_1$. For very long maturities, the forward rate will tend to β_0 (all the exponentials tend to zero, and application of l'Hopital ensures that the terms that are also multiplied by the maturity tend to zero).

The parameters for the Svensson (1994) model can be obtained from the ECB data portal for the Euro area, either for all bonds or only for AAA-rated bonds. The yields on AAA-rated bonds are used to determine the term structure in this thesis.[9] As an illustration, the three-month forward rate that followed from the Svensson (1994) model parameters is plotted. This can be found in Figure 3.1. It is clear that the ZLB does not hold for this time period. This should be considered when choosing a term

[9]Available via the ECB

structure model. Since this data does not go far enough into the past, it starts somewhere in 2004. Historical 3 month EURIBOR rates were used in the calibration of the deposit volume model, where they serve as a proxy for the market rate. This data was obtained from 'Het Financieele Dagblad'.[10]

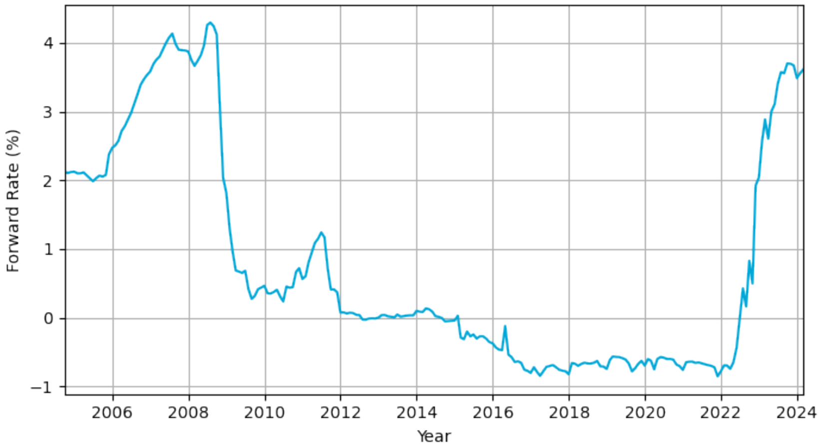


Figure 3.1: Three-month EURIBOR forward rate. This rate follows from the Svensson model parameters for AAA-rated Euro bonds from 2004 to the start of 2024.

3.2 Deposit Rate Data

For historical data on the deposit rate on NMDs, the data archive of DNB can be used. There, aggregate, volume-weighted deposit rate data for the Dutch banking sector is available with monthly frequency. These rates are used to calibrate the deposit volume model and are plotted in Figure 3.2.[11]

In theory, the interest rates of most individual banks are also publicly available. These rates could be used when calibrating the model for a specific bank. This is not done in this thesis, as the data on individual banks is not granular enough to provide a reasonable number of data points for analysis. This is mainly due to the (lack of) publicly available deposit volume data. A variable deposit rate, based on the amount of volume in a single account, which was awarded by some Dutch banks, combined with the lack of the volume data to determine what part of the volume receives what rate, makes this a rather uncertain exercise.

When this data is available to the experimenter, carrying out the same analysis could potentially lead to interesting (and more certain) results.

3.3 Deposit Volume Data

On deposit volume, the available data is limited to aggregate data from DNB,[12] this data has monthly frequency, but is split out according to the type of instrument. The data concerning NMD volumes can be found as 'Deposits redeemable at notice', which is split out between 'The Netherlands', the 'Euro Area' and the 'Rest of the World', and further split out in 'Households', 'Non-financial private corporations', 'Non-financial public corporations' and 'Other financial institutions.'

The data that is of interest to this thesis is the data from 'Households' and 'Non-financial corporations' (both private and public) in the Netherlands. This data is available from December 1998 up to January 2024. This data is plotted in Figure 3.3, where only the data is plotted where when there is also deposit rate data available, which means the plot starts in 2003. In general, an increasing trend can be observed, with a notable dip around the 2008 financial crisis and an increase around the Covid-19 period (2020 - 2022).

In addition to this monthly aggregate data, DNB provides insights in data from individual banks on

[10] Since it is very similar to the data in Figure 3.1, this data is not plotted. It can be accessed via FD
[11] See data from DNB (Table 5.2.7.1: 'MFI households deposits and loans, interest rates, adjusted for breaks (Month)', row: 'Overnight deposits - Total')
[12] DNB Data Portal Table 5.2.5

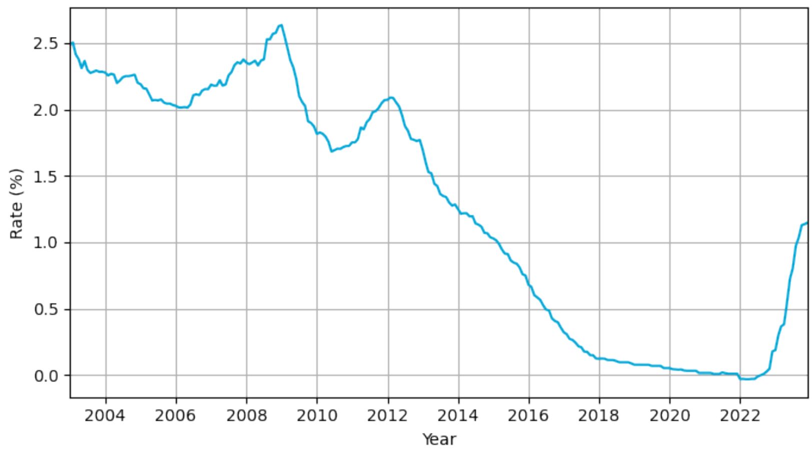


Figure 3.2: Volume weighted deposit rate data for the Dutch banking sector. This rate is monthly data from DNB from 2004 to the end of 2023.

a semi-annual basis.[13] This data is quite sparse, but may be useful for checking if relations found for the aggregate deposit volume hold for the individual banks as well. This data is available for 29 banks from the second half of 2014 onwards. The split in households and location lacks in this data source, so that the data from the Netherlands and the Euro area are combined. The deposit volume data for a few larger banks is plotted in Figure 3.4, and for a few smaller banks in Figure 3.5.

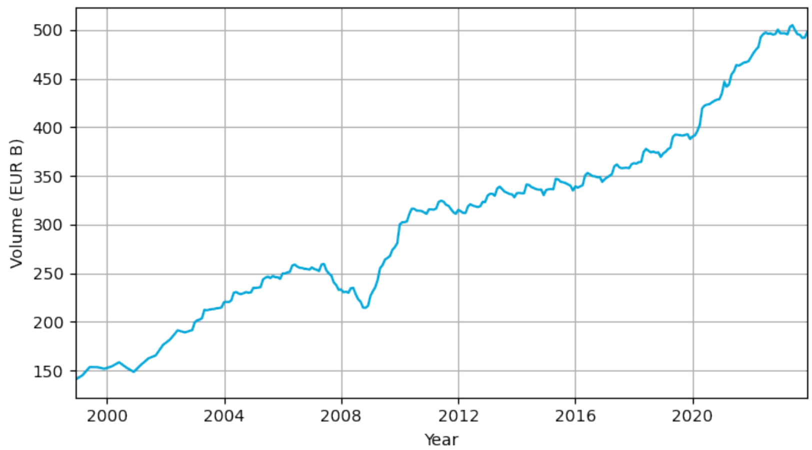


Figure 3.3: Aggregated deposit volume data for the Dutch banking sector. This volume is monthly data from DNB from 2004 to the end of 2023.

3.4 Limitations of publicly available data

While the data on the term structure is readily available to the public via the data portal of the ECB,[9] which provides the term structure on a daily basis (weekday). The availability of deposit rate, and especially deposit volume data is much more limited. With aggregate data being available on a monthly basis, and individual bank data being available only on a semi-annual basis. In this section, the effects of the limited data availability on the results of this thesis are explicitly highlighted, and it is explained what can be gained from access to more granular data.

A direct consequence of using aggregate data is that the model essentially reduces to a model for one large bank, that serves the whole of the Dutch market. This means that the dynamic of customers switching banks within the Dutch ecosystem is essentially lost, since volume moving from one Dutch

[13] DNB Data Portal Table 5.15

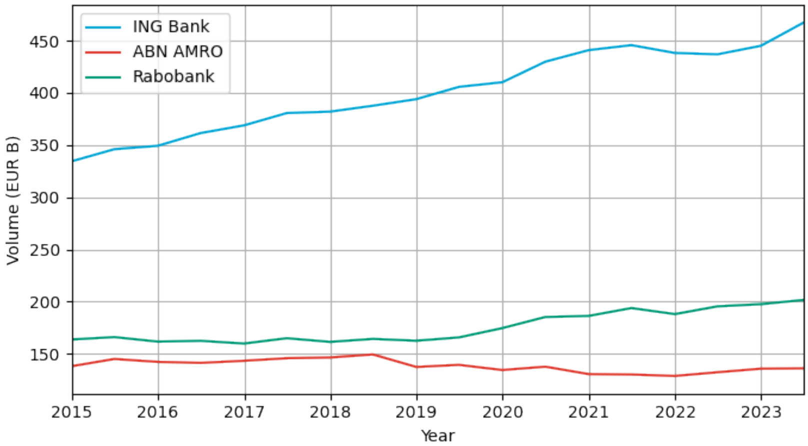


Figure 3.4: Household deposit volume data for a few large Dutch banks. This is half-yearly data from DNB from the end of 2014 to half-way 2023.

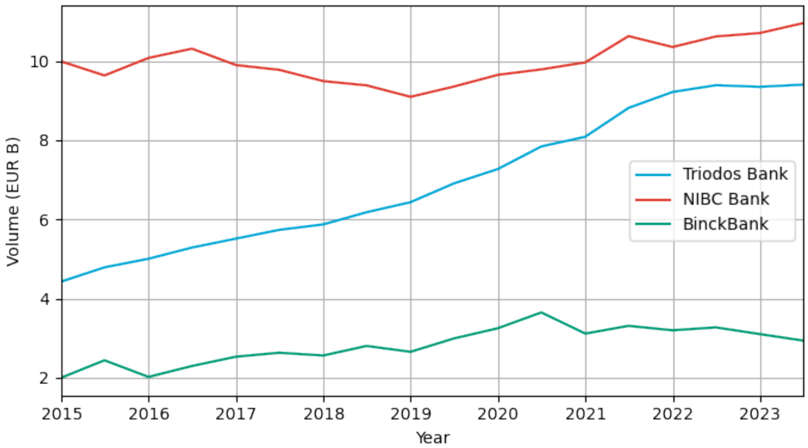


Figure 3.5: Household deposit volume data for a few smaller Dutch banks. This is half-yearly data from DNB from the end of 2014 to half-way 2023.

bank to another Dutch bank does not change the total aggregate volume. The only way that volume is lost, is when customers either spend their money, switch to a bank that is not regulated by DNB or put their money towards an alternative investment, such as long term savings or the stock market. This is suspected to limit the effect of the deposit rate, when compared to the effect that would be found with data for a single bank.

A second point is that the deposit rate loses some properties, stated in Section 2, due to it being a volume weighted average. This results in it losing the property of being only at set levels and changing in steps. Only when a bank with significant market share or a group of banks with significant market share changes its rate, a sudden change in the aggregate deposit rate should be observed. The rate being at round numbers is lost due to the aggregation.

Finally, monthly data might not be granular enough to observe all effects occurring within a month. Ideally, data with a daily frequency would be used to calibrate the model. Even though noise would potentially result in aggregation to weekly or monthly data being a requirement anyway.

All in all, the results, presented in this thesis, might be limited in significance due to the limited data availability. This, however, does not mean that the techniques and methodology, used in this thesis, are of limited significance as well. Throughout the thesis, it will be discussed what potentially changes when the methods are applied to more granular data. Higher data availability will likely yield better results for, in particular, the deposit volume model calibration and the deposit rate optimization.

4
Method

In this section, the methods used in this thesis will be worked out in detail. The structure of this section is as follows. Firstly, the calibration of a term structure model will be discussed. In the second part, the calibration of the deposit volume model is discussed. An optimization procedure for choosing a deposit rate model is set out in the third part of this section. The section is concluded with the description of our method for risk analysis, based on quantiles of the deposit volume, conditioned on market rate scenarios.

NMDs are thus considered from a bank's perspective. In the risk analysis, attention is given to the lower quantiles of the distribution of the deposit volume. With NMDs constituting an important part of a banks funding, it is important to focus on 'worst-case'-scenarios, where a bank's liquidity might be compromised as a result of decreasing NMD volume. The deposit rate optimization exercise on the other hand, focusses more on the potential of an NMD portfolio to generate an income for the bank. A similar optimization could theoretically be carried out to minimize the risks in 'worst-case'-scenarios, at relatively large computational cost.

4.1 Term Structure Models

We implement a general structure, capable of handling interest rate models for which the forward rate is analytically available. The procedure we follow is set out by De Rossi (2010) for the CIR (Cox et al., 1985) model. How to adjust the procedure for other term structure models is discussed briefly at the end of this section.

The main advantage of the CIR model is the lower bound that is enforced by the dynamics of the model. This lower bound is zero in the 'vanilla' model, but can be shifted using either a constant shift or by shifting the term structure with a deterministic function. The latter, while more flexible, makes the model less tractable. Determining the evolution of such a function for future times is also a difficult task. We therefore implement a deterministic shift in the CIR model used in this thesis (this is required due to the market rate dipping below the ZLB in our data).

Since the model only features four parameters (in addition to the shift), namely a, θ, σ and λ, which denotes the market price of risk. This model is less flexible than other potential models (mainly the multifactor models), due to the limited number of free parameters. This comes both as an advantage and a disadvantage, because it makes the model more tractable, but the fit to the data is most likely poorer than a model with more free parameters.

The implementation of multifactor models is discussed briefly at the end of this subsection. There, models with two or three factors are considered. While adding more factors would probably improve the in-sample model fit, in general, two or three factors are sufficient to fit to most term structures with reasonable accuracy ($> 90\%$) (Brigo & Mercurio, 2007; Jamshidian & Zhu, 1997).

4.1.1 Cox et al. (1985) model with deterministic shift

The implementation used for the CIR model is set out by De Rossi (2010). His implementation uses calibration to historical forward rates on ZCBs. The calibration process can also handle (Gaussian) measurement errors in the forward rates (for perfect measurements, a different method would be more suitable).

CIR model calibration

This section follows the paper by De Rossi (2010). We refer to that paper for a more detailed explanation of the steps.

The general idea of the approach by De Rossi (2010) is to do a maximum likelihood estimation on the model parameters via the unobserved short rate. We start with a time series of forward rate vectors with forward rates for different maturities with i.i.d. Gaussian observation errors ($\mathcal{N}(0,h)$, where h denotes the variance). Mathematically, the observations are of the following form:

$$
\begin{aligned}
\boldsymbol{t} &= \{t_n \,|\, 1 \le n \le N, t_n - t_{n-1} = \delta\} && \text{- observation times,}\\
\boldsymbol{\tau} &= \{\tau_m \,|\, 1 \le m \le M\} && \text{- maturities,}\\
\boldsymbol{f^o} &= \{\boldsymbol{f}^o_{t_n} = (f^o_{\tau_1,t_n}, f^o_{\tau_2,t_n}, \dots, f^o_{\tau_M,t_n})' \,|\, 0 \le n \le N\} && \text{- observations.}
\end{aligned}
\tag{4.1}
$$

In Equation 4.1, the first set denotes the set of observation times, which are spaced evenly through time. The second set is the set of maturities of the observed bonds. The third set contains the observations as vectors (the prime denotes the transpose). In general, the observation times being equally spaced in time and the observed maturities being the same for every observation time are not required for the model, but do simplify the implementation considerably.

For the CIR model, the forward rate of a bond with maturity τ is given by (Brigo & Mercurio, 2007; De Rossi, 2010):

$$
\begin{aligned}
f_{\tau,t} &= -A(\tau) + B(\tau) r_t,\\
A(\tau) &= \frac{2a}{\tau\sigma^2} \log\left\{\frac{2\gamma \exp(\tau(a+\lambda+\gamma)/2)}{2\gamma + (a+\lambda+\gamma)(\exp(\gamma\tau)-1)}\right\},\\
B(\tau) &= \frac{1}{\tau}\frac{2(\exp(\gamma\tau)-1)}{2\gamma + (a+\lambda+\gamma)(\exp(\gamma\tau)-1)},\\
\gamma &= \sqrt{(a+\lambda)^2 + 2\sigma^2}.
\end{aligned}
\tag{4.2}
$$

In Equation 4.2, a, θ, σ and λ are the usual parameters for the CIR model. The relation between the observed forward rate and the 'true' forward rate is.

$$
f^o_{\tau,t} = f_{\tau,t} + \epsilon_t \qquad \epsilon_t \sim \mathcal{N}(0,h). \tag{4.3}
$$

In Equation 4.3, $\mathcal{N}(\mu,\eta^2)$ denotes a Gaussian distribution with mean μ and variance η^2.

The ML estimation that will be carried out is now given by De Rossi (2010):

$$
\begin{aligned}
\underset{h,a,\theta,\sigma,\lambda}{\arg\max}\, p(\boldsymbol{f^o}) &= \underset{h,a,\theta,\sigma,\lambda}{\arg\max} \int p(\boldsymbol{f^o}, \boldsymbol{r}) d\boldsymbol{r}\\
&= \underset{h,a,\theta,\sigma,\lambda}{\arg\max} \int p(\boldsymbol{f^o}|\boldsymbol{r}) p(\boldsymbol{r}) d\boldsymbol{r}\\
&= \underset{h,a,\theta,\sigma,\lambda}{\arg\max} \int \prod_{n=1}^{N} p(\boldsymbol{f^o_{t_n}}|r_{t_n}) \prod_{n=2}^{N} p(r_{t_n}|r_{t_{n-1}}) p(r_{t_1}) d\boldsymbol{r}.
\end{aligned}
\tag{4.4}
$$

This integral is evaluated using Monte Carlo integration. We make use of the properties of the CIR model, namely that

$$
\begin{aligned}
p(r_{t_1}) &= \text{gamma}\left(r_{t_1}; \frac{2a}{\sigma^2}, \frac{2a\theta}{\sigma^2}\right),\\
p(r_{t_n}|r_{t_{n-1}}) &= \frac{4a}{\sigma^2(1-\exp(-a\delta))}\chi^2\left(\frac{4ar_{t_n}}{\sigma^2(1-\exp(-a\delta))}; \frac{2a\theta}{\sigma^2}, \frac{4ar_{t_{n-1}}\exp(-a\delta)}{\sigma^2(1-\exp(-a\delta))}\right),\\
\text{gamma}(x;\beta,\alpha) &= \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x},\\
\chi^2(x;k,l) &= \frac{1}{2}e^{-(x+l)/2}\left(\frac{x}{l}\right)^{k/4-1/2} I_{k/2-1}(\sqrt{lx}).
\end{aligned}
\tag{4.5}
$$

In Equation 4.5, $\Gamma(x)$ denotes the gamma function and $I_y(x)$ denotes the modified Bessel function of the first kind of order y (see, for instance, (Abramowitz & Stegun, 1968)).

The final factor in the density function, $p(\boldsymbol{f}^o_{t_n}|r_{t_n})$, we can treat as an importance density from which we will draw possible values for $\boldsymbol{r}$ (as is done by De Rossi (2010)). This can be seen as follows

$$
\begin{aligned}
\phi_{t_n}(x) = p(\boldsymbol{f}^o_{t_n}|r_{t_n}=x) &= \prod_{m=1}^{M}\frac{1}{\sqrt{2\pi h}}\exp\left(-\frac{[f_{\tau_m,t_n}+A(\tau_m)-B(\tau_m)x]^2}{2h}\right)\\
&= \frac{1}{(2\pi h)^{M/2}}\exp(-q_1x^2-q_{2,t_n}x-q_{3,t_n})\\
&= \frac{1}{(2\pi h)^{M/2}}\exp\left(-q_1\left[x^2+\frac{q_{2,t_n}}{q_1}x+\frac{q_{3,t_n}}{q_1}\right]\right)\\
&= \frac{1}{(2\pi h)^{M/2}}\exp\left(-q_1\left[\left\{x+\frac{q_{2,t_n}}{2q_1}\right\}^2-\frac{q_{2,t_n}^2}{4q_1^2}+\frac{q_{3,t_n}}{q_1}\right]\right)\\
&= \frac{1}{(2\pi h)^{M/2}}\exp\left(\frac{q_{2,t_n}^2}{4q_1}-q_{3,t_n}\right)\exp\left(-q_1\left\{x+\frac{q_{2,t_n}}{2q_1}\right\}^2\right)\\
&= \frac{1}{(2\pi h)^{M/2}}\exp\left(\frac{q_{2,t_n}^2}{4q_1}-q_{3,t_n}\right)\exp\left(-\frac{1}{2}\left\{\frac{x+q_{2,t_n}/(2q_1)}{1/\sqrt{2q_1}}\right\}^2\right)\\
&= \frac{1}{(2\pi h)^{M/2}}\exp\left(\frac{q_{2,t_n}^2}{4q_1}-q_{3,t_n}\right)\sqrt{\frac{\pi}{q_1}}\frac{\sqrt{2q_1}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left\{\frac{x+q_{2,t_n}/(2q_1)}{1/\sqrt{2q_1}}\right\}^2\right),
\end{aligned}
\tag{4.6}
$$

with

$$
\begin{aligned}
q_1 &= \frac{1}{2h}\sum_{m=1}^{M}B^2(\tau_m),\\
q_{2,t_n} &= -\frac{1}{h}\sum_{m=1}^{M}[f^o_{\tau_m,t_n}+A(\tau_m)]B(\tau_m),\\
q_{3,t_n} &= \frac{1}{2h}\sum_{m=1}^{M}[f^o_{\tau_m,t_n}+A(\tau_m)]^2.
\end{aligned}
\tag{4.7}
$$

We note that the latter two factors in Equation 4.6 define a Gaussian distribution density function with mean $q_{2,t_n}/(2q_1)$ and variance $1/(2q_1)$. This means that we can sample candidate values of r_{t_n} from this distribution by sampling from the respective normal distribution. An important note is that we discard any negative draws, as the CIR model does not permit negative values for the short rate. [14]

We can now evaluate the integral in Equation 4.4 by drawing sampling the distribution Equation 4.6, assigning weights to the samples according to the distribution functions associated with the CIR model

[14] We can turn $\phi_{t_n}(x)$ into a proper distribution function by normalizing: $\overline{\phi_{t_n}}(x) = \phi_{t_n}(x)/\int_0^\infty \phi_{t_n}(y)dy$

and evaluating this weighted sum. We take $\bar{n}$ samples. The weights of the different samples are constructed as follows. We denote

$$K_{t_n} = \int_0^\infty \phi_{t_n}(x)dx. \tag{4.8}$$

The weight of the draw $r_{t_n}^{(i)}$ is then given by

$$\begin{aligned} w_{t_1}^{(i)} &= K_{t_1} p\big(r_{t_1}^{(i)}\big), \\ w_{t_n}^{(i)} &= K_{t_n} p\big(r_{t_n}^{(i)} | r_{t_{n-1}}^{(i)}\big) \frac{w_{t_{n-1}}^{(i)}}{\sum_{j=1}^{\bar{n}} w_{t_{n-1}}^{(j)}}, n = 2, \dots, N. \end{aligned} \tag{4.9}$$

Often, the weights of a few samples dominate, while the other weights are relatively close to zero. In order to remove this degeneracy, we use resampling. The resampling procedure that is used, is the same as used by De Rossi (2010), and is the procedure from Kitagawa (1996). To determine is resampling is necessary, the following measure of degeneracy is defined:

$$\tilde{n}_{t_n} = \frac{\left(\sum_{i=1}^{\bar{n}} w_{t_n}^{(i)}\right)^2}{\sum_{i=1}^{\bar{n}} \left(w_{t_n}^{(i)}\right)^2}. \tag{4.10}$$

When there is degeneracy, evaluating Equation 4.10 will yield a value for $\tilde{n}_{t_n}$ close to 1, while a value of $\tilde{n}_{t_n}$ close to $\bar{n}$ indicates that no degeneracy is present.

The sampling procedure is as follows (this is the procedure used by De Rossi (2010), but is repeated here for convenience)

Importance sampling Monte Carlo filter

- Get $\bar{n}$ samples for $r_{t_1}^{(i)}$ from the density ϕ_{t_1},
- Repeat the following steps for $n = 1, \dots, N-1$:
    - Compute the weights $w_{t_n}^{(i)}$,
    - Calculate $\tilde{n}_{t_n}$, if $\tilde{n}_{t_n} < \bar{n}/2$, resample using the procedure of Kitagawa (1996) and recalculate the weights for all previous, n
    - Get $\bar{n}$ samples for $r_{t_n}^{(i)}$ from the density ϕ_{t_n},

At the end of this procedure, the weights for every t_n and every i have been calculated. The value of the likelihood function is now given by

$$p(\boldsymbol{f}^o) = \frac{1}{\bar{n}} \prod_{n=1}^{N} \sum_{i=i}^{\bar{n}} w_{t_n}^{(i)}. \tag{4.11}$$

We now carry out the maximum likelihood estimation using a genetic algorithm (GA), as by De Rossi (2010). The genes of the algorithm are h, a, λ, σ and θ. For the fitness function, we use the logarithm of Equation 4.11, floored at zero. We use a roulette wheel algorithm for picking solutions that survive to the next generation, so that the probability of survival is proportional to the log-likelihood associated with the solution. Additionally, the best performing solution survives to the next generation. We apply crossover and mutation to obtain new solutions (chromosomes). After a predetermined amount of generations, the algorithm is terminated. The chromosome with the highest associated likelihood then is the final ML estimate. For the implementation of the GA, we make use of the PyGAD package (Gad, 2021).

Resampling procedure
Due to the high potential degeneracy in the estimates, a resampling step is sometimes needed. The goal of such a resampling step is to ensure that the weights of the different estimates (often called particles in literature (De Rossi, 2010, for example)) are similar. In this way, multiple particles have an impact on the final estimate, rather than one or a few dominant particles. The resampling approach used in this thesis is that of Kitagawa (1996). This approach is used by De Rossi (2010) as well. We briefly describe the procedure here for completeness. We start with a vector of weights $\boldsymbol{w}$ of length $\bar{n}$, in which we assume that some degeneracy is found.

The procedure is started by computing what is called the 'cumulative distribution function of the weights' (De Rossi, 2010, p. 8). This is defined as, with normalized weights $\omega_i = \frac{w_i}{\|\boldsymbol{w}\|_1}$, and setting $W_0 = 0$:

$$W_i = W_{i-1} + \omega_i \quad \text{for } 1 \leq i \leq \bar{n}. \tag{4.12}$$

This cumulative distribution function has large jumps at a few points, and is (almost) constant for large periods of time due to the degeneracy in the vector $\boldsymbol{w}$. After resampling, we want to end up with a cumulative distribution function that rises more steadily. In order to obtain such a resampled vector, we start by defining a dummy cumulative distribution function $\hat{W}_i$, with jumps of size $\frac{1}{\bar{n}}$, that start at with $\hat{W}_1$ as a random sample from the interval $\left(0, \frac{1}{\bar{n}}\right)$ (corresponding to step (a-D) Kitagawa, 1996, p. 23). We now create the new vector $\hat{\boldsymbol{w}}$ with the following algorithm.

Resampling algorithm (De Rossi, 2010; Kitagawa, 1996)

1. Set indices $i = 1$ and $j = 1$
2. We now fill the new vector using the following procedure
3.
   - Check the inequality $W_i \geq \hat{W}_i$.
   - If it holds, we set $\hat{w}_j = w_i$ and increment j
   - If it does not hold, we increment i

At the end of this procedure (once we have obtained a value for all indices $1 \leq j \leq \bar{n}$). We have obtained a new sample, which does not exhibit degeneracy. This resampling does require us to calculate the weights of all particles, for all steps, again.

Extension of the method to other term structure models
The extension of the outlined procedure to other term structure models is relatively straight forward. Where the method above is set out for the CIR model, the used GA can handle an arbitrary number of parameters. Extension to different models is therefore equivalent to a change in the formulas specifying the forward rate.

4.2 Deposit Volume

An integral part of this thesis is the modelling of the deposit volume. We will specify a general candidate model and refine it step by step, we will check the influence of both the market rate and the deposit rate for values at the same time point and for lagged values, to check for a possibly 'slow' response from consumers.

The way we model the deposit volume is governed by the availability of the data. Since the available data has monthly frequency, this means that we model the rate as a time series with a delta of 1 month between the observations.

4.2.1 Linear regression model

We model the evolution of the logarithm of the deposit volume (to ensure positivity of the volume) and assume that this linearly depends on current and lagged market and deposit rates (or quantities derived from this). We also (possibly) include a time component as a proxy for the effects of macroeconomic trends (inflation, GDP growth etc.). This time component has been implemented in literature before (Jarrow & Van Deventer, 1998).

The implemented model will be of the following form:

$$\begin{aligned} \log(V(t_n)) - \log(V(t_{n-1})) &= \boldsymbol{\alpha}'\boldsymbol{\rho}(t_n) + \gamma\epsilon_{t_n}, \\ \boldsymbol{t} = \left(t_n\right)_{0\leq n\leq n_{\text{max}}}, &\quad \forall 0 \leq n \leq n_{\text{max}} - 1 : t_{n+1} - t_n = \Delta t. \end{aligned} \tag{4.13}$$

In Equation 4.13, the vector $\boldsymbol{\alpha}$ contains the coefficients of the model parameters, which are contained in the vector $\boldsymbol{\rho}(t_n)$ (this vector can also contain a one, so that the model has an intercept). ϵ_{t_n} are error processes with variance 1, assumed to be independent and identically distributed (i.i.d.), scaled by the parameter $\gamma \geq 0$.

Calibration

In order to decide which terms to include in the vector $\boldsymbol{\rho}$, we conduct the following procedure (called 'mixed-selection' (James et al., 2023, p. 87)). The candidate variables for being included in the model are the market rate $r(t_{n-m_r})$, the deposit rate $i(t_{n-m_i})$ (where $m \geq 0$ determines the lag), the change in the market rate $dr(t_n) \equiv r(t_n) - r(t_{n-1})$, the change in deposit rate $di(t_n) \equiv i(t_n) - i(t_{n-1})$ (these could be lagged as well), the difference between the market rate and the deposit rate ($s(t_{n-m_s}) = r(t_{n-m_s}) - i(t_{n-m_s})$) and the change in this quantity ($ds(t_n) = s(t_n) - s(t_{n-1})$) (these could also be lagged), and the time t_n as proxy for macroeconomic effects.

The calibration approach is then as follows (adapted from James et al. (2023, p. 87)), where all the fits are carried out using an Ordinary Least-Squares (OLS) approach:

Determining the parameters and coefficients of the linear Deposit Volume Model

- Start with a model containing only the intercept and no terms in $\boldsymbol{\rho}(t_n)$.
- Repeat the following steps until all variables in the model have a p-value below p_c, and all variables outside the model have a p-value above p_c.

  Note: the p-value corresponds to the probability of the null hypothesis (H_0: The coefficient corresponding to the considered variable is equal to 0.) being true.

  - Calculate p-values for all candidate parameters and model parameters. *Note: This is the point where the looping condition is checked.*
  - Add the parameter from the candidate parameters with the lowest p-value, that is also below p_c, to the model. This p-value is determined via a t-test on the hypothesis that the coefficient of the variable in a linear OLS model for the residuals from the considered model is equal to zero. Rejection of this hypothesis means we add the variable to the model. If there is no parameter satisfying this, move on to the next step.
  - Remove the parameter with the highest p-value, that is also above p_c, from the model. This p-value is also calculated from a t-test, considering only the influence of that certain variable. If there is no parameter satisfying this, move on to the next step.

Note: for all tests conducted in this procedure, it is assumed that the residuals follow a normal distribution. If this is not the case, these tests are not valid and the model should be altered (by including a variable that makes the residual a normal distribution, by transforming the dependent variable, or by removing outliers). See the subsequent section on the method employed in this thesis, when the residuals are found to exhibit serial correlation.

After this procedure, we have obtained a fitted model for the evolution of the deposit volume. Checking the p-values of the parameters in the model at every step helps to avoid including redundant variables in the model (James et al., 2023, p. 87).[15]

During the calibration procedure, we will also keep track of the evolution of the R^2 parameter, which can be somewhat loosely interpreted as the part of the total variation in the data, that is explained by the model. This parameter can be calculated for each intermediate model. A high R^2-value for an explanatory variable indicates that the including this in the model This value can only rise when adding extra parameters, but a small increase would be an indication that the added parameter has little significance, or that the effect of this parameter is included in one of the other parameters. When carrying out the above procedure, the focus should lie on adding variables for which a model with only that variable and a constant has a high R^2-value. Adding variables for which this quantity is low, is very likely to increase the p-values for all variables in the considered model, and should thus be avoided. While it is not strictly necessary to keep track of this quantity during the procedure, calculating R^2-values associated with all considered variables before starting the procedure could give an indication of which variables should be included in the final model.

Another way in which we can somewhat guess which parameters will be in the final model, before starting the procedure above, is by determining the correlation between all pairs of considered variables and all considered variables and the evolution of the logarithm of the deposit volume. The final model is unlikely to contain a pair of highly correlated variables, as these likely 'explain' the same variation in the dependent variable. This results in a large uncertainty in their coefficients when both are in a model, which in turn increases their individual p-values that are likely to lie above the threshold of 0.05.

In order to avoid including variables that are highly correlated, we check the correlation between the considered variables in our data. The magnitude of the correlation between any two variables then gives an indication whether including both in a model will result in a highly collinear model. The correlation between the considered independent variables and the difference in the logarithm of the volume ($d \log V(t_n)$) also gives an indication which variables have the highest explanatory power.

In the end, the goal is to end up with a model that contains only terms with individual p-values under 0.05, where the collinearity of the independent variables is low. This is somewhat ensured by all p-values being below 0.05, as including two independent variables with high correlation results generally results in a large p-value for at least one of them. The model should include at least the market rate and the deposit rate or a quantity derived from these. Including both is vital for the following steps in the analysis. Of course, this should not come at the cost of model significance or result in high collinearity.

Determining the error process ϵ

After the calibration of the model, we are left with a vector of residuals:

$$\hat{\boldsymbol{\epsilon}} = \Delta \log(V(\boldsymbol{t})) - \hat{f}(\boldsymbol{t}). \tag{4.14}$$

In Equation 4.14, $\hat{f}(\boldsymbol{t})$ is the fitted model. The individual residuals are given by;

$$\begin{aligned} &\gamma\epsilon_{t_n} = \log V(t_n) - \log V(t_{n-1}) - \hat{\alpha}\rho(t_n), \\ &\text{for } 1 \leq n \leq n_{\text{max}}. \end{aligned} \tag{4.15}$$

We assume that the entries in this vector are samples of some zero-mean (we can assume this because of the intercept in the model, if the model does not contain an intercept, this should be checked), random process with unit variance, scaled by the (for now unknown) constant γ.

An estimate for γ ($\hat{\gamma}$) is defined as:

$$\hat{\gamma} = ||\gamma\boldsymbol{\epsilon}||_2. \tag{4.16}$$

In Equation 4.16, $||\cdot||_2$ denotes the Euler-norm. This is an estimate because the inclusion of a constant in the model guarantees that the $\gamma\epsilon_{t_n}$ have mean zero, letting the ϵ_{t_n} have unit variance than guarantees the result.

Finally, we can check whether the processes ϵ_{t_n} are i.i.d (and what distribution they follow). We can do this in two steps.

[15] In a model with $r(t_n)$, $r(t_{n-1})$, and $dr(t_n)$ for example, it is very likely that one of these variables is redundant.

Kolmogorov-Smirnov test (Bijma et al., 2017, p. 139-141) The Kolmogorov-Smirnov test is a test for assessing if a set of sample data is from some continuous distribution. For this, the difference between the empirical cumulative density function (CDF) of the samples and the analytical CDF of the continuous distribution. For samples $X_1, X_2, ..., X_n$ from unknown distribution with CDF denoted by f. The empirical CDF of the data is given by:

$$f_n^{\text{(emp)}}(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}. \tag{4.17}$$

In Equation 4.17, n is the number of samples and X_i is the value of the i-th sample. $1_{\{\text{condition}\}}$ is the indicator function that is equal to one when the condition is satisfied, and equal to zero otherwise. We denote the analytical CDF we compare to by f_0. For large n, the empirical CDF will converge to the true CDF by the law of large numbers. The null-hypothesis for this test can now be formulated as $H_0 : f = f_0$. We then evaluate the following statistic:

$$T = \sup_{x\in\mathbb{R}} \left|f_n^{\text{(emp)}}(x) - f_0(x)\right|. \tag{4.18}$$

For large values of T, H_0 is rejected. A p-value for this can be calculated from the distribution of T under H_0. This distribution is the same for all continuous CDFs.

First, using a Kolmogorov-Smirnov test (Bijma et al., 2017, p. 139-141), we can determine if the residuals follow a standard normal process. From this, a p-value is obtained which corresponds to the probability that the observed samples of ϵ_{t_n} are drawn from a standard normal process ($\mathcal{N}(0,1)$).

If we obtain a low p-value, this would indicate that the ϵ_{t_n} are distributed following a different process. We can find an approximate cumulative distribution function (CDF) for the process from the empirical results. After this, we can try to match the CDF to the CDF of some continuous distribution, or we can use this empirical CDF in the experiments.

We could also check whether the ϵ_{t_n} are independent. This can be checked by constructing a lag-plot. In such a plot, the value of the next error is plotted against the value of the current error. If there is no structure in this plot. It can be inferred that these errors are independent. More analytically, we can look at the covariance between subsequent errors. A value of (or very close to) zero would point towards independence. We can make this more explicit using the following test.

Durbin-Watson test A more rigorous approach to test for sequential correlation (error autocorrelation) is by conducting a so called Durbin-Watson test (Durbin & Watson, 1950, 1951, 1971). The statistic for this test is given by

$$DW = \frac{\sum_{i=1}^{n}(\epsilon_{t_i} - \epsilon_{t_{i-1}})^2}{\sum_{i=0}^{n}\epsilon_{t_i}^2} \tag{4.19}$$

When the errors are uncorrelated, this statistic should evaluate to about 2, with values significantly below 2 indicating positive autocorrelation and above 2 indicating negative autocorrelation. Critical values for testing are given in Durbin and Watson (1951).

When the Durbin-Watson test rejects the null hypothesis that the residuals do not exhibit serial correlation, we must account for this correlation and develop a new model with uncorrelated residuals. To this end, we employ the procedure of Cochrane and Orcutt (1949) and turn our model into an $AR(1)$-model.

The procedure turn a model of the form $y_i = \boldsymbol{a}_1\boldsymbol{x}_i + \epsilon_1$ into a model of the form $y_i - by_{i-1} = \boldsymbol{a}_2(\boldsymbol{x}_i - b\boldsymbol{x}_{i-1}) + \epsilon_2$. Here y is the dependent variable and $\boldsymbol{x}$ is a vector of the independent variables, b is the serial correlation between residuals ϵ_1. We can now check, again using a Durbin and Watson (1950) test, whether the residuals ϵ_2 are correlated. We also need to check whether the coefficients of all independent variables in the second model still have an acceptable p-value, since this changes when transforming the model. This model can be rewritten to an

After the calibration and determining the distribution of the residuals, we will have obtained a model for the evolution of the deposit volume given the market rate and the deposit rate. We can try to fit a distribution to the residuals, but if this is unfeasible, the empirical distribution of the errors can also be used for the subsequent experiments.

4.3 Deposit Rate Optimization

The deposit rate is an important factor in the evolution of the deposit volume. For banks, the deposit rate is a tool to influence the volume, and a driver of the bank's result from their portfolio of NMDs. It is therefore a natural step to look for an optimal approach for setting the deposit rate. In Section 2, characteristics, observed in the deposit rate, have been set out. These characteristics were taken into account in defining the possible deposit rate models.

For the optimization, we must first define a quantity to optimize. An obvious candidate is the expected profit or expected value of the NMD portfolio of a bank. The value of an NMD portfolio is often defined as the expectation of the volume, multiplied by the difference between the market and deposit rate, over a set period of time, where payouts are assumed on a monthly basis and future cashflows are discounted by the market rate. This corresponds to the expected, discounted profit on the portfolio over a set period.

Another quantity that could be optimized for, is size of the deposit volume portfolio, more specifically, we define an optimal volume and minimize the difference between the expected volume and this optimal volume over time, where we only consider differences where the actual volume is lower than the target. The reasoning behind such an approach would be that, in practice, most of the banks funding through NMDs is locked up in longer term investments. For a bank's stability, it is therefore important that the volume does not decrease too much below certain thresholds.

In mathematical terms, these quantities can be written down as (for the profit):

$$W(\tau) = \sum_{0<t_n<\tau} \frac{V(t_n)}{\prod_{i=1}^{n}(1+r(t_i))^{\delta}}\left[(1+r(t_n))^{\delta} - (1+i(t_n))^{\delta}\right]. \tag{4.20}$$

In Equation 4.20, τ is the timeframe over which we calculate the profit and δ is the difference between subsequent values of t_i in years $\left(\frac{1}{12}\text{ in our case}\right)$. This equation is similar to the valuation equation by Jarrow and Van Deventer (1998, Equation 6).

Here, we assume that the interest is awarded on a monthly basis. This is often the case for the deposit rate, for the market rate or short rate, continuous compounding is the norm. When δ is small enough, the difference between the two is minimal. Furthermore, continuous compounding would require us to calculate the short rate more frequently, increasing the complexity of the calculations.

A quantity that expresses the stability of the deposit volume, would be, for instance:

$$S(\tau) = \sum_{0<t_n<\tau} \frac{\min\{0, V(t_n) - V^*(t_n)\}}{\prod_{i=0}^{n}(1+r(t_i))^{\delta}}. \tag{4.21}$$

In Equation 4.21, $V^*(t_n)$ denotes some target volume. For this volume, we could take a percentage of the current volume, with a trend to account for the general increase in volume. The minimum is implemented here, since having the volume be above the target is significantly less problematic than having the volume below the target. The maximum value for this quantity is thus 0.

The goal is now to find an optimal deposit rate model, according to either one of the criteria Equation 4.20, or the stability Equation 4.21, where we maximize one or a combination of the two. In order to find such a model, we must define a class of possible deposit rate models. Then, we carry out an optimization procedure to find a model that is optimal in this sense.

In defining this class, we take the characteristics, as set out in Section 2, into account. While optimizing over a class of general functions is a complicated procedure. If we can define several subclasses of functions, that differ only in their parameters, we can optimize over these subclasses and then find the best optimum to find the optimum of the whole class. When possible, we can even define deposit rate models as linear combinations of models of these subclasses. Then, the optimization can be carried out in a single procedure, although this procedure is more elaborate.

We define several subclasses of possible deposit rate models. For the whole of this analysis, we assume that there is a ZLB on the deposit rate, which corresponds to what is observed in practice. Firstly, we have models with a constant bank margin μ. The formula for the deposit rate is then given by:

$$i(t) = \max\{0, r(t) - \mu\}. \tag{4.22}$$

The second subclass of deposit rate models, are models where the deposit rate is a set fraction α of the market rate. These models are given by:

$$i(t) = \max\{0, \alpha r(t)\}. \tag{4.23}$$

A more complicated subclass, that aims to capture incomplete pass-through and some delay in the deposit rate setting, would be given by:

$$i(t_n) = \max\left[0, \alpha r(t_{n-1}) + \begin{cases} \beta dr_1(t_n), & dr_1(t_n) \geq 0, \\ \gamma dr_1(t_n), & dr_1(t_n) < 0. \end{cases}\right] \tag{4.24}$$

In Equation 4.24, we use the market rate one time-step (month) back, as well as the difference between this and the current market rate. The asymmetric pass-through arises when the parameters β and γ are not equal.

Since an important objective for the bank is the volume not decreasing below a certain target volume, and the volume is a known quantity, it is a natural step to include the volume in the setting of the deposit rate. In order to investigate whether this has a positive effect, we define a deposit rate model that contains two free parameters and sets the deposit rate as a function of the market rate and the deposit volume. This model is defined as:

$$i(t_n) = \max\left\{\left(\theta_1 + \theta_2 \frac{V(t_n) - V^*(t_n)}{V^*(t_n)}\right) r(t_n), 0\right\}. \tag{4.25}$$

In Equation 4.25, the θ_i are the parameters that will be determined via the optimization procedure, while $V^*(t_n)$ is the target volume at time t_n, which is the same as in Equation 4.21. This model was designed to be similar to the constant relative margin model, but incorporate a correction on the margin, depending on the relative difference between the actual volume and the target volume.

The optimization of the deposit rate model can be carried out using methods from the field of stochastic approximation. These are techniques that are designed to optimize functions of the form $f(\boldsymbol{\theta}) = \mathbb{E}[F(\boldsymbol{\theta}, \boldsymbol{\chi})]$, where $\boldsymbol{\chi}$ denotes a vector of random variables. In our case, the variables in the vector $\boldsymbol{\theta}$ are the variables in the different deposit rate models (μ, α, β, γ, θ_1 or θ_2, depending on the model that is optimized). The random processes in $\boldsymbol{\chi}$ are the Brownian motion in the TSM and the residual part in the volume model. The Brownian motion enters the model via the market rate, deposit rate and deposit volume, since the latter two depend on the former; the residual part of the volume enters the model via the deposit volume, and via the deposit rate in case of a volume dependent deposit rate model.

We follow the procedure, set out in the book Kushner and Yin (2003, Section 1.1.4). There, a minimization problem is solved using the Robbins-Monro algorithm, which is able to find roots of functions. The minimization then essentially corresponds to finding the root of the derivative of a function. Since we are in multidimensional space, we aim to find a zero of the gradient. The stochastic approximation procedure is set out in the following part. This follows Kushner and Yin (2003, Section 1.1.4), but we apply their general steps to our specific case. We argue that a maximization is equivalent to minimization of the negative of a function, we can therefore use the same procedure.

The Robbins-Monro algorithm

The Robbins-Monro algorithm (Kushner & Yin, 2003; Robbins & Monro, 1951), is a method for finding the root of an equation, using stochastic approximation. This essentially means that the algorithm is capable of finding roots, even if the function of which we try to find the root is an expectation and its derivative is unknown. The algorithm works as follows: (for a proof, we refer to Appendix B or (Robbins & Monro, 1951)).

The algorithm can solve problems of the form:

$$f(\boldsymbol{\theta}) = 0. \tag{4.26}$$

In Equation 4.26, $f(\boldsymbol{\theta})$ is allowed to be an expectation, i.e. $f(\boldsymbol{\theta}) = \mathbb{E}[F(\boldsymbol{\theta}, \boldsymbol{\chi})]$, with $\boldsymbol{\chi}$ being a vector of random variables, over which the expectation is taken.

The power of the algorithm lies in its ability to solve this type of problem, without empirically calculating the expectation for every tested value of $\boldsymbol{\theta}$. Instead, it uses a recursive procedure, that, under certain conditions, converges to the root $\bar{\boldsymbol{\theta}}$.

For each step, we calculate a new estimate $\boldsymbol{\theta}_{k+1}$. The equation we evaluate is:

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k - \epsilon_k F(\boldsymbol{\theta}_k, \boldsymbol{\chi}_k), \tag{4.27}$$

where $\boldsymbol{\chi}_k$ denotes a realization of the random variable $\boldsymbol{\chi}$.

In Equation 4.27, the ϵ_k must satisfy:

$$\begin{aligned} \epsilon_k &> 0, \quad \forall k \in \mathbb{N} \cup \{0\}, \\ \lim_{k\to\infty} \epsilon_k &= 0, \\ \sum_{k=0}^{\infty} \epsilon_k &= \infty, \\ \sum_{k=0}^{\infty} \epsilon_k^2 &< \infty. \end{aligned} \tag{4.28}$$

The final condition in Equation 4.28 can be weakened (Kushner & Yin, 2003, p. 3-4), but this is outside the scope of this thesis. The choice of sequence $\{\epsilon_k\}$ governs the performance of the algorithm. Trying several sequences and empirically finding the best performing sequence is often the chosen approach.

Due to the nature of the function $f(\boldsymbol{\theta})$, namely that we have $f(\boldsymbol{\theta}) < 0$ if $\boldsymbol{\theta} < \bar{\boldsymbol{\theta}}$ and $f(\boldsymbol{\theta}) > 0$ if $\boldsymbol{\theta} > \bar{\boldsymbol{\theta}}$ (component wise). The next estimate is expected to move towards the root at $\bar{\boldsymbol{\theta}}$. The nature of the sequence $(a_n)_{n\geq 0}$ ensures that the algorithm converges. For the exact requirements for convergence, we refer to the proof in Appendix B. If we look for a minimum of a function, then its gradient satisfies all the above requirements in the neighbourhood of a (local) minimum $\bar{\boldsymbol{\theta}}$.

If we calculate the sequence $\{\boldsymbol{\theta}_k\}$, this sequence will converge to the (local) minimum $\bar{\boldsymbol{\theta}}$. The stopping criterion that we will use for this iterative scheme is the difference in subsequent values of $\boldsymbol{\theta}_k$ becoming small.

Before we can use this procedure, we require that the gradient of the function we want to optimize, with respect to the variables we want to optimize for, is available. The functions that we want to optimize for are given in Equation 4.20 and Equation 4.21. These contain several components for which taking the derivative is non-trivial or that are not differentiable at all. For instance, when the deposit rate follows a model with a max-function, the derivative is not defined at the point where the maximum 'switches', for example, in Equation 4.22, at the point $\mu = r$.

A possible workaround here is to approximate this maximum function with some function that is differentiable everywhere. This can be done, by considering a Fourier series expansion for example. Another way is to make use of a numerical approach, where instead of a proper derivative, we use a forward, backward or midpoint scheme to approximate the derivative.

Even if the model is strictly speaking differentiable, the functions are too elaborate to evaluate the gradient analytically due to them containing a sum and the elaborate nature of the volume calculation (this is again a sum, so that we have a nested sum in the final expression). We can, however, use automatic differentiation in Python to evaluate these gradients. Using this procedure, the step-by-step plan for optimizing the deposit rate model is as follows. Note that finding the maximum of a function is the same as finding the minimum of the negative version of that function, i.e. $\max\{f\} = -\min\{-f\}$, and this is equivalent to finding the root of the gradient of f.

Input: The term structure rates for all time steps $(r(t_n))_{0\leq n\leq \tilde{n}}$, the deposit model parameters $\boldsymbol{\rho}$, the deposit rate model type, the quantity to optimize, the initial guess $\boldsymbol{\theta}_0$ for the parameter vector $\boldsymbol{\theta}$, and a parameter determining the stopping criterion ξ.
Setup: Set $n \leftarrow 0$; Define the sequence $a_{n_{n\geq 0}}$ with $a_n = \frac{1}{n+1}$.
Iterate until $\|\boldsymbol{\theta}_{n+1} - \boldsymbol{\theta}_n\|_2 < \xi$ or a set number of iterations is reached:

- calculate a value of the stochastic objective function $F(\boldsymbol{\theta_n}, \boldsymbol{\chi_n})$, and calculate its derivative with respect to $\boldsymbol{\theta}$ at $\boldsymbol{\theta}_n$ (denoted by $\boldsymbol{g}(\boldsymbol{\theta}_n)$) if we want to locate a minimum; if we try to find the maximum, multiply the function by -1 first.
- Set the new estimate $\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n - a_n\boldsymbol{g}(\boldsymbol{\theta}_n)$.[16]

Result: Parameter vector $\bar{\boldsymbol{\theta}}$ for which the objective function is (locally) minimized/maximized.

Specifically for our case, we optimize over a combination of the profit and the shortage. The objective function is thus

$$F(\boldsymbol{\theta}, \boldsymbol{\chi}) = cW(\tau, \boldsymbol{\theta}, \boldsymbol{\chi}) + S(\tau, \boldsymbol{\theta}, \boldsymbol{\chi}). \tag{4.29}$$

In Equation 4.29, $c \in \mathbb{R}$ and $W(\tau, \boldsymbol{\theta}, \boldsymbol{\chi})$ and $S(\tau, \boldsymbol{\theta}, \boldsymbol{\chi})$ are as in Equation 4.20 and Equation 4.21 respectively. Here, we explicitly denote their dependence on the deposit rate model parameters and stochastic processes. We thus have

$$f(\boldsymbol{\theta}) = \mathbb{E}_{\boldsymbol{\chi}}[F(\boldsymbol{\theta}, \boldsymbol{\chi})] = \mathbb{E}_{\boldsymbol{\chi}}[cW(\tau, \boldsymbol{\theta}, \boldsymbol{\chi}) + S(\tau, \boldsymbol{\theta}, \boldsymbol{\chi})]. \tag{4.30}$$

An important note is that, with this procedure, our final result could be the parameter vector for only a local minimum/maximum, rather than a global minimum/maximum. In order to mitigate for this effect, we repeat the iteration with several starting vectors $\boldsymbol{\theta}_0$. We then take the value with the lowest/highest objective function value of these as the 'true' minimum/maximum.

We conduct this procedure for the constant absolute margin deposit rate model, as defined in the previous section. After conducting this procedure for all deposit rate model types, we have found an optimal deposit rate model. In the following section, we describe a procedure for conducting a risk analysis on an NMD portfolio. Using this procedure, we can compare the optimized models on the involved risks.

4.4 Risk Analysis

Now, we are in a position to do a risk analysis on the deposit volume. We use the obtained models for the deposit volume, deposit rate and term structure. For this analysis, we use the optimal constant absolute margin model. Firstly, we can look at the distribution for the deposit volume, where we are interested in the quantiles at the lower end of the distribution. We define these quantiles as:

$$Q_\alpha(V_t) = \sup\{x | \mathbb{P}(V_t < x) < \alpha\}. \tag{4.31}$$

In Equation 4.31, α is a number between zero and one. The quantile $Q_\alpha(V_t)$ denotes a volume that is lower than the volume V_t with probability $1-\alpha$. In other words, with probability $1-\alpha$, we have $V_t > Q_\alpha(V_t)$. We can also define a conditional quantile:

$$Q_\alpha(V_t|y) = \sup\{x | \mathbb{P}(V_t < x|y) < \alpha\}. \tag{4.32}$$

In Equation 4.32, we consider probabilities conditioned on y, where y denotes an event. In the case of this thesis, the events y are the conditionings on quantities associated with the market rate r. These quantiles can be computed from the results of simulations. The results of these simulations can be used to construct a CDF for the deposit volume. Q_α can then be found as the α-quantile of this empirical CDF.

The number of simulations that are carried out, is important for the validity of the computed quantiles. A low number of simulations will mean that the computed quantile is rather uncertain.

With our simulations, we calculate the quantiles of several quantities. Firstly, we calculate the quantiles for the deposit volume at a certain time point (1 year, 5 years and 10 years). Additionally, we look at some quantiles of associated quantities (the minimum and maximum of the volume within the timeframe) and those quantities conditioned on various rate scenarios. For example, if we save the market rate data over the time period, we can investigate the relationship between the average rate and the quantiles, or the minimum and maximum of the rate and the quantiles.

The calculations in the experiments will be carried out as follows:

Model simulation procedure

1. Fit the term structure model to a range of forward rate data.
2. Use the short rate obtained in the fitting of the term structure, along with the historical deposit rate and deposit volume to fit the deposit rate model to the data.
3. Carry out an Euler-Maruyama scheme to obtain the short rate up until time T.[17]
4. Use this short rate, along with a rule for setting the deposit rate, to obtain the changes in the logarithm of the deposit volume. At each volume calculation step, add a random sample of the residuals from the fitted volume model to incorporate randomness in the model.
5. Deduce the evolution of the volume (up to time T) from these differences in the logarithm.
6. Repeat the previous three steps until enough data points for the intended risk level are obtained.

Denoting the number of simulations by N_{sim} and the desired precision of the α in the quantile by $\Delta\alpha$ (which can be seen as a minimum significant difference between values of α), the relation we aim to satisfy is $N_{\text{sim}} > \frac{10}{\Delta\alpha}$. The numerator of this fraction (10) is the expected number of simulation values in each 'bin', if we were to divide the interval $[0, 1]$ in bins with width $\Delta\alpha$ and N_{sim} is minimal.[18]

The procedure that is used in the experiments is to simulate the evolution of the deposit volume $10,000,000$ times. With these simulations, we can calculate quantiles for the following quantities:

$$\begin{aligned}
&\bullet\quad V(t_n),\\
&\bullet\quad H(t_n) = \max\{V(t_\nu) | 0 \leq \nu \leq n\},\\
&\bullet\quad L(t_n) = \min\{V(t_\nu) | 0 \leq \nu \leq n\},
\end{aligned} \tag{4.33}$$

conditioned on the quantities:

$$\begin{aligned}
&\bullet\quad \overline{r}(t_n) = \max\{r(t_\nu) | 0 \leq \nu \leq n\},\\
&\bullet\quad \underline{r}(t_n) = \min\{r(t_\nu) | 0 \leq \nu \leq n\},\\
&\bullet\quad \overline{\underline{r}}(t_n) = \overline{r}(t_n) - \underline{r}(t_n),\\
&\bullet\quad \hat{r}(t_n) = \frac{1}{n+1}\sum_{\nu=0}^{n} r(t_\nu),\\
&\bullet\quad \tilde{r}(t_n) = \frac{1}{n}\sum_{\nu=1}^{n} |r(t_\nu) - r(t_{\nu-1})|.
\end{aligned} \tag{4.34}$$

Here, $H(t_n)$ and $L(t_n)$ are the maximum and minimum volume over the considered time period respectively. $\overline{r}(t_n)$, $\underline{r}(t_n)$, $\overline{\underline{r}}(t_n)$, $\hat{r}(t_n)$ and $\tilde{r}(t_n)$ are the maximum market rate, the minimum market

[18] A $\Delta\alpha$ of 0.01% would mean that alpha can be varied in steps of 0.01%. This requires that $N_{\text{sim}} > 100,000$.

rate, the maximum absolute rate spread, the mean rate and the mean (monthly), absolute, rate change respectively.

The conditioning on these quantities, and especially on quantiles of these quantities, can be used as a type of risk analysis. For instance, conditioning on the minimum short rate being in the lowest 1% would correspond to assuming a scenario with low market rates. This approach is different from adding shocks to the model, which is more commonplace and implemented in guidelines (EBA, 2022b). This conditioning approach has the advantage of being more flexible, as it is easier to test for multiple quantiles, or for a range of quantiles at once. It is however only able to simulate scenarios that are in the model. With this, we mean that it is implicitly assumed that the model holds in extreme cases as well. Risk analysis by adding shocks to the model is able to investigate scenarios that are outside the normal model.

Comparing the values of the quantiles for various different conditionings yields insights in the stability of the NMD portfolio. For instance, if the lowest quantiles shift considerably when changing the considered deposit rate quantile, this would be an indication of instability of the NMD portfolio when market rates are around that point.

4.4.1 Comparing Deposit Rate Models

With an approach slightly adapted from the previous, we are in a position to compare the risks involved with optima for the different types of deposit rate models, defined in section 4.3. For this comparison, we look at expectations for the profit (Equation 4.20) and the shortage (Equation 4.21), as well as the minimum volume over a certain time period. To this end, we simulate numerous market rate scenarios.

For each scenario, we calculate the deposit rate. This is straight forward for the constant absolute and relative margin models. For the volume dependent model, we calculate this in conjunction with the deposit volume. Several deposit volume evolutions are simulated for each rate scenario. These will serve as the base scenarios for our analysis. For the volume dependent deposit rate model, the corresponding deposit rates are calculated at this step.

We run 100 market rate simulations. For each of these, and for each of the deposit rate models, we carry out 10 000 deposit volume simulations. We use these simulations to calculate expected values for both optimization criteria and the minimum volume. The results for this can then be plotted in the form of a scatter plot, where we set out the mean market rate $\hat{r}(t_n)$ on the x-axis to be able to compare the deposit rate models in different interest rate scenarios. Since we use the same market rate scenarios for each of the deposit rate models, we can compare their performance by examining these scatter plots. Furthermore, we can investigate the performance of these models under different market rate conditions and see whether a certain deposit rate model's performance is highly affected by the market rate.

5
Results & Discussion

This section presents the results that were obtained by following the approach as set out in Section 4. The obtained results are also discussed here. The section starts with the results of the calibration of the Term Structure Model (TSM). After this, the results for the calibration of the deposit volume model are presented. Subsequently, the results of the optimization of the different types of deposit rate models is presented and discussed. The risk analysis of the Dutch NMD portfolio is set out next. This section finishes with a comparison of the performance of the different optimal deposit rate models.

All experiments were conducted in Python, using the packages *NumPy, SciPy, StatsModels* and *Pandas*. The *@jit* decorator from *Numba* was used to accelerate some functions in the program. For the implementation of the genetic algorithm, the *PyGAD* (Gad, 2021) was used. For all other optimization procedures (OLS, calculating p-values and R^2-values), the *SciPy* and *StatsModels* libraries were used. The package *Autograd* was used for automatic derivation in the deposit rate model optimization procedure.

5.1 Term Structure

A TSM is fitted to historical observations of forward rates. For the calibration, the parameters for a Svensson (1994) model forward rate curve, as available from the ECB, were used. These forward rate curves were used to obtain forward rates for the maturities of 3 months, 6 months, 1 year, 2 years, 5 years, 10 years and 20 years. The forward rates for these different maturities were used for the calibration of the CIR model, as described in the previous section (Section 4).

In order to make the data compatible with the CIR model, a shift is necessary since the CIR model does not allow negative rates. The forward rate data is shifted upwards by $\Delta r = 100\text{bp}$. The rates that the fitted model will output are shifted back by the same amount. This means that it is effectively assumed $r + \Delta r$ follows a CIR model. The appropriate shift was determined by observing the 3 month EURIBOR forward rate from historical data. In the considered time period, this rate goes below zero, but stays well above -100bp. Alternatively, since the genetic algorithm approach is very general, the shift could also be considered as a parameter in the maximum likelihood procedure. Since the term structure model is not the main focus of this thesis, this complication was not explored. A comparison of different TSMs and their influence on the subsequent optimizations and risk analysis could be the subject of further research.

The fitted parameters of the CIR model can be found in Table 5.1. Here, the parameters α, θ and σ are for the CIR model in the real-world measure. The parameter λ is used to transfer between this measure and the risk-neutral measure. This can be observed in Section 4, where the expression for the forward rate (Equation 4.2) does contain the parameter λ, while the expression for the short rate the next time step over (Equation 4.5), does not contain this parameter. The calibrated values of the CIR model seem reasonable, predicting a long term short rate of $\theta - \Delta r = 2.6\%$, especially regarding the long period of near zero market rates in the sample. While the σ parameter may seem large since it is in the same order as θ, the actual volatility is limited due to the order of the short rate being around 10^{-2} and the multiplication with the square root of the short rate in the volatility term in the CIR model.

Table 5.1: Calibrated coefficients of the CIR model. The coefficients in this table are for a CIR model that models the rates shifted upwards by 100bp and are coefficients in the real world measure.

h	α (per year)	λ (per year)	θ	σ (per year)
$9.93 \cdot 10^{-3}$	$3.35 \cdot 10^{-1}$	$-8.60 \cdot 10^{-2}$	$3.63 \cdot 10^{-2}$	$2.35 \cdot 10^{-2}$

The values in Table 5.1 were used for all subsequent experiments. These are all forecasting experiments, meaning that the parameters that are used are those that are in the model in the real world measure. These are α, θ and σ.

5.2 Deposit Volume

In this section, the model Equation 4.13 is specified by finding the variables in the vector $\boldsymbol{\rho}$. Firstly, it is determined which (lagged) rates should be considered for the model. Subsequently, a deposit volume model is obtained via ordinary least squares. Since the analysis and the forecasting relies on the residuals of this fitted model to be independent, the section concludes with an analysis of these residuals and measures taken to ensure that the residuals in the final model are sequentially independent, and random sampling can be used in the subsequent simulations of the volume.

To determine which variables should be considered for inclusion in the volume model, p-values and R^2-values are calculated for both the market rate and the deposit rate, as well as quantities derived from these. These are the p-values and R^2-values for a model with only a constant and the considered variable. The p-values thus corresponds to the probability of both the constant and the coefficient associated with the considered variable having opposite sign, assuming normally distributed residuals. This experiment is carried out to give an indication of which variable, and more specifically, which lag of a variable should be considered for the total model. Here, lower p-values and higher R^2 values are to be preferred. The results for this analysis are presented in Figures 5.1 to 5.3.

Figure 5.1 contains the results for models with the (lagged) market or deposit rate included directly, from which it can be seen that including the market rate r without any lag yields the lowest p-value and highest R^2-value, while the p-values and R^2-values for the deposit rate are rather low and high respectively. From this, it can be seen that adding a deposit rate term to the model probably does not result in a model with a better fit, if a market rate term is include, the lag that yields the best fitting model is likely to be a lag of zero.

Results for an analogous experiment, but for the change in both the market rate and the deposit rate are plotted in Figure 5.2. Here, the lag means that the difference between points further apart in time is considered. From these results, it can be seen that both the difference in the market rate, and the difference in the deposit rate yield models with low p-values and significant R^2-values. It is interesting to see that for the market rate difference, longer lags seem to yield better results, while for the deposit rate, the results for shorter lags are better. This could indicate that the adjustments of customers or banks are somewhat lagged. It could also be the result of the longer lag period decreasing the relative noise component in the market rate difference. For the deposit rate, the results in Figure 5.2 indicate that the difference between deposit rates with a lag of a single month is the best predictor of deposit volume changes.

Finally, Figure 5.3 contains the results for models with the spread between the market and deposit rate, as well as the difference between this spread. Here, the spread is defined as $s(t_n) = r(t_n) - i(t_n)$. From these results, it can be seen that the highest R^2-values and lowest p-values are found for the non-lagged version of the spread and the difference in spread over a period of around 6 months. The latter could again be due to the increased signal-to-noise ratio (SNR) from the longer period between spreads. The results for the spread are very similar to the results for the market rate models, which is not surprising, considering the deposit rate was close to zero for a large part of the considered data.

In addition to the previous experiment, it is useful to consider the correlation between different variables that are considered for the model. This is because including highly correlated variables can lead to an increase in the p-values associated with these parameters. Here the considered p-value of the parameter is the probability that the sign of the coefficient associated with that parameter has opposite sign. Additionally, including highly correlated variables can result in little to no increase in model fit, since they could explain the same part of the variation in the volume. To give an idea of the correlation between the considered variables, the correlation between any pair of them is calculated and plotted in Figure 5.4. In Figure 5.4, the absolute value of the correlation coefficient is visualized with a colour gradient, where yellow means high correlation and purple means low correlation. Included are all considered variables with their lags, the time t and the change in the logarithm of the deposit volume, which is the quantity that is modelled.

From Figure 5.4, the obvious unit correlation of variables with themselves and with their lagged variants, as well as the symmetry of the matrix can be observed. More interesting observations are that the time t does not correlate well with the change in deposit volume in the historical data, where it is often found in literature as a proxy for macroeconomic effects, and that the spread and market rate show high correlation. This is well explained by the deposit rate being close to zero in a large part of the historical data. An important takeaway from this matrix is, that it is probably not beneficial to include different lags of the same variable in the model and that including both the market rate and the spread in a model is also likely not beneficial. A model where this is done likely has uncertain coefficients and does not have a much higher total model R^2-value, compared to the same model with one of these highly correlated variables removed. A final takeaway from this matrix can be seen from the correlation of the variables with the change in the deposit volume, since this is quite low across the board, a large part of the evolution of the volume is likely captured in the residuals, which would be indicated by a low total model R^2-value.

Keeping this in mind, and carrying out the procedure as set out in the method. A model with a constant term, the difference between the current market rate and the market rate three months before, and the spread between the current market rate and deposit rate is obtained. The coefficients of this model, along with the associated standard errors, t-statistics, p-values, again those of the individual variable, and the confidence intervals can be found in Table 5.2. An important note here is that this analysis assumes normally distributed residuals. This assumption is checked next.

In order to visually inspect if serial correlation is present in the residuals, and to visualize the distribution of the residuals, the residual distribution is plotted in Figure 5.5a and a lag-plot is plotted in Figure 5.5b. There seems to be one outlier in the residuals around $0.06 \log$ EUR M. A first look at the lag-plot Figure 5.5b does not show much structure in the plot, indicating that the serial correlation between the residuals is not extreme.

A Durbin-Wattson test was carried out in order to assess this more rigorously. This test yielded a Durbin-Watson test statistic of 1.66. This value is below the critical value of 1.74 for the number of model variables (2) and observations in the model (232) and sample respectively at confidence level 0.05. Meaning that the hypothesis of no serial correlation cannot be rejected, and the subsequent analysis should account for this correlation.

This serial correlation is accounted for by carrying out the Cochrane-Orcutt procedure, as described in Section 4. The correlation coefficient between the residuals is found to be $b = 0.17$. Using this, and rewriting the model in $AR(1)$-form result in a model with coefficients as given in Table 5.3. These values only differ slightly from those in Table 5.2 and, most importantly, all p-values remain below 0.05. The residuals of this model are plotted in Figure 5.6a, where again only a small difference as compared to the Figure 5.5a is observed. For completeness, the lag-plot of the residuals after the Cochrane-Orcutt procedure is plotted in Figure 5.6b.

The Durbin-Watson statistic is calculated for this model as well, to ensure that the serial correlation is removed. This results in a Durbin-Watson statistic of 1.98, which is well above the critical value. The residuals can thus be assumed to be independent. This ensures that for the simulations, the residuals can be sampled at random from the empirical distribution in Figure 5.6a.

Table 5.2: Coefficients of the fitted deposit volume model before the Cochrane-Orcutt procedure. The coefficients are given with their associated t statistic, p-value and confidence interval.

	Coefficient	Standard Error	t-statistic	p-value	95% Confidence Interval	
constant	3.7 $\times 10^{-3}$	1 $\times 10^{-3}$	4.73	0.00	2 $\times 10^{-3}$	5 $\times 10^{-3}$
$dr_3(t_n)$	-0.72	0.21	-3.45	0.00	-1.13	-0.31
$s(t_n)$	-0.19	0.07	-2.50	0.01	-0.34	-0.04

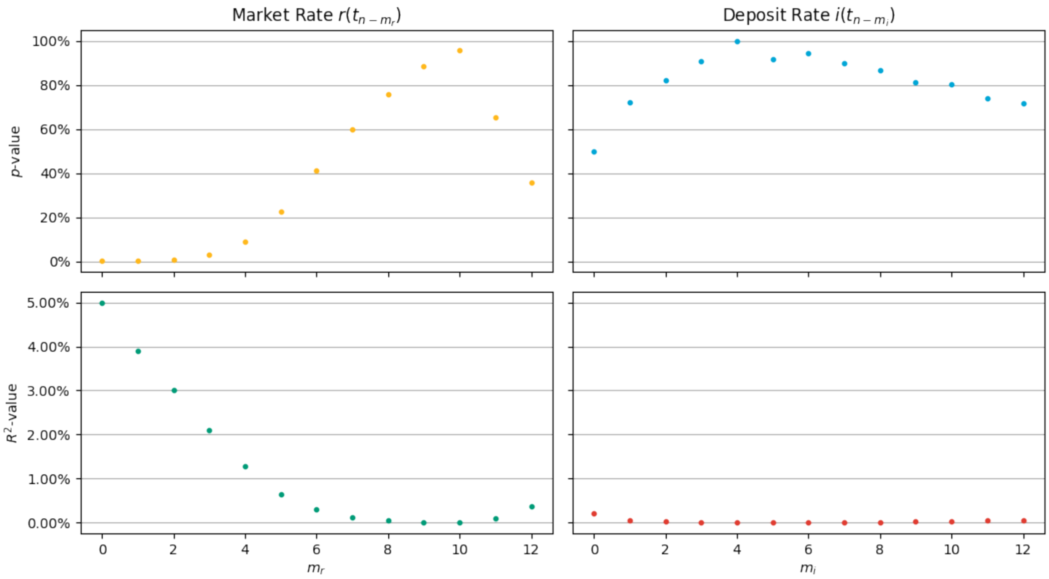


Figure 5.1: Comparison of the p- and R^2-values for different (lagged) market and deposit rates. The p-values and R^2-values correspond to the values associated with a linear model with only the considered variable and a constant term.

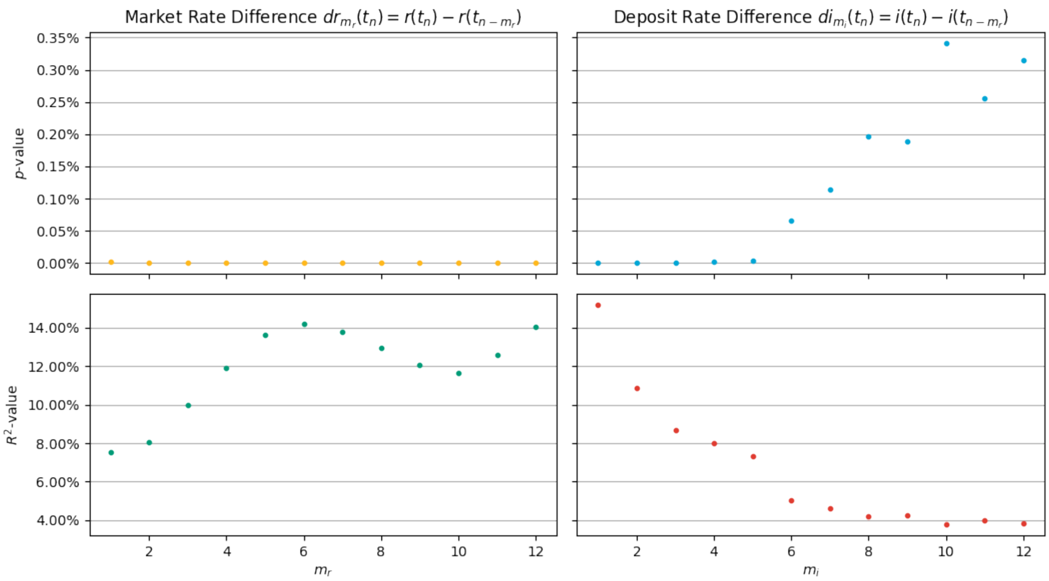


Figure 5.2: Comparison of the p- and R^2-values for different (lagged) market and deposit rate differences. The p-values and R^2-values correspond to the values associated with a linear model with only the considered variable and a constant term.

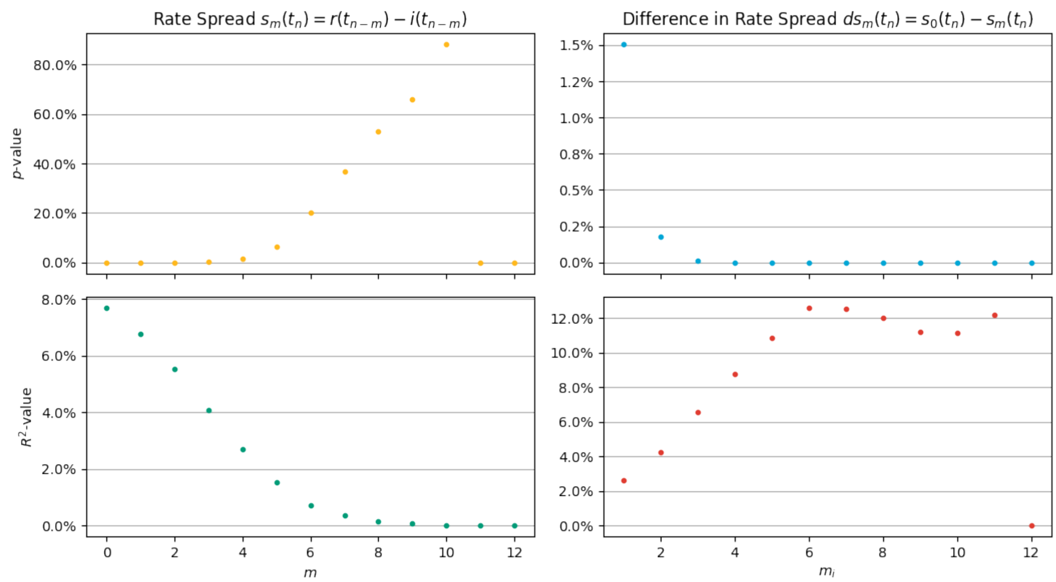


Figure 5.3: Comparison of the p- and R^2-values for different (lagged) spreads between the market and deposit rate and the differences in this quantity. The p-values and R^2-values correspond to the values associated with a linear model with only the considered variable and a constant term.

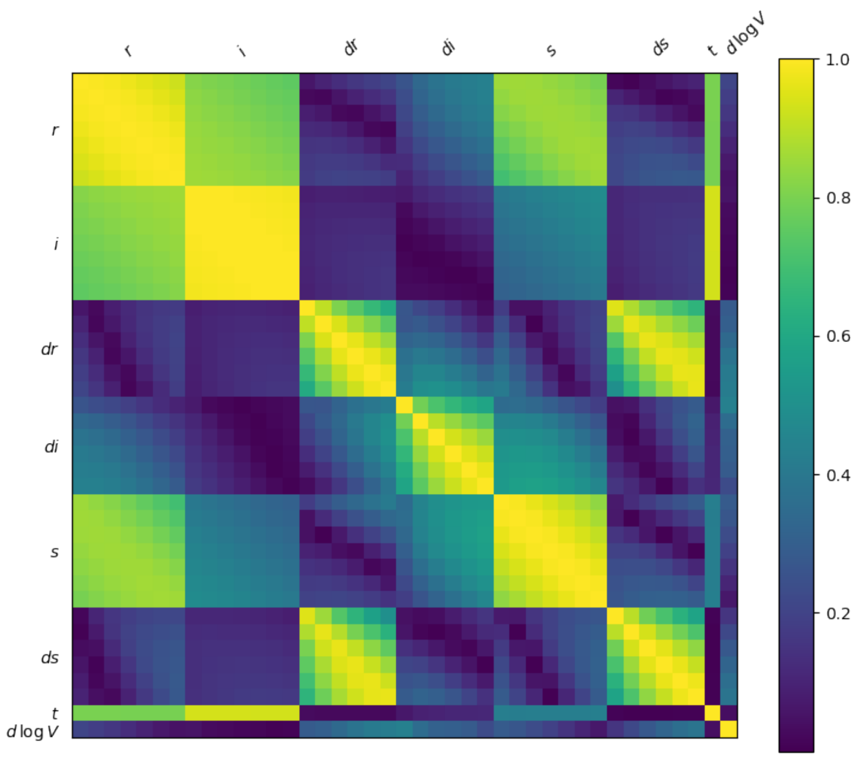
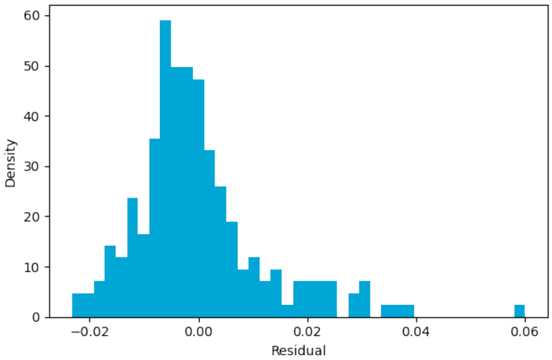
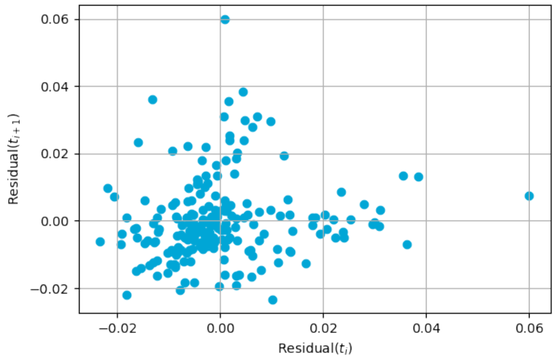


Figure 5.4: The correlation matrix for the variables, considered for inclusion in the deposit volume model. High (anti-)correlation is displayed as a yellow colour, while low correlation is displayed as a dark-purple colour. The variables are grouped and labelled without subscript, i.e. the label r is put in the middle of the cluster of (lagged) market rates.

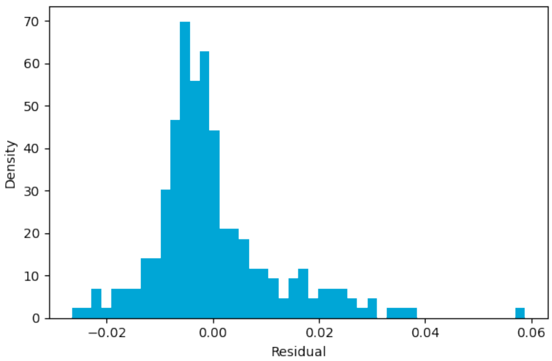


(a) **Empirical distribution of Deposit Volume Model residuals.**

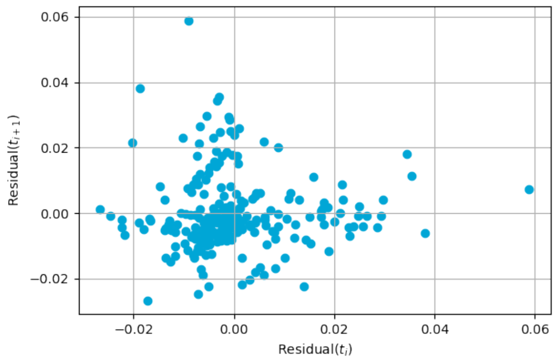


(b) **Lag-plot of the deposit volume model residuals.**

Figure 5.5: Residual distribution and lag-plot before the Cochrane-Orcutt procedure. The left plot displays the empirical distribution of the residuals, while the right plot shows a lag-plot of the residuals.



(a) **Empirical distribution of Deposit Volume Model residuals.**



(b) **Lag-plot of the deposit volume model residuals.**

Figure 5.6: Residual distribution and lag-plot after the Cochrane-Orcutt procedure. The left plot displays the empirical distribution of the residuals, while the right plot shows a lag-plot of the residuals.

Table 5.3: Coefficients of the fitted deposit volume model after the Cochrane-Orcutt procedure. The coefficients are given with their associated t statistic, p-value and confidence interval. The value of b that follows from this procedure is 0.17.

	Coefficient	Standard Error	t-statistic	p-value	95% Confidence Interval	
constant	3.1×10^{-3}	1×10^{-3}	4.01	0.00	2×10^{-3}	5×10^{-3}
$dr_3(t_n)$	-0.73	0.24	-2.99	0.00	-1.21	-0.25
$s(t_n)$	-0.18	0.09	-2.08	0.04	-0.36	-0.01

5.3 Deposit Rate Model Optimization

Here, the results of the optimization of the different deposit rate models (Equation 4.22, Equation 4.23, Equation 4.24 and Equation 4.25), as found in section 4.3, are presented. These optimized models are compared using the risk analysis framework in the final part of this section. When optimizing for the 'profit' (Equation 4.20) alone, for all models it is obtained that the optimal deposit rate is a deposit rate that is always zero. This is due to the restriction that the ZLB imposes on the deposit rate, otherwise, the optimization would suggest deposit rates below zero as optimal.

For this reason, and since, in practise, a large part of the deposit volume of a bank might be locked up in longer term investments, the optimization is also carried out over the combination of the 'profit' and 'shortage' (Equation 4.21). This combination ensures that the optimal models aim to keep the volume above or around the target volume, which was set at 80% of the starting volume, which was subsequently increased by the market rate. The optimization criterion is thus defined as:

$$A(\tau) = cW(\tau) + S(\tau). \tag{5.1}$$

For the optimization, only the relative weights of the components are of importance, it is therefore enough to include only one constant $c \in \mathbb{R}^+$ in Equation 5.1.

In the optimization procedure, the sequence $(a_n)_{n\geq 0}$ can be chosen somewhat freely. Multiplication of the sequence $(\frac{1}{n+1})_{n\geq 0}$ by a constant still results in a sequence that satisfies the required properties, as stated in section 4.3. Furthermore, only increasing n when the result of the derivative changes sign will still leave the convergence intact, but speed it up significantly. This principle was used for all subsequent optimization. For the optimizations of multiple variables, each variable was assigned its own sequence $(a_n^{(i)})_{n\geq 0}$, which still keeps convergence intact.

The results for the optimization of the constant absolute margin model are shown in Figure 5.7. The optimization was carried out multiple times with different starting values to show that the convergence is unique, as long as the initial guess is somewhat reasonable. For example, setting the initial margin very high yields no convergence, since the deposit rate is then always set to zero for 'normal' market rate values and the derivative with respect to the parameter μ around that value is then zero in probability. For this constant margin model, it can be seen that there is convergence to a constant margin of $\mu = 0.008$, meaning that the optimal deposit rate model sets the deposit rate 80bp lower than the market rate, or zero when the ZLB is reached. The sequence that was used for this optimization was $(a_n)_{n\geq 0} = (\frac{1}{10^{10}(n+1)})_{n\geq 0}$.

Figure 5.8 contains the results for the optimization of the constant relative margin model. Again the optimization procedure was carried out multiple times in order to show unique convergence. Again for reasonable, i.e. positive, initial guesses of α. The optimal value that was found from these simulations is $\alpha = 0.7$, meaning that the optimal deposit rate for this model is around 70% of the market rate when the market rate is positive. For a negative market rate, the deposit rate should be zero due to the ZLB. The sequence that was used for this optimization was $(a_n)_{n\geq 0} = (\frac{4}{10^{8}(n+1)})_{n\geq 0}$.

Another model that was calibrated, was a deposit rate model containing asymmetric pass-through. This in light of the deposit rate characteristics that were set out in Section 2. The base for this model is the constant relative margin model. It was found that the optimal value for the parameter α in Equation 4.24 is the same as the regular constant relative margin model, which is somewhat expected. For the other parameters (β and γ), it was found that these remain at their starting values, no matter what these starting values are. As an illustration of this, Figure 5.9 is included. This indicates that the influence of this asymmetric adjustment on the objective, which is the combination of the 'profit' and the 'shortage' is limited. For this optimization the sequences that were used were $(a_n^{(\alpha)})_{n\geq 0} = \left(\frac{4}{10^8(n+1)}\right)_{n\geq 0}$, $(a_n^{(\beta)})_{n\geq 0} = \left(\frac{1}{10^9(n+1)}\right)_{n\geq 0}$ and $(a_n^{(\gamma)})_{n\geq 0} = \left(\frac{1}{10^9(n+1)}\right)_{n\geq 0}$.

The final model that was optimized, was a model that takes the volume as an input, in addition to the market rate. This model (Equation 4.25) is constructed similarly to the constant relative margin model, but here, the margin depends on the volume itself. Intuitively, the multiplier should be lower when the volume is (far) above the target and higher when the volume is closer to or below the target volume. The results for the optimization of this model, for several starting values of (θ_1, θ_2) can be found in Figure 5.10a and Figure 5.10b for θ_1 and θ_2 respectively. From Figure 5.10a, it can be seen that this parameter converges to a value around $\theta_1 = 2.5$, while from Figure 5.10b, it can be seen that the parameter θ_2 does not converge well, but that the parameter is quite stable when choosing the starting value $(\theta_1, \theta_2) = (2.5, -40)$. It can however be seen that the parameter does tend to go towards a large negative value. In practice, this means that the optimal form of the model Equation 4.25 is a model that keeps the deposit rate at zero, unless the volume is slightly above or below the target volume, where the model drastically increases the deposit rate in an effort to increase the volume. This result is expected when considering the results for the optimization without considering the 'shortage'. There the optimal deposit rate was found to be zero. This volume dependent model essentially keeps the rate at zero, as long as it avoids the volume being below the target. In this way, the influence of the 'shortage' on the optimization criterion $A(\tau)$ is limited. For this optimization, the sequences that were used were $(a_n^{(\theta_1)})_{n\geq 0} = \left(\frac{1}{10^6(n+1)}\right)_{n\geq 0}$ and $(a_n^{(\theta_2)})_{n\geq 0} = \left(\frac{1}{10^3(n+1)}\right)_{n\geq 0}$.

Some important considerations with these optimizations arise due to the nature of the data and the model derivation. Firstly, since the data that was used is aggregate data, the whole Dutch banking sector is essentially modelled as a single bank. The effects of customers switching banks within the Dutch banking sector is thus not observed. These optimal values therefore likely do not hold for individual banks, but rather for the Dutch banking sector as a whole. The optimization procedure, and the procedure for deriving a volume model can however be used on single bank data, which would give valid results for the considered bank.

The second consideration, which mainly revolves around the final deposit rate model Equation 4.25, is that the data to which the volume model is calibrated does not contain periods of extremely high deposit rates. Deposit volume evolution when the deposit rate is much larger than market rate is therefore not necessarily as in the model, however, the desired effect of consumers depositing more volume at the bank is still the logical effect, as the yield they then receive on their deposit is very high.

In the subsequent risk analysis, the optimal constant absolute margin model is used. A comparison between the optimal constant absolute margin, constant relative margin and volume dependent deposit rate models can be found in the final part of this section.

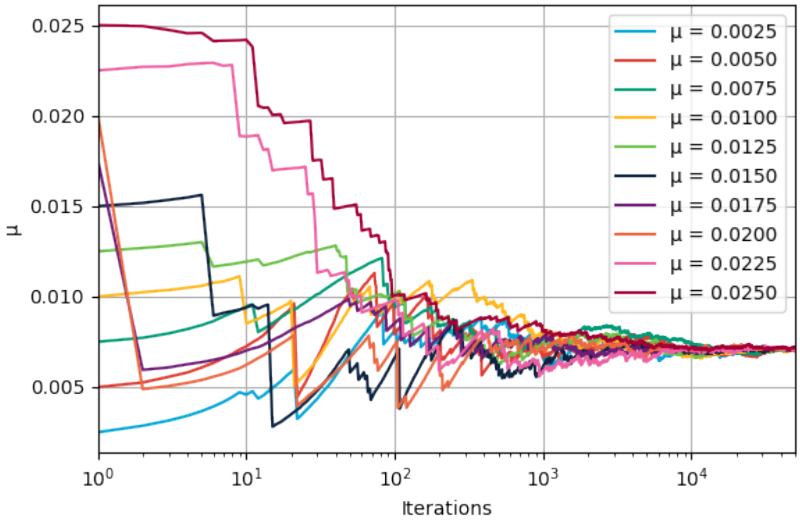


Figure 5.7: Robbins-Monro optimization for the constant absolute margin model for several initial guesses. Plot of the evolution of the estimate for the parameter μ, for which the combination of profit and shortage is optimized. Convergence generally becomes slow. The results indicate an optimum value for μ around 0.008.

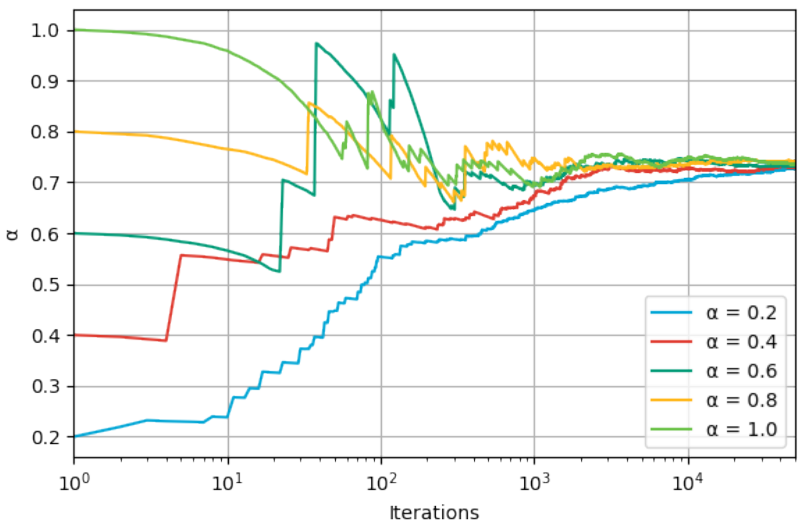


Figure 5.8: Robbins-Monro optimization for the constant relative margin model for several initial guesses. Plot of the evolution of the estimate for the parameter α, for which the combination of profit and shortage is optimized. Convergence generally becomes slow. The results indicate an optimum value for α between 0.6 and 0.8.

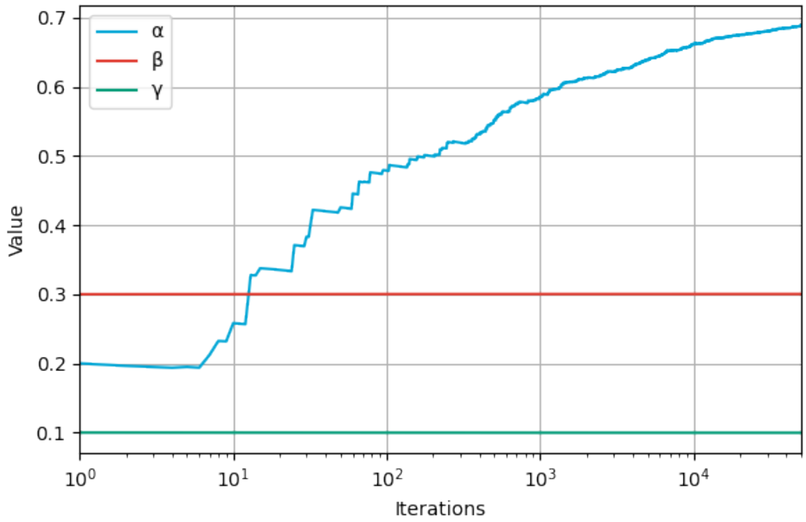
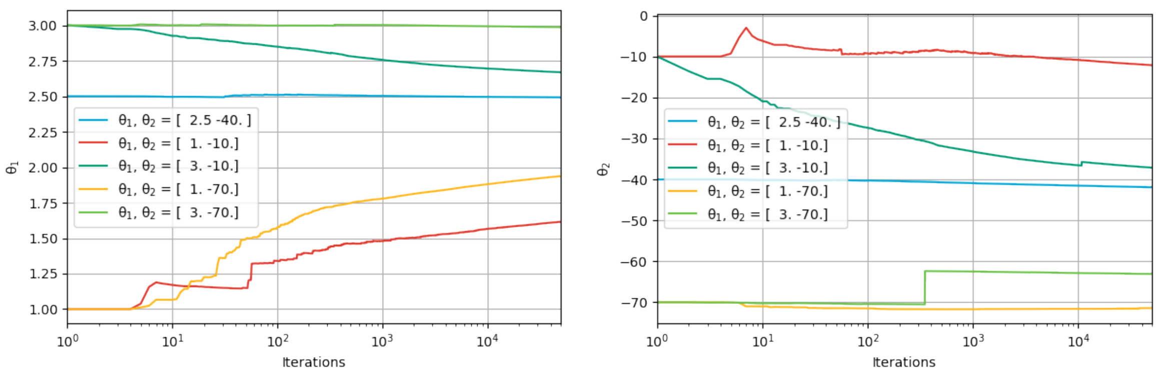


Figure 5.9: Robbins-Monro optimization for the asymmetric pass-through model for a single set of initial guesses. Plot of the evolution of the parameters of the asymmetric model for a particular set of starting values. α behaves similarly to the constant fraction model, while the values of the other parameters do not change much, indicating that their influence is minimal.



(a) θ_1

(b) θ_2

Figure 5.10: Robbins-Monro optimization for the volume dependent model for several pairs of initial guesses. Plots of the evolution of the parameters θ_1 (left) and θ_2 (right) of the volume dependent model for five sets of starting values. The plots show no clear convergence, but the initial guess $(\theta_1, \theta_2) = (2.5, -40)$ seems rather stable.

5.4 Risk Analysis

Using the obtained models for the market rate and the deposit volume, along with the optimized deposit rate model for a constant absolute margin, a risk analysis was conducted for the deposit volume over a period of 10 years. To this end, several paths of the volume evolution were simulated using the obtained term structure model and deposit volume model. For the deposit rate, the optimal constant absolute margin model, with $\mu = 0.008$, was used.

The random process in the term structure model is Brownian motion, while for the random process in the deposit volume model, the residuals that were obtained from the deposit volume model calculations are sampled uniformly. This can be done, since the residuals of the deposit volume model are not serially correlated. For the risk analysis, the volume evolution paths were conditioned on the quantities set out in Equation 4.34. The results presented here are conditioned on the minimum of the market rate being in the lowest quantiles.

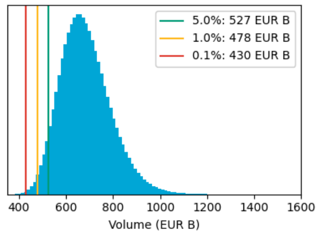
This section presents the results for the final volume, minimum volume and maximum volume for a period of 10 years. Both unconditioned, and conditioned on the minimum of the market rate over the simulated period being in the lowest 0.1%, 1% and 5% quantiles, which are -43bp, -23bp and -2bp respectively. These results can be found in Figure 5.11, Figure 5.12, Figure 5.13, and Figure 5.14 for the lowest 0.1%, 1% and 5% quantiles respectively. In all these figures, sub-figure 'a' shows the distribution of the volume at the end of the considered period, while sub-figures 'b' and 'c' show the distributions of the minimum and maximum attained volume over the considered time period respectively. The same analysis was carried out for a period of one year and five years. The results for these periods can be found in Appendix C.

For this risk analysis, it is important to look at differences between the distributions for different conditionings. In order to facilitate this, the plots for the different quantiles are placed in a grid, where the scale of the x-axis is the same for all plots in a column. In this way comparison of the different quantiles can be done visually. These quantiles of the volume distributions are plotted as vertical lines, where the red line denotes the 0.1% quantile, the yellow line denotes the 1% quantile and the green line denotes the 5% quantile. In each of the figures, the value of the quantiles is included in the legend of the figure.

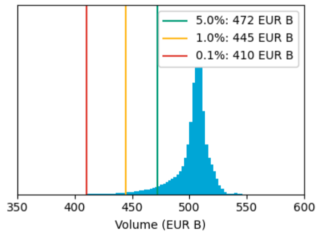
From the quantiles in the figures Figures 5.11b, 5.12b, 5.13b and 5.14b it can be seen that the quantiles of the distribution increase for stricter conditionings. This indicates that a low interest rate environment does, somewhat surprisingly, not result in lower deposit volumes. This can be explained by the spread between the market rate and the deposit rate being small when the market rate is close to zero, or even negative when the market rate is below zero, due to the ZLB on the deposit rate. In both situations, saving and receiving the deposit rate becomes more attractive to customers, which would explain why the deposit volume does not decrease.

Similar results can be seen in the figures for the final volume in Figures 5.11a, 5.12a, 5.13a and 5.14a and for the maximum volume in Figures 5.11c, 5.12c, 5.13c and 5.14c. Additionally, when the shapes of the distributions are examined, there seems to be little to no difference in the shape of the distributions for the different conditioning. The distribution of the volume after the considered period, as well as the distribution of the maximum volume over the considered period, are shaped like a bell-curve, but extend further towards the right. This is similar to a log-normal distribution, which would be expected if the residuals of the deposit volume were normally distributed and the dependence on the parameters is only linear. While the residuals of the differences in the log-volume do not follow a normal distribution, due to the central limit theorem, their sum will approach a normal distribution, which explains the log-normal like shape of the resulting graph.

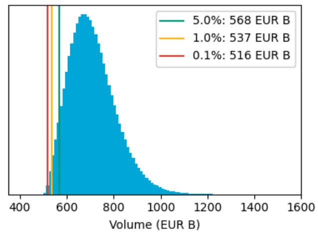
The distributions of the minimum volume also have similar shapes. These distributions have a large peak around 500 EUR B, and extend towards the left side, where the lowest quantiles lie reasonably far from the peak. The peak is because the starting volume lies around 500 EUR B. The minimum of the volume over the whole timeframe cannot exceed the initial volume. This, combined with the general growth of the deposit volume results in a peak at that value. Since there is a significant difference between the quantiles in Figures 5.11b, 5.12b, 5.13b and 5.14b and Figures 5.11a, 5.12a, 5.13a and 5.14a, it can be deduced that the minimum volume in the sample is regularly attained somewhere in the considered period, rather than at the end.



(a) Distribution of the final volume

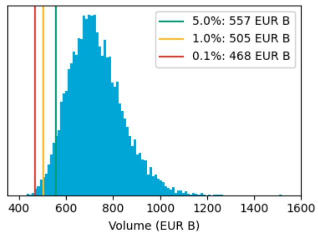


(b) Distribution of the minimum volume

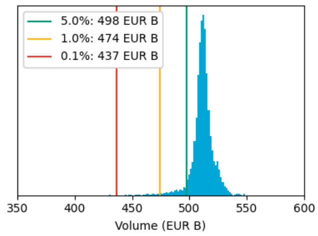


(c) Distribution of the maximum volume

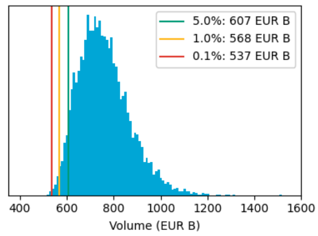
Figure 5.11: Deposit volume distributions for a ten-year period. Plots of the simulation results for the final volume after ten years, the minimum volume over ten years and the maximum volume over ten years.



(a) Distribution of the final volume

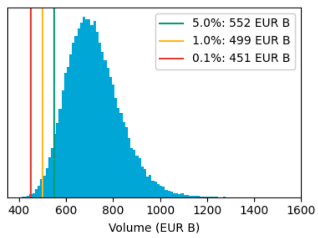


(b) Distribution of the minimum volume

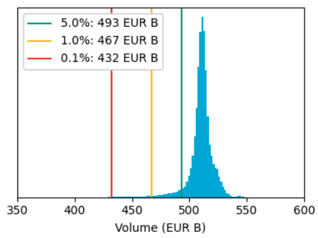


(c) Distribution of the maximum volume

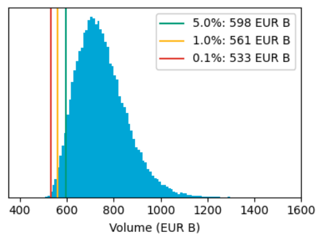
Figure 5.12: Deposit volume distributions for a ten-year period, conditioned on the minimum market rate being in the lowest 0.1%-quantile. Plots of the simulation results for the final volume after ten years, the minimum volume over ten years and the maximum volume over ten years.



(a) Distribution of the final volume

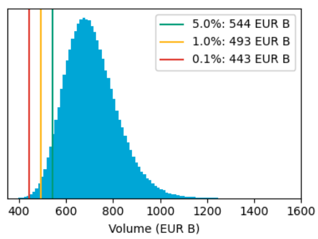


(b) Distribution of the minimum volume

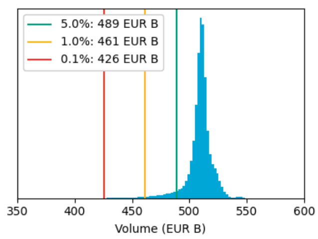


(c) Distribution of the maximum volume

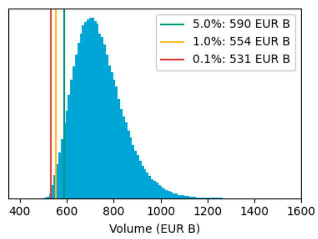
Figure 5.13: Deposit volume distributions for a ten-year period, conditioned on the minimum market rate being in the lowest 1%-quantile. Plots of the simulation results for the final volume after ten years, the minimum volume over ten years and the maximum volume over ten years.



(a) Distribution of the final volume



(b) Distribution of the minimum volume



(c) Distribution of the maximum volume

Figure 5.14: Deposit volume distributions for a ten-year period, conditioned on the minimum market rate being in the lowest 5%-quantile. Plots of the simulation results for the final volume after ten years, the minimum volume over ten years and the maximum volume over ten years.

5.5 Deposit Rate Model Comparison

Finally, the different optimal deposit rate models are compared on their performance in different interest rate scenarios. The parameters used for the model are those that were obtained from the deposit rate model optimization. For the constant absolute margin model, this is $\mu = 0.008$, for the constant relative margin model, this is $\alpha = 0.73$, and for the volume dependent model, the parameters that were used are $\theta_1 = 2.5$ and $\theta_2 = -45$.

These models were compared by simulating 100 market rate scenarios. For each of these scenarios, and each of the deposit rate models, the deposit volume and deposit rate were simulated 10,000 times. In this way, the different models can be compared on equal market rate scenarios, and the large number of simulations allows for the calculation of quantiles of the minimum volume, the profit, shortage and objective. In this way, the models can be compared not only on their expected performance, but also in worst case scenarios.

The results for the comparison of the different models are presented in Figures 5.15 to 5.18. Each figure is structured the same, with sub-figure 'a' containing the results for the expected value of the different quantities, while sub-figures 'b', 'c' and 'd' contain the results for the lowest 5%, 1% and 0.1% respectively. All comparisons here are over a period of 10 years, results for a period of 5 years and 1 year can be found in Appendix D. On the x-axis of the figure, the mean market rate over the whole 10-year period is set out as the dependent variable.

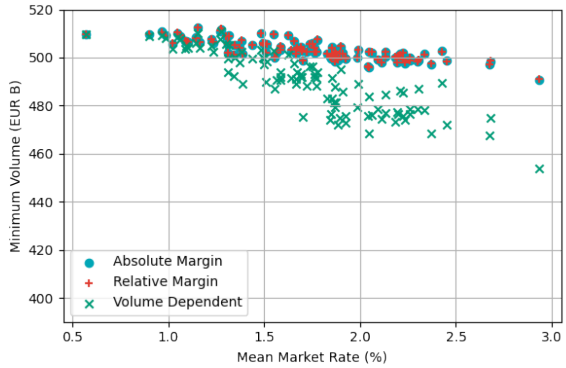
Firstly, the minimum deposit volume, reached at some time is the considered period of 10 years is compared between the different models in Figure 5.15. From all figures here, it can be seen that the constant absolute margin and constant relative margin model perform very similar. The performance for the volume dependent model is significantly different, Figure 5.15a shows that the expected minimum deposit volume is lower for this model than for the other two, while for the most restrictive quantiles, the minimum volume for this model is similar, or even larger than for the other two models, as can be inferred from Figure 5.15d. This indicates that this model aims to keep the volume closer to the target volume, which is below the starting volume, but also makes sure that the volume does not drop very far below this target. This can be seen from the limited decrease in the values attained for the volume dependent model, when decreasing the quantile.

The second set of plots (Figure 5.16) contains a comparison of the profit (Equation 4.20) for the different models, where both the expected values and the quantiles of these are compared. Again, the performance of the constant absolute margin model and constant relative margin model is similar, while the expected profit for the volume dependent model is higher than the expected profit for the models depending only on the market rate, for the more restrictive quantiles and especially high interest rate scenarios, the profit for the volume dependent model is lower than for the other two models, as can be seen in Figures 5.16c and 5.16d. In the latter, even negative values for the profit are reached, meaning that the market rate is below the deposit rate in a significant part of some simulations. In all figures, there is a drop in the profit for the lower market rate scenarios. This drop is probably caused by the ZLB minimizing the difference between the deposit rate and market rate when the latter is close to zero.

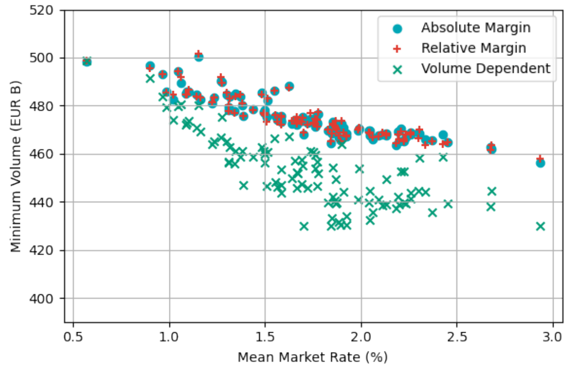
Additionally, a comparison is made on the shortage for each of the models in Figure 5.17. Again, both the expected value, and the 5%-, 1%- and 0.1%-quantiles are compared in Figures 5.17a to 5.17d respectively. From these plots, it can be seen that the volume dependent model outperforms the other two models when looking at the expectation (Figure 5.17a) and lowest two quantiles, as there, the shortages for the volume dependent model lie quite consistently above the shortages for the other two models. For the 5%-quantile, this is not as clear and the models seem to perform similarly.

Finally, a comparison of the objective function value is presented in Figure 5.18. This is the quantity that is used in the optimization of all models. The results for the expectation (Figure 5.18a) show again that the expected profit for the volume dependent model is higher than for the other two models. The influence of the shortage here can be seen from the decrease in objective function value in high market rate scenarios. When looking at the quantiles in Figures 5.18b to 5.18d, it can be seen that the influence of the shortage dominates, as the graphs are quite similar to those in Figure 5.17, where only the shortage is considered. This indicates that in these scenarios, the optimal deposit rate models, and especially the ones only depending on the market rate, do not keep the volume above the target volume. It is however clear from this comparison that the volume dependent model outperforms the non-volume dependent models. In all sub-figures of Figure 5.18, the volume dependent objective values generally lie above the objective values for the other two models.

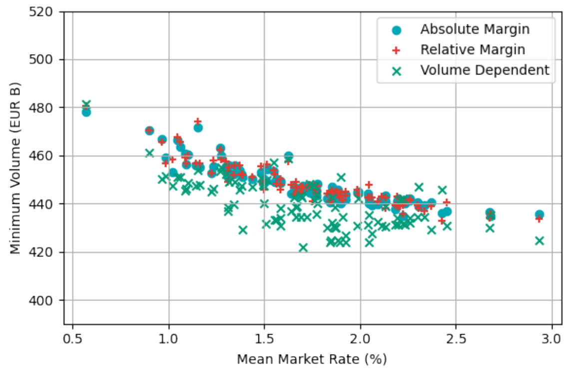
An important side note with this analysis, is that the historical data sample contains mainly market



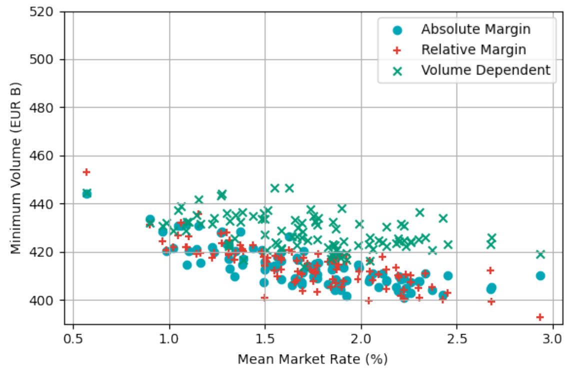
(a) Expected



(b) 5%-quantile



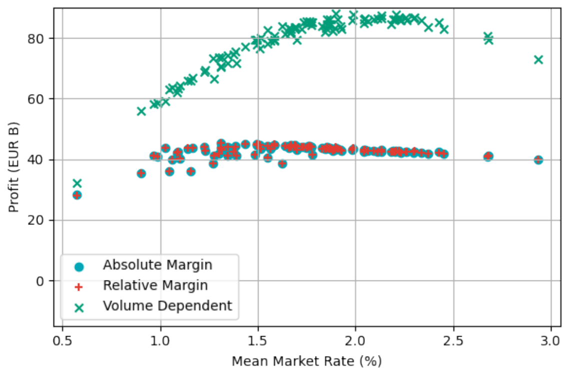
(c) 1%-quantile



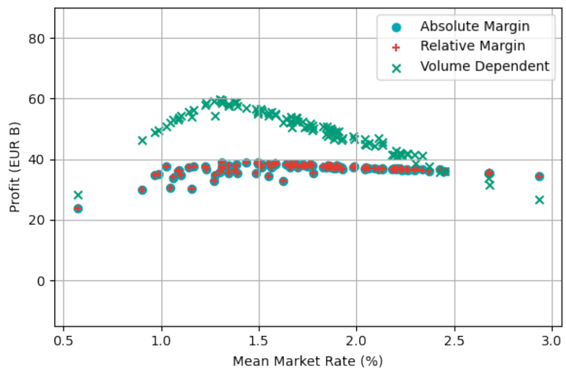
(d) 0.1%-quantile

Figure 5.15: Comparison of the different optimal deposit rate models on the minimum volume attained in a period of 10 years. Scatter plots comparing the expected minimum volume and lowest quantiles of this minimum volume for the three optimized deposit rate models for 100 market rate scenarios over a 10-year period.

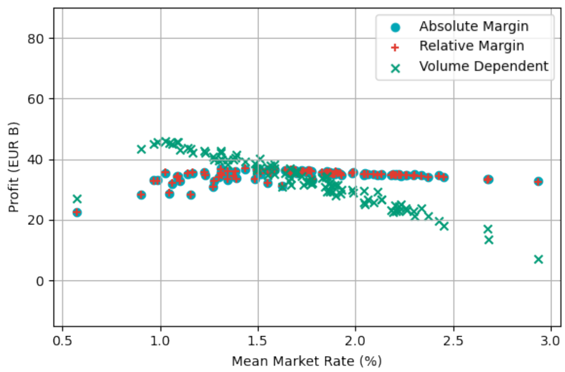
rates close to zero, while the simulated market rates lie significantly higher than this. The volume evolution that is predicted by the model is thus somewhat uncertain, and additional effects, that are not in the historical data might present themselves when market rates are around these levels. Additionally, the deposit rates for the volume dependent model can exceed the market rate, even when the market rate is positive. This is again not observed in the historical data sample, so the evolution of the deposit volume then might also be not as in the model. However, a similar effect, namely the deposit volume increasing, will probably be accomplished, since the goal of such a deposit rate would be to increase the volume and the deposit rate is then an especially attractive option.



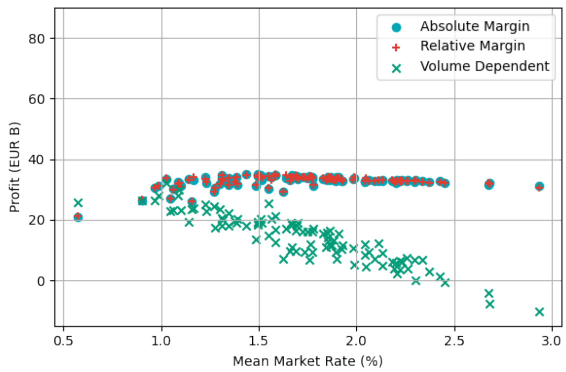
(a) Expected



(b) 5%-quantile

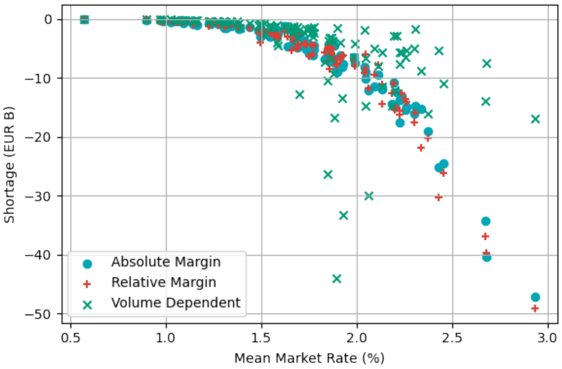


(c) 1%-quantile

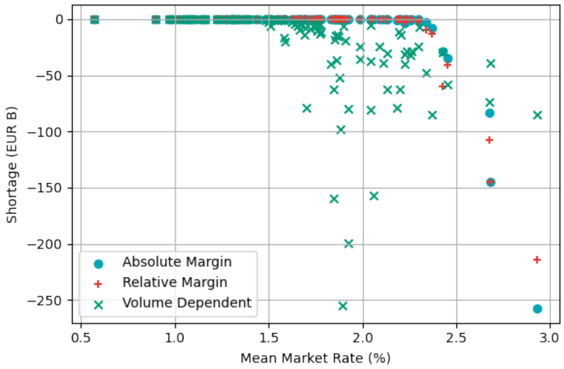


(d) 0.1%-quantile

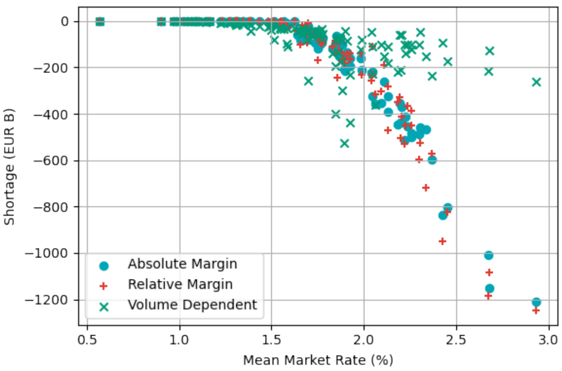
Figure 5.16: Comparison of the different optimal deposit rate models on the profit over a period of 10 years. Scatter plots comparing the expected profit and lowest quantiles of this for the three optimized deposit rate models for 100 market rate scenarios over a 10-year period.



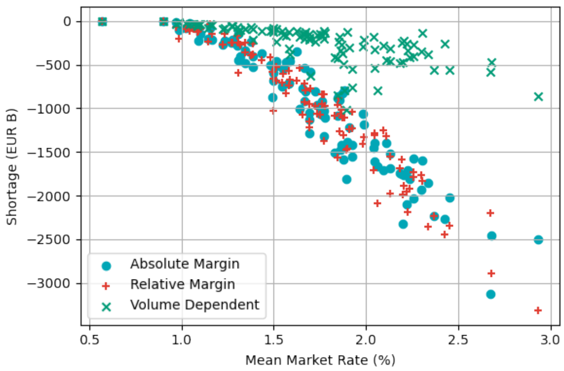
(a) Expected



(b) 5%-quantile

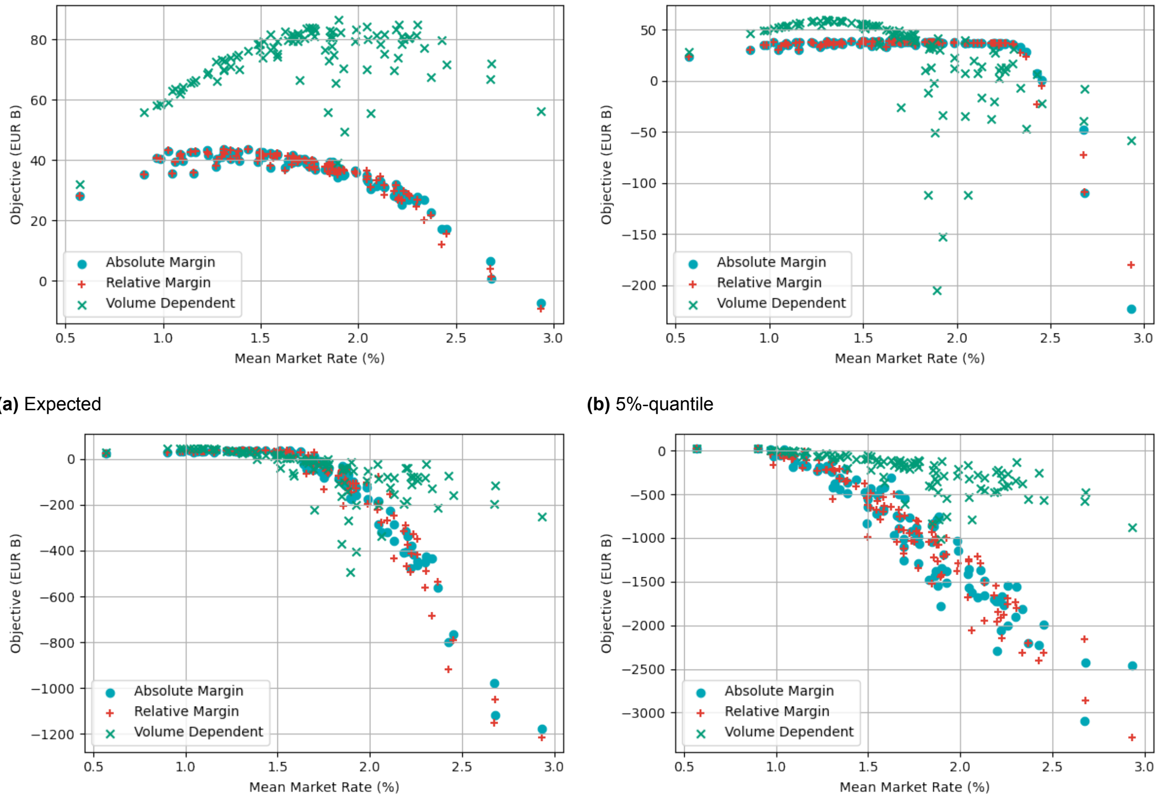


(c) 1%-quantile



(d) 0.1%-quantile

Figure 5.17: Comparison of the different optimal deposit rate models on the shortage over a period of 10 years. Scatter plots comparing the expected shortage and lowest quantiles of this for the three optimized deposit rate models for 100 market rate scenarios over a 10-year period.



(a) Expected

(b) 5%-quantile

(c) 1%-quantile

(d) 0.1%-quantile

Figure 5.18: Comparison of the different optimal deposit rate models on the objective over a period of 10 years. Scatter plots comparing the expected objective, sum of profit and shortage, and lowest quantiles of this for the three optimized deposit rate models for 100 market rate scenarios over a 10-year period.

6 Conclusion

With the analyses from the previous chapter, some conclusions on the risks in NMDs and the optimal setting of the deposit rate can be drawn. This work contributes to the literature by describing a way of risk analysis that remains in the model, more explicitly, no external 'shocks' are added to the model, but the edge cases of the model are explored by conditioning on extreme quantiles of distributions of variables, that enter as independent variables in another sub-model. Additionally, a method for finding an optimal deposit rate model is described and carried out.

Firstly, it can be concluded that purely from a volume perspective, the market rate scenarios that result in the lowest deposit volume are not the scenarios where the deposit rate is the lowest. Rather, scenarios with moderate market rate seem to yield the lowest deposit volumes. This is caused by, on the one hand the ZLB on the deposit rate, which means that when the market rate is close to zero, the deposit rate is not much lower (in absolute sense), which results in savings being comparable in yield to bonds, and by proxy, other alternative investments (or even better when the market rate dips below zero). High market rates on the other hand result in saving becoming relatively more attractive (when considering a fixed margin deposit rate model), since the deposit rate is a larger and larger fraction of the market rate in that case. With moderate market rates, the attractiveness of saving is relatively the lowest. An important point of discussion here, is that the considered period does not contain periods of very high market rates, and mostly consists of periods of low to moderate market rates. The dynamics at high market rates are thus outside the range of the data that was used to calibrate the model, and other characteristics, that do not exist in the considered data, might become apparent.

From the analysis on the deposit rate model optimization, it can be concluded that, if done collectively, keeping the deposit rate at zero is optimal when only optimizing for the profit over a ten-year period. The main sidenote with this, is that the model essentially considers the whole banking sector in the Netherlands as a single bank. So while, for the banking sector as a whole this conclusion may hold, if only a single bank keeps its rate low, depositors may move away from this bank towards another bank in the Dutch sector. These migrations are not captured in the data, since only have aggregate data was used, and therefore the model does not account for this.

If the optimization is only for the volume staying above a target volume, where a penalty is introduced whenever the volume is below this target, it was found that the optimal deposit rate is to set it as high as possible, and therefore also higher than the market rate. This makes sense, since this minimizes the probability of the volume going below the target. Of course, implementing such a deposit rate is nonsensical, since this would mean that a bank would always lose money on its NMD portfolio.

Optimizing over both these quantities simultaneously, yields optimal values that are more sensical. One of the quantities that is optimized for, is assigned a weight, which determines the relative importance of the two quantities. In practice, this is an input of the optimization procedure. It is found that the algorithm converges to an optimal value, but that the convergence is rather slow. From the results of this thesis, a good approach seems to run the algorithm for a set amount of iterations, and then restart the algorithm with the value obtained from the previous optimization as a starting value.

For a choice of equal weighting and choice of target volume of 0.8 times the starting volume, increased with the market rate. The optimal constant margin $\mu = 0.008$ was found and the optimal constant fraction of $\alpha = 0.73$ was found. As with the risk analysis, the convergence of the model is expected to improve when data of an individual bank is available. This would therefore be a suggestion for further research. Another open point is that the optimization in this thesis was done over a predetermined set of deposit rate models. Optimizing over a possible deposit rate model space, after properly defining such a space, could potentially result in a more optimal model, in the sense that the expected objective value for such a model is higher than for all three model types considered in this thesis. Additionally, the considered models do not account for several of the deposit rate characteristics, such as it being set at round values. It would be interesting to see whether such a restriction has an effect on either the risk analysis or the optimal model.

When comparing these models to a model that takes the size of the volume, compared to some target volume into account, a model was found that outperforms both models that only consider the current market rate. This is accomplished by keeping the volume around the target and setting the deposit rate to zero whenever the volume is sufficiently over the target volume. With this model, however, it should be noted that the deposit rate can be set very high, and is not necessarily below the market rate, even when the market rate is positive. Since similar situations have not been observed in the data, the behaviour of the volume in such scenarios could be influenced by other, not observed factors.

In practice, the deposit rate model optimization and the optimal models that can be obtained using this method can serve as a way to challenge existing methods for setting the deposit rate. When the objective of a bank is clear and can be expressed mathematically, this method can be used to find an optimal model of some form. When this optimal model yields a deposit rate that significantly differs from the deposit rate that is set out with current methods, this might be a reason to consider adjusting the deposit rate more towards the, according to the model, optimal value.

The results obtained in this thesis correspond well with those in the concept report of the ACM (2024). Both this thesis and the ACM concept report point towards little inclination of depositors to switch banks, allowing banks to set low deposit rates. Implementation of the suggestions in the ACM concept report might result in more depositors switching banks, more competition among banks and thus higher optimal deposit rates.

Extension to other markets, and calibration to more granular, individual bank data are natural next steps for subsequent research. The approach in this thesis is still applicable in both scenarios. While it is expected that the results for other markets are mostly similar, especially for other European nations, calibration to more granular and individual bank data could uncover effects that have remained unobserved in this thesis, such as the movement of consumers between banks within the national market.

A

Bond price is risk neutral for all Vasicek and CIR models

In this appendix, we will prove that all Vacisek and CIR models give rise to a risk neutral bond price. This is immediate by the construction of a Doob martingale. It is however still interesting to derive the SDEs governing the evolution of the bond prices. We start with the definition of the zero coupon bond price.

Definition A.1. The price $P(t,T)$ of a Zero Coupon Bond (ZCB) maturing at time T, at time $t \leq T$ is given by

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(s)ds}\right]$$

where $r(t)$ is the short rate at time t.

We now state the following lemma defining the Doob martingale.

Lemma A.2 (Doob martingale)**.** *Let X be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_t \subset \mathcal{F}$ denote the filtration up till time t. Then the sequence of random variables $Y_t = \mathbb{E}[X|\mathcal{F}_t]$ defines a martingale. That is, for all $s \leq t$, we have $\mathbb{E}[Y_t|Y_s] = \mathbb{E}[Y_t|\mathcal{F}_s] = Y_s$.*

Proof. See Doob (1940, Theorem 3.7). □

Applying Lemma A.2 to the discounted bond price $(P(t,T)/D(t)$ with $D(t) = \exp\left[\int_0^t -r(u)du\right])$, we can conclude that the discounted bond price is a martingale and thus risk neutral.

We now derive the same result, but additionally the governing SDEs for both the Vasicek (1977) and the Cox et al. (1985) model.

A.1 Vasicek (1977) model

The Vasicek (1977) model belongs to the class of affine term structure models. This means that the price of a Zero Coupon Bond (ZCB) is explicit and of the form:

$$P(t,T) = A(t,T)e^{B(t,T)r(t)}, \tag{A.1}$$

with $P(t,T)$ being the price of a bond at time t, maturing at time T and $r(t)$ being the short rate at time t. In case of the Vasicek model, we have:

$$\begin{aligned} A(t,T) &= \exp\left\{\left(\theta - \frac{\sigma^2}{2a^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4a}B(t,T)^2\right\} \\ B(t,T) &= \frac{1}{a}\left[1 - e^{-a(T-t)}\right], \end{aligned} \tag{A.2}$$

where a, θ and σ are the parameters of the Vasicek (1977) model, as defined in Equation 2.5.

We first state the definition of an arbitrage free product and state Itô's lemma, which we will use in the subsequent proof.

Definition A.3. The price of a product or derivative S is said to be *risk neutral* if the SDE governing S is of the form:

$$dS(t) = r(t)Sdt + \eta(t)dW(t),$$

where $r(t)$ is the risk-free rate at time t and $\eta(t)$ is some function.

Lemma A.4 (Itô's Formula)**.** *Given an SDE*

$$dX = \bar{\mu}dt + \bar{\sigma}dW(t),$$

and a smooth function $f(t, X)$. The SDE for the function $f(t, X)$ is given by

$$df(t,X) = \left[f_t(t,X) + \bar{\mu}f_X(t,X) + \frac{\bar{\sigma}^2}{2}f_{XX}(t,X)\right]dt + \bar{\sigma}f_X(t,X)dW(t),$$

where the subscripts denote the partial derivative with respect to that variable.

Proof. See, for instance (Øksendal, 1998) □

We will now use Lemma A.4 to prove that the price of a ZCB in the Vasicek model is always risk neutral.

Theorem A.5. *The price of a Zero Coupon Bond in the Vasicek (1977) model is risk neutral for all values of a, θ and σ, with $a \neq 0$.*

Proof. We will use Itô's Formula with the smooth function given by Equation A.1 with $X = r$ and Equation A.2. First, we notice that A and B do not depend on r. Furthermore, since $a \neq 0$, both A and B exist. We can now write down the partial derivatives of the ZCB price.

$$P_t = (A_t - AB_t)e^{-Br},$$

$$P_r = -ABe^{-Br},$$

$$P_{rr} = AB^2e^{-Br}.$$

Now, using that $\bar{\mu} = -a(\theta - r)$ and $\bar{\sigma} = \sigma$, we can write

$$dP = (A_t - AB_tr + a(r-\theta)AB + \frac{\sigma^2}{2}AB^2)e^{-Br}dt - \sigma ABe^{-Br}dW. \tag{A.3}$$

We now work out the derivatives of A and B with respect to t. We obtain

$$A_t = \left[\left(\theta - \frac{\sigma^2}{2a^2}\right)(B_t + 1) - \frac{\sigma^2}{2a}BB_t\right]A,$$

$$B_t = -e^{-a(T-t)}.$$

Substituting these, we obtain

$$dP = \left[\left(\theta - \frac{\sigma^2}{2a^2}\right)A + \left(\theta - \frac{\sigma^2}{2a^2}\right)AB_t - \frac{\sigma^2}{2a}ABB_t - AB_tr + a(r-\theta)AB + \frac{\sigma^2}{2}AB^2\right]e^{-Br}dt - ABe^{-Br}dW$$

$$
\begin{aligned}
dP = \quad & \left[\theta - \frac{\sigma^2}{2a^2} + \left(\theta - \frac{\sigma^2}{2a^2}\right)B_t - \frac{\sigma^2}{2a}BB_t - B_t r + a(r-\theta)B + \frac{\sigma^2}{2}B^2\right]Pdt - BPdW \\
= \quad & \left[\theta - \frac{\sigma^2}{2a^2} - \left(\theta - \frac{\sigma^2}{2a^2}\right)e^{-a(T-t)} + \frac{\sigma^2}{2a^2}\left[1-e^{-a(T-t)}\right]e^{-a(T-t)} + re^{-a(T-t)} + \right. \\
& \left. (r-\theta)\left[1-e^{-a(T-t)}\right] + \frac{\sigma^2}{2a^2}\left[1-2e^{-a(T-t)}+e^{-2a(T-t)}\right]\right]Pdt - BPdW \\
= \quad & \left[\big(\underbrace{1-e^{-a(T-t)}-1+e^{-a(T-t)}}_{=0}\big)\theta + \right. \\
& \big(\underbrace{e^{-a(T-t)}-1+e^{-a(T-t)}-e^{-2a(T-t)}+1-2e^{-a(T-t)}+e^{-2a(T-t)}}_{=0}\big)\frac{\sigma^2}{2a^2} + \\
& \left.\big(\underbrace{e^{-a(T-t)}+1-e^{-a(T-t)}}_{=1}\big)r\right]Pdt - BPdW \\
= \quad & rPdt - BPdW
\end{aligned}
$$

We conclude that the ZCB price in the Vasicek (1977) model is risk neutral for all a, θ, σ, with $a \neq 0$. □

A.2 Cox et al. (1985) model

We now demonstrate the same for the CIR model. This result is stated by Brigo and Mercurio (2007), but a derivation is omitted. We include this here for completeness. We proceed in the same fashion as for the Vasicek model. The CIR model is also an ATSM. In this case, we have

$$
\begin{aligned}
A(t,T) &= \left[\frac{2h\exp\{(a+h)(T-t)/2\}}{2h+(a+h)(\exp\{(T-t)h\}-1)}\right]^{2a\theta/\sigma^2}, \\
B(t,T) &= \frac{2(\exp\{(T-t)h\}-1)}{2h+(a+h)(\exp\{(T-t)h\}-1)}, \\
h &= \sqrt{a^2+2\sigma^2}.
\end{aligned} \tag{A.4}
$$

We will now prove the following theorem:

Theorem A.6. *The price of a Zero Coupon Bond in the Cox et al. (1985) Model is risk neutral for all values of a, θ and σ, with $a \neq 0$.*

Proof. The proof of this theorem follows the same structure as Theorem A.5. We start form Equation A.3 (with $\bar{\sigma} = \sigma\sqrt{r}$ instead) and calculate the derivatives of Equation A.4. We obtain:

$$A_t = -2a\theta\frac{1+\exp\{(T-t)h\}}{N}A,$$

$$B_t = -\frac{4h^2\exp\{(T-t)h\}}{N^2},$$

$$N = h - a + (h+a)\exp\{(T-t)h\}$$

Substituting these yields

$$
\begin{aligned}
dP = \quad & \left(-2a\theta\frac{1+\exp\{(T-t)h\}}{N}A + \frac{4h^2\exp\{(T-t)h\}}{N^2}Ar + a(r-\theta)BA + \frac{\sigma^2 r}{2}B^2A\right)e^{-Br}dt - \\
& \sigma\sqrt{r}BAe^{-Br}dW
\end{aligned}
$$

$$
\begin{aligned}
dP = \; & \left(-2a\theta \frac{1+\exp\{(T-t)h\}}{N} + \frac{4h^2\exp\{(T-t)h\}}{N^2} r + a(r-\theta)B + \frac{\sigma^2 r}{2}B^2 \right) Pdt - \\
& \sigma\sqrt{r}BPdW \\
= \; & \left(-2a\theta \frac{1+\exp\{(T-t)h\}}{N} + \frac{4h^2\exp\{(T-t)h\}}{N^2} r + \right. \\
& \left. a(r-\theta)\frac{2(\exp\{(T-t)h\}-1)}{N} + \frac{2\sigma^2\exp\{2(T-t)h\} - 4\sigma^2\exp\{(T-t)h\} + 2\sigma^2}{N^2} r \right) Pdt - \\
& \sigma\sqrt{r}BPdW \\
= \; & \left(\frac{4h^2\exp\{(T-t)h\}}{N^2} r - \frac{2a + 2a\exp\{(T-t)h\}}{N}\theta + \frac{2a\exp\{(T-t)h\} - 2a}{N} r - \right. \\
& \left. \frac{2a\exp\{(T-t)h\} - 2a}{N}\theta + \frac{2\sigma^2\exp\{2(T-t)h\} - 4\sigma^2\exp\{(T-t)h\} + 2\sigma^2}{N^2} r \right) Pdt - \\
& \sigma\sqrt{r}BPdW \\
= \; & \left(\frac{4h^2\exp\{(T-t)h\} + 2\sigma^2\exp\{2(T-t)h\} - 4\sigma^2\exp\{(T-t)h\} + 2\sigma^2}{N^2} r + \right. \\
& \left. \frac{2a\exp\{(T-t)h\} - 2a}{N} r - \frac{2a - 2a\exp\{(T-t)h\} + 2a\exp\{(T-t)h\} - 2a}{N}\theta \right) Pdt - \\
& \sigma\sqrt{r}BPdW \\
= \; & \left(\frac{4h^2\exp\{(T-t)h\} + 2\sigma^2\exp\{2(T-t)h\} - 4\sigma^2\exp\{(T-t)h\} + 2\sigma^2}{N^2} r + \right. \\
& \frac{2a^2 - 2ah - 4a^2\exp\{(T-t)h\} + 2ah\exp\{2(T-t)h\} + 2a^2\exp\{2(T-t)h\}}{N^2} r - \\
& \left. \underbrace{\frac{2a - 2a\exp\{(T-t)h\} + 2a\exp\{(T-t)h\} - 2a}{N}}_{=0} \theta \right) Pdt - \sigma\sqrt{r}BPdW \\
= \; & \left(\frac{4h^2\exp\{(T-t)h\} + 2\sigma^2\exp\{2(T-t)h\} - 4\sigma^2\exp\{(T-t)h\} + 2\sigma^2}{h^2 - 2ah + a^2 + 4\sigma^2\exp\{(T-t)h\} + (h^2 + 2ah + a^2)\exp\{2(T-t)h\}} r + \right. \\
& \left. \frac{2a^2 - 2ah - 4a^2\exp\{(T-t)h\} + 2ah\exp\{2(T-t)h\} + 2a^2\exp\{2(T-t)h\}}{h^2 - 2ah + a^2 + 4\sigma^2\exp\{(T-t)h\} + (h^2 + 2ah + a^2)\exp\{2(T-t)h\}} r \right) Pdt - \\
& \sigma\sqrt{r}BPdW \\
= \; & \left(\frac{2\sigma^2 + 2a^2 - 2ah}{h^2 - 2ah + a^2 + 4\sigma^2\exp\{(T-t)h\} + (h^2 + 2ah + a^2)\exp\{2(T-t)h\}} r + \right. \\
& \frac{(4h^2 - 4\sigma^2 - 4a^2)\exp\{(T-t)h\}}{h^2 - 2ah + a^2 + 4\sigma^2\exp\{(T-t)h\} + (h^2 + 2ah + a^2)\exp\{2(T-t)h\}} r + \\
& \left. \frac{(2ah + 2a^2 + 2\sigma^2)\exp\{2(T-t)h\}}{h^2 - 2ah + a^2 + 4\sigma^2\exp\{(T-t)h\} + (h^2 + 2ah + a^2)\exp\{2(T-t)h\}} r \right) Pdt - \\
& \sigma\sqrt{r}BPdW \\
= \; & \left(\frac{h^2 - a^2 + 2a^2 - 2ah}{h^2 - 2ah + a^2 + 4\sigma^2\exp\{(T-t)h\} + (h^2 + 2ah + a^2)\exp\{2(T-t)h\}} r + \right. \\
& \frac{(4a^2 + 8\sigma^2 - 4\sigma^2 - 4a^2)\exp\{(T-t)h\}}{h^2 - 2ah + a^2 + 4\sigma^2\exp\{(T-t)h\} + (h^2 + 2ah + a^2)\exp\{2(T-t)h\}} r + \\
& \left. \frac{(2ah + 2a^2 + h^2 - a^2)\exp\{2(T-t)h\}}{h^2 - 2ah + a^2 + 4\sigma^2\exp\{(T-t)h\} + (h^2 + 2ah + a^2)\exp\{2(T-t)h\}} r \right) Pdt - \\
& \sigma\sqrt{r}BPdW
\end{aligned}
$$

$$\begin{aligned} dP &= \left(\underbrace{\frac{h^2+a^2-2ah+4\sigma^2\exp\{(T-t)h\}(2ah+a^2+h^2)\exp\{2(T-t)h\}}{h^2-2ah+a^2+4\sigma^2\exp\{(T-t)h\}+\left(h^2+2ah+a^2\right)\exp\{2(T-t)h\}}}_{=1} r\right)Pdt- \\ &\quad \sigma\sqrt{r}BPdW \\ &= rPdt-\sigma\sqrt{r}BPdW \end{aligned}$$

We conclude that the ZCB price in the Cox et al. (1985) model is risk neutral for all a, θ, σ, with $\sigma \neq 0$. The case with $\sigma = 0$ is clear from the proof for the Vasicek (1977) model. □

B

Proof of the Robbins-Monro algorithm

In this appendix, we will present a proof of the convergence of the Robbins-Monro algorithm (Robbins & Monro, 1951). We state the proof for the convergence of the algorithm for multivariate function, which was first proven by Blum (1954). Ruppert (1985) later provided a proof for a larger class of functions. In the following, we will adapt the proof by Blum (1954) for finding the maximum of an objective function. We will prove the result for the case where the objective function only has one maximum, and no other local maxima.

Definition B.1 (Type $\frac{1}{n}$ sequence (Robbins & Monro, 1951))**.** A sequence $(a_n)_{n\geq 0}$ is of type $\frac{1}{n}$ if there exist two constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that, for all $n \geq 0$,

$$\frac{c_1}{n+1} \leq a_n \leq \frac{c_2}{n+1}.$$

We denote the objective function by $w(\boldsymbol{\theta})$, which is maximal for $\bar{\boldsymbol{\theta}}$. Furthermore, w is an expectation, i.e. $w(\boldsymbol{\theta}) = \mathbb{E}_\chi[W(\boldsymbol{\theta}, \chi)]$. Additionally, we assume that the gradient and the Jacobian with respect to the variable $\boldsymbol{\theta}$ exist on the whole domain. We denote the gradients by $f(\boldsymbol{\theta})$ and $F(\boldsymbol{\theta}, \chi)$ and the Jacobians by $j(\boldsymbol{\theta})$ and $J(\boldsymbol{\theta}, \chi)$. We also define a positive sequence $(a_n)_{n\geq 0}$. Finally, we assume that the following conditions hold:

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n = \infty; \\
&\sum_{n=0}^{\infty} a_n < \infty; \\
&w(\boldsymbol{\theta}) \leq 0 \\
&\forall \epsilon > 0: \qquad \inf_{\epsilon \leq \|\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}\|_2} f(\boldsymbol{\theta})' \mathbb{E}_\chi[F(\boldsymbol{\theta}, \chi)] > 0; \\
&\qquad\qquad\qquad \inf_{\epsilon \leq \|\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}\|_2} |f(\boldsymbol{\theta}) - f(\bar{\boldsymbol{\theta}})| > 0; \\
&\forall a \in \mathbb{R} \qquad \mathbb{E}_\chi[F(\boldsymbol{\theta}, \chi)]' j(\boldsymbol{\theta} + ba\mathbb{E}_\chi[F(\boldsymbol{\theta}, \chi)]) \mathbb{E}_\chi[F(\boldsymbol{\theta}, \chi)] \leq V < \infty.
\end{aligned} \tag{B.1}$$

Theorem B.2 (Convergence of the Robbins-Monro Algorithm for multivariate functions (Blum, 1954, Theorem 1))**.** *Let w, W, f, F, j, J be as defined above and let $(a_n)_{n\geq 0}$ be a sequence, that satisfy Equation B.1. Then, for all values $\boldsymbol{\theta}_0$, the sequence $(\boldsymbol{\theta}_n)_{n\geq 0}$, defined recursively as:*

$$\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n - a_n F(\boldsymbol{\theta}_n, \chi),$$

converges almost surely to $\bar{\boldsymbol{\theta}}$.

Proof. For any real number a, we have, by Taylor's theorem:

$$w(\boldsymbol{\theta} + aF(\boldsymbol{\theta}, \chi)) = w(\boldsymbol{\theta}) - af(\boldsymbol{\theta})'F(\boldsymbol{\theta}, \chi) + \frac{a^2}{2} F(\boldsymbol{\theta}, \chi)' j(\boldsymbol{\theta} - baF(\boldsymbol{\theta}, \chi)) F(\boldsymbol{\theta}, \chi).$$

for some $b \in [0,1]$. If we take expectations on both sides, we find

$$\mathbb{E}[w(\boldsymbol{\theta} + aF(\boldsymbol{\theta},\boldsymbol{\chi}))] = w(\boldsymbol{\theta}) - af(\boldsymbol{\theta})'\mathbb{E}_{\chi}[F(\boldsymbol{\theta},\boldsymbol{\chi})] + \frac{a^2}{2}\mathbb{E}_{\chi}[F(\boldsymbol{\theta},\boldsymbol{\chi})'j(\boldsymbol{\theta} - baF(\boldsymbol{\theta},\boldsymbol{\chi}))F(\boldsymbol{\theta},\boldsymbol{\chi})].$$

Now, we use the sequence $(a_n)_{n\geq 0}$ and set $\boldsymbol{\theta}_0$, and carry out the recursive procedure $\boldsymbol{\theta}_{n+1} = \theta_n - a_n F(\boldsymbol{\theta}_n,\boldsymbol{\chi})$. If we take the conditional expectation of this in light of the expectation above, we find, denoting draws from $W(\boldsymbol{\theta}_n,\boldsymbol{\chi})$ by W_n:

$$\mathbb{E}_{\chi}[W(\theta_{n+1},\chi)|W_0,\dots,W_n] = W_n - a_n f(\boldsymbol{\theta_n})'\mathbb{E}_{\chi}[F(\boldsymbol{\theta_n},\boldsymbol{\chi})|W_0,\dots,W_n] +$$

$$\frac{a^2}{2}\mathbb{E}_{\chi}[F(\boldsymbol{\theta_n},\boldsymbol{\chi})'j(\boldsymbol{\theta_n} - ba_n F(\boldsymbol{\theta_n},\boldsymbol{\chi}))F(\boldsymbol{\theta_n},\boldsymbol{\chi})|W_0,\dots,W_n].$$

From the conditions in Equation B.1, we have $f(\boldsymbol{\theta_n})'\mathbb{E}_{\chi}[F(\boldsymbol{\theta_n},\boldsymbol{\chi})|W_0,\dots,W_n] \geq 0$ and $\mathbb{E}_{\chi}[F(\boldsymbol{\theta_n},\boldsymbol{\chi})'j(\boldsymbol{\theta_n} + baF(\boldsymbol{\theta_n},\boldsymbol{\chi}))F(\boldsymbol{\theta_n},\boldsymbol{\chi})|W_0,\dots,W_n] \leq V$. Both almost surely and for all $n \geq 0$.

From this, we conclude that $\mathbb{E}_{\chi}[W(\boldsymbol{\theta}_{n+1},\boldsymbol{\chi}) - W_n|W_0,\dots,W_n] \leq \frac{a_n^2}{2}V$ almost surely. From Equation B.1, we can assume that $V \geq 0$. Thus, clearly, the series of random variables $\sum_{n=0}^{\infty} W_n$ converges almost surely (by Blum, 1954, Section 3).

If we now take expectations, we find that

$$\mathbb{E}_{\chi[W(\boldsymbol{\theta}_{n+1},\boldsymbol{\chi})]} = f(\boldsymbol{\theta}_0) - \sum_{j=1}^{n} a_j f(\boldsymbol{\theta_j})'\mathbb{E}_{\chi}[F(\boldsymbol{\theta_j},\boldsymbol{\chi})] + \sum_{j=0}^{n}\frac{a_j^2}{2}\mathbb{E}_{\chi}[F(\boldsymbol{\theta},\boldsymbol{\chi})'j(\boldsymbol{\theta} - baF(\boldsymbol{\theta},\boldsymbol{\chi}))F(\boldsymbol{\theta},\boldsymbol{\chi})],$$

with $\mathbb{E}_{\chi[W(\boldsymbol{\theta}_{n+1},\boldsymbol{\chi})]} \geq 0$, $f(\boldsymbol{\theta_j})'\mathbb{E}_{\chi}[F(\boldsymbol{\theta_j},\boldsymbol{\chi})] \leq 0$ and $\mathbb{E}_{\chi}[F(\boldsymbol{\theta},\boldsymbol{\chi})'j(\boldsymbol{\theta} - ba_n F(\boldsymbol{\theta},\boldsymbol{\chi}))F(\boldsymbol{\theta},\boldsymbol{\chi})] \leq V$ for all $n \geq 0$.

Since the second sum converges, is non-negative and the series $\sum_{n=0}^{\infty} a_n$ diverges, we must have that

$$\limsup_{n\to\infty} f(\boldsymbol{\theta_j})'\mathbb{E}_{\chi}[F(\boldsymbol{\theta_j},\boldsymbol{\chi})] = 0,$$

and

$$\liminf_{n\to\infty} |f(\boldsymbol{\theta_j})'\mathbb{E}_{\chi}[F(\boldsymbol{\theta_j},\boldsymbol{\chi})]| = 0.$$

We can now define a sequence $(n_k)_{k\geq 0}$ such that $\lim_{k\to\infty} f(\boldsymbol{\theta_{n_k}})'\mathbb{E}_{\chi}[F(\boldsymbol{\theta_{n_k}},\boldsymbol{\chi})] = 0$. Then this expectation converges to zero in probability and there is a further subsequence that converges almost surely. It now follows that the corresponding subsequence for $\boldsymbol{\theta_n}$ also converges almost surely to $\bar{\boldsymbol{\theta}}$ and, due to continuity of w, the corresponding sequence in $\mathbb{R}$ converges to the value at the maximum. □

We will use sequences of type $\frac{1}{n}$ in our optimization procedure.

In our setup, we cannot be certain that the maximum is unique, or that there are no other unique maxima present. We can also not be sure that all the conditions in Equation B.1 hold. We suspect that the algorithm still converges, as long as the number of maxima is finite. If the number of maxima is not finite, the algorithm could keep diverging and not end up in one of the maxima. We suspect that starting in the neighbourhood of one of the local maxima would yield convergence to that maximum. Using different values for $\boldsymbol{\theta}_0$ would then give the best chance of encountering the true maximum in one of the optimizations.

C

Risk analysis results for different periods

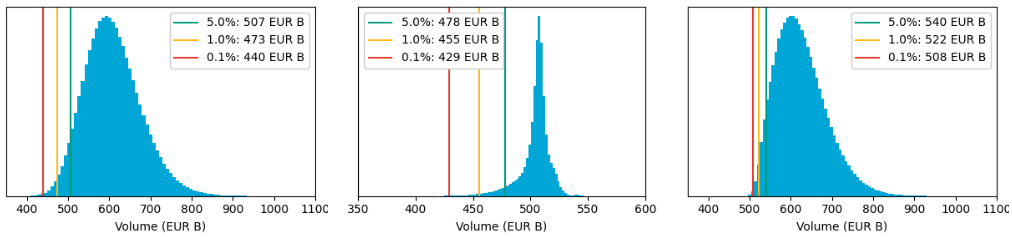
This appendix contains the results for the risk analysis simulation on alternative time periods. It follows the same structure as section 5.4, but considers a shorter period of interest. We consider the periods 5 years and 1 year. For each of the periods, we plot the unconditioned results, as well as the results, conditioned on the minimum of the short rate being in the lowest 0.1%-, 1%-, and 5%-quantiles.

C.1 Five-year period

The unconditional results for the final deposit volume after, the minimum volume over, and the maximum volume over five years can be found, unconditioned, as well as for the minimum of the short rate being in the lowest 0.1%-, 1%-, and 5%-quantiles (-43bp, -22bp, and 0bp respectively) in Figure C.1, Figure C.2, Figure C.3, and Figure C.4 respectively.

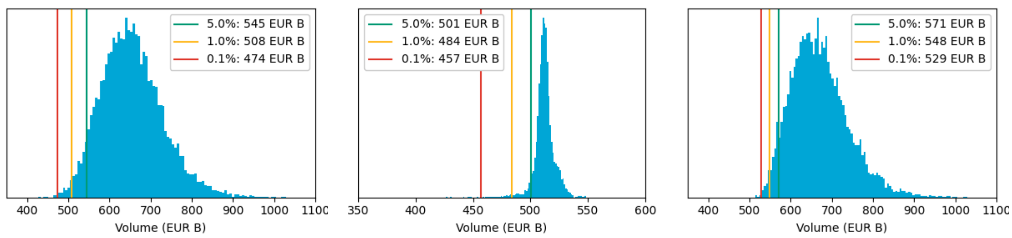
C.2 One-year period

The unconditional results for the final deposit volume after, the minimum volume over, and the maximum volume over one year can be found, unconditioned, as well as for the minimum of the short rate being in the lowest 0.1%-, 1%-, and 5%-quantiles (-34bp, -9bp, and 17bp respectively) in Figure C.5, Figure C.6, Figure C.7, and Figure C.7 respectively.



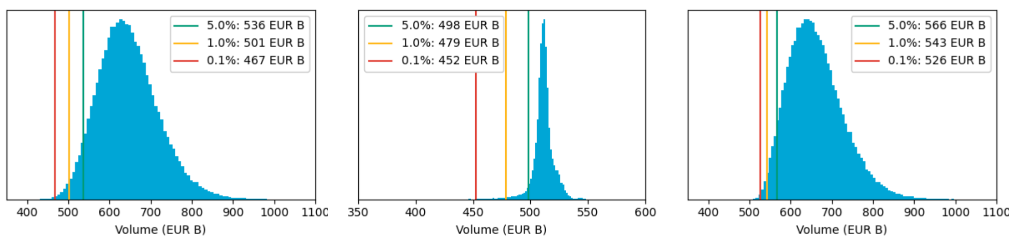
(a) Distribution of the final volume **(b)** Distribution of the minimum volume **(c)** Distribution of the maximum volume

Figure C.1: Deposit volume distributions for a five-year period. Plots of the simulation results for the final volume after five years, the minimum volume over five years and the maximum volume over five years.



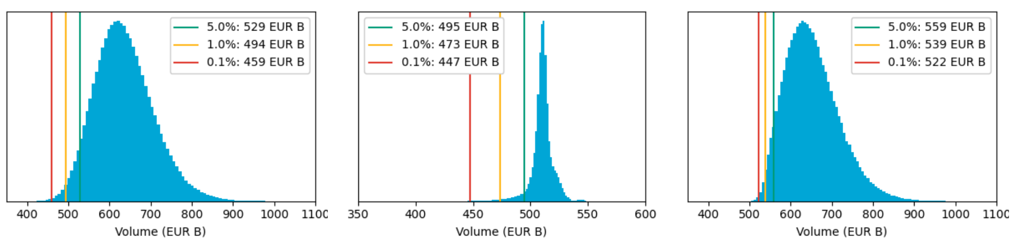
(a) Distribution of the final volume **(b)** Distribution of the minimum volume **(c)** Distribution of the maximum volume

Figure C.2: Deposit volume distributions for a five-year period, conditioned on the minimum market rate being in the lowest 0.1%-quantile. Plots of the simulation results for the final volume after five years, the minimum volume over five years and the maximum volume over five years.



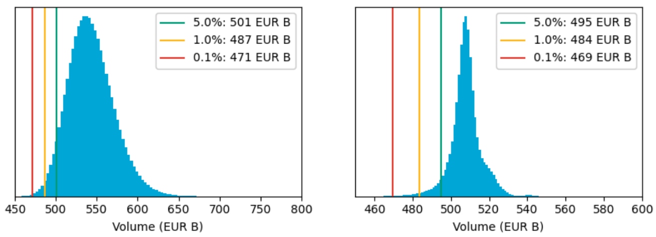
(a) Distribution of the final volume **(b)** Distribution of the minimum volume **(c)** Distribution of the maximum volume

Figure C.3: Deposit volume distributions for a five-year period, conditioned on the minimum market rate being in the lowest 1%-quantile. Plots of the simulation results for the final volume after five years, the minimum volume over five years and the maximum volume over five years.



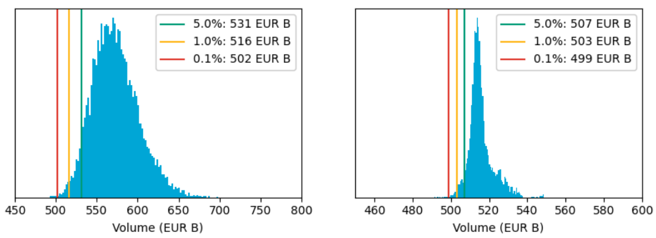
(a) Distribution of the final volume **(b)** Distribution of the minimum volume **(c)** Distribution of the maximum volume

Figure C.4: Deposit volume distributions for a five-year period, conditioned on the minimum market rate being in the lowest 5%-quantile. Plots of the simulation results for the final volume after five years, the minimum volume over five years and the maximum volume over five years.



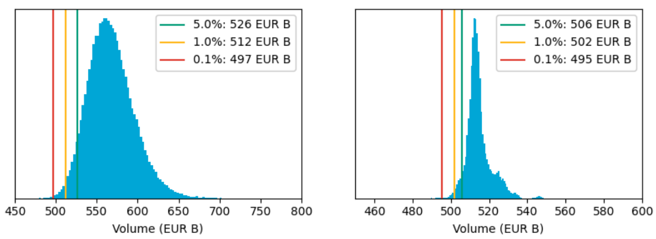
(a) Distribution of the final volume **(b)** Distribution of the minimum volume **(c)** Distribution of the maximum volume

Figure C.5: Deposit volume distributions for a one-year period. Plots of the simulation results for the final volume after one year, the minimum volume over one year and the maximum volume over one year.



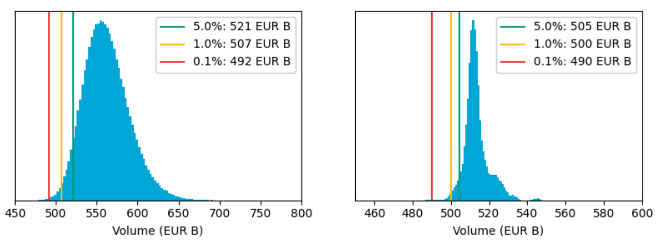
(a) Distribution of the final volume **(b)** Distribution of the minimum volume **(c)** Distribution of the maximum volume

Figure C.6: Deposit volume distributions for a one-year period, conditioned on the minimum market rate being in the lowest 0.1%-quantile. Plots of the simulation results for the final volume after one year, the minimum volume over one year and the maximum volume over one year.



(a) Distribution of the final volume **(b)** Distribution of the minimum volume **(c)** Distribution of the maximum volume

Figure C.7: Deposit volume distributions for a one-year period, conditioned on the minimum market rate being in the lowest 1%-quantile. Plots of the simulation results for the final volume after one year, the minimum volume over one year and the maximum volume over one year.



(a) Distribution of the final volume **(b)** Distribution of the minimum volume **(c)** Distribution of the maximum volume

Figure C.8: Deposit volume distributions for a one-year period, conditioned on the minimum market rate being in the lowest 5%-quantile. Plots of the simulation results for the final volume after one year, the minimum volume over one year and the maximum volume over one year.

D

Deposit rate optimization results for different periods

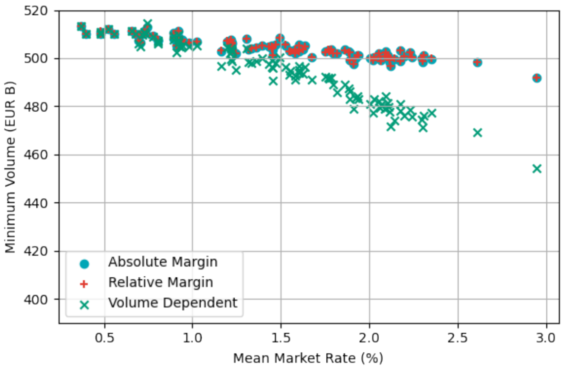
In this appendix, the results for the deposit rate model comparison are presented, but here, simulations over shorter periods are considered. The considered periods are a five-year period and a one-year period. The results are presented in figures analogous to how the results for the simulations over the ten-year period were presented.

D.1 Five-year period

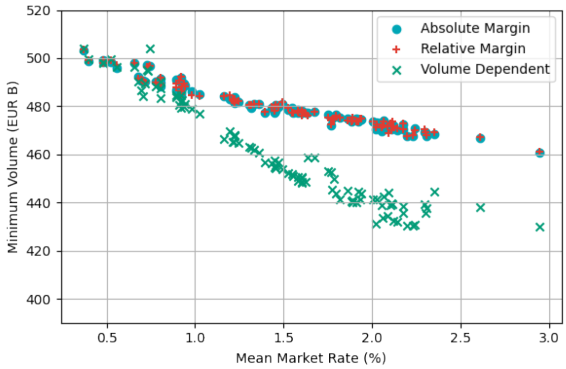
The results for the five-year period are presented in Figures D.1 to D.4. These results are quite similar to the results for the ten-year period. However, for high interest rate scenarios, the volume dependent model performs worse over this period of five years, compared to the two other models. This could be because the aggressive deposit rate setting in the volume dependent model lowers the volume faster, when compared to the other two models. Five years is then simply not enough time for the volume to move below the target for the other deposit rate models.

D.2 One-year period

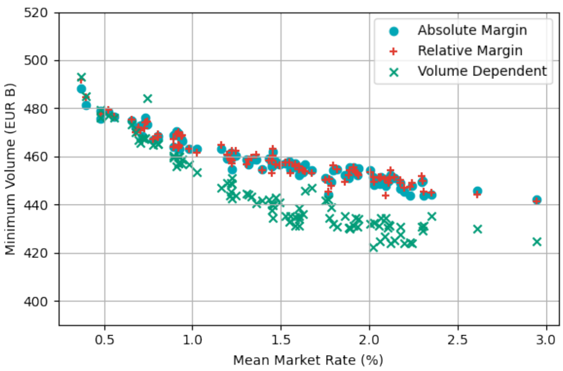
We also present the results for simulations of a one-year period in this appendix. These results can be found in Figures D.5 to D.8. These results are not very interesting, as it is evident that there is not enough time for any of the models to have any shortage. As is the case with the other two time periods, the aggressive deposit rate setting in the volume dependent model is shown to result in a higher profit than for both other models.



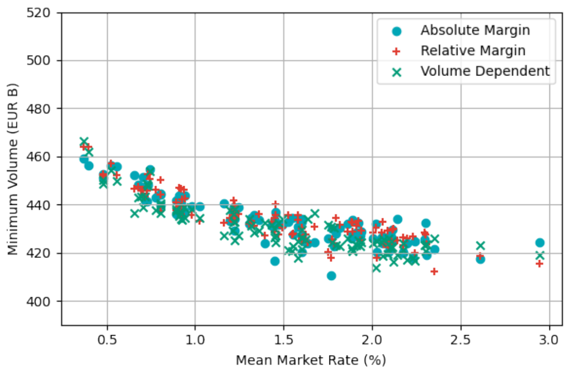
(a) Expected



(b) 5%-quantile

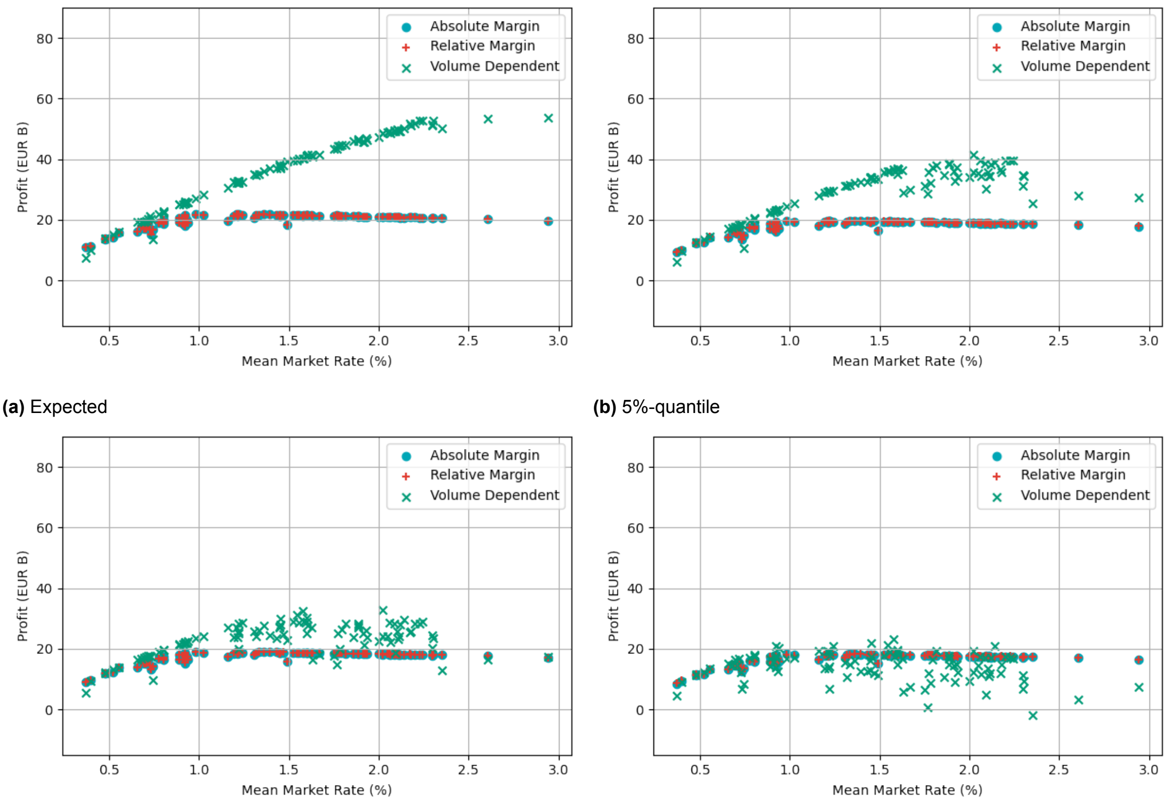


(c) 1%-quantile



(d) 0.1%-quantile

Figure D.1: Comparison of the different optimal deposit rate models on the minimum volume attained in a period of 5 years. Scatter plots comparing the expected minimum volume and lowest quantiles of this minimum volume for the three optimized deposit rate models for 100 market rate scenarios over a 5-year period.



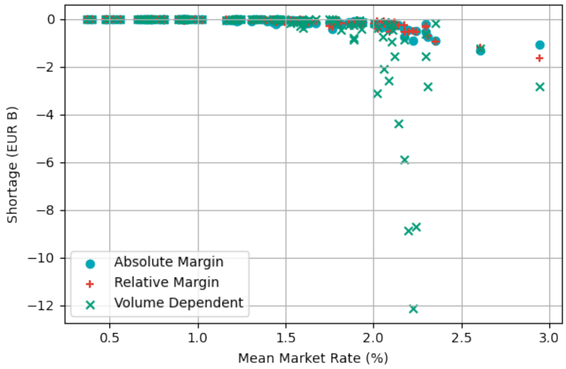
(a) Expected

(b) 5%-quantile

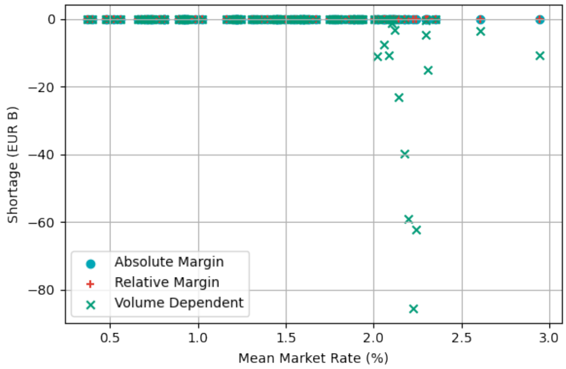
(c) 1%-quantile

(d) 0.1%-quantile

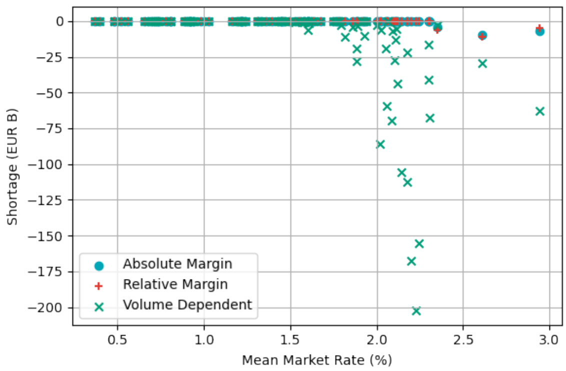
Figure D.2: Comparison of the different optimal deposit rate models on the profit over a period of 5 years. Scatter plots comparing the expected profit and lowest quantiles of this for the three optimized deposit rate models for 100 market rate scenarios over a 5-year period.



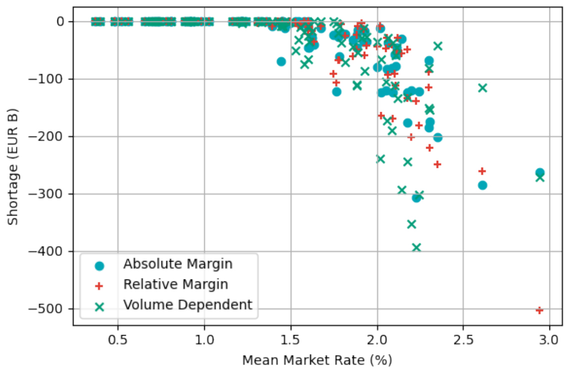
(a) Expected



(b) 5%-quantile

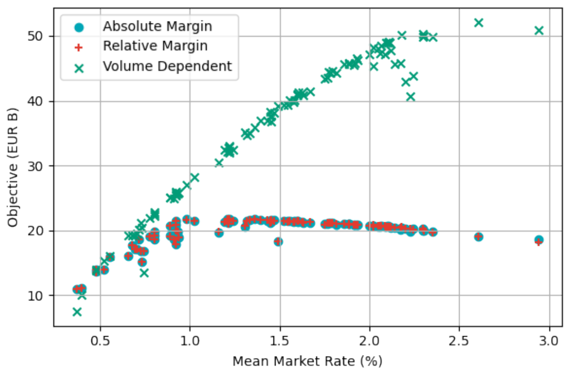


(c) 1%-quantile

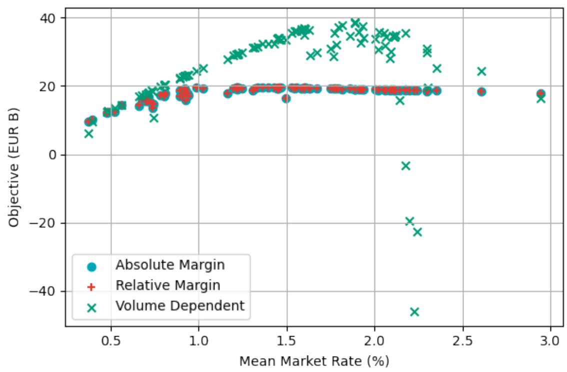


(d) 0.1%-quantile

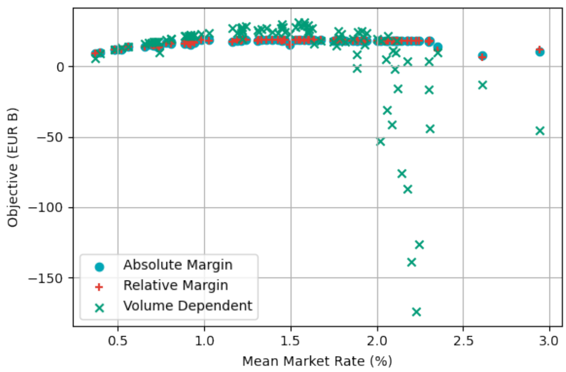
Figure D.3: Comparison of the different optimal deposit rate models on the shortage over a period of 5 years. Scatter plots comparing the expected shortage and lowest quantiles of this for the three optimized deposit rate models for 100 market rate scenarios over a 5-year period.



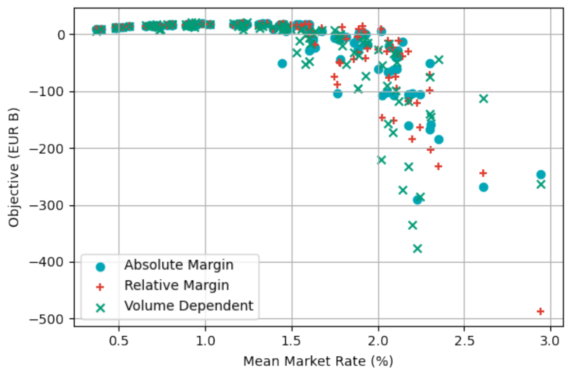
(a) Expected



(b) 5%-quantile

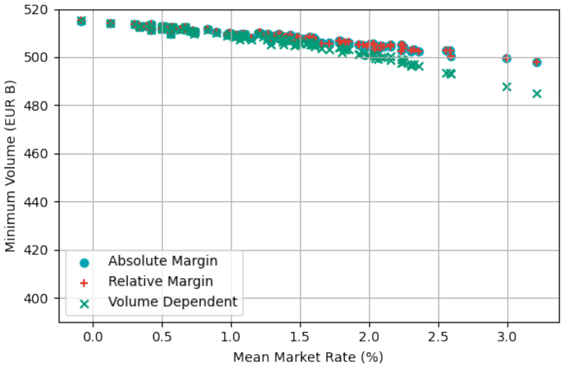


(c) 1%-quantile

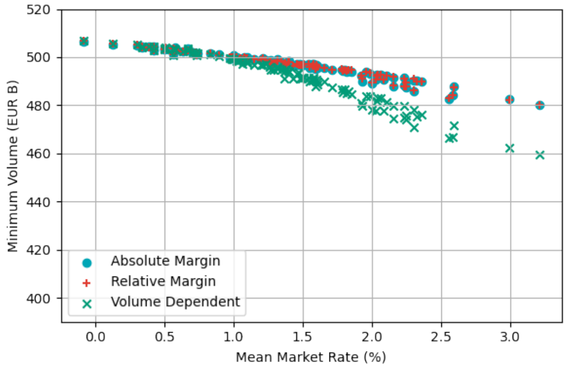


(d) 0.1%-quantile

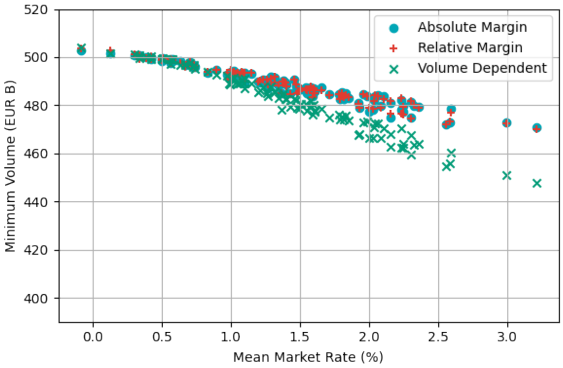
Figure D.4: Comparison of the different optimal deposit rate models on the objective over a period of 5 years. Scatter plots comparing the expected objective, sum of profit and shortage, and lowest quantiles of this for the three optimized deposit rate models for 100 market rate scenarios over a 5-year period.



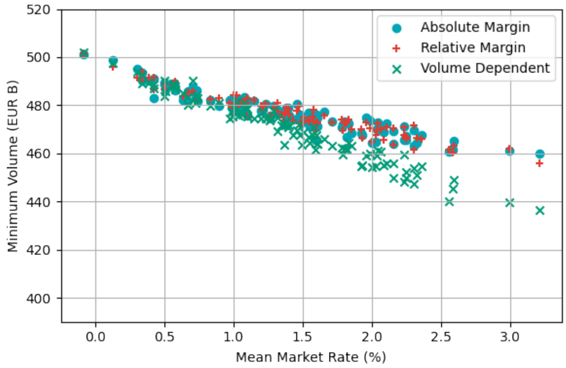
(a) Expected



(b) 5%-quantile

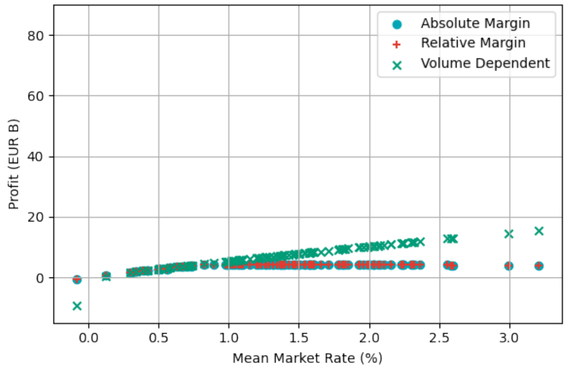


(c) 1%-quantile

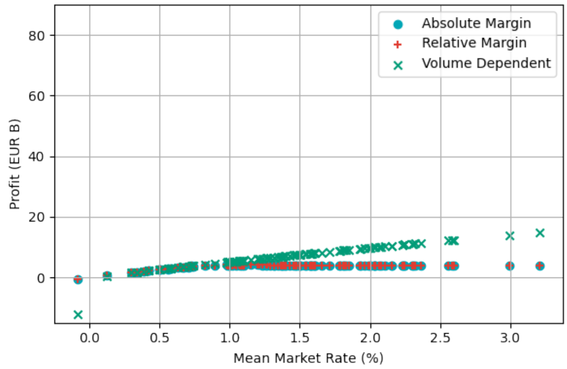


(d) 0.1%-quantile

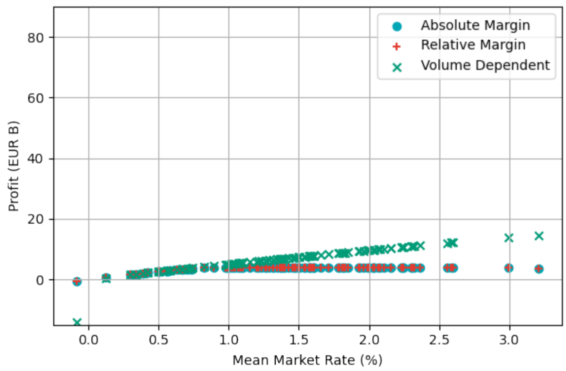
Figure D.5: Comparison of the different optimal deposit rate models on the minimum volume attained in a period of 1 years. Scatter plots comparing the expected minimum volume and lowest quantiles of this minimum volume for the three optimized deposit rate models for 100 market rate scenarios over a 1-year period.



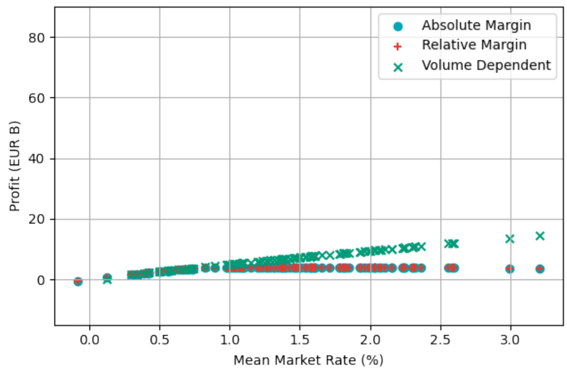
(a) Expected



(b) 5%-quantile

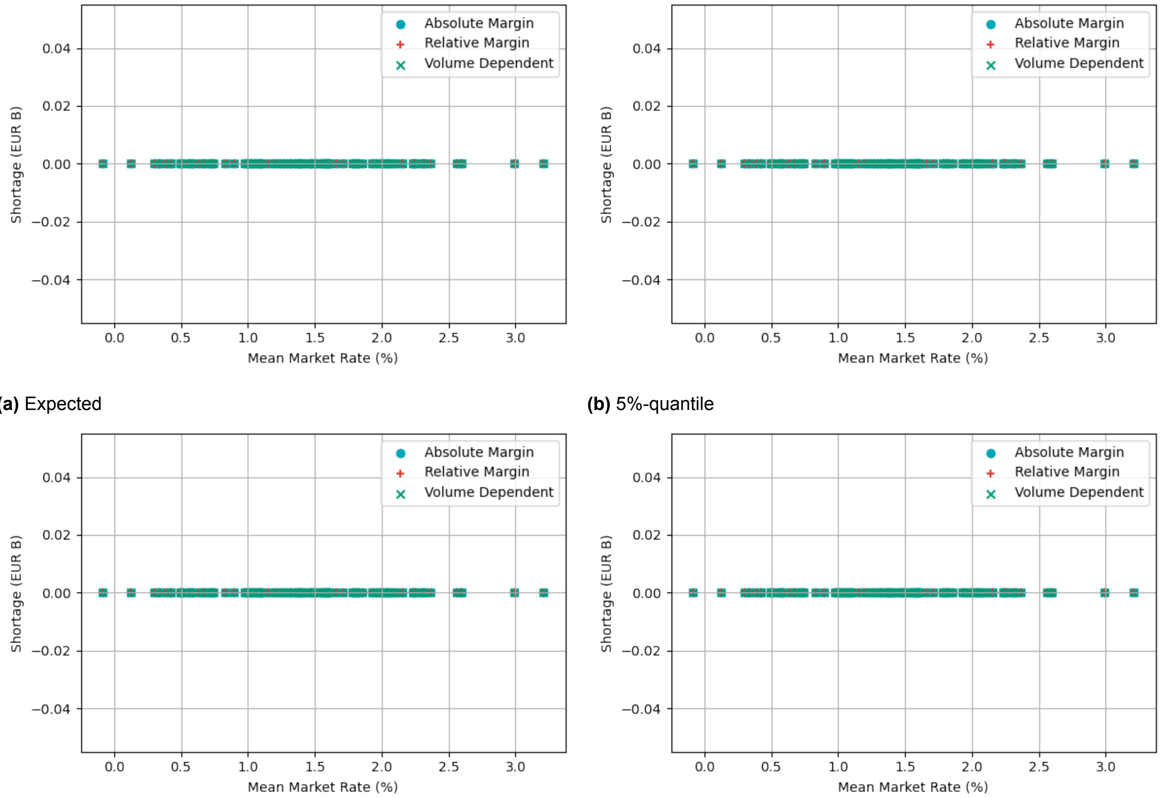


(c) 1%-quantile



(d) 0.1%-quantile

Figure D.6: Comparison of the different optimal deposit rate models on the profit over a period of 1 years. Scatter plots comparing the expected profit and lowest quantiles of this for the three optimized deposit rate models for 100 market rate scenarios over a 1-year period.



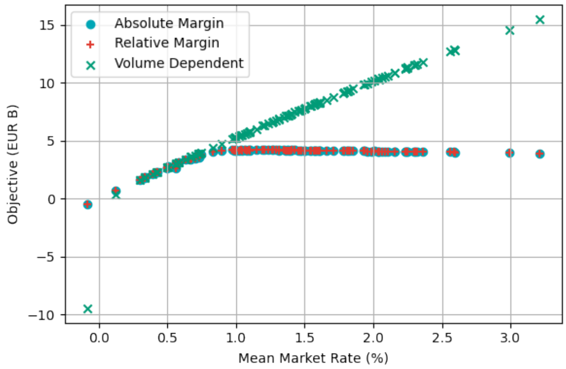
(a) Expected

(b) 5%-quantile

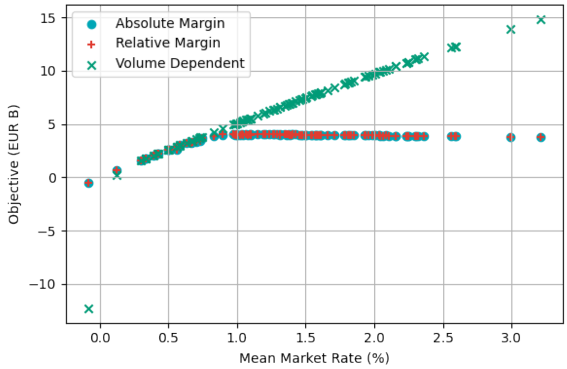
(c) 1%-quantile

(d) 0.1%-quantile

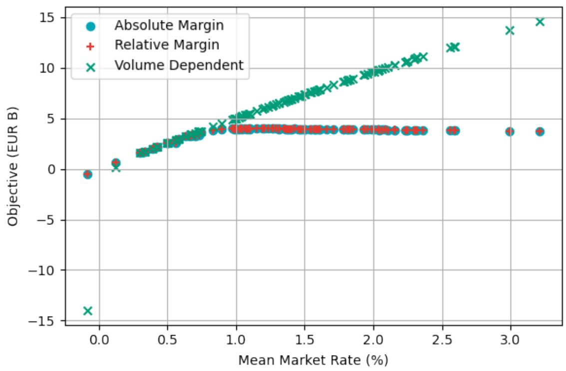
Figure D.7: Comparison of the different optimal deposit rate models on the shortage over a period of 1 years. Scatter plots comparing the expected shortage and lowest quantiles of this for the three optimized deposit rate models for 100 market rate scenarios over a 1-year period.



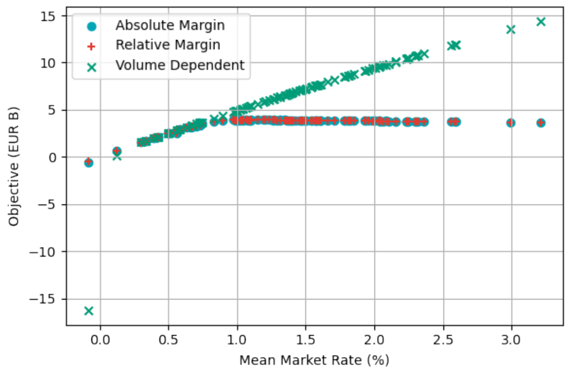
(a) Expected



(b) 5%-quantile



(c) 1%-quantile



(d) 0.1%-quantile

Figure D.8: Comparison of the different optimal deposit rate models on the objective over a period of 1 years. Scatter plots comparing the expected objective, sum of profit and shortage, and lowest quantiles of this for the three optimized deposit rate models for 100 market rate scenarios over a 1-year period.

References

Abramowitz, M., & Stegun, I. A. (1968). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. US Government printing office.

ACM. (2024, May). *Concept Rapport: Concurrentie op de Nederlandse spaarmarkt* (tech. rep.). Autoriteit Consument en Markt.

Altavilla, C., Burlon, L., Giannetti, M., & Holton, S. (2022). Is there a zero lower bound? The effects of negative policy rates on banks and firms. *Journal of Financial Economics*, *144*(3), 885–907. https://doi.org/10.1016/j.jfineco.2021.06.032

Andries, N., & Billon, S. (2016). Retail bank interest rate pass-through in the euro area: An empirical survey. *Economic Systems*, *40*(1), 170–194. https://doi.org/10.1016/j.ecosys.2015.06.001

Bardenhewer, M. M. (2006, October). Modeling Non-maturing Products. In *Liquidity risk measurement and management* (pp. 220–256). https://doi.org/10.1002/9781118390399.ch10

Barndorff-Nielsen, O. E. (1995). *Normal inverse Gaussian processes and the modelling of stock returns* (tech. rep.). Department of Mathematical Sciences, Aarhus University.

BCBS. (2016). Basel Committee on Banking Supervision - Interest rate risk in the banking book. In *Standards*. https://www.bis.org/basel_framework/

Bijma, F., Jonker, M., Vaart, A., & Erné, R. (2017, May). *An Introduction to Mathematical Statistics*. Amsterdam University Press. https://doi.org/10.1515/9789048536115

Black, F. (1995). Interest Rates as Options. *The Journal of Finance*, *50*(5), 1371. https://doi.org/10.2307/2329320

Blöchlinger, A. (2015). Identifying, valuing and hedging of embedded options in non-maturity deposits. *Journal of Banking and Finance*, *50*, 34–51. https://doi.org/10.1016/j.jbankfin.2014.09.013

Blöchlinger, A. (2021). Interest rate risk in the banking book: A closed-form solution for non-maturity deposits. *Journal of Banking and Finance*, *125*. https://doi.org/10.1016/j.jbankfin.2021.106080

Blum, J. R. (1954). Multidimensional Stochastic Approximation Methods. *The Annals of Mathematical Statistics*, *25*(4), 737–744. https://doi.org/10.1214/aoms/1177728659

Brigo, D., & Mercurio, F. (2007). *Interest Rate Models - Theory and Practice* (M. Avellaneda, G. Barone-Adesi, M. Broadie, M. H. A. Davis, E. Derman, C. Klüppelberg, E. Kopp, & W. Schachermayer, Eds.; 2nd ed.). Springer - Verlag. https://doi.org/10.1007/978-3-540-34604

Carfora, M. F., Cutillo, L., & Orlando, A. (2014). Modeling the European Central Bank official rate: a stochastic approach. *Journal of Applied Quantitative Methods*, *9*(3). https://core.ac.uk/display/54089336?utm_source=pdf&utm_medium=banner&utm_campaign=pdf-decoration-v1

Castagna, A., & Scaravaggi, A. (2017, December). *A Benchmark Framework for Non Maturing Deposits: An Application to Public Data Available from Banca d'Italia* (tech. rep.). https://ssrn.com/abstract=3090427

Chen, R.-R., & Scott, L. (2003). Multi-Factor Cox-Ingersoll-Ross Models of the Term Structure: Estimates and Tests from a Kalman Filter Model. *The Journal of Real Estate Finance and Economics*, *27*(2), 143–172. https://doi.org/10.1023/A:1024736903090

Cochrane, D., & Orcutt, G. H. (1949). Application of Least Squares Regression to Relationships Containing Auto- Correlated Error Terms. *Journal of the American Statistical Association*, *44*(245), 32. https://doi.org/10.2307/2280349

Cour-Thimann, P., & Jung, A. (2021). Interest-rate setting and communication at the ECB in its first twenty years. *European Journal of Political Economy*, *70*, 102039. https://doi.org/10.1016/j.ejpoleco.2021.102039

Cox, J. C., Ingersoll, J. E., & Ross, S. A. (1985). A Theory of the Term Structure of Interest Rates. *Econometrica*, *53*(2), 385–407. https://doi.org/10.2307/1911242

De Rossi, G. (2010). Maximum Likelihood Estimation of the Cox–Ingersoll–Ross Model Using Particle Filters. *Computational Economics*, *36*(1), 1–16. https://doi.org/10.1007/s10614-010-9208-0

de Bondt, G. J. (2005). Interest Rate Pass-Through: Empirical Results for the Euro Area. *German Economic Review*, *6*(1), 37–78. https://doi.org/10.1111/j.1465-6485.2005.00121.x de Jong, F., & Wielhouwer, J. (2003). The Valuation and Hedging of Variable Rate Savings Accounts. *ASTIN Bulletin*, *33*(2), 383–397. https://doi.org/10.2143/AST.33.2.503699

Doob, J. L. (1940). Regularity Properties of Certain Families of Chance Variables. *Transactions of the American Mathematical Society*, *47*(3), 455–486. https://doi.org/10.2307/1989964

Dothan, L. (1978). On the term structure of interest rates. *Journal of Financial Economics*, *6*(1), 59–69. https://doi.org/10.1016/0304-405X(78)90020-X

Dueker, M. J. (2000). Are prime rate changes asymmetric? *Review-Federal Reserve Bank of Saint Louis*, *82*(5), 33–40. https://doi.org/10.20955/r.82.33-40

Durbin, J., & Watson, G. S. (1950). Testing for Serial Correlation in Least Squares Regression: I. *Biometrika*, *37*(3/4), 409. https://doi.org/10.2307/2332391

Durbin, J., & Watson, G. S. (1951). Testing for Serial Correlation in Least Squares Regression. II. *Biometrika*, *38*(1/2), 159. https://doi.org/10.2307/2332325

Durbin, J., & Watson, G. S. (1971). Testing for Serial Correlation in Least Squares Regression. III. *Biometrika*, *58*(1), 1. https://doi.org/10.2307/2334313

EBA. (2022a, October). *European Banking Authority: Final Draft RTS on Standardised Methodologies on IRRBB* (tech. rep.). https://www.eba.europa.eu/sites/default/documents/files/document_library/Publications/Draft%20Technical%20Standards/2022/EBA-RTS-2022-09%20RTS%20on%20SA/1041755/Final%20draft%20RTS%20on%20SA.pdf

EBA. (2022b, October). *European Banking Authority: Final Drafts RTS on Supervisory Outlier Tests* (tech. rep.). https://www.eba.europa.eu/sites/default/documents/files/document_library/Publications/Draft%20Technical%20Standards/2022/EBA-RTS-2022-10%20RTS%20on%20SOTs/1041756/Final%20draft%20RTS%20on%20SOTs.pdf

EBA. (2022c, October). *European Banking Authority: Final Report on the Management of Interest Rate Risk and Credit Spread Risk Arising from Non-Trading Book Activities* (tech. rep.). https://www.eba.europa.eu/sites/default/documents/files/document_library/Publications/Guidelines/2022/EBA-GL-2022-14%20GL%20on%20IRRBB%20and%20CSRBB/1041754/Guidelines%20on%20IRRBB%20and%20CSRBB.pdf

Frachot, A. (2001, October). *A Note on Behavioral Models for Managing Optionality in Banking Books* (tech. rep.). Groupe de Recherche Opérationnelle, Crédit Lyonnais. Paris.

Gad, A. F. (2021). PyGAD: An Intuitive Genetic Algorithm Python Library. http://arxiv.org/abs/2106.06158

Hannan, T. H., & Berger, A. N. (1991). The Rigidity of Prices: Evidence from the Banking Industry. *The American Economic Review*, *81*(4), 938–945. http://www.jstor.org/stable/2006653

Heath, D., Jarrow, R., & Morton, A. (1992). Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica*, *60*(1), 77. https://doi.org/10.2307/2951677

Hull, J., & White, A. (1990). Pricing Interest-Rate-Derivative Securities. *The Review of Financial Studies*, *3*(4), 573–592. https://doi.org/10.1093/rfs/3.4.573

Hull, J., & White, A. (1993). *One-Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities* (tech. rep. No. 2). https://doi.org/10.2307/2331288

Hull, J., & White, A. (1994). Numerical procedures for implementing Term Structure Models II: Two-Factor Models. *The Journal of Derivatives*, *2*(2), 37–48. https://doi.org/10.3905/jod.1994.407908

Hutchison, D. E., & Pennacchi, G. G. (1996). Measuring Rents and Interest Rate Risk in Imperfect Financial Markets: The Case of Retail Bank Deposits. *The Journal of Financial and Quantitative Analysis*, *31*(3), 399–417. https://doi.org/10.2307/2331398

James, G., Witten, D., Hastie, T., Tibshirani, R., & Taylor, J. (2023, June). *An Introduction to Statistical Learning* (1st ed.). Springer International Publishing. https://doi.org/10.1007/978-3-031-38747-0

Jamshidian, F., & Zhu, Y. (1997). Scenario Simulation: Theory and methodology. *Finance and Stochastics*, *1*(1), 43–67. https://doi.org/10.1007/s007800050016

Jarrow, R. A., & Van Deventer, D. R. (1998). *The arbitrage-free valuation and hedging of demand deposits and credit card loans* (tech. rep.). https://doi.org/10.1016/S0378-4266(97)00058-7

Kahn, C., Pennacchi, G., & Sopranzetti, B. (1999). Bank Deposit Rate Clustering: Theory and Empirical Evidence. *The Journal of Finance*, *54*(6), 2185–2214. https://doi.org/10.26509/frbc-wp-199604

Kalkbrener, M., & Willing, J. (2004). Risk management of non-maturing liabilities. *Journal of Banking and Finance*, *28*(7), 1547–1568. https://doi.org/10.1016/S0378-4266(03)00131-6

Kitagawa, G. (1996). Monte Carlo Filter and Smoother for Non-Gaussian Nonlinear State Space Models. *Journal of Computational and Graphical Statistics*, *5*(1), 1. https://doi.org/10.2307/1390750

Kushner, H. J., & Yin, G. G. (2003). *Stochastic Approximation and Recursive Algorithms and Applications* (B. Rozovskii & M. Yor, Eds.; 2nd ed.). Springer.

Longstaff, F. A., & Schwartz, E. S. (1992). Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model. *The Journal of Finance*, *47*(4), 1259. https://doi.org/10.2307/2328939

Marena, M., Romeo, A., & Semeraro, P. (2023). Non-maturing deposits modelling in a Ornstein-Uhlenbeck framework. *Applied Stochastic Models in Business and Industry*, *39*(4), 539–548. https://doi.org/10.1002/asmb.2766

Mester, L. J., & Saunders, A. (1995). When does the prime rate change? *Journal of Banking & Finance*, *19*(5), 743–764. https://doi.org/10.1016/0378-4266(94)00090-P

Nehls, H. (2006). The Interest Rate Pass-Through in German Banking Groups. *226*(4), 463–480. https://doi.org/10.1515/jbnst-2006-0407

Nelson, C. R., & Siegel, A. F. (1987). Parsimonious Modeling of Yield Curves. *The Journal of Business*, *60*(4), 473–489. https://doi.org/10.1086/296409

Neumark, D., & Sharpe, S. A. (1992). Market Structure and the Nature of Price Rigidity: Evidence from the Market for Consumer Deposits. *The Quarterly Journal of Economics*, *107*(2), 657–680. https://doi.org/10.2307/2118485

Nyström, K. (2008). On deposit volumes and the valuation of non-maturing liabilities. *Journal of Economic Dynamics and Control*, *32*(3), 709–756. https://doi.org/10.1016/j.jedc.2007.03.004

O'Brien, J. M. (2001). Estimating the Value and Interest Rate Risk of Interest-Bearing Transactions Deposits. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.256712

Øksendal, B. (1998). *Stochastic Differential Equations: An Introduction with Applications* (5th ed.). Springer-Verlag.

Orlando, G., & Bufalo, M. (2021). Interest rates forecasting: between Hull and White and the CIR#. How to make a single factor model work. *Journal of Forecasting*, *40*(8), 1566–1580. https://doi.org/10.1002/for.2783

Paraschiv, F. (2011, May). *Modeling Client Rate and Volumes of Non-maturing Savings Accounts* [Doctoral dissertation, University of St. gallen].

Paraschiv, F., & Schürle, M. (2010, May). *Modeling client rate and volumes of non-maturing accounts*, University of St.Gallen. https://api.semanticscholar.org/CorpusID:156140310

Robbins, H., & Monro, S. (1951). A Stochastic Approximation Method. *The Annals of Mathematical Statistics*, *22*(3), 400–407. https://doi.org/10.1214/aoms/1177729586

Rosen, R. J. (2002). What Goes Up Must Come Down? Asymmetries and Persistence in Bank Deposit Rates. *Journal of Financial Services Research*, *21*(3), 173–193. https://doi.org/10.1023/A:1015085826129

Ruppert, D. (1985). A Newton-Raphson Version of the Multivariate Robbins-Monro Procedure. *The Annals of Statistics*, *13*(1), 236–245. https://doi.org/10.1214/aos/1176346589

Sander, H., & Kleimeier, S. (2002). Asymmetric Adjustment of Commercial Bank Interest Rates in the Euro Area: An Empirical Investigation into Interest Rate Pass-Through. *Credit and Capital Markets - Kredit und Kapital*, *35*(2), 161–192. https://doi.org/10.3790/ccm.35.2.161

Selvaggio, R. D. (1996). Using the OAS methodology to value and hedge commercial bank retail demand deposit premiums. In F. J. Fabozzi & A. Konishi (Eds.), *The handbook of asset/liability management* (pp. 363–373). McGraw-Hill.

Sørensen, C. K., & Werner, T. (2006, January). *Bank interest rate pass-through in the Euro area: a cross country comparison*, European Central Bank.

Svensson, L. E. (1994, September). *Estimating and Interpreting Forward Interest Rates: Sweden 1992 - 1994* (tech. rep.). National Bureau of Economic Research. Cambridge, MA. https://doi.org/10.3386/w4871

Ulate, M. (2021). Alternative models of interest rate pass-through in normal and negative territory. *International Journal of Central Banking*, *17*(1), 3–34. https://doi.org/10.24148/wp2020-31

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, *5*(2), 177–188. https://doi.org/10.1016/0304-405X(77)90016-2