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Full Length Article

An intersection representation for a class of anisotropic vector-valued function spaces[☆]

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Abstract

The main result of this paper is an intersection representation for a class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg and Netrusov (2007), which includes weighted anisotropic mixed-norm Besov and Lizorkin–Triebel spaces. In the special case of the classical Lizorkin–Triebel spaces, the intersection representation gives an improvement of the well-known Fubini property. The main result has applications in the weighted L_q-L_p-maximal regularity problem for parabolic boundary value problems, where weighted anisotropic mixed-norm Lizorkin–Triebel spaces occur as spaces of boundary data.

MSC: primary 46E35; 46E40; secondary 46E30

Keywords: Anisotropic; Axiomatic approach; Banach space-valued functions and distributions; Difference norm; Fubini property; Intersection representation; Maximal function; Quasi-Banach function space

1. Introduction

The motivation for this paper comes from the L_q-L_p-maximal regularity problem for fully inhomogeneous parabolic boundary value problems, see [15,36,37]. In such problems,

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Lizorkin–Triebel spaces have turned out to naturally occur in the description of the sharp regularity of the boundary data. This goes back to [59] in the special case that $1 < p \leq q < \infty$ for second order problems with special boundary conditions and was later extended in [15] to the full range $q, p \in (1, \infty)$ for the more general setting of vector-valued parabolic boundary value problems with boundary conditions of Lopatinskii–Shapiro type. The inevitability of Lizorkin–Triebel spaces for a correct description of the boundary data was reaffirmed in [27,28], but in a different form on the function space theoretic side.

On the one hand, in [15,59] the parabolic anisotropic regularity of the boundary data is described by means of an intersection of two function space-valued function spaces, in which the Lizorkin–Triebel space appears as an isotropic vector-valued Lizorkin–Triebel space describing the sharp temporal regularity. On the other hand, in [27,28] the anisotropic structure is dealt with more directly through a Fourier analytic approach, leading to anisotropic mixed-norm Lizorkin–Triebel spaces. A link between the two approaches was obtained in [16, Proposition 3.23], by comparing the trace result [28, Theorem 2.2] with a trace result from [5,6]: for every $q, p \in (1, \infty)$, $a, b \in (0, \infty)$ and $s \in (0, \infty)$, there is the intersection representation

$$F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^n \times \mathbb{R}) = F^{s/b}_{q,p}(\mathbb{R}; L_p(\mathbb{R}^n)) \cap L_q(\mathbb{R}; B^{s/a}_{p,p}(\mathbb{R}^n)). \tag{1}$$

The anisotropic mixed-norm Lizorkin–Triebel space $F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n \times \mathbb{R})$ for $s \in \mathbb{R}$, $r \in [1, \infty]$, is defined analogously to the classical isotropic Lizorkin–Triebel space $F^s_{p,r}(\mathbb{R}^d)$, but with an underlying Littlewood–Paley decomposition of $\mathbb{R}^n \times \mathbb{R}$ that is adapted to the (a, b)-anisotropic scalings $\{\delta^{(a,b)}_\lambda : \lambda \in (0, \infty)\}$ given by

$$\delta^{(a,b)}_\lambda(\xi, \tau) = (\lambda^a \xi, \lambda^b \tau), \qquad (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}. \tag{2}$$

Intuitively the dilation structure (2) causes a decay behavior on the Fourier side at different rates in the two components of $\mathbb{R}^n \times \mathbb{R}$ in such a way that smoothness $s \in (0, \infty)$ with respect to the anisotropy (a, b) corresponds to smoothness s/a in the spatial direction and smoothness s/b in the time direction. One way to look at the intersection representation (1) is as a way to make this intuition precise for $F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n \times \mathbb{R})$ in the special case that $r = p$.

It is the goal of this paper to provide a more systematic approach to the intersection representation (1) and obtain more general versions of it, covering the weighted Banach space-valued setting. In order to do so, we introduce a new class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg&Netrusov [24], which includes Banach space-valued weighted anisotropic mixed-norm Besov and Lizorkin–Triebel spaces (see Section 3).

The main result of this paper is an intersection representation for this new class of anisotropic function spaces (see Section 5), from which the following theorem can be obtained as a special case (see Example 5.8):

Theorem 1.1. *Let $a, b \in (0, \infty)$, $p, q \in (1, \infty)$, $r \in [1, \infty]$ and $s \in (0, \infty)$. Then*

$$F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n \times \mathbb{R}^m) = \mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m; L_p(\mathbb{R}^n)) \cap L_q(\mathbb{R}^m; F^{s/a}_{p,r}(\mathbb{R}^n)), \tag{3}$$

where, for $E = L_p(\mathbb{R}^n)$ and $\sigma \in \mathbb{R}$,

$$\mathbb{F}^{\sigma}_{q,r}(\mathbb{R}^m; E) = \left\{ f \in \mathcal{S}'(\mathbb{R}^m; E) : (2^{k\sigma} S_k f)_k \in L_q(\mathbb{R}^n; E[\ell_r(\mathbb{N})]) \right\}$$

with $(S_k)_{k\in\mathbb{N}}$ the Fourier multiplier operators induced by a Littlewood–Paley decomposition of $\mathbb{R}^m$ and where

$$E[\ell_r(\mathbb{N})] = \left\{ (f_n)_n \in E^{\mathbb{N}} : \|(f_n)_n\|_{\ell_r(\mathbb{N})} \in E \right\}.$$

In the case $p = r$, Fubini's theorem yields $\mathbb{F}^{s/b}_{q,p}(\mathbb{R}^m; L_p(\mathbb{R}^n)) = F^{s/b}_{q,p}(\mathbb{R}^m; L_p(\mathbb{R}^n))$ and $F^{s/a}_{p,p}(\mathbb{R}^n) = B^{s/a}_{p,p}(\mathbb{R}^n)$, and from (3) we obtain an extension of the intersection representation (1) to decompositions $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m$:

$$F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^n \times \mathbb{R}^m) = F^{s/b}_{q,p}(\mathbb{R}^m; L_p(\mathbb{R}^n)) \cap L_q(\mathbb{R}^m; B^{s/a}_{p,p}(\mathbb{R}^n)).$$

In the special case that $a = b$ and $p = q$, the latter can be viewed as a special instance of the Fubini property. In fact, the main result of this paper, Theorems 5.1/5.4, extends the well-known Fubini property for the classical Lizorkin–Triebel spaces $F^s_{p,q}(\mathbb{R}^d)$ (see [57, Section 4] and the references given therein), see Remark 5.5. However, as seen in Theorem 1.1, the availability of Fubini's theorem is not required for intersection representations, it should just be thought of as a powerful tool to simplify the function spaces that one has to deal with in the special case that some of the parameters coincide.

As a special case of the general intersection representation from Section 5 we also obtain intersection representations for anisotropic mixed-norm Besov spaces (see Example 5.9). An intersection representation for anisotropic Besov spaces for which the integrability parameter coincides with the microscopic parameter can be found in [1, Theorem 3.6.3].

Let us now give an alternative viewpoint of (3) in order to motivate and provide some intuition for the function space theoretic framework of this paper. First of all, the isotropic $\mathbb{F}^{s/b}_{q,r}$ and $F^{s/a}_{p,r}$ on the right-hand side of (3) could be viewed as the anisotropic $\mathbb{F}^{s,b}_{q,r}$ and $F^{s,a}_{p,r}$:

$$F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n \times \mathbb{R}^m) = \mathbb{F}^{s,b}_{q,r}(\mathbb{R}^m; L_p(\mathbb{R}^n)) \cap L_q(\mathbb{R}^m; F^{s,a}_{p,r}(\mathbb{R}^n)). \tag{4}$$

As already mentioned above, in this paper we will introduce a new class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg&Netrusov [24]. This class of function spaces will be defined in such a way that each of the three spaces in (4) is naturally contained in it. In particular, this will allow us to treat the three function spaces in (4) in the same way from a conceptual point of view.

In order to elaborate a bit on the latter, let us write $d = n + m$, $A = aI_n$, $B = bI_m$, $\boldsymbol{A} = (A, B)$ and let $\mathbb{R}^d$ be (n, m)-decomposed, i.e. $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m$, with (n, m)-anisotropy $\boldsymbol{A}$ and the induced one-parameter group of expansive dilations $(\boldsymbol{A}_t)_{t\in\mathbb{R}_+}$:

$$\boldsymbol{A}_t(x, y) = (A_t x, B_t y) = (t^a x, t^b y), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^d.$$

Let

$$\begin{aligned} E &= L_{(p,q)}(\mathbb{R}^n \times \mathbb{R}^m)[\ell^s_r(\mathbb{N})] \\ &= \left\{ (f_n)_{n\in\mathbb{N}} \in L_0(\mathbb{R}^d \times \mathbb{N}) \ : \left(\sum_{n=0}^{\infty} 2^{ns} |f_n|^r \right)^{1/r} \in L_{(p,q)}(\mathbb{R}^n \times \mathbb{R}^m) \right\}, \end{aligned}$$

where we use the natural identification $L_0(\mathbb{R}^d)^{\mathbb{N}} \simeq L_0(\mathbb{R}^d \times \mathbb{N})$; here, given a measure space $(S, \mathscr{A}, \mu)$, $L_0(S)$ stands for the space of equivalence classes of measurable functions from S to $\mathbb{C}$. Let $E_{(n,m);1}$ and $E_{(n,m);2}$ denote E viewed as Banach function space on $\mathbb{R}^m \times \mathbb{N} \times \mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{N} \times \mathbb{R}^m$, respectively. Let $(S^{\boldsymbol{A}}_n)_{n\in\mathbb{N}}$ be a Littlewood–Paley decomposition of $\mathbb{R}^d$ with respect to the dilation structure $(\boldsymbol{A}_t)_{t\in\mathbb{R}_+}$ induced by the anisotropy $\boldsymbol{A}$, let $(S^A_n)_{n\in\mathbb{N}}$ be a Littlewood–Paley decomposition of $\mathbb{R}^m$ with respect to the dilation structure $(A_t)_{t\in\mathbb{R}_+}$ induced by the anisotropy A and let $(S^B_n)_{n\in\mathbb{N}}$ be a Littlewood–Paley decomposition of $\mathbb{R}^n$ with respect to the dilation structure $(B_t)_{t\in\mathbb{R}_+}$ induced by the anisotropy B; see Definition 3.18 in the main text.

With the just introduced notation, $F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n \times \mathbb{R}^m)$ coincides with the space

$$Y^{\boldsymbol{A}}(E) = \{f \in \mathcal{S}'(\mathbb{R}^d) : (S^{\boldsymbol{A}}_n f)_n \in E\},$$

$\mathbb{F}^{s,b}_{q,r}(\mathbb{R}^m; L_p(\mathbb{R}^n))$ can be naturally identified with the space

$$Y^B(E_{(n,m);2}) = \{f \in L_0(\mathbb{R}^n; \mathcal{S}'(\mathbb{R}^m)) : (S^B_n f)_n \in E_{(n,m);2}\},$$

and $L_q(\mathbb{R}^m; F^{s,a}_{p,r}(\mathbb{R}^n))$ can be naturally identified with the space

$$Y^A(E_{(n,m);1}) = \{f \in L_0(\mathbb{R}^m; \mathcal{S}'(\mathbb{R}^n)) : (S^A_n f)_n \in E_{(n,m);1}\},$$

so that (4) takes the form

$$Y^{\boldsymbol{A}}(E) = Y^B(E_{(n,m);2}) \cap Y^A(E_{(n,m);1}). \tag{5}$$

Each of the spaces $Y^{\boldsymbol{A}}(E)$, $Y^B(E_{(n,m);2})$ and $Y^A(E_{(n,m);1})$ is defined as a subspace of $L_0(S; \mathcal{S}'(\mathbb{R}^N))$ for some σ-finite measure space $(S, \mathscr{A}, \mu)$, in terms of an anisotropy on $\mathbb{R}^N$ and a Banach function space on $\mathbb{R}^N \times \mathbb{N} \times S$, where we take the trivial measure space $(S, \mathscr{A}, \mu) = (\{0\}, \{\varnothing, \{0\}\}, \#)$ in case of $Y^{\boldsymbol{A}}(E)$ above. Furthermore, we view the Euclidean space $\mathbb{R}^N$ as being decomposed as $\mathbb{R}^N = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell}$ with $\ell \in \mathbb{N}_1$ and $\mathpzc{d} = (d_1, \ldots, d_\ell) \in (\mathbb{N}_1)^\ell$, $d_1 + \cdots + d_\ell = N$, where we take $\ell = 2$ in case of $Y^{\boldsymbol{A}}(E)$ above and take $\ell = 1$ in cases of $Y^B(E_{(n,m);2})$ and $Y^A(E_{(n,m);1})$ above. This viewpoint naturally leads us to extend the axiomatic approach to function spaces by Hedberg&Netrusov [24] to the anisotropic mixed-norm setting in which there additionally is some extra underlying measure space $(S, \mathscr{A}, \mu)$. This will give us a general framework that is well suited for a systematic treatment of intersection representations as in Theorem 1.1 as well as extensions to a Banach space-valued setting with Muckenhoupt weights.

One of the main ingredients in the proof of Theorem 1.1 (and the more general intersection representations) is a characterization by differences. For a function $f \in \mathbb{R}^d \to \mathbb{C}$, $h \in \mathbb{R}^d$ and an integer $M \geq 1$, we write

$$\Delta_h f(x) = f(x+h) - f(x), \qquad x \in \mathbb{R}^d,$$

and

$$\Delta_h^M f(x) = \underbrace{\Delta_h \cdots \Delta_h}_{M \text{ times}} f(x) = \sum_{j=0}^{M} (-1)^j \binom{M}{j} f(x + (M-j)h), \quad x \in \mathbb{R}^d.$$

For the special case of the anisotropic mixed-norm Lizorkin–Triebel space $F^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell})$, the difference norm characterization takes the form of Theorem 1.2.

Before we state it, let us introduce some notation. Let $\ell \in \mathbb{N}_1$, $\mathpzc{d} \in (\mathbb{N}_1)^\ell$ with $d_1 + \cdots + d_\ell = d$, $\boldsymbol{a} \in (0,\infty)^\ell$, $\boldsymbol{p} \in (0,\infty)^\ell$, $q \in [1,\infty]$ and $s \in \mathbb{R}$. We put

$$F^{s,(\boldsymbol{a};\mathpzc{d})}_{\boldsymbol{p},q}(\mathbb{R}^d) := F^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell})$$

and

$$L_{(\boldsymbol{p};\mathpzc{d})}(\mathbb{R}^d) := L_{p_\ell}(\mathbb{R}^{d_\ell})[\ldots[L_{p_1}(\mathbb{R}^{d_1})]\ldots]$$
$$= \left\{ f \in L_0(\mathbb{R}^d) : \left(\int_{\mathbb{R}^{d_\ell}} \left(\ldots \left(\int_{\mathbb{R}^{d_1}} |f(x)|^{p_1} \,\mathrm{d}x_1 \right)^{p_2/p_1} \ldots \right)^{p_\ell/p_{\ell-1}} \mathrm{d}x_\ell \right)^{1/p_\ell} < \infty \right\}.$$

Theorem 1.2. *Let $\ell \in \mathbb{N}_1$, $\mathpzc{d} \in (\mathbb{N}_1)^\ell$ with $\mathpzc{d}_1 + \cdots + \mathpzc{d}_\ell = d$, $\boldsymbol{a} \in (0,\infty)^\ell$, $\boldsymbol{p} \in (1,\infty)^\ell$, $q \in [1,\infty]$ and $s \in (0,\infty)$. Let $\boldsymbol{\varphi} \in [1,\infty)^\ell$ and $M \in \mathbb{N}$ satisfy $s > \sum_{j=1}^{\ell} a_j \mathpzc{d}_j (1-\varphi_j^{-1})$ and $M \min\{a_1\mathpzc{d}_1, \ldots, a_\ell \mathpzc{d}_\ell\} > s$. For all $f \in L_{(\boldsymbol{p};\mathpzc{d})}(\mathbb{R}^d)$ there is the two sided estimate*

$$\|f\|_{F^{s,(\boldsymbol{a};\mathpzc{d})}_{\boldsymbol{p},q}(\mathbb{R}^d)} \eqsim \|f\|_{L_{(\boldsymbol{p};\mathpzc{d})}(\mathbb{R}^d)} + \|(2^{ns} d^{(\boldsymbol{a};\mathpzc{d}),\boldsymbol{\varphi}}_{M,n}(f))_{n\geq 1}\|_{L_{(\boldsymbol{p};\mathpzc{d})}(\mathbb{R}^d)[\ell_q(\mathbb{N}_1)]},$$

where

$$d^{(\boldsymbol{a};\mathpzc{d}),\boldsymbol{\varphi}}_{M,n}(f)(x) := 2^{n\sum_{j=1}^{\ell} a_j \mathpzc{d}_j \varphi_j^{-1}} \left\| h \mapsto 1_{B_{\mathbb{R}^{\mathpzc{d}_1}}(0,2^{-na_1})\times\cdots\times B_{\mathbb{R}^{\mathpzc{d}_\ell}}(0,2^{-na_\ell})}(h)\Delta_h^M f(x) \right\|_{L_{(\boldsymbol{\varphi};\mathpzc{d})}(\mathbb{R}^d)}.$$

The implicit constants in this two-sided estimate, which is in (modified) Vinogradov notation for estimates (see the end of this introduction on notation and conventions), only depends on $\mathpzc{d}$, $\boldsymbol{a}$, $\boldsymbol{p}$, q and s.

As a special case of the general difference norm results in this paper (see Section 4), we also have a corresponding version of Theorem 1.2 for $\mathbb{F}^{s,(\boldsymbol{a};\mathpzc{d})}_{\boldsymbol{p},q}(\mathbb{R}^d;E)$. In connection to (the proof of) Theorem 1.1, this especially includes $\mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m;L_p(\mathbb{R}^n))$.

Theorem 1.2 is an extension of the difference norm characterization contained in [24, Theorem 1.1.14] to the anisotropic mixed norm setting, restricted to the special case of Lizorkin–Triebel spaces in the parameter range $\boldsymbol{p} \in (1,\infty)^\ell$, $q \in [1,\infty]$. However, the range $\boldsymbol{p} \in (0,\infty)^\ell$, $q \in (0,\infty]$ are also covered by the general difference norm results in Section 4 for the axiomatic setting considered in this paper. In fact, we cover weighted anisotropic mixed-norm Banach space-valued Besov and Lizorkin–Triebel spaces (both in the normed and quasi-normed parameter ranges). Related estimates involving differences in the isotropic case can be found in e.g. [53,55,56].

The following duality result is a special case of a more general duality result from this paper for our abstract class of anisotropic vector-valued function spaces (see Theorem 6.3 and Example 6.4).

Theorem 1.3. *Let X be a Banach space, $\mathpzc{d} \in (\mathbb{N}_1)^\ell$ with $\mathpzc{d}_1 + \cdots + \mathpzc{d}_\ell = d$, $\boldsymbol{a} \in (0,\infty)^\ell$, $\boldsymbol{p} \in (1,\infty)^\ell$, $q \in (1,\infty)$ and $s \in \mathbb{R}$. Viewing*

$$[F^{s,(\boldsymbol{a},\mathpzc{d})}_{\boldsymbol{p},q}(\mathbb{R}^d;X)]^* \hookrightarrow \mathcal{S}'(\mathbb{R}^d;X^*)$$

under the natural pairing (induced by $\mathcal{S}'(\mathbb{R}^d;X^*) = [\mathcal{S}(\mathbb{R}^d;X)]'$, see [2, Theorem 1.3.1]), there is the identity

$$[F^{s,(\boldsymbol{a},\mathpzc{d})}_{\boldsymbol{p},q}(\mathbb{R}^d;X)]^* = F^{-s,(\boldsymbol{a},\mathpzc{d})}_{\boldsymbol{p}',q'}(\mathbb{R}^d;X^*)$$

with an equivalence of norms.

Duality results for the classical isotropic Besov and Lizorkin–Triebel spaces can be found in [55, Section 2.11.2]. In the Banach space-valued setting, [2, Theorem 2.3.1] is a duality result for Besov spaces. There the underlying Banach space is assumed to be reflexive or to have a separable dual space, except for the case $p = \infty$ (see [2, Remark 2.3.2]). In this paper we obtain a partial extension of [2, Theorem 2.3.1] to the weighted mixed-norm setting with no assumptions on the Banach space (see Example 6.4).

The following result is a sum representation for anisotropic mixed-norm Lizorkin–Triebel spaces of negative smoothness, which is a dual version to the intersection representation Theorem 1.1.

Corollary 1.4. *Let $a, b \in (0, \infty)$, $p, q \in (1, \infty)$, $r \in (1, \infty)$ and $s \in (-\infty, 0)$. Then*

$$F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n \times \mathbb{R}^m) = \mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m; L_p(\mathbb{R}^n)) + L_q(\mathbb{R}^m; F^{s/a}_{p,r}(\mathbb{R}^n)), \tag{6}$$

where $\mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m; L_p(\mathbb{R}^n))$ is as defined in Theorem 1.1.

The above sum representation is an easy consequence of the intersection representation, the duality results and some basic functional analysis on duals of intersections. A sum representation for anisotropic Besov spaces for which the integrability parameter coincides with the microscopic parameter can be found in [1, Theorem 3.6.6].

Note that in the special case $r = p$, (6) reduces to

$$F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^n \times \mathbb{R}^m) = F^{s/b}_{q,p}(\mathbb{R}^m; L_p(\mathbb{R}^n)) + L_q(\mathbb{R}^m; F^{s/a}_{p,p}(\mathbb{R}^n))$$

by Fubini's theorem.

Overview

This paper is organized as follows.

- *Section* 2: We discuss the necessary preliminaries on anisotropy and decomposition, quasi-Banach function spaces, vector-valued functions and distributions, and UMD Banach spaces.
- *Section* 3: We introduce a new class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg&Netrusov [24] and discuss some basic properties of these function spaces. In particular, in Definition 3.15 we define the spaces $Y^{\boldsymbol{A}}(E; X) \subset L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ for 'admissable' quasi-Banach function spaces E on $\mathbb{R}^d \times \mathbb{N} \times S$ in the sense of Definition 3.1. Proposition 3.19 gives a characterization of $Y^{\boldsymbol{A}}(E; X)$ in terms of Littlewood–Paley decompositions, which is how Besov and Lizorkin–Triebel spaces are usually defined to begin with. Example 3.20 then subsequently gives some concrete examples of $Y^{\boldsymbol{A}}(E; X)$, including Besov and Lizorkin–Triebel spaces in different generalities.
- *Section* 4: We derive several estimates for the spaces of measurable functions $YL^{\boldsymbol{A}}(E; X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E; X)$, including estimates involving differences. The spaces $YL^{\boldsymbol{A}}(E; X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E; X)$ are defined in Definitions 3.11 and 3.12, but coincide with $Y^{\boldsymbol{A}}(E; X)$ under the conditions of Theorem 3.22. In particular, we obtain difference norm characterizations for $Y^{\boldsymbol{A}}(E; X)$ in Corollary 4.7 and Theorem 4.8. The latter covers Theorem 1.2 as a special case.
- *Section* 5: Using the difference norm estimates from Section 4, we obtain intersection representations for $Y^{\boldsymbol{A}}(E; X)$ in the spirit of (5) in Corollary 5.3 and Theorem 5.4 (as well as intersection representations for $YL^{\boldsymbol{A}}(E; X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E; X)$). In Examples 5.6 and 5.7 we formulate the intersection representations for concrete choices of E, which in particular include the Besov and Lizorkin–Triebel cases. Example 5.6 covers Theorem 1.1 as a special case.
- *Section* 6: We present a duality result for $Y^{\boldsymbol{A}}(E; X)$ in Theorem 6.3, for which we give concrete examples in Example 6.4. The latter includes Theorem 1.3.
- *Section* 7: Combining the intersection representation from Section 5 with the duality result from Section 6, we obtain a sum representation for $Y^{\boldsymbol{A}}(E; X)$ in Corollary 7.1.

Notation and convention.

We write: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $\mathbb{N}_k = \{k, k+1, k+2, k+3, \ldots\}$ for $k \in \mathbb{N}$, $\hat{f} = \mathcal{F}f$, $\mathbb{Z}_{<0} = \{\ldots, -3, -2, -1\}$, $\check{f} = \mathcal{F}^{-1}f$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, $\ell^s_p(\mathbb{N}) = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{n=0}^{\infty} 2^{ns}|a_n|^p < \infty\}$. Furthermore, $\lfloor x \rfloor \in \mathbb{N}$ denotes the least integer part of $x \in [0, \infty)$. Given a quasi-Banach space Y, we denote by $\mathcal{B}(Y)$ the space of bounded linear operators on Y and we write $B_Y = \{y \in Y : \|y\| \leq 1\}$ for the closed unit ball in Y. Throughout the paper, we work over the field of complex scalars and fix a Banach space X and a σ-finite measure space $(S, \mathscr{A}, \mu)$. Given two topological vector spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the linear inclusion mapping of X into Y is continuous and we write $X \overset{d}{\hookrightarrow} Y$ if $X \hookrightarrow Y$ and X is dense in Y.

We use (modified) Vinogradov notation for estimates: $a \lesssim b$ means that there exists a constant $C \in (0, \infty)$ such that $a \leq Cb$; $a \lesssim_{p,P} b$ means that there exists a constant $C \in (0, \infty)$, only depending on p and P, such that $a \leq Cb$; $a \eqsim b$ means $a \lesssim b$ and $b \lesssim a$; $a \eqsim_{p,P} b$ means $a \lesssim_{p,P} b$ and $b \lesssim_{p,P} a$.

We will frequently write something like $\overset{(*)}{\leq}$ or $\overset{(*)}{\lesssim}$, where $(*)$ for instance refers to an equation, to indicate that we use $(*)$ to get $\leq$ or $\lesssim$, respectively.

2. Preliminaries

2.1. Anisotropy and decomposition

2.1.1. Anisotropy on $\mathbb{R}^d$

An *anisotropy* on $\mathbb{R}^d$ is a real $d \times d$ matrix A with spectrum $\sigma(A) \subset \mathbb{C}_+$. An anisotropy A on $\mathbb{R}^d$ gives rise to a one-parameter group of expansive dilations $(A_t)_{t\in\mathbb{R}_+}$ given by

$$A_t = t^A = \exp[A \ln(t)], \quad t \in \mathbb{R}_+,$$

where $\mathbb{R}_+$ is considered as multiplicative group.

In the special case $A = \mathrm{diag}(\boldsymbol{a})$ with $\boldsymbol{a} = (a_1, \ldots, a_d) \in (0, \infty)^d$, the associated one-parameter group of expansive dilations $(A_t)_{t\in\mathbb{R}_+}$ is given by

$$A_t = \exp[A \ln(t)] = \mathrm{diag}(t^{a_1}, \ldots, t^{a_d}), \quad t \in \mathbb{R}_+$$

Given an anisotropy A on $\mathbb{R}^d$, an A-homogeneous distance function is a Borel measurable mapping $\rho : \mathbb{R}^d \longrightarrow [0, \infty)$ satisfying

(i) $\rho(x) = 0$ if and only if $x = 0$ (*non-degenerate*);
(ii) $\rho(A_t x) = t\rho(x)$ for all $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ ($(A_t)_{t\in\mathbb{R}_+}$-*homogeneous*);
(iii) there exists $c \in [1, \infty)$ so that $\rho(x+y) \leq c(\rho(x)+\rho(y))$ for all $x, y \in \mathbb{R}^d$ (*quasi-triangle inequality*). The smallest such c is denoted c_ρ.

Any two homogeneous quasi-norms ρ_1, ρ_2 associated with an anisotropy A on $\mathbb{R}^d$ are equivalent in the sense that

$$\rho_1(x) \eqsim_{\rho_1,\rho_2} \rho_2(x), \qquad x \in \mathbb{R}^d.$$

If ρ is a quasi-norm associated with an anisotropy A on $\mathbb{R}^d$ and λ denotes the Lebesgue measure on $\mathbb{R}^d$, then $(\mathbb{R}^d, \rho, \lambda)$ is a space of homogeneous type.

Given an anisotropy A on $\mathbb{R}^d$, we define the quasi-norm ρ_A associated with A as follows: we put $\rho_A(0) := 0$ and for $x \in \mathbb{R}^d \setminus \{0\}$ we define $\rho_A(x)$ to be the unique number

$\rho_A(x) = \lambda \in (0, \infty)$ for which $A_{\lambda^{-1}}x \in S^{d-1}$, where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. Then

$$\rho_A(x) := \min\{\lambda > 0 : |A_{\lambda^{-1}}x| \leq 1\}, \qquad x \neq 0.$$

The quasi-norm ρ_A is C^∞ on $\mathbb{R}^d \setminus \{0\}$. We write

$$B^A(x,r) := B_{\rho_A}(x,r) = \{y \in \mathbb{R}^d : \rho_A(x-y) \leq r\}, \qquad x \in \mathbb{R}^d, r \in (0,\infty).$$

We furthermore write $c_A := c_{\rho_A}$.

Given an anisotropy A on $\mathbb{R}^d$, we write

$$\lambda^A_{\min} := \min\{\mathrm{Re}\,(\lambda) : \lambda \in \sigma(A)\}, \quad \lambda^A_{\max} := \max\{\mathrm{Re}\,(\lambda) : \lambda \in \sigma(A)\}.$$

Note that $0 < \lambda^A_{\min} \leq \lambda^A_{\max} < \infty$. Given $\varepsilon \in (0, \lambda^A_{\min})$, it holds that

$$\begin{aligned} t^{\lambda^A_{\min}-\varepsilon}|x| \quad &\lesssim_\varepsilon |A_t x| \quad \lesssim_\varepsilon t^{\lambda^A_{\max}+\varepsilon}|x|, \qquad |t| \geq 1, \\ t^{\lambda^A_{\max}+\varepsilon}|x| \quad &\lesssim_\varepsilon |A_t x| \quad \lesssim_\varepsilon t^{\lambda^A_{\min}-\varepsilon}|x|, \qquad |t| \leq 1, \end{aligned}$$

and

$$\begin{aligned} t^{1/(\lambda^A_{\max}+\varepsilon)}\rho_A(x) \quad &\lesssim_\varepsilon \rho_A(tx) \quad \lesssim_\varepsilon t^{1/(\lambda^A_{\min}-\varepsilon)}\rho_A(x), \qquad |t| \geq 1, \\ t^{1/(\lambda^A_{\min}-\varepsilon)}\rho_A(x) \quad &\lesssim_\varepsilon \rho_A(tx) \quad \lesssim_\varepsilon t^{1/(\lambda^A_{\max}+\varepsilon)}\rho_A(x), \qquad |t| \leq 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} \rho_A(x)^{\lambda^A_{\min}-\varepsilon} \quad &\lesssim_\varepsilon |x| \quad \lesssim_\varepsilon \rho_A(x)^{\lambda^A_{\max}+\varepsilon}, \qquad |x| \geq 1, \\ \rho_A(x)^{\lambda^A_{\max}+\varepsilon} \quad &\lesssim_\varepsilon |x| \quad \lesssim_\varepsilon \rho_A(x)^{\lambda^A_{\min}-\varepsilon}, \qquad |x| \leq 1, \end{aligned}$$

2.1.2. $\mathpzc{d}$-Decompositions and anisotropy

Let $\mathpzc{d} = (\mathpzc{d}_1, \ldots, \mathpzc{d}_\ell) \in (\mathbb{N}_1)^\ell$ be such that $d = |\mathpzc{d}|_1 = \mathpzc{d}_1 + \cdots + \mathpzc{d}_\ell$. The decomposition

$$\mathbb{R}^d = \mathbb{R}^{\mathpzc{d}_1} \times \cdots \times \mathbb{R}^{\mathpzc{d}_\ell}.$$

is called the *$\mathpzc{d}$-decomposition* of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ we accordingly write $x = (x_1, \ldots, x_\ell)$ and $x_j = (x_{j,1}, \ldots, x_{j,\mathpzc{d}_j})$, where $x_j \in \mathbb{R}^{\mathpzc{d}_j}$ and $x_{j,i} \in \mathbb{R}$ ($j = 1, \ldots, \ell; i = 1, \ldots, \mathpzc{d}_j$). We also say that we view $\mathbb{R}^d$ as being *$\mathpzc{d}$-decomposed*. Furthermore, for each $k \in \{1, \ldots, \ell\}$ we define the inclusion map

$$\iota_k = \iota_{[\mathpzc{d};k]} : \mathbb{R}^{\mathpzc{d}_k} \longrightarrow \mathbb{R}^n, \; x_k \mapsto (0, \ldots, 0, x_k, 0, \ldots, 0), \tag{7}$$

and the projection map

$$\pi_k = \pi_{[\mathpzc{d};k]} : \mathbb{R}^n \longrightarrow \mathbb{R}^{\mathpzc{d}_k}, \; x = (x_1, \ldots, x_\ell) \mapsto x_k.$$

A $\mathpzc{d}$-anisotropy is a tuple $\boldsymbol{A} = (A_1, \ldots, A_\ell)$ with each A_j an anisotropy on $\mathbb{R}^{\mathpzc{d}_j}$. A $\mathpzc{d}$-anisotropy $\boldsymbol{A}$ gives rise to a one-parameter group of expansive dilations $(\boldsymbol{A}_t)_{t \in \mathbb{R}_+}$ given by

$$\boldsymbol{A}_t x = (A_{1,t}x_1, \ldots, A_{\ell,t}x_l), \qquad x \in \mathbb{R}^d, t \in \mathbb{R}_+,$$

where $A_{j,t} = \exp[A_j \ln(t)]$. Note that $\boldsymbol{A}^\oplus := \oplus_{j=1}^\ell A_j$ is an anisotropy on $\mathbb{R}^d$ with $\boldsymbol{A}^\oplus_t = \boldsymbol{A}_t$ for every $t \in \mathbb{R}_+$. We define the $\boldsymbol{A}^\oplus$-homogeneous distance function $\rho_{\boldsymbol{A}}$ by

$$\rho_{\boldsymbol{A}}(x) := \max\{\rho_{A_1}(x_1), \ldots, \rho_{A_\ell}(x_\ell)\}, \qquad x \in \mathbb{R}^d.$$

We write

$$B^{\boldsymbol{A}}(x,R) := B_{\rho_{\boldsymbol{A}}}(x,R), \qquad x \in \mathbb{R}^d, R \in (0,\infty),$$

and

$$B^{\boldsymbol{A}}(x, \boldsymbol{R}) := B^{A_1}(x_1, R_1) \times \cdots \times B^{A_\ell}(x_\ell, R_\ell), \qquad x \in \mathbb{R}^d, \boldsymbol{R} \in (0, \infty)^\ell.$$

Note that $B^{\boldsymbol{A}}(x, R) = B^{\boldsymbol{A}}(x, \boldsymbol{R})$ when $\boldsymbol{R} = (R, \dots, R)$.

2.2. *Quasi-Banach function spaces*

For the theory of quasi-Banach spaces, or more generally, F-spaces, we refer the reader to [29,30].

Let Y be a vector space. A semi-quasi-norm is a mapping $p : Y \longrightarrow [0, \infty)$ with the following two properties:

- *Homogeneity.* $p(\lambda y) = |\lambda| \cdot p(y)$ for all $y \in Y$ and $\lambda \in \mathbb{C}$.
- *Quasi-triangle inequality.* There exists a finite constant $c \geq 1$ such that, for all $y, z \in Y$,

$$p(y + z) \leq c[p(y) + p(z)].$$

A quasi-norm is a semi-quasi-norm p that satisfies:

- *Definiteness.* If $y \in Y$ satisfies $p(y) = 0$, then $y = 0$.

Let Y be a vector space and $\kappa \in (0, 1]$. A κ-norm is a function $||| \cdot ||| : Y \longrightarrow [0, \infty)$ with the following three properties:

- *Homogeneity.* $|||\lambda y||| = |\lambda| \cdot |||y|||$ for all $y \in Y$ and $\lambda \in \mathbb{C}$.
- *κ-triangle inequality.* For all $y, z \in Y$,

$$|||y + z|||^\kappa \leq |||y|||^\kappa + |||z|||^\kappa .$$

- *Definiteness.* If $y \in Y$ satisfies $|||y||| = 0$, then $y = 0$.

Note that every κ-norm is a quasi-norm. The Aoki–Rolewitz theorem [3,46] says that, conversely, given a quasi-normed space $(Y, \|\cdot\|)$ there exist $r \in (0, 1]$ and an r-norm $||| \cdot |||$ on Y that is equivalent to $\|\cdot\|$.

Let Y be a quasi-Banach space with a quasi-norm that is equivalent to some κ-norm, $\kappa \in (0, 1]$. If $(y_n)_n \subset Y$ satisfies $\sum_{n=0}^\infty \|y_n\|_Y^\kappa < \infty$, then $\sum_{n \in \mathbb{N}} y_n$ converges in Y and $\| \sum_{n=0}^\infty y_n \|_Y \lesssim \left(\sum_{n=0}^\infty \|y_n\|_Y^\kappa \right)^{1/\kappa}$.

Let $(T, \mathscr{B}, \nu)$ be a σ-finite measure space. A quasi-Banach function space F on T is an order ideal in $L_0(T)$ that has been equipped with a quasi-Banach norm $\|\cdot\|$ with the property that $\| \, |f| \, \| = \|f\|$ for all $f \in F$.

A quasi-Banach function space F on T has the Fatou property if and only if, for every increasing sequence $(f_n)_{n \in \mathbb{N}}$ in F with supremum f in $L_0(T)$ and $\sup_{n \in \mathbb{N}} \|f_n\|_F < \infty$, it holds that $f \in F$ with $\|f\|_F = \sup_{n \in \mathbb{N}} \|f_n\|_F$.

Lemma 2.1. *Let V be a quasi-normed space continuously embedded into a complete topological vector space W. Suppose that V has the* Fatou property *with respect to W, i.e. for all $(v_n)_{n \in \mathbb{N}} \subset V$ the following implication holds:*

$$\lim_{n \to \infty} v_n = v \text{ in } W, \; \liminf_{n \to \infty} \|v_n\|_V < \infty \implies v \in V, \; \|f\|_V \leq \liminf_{n \to \infty} \|f_n\|_V .$$

Then V is complete.

2.3. Vector-valued functions and distributions

As general reference to the theory of vector-valued distributions we mention [2] and [51].

Let G be a topological vector space. The space of G-valued tempered distributions $\mathcal{S}'(\mathbb{R}^d; G)$ is defined as $\mathcal{S}'(\mathbb{R}^d; G) := \mathcal{L}(\mathcal{S}(\mathbb{R}^d), G)$, the space of continuous linear operators from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ to G. In this chapter we equip $\mathcal{S}'(\mathbb{R}^d; G)$ with the topology of pointwise convergence. Standard operators (derivatives, Fourier transform, convolution, etc.) on $\mathcal{S}'(\mathbb{R}^d; G)$ can be defined as in the scalar case.

By a combination of [2, Theorem 1.4.3] and (the proof of) [2, Lemma 1.4.6], the space of finite rank operators $\mathcal{S}'(\mathbb{R}^d) \otimes G$ is sequentially dense in $\mathcal{S}'(\mathbb{R}^d; G)$. Furthermore, as a consequence of the Banach–Steinhaus Theorem (see [48, Theorem 2.8]), if G is sequentially complete, then so is $\mathcal{S}'(\mathbb{R}^d; G)$.

Given a quasi-Banach space X, denote by $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^d; X)$ the space of slowly increasing smooth functions on $\mathbb{R}^d$. This means that $f \in \mathscr{O}_{\mathrm{M}}(\mathbb{R}^d; X)$ if and only if $f \in C^{\infty}(\mathbb{R}^d; X)$ and, for each $\alpha \in \mathbb{N}^d$, there exist $m_\alpha \in \mathbb{N}$ and $c_\alpha > 0$ such that

$$\|D^\alpha f(x)\|_X \leq c_\alpha (1 + |x|^2)^{m_\alpha}, \qquad x \in \mathbb{R}^d.$$

The topology of $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^d; X)$ is induced by the family of semi-quasi-norms

$$p_{\phi,\alpha}(f) := \|\phi D^\alpha f\|_\infty, \qquad \phi \in \mathcal{S}(\mathbb{R}^d), \alpha \in \mathbb{N}^d.$$

Let $(T, \mathscr{B}, \nu)$ be a σ-finite measure space and let G be a topological vector space. We define $L_0(T; G)$ as the space as of all ν-a.e. equivalence classes of ν-strongly measurable functions $f : T \to G$. Suppose there is a system $\mathcal{Q}$ of semi-quasi-norms generating the topology of G. We equip $L_0(T; G)$ with the topology generated by the translation invariant pseudo-metrics

$$\rho_{B,q}(f, g) := \int_B (q(f - g) \wedge 1) \,\mathrm{d}\nu, \qquad B \in \mathscr{B}, \nu(B) < \infty, q \in \mathcal{Q}.$$

This topological vector space topology on $L_0(T; G)$ is independent of $\mathcal{Q}$ and is called the topology of convergence in measure. Note that $L_0(T) \otimes G$ is sequentially dense in $L_0(T; G)$ as a consequence of the dominated convergence theorem and the definitions.

If G is an F-space, then $L_0(T; G)$ is an F-space as well. Here we could for example take $G = L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)$ with $\boldsymbol{r} \in (0, \infty]^\ell$ and X a Banach space, where

$$L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d) = \left\{ f \in L_0(\mathbb{R}^d) : f 1_B \in L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d), B \subset \mathbb{R}^d \text{ bounded Borel} \right\}$$

and

$$L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d) = L_{r_\ell}(\mathbb{R}^{d_\ell})[\ldots[L_{r_1}(\mathbb{R}^{d_1})]\ldots].$$

Let X be a Banach space. Then $L_0(T) \otimes \mathcal{S}'(\mathbb{R}^d) \otimes X$ is sequentially dense in both of $L_0(T; \mathcal{S}'(\mathbb{R}^d; X))$ and $\mathcal{S}'(\mathbb{R}^d; L_0(T; X))$, while the two induced topologies on $L_0(T) \otimes \mathcal{S}'(\mathbb{R}^d) \otimes X$ coincide. Therefore, we can naturally identify

$$L_0(T; \mathcal{S}'(\mathbb{R}^d; X)) \cong \mathcal{S}'(\mathbb{R}^d; L_0(T; X)).$$

A function $g : T \longrightarrow X^*$ is called $\sigma(X^*, X)$-*measurable* (or X-*weakly measurable*) if $\langle x, g \rangle : T \longrightarrow \mathbb{C}$ is measurable for all $x \in X$. We denote by $L^0(T; X^*, \sigma(X^*, X))$ the vector space of all μ-a.e. equivalence classes of $\sigma(X^*, X)$-*measurable* functions $g : T \longrightarrow X^*$.

As in [44], we may define the *abstract norm* $\vartheta : L_0(T; X^*, \sigma(X^*, X)) \longrightarrow L_0(T)$ by

$$\vartheta(g) := \sup\{ |\langle x, g \rangle| : x \in B_X \}, \qquad g \in L_0(T; X^*, \sigma(X^*, X)).$$

Note that $L_0(T; X^*) \subset L_0(T; X^*, \sigma(X^*, X))$ and that $\vartheta(g) = \|g\|_{X^*}$ for all $g \in L_0(T; X^*)$.

We equip $L_0(T; X^*, \sigma(X^*, X))$ with the topology generated by the system of translation invariant pseudo-metrics

$$\rho_B(f, g) := \int_B (\vartheta(f-g) \wedge 1) \mathrm{d}\nu, \qquad B \in \mathscr{B}, \nu(B) < \infty.$$

In this way, $L_0(T; X^*, \sigma(X^*, X))$ becomes a topological vector space.

For a Banach function space E on T we define $E(X^*, \sigma(X^*, X))$ by

$$E(X^*, \sigma(X^*, X)) := \{f \in L_0(T; X^*, \sigma(X^*, X)) : \vartheta(f) \in E\}.$$

Endowed with the norm

$$\|f\|_{E(X^*,\sigma(X^*,X))} := \|\vartheta(f)\|_E,$$

$E(X^*, \sigma(X^*, X))$ becomes a Banach space.

Let E be a Banach function space on T with an order continuous norm and a weak order unit (i.e. an element $\xi \in E$ with $\xi > 0$ pointwise a.e.). Then (see [44])

$$[E(X)]^* = E^\times(X^*, \sigma(X^*, X))$$

under the natural pairing, where $E^\times$ is the Köthe dual of E given by

$$E^\times = \{g \in L_0(T) : \forall f \in E, fg \in L_1(T)\}, \quad \|g\|_{E^\times} = \sup_{f \in E, \|f\|_E \leq 1} \int_T fg \, \mathrm{d}\nu.$$

Moreover, if X^* has the Radon–Nykodým property with respect to ν, then

$$[E(X)]^* = E^\times(X^*, \sigma(X^*, X)) = E^\times(X^*).$$

3. Definitions and basic properties

Suppose that $\mathbb{R}^d$ is $\mathpzc{d}$-decomposed with $\mathpzc{d} \in (\mathbb{N}_1)^\ell$ and let $\boldsymbol{A} = (A_1, \dots, A_\ell)$ be a $\mathpzc{d}$-anisotropy. Let X be a Banach space, $(S, \mathscr{A}, \mu)$ a σ-finite measure space, $\varepsilon_+, \varepsilon_- \in \mathbb{R}$ and $\boldsymbol{r} \in (0, \infty)^\ell$.

For $j \in \{1, \dots, \ell\}$, we define the maximal function operator $M^{A_j}_{r_j;[\mathpzc{d};j]}$ on $L_0(S \times \mathbb{R}^d)$ by

$$M^{A_j}_{r_j;[\mathpzc{d};j]}(f)(s, x) := \sup_{\delta>0} \fint_{B^{A_j}(0,\delta)} |f(s, x + \iota_{[\mathpzc{d};j]} y_j)| \, \mathrm{d}y_j,$$

where $\iota_{[\mathpzc{d};j]} : \mathbb{R}^{d_j} \to \mathbb{R}^d$ is the inclusion mapping from (7). We define the maximal function operator $M^{\boldsymbol{A}}_{\boldsymbol{r}}$ by iteration:

$$M^{\boldsymbol{A}}_{\boldsymbol{r}}(f) := M^{A_\ell}_{r_\ell;[\mathpzc{d};\ell]}(\dots(M^{A_1}_{r_1;[\mathpzc{d};1]}(f))\dots).$$

We write $M^{\boldsymbol{A}} := M^{\boldsymbol{A}}_{\boldsymbol{1}}$.

The following definition is an extension of [24, Definition 1.1.1] to the anisotropic setting with some extra underlying measure space $(S, \mathscr{A}, \mu)$. The extra measure space provides the right setting for intersection representations, see Section 5.

Definition 3.1. We define $\mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ as the set of all quasi-Banach function spaces E on $\mathbb{R}^d \times \mathbb{N} \times S$ with the Fatou property for which the following two properties are fulfilled:

(a) S_+, S_-, the left respectively right shift on $\mathbb{N}$, are bounded on E with

$$\|(S_+)^k\|_{\mathcal{B}(E)} \lesssim 2^{-\varepsilon_+ k} \quad \text{and} \quad \|(S_-)^k\|_{\mathcal{B}(E)} \lesssim 2^{\varepsilon_- k}, \qquad k \in \mathbb{N}.$$

(b) $M_{\boldsymbol{r}}^{\boldsymbol{A}}$ is bounded on E:

$$\|M_{\boldsymbol{r}}^{\boldsymbol{A}}(f_n)\|_E \lesssim \|(f_n)\|_E, \qquad (f_n) \in E.$$

We similarly define $\mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r})$ without the presence of $(S, \mathscr{A}, \mu)$, or equivalently, $\mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}) = \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (\{0\}, \{\varnothing, \{0\}\}, \#))$.

Remark 3.2. Note that $\varepsilon_+ \leq \varepsilon_-$ when $E \neq \{0\}$, which can be seen by considering $(S_+)^k \circ (S_-)^k$, $k \in \mathbb{N}$.

Remark 3.3. Note that

$$\mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu)) \subset \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \tilde{\boldsymbol{r}}, (S, \mathscr{A}, \mu)), \quad \boldsymbol{r} \geq \tilde{\boldsymbol{r}}.$$

Example 3.4. Suppose that $\ell = 1$ and $\boldsymbol{A} = A = I_d$, so that we are in the classical isotropic setting. Then $\boldsymbol{r} = r \in (0, \infty)$ and

$$M_{\boldsymbol{r}}^{\boldsymbol{A}}(f)(x) = M_r(f)(x) = \sup_{\delta>0} \Big(\fint_{B(0,\delta)} |f(x+y)|^r \,\mathrm{d}y \Big)^{1/r}$$

on $L_0(\mathbb{R}^d)$. By the Fefferman–Stein vector-valued maximal inequality (see e.g. [55, Section 1.2.3]) and the Hardy–Littlewood maximal inequality, we thus obtain the following examples.

(i) Let $p \in (0, \infty)$, $q \in (0, \infty]$ and $s \in \mathbb{R}$. If $r \in (0, \infty)$ is such that $r < p \wedge q$, then

$$E = L_p(\mathbb{R}^d)[\ell_q^s(\mathbb{N})] \in \mathcal{S}(s, s, I_d, r).$$

(ii) Let $p \in (0, \infty)$, $q \in (0, \infty]$ and $s \in \mathbb{R}$. If $r \in (0, \infty)$ is such that $r < p$, then

$$E = \ell_q^s(\mathbb{N})[L_p(\mathbb{R}^d)] \in \mathcal{S}(s, s, I_d, r).$$

The following example generalizes the previous example to the anisotropic weighted mixed-norm setting. Furthermore, it also goes beyond the case of a trivial underlying measure space $(S, \mathscr{A}, \mu)$.

Example 3.5. Let us give some concrete choices of $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. Condition (b) in Definition 3.1 can be covered by means of the lattice Hardy–Littlewood maximal function operator: if F is a UMD Banach function space on S, A an anisotropy, $p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^d, A)$ then (see [8,18,19,47,54])

$$Mf(x) := \sup_{\delta>0} \fint_{B^A(x,\delta)} |f(y)|\, dy$$

defines a bounded sublinear operator on $L_p(\mathbb{R}^d, w; F) = L_p(\mathbb{R}^d, w)[F]$. The latter induces a bounded sublinear operator on $L_p(\mathbb{R}^d, w)[F[\ell_\infty]]$ in the natural way. Let us furthermore remark that the mixed-norm space $F[G]$ of two UMD Banach function spaces F and G is again a UMD Banach function space (see [47, page 214]). This leads to the following examples:

(i) Let $\boldsymbol{p} \in (0, \infty)^\ell$, $q \in (0, \infty]$, $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_\infty(\mathbb{R}^{d_j}, A_j)$ and $s \in \mathbb{R}$. If $\boldsymbol{r} \in (0, \infty)^\ell$ is such that $r_j < p_1 \wedge \cdots \wedge p_j \wedge q$ for $j = 1, \ldots, \ell$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j/r_j}(\mathbb{R}^{d_j}, A_j)$, then

$$E = L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})[\ell_q^s(\mathbb{N})] \in \mathcal{S}(s, s, \boldsymbol{A}, \boldsymbol{r}).$$

(ii) Let $\boldsymbol{p} \in (0,\infty)^\ell$, $q \in (0,\infty]$, $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_\infty(\mathbb{R}^{d_j}, A_j)$ and $s \in \mathbb{R}$. If $\boldsymbol{r} \in (0,\infty)^\ell$ is such that $r_j < p_1 \wedge \cdots \wedge p_j$ for $j = 1, \ldots, \ell$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j/r_j}(\mathbb{R}^{d_j}, A_j)$, then

$$E = \ell_q^s(\mathbb{N})[L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})] \in \mathcal{S}(s, s, \boldsymbol{A}, \boldsymbol{r}).$$

(iii) Let $\boldsymbol{p} \in (0,\infty)^\ell$, $q \in (0,\infty]$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_\infty(\mathbb{R}^{d_j}, A_j)$, $s \in \mathbb{R}$ and F a quasi-Banach function space on S. If $\boldsymbol{r} \in (0,\infty)^\ell$ is such that $r_j < p_1 \wedge \cdots \wedge p_j \wedge q$ for $j = 1, \ldots, \ell$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j/r_j}(\mathbb{R}^{d_j}, A_j)$ and $F^{[\boldsymbol{r}_{\max}]}$ is a UMD Banach function space, where

$$F^{[r]} := \{f \in L_0(S) : |f|^{1/r} \in F\}, \qquad \|f\|_{F^{[r]}} := \||f|^{1/r}\|_F^r,$$

then

$$E = L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})[F[\ell_q^s(\mathbb{N})]] \in \mathcal{S}(s, s, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu)).$$

Remark 3.6. Note that we can take $\boldsymbol{r} = \boldsymbol{1}$ in Example 3.5 when, in each of the corresponding examples:

(i) $\boldsymbol{p} \in (1,\infty)^\ell$, $q \in (1,\infty]$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j}(\mathbb{R}^{d_j}, A_j)$;
(ii) $\boldsymbol{p} \in (1,\infty)^\ell$, $q \in (0,\infty]$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j}(\mathbb{R}^{d_j}, A_j)$;
(iii) $\boldsymbol{p} \in (1,\infty)^\ell$, $q \in (1,\infty]$, $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j}(\mathbb{R}^{d_j}, A_j)$ and F is a UMD Banach function space.

For a quasi-Banach function space E on $\mathbb{R}^d \times \mathbb{N} \times S$ we define the quasi-Banach function space $E_\otimes^{\boldsymbol{A}}$ on S by

$$\|f\|_{E_\otimes^{\boldsymbol{A}}} := \|1_{B^{\boldsymbol{A}}(0,1)\times\{0\}} \otimes f\|_E, \qquad f \in L_0(S).$$

Note that $E_\otimes^{\boldsymbol{A}} \cong \mathbb{C}$ in case that $(S, \mathscr{A}, \mu) = (\{0\}, \{\varnothing, \{0\}\}, \#)$.

Example 3.7. In the situation of Example 3.5(iii), $E_\otimes^{\boldsymbol{A}} = F$ with

$$\|f\|_{E_\otimes^{\boldsymbol{A}}} = \|1_{B^{\boldsymbol{A}}(0,1)}\|_{L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})} \|f\|_F, \qquad f \in F.$$

Let $\boldsymbol{p} \in (0,\infty)^\ell$ and $w : [1,\infty)^\ell \to (0,\infty)$. We define the quasi-Banach function space

$$B_{\boldsymbol{A}}^{\boldsymbol{p},w} := \Big\{ f \in L_0(S) : \sup_{\boldsymbol{R} \in [1,\infty)^\ell} w(\boldsymbol{R}) \|f\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,\boldsymbol{R}))} < \infty \Big\} \tag{8}$$

which is an extension of (a slight variant of) the space B^p considered by Beurling in [7] (see [45]).

Let $\boldsymbol{p}, \boldsymbol{q} \in (0,\infty)^\ell$. We define $w_{\boldsymbol{A},\boldsymbol{q}} : [1,\infty)^\ell \to \mathbb{R}_+$ by

$$w_{\boldsymbol{A},\boldsymbol{q}}(\boldsymbol{R}) := \boldsymbol{R}^{-\mathrm{tr}(\boldsymbol{A})\boldsymbol{q}^{-1}} = \prod_{j=1}^{\ell} R_j^{-\mathrm{tr}(A_j)/q_j}, \qquad \boldsymbol{R} \in [1,\infty)^\ell.$$

The quasi-Banach function space $B_{\boldsymbol{A}}^{\boldsymbol{p},w_{\boldsymbol{A},\boldsymbol{q}}} \hookrightarrow L_{\boldsymbol{p},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d)$ introduced in (8) will be convenient to formulate some of the estimates we will obtain. Note that, if $\boldsymbol{p} \in [1,\infty)^\ell$, then

$$B_{\boldsymbol{A}}^{\boldsymbol{p},w_{\boldsymbol{A},\boldsymbol{q}}}(X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X).$$

Lemma 3.8. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and $\lambda \in (-\infty, \varepsilon_+)$. For $F = (f_n)_n \in E$ and $g := \sum_{n=0}^{\infty} 2^{n\lambda}|f_n|$ we have*

$$\|(\delta_{0,n} g)_n\|_E \lesssim \|F\|_E. \tag{9}$$

Moreover, $g \in E^{A}_{\otimes}[B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}] \hookrightarrow E^{A}_{\otimes}[L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d)]$ *with*

$$\|g\|_{E^{A}_{\otimes}[B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}]} \lesssim \|F\|_E. \tag{10}$$

Remark 3.9. Suppose that $\varepsilon_+ > 0$ and $\lambda \in (0, \varepsilon_+)$ in Lemma 3.8. Let $\kappa \in (0, 1]$ with $\kappa \leq \boldsymbol{r}_{\min}$ be such that $\|\cdot\|_E$ is a equivalent to a κ-norm. Then, in particular, $2^{n\lambda} f_n \in E^{A}_{\otimes}[B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}]$ with $\|2^{n\lambda} f_n\|_{E^{A}_{\otimes}[B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}]} \lesssim \|F\|_E$, so that

$$\sum_{n=0}^{\infty} \|f_n\|^{\kappa}_{E^{A}_{\otimes}[B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}]} = \sum_{n=0}^{\infty} 2^{-n\lambda\kappa} \|2^{n\lambda} f_n\|^{\kappa}_{E^{A}_{\otimes}[B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}]} \lesssim \sum_{n=0}^{\infty} 2^{-n\lambda\kappa} \|F\|_E \lesssim \|F\|_E.$$

Remark 3.10. Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. Similarly to the proof of Lemma 3.8 (but simpler) it can be shown that

$$E_i \hookrightarrow E^{A}_{\otimes}[B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}].$$

Proof of Lemma 3.8. This can be shown similarly to [24, Lemma 1.1.4]. Let us just provide the details for (10). As $|B^{A_j}(x_j, R_j)| \eqsim R_j^{\mathrm{tr}(A_j)/r_j}$, $j = 1, \dots, \ell$, for any $x \in \mathbb{R}^d$ and $\boldsymbol{R} \in (0, \infty)^{\ell}$, we have

$$1_{B^{A}(0,\boldsymbol{R})} \otimes \|g\|_{L_{\boldsymbol{r},\mathscr{d}}(B^{A}(0,\boldsymbol{R}))} \lesssim \prod_{j=1}^{\ell} R_j^{\mathrm{tr}(A_j)/r_j} M^{\boldsymbol{A}}_{\boldsymbol{r}}(g), \qquad \boldsymbol{R} \in [1, \infty)^{\ell}.$$

Therefore,

$$1_{B^{A}(0,\boldsymbol{1})} \otimes w_{A,\boldsymbol{r}}(\boldsymbol{R}) \|g\|_{L_{\boldsymbol{r},\mathscr{d}}(B^{A}(0,\boldsymbol{R}))} \lesssim M^{\boldsymbol{A}}_{\boldsymbol{r}}(g), \qquad \boldsymbol{R} \in [1, \infty)^{\ell},$$

so that

$$1_{B^{A}(0,\boldsymbol{1})} \otimes \|g\|_{B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}} \lesssim M^{\boldsymbol{A}}_{\boldsymbol{r}}(g).$$

It thus follows that

$$\|g\|_{E^{A}_{\otimes}[B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}]} = \left\| 1_{B^{A}(0,\boldsymbol{1}) \times \{0\}} \otimes \|g\|_{B^{\boldsymbol{r},w_{A,\boldsymbol{r}}}_{\boldsymbol{A}}} \right\|_E \lesssim \|M^{\boldsymbol{A}}_{\boldsymbol{r}}(\delta_{0,n} g)_n\|_E.$$

Using the boundedness of $M^{\boldsymbol{A}}_{\boldsymbol{r}}$ on E in combination with (9) we obtain the desired estimate (10). □

Having introduced the classes of 'admissible' quasi-Banach function spaces in Definition 3.1 and having discussed some basic properties of these, let us now proceed with the associated function spaces. Let us for introductory purposes first have a look at the classical isotropic Lizorkin–Triebel and Besov spaces.

In the setting of Example 3.4, we would like to associate to $E = L_p(\mathbb{R}^d)[\ell^s_q(\mathbb{N})]$ and $E = \ell^s_q(\mathbb{N})[L_p(\mathbb{R}^d)]$ the classical Lizorkin–Triebel space $Y(E) = F^s_{p,q}(\mathbb{R}^d)$ and the classical Besov space $Y(E) = B^s_{p,q}(\mathbb{R}^d)$, respectively.

A standard way to define the Lizorkin–Triebel and Besov spaces is by means of a smooth resolution of unity/Littlewood–Paley decomposition, as in [55, Section 2.3.1, Definition 2]. However, there are many other ways. For instance, $F^s_{p,q}(\mathbb{R}^d)$ and $B^s_{p,q}(\mathbb{R}^d)$ could alternatively be defined through the Nikol'skij representations as in [55, Section 2.5.2] (also see the references therein), which may be characterized as a "decomposition of the given distribution by entire analytic functions of exponential type". This decomposition is a representation as a series of entire analytic functions of exponential type whose spectra lie in dyadic annuli. The annuli can be even replaced by balls when $s > d(\frac{1}{r}-1)_+$, where r is as in Example 3.4, see [49, Section 2.3.2], [28, Section 3.6] or [24, Proposition 1.1.12]. Moreover, in the latter situation, $F^s_{p,q}(\mathbb{R}^d)$ and $B^s_{p,q}(\mathbb{R}^d)$ consist of regular distributions and the series not only converges in a distributional sense (in $\mathcal{S}'$) but also in a measure theoretic sense (in $L_{1,\mathrm{loc}}$). The characterization through the series representation with the dyadic ball condition and the convergence in a measure theoretic sense, valid under the restriction $s > d(\frac{1}{r}-1)_+$, has turned out to be quite useful. Such a description is taken as the definition of the spaces of measurable functions $FL^s_{p,q}(\mathbb{R}^d)$ and $BL^s_{p,q}(\mathbb{R}^d)$ for $s \in (0,\infty)$, so that $F^s_{p,q}(\mathbb{R}^d) = FL^s_{p,q}(\mathbb{R}^d)$ and $B^s_{p,q}(\mathbb{R}^d) = BL^s_{p,q}(\mathbb{R}^d)$ when $s > d(\frac{1}{r}-1)_+$. As is mentioned in [24, page 9], the spaces $FL^s_{p,q}(\mathbb{R}^d)$ and $BL^s_{p,q}(\mathbb{R}^d)$ have been less studied in the range $s \leq d(\frac{1}{r}-1)_+$, where they do not coincide with the Lizorkin–Triebel and Besov spaces, but see [42,43].

We will associate to $E = L_p(\mathbb{R}^d)[\ell^s_q(\mathbb{N})]$ and $E = \ell^s_q(\mathbb{N})[L_p(\mathbb{R}^d)]$ the spaces of distributions $Y(E) = F^s_{p,q}(\mathbb{R}^d)$ and $Y(E) = B^s_{p,q}(\mathbb{R}^d)$, respectively, through the Nikol'skij representation discussed above. We will furthermore associate to these choices of E, under the restriction that $s \in (0,\infty)$, the respective spaces of measurable functions $YL(E) = FL^s_{p,q}(\mathbb{R}^d)$ and $YL(E) = BL^s_{p,q}(\mathbb{R}^d)$.

Let us now turn back to the general setting. In Definitions 3.11 and 3.12 we will define the spaces $YL^{\boldsymbol{A}}(E;X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E;X)$, respectively, which are both generalizations of $YL(E)$ from [24, Definition 1.1.15] to our setting. The difference between $YL^{\boldsymbol{A}}(E;X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ will be a matter of the X-valued setting. Whereas $YL^{\boldsymbol{A}}(E;X)$ will be defined in a more straightforward way, simply replacing E by $E(X)$ compared to the scalar-valued setting, the definition of $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ will be more technical, involving testing with functionals $x^* \in X^*$ in combination with, and in interplay with, some kind of domination. The motivation for the more technical space $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ comes from Remark 4.5 on estimates involving differences.

In Definition 3.15 we will define the space $Y^{\boldsymbol{A}}(E;X)$ through a Nikol'skij representation type of approach, which is a generalization of $Y(E)$ from [24, Definition 1.1.16] to our setting. The equivalent Littlewood–Paley description will follow in Proposition 3.19. Concrete examples will be given Example 3.20, which includes the classical Lizorkin–Triebel and Besov spaces discussed above. Furthermore, in Theorem 3.22 we will see that, under a suitable restriction, $Y^{\boldsymbol{A}}(E;X)$ coincides with $YL^{\boldsymbol{A}}(E;X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E;X)$.

Definition 3.11. Suppose that $\varepsilon_+, \varepsilon_- > 0$ and let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. We define $YL^{\boldsymbol{A}}(E;X)$ as the space of all $f \in L_0(S; L_{\boldsymbol{r},\boldsymbol{d},\mathrm{loc}}(\mathbb{R}^d;X))$ which have a representation

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; L_{\boldsymbol{r},\boldsymbol{d},\mathrm{loc}}(\mathbb{R}^d;X))$$

with $(f_n)_n \subset L_0(S; \mathcal{S}'(\mathbb{R}^d;X))$ satisfying the spectrum condition

$$\operatorname{supp} \hat{f}_n \subset \overline{B}^{\boldsymbol{A}}(0, 2^{n+1}), \qquad n \in \mathbb{N},$$

and $(f_n)_n \in E(X)$. We equip $YL^{\boldsymbol{A}}(E;X)$ with the quasinorm

$$\|f\|_{YL^{\boldsymbol{A}}(E;X)} := \inf \|(f_n)\|_{E(X)},$$

where the infimum is taken over all representations as above. We write $YL^{\boldsymbol{A}}(E) := YL^{\boldsymbol{A}}(E;\mathbb{C})$.

Definition 3.12. Suppose that $\varepsilon_+, \varepsilon_- > 0$ and let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. We define $\widetilde{YL}^{\boldsymbol{A}}(E; X)$ as the space of all $f \in L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X))$ for which there exists $(g_n)_n \in E_+$ such that, for all $x^* \in X^*$, $\langle f, x^* \rangle$ has a representation

$$\langle f, x^* \rangle = \sum_{n=0}^{\infty} f_{x^*,n} \quad \text{in} \quad L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d))$$

with $(f_{x^*,n})_n \subset L_0(S; \mathcal{S}'(\mathbb{R}^d))$ satisfying the spectrum condition

$$\operatorname{supp} \hat{f}_{x^*,n} \subset \overline{B}^{\boldsymbol{A}}(0, 2^{n+1}), \qquad n \in \mathbb{N},$$

and the domination $|f_{x^*,n}| \leq \|x^*\| g_n$. We equip $\widetilde{YL}^{\boldsymbol{A}}(E; X)$ with the quasinorm

$$\|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} := \inf \|(g_n)\|_E,$$

where the infimum is taken over all $(g_n)_n$ as above. We write $\widetilde{YL}^{\boldsymbol{A}}(E) := \widetilde{YL}^{\boldsymbol{A}}(E; \mathbb{C})$.

Remark 3.13. Note that $\widetilde{YL}^{\boldsymbol{A}}(E) = YL^{\boldsymbol{A}}(E)$.

Remark 3.14. Suppose that $\varepsilon_+, \varepsilon_- > 0$ and let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. Then the following statements hold:

(i) $YL^{\boldsymbol{A}}(E; X) \subset \widetilde{YL}^{\boldsymbol{A}}(E; X)$ with equality of norms.

(ii) Let $f \in YL^{\boldsymbol{A}}(E; X)$ with $(f_n)_n$ as in Definition 3.11 with $\|(f_n)_n\|_{E(X)} \leq 2\|f\|_{YL^{\boldsymbol{A}}(E;X)}$. Let $\tilde{\boldsymbol{r}} \in (0, \infty)^\ell$ be such that

$$E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \tilde{\boldsymbol{r}}, (S, \mathscr{A}, \mu)). \tag{11}$$

Then, by Remark 3.9, as

$$E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X)) \hookrightarrow L_0(S; L_{\tilde{\boldsymbol{r}},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)) \hookrightarrow L_0(S; L_{\boldsymbol{r} \tilde{\wedge} \boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)),$$

there is the convergence $f = \sum_{n=0}^{\infty} f_n$ in $E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X))$ with

$$\|f\|_{E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X))} \lesssim \|(f_n)_n\|_{E(X)} \leq 2\|f\|_{YL^{\boldsymbol{A}}(E;X)}.$$

In particular, $YL^{\boldsymbol{A}}(E; X)$ does not depend on $\boldsymbol{r}$ and

$$YL^{\boldsymbol{A}}(E; X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{\boldsymbol{r}, w_{\boldsymbol{A},\boldsymbol{r}}}(X)).$$

(iii) Let $f \in \widetilde{YL}^{\boldsymbol{A}}(E; X)$ with $(g_n)_n \in E_+$ and $\{f_{x^*,n}\}_{(x^*,n)}$ as in Definition 3.12 with $\|(g_n)_n\|_E \leq 2\|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$. Let $\tilde{\boldsymbol{r}} \in (0, \infty)^\ell$ satisfy (11). Then $\|f\|_X \leq \sum_{n=0}^{\infty} g_n$, so that $f \in E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X)) \subset L_0(S; L_{\tilde{\boldsymbol{r}},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X))$ with

$$\|f\|_{E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X))} \lesssim \|(g_n)_n\|_E \leq 2\|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$$

by Remark 3.9. By (ii) it furthermore holds that

$$\langle f, x^* \rangle = \sum_{n=0}^{\infty} f_{x^*,n} \quad \text{in} \quad L_0(S; L_{\tilde{\boldsymbol{r}},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d)).$$

Therefore, $\widetilde{YL}^{\boldsymbol{A}}(E; X)$ does not depend on $\boldsymbol{r}$ and

$$\widetilde{YL}^{\boldsymbol{A}}(E; X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{\boldsymbol{r}, w_{\boldsymbol{A},\boldsymbol{r}}}(X)).$$

Definition 3.15. Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. We define $Y^{\boldsymbol{A}}(E; X)$ as the space of all $f \in L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ which have a representation

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$$

with $(f_n)_n \subset L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ satisfying the spectrum condition

$$\begin{aligned} &\operatorname{supp} \hat{f}_0 \subset \overline{B}^{\boldsymbol{A}}(0, 2) \\ &\operatorname{supp} \hat{f}_n \subset \overline{B}^{\boldsymbol{A}}(0, 2^{n+1}) \setminus B^{\boldsymbol{A}}(0, 2^{n-1}), \qquad n \geq 1, \end{aligned}$$

and $(f_n)_n \in E(X)$. We equip $Y^{\boldsymbol{A}}(E; X)$ with the quasinorm

$$\|f\|_{Y^{\boldsymbol{A}}(E;X)} := \inf \|(f_n)\|_{E(X)},$$

where the infimum is taken over all representations as above.

Example 3.16. In the setting of Example 3.4,

$$Y^{\boldsymbol{A}}(E) = Y^{I_d}(E) = \begin{cases} F^s_{p,q}(\mathbb{R}^d), & \text{if } E = L_p(\mathbb{R}^d)[\ell^s_q(\mathbb{N})], \\ B^s_{p,q}(\mathbb{R}^d), & \text{if } E = \ell^s_q(\mathbb{N})[L_p(\mathbb{R}^d)], \end{cases}$$

see for instance [55, Section 2.5.2].

More examples will be given in Example 3.20, after the Littlewood–Paley description given in Proposition 3.19.

Proposition 3.17. *Suppose that $\varepsilon_+, \varepsilon_- > 0$ and let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. Then $YL^{\boldsymbol{A}}(E; X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E; X)$ are quasi-Banach spaces with*

$$YL^{\boldsymbol{A}}(E; X) \subset \widetilde{YL}^{\boldsymbol{A}}(E; X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{\boldsymbol{r}, \boldsymbol{w}_{\boldsymbol{A}, \boldsymbol{r}}}_{\boldsymbol{A}}; X),$$

where $YL^{\boldsymbol{A}}(E; X)$ is a closed subspace of $\widetilde{YL}^{\boldsymbol{A}}(E; X)$.

Proof. By Remark 3.14,

$$YL^{\boldsymbol{A}}(E; X), \widetilde{YL}^{\boldsymbol{A}}(E; X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{\boldsymbol{r}, \boldsymbol{w}_{\boldsymbol{A}, \boldsymbol{r}}}_{\boldsymbol{A}}; X). \tag{12}$$

That $YL^{\boldsymbol{A}}(E; X) \subset \widetilde{YL}^{\boldsymbol{A}}(E; X)$ with $\|f\|_{YL^{\boldsymbol{A}}(E;X)} = \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$ for all $f \in YL^{\boldsymbol{A}}(E; X)$ follows easily from the definitions. So it remains to be shown that $YL^{\boldsymbol{A}}(E; X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E; X)$ are complete.

Let us first treat $YL^{\boldsymbol{A}}(E; X)$. To this end, let the subspace $E(X)_{\boldsymbol{A}}$ of $E(X)$ be defined by

$$E(X)_{\boldsymbol{A}} := \left\{ (f_n)_n \in E(X) : f_n \in L_0(S; \mathcal{S}'(\mathbb{R}^d; X)),\ \operatorname{supp} \hat{f}_n \subset \overline{B}^{\boldsymbol{A}}(0, 2^{n+1}) \right\}$$

By Lemma 3.8,

$$\Sigma : E(X)_{\boldsymbol{A}} \longrightarrow E^{\boldsymbol{A}}_{\otimes}[L_{\boldsymbol{r}}(\mathbb{R}^d, \boldsymbol{w})](X) \hookrightarrow L_0(S; L_{\boldsymbol{r}, \mathscr{d}, \mathrm{loc}}(\mathbb{R}^d; X)),\ (f_n)_n \mapsto \sum_{n=0}^{\infty} f_n$$

is a well-defined continuous linear mapping. As

$$YL^{\boldsymbol{A}}(E; X) \simeq E(X)_{\boldsymbol{A}} / \ker(\Sigma) \quad \text{isometrically},$$

it suffices to show that $E(X)_{\boldsymbol{A}}$ is complete.

In order to show that $E(X)_{\boldsymbol{A}}$ is complete, we prove that it is a closed subspace of the quasi-Banach space $E(X)$. Put $w(x) := \prod_{j=1}^{\ell}(1+\rho_{A_j}(x_j))^{\mathrm{tr}(A_j)/r_j}$. Then it is enough to show that, for each $k \in \mathbb{N}$,

$$E(X)_{\boldsymbol{A}} \longrightarrow L_0(S; BC(\mathbb{R}^d, w; X)),\ (f_n)_n \mapsto f_k, \tag{13}$$

continuously, where $BC(\mathbb{R}^d, w; X) = \{h \in C(\mathbb{R}^d; X) : wh \in L_\infty(\mathbb{R}^d; X)\}$. Indeed, $BC(\mathbb{R}^d, w; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X)$.

In order to establish (13), let $(f_n)_n \in E(X)_{\boldsymbol{A}}$. By Corollary A.2,

$$\sup_{z \in B^{\boldsymbol{A}}(0,2^{-n})} \|f_n\|_X \lesssim M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f_n\|_x)(x),$$

so that

$$\begin{aligned}\|f_n(x)\|_X &\lesssim \inf_{z \in B^{\boldsymbol{A}}(0,2^{-n})} M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f_n\|_X)(x+z) \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}} \left\| M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f_n\|_X)\right\|_{L_{\boldsymbol{r},d}(B^{\boldsymbol{A}}(x,2^{-n}))}.\end{aligned}$$

For $\boldsymbol{R} \in [1,\infty)^\ell$ we can thus estimate

$$\begin{aligned}\sup_{z \in B^{\boldsymbol{A}}(0,\boldsymbol{R})} \|f_n(x)\|_X &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}} \left\| M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f_n\|_X)\right\|_{L_{\boldsymbol{r},d}(B^{\boldsymbol{A}}(0,\boldsymbol{c}_{\boldsymbol{A}}[\boldsymbol{R}+2^{-n}\boldsymbol{1}]))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}} \left\| M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f_n\|_X)\right\|_{L_{\boldsymbol{r},d}(B^{\boldsymbol{A}}(0,2\boldsymbol{c}_{\boldsymbol{A}}\boldsymbol{R}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}} \inf_{z \in B^{\boldsymbol{A}}(0,\boldsymbol{R})} \left\| M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f_n\|_X)\right\|_{L_{\boldsymbol{r},d}(B^{\boldsymbol{A}}(0,2\boldsymbol{c}_{\boldsymbol{A}}(\boldsymbol{c}_{\boldsymbol{A}}+\boldsymbol{1})\boldsymbol{R}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} \inf_{z \in B^{\boldsymbol{A}}(0,\boldsymbol{R})} M_{\boldsymbol{r}}^{\boldsymbol{A}}(M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f_n\|_X))(z).\end{aligned} \tag{14}$$

The latter implies that

$$1_{B^{\boldsymbol{A}}(0,\boldsymbol{R})} \otimes \|f_n\|_{L_\infty(B^{\boldsymbol{A}}(0,\boldsymbol{R});X)} \lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} M_{\boldsymbol{r}}^{\boldsymbol{A}}(M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f_n\|_X))$$

for $\boldsymbol{R} \in [1,\infty)^\ell$. It thus follows that

$$\begin{aligned}\|f_n\|_{E_\otimes^{\boldsymbol{A}}(L_\infty(B^{\boldsymbol{A}}(0,\boldsymbol{R});X))} &\leq \left\| 1_{B^{\boldsymbol{A}}(0,\boldsymbol{R})\times\{0\}} \otimes \|f_n\|_{L_\infty(B^{\boldsymbol{A}}(0,\boldsymbol{R});X)}\right\|_E \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} \|(\delta_{0,k} M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f_n\|_X))_k\|_E \\ &\lesssim 2^{n(\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}-\varepsilon_+)} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} \|(h_k)_k\|_{E(X)}.\end{aligned}$$

Let us finally prove that $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ is complete. To this end, let $\kappa \in (0,1]$ with $\kappa \leq \boldsymbol{r}_{\min}$ be such that $\|\cdot\|_E$ is equivalent to a κ-norm. Then $\|\cdot\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$ and $\|\cdot\|_{E_\otimes^{\boldsymbol{A}}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})](X)}$ are equivalent to κ-norms as well. It suffices to show that, if $(f^{(k)})_{k\in\mathbb{N}} \subset \widetilde{YL}^{\boldsymbol{A}}(E;X)$ satisfies $\sum_{k=0}^\infty \|f^{(k)}\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}^\kappa < \infty$, then $\sum_{k=0}^\infty f^{(k)}$ is a convergent series in $\widetilde{YL}^{\boldsymbol{A}}(E;X)$. So fix such a $(f^{(k)})_{k\in\mathbb{N}}$. As a consequence of (12),

$$\sum_{k=0}^\infty \|f^{(k)}\|_{E_\otimes^{\boldsymbol{A}}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})]}^\kappa \lesssim \sum_{k=0}^\infty \|f^{(k)}\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}^\kappa < \infty.$$

As $E_\otimes^{\boldsymbol{A}}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})]$ is a quasi-Banach space with a κ-norm, $\sum_{k=0}^\infty f^{(k)}$ converges to some F in $E_\otimes^{\boldsymbol{A}}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})]$. To finish the proof, we show that $F \in \widetilde{YL}^{\boldsymbol{A}}(E;X)$ with convergence $F = \sum_{k=0}^\infty f^{(k)}$ in $\widetilde{YL}^{\boldsymbol{A}}(E;X)$.

For each $k \in \mathbb{N}$ there exists $(g_n^{(k)})_n \in E_+$ with $\|(g_n^{(k)})_n\|_E \leq 2\|f^{(k)}\|_{\widetilde{YL}^{A}(E;X)}$ such that, for every $x^* \in X^*$, $\langle f^{(k)}, x^*\rangle$ has the representation

$$\langle f^{(k)}, x^*\rangle = \sum_{n=0}^{\infty} f_{x^*,n}^{(k)} \quad \text{in} \quad L_0(S; L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$$

for some $(f_{x^*,n}^{(k)})_n \in E_{\boldsymbol{A}}$ with $|f_{x^*,n}^{(k)}| \leq \|x^*\| g_n^{(k)}$. By Remark 3.14,

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \|f_{x^*,n}^{(k)}\|^{\kappa}_{E_{\otimes}^{\boldsymbol{A}}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})]} \lesssim \sum_{k=0}^{\infty} \|f^{(k)}\|^{\kappa}_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} < \infty.$$

As $E_{\otimes}^{\boldsymbol{A}}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})] \hookrightarrow L_0(S; L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d)) \hookrightarrow L_0(S \times \mathbb{R}^d)$ is a quasi-Banach space with a κ-norm, we thus find that $F = \sum_{n=0}^{\infty} F_{x^*,n}$ in $L_0(S; L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$ with $F_{x^*,n} := \sum_{k=0}^{\infty} f_{x^*,n}^{(k)}$ in $L_0(\mathbb{R}^d \times S)$ satisfying $|F_{x^*,n}| \leq \sum_{k=0}^{\infty} |f_{x^*,n}^{(k)}| \leq \|x^*\| \sum_{k=0}^{\infty} g_n^{(k)}$. As $E_{\boldsymbol{A}}$ is a closed subspace of the quasi-Banach function space E on $\mathbb{R}^d \times \mathbb{N} \times S$ with κ-norm, it follows from

$$\sum_{k=0}^{\infty} \|(f_{x^*,n}^{(k)})_n\|_E^{\kappa} \leq \|x^*\|^{\kappa} \sum_{k=0}^{\infty} \|f^{(k)}\|^{\kappa}_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} < \infty$$

that $(F_{x^*,n})_n = \sum_{k=0}^{\infty} f_{x^*,n}^{(k)}$ in E and thus that $(F_{x^*,n})_n \in E_{\boldsymbol{A}}$. Moreover, $G_n := \sum_{k=0}^{\infty} g_n^{(k)}$ defines $(G_n)_n \in E_+$ with

$$\|(G_n)_n\|_E^{\kappa} \leq \sum_{k=0}^{\infty} \|(g_n^k)_n\|_E^{\kappa} \leq 2\sum_{k=0}^{\infty} \|f^{(k)}\|^{\kappa}_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$$

and $|F_{x^*,n}| \leq \|x^*\| G_n$. This shows that $F \in \widetilde{YL}^{\boldsymbol{A}}(E;X)$ with convergence $F = \sum_{k=0}^{\infty} f^{(k)}$ in $\widetilde{YL}^{\boldsymbol{A}}(E;X)$. □

The content of the following proposition is a Littlewood–Paley characterization for $Y^{\boldsymbol{A}}(E;X)$. Before we state it, we first need to introduce the set $\Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ of all $\boldsymbol{A}$-anisotropic Littlewood–Paley sequences $\varphi = (\varphi_n)_{n\in\mathbb{N}}$.

Definition 3.18. For $0 < \gamma < \delta < \infty$ we define $\Phi_{\gamma,\delta}^{\boldsymbol{A}}(\mathbb{R}^d)$ as the set of all sequences $\varphi = (\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ that can be constructed in the following way: given $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$0 \leq \hat{\varphi}_0 \leq 1, \;\; \hat{\varphi}_0(\xi) = 1 \text{ if } \rho_{\boldsymbol{A}}(\xi) \leq \gamma, \;\; \hat{\varphi}_0(\xi) = 0 \text{ if } \rho_{\boldsymbol{A}}(\xi) \geq \delta,$$

$(\varphi_n)_{n\geq 1} \subset \mathcal{S}(\mathbb{R}^d)$ is obtained through

$$\hat{\varphi}_n = \hat{\varphi}_1(\boldsymbol{A}_{2^{-n+1}}\,\cdot) = \hat{\varphi}_0(\boldsymbol{A}_{2^{-n}}\,\cdot) - \hat{\varphi}_0(\boldsymbol{A}_{2^{-n+1}}\,\cdot), \qquad n \geq 1.$$

We define $\Phi^{\boldsymbol{A}}(\mathbb{R}^d) := \bigcup_{0<\gamma<\delta<\infty} \Phi_{\gamma,\delta}^{\boldsymbol{A}}(\mathbb{R}^d)$.

Let $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi_{\gamma,\delta}^{\boldsymbol{A}}(\mathbb{R}^d)$. Then $\sum_{n=0}^{\infty} \hat{\varphi}_n = 1$ in $\mathscr{O}_M(\mathbb{R}^d)$ with

$$\operatorname{supp}\hat{\varphi}_0 \subset \{\xi : \rho_{\boldsymbol{A}}(\xi) \leq \gamma\}, \qquad \operatorname{supp}\hat{\varphi}_n \subset \{\xi : 2^{n-1}\gamma \leq \rho_{\boldsymbol{A}}(\xi) \leq 2^n\delta\}, \quad n \geq 1,$$

To φ we associate the family of convolution operators $(S_n)_{n\in\mathbb{N}} = (S_n^{\varphi})_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X), \check{\mathcal{E}}'(\mathbb{R}^d;X))$ given by

$$S_n f = S_n^{\varphi} f := \varphi_n * f = \mathcal{F}^{-1}[\hat{\varphi}_n \hat{f}].$$

Proposition 3.19. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ with associated sequence of convolution operators $(S_n)_{n\in\mathbb{N}}$. Then*

$$Y^{\boldsymbol{A}}(E; X) = \{f \in L_0(S; \mathcal{S}'(\mathbb{R}^d; X)) : (S_n f)_n \in E(X)\}$$

with

$$\|f\|_{Y^{\boldsymbol{A}}(E;X)} \eqsim \|(S_n f)_n\|_{E(X)}.$$

Before we go the proof of Proposition 3.19, let us first consider the following.

Example 3.20. In the following three points we let the notation be as in Examples 3.5(i), 3.5(ii) and 3.5(iii), respectively. We define:

(i) $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X) := Y^{\boldsymbol{A}}(E; X)$ for $E = L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})[\ell^s_q(\mathbb{N})]$;

(ii) $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X) := Y^{\boldsymbol{A}}(E; X)$ for $E = \ell^s_q(\mathbb{N})[L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})]$;

(iii) $\mathbb{F}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X) := Y^{\boldsymbol{A}}(E; X)$ for $E = L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})[F[\ell^s_q(\mathbb{N})]]$.

Restricting to special cases we find, in view of Proposition 3.19, B- and F-spaces that have been studied in the literature:

(i)&(ii):
(a) In case $\ell = 1$, $w = 1$ and $X = \mathbb{C}$, $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ and $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ reduce to the anisotropic Besov and Lizorkin–Triebel spaces considered in e.g. [14,17]. The latter are special cases of the anisotropic spaces from the more general [4,9,10] by taking 2^A as the expansive dilation in the approach there.
(b) In case $\ell = d$, $\boldsymbol{A} = \mathrm{diag}(\boldsymbol{a})$ with $\boldsymbol{a} \in (0, \infty)$, $\boldsymbol{w} = \boldsymbol{1}$ and $X = \mathbb{C}$, $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ and $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ reduce to the anisotropic mixed-norm Besov and Lizorkin–Triebel spaces considered in e.g. [27,28].
(c) In case $\boldsymbol{A} = (a_1 I_{d_1}, \ldots, a_\ell I_{d_\ell})$ with $\boldsymbol{a} \in (0, \infty)$, $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ and $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ reduce to the anisotropic weighted mixed-norm Besov and Lizorkin–Triebel spaces considered in [33,36].
(d) In case $\ell = 1$ and $A = I$, $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ and $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ reduce to the weighted Besov and Lizorkin–Triebel spaces considered in e.g. [11–13,20–23,35,52] ($X = \mathbb{C}$) and [39–41] (X a general Banach space). In the case $w = 1$ these further reduces to the classical Besov and Lizorkin–Triebel spaces (see e.g. [50,55,56]).

(iii):
(a) In case $\ell = 1$, $A = I$, $p \in (1, \infty)$, $q \in [1, \infty]$, $w = 1$, F is a UMD Banach function space and $X = \mathbb{C}$, $\mathbb{F}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X)$ reduces to a special case of the generalized Lizorkin–Triebel spaces considered in [32].
(b) In case $\ell = 1$, $A = I$, $p \in (1, \infty)$, $q = 2$, $w \in A_p(\mathbb{R}^d)$, F is a UMD Banach function space and X is a Hilbert space, $\mathbb{F}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X)$ coincides with the weighted Bessel potential space $H^s_p(\mathbb{R}^d, w; F(X))$ (which can be seen as a special case of [41, Proposition 3.2] through the use of the Khintchine–Maurey theorem (see e.g. [26, Theorem 7.2.13])).

The proof of Proposition 3.19 basically only consists of proving the estimate in the following lemma. We have extracted it as a lemma as it is interesting on its own. A consequence of the lemma for instance is that the spectrum condition in Definition 3.15 could be slightly modified.

Lemma 3.21. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$, $c \in (1, \infty)$ and $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ with associated sequence of convolution operators $(S_n)_{n\in\mathbb{N}}$. For all $f \in L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ which* *have a representation*

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$$

with $(f_n)_n \subset L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ *satisfying the spectrum condition*

$$\operatorname{supp} \hat{f}_0 \subset \overline{B}^{\boldsymbol{A}}(0, c)$$
$$\operatorname{supp} \hat{f}_n \subset \overline{B}^{\boldsymbol{A}}(0, c2^n) \setminus B^{\boldsymbol{A}}(0, c^{-1}2^n), \qquad n \geq 1,$$

there is the estimate

$$\|(S_n f)_n\|_{E(X)} \lesssim \|(f_n)_n\|_{E(X)}.$$

Proof. This can be established as in [33, Lemma 5.2.10] (also see [55, Section 2.3.2] and [58, Section 15.5]), using a combination of Corollary A.2 and Lemma A.3. □

Proof of Proposition 3.19. Let $f \in Y^{\boldsymbol{A}}(E; X)$. Take $(f_n)_n$ as in Definition 3.15 with $\|(f_n)_n\|_{E(X)} \leq 2\|f\|_{Y^{\boldsymbol{A}}(E;X)}$. Lemma 3.21 (with $c = 2$) then gives

$$\|(S_n f)_n\|_{E(X)} \lesssim \|(f_n)_n\|_{E(X)} \leq 2\|f\|_{Y^{\boldsymbol{A}}(E;X)}.$$

For the reverse direction, let $f \in L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ be such that $(S_n f)_n \in E(X)$. Pick $\psi = (\psi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ such that

$$\operatorname{supp} \hat{\psi}_0 \subset \overline{B}^{\boldsymbol{A}}(0, 2), \qquad \operatorname{supp} \hat{\psi}_n \subset \overline{B}^{\boldsymbol{A}}(0, 2^{n+1}) \setminus B^{\boldsymbol{A}}(0, 2^{n-1}), \quad n \geq 1,$$

and let $(T_n)_{n\in\mathbb{N}}$ denote the associated sequence of convolution operators. Then

$$\operatorname{supp} \widehat{T_0 f} \subset \overline{B}^{\boldsymbol{A}}(0, 2), \qquad \operatorname{supp} \widehat{T_n f} \subset \overline{B}^{\boldsymbol{A}}(0, 2^n) \setminus B^{\boldsymbol{A}}(0, 2^{n-1}), \quad n \geq 1, \tag{15}$$

Picking $c \in (1, \infty)$ such that

$$\operatorname{supp} \hat{\varphi}_0 \subset \overline{B}^{\boldsymbol{A}}(0, c), \qquad \operatorname{supp} \hat{\varphi}_n \subset \overline{B}^{\boldsymbol{A}}(0, c2^n) \setminus B^{\boldsymbol{A}}(0, c^{-1}2^n), \quad n \geq 1,$$

we furthermore have

$$\operatorname{supp} \widehat{S_n f} \subset \overline{B}^{\boldsymbol{A}}(0, c), \qquad \operatorname{supp} \widehat{S_n f} \subset \overline{B}^{\boldsymbol{A}}(0, c2^n) \setminus B^{\boldsymbol{A}}(0, c^{-1}2^n), \quad n \geq 1.$$

As $f = \sum_{n=0}^{\infty} S_n f$ in $L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$, Lemma 3.21 gives

$$\|(T_n f)_n\|_{E(X)} \lesssim \|(S_n f)_n\|_{E(X)}.$$

Since $f = \sum_{n=0}^{\infty} S_n f$ in $L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ with (15), it follows that $f \in Y^{\boldsymbol{A}}(E; X)$ with

$$\|f\|_{Y^{\boldsymbol{A}}(E;X)} \leq \|(T_n f)_n\|_{E(X)} \lesssim \|(S_n f)_n\|_{E(X)}. \qquad \square$$

Theorem 3.22. *Let* $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. *Suppose that* $\varepsilon_+ > \operatorname{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{1})_+$, *where* $\operatorname{tr}(\boldsymbol{A})$ *is the component-wise trace of* $\boldsymbol{A}$ *given by* $\operatorname{tr}(\boldsymbol{A}) := (\operatorname{tr}(A_1), \ldots, \operatorname{tr}(A_\ell))$. *Then*

$$\widetilde{YL}^{\boldsymbol{A}}(E; X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{1, w_{\boldsymbol{A}, \boldsymbol{r}\wedge \boldsymbol{1}}}_{\boldsymbol{A}}(X)) \hookrightarrow L_0(S; L_{\boldsymbol{1}\wedge\boldsymbol{r}, \mathpzc{d}, \mathrm{loc}}(\mathbb{R}^d; X)) \tag{16}$$

and

$$\begin{aligned} Y^{\boldsymbol{A}}(E; X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{1, w_{\boldsymbol{A}, \boldsymbol{r}\wedge \boldsymbol{1}}}_{\boldsymbol{A}}(X)) &\hookrightarrow \mathcal{S}'(\mathbb{R}^d; E^{\boldsymbol{A}}_{\otimes}(X)) \\ &\hookrightarrow \mathcal{S}'(\mathbb{R}^d; L_0(S; X)) = L_0(S; \mathcal{S}'(\mathbb{R}^d; X)) \end{aligned} \tag{17}$$

and there is the identity

$$Y^{\boldsymbol{A}}(E;X) = YL^{\boldsymbol{A}}(E;X) = \widetilde{YL}^{\boldsymbol{A}}(E;X). \tag{18}$$

Remark 3.23. Note that the condition $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \mathbf{1})_+$ is for instance fulfilled when $\boldsymbol{r} \geq \mathbf{1}$.

We will use the following lemma in the proof of Theorem 3.22.

Lemma 3.24. *Let the notations and assumptions be as in Theorem 3.22. Let $c \in (0,\infty)$. If*

$$(f_n)_n \in E(X)_{\boldsymbol{A},c} := \left\{ (h_n)_n \in E(X) : h_n \in L_0(S;\mathcal{S}'(\mathbb{R}^d;X)),\ \mathrm{supp}\,\hat{h}_n \subset \overline{B}^{\boldsymbol{A}}(0, c2^{n+1}) \right\},$$

then $\sum_{n\in\mathbb{N}} f_n$ is a convergent series in $L_0(S; B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge\mathbf{1}}}_{\boldsymbol{A}}(X))$ with

$$\Big\| \sum_{n=0}^{\infty} f_n \Big\|_{E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge\mathbf{1}}}_{\boldsymbol{A}}(X))} \leq \Big\| \sum_{n=0}^{\infty} \|f_n\|_X \Big\|_{E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge\mathbf{1}}}_{\boldsymbol{A}})} \lesssim \|(f_n)_n\|_{E(X)}.$$

Proof. It suffices to prove the second estimate. We may without loss of generality assume that $\boldsymbol{r} \in (0,1]^\ell$. Choose $\kappa > 0$ such that $E^{\boldsymbol{A}}_{\otimes}$ has a κ-norm. For simplicity of notation we only present the case $\ell = 2$ and $c = 1$, the general case being the same.

Let $(f_n)_n \in E(X)_{\boldsymbol{A}}$. Let $\boldsymbol{R} \in [1,\infty)^2$. As a consequence of the Paley–Wiener–Schwartz theorem,

$$\check{\mathcal{E}}'_{\overline{B}^{\boldsymbol{A}}(0,2^n)}(\mathbb{R}^d;X) \hookrightarrow C^\infty(\mathbb{R}^{d_2}; \check{\mathcal{E}}'_{\overline{B}^{A_1}(0,2^n)}(\mathbb{R}^{d_1};X)) \cap C^\infty(\mathbb{R}^{d_1}; \check{\mathcal{E}}'_{\overline{B}^{A_2}(0,2^n)}(\mathbb{R}^{d_2};X)).$$

In particular, as in (14) we find that

$$\|f_n(x_1,z_2)\|_X \lesssim (2^n R_1)^{\mathrm{tr}(A_1)/r_1} M^{A_1}_{r_1;[d;1]}(M^{A_1}_{r_1,[d;1]}(\|f_n\|_X))(y_1,z_2) \tag{19}$$

for all $x_1, y_1 \in B^{A_1}(0,R_1)$ and $z_1 \in \mathbb{R}^{d_1}$, and

$$\|f_n(z_1,x_2)\|_X \lesssim (2^n R_2)^{\mathrm{tr}(A_2)/r_2} M^{A_2}_{r_2;[d;2]}(M^{A_2}_{r_2,[d;2]}(\|f_n\|_X))(z_1,y_2) \tag{20}$$

for all $x_2, y_2 \in B^{A_2}(0,R_2)$ and $z_2 \in \mathbb{R}^{d_2}$.

Then, for $z \in B^{\boldsymbol{A}}(0,\boldsymbol{R})$,

$$\begin{aligned}
&\int_{B^{\boldsymbol{A}}(0,\boldsymbol{R})} \|f_n(x)\|_X \,\mathrm{d}x \\
&\quad = \int_{B^{A_2}(0,R_2)} \int_{B^{A_1}(0,R_1)} \|f_n(x_1,x_2)\|_X \,\mathrm{d}x_1\mathrm{d}x_2 \\
&\quad \overset{(19)}{\lesssim} ((2^n R_1)^{\mathrm{tr}(A_1)/r_1})^{1-r_1} \int_{B^{A_2}(0,R_2)} M^{A_1}_{r_1;[d;1]}(M^{A_1}_{r_1;[d;1]}(\|f_n(\,\cdot\,,x_2)\|_X))(z_1)^{r_1-1} \\
&\qquad\quad \cdot \int_{B^{A_1}(0,R_1)} \|f_n(x_1,x_2)\|_X^{r_1} \,\mathrm{d}x_1\mathrm{d}x_2 \\
&\quad \lesssim 2^{n\mathrm{tr}(A_1)(1-r_1)/r_1} R_1^{\mathrm{tr}(A_1)/r_1} \int_{B^{A_2}(0,R_2)} M^{A_1}_{r_1;[d;1]}(M^{A_1}_{r_1;[d;1]}(\|f_n(\,\cdot\,,x_2)\|_X))(z_1)\mathrm{d}x_2 \\
&\quad \overset{(20)}{\lesssim} 2^{n\left(\mathrm{tr}(A_1)(1-r_1)/r_1 + \mathrm{tr}(A_2)(1-r_2)/r_2\right)} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} \\
&\qquad\quad \cdot M^{A_1}_{r_1;[d;1]} M^{A_1}_{r_1;[d;1]} M^{A_2}_{r_2;[d;2]} M^{A_2}_{r_2;[d;2]}(\|f_n\|_X)(z_1,z_2)^{1-r_2}
\end{aligned}$$

$$\cdot\ M^{A_2}_{r_2;[d;1]} M^{A_2}_{r_2;[d;2]} M^{A_1}_{r_1;[d;2]} M^{A_1}_{r_1;[d;1]}(\|f_n\|_X)(z_1,z_2)^{r_2}$$
$$\leq 2^{n(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} [M^{\boldsymbol{A}}_{\boldsymbol{r}}]^4(\|f_n\|_X)(z).$$

This implies that

$$1_{B^A(0,\boldsymbol{R})} \otimes \int_{B^A(0,\boldsymbol{R})} \sum_{n=0}^{\infty} \|f_n(x)\|_X \,\mathrm{d}x \lesssim \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} \sum_{n=0}^{\infty} 2^{n(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})}[M^{\boldsymbol{A}}_{\boldsymbol{r}}]^4(\|f_n\|_X).$$

Since $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{1})_+$, it follows that

$$\left\| \sum_{n=0}^{\infty} \|f_n\|_X \right\|_{E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A}},\boldsymbol{r}\wedge\boldsymbol{1}}_{\boldsymbol{A}})} \overset{(9)}{\lesssim} \|([M^{\boldsymbol{A}}_{\boldsymbol{r}}]^4(\|f_n\|_X))_n\|_E$$
$$\lesssim \|(f_n)\|_{E(X)}. \quad \square$$

Proof of Theorem 3.22. We may without loss of generality assume that $\boldsymbol{r} \in (0,1]^{\ell}$.

As $L_0(S; B^{1,w_{\boldsymbol{A}},\boldsymbol{r}\wedge\boldsymbol{1}}_{\boldsymbol{A}}(X)) \hookrightarrow L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$, the first inclusion in (17) follows from Lemma 3.24. So in (17) it remains to prove the second inclusion. To this end, let us first note that

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{B}(B^{1,w_{\boldsymbol{A}},\boldsymbol{r}\wedge\boldsymbol{1}}_{\boldsymbol{A}}(X), X),\ \phi \mapsto \langle\,\cdot\,,\phi\rangle.$$

This induces

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{B}(E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A}},\boldsymbol{r}\wedge\boldsymbol{1}}_{\boldsymbol{A}}(X)), E^{\boldsymbol{A}}_{\otimes}(X)),\ \phi \mapsto \langle\,\cdot\,,\phi\rangle.$$

Therefore, $f \mapsto [\phi \mapsto \langle f,\phi\rangle]$ is a continuous linear operator from $E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A}},\boldsymbol{r}\wedge\boldsymbol{1}}_{\boldsymbol{A}}(X))$ to $\mathcal{L}(\mathcal{S}(\mathbb{R}^d); E^{\boldsymbol{A}}_{\otimes}(X))$, which is a reformulation of the required inclusion.

As $L_0(S; B^{1,w_{\boldsymbol{A}},\boldsymbol{r}\wedge\boldsymbol{1}}_{\boldsymbol{A}}) \hookrightarrow L_0(S; L_{\boldsymbol{r},d,\mathrm{loc}}(\mathbb{R}^d))$, the inclusion

$$Y^{\boldsymbol{A}}(E) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A}},\boldsymbol{r}\wedge\boldsymbol{1}}_{\boldsymbol{A}})$$

follows from Lemma 3.24. We thus get a continuous bilinear mapping

$$\widetilde{YL}^{\boldsymbol{A}}(E,X) \times X^* \longrightarrow YL^{\boldsymbol{A}}(E) \hookrightarrow L_0(S; \mathcal{S}'(\mathbb{R}^d)),\ (f,x^*) \mapsto \langle f,x^*\rangle.$$

and a continuous linear mapping

$$\widetilde{YL}^{\boldsymbol{A}}(E,X) \longrightarrow L_0(S; \mathcal{S}'(\mathbb{R}^d; X^{**})),\ f \mapsto T_f, \tag{21}$$

defined by

$$\langle x^*, T_f(\phi)\rangle := \langle f,x^*\rangle(\phi), \qquad \phi \in \mathcal{S}(\mathbb{R}^d), x^* \in X^*.$$

Let us now show that $f \mapsto T_f$ (21) restricts to a bounded linear mapping

$$\widetilde{YL}^{\boldsymbol{A}}(E,X) \longrightarrow Y^{\boldsymbol{A}}(E; X^{**}),\ f \mapsto T_f. \tag{22}$$

To this end, let $f \in \widetilde{YL}^{\boldsymbol{A}}(E;X)$ and put $F := T_f$. Let $(g_n)_n$ and $(f_{x^*,n})_{(x^*,n)}$ be as in Definition 3.12 with $\|(g_n)_n\|_E \leq 2\|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$. It will convenient to put $g_n := 0$ and $f_{x^*,n} := 0$ for $n \in \mathbb{Z}_{<0}$. By Lemma 3.24, as $(f_{x^*,n})_n \in E_{\boldsymbol{A}}$ and $B^{1,w_{\boldsymbol{A}},\boldsymbol{r}\wedge\boldsymbol{1}}_{\boldsymbol{A}} \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$,

$$\langle f,x^*\rangle = \sum_{k=0}^{\infty} f_{x^*,k} \quad \text{in} \quad L_0(S; B^{1,w_{\boldsymbol{A}},\boldsymbol{r}\wedge\boldsymbol{1}}_{\boldsymbol{A}}) \hookrightarrow L_0(S; \mathcal{S}'(\mathbb{R}^d)), \qquad x^* \in X^*.$$

Now let $(S_n)_{n\in\mathbb{N}}$ be as in Proposition 3.19. There exists $h \in \mathbb{N}$ independent of f such that $S_n f_{x^*,k} = 0$ for all $x^* \in X^*$, $n \in \mathbb{N}$ and $k \in \mathbb{Z}_{<n-h}$. Let $x^* \in X^*$. Then

$$\begin{aligned}\langle x^*, S_n F\rangle &= S_n\langle x^*, F\rangle = S_n\langle f, x^*\rangle = S_n \sum_{k=0}^{\infty} f_{x^*,k} = \sum_{k=0}^{\infty} S_n f_{x^*,k} \\ &= \sum_{k=n-h}^{\infty} S_n f_{x^*,k} = \sum_{k=0}^{\infty} S_n f_{x^*,k+n-h}\end{aligned}$$

with convergence in $L_0(S;\mathcal{S}'(\mathbb{R}^d))$. Together with Corollary A.6, this implies the pointwise estimates

$$\begin{aligned}|\langle x^*, S_n F\rangle| &\leq \sum_{k=0}^{\infty} |S_n f_{x^*,k+n-h}| \lesssim \sum_{k=0}^{\infty} 2^{(k-h)+\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\mathbf{1})} M_{\boldsymbol{r}}^{\boldsymbol{A}}(f_{k+n-h,x^*}) \\ &\leq \|x^*\| \sum_{k=0}^{\infty} 2^{(k-h)+\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\mathbf{1})} M_{\boldsymbol{r}}^{\boldsymbol{A}}(g_{k+n-h}).\end{aligned}$$

Taking the supremum over $x^* \in X^*$ with $\|x^*\| \leq 1$, we obtain

$$\|S_n F\|_{X^{**}} \leq \sum_{k=0}^{\infty} 2^{(k-h)+\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\mathbf{1})} M_{\boldsymbol{r}}^{\boldsymbol{A}}(g_{k+n-h}).$$

Picking $\kappa > 0$ such that E has a κ-norm, we find that

$$\begin{aligned}\|(S_n F)_n\|_{E(X^{**})}^{\kappa} &= \left\|(\|S_n f\|_{X^{**}})_n\right\|_E^{\kappa} \\ &\lesssim \sum_{k=0}^{\infty} 2^{\kappa(k-h)+\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\mathbf{1})} \left\| M_{\boldsymbol{r}}^{\boldsymbol{A}}(g_{k+n-h})_n\right\|_E^{\kappa}\end{aligned}$$

Since

$$\begin{aligned}\left\|M_{\boldsymbol{r}}^{\boldsymbol{A}}(g_{k+n-h})_n\right\|_E = \left\|(g_{k+n-h})_n\right\|_E &\lesssim \begin{cases}\left\|(S_-)^{h-k}(g_n)_n\right\|_E, & k \leq h,\\ \left\|(S_+)^{k-h}(g_{k+n-h})_n\right\|_E, & k \geq h,\end{cases} \\ &\lesssim \left(2^{\varepsilon_-(h-k)_+} + 2^{-\varepsilon_+(k-h)_+}\right)\|(g_n)_n\|_E \\ &\lesssim 2^{-\varepsilon_+(k-h)_+}\|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}\end{aligned}$$

for all $k \in \mathbb{N}$, it follows that

$$\|(S_n F)_n\|_{E(X^{**})}^{\kappa} \lesssim \sum_{k=0}^{\infty} 2^{\kappa(k-h)_+\left(\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\mathbf{1})-\varepsilon_+\right)}\|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}^{\kappa}.$$

As $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\mathbf{1})$, we find that $\|(S_n F)_n\|_{E(X^{**})} \lesssim \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$ and thus that $F \in Y^{\boldsymbol{A}}(E;X^{**})$ with $\|F\|_{Y^{\boldsymbol{A}}(E;X^{**})} \lesssim \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$ (see Proposition 3.19). So we obtain the desired (22).

Next we prove that

$$\widetilde{YL}^{\boldsymbol{A}}(E;X) \hookrightarrow Y^{\boldsymbol{A}}(E;X). \tag{23}$$

So let $f \in \widetilde{YL}^{\boldsymbol{A}}(E;X)$. A combination of (22) and (17) gives that $F := T_f \in L_0(S;X^{**})$. Since $f \in L_0(S;L_{\boldsymbol{r},d,\mathrm{loc}}(\mathbb{R}^d;X))$ with $\langle x^*, F\rangle = \langle f, x^*\rangle$ for every $x^* \in X^*$, it follows that

$$f = F \in L_0(S;B_{\boldsymbol{A}}^{1,w_{A,\boldsymbol{r}\wedge\mathbf{1}}}(X^{**})) \cap L_0(S;L_{\boldsymbol{r},d,\mathrm{loc}}(\mathbb{R}^d;X)) \subset L_0(S;B_{\boldsymbol{A}}^{1,w_{A,\boldsymbol{r}\wedge\mathbf{1}}}(X)).$$

Therefore, by boundedness of (22),

$$\widetilde{YL}^{\boldsymbol{A}}(E;X) \hookrightarrow \left\{g \in Y^{\boldsymbol{A}}(E;X^{**}) : g \in L_0(S;\mathcal{S}'(\mathbb{R}^d;X))\right\} = Y^{\boldsymbol{A}}(E;X). \quad \square$$

For a quasi-Banach function space E on $\mathbb{R}^d \times \mathbb{N} \times S$ and a number $\sigma \in \mathbb{R}$ we define the quasi-Banach function space E^σ on $\mathbb{R}^d \times \mathbb{N} \times S$ by

$$\|(f_n)_n\|_{E^\sigma} := \|(2^{n\sigma} f_n)_n\|_E, \qquad (f_n)_n \in L_0(\mathbb{R}^d \times \mathbb{N} \times S).$$

Note that $E^\sigma \in \mathcal{S}(\varepsilon_+ + \sigma, \varepsilon_- + \sigma, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ when $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$.

Proposition 3.25. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and $\sigma \in \mathbb{R}$. Let $\psi \in \mathscr{O}_{\mathrm{M}}(\mathbb{R}^d)$ be such that $\psi(\xi) = \rho_{\boldsymbol{A}}(\xi)$ for $\rho_{\boldsymbol{A}}(\xi) \geq 1$ and $\psi(\xi) \neq 0$ for $\rho_{\boldsymbol{A}}(\xi) \leq 1$. Then $\phi(D) \in \mathcal{L}(L_0(S;\mathcal{S}'(\mathbb{R}^d;X)))$ restricts to an isomorphism*

$$\phi(D) : Y^{\boldsymbol{A}}(E^\sigma;X) \xrightarrow{\simeq} Y^{\boldsymbol{A}}(E;X).$$

Proof. Using Proposition 3.19 and Lemma A.3, this can be proved as [33, Lemma 5.2.28] (also see [55, Theorem 2.3.8]). $\square$

Proposition 3.26. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. Then*

$$Y^{\boldsymbol{A}}(E;X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; E^{\boldsymbol{A}}_{\otimes}(X)) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; L_0(S;X)) = L_0(S;\mathcal{S}'(\mathbb{R}^d;X))$$

and $Y^{\boldsymbol{A}}(E;X)$, when equipped with an equivalent quasi-norm from Proposition 3.19, has the Fatou property with respect to $L_0(S;\mathcal{S}'(\mathbb{R}^d;X))$. As a consequence (see Lemma 2.1), $Y^{\boldsymbol{A}}(E;X)$ is a quasi-Banach space.

Proof. The chain of inclusions follow from a combination of Theorem 3.22 and Proposition 3.25.

In order to establish the Fatou property, suppose that $Y^{\boldsymbol{A}}(E;X)$ has been equipped with an equivalent quasi-norm from Proposition 3.19. Let $f_k \to f$ in $L_0(S;\mathcal{S}'(\mathbb{R}^d;X))$ with $\liminf_{k\to\infty} \|f_k\|_{Y^{\boldsymbol{A}}(E;X)} < \infty$. Then

$$S_n f = \lim_{k\to\infty} S_n f_k \quad \text{in} \quad L_0(S;\mathscr{O}_M(\mathbb{R}^d;X)) \hookrightarrow L_0(S;L_{1,\mathrm{loc}}(\mathbb{R}^d;X)) \hookrightarrow L_0(\mathbb{R}^d \times S;X),$$

so that

$$(S_n f)_{n\in\mathbb{N}} = \lim_{k\to\infty} (S_n f_k)_{n\in\mathbb{N}} \quad \text{in} \quad L_0(\mathbb{R}^d \times S;X).$$

By passing to a suitable subsequence we may without loss of generality assume that $(S_n f_k)_{n\in\mathbb{N}} \to (S_n f)_{n\in\mathbb{N}}$ pointwise a.e. as $k \to \infty$. Using the Fatou property of E, we find

$$\begin{aligned}\|f\|_{Y^{\boldsymbol{A}}(E;X)} &= \left\|(\|S_n f\|_X)_n\right\|_E = \left\|\liminf_{k\to\infty}(\|S_n f_k\|_X)_n\right\|_E \\ &\leq \liminf_{k\to\infty} \left\|(\|S_n f_k\|_X)_n\right\|_E = \liminf_{k\to\infty} \|f_k\|_{Y^{\boldsymbol{A}}(E;X)}. \quad \square\end{aligned}$$

Proposition 3.27. *Let $F \in \mathcal{S}(0, 0, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$, $s \in \mathbb{R}$ and $\lambda \in (0,\infty)$. Suppose that there exists a constant $C \in [1,\infty)$ such that, $\|(f_{j(n)})_{n\in\mathbb{N}}\|_F \leq C\|(f_n)_{n\in\mathbb{N}}\|_F$ for all $\{f_n\}_{n\in\mathbb{N}\cup\{*\}} \subset F$ with $f_* = 0$ and mappings $j : \mathbb{N} \to \mathbb{N} \cup \{*\}$ with the property that $\# j^{-1}(k) \leq 1$ for every $k \in \mathbb{N}$. Then*

$$Y^{\boldsymbol{A}}(F^s;X) = Y^{\lambda\boldsymbol{A}}(F^{\lambda s};X)$$

with an equivalence of quasi-norms.

The following lemma constitutes the main step in the proof of Proposition 3.27.

Lemma 3.28. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, 0, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$, $s \in \mathbb{R}$ and $\lambda \in (0, \infty)$. Set $h := \lfloor \frac{1}{\lambda} \rfloor + 2$. For all $f \in L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ of the form*

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$$

with $(f_n)_{n \in \mathbb{Z}} \subset L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ satisfying the spectrum condition

$$\begin{cases} \operatorname{supp} \hat{f}_0 & \subset \overline{B}^{\lambda \boldsymbol{A}}(0, 2), \\ \operatorname{supp} \hat{f}_n & \subset \overline{B}^{\lambda \boldsymbol{A}}(0, 2^{n+1}) \setminus B^{\lambda \boldsymbol{A}}(0, 2^{n-1}), \qquad n \geq 1, \end{cases} \tag{24}$$

and $f_n = 0$ for $n \in \mathbb{Z}_{<0}$, there is the estimate

$$\| f \|_{Y^{\boldsymbol{A}}(E^s; X)} \lesssim \sum_{m=-h}^{h} \left\| (2^{\lambda s \lfloor \frac{n}{\lambda} \rfloor} f_{m + \lfloor \frac{n}{\lambda} \rfloor})_n \right\|_{E(X)}.$$

Proof. Let $\varphi = (\varphi_n)_{n \in \mathbb{N}} \in \Phi^{\boldsymbol{A}}_{1,2}(\mathbb{R}^d)$ with associated sequence of convolution operators $(S_n)_{n \in \mathbb{N}}$.

In view of the spectrum conditions of $(\varphi_n)_{n \in \mathbb{N}}$ and $(f_n)_{n \in \mathbb{N}}$ and the fact that $\rho_{\lambda \boldsymbol{A}} = \rho_{\boldsymbol{A}}^{\lambda}$, it holds true that $S_n f_k = 0$ for every $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ satisfying $|k - \lfloor \frac{n}{\lambda} \rfloor| \leq \lfloor \frac{1}{\lambda} \rfloor + 2$. Since

$$S_n f = S_n \left(\sum_{k=0}^{\infty} f_k \right) = \sum_{k=0}^{\infty} S_n f_k \quad \text{in} \quad L_0(S; \mathcal{S}'(\mathbb{R}^d; X)),$$

it follows that

$$S_n f = \sum_{m=-h}^{h} S_n f_{m + \lfloor \frac{n}{\lambda} \rfloor}, \qquad n \in \mathbb{N}.$$

As

$$\operatorname{supp}(\hat{f}_{m + \lfloor \frac{n}{\lambda} \rfloor}) \subset \overline{B}^{\lambda \boldsymbol{A}}(0, 2^{m + \lfloor \frac{n}{\lambda} \rfloor}) \subset \overline{B}^{\lambda \boldsymbol{A}}(0, 2^{h + \frac{n}{\lambda}}) = \overline{B}^{\boldsymbol{A}}(0, 2^{\lambda h + n})$$

for all $n \in \mathbb{N}$ and $m \in \{-h, \dots, h\}$, a combination of Proposition 3.19, Corollary A.2 and Lemma A.3 thus yields that

$$\| f \|_{Y^{\boldsymbol{A}}(E^s; X)} \lesssim \| (S_n f)_n \|_{E^s(X)} \lesssim \sum_{m=-h}^{h} \| (f_{m + \lfloor \frac{n}{\lambda} \rfloor})_n \|_{E^s(X)}.$$

The desired estimate finally follows from the observation that $2^{ns} \eqsim 2^{\lambda \lfloor \frac{n}{\lambda} \rfloor}$ for all $n \in \mathbb{N}$. □

Proof of Proposition 3.27. It suffices to show that $Y^{\lambda \boldsymbol{A}}(F^{\lambda s}; X) \hookrightarrow Y^{\boldsymbol{A}}(F^s; X)$, the reverse inclusion also being of this form (for suitable choices of parameters). Let $f \in Y^{\lambda \boldsymbol{A}}(F^{\lambda s}; X)$. Then f has a representation as a convergence series

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$$

with $(f_n)_{n \in \mathbb{N}} \subset L_0(S; \mathcal{S}'(\mathbb{R}^d; X))$ satisfying the spectrum condition (24) and $\| (f_n)_n \|_{F^{\lambda s}(X)} \leq 2 \| f \|_{Y^{\lambda \boldsymbol{A}}(F^{\lambda s}; X)}$. Set $f_n := 0$ for $n \in \mathbb{Z}_{<0}$. The assumptions on F and the observation that

$\#\{n : \lfloor \frac{n}{\lambda} \rfloor = k\} \leq \lfloor \lambda \rfloor + 1$ for all $k \in \mathbb{N}$, give us the estimates

$$\left\| (2^{\lambda s \lfloor \frac{n}{\lambda} \rfloor} f_{m+\lfloor \frac{n}{\lambda} \rfloor})_n \right\|_{F(X)} \leq C(\lfloor \lambda \rfloor + 1) \| (2^{\lambda s k} f_k)_k \|_{F(X)}, \qquad m \in \mathbb{Z}.$$

An application of Lemma 3.28 finishes the proof. □

Example 3.29. In the setting of Example 3.20, Proposition 3.27 yields:

(i) $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X) = F^{\lambda s,\lambda \boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$,

(ii) $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X) = B^{\lambda s,\lambda \boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$,

(iii) $\mathbb{F}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X) = \mathbb{F}^{\lambda s,\lambda \boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X)$,

with an equivalence of quasi-norms depending on $\lambda \in (0, \infty)$. In particular, in the special case that $\boldsymbol{A} = a\boldsymbol{I}_{\mathpzc{d}} = a(I_{d_1}, \ldots, I_{d_\ell})$ for some $a \in (0, \infty)$, taking $\lambda = 1/a$ yields a description as an isotropic space.

4. Difference norms

In this section we derive several estimates for $YL^{\boldsymbol{A}}(E; X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E; X)$, as well as for $Y^{\boldsymbol{A}}(E; X)$. The main interest lies in the estimates involving differences, as these form the basis for the intersection representation in Section 5.

4.1. Some notation

Let X be a Banach space. For each $M \in \mathbb{N}_1$ and $h \in \mathbb{R}^d$ we define difference operator Δ^M_h on $L_0(\mathbb{R}^d; X)$ by

$$\Delta^M_h := (L_h - I)^M = \sum_{i=0}^{M} (-1)^i \binom{M}{i} L_{(M-i)h},$$

where L_h denotes the left translation by h.

For $N \in \mathbb{N}$ we denote by $\mathcal{P}^d_N$ the space of polynomials of degree at most N on $\mathbb{R}^d$. We write $\mathcal{P}^d_N(\mathbb{Q}) \subset \mathcal{P}^d_N$ for the subset of polynomials having rational coefficients.

Let $M \in \mathbb{N}_1$. Let $F = L_{\boldsymbol{p},\mathpzc{d}} = L_{\boldsymbol{p},\mathpzc{d}}(\mathbb{R}^d)$ with $\boldsymbol{p} \in (0, \infty)^\ell$. Let $B \subset \mathbb{R}^d$ be a bounded Borel set of non-zero measure. For $f \in L_0(\mathbb{R}^d)$ we define

$$\mathcal{E}_M(f, B, F) := \inf_{\pi \in \mathcal{P}^d_{M-1}} \| (f - \pi) 1_B \|_F = \inf_{\pi \in \mathcal{P}^d_{M-1}(\mathbb{Q})} \| (f - \pi) 1_B \|_F$$

and

$$\overline{\mathcal{E}}_M(f, B, F) := \frac{\mathcal{E}_M(f, B, F)}{\mathcal{E}_M(1, B, F)}.$$

We define the collection of dyadic anisotropic cubes $\{Q^{\boldsymbol{A}}_{n,k}\}_{(n,k) \in \mathbb{Z} \times \mathbb{Z}^d}$ by

$$Q^{\boldsymbol{A}}_{n,k} := \boldsymbol{A}_{2^{-n}} \left([0, 1)^d + k \right).$$

For $b \in (0, \infty)$ we define $\{Q^{\boldsymbol{A}}_{n,k}(b)\}_{(n,k) \in \mathbb{Z} \times \mathbb{Z}^d}$ by

$$Q^{\boldsymbol{A}}_{n,k}(b) := \boldsymbol{A}_{2^{-n}} \left([0, 1)^d(b) + k \right),$$

where $[0, 1)^d(b)$ is the cube concentric to $[0, 1)^d$ with sidelength b:

$$[0, 1)^d(b) := \left[\frac{1-b}{2}, \frac{1+b}{2} \right)^d.$$

We furthermore define the corresponding families of indicator functions $\{\chi^{\boldsymbol{A}}_{n,k}\}_{(n,k)\in\mathbb{Z}\times\mathbb{Z}^d}$ and $\{\chi^{\boldsymbol{A},b}_{n,k}\}_{(n,k)\in\mathbb{Z}\times\mathbb{Z}^d}$:

$$\chi^{\boldsymbol{A}}_{n,k} := 1_{Q^{\boldsymbol{A}}_{n,k}} \qquad \text{and} \qquad \chi^{\boldsymbol{A},b}_{n,k} := 1_{Q^{\boldsymbol{A}}_{n,k}(b)}.$$

Definition 4.1. Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. We define $y^{\boldsymbol{A}}(E)$ as the space of all $(s_{n,k})_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} \subset L_0(S)$ for which $(\sum_{k\in\mathbb{Z}^d} s_{n,k}\chi^{\boldsymbol{A}}_{n,k})_{n\in\mathbb{N}} \in E$. We equip $y^{\boldsymbol{A}}(E)$ with the quasi-norm

$$\|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)} := \Big\|\Big(\sum_{k\in\mathbb{Z}^d} s_{n,k}\chi^{\boldsymbol{A}}_{n,k}\Big)_n\Big\|_E.$$

Definition 4.2. Let F be a quasi-Banach function space on the σ-finite measure space $(T, \mathscr{B}, \nu)$. We define $\mathscr{F}_{\mathrm{M}}(X^*; F)$ as the space of all $\{F_{x^*}\}_{x^*\in X^*} \subset L_0(T)$ for which there exists $G \in F_+$ such that $|F_{x^*}| \leq \|x^*\| G$ for all $x^* \in X^*$. We equip $\mathscr{F}_{\mathrm{M}}(X^*; F)$ with the quasi-norm

$$\|\{F_{x^*}\}_{x^*}\|_{\mathscr{F}_{\mathrm{M}}(X^*;F)} := \inf \|G\|_F,$$

where the infimum is taken over all majorants G as above.

In the special case that $F = E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ in the above definition, it will be convenient to view $\mathscr{F}_{\mathrm{M}}(X^*; E)$ as the space of all $\{g_{x^*,n}\}_{(x^*,n)\in X^*\times\mathbb{N}} \subset L_0(S)$ for which there exists $(g_n)_n \in E_+$ such that $|g_{x^*,n}| \leq \|x^*\| g_n$, equipped with the quasi-norm

$$\|\{g_{x^*,n}\}_{(x^*,n)}\|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} := \inf \|(g_n)_n\|_E,$$

where the infimum is taken over all majorants $(g_n)_n$ as above.

Note that the corresponding properties from Definition 3.1 for $\mathscr{F}_{\mathrm{M}}(X^*; E)$ are inherited from E.

Definition 4.3. Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. We define $\widetilde{y}^{\boldsymbol{A}}(E; X)$ as the space of all $(s_{x^*,n,k})_{(x^*,n,k)\in X^*\times\mathbb{N}\times\mathbb{Z}^d} \subset L_0(S)$ for which $(\sum_{k\in\mathbb{Z}^d} s_{x^*,n,k}\chi^{\boldsymbol{A}}_{n,k})_{n\in\mathbb{N}} \in \mathscr{F}_{\mathrm{M}}(X^*; E)$. We equip $\widetilde{y}^{\boldsymbol{A}}(E; X)$ with the quasi-norm

$$\|(s_{x^*,n,k})_{(n,k)}\|_{\widetilde{y}^{\boldsymbol{A}}(E;X)} := \Big\|\Big(\sum_{k\in\mathbb{Z}^d} s_{x^*,n,k}\chi^{\boldsymbol{A}}_{n,k}\Big)_n\Big\|_{\mathscr{F}_{\mathrm{M}}(X^*;E)}.$$

4.2. *Statements of the results*

The first two main results of this section, Theorems 4.4 and 4.6, contain estimates for $YL^{\boldsymbol{A}}(E; X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E; X)$, respectively, involving differences, as well as atoms and oscillations, in the general case $\boldsymbol{r} \in (0,\infty)^\ell$. The third main result of this section, Theorem 4.8, provides estimates for $Y^{\boldsymbol{A}}(E; X) = YL^{\boldsymbol{A}}(E; X) = \widetilde{YL}^{\boldsymbol{A}}(E; X)$ involving differences in the special case that $\boldsymbol{r} = 1$ (in which case, indeed, $Y^{\boldsymbol{A}}(E; X) = YL^{\boldsymbol{A}}(E; X) = \widetilde{YL}^{\boldsymbol{A}}(E; X)$ by Theorem 3.22 (and Remark 3.23)); some things simplify here when $\boldsymbol{r} = 1$.

Theorem 4.4. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and suppose that $\varepsilon_+, \varepsilon_- > 0$. Let $\boldsymbol{p} \in (0,\infty)^\ell$ and $M \in \mathbb{N}$ satisfy $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$ and $M\lambda^{\boldsymbol{A}}_{\min} > \varepsilon_-$, where $\mathrm{tr}(\boldsymbol{A}) = (\mathrm{tr}(A_1), \ldots, \mathrm{tr}(A_\ell))$. Given $f \in L_0(S; L_{\boldsymbol{r},d}(\mathbb{R}^d; X))$, consider the following statements:*

(i) $f \in YL^{\boldsymbol{A}}(E; X)$.

(ii) *There exist* $(s_{n,k})_{(n,k)} \in y^{\boldsymbol{A}}(E)$ *and* $(b_{n,k})_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} \subset L_0(S; C_c^M([-1,2]^d))$ *with* $\|b_{n,k}\|_{C_b^M} \leq 1$ *such that, setting* $a_{n,k} := b_{n,k}(\boldsymbol{A}_{2^n} \cdot -k)$, f *has the representation*

$$f = \sum_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} s_{n,k}a_{n,k} \qquad \text{in} \qquad L_0(S; L_{\boldsymbol{p},d,\mathrm{loc}}(\mathbb{R}^d; X)). \tag{25}$$

(iii) $f \in E_0(X) \cap L_0(S; L_{\boldsymbol{p},d,\mathrm{loc}}(\mathbb{R}^d; X))$ *and* $(d_M^{\boldsymbol{A},\boldsymbol{p}}(f)_n)_{n\geq 1} \in E(\mathbb{N}_1)$*, where*

$$d_{M,n}^{\boldsymbol{A},\boldsymbol{p}}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto \Delta_z^M f \right\|_{L_{\boldsymbol{p},d}(B^{\boldsymbol{A}}(0,2^{-n});X)}, \qquad n \in \mathbb{N}.$$

For these statements, there is the chain of implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii)*. Moreover, there are the following estimates:*

$$\|f\|_{E_0(X)} + \|(d_{M,n}^{\boldsymbol{A},\boldsymbol{p}}(f))_{n\geq 1}\|_{E(\mathbb{N}_1)} \lesssim \|f\|_{YL^{\boldsymbol{A}}(E;X)} \eqsim \|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)}.$$

Remark 4.5. Theorem 4.4 is partial extension of [24, Theorem 1.1.14], which is concerned with $YL(E)$ with $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, I, \boldsymbol{r})$. That result actually extends completely to the anisotropic scalar-valued setting $YL^{\boldsymbol{A}}(E)$ with $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r})$. However, in the general Banach space-valued case there arises a difficulty due to the unavailability of the Whitney inequality [24, (1.2.2)/Theorem A.1] (see [60,61]) and the derived Lemma 4.12. We overcome this issue in Theorem 4.6 by extending [24, Theorem 1.1.14] to $\widetilde{YL}^{\boldsymbol{A}}(E; X)$ instead of $YL^{\boldsymbol{A}}(E; X)$ (recall Remark 3.13). This was actually the motivation for introducing the space $\widetilde{YL}^{\boldsymbol{A}}(E; X)$, which is connected to $YL^{\boldsymbol{A}}(E; X)$ and $Y^{\boldsymbol{A}}(E; X)$ through Theorem 3.22.

Theorem 4.6. *Let* $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ *and suppose that* $\varepsilon_+, \varepsilon_- > 0$*. Let* $\boldsymbol{p} \in (0,\infty)^\ell$ *and* $M \in \mathbb{N}$ *satisfy* $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$ *and* $M\lambda_{\min}^{\boldsymbol{A}} > \varepsilon_-$*. Given* $f \in L_0(S; L_{\boldsymbol{r},d}(\mathbb{R}^d; X))$*, consider the following statements:*

(I) $f \in \widetilde{YL}^{\boldsymbol{A}}(E; X)$.

(II) *There exist* $(s_{x^*,n,k})_{(n,k)} \in \tilde{y}^{\boldsymbol{A}}(E; X)$ *and* $(b_{x^*,n,k})_{(x^*,n,k)\in X^*\times\mathbb{N}\times\mathbb{Z}^d} \subset L_0(S; C_c^M([-1,2]^d))$ *with* $\|b_{x^*,n,k}\|_{C_b^M} \leq 1$ *such that, setting* $a_{x^*,n,k} := b_{x^*n,k}(\boldsymbol{A}_{2^n} \cdot -k)$*, for all* $x^* \in X^*$*,* $\langle f, x^*\rangle$ *has the representation*

$$\langle f, x^*\rangle = \sum_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} s_{x^*,n,k}a_{x^*,n,k} \qquad \text{in} \qquad L_0(S; L_{\boldsymbol{p},d,\mathrm{loc}}(\mathbb{R}^d)).$$

(III) $f \in E_0(X) \cap L_0(S; L_{\boldsymbol{p},d,\mathrm{loc}}(\mathbb{R}^d; X))$ *and*

$$\{d_{M,x^*,n}^{\boldsymbol{A},\boldsymbol{p}}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq 1}} \in \mathscr{F}_{\mathrm{M}}(X^*; E(\mathbb{N}_1)),$$

where

$$d_{M,x^*,n}^{\boldsymbol{A},\boldsymbol{p}}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto \Delta_z^M \langle f, x^*\rangle \right\|_{L_{\boldsymbol{p},d}(B^{\boldsymbol{A}}(0,2^{-n}))}, \qquad n \in \mathbb{N}.$$

(IV) $f \in E_0(X) \cap L_0(S; L_{\boldsymbol{p},d,\mathrm{loc}}(\mathbb{R}^d; X))$ *and*

$$\{\mathcal{E}_{M,x^*,n}^{\boldsymbol{A},\boldsymbol{p}}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_1} \in \mathscr{F}_{\mathrm{M}}(X^*; E(\mathbb{N}_1)),$$

where

$$\mathcal{E}_{M,x^*,n}^{\boldsymbol{A},\boldsymbol{p}}(f)(x) := \overline{\mathcal{E}}_M(\langle f, x^*\rangle, B^{\boldsymbol{A}}(x, 2^{-n}), L_{\boldsymbol{p},d}), \qquad x^* \in X^*, n \in \mathbb{N}.$$

(V) $f \in E_0(X)$ *and there is* $\{\pi_{x^*,n,k}\}_{(x^*,n,k)\in X^*\times\mathbb{N}_1\times\mathbb{Z}} \in \mathcal{P}^d_{M-1}$ *such that*

$$g_{x^*,n} := \sum_{k\in\mathbb{Z}^d} |\langle f, x^*\rangle - \pi_{x^*,n,k}|\, 1_{Q^{\boldsymbol{A}}_{n,k}(3)}, \qquad n \geq 1,$$

satisfies $\{g_{x^*,n}\}_{(x^*,n)\in X^*\times\mathbb{N}_1} \in \mathscr{F}_{\mathrm{M}}(X^*; E(\mathbb{N}_1))$.

For $f \in L_0(S; L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d; X))$ *it holds that* (V) $\Rightarrow$ (I) $\Leftrightarrow$ (II) $\Rightarrow$ (III) & (IV) *with corresponding estimates*

$$\begin{aligned}\|f\|_{E_0(X)} + \|(d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f))_{(x^*,n)}\|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} + \|\{\mathcal{E}^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_1}\|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq 1}))} \\ \lesssim \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} \eqsim \|(s_{x^*,n,k})_{(x^*,n,k)}\|_{\widetilde{y}^{\boldsymbol{A}}(E)} \\ \lesssim \|f\|_{E_0(X)} + \left\|\{g_{x^*,n}\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq 1}}\right\|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_1))}.\end{aligned}$$

Moreover, for f *of the form* $f = \sum_{i\in I} 1_{S_i} \otimes f^{[i]}$ *with* $(S_i)_{i\in I} \subset \mathscr{A}$ *a countable family of mutually disjoint sets and* $(f^{[i]})_{i\in I} \in L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)$*, it holds that* (I), (II), (III), (IV), *and* (V) *are equivalent statements and there are the corresponding estimates*

$$\begin{aligned}\|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} &\eqsim \|(s_{x^*,n,k})_{(x^*,n,k)}\|_{\widetilde{y}^{\boldsymbol{A}}(E)} \\ &\eqsim \|f\|_{E_0(X)} + \left\|\{d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq 1}}\right\|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_1))} \\ &\eqsim \|f\|_{E_0(X)} + \left\|(\mathcal{E}^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_n\right\|_E \\ &\eqsim \|f\|_{E_0(X)} + \left\|\{g_{x^*,n}\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq 1}}\right\|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_1))}.\end{aligned}$$

Corollary 4.7. *Let* $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ *and suppose that* $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{1})_+$. *Let* $\boldsymbol{p} \in (0,\infty]^\ell$ *and* $M \in \mathbb{N}$ *satisfy* $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$ *and* $M\lambda^{\boldsymbol{A}}_{\min} > \varepsilon_-$. *Then, for each* $f \in L_0(S; L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d; X))$ *of the form* $f = \sum_{i\in I} 1_{S_i} \otimes f^{[i]}$ *with* $(S_i)_{i\in I} \subset \mathscr{A}$ *a countable family of mutually disjoint sets and* $(f^{[i]})_{i\in I} \in L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)$,

$$\|f\|_{Y^{\boldsymbol{A}}(E;X)} \eqsim \|f\|_{YL^{\boldsymbol{A}}(E;X)} \eqsim \|f\|_{E_0(X)} + \|(d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_{n\geq 1}\|_{E(\mathbb{N}_1)}.$$

Theorem 1.2 from the introduction can be obtained as a special case of the following theorem.

Theorem 4.8. *Let* $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{1}, (S, \mathscr{A}, \mu))$ *and suppose that* $\varepsilon_+, \varepsilon_- > 0$. *Let* $\boldsymbol{p} \in [1,\infty]^\ell$ *and* $M \in \mathbb{N}$ *satisfy* $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{1} - \boldsymbol{p}^{-1})$ *and* $M\lambda^{\boldsymbol{A}}_{\min} > \varepsilon_-$. *Write*

$$I^{\boldsymbol{A}}_{M,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})} \int_{B^{\boldsymbol{A}}(0,2^{-n})} \Delta^M_z f\, dz, \qquad f \in L_0(S; L_{1,\mathrm{loc}}(\mathbb{R}^d; X)).$$

Then

$$\begin{aligned}\|f\|_{Y^{\boldsymbol{A}}(E;X)} \eqsim \|f\|_{YL^{\boldsymbol{A}}(E;X)} &\eqsim \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} \\ &\eqsim \|f\|_{E_0(X)} + \|(I^{\boldsymbol{A}}_{M,n}(f))_{n\geq 1}\|_{E(\mathbb{N}_1;X)} \\ &\eqsim \|f\|_{E_0(X)} + \|(d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_{n\geq 1}\|_{E(\mathbb{N}_1;X)}\end{aligned}$$

for all $f \in E_0(X) \hookrightarrow E_i \hookrightarrow E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}](X)$ *(see Remark 3.10).*

Remark 4.9. Recall from Example 3.20 that, in case $\ell = 1$, $A = I$, $p \in (1,\infty)$, $q = 2$, $w \in A_p(\mathbb{R}^d)$, F is a UMD Banach function space and X is a Hilbert space, $\mathbb{F}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X)$ coincides with the weighted vector-valued Bessel potential space $H^s_p(\mathbb{R}^d, w; F(X))$.

Theorem 4.8 thus especially gives a difference norm characterization for $H^s_p(\mathbb{R}^d, w; F(X))$ (cf. [34, Remark 4.10]).

Proposition 4.10. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and suppose that $\varepsilon_+, \varepsilon_- > 0$. Let $c \in \mathbb{R}$. Let $\boldsymbol{p} \in (0, \infty]^\ell$ and $M \in \mathbb{N}$ satisfy $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$ and $M > \varepsilon_-$. Then*

$$\|\{d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,n}(f)\}_n\|_{E(X)} \lesssim \|f\|_{YL^{\boldsymbol{A}}(E;X)}, \qquad f \in L_0(S; L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d; X)),$$

and

$$\|\{d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,x^*,n}(f)\}_{(x^*,n)}\|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} \lesssim \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}, \qquad f \in L_0(S; L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d; X)),$$

where

$$d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto L_{cz}\Delta^M_z f \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n});X))}$$

and

$$d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,x^*,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto L_{cz}\Delta^M_z \langle f, x^*\rangle \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))}.$$

4.3. Some lemmas

Lemma 4.11. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. Put $C := \max_{x\in[0,1]^d} \rho_{\boldsymbol{A}}(x) \in [1, \infty)$. Then, for each $(s_{n,k})_{(n,k)} \in y^{\boldsymbol{A}}(E)$,*

$$\|s_{n,k}\|_{E^{\boldsymbol{A}}_{\otimes}} \lesssim_{n,\boldsymbol{A},\boldsymbol{r}} (C + \rho_{\boldsymbol{A}}(k))^{\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}}, \qquad (n, k) \in \mathbb{N} \times \mathbb{Z}^d.$$

Proof. Fix $(i, l) \in \mathbb{N} \times \mathbb{Z}^d$. By Remark 3.10, $E_i \hookrightarrow E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}]$, so that

$$\begin{aligned}\|s_{i,l}\|_{E^{\boldsymbol{A}}_{\otimes}} \|\chi^{\boldsymbol{A}}_{i,l}\|_{B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}} &= \|s_{i,l}\chi^{\boldsymbol{A}}_{i,l}\|_{E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}]} \lesssim_i \|s_{i,l}\chi^{\boldsymbol{A}}_{i,l}\|_{E_i} \\ &\leq \left\| \Big(\sum_{k\in\mathbb{Z}^d} s_{n,k}\chi^{\boldsymbol{A}}_{n,k}\Big)_n \right\|_E = \|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)}.\end{aligned} \tag{26}$$

Let $\boldsymbol{R} = (R, \dots, R) \in [1, \infty)^\ell$ be given by $R := c_{\boldsymbol{A}}(C + \rho_{\boldsymbol{A}}(l))$. Then

$$\rho_{\boldsymbol{A}}(x + l) \leq c_{\boldsymbol{A}}(\rho_{\boldsymbol{A}}(x) + \rho_{\boldsymbol{A}}(l)) \leq c_{\boldsymbol{A}}(C + \rho_{\boldsymbol{A}}(l)) = R \leq 2^i R, \qquad x \in [0, 1]^d.$$

Therefore,

$$\mathrm{supp}\,(\chi^{\boldsymbol{A}}_{i,l}) = \boldsymbol{A}_{2^{-i}}([0, 1]^d + l) \subset B^{\boldsymbol{A}}(0, \boldsymbol{R}).$$

As a consequence,

$$[c_{\boldsymbol{A}}(C + \rho_{\boldsymbol{A}}(l))]^{-\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}} \|\chi^{\boldsymbol{A}}_{i,l}\|_{L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d)} \leq \|\chi^{\boldsymbol{A}}_{i,l}\|_{B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}} \tag{27}$$

Observing that $\|\chi^{\boldsymbol{A}}_{i,l}\|_{L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d)} = c_{i,\boldsymbol{A},\boldsymbol{r}}$, a combination of (26) and (27) gives the desired result. □

Lemma 4.12. *Let $p \in (0, \infty]$ and $M \in \mathbb{N}_1$. Then there is a constant $C = C_{M,p,d}$ such that, if $f \in L_{p,\mathrm{loc}}(\mathbb{R}^d)$ and $Q = \boldsymbol{A}_\lambda([0, 1)^d + b)$ with $\lambda \in (0, \infty)$ and $b \in \mathbb{R}^d$, then there is $\pi \in \mathcal{P}^d_{M-1}$*

satisfying (with the usual modification if $p = \infty$):

$$|f - \pi|\, 1_Q \leq C \left(\fint_{B^{\boldsymbol{A}}(0,\lambda)} |\Delta_z^M f|^p \, dz \right)^{1/p} + C \left(\fint_{B^{\boldsymbol{A}}(0,\lambda)} \fint_{Q(2)} |\Delta_z^M f|^p \, dy \, dz \right)^{1/p}.$$

Proof. The case $\lambda = 1$ is contained in [24, Lemma 1.2.1], from which the general case can be obtained by a scaling argument. □

From Lemma 4.13 to Corollary 4.15 we will actually only use Corollary 4.15 in the scalar-valued case in the proof of Theorem 4.6. However, although the scalar-valued case is easier, we have decided to present it in this way as it could be useful for potential extensions of Theorem 4.4 along these lines. In the latter the main obstacle is Lemma 4.12.

We write $\mathcal{P}_N^d(X) \simeq X^{M_{N,d}}$, where $M_{N,d} := \#\{\alpha \in \mathbb{N}^d : |\alpha| \leq M\}$, for the space of X-valued polynomials of degree at most N on $\mathbb{R}^d$.

Lemma 4.13. *Let $(T, \mathscr{B}, \nu)$ a measure space, $\mathbb{F} \subset L_2(T)$ a finite dimensional subspace, $\mathbb{E} \subset L_0(T; X)$ a topological vector space with $\mathbb{F} \otimes X \subset \mathbb{E}$ such that*

$$\mathbb{F} \times X \longrightarrow \mathbb{E},\ (p, f) \mapsto f \otimes x,$$

and

$$\mathbb{F} \times \mathbb{E} \longrightarrow L_1(T; X), (f, g) \mapsto fg,$$

are well-defined bilinear mappings that are continuous with respect to the second variable. Then $\mathbb{F} \otimes X$ is a complemented subspace of $\mathbb{E}$.

Proof. Choose an orthogonal basis $b_1, \ldots, b_n$ of the finite dimensional subspace $\mathbb{F}$ of $L_2(T)$. Then

$$\pi : \mathbb{E} \longrightarrow \mathbb{E},\ g \mapsto \sum_{i=1}^{n} \left[\int_T b_i(t) g(t) \mathrm{d}\nu(t) \right] \otimes b_i,$$

is a well-defined continuous linear mapping on $\mathbb{E}$, which is a projection onto the linear subspace $\mathbb{F} \otimes X \subset \mathbb{E}$. □

Corollary 4.14. *If $\mathbb{E}$ in Lemma 4.13 is an F-space, then so is $(\mathbb{F} \otimes X, \tau_{\mathbb{E}})$. As a consequence, if τ is a topological vector space topology on $\mathbb{F} \otimes X$ with $(\mathbb{F} \otimes X, \tau_{\mathbb{E}}) \hookrightarrow (\mathbb{F} \otimes X, \tau)$, then the latter is in fact a topological isomorphism.*

Corollary 4.15. *Let $B = [-1, 2]^d$, $N \in \mathbb{N}$ and $q \in [1, \infty)$. Set $B_{n,k} := \boldsymbol{A}_{2^{-n}}(B + k)$ for $(n, k) \in \mathbb{N} \times \mathbb{Z}^d$. Then*

$$\|\pi(\boldsymbol{A}_{2^{-n}} \cdot + k)\|_{C_b^N(B;X)} \lesssim 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})/q} \|\pi\|_{L_q(B_{n,k};X)}, \qquad \pi \in \mathcal{P}_N^d(X), (n, k) \in \mathbb{N} \times \mathbb{Z}^d.$$

Proof. Let us first note that a substitution gives

$$\|\pi(\boldsymbol{A}_{2^{-n}} \cdot + k)\|_{L_q(B;X)} = 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})/q} \|\pi\|_{L_q(B_{n,k};X)},$$

while $\pi(\boldsymbol{A}_{2^{-n}} \cdot +k) \in \mathcal{P}^d_N(X)$. Applying Corollary 4.14 to $\mathbb{F} = \mathcal{P}^d_N$, viewed as finite dimensional subspace of $L_2(B)$, and $\mathbb{E} = C^N_n(B;X)$ and τ the topology on $\mathcal{P}_N(X) = \mathbb{F} \otimes X$ induced from $L_q(B;X)$, we obtain the desired result. □

Lemma 4.16. *Let $q, p \in (0,\infty)$, $q \leq p$, $b \in (0,\infty)$ and $M \in \mathbb{N}_1$. Let $f \in L_{p,\mathrm{loc}}(\mathbb{R}^d)$ and let $\{\pi_{n,k}\}_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} \subset \mathcal{P}^d_{M-1}$ such that*

$$\|f - \pi_{n,k}\|_{L_q(Q^{\boldsymbol{A}}_{n,k}(b))} \leq 2\mathcal{E}_M(f, Q^{\boldsymbol{A}}_{n,k}(b), L_q),$$

and let $\{\phi_{n,k}\}_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} \subset L_\infty(\mathbb{R}^d)$ be such that $\operatorname{supp}\phi_{n,k} \subset Q^{\boldsymbol{A}}_{n,k}(b)$, $\sum_{k\in\mathbb{Z}^d}\phi_{n,k} \equiv 1$, and $\|\phi_{n,k}\|_{L_\infty} \leq 1$. Then, for $(f_n)_{n\in\mathbb{N}} \subset L_0(S)$ defined by

$$f_n := \sum_{k\in\mathbb{Z}^d} \pi_{n,k}\phi_{n,k},$$

there is the convergence $f = \lim_{n\to\infty} f_n$ almost everywhere and in $L_{p,\mathrm{loc}}$.

Proof. This can be proved as in [24, Lemma 1.2.3]. □

Lemma 4.17. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$, $b \in (0,\infty)$ and suppose that $\varepsilon_+, \varepsilon_- > 0$. Let $\boldsymbol{p} \in (0,\infty]^\ell$ satisfy $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$. Define the sublinear operator*

$$T^{\boldsymbol{A}}_{\boldsymbol{p}} : L_0(S)^{\mathbb{N}\times\mathbb{Z}^d} \longrightarrow L_0(S;[0,\infty])^{\mathbb{N}\times\mathbb{Z}^d}, \qquad (s_{n,k})_{(n,k)} \mapsto (t_{n,k})_{(n,k)},$$

by

$$t_{n,k} := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{p}^{-1}} \Big\| \sum_{m,l} |s_{m,l}|\chi^{\boldsymbol{A}}_{m,l} \Big\|_{L_{\boldsymbol{p},\mathscr{d}}}$$

and the sum is taken over all indices $(m,l) \in \mathbb{N}\times\mathbb{Z}^d$ such that $Q^{\boldsymbol{A}}_{m,l} \subset Q^{\boldsymbol{A}}_{n,k}(b)$ and $m \geq n$. Then $T^{\boldsymbol{A}}_{\boldsymbol{p}}$ restricts to a bounded sublinear operator on $y^{\boldsymbol{A}}(E)$.

Proof. Let $(s_{n,k})_{(n,k)} \in y^{\boldsymbol{A}}(E)$ and $(t_{n,k})_{(n,k)} = T^{\boldsymbol{A}}_{\boldsymbol{p}}[(s_{n,k})_{(n,k)}] \in L_0(S;[0,\infty])^{\mathbb{N}\times\mathbb{Z}^d}$. We need to show that $\|(t_{n,k})\|_{y^{\boldsymbol{A}}(E)} \lesssim \|(s_{n,k})\|_{y^{\boldsymbol{A}}(E)}$. Here we may without loss of generality assume that $s_{n,k} \geq 0$ for all (n,k).

Set

$$\delta := \frac{1}{2}\left(\varepsilon_+ - \mathrm{tr}(A)\cdot(\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})\right) \in (0,\infty).$$

Define

$$g_m := \sum_{l\in\mathbb{Z}^d} s_{m,l}\chi^{\boldsymbol{A}}_{m,l} \in L_0(S), \qquad m \in \mathbb{N}.$$

Then

$$t_{n,k} \leq 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \Big\| \sum_{m=n}^{\infty} g_m \Big\|_{L_{\boldsymbol{p},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b))}. \tag{28}$$

As the right-hand side is increasing in $\boldsymbol{p}$ by Hölder's inequality, it suffices to consider the case $\boldsymbol{p} \geq \boldsymbol{r}$.

Several applications of the elementary embedding

$$\ell^{s_0}_{q_0}(\mathbb{N}) \hookrightarrow \ell^{s_1}_{q_1}(\mathbb{N}), \qquad s_0 > s_1, q_0, q_1 \in (0,\infty],$$

in combination with Fubini's theorem yield that

$$\Big\| \sum_{m=n}^{\infty} g_m \Big\|_{L_{\boldsymbol{p},\mathpzc{d}}(Q^{\boldsymbol{A}}_{n,k}(b))} \lesssim \sum_{m=n}^{\infty} 2^{(m-n)\delta} \|g_m\|_{L_{\boldsymbol{p},\mathpzc{d}}(Q^{\boldsymbol{A}}_{n,k}(b))}. \tag{29}$$

In order to estimate the summands on the right-hand side of (28), we will use the following fact. Let $(T_1, \mathscr{B}_1, \nu_1), \dots, (T_\ell, \mathscr{B}_\ell, \nu_\ell)$ be σ-finite measure spaces and let $I_1, \dots, I_\ell$ be countable sets. Put $T = T_1 \times \cdots \times T_\ell$ and $I = I_1 \times \cdots \times I_\ell$. Let $(c_{\boldsymbol{i}})_{\boldsymbol{i} \in I} \subset \mathbb{C}$ and, for each $j \in \{1, \dots, \ell\}$, let $(A^{(j)}_{i_j \in I_j}) \subset \mathscr{B}_j$ be a sequence of mutually disjoint sets. Then

$$\| \sum_{\boldsymbol{i} \in I} c_{\boldsymbol{i}} 1_{A^{(1)}_{i_1} \times \cdots \times A^{(\ell)}_{i_\ell}} \|_{L_{\boldsymbol{p}}(T)} \leq \left(\sup_{\boldsymbol{i} \in I} \prod_{j=1}^{\ell} |A^{(j)}_{i_j}|^{\frac{1}{p_j} - \frac{1}{r_j}} \right) \| \sum_{\boldsymbol{i} \in I} c_{\boldsymbol{i}} 1_{A^{(1)}_{i_1} \times \cdots \times A^{(\ell)}_{i_\ell}} \|_{L_{\boldsymbol{r}}(T)}. \tag{30}$$

Indeed,

$$\begin{aligned}
&\| \sum_{\boldsymbol{i} \in I} c_{\boldsymbol{i}} 1_{A^{(1)}_{i_1} \times \cdots \times A^{(\ell)}_{i_\ell}} \|_{L_{\boldsymbol{p}}(T)} \\
&= \left(\sum_{i_\ell \in I_\ell} |A^{(\ell)}_{i_\ell}| \left(\dots \left(\sum_{i_1 \in I_1} |A^{(1)}_{i_1}|\, |c_{\boldsymbol{i}}|^{p_1} \right)^{p_2/p_1} \dots \right)^{p_\ell/p_{\ell-1}} \right)^{1/p_\ell} \\
&\leq \left(\sum_{i_\ell \in I_\ell} |A^{(\ell)}_{i_\ell}|^{r_\ell/p_\ell} \left(\dots \left(\sum_{i_1 \in I_1} |A^{(1)}_{i_1}|^{r_1/p_1}\, |c_{\boldsymbol{i}}|^{r_1} \right)^{r_2/r_1} \dots \right)^{r_\ell/r_{\ell-1}} \right)^{1/r_\ell} \\
&\leq \left(\sup_{\boldsymbol{i} \in I} \prod_{j=1}^{\ell} |A^{(j)}_{i_j}|^{\frac{1}{p_j} - \frac{1}{r_j}} \right) \left(\sum_{i_\ell \in I_\ell} |A^{(\ell)}_{i_\ell}| \left(\dots \left(\sum_{i_1 \in I_1} |A^{(1)}_{i_1}|\, |c_{\boldsymbol{i}}|^{r_1} \right)^{r_2/r_1} \dots \right)^{r_\ell/r_{\ell-1}} \right)^{1/r_\ell} \\
&= \left(\sup_{\boldsymbol{i} \in I} \prod_{j=1}^{\ell} |A^{(j)}_{i_j}|^{\frac{1}{p_j} - \frac{1}{r_j}} \right) \| \sum_{\boldsymbol{i} \in I} c_{\boldsymbol{i}} 1_{A^{(1)}_{i_1} \times \cdots \times A^{(\ell)}_{i_\ell}} \|_{L_{\boldsymbol{r}}(T)},
\end{aligned}$$

where we used $\boldsymbol{p} \geq \boldsymbol{r}$ in the first inequality.

Let us now use the above fact to estimate $\|g_m\|_{L_{\boldsymbol{p},\mathpzc{d}}(Q^{\boldsymbol{A}}_{n,k}(b))}$:

$$\begin{aligned}
\|g_m\|_{L_{\boldsymbol{p},\mathpzc{d}}(Q^{\boldsymbol{A}}_{n,k}(b))} &\leq \Big\| \sum_{l \in \mathbb{Z}^d : Q^{\boldsymbol{A}}_{m,l} \cap Q^{\boldsymbol{A}}_{n,k}(b) \neq \varnothing} s_{m,l} \chi^{\boldsymbol{A}}_{m,l} \Big\|_{L_{\boldsymbol{p},\mathpzc{d}}(\mathbb{R}^d)} \\
&\overset{(30)}{\leq} 2^{-m \operatorname{tr}(\boldsymbol{A}) \cdot (\boldsymbol{p}^{-1} - \boldsymbol{r}^{-1})} \Big\| \sum_{l \in \mathbb{Z}^d : Q^{\boldsymbol{A}}_{m,l} \cap Q^{\boldsymbol{A}}_{n,k}(b) \neq \varnothing} s_{m,l} \chi^{\boldsymbol{A}}_{m,l} \Big\|_{L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d)} \\
&\leq 2^{-m \operatorname{tr}(\boldsymbol{A}) \cdot (\boldsymbol{p}^{-1} - \boldsymbol{r}^{-1})} \|g_m\|_{L_{\boldsymbol{r},\mathpzc{d}}(Q^{\boldsymbol{A}}_{n,k}(b+2))} \\
&= 2^{(m-n)((\varepsilon_+ - 2\delta)) - n \operatorname{tr}(\boldsymbol{A}) \cdot (\boldsymbol{p}^{-1} - \boldsymbol{r}^{-1})} \|g_m\|_{L_{\boldsymbol{r},\mathpzc{d}}(Q^{\boldsymbol{A}}_{n,k}(b+2))}
\end{aligned} \tag{31}$$

Putting (28), (29) and (31) together, we obtain

$$
\begin{aligned}
t_{n,k}\chi^{\boldsymbol{A}}_{n,k} &\leq \sum_{m=n}^{\infty} 2^{(m-n)((\varepsilon_+-\delta))+n\operatorname{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}} \|g_m\|_{L_{\boldsymbol{r},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b+2))}\chi^{\boldsymbol{A}}_{n,k} \\
&\lesssim_{b,\boldsymbol{A},\boldsymbol{r}} \sum_{m=n}^{\infty} 2^{(m-n)(\varepsilon_+-\delta)} M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_m).
\end{aligned} \tag{32}
$$

Since

$$
\left(\sum_{m=n}^{\infty} 2^{(m-n)(\varepsilon_+-\delta)} M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_m)\right)_{n\in\mathbb{N}} = \sum_{i=0}^{\infty} 2^{i(\varepsilon_+-\delta)}(S_+)^i M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[(g_n)_{n\in\mathbb{N}}\right],
$$

it follows that $(t_{n,k}) \in y^{\boldsymbol{A}}(E)$ with

$$
\begin{aligned}
\|(t_{n,k})\|^{\kappa}_{y^{\boldsymbol{A}}(E)} &= \Big\|\Big(\sum_{k\in\mathbb{Z}^d} t_{n,k}\chi^{\boldsymbol{A}}_{n,k}\Big)_n\Big\|^{\kappa}_E \\
&\lesssim \sum_{i=0}^{\infty} 2^{\kappa i((\varepsilon_+-\delta))} \|(S_+)^i M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[(g_n)_n\right]\|^{\kappa}_E \\
&\lesssim \sum_{i=0}^{\infty} 2^{-\kappa i\delta} \|(g_n)_n\|^{\kappa}_E \lesssim \|(g_n)_n\|^{\kappa}_E \\
&= \|(s_{n,k})\|^{\kappa}_{y^{\boldsymbol{A}}(E)},
\end{aligned} \tag{33}
$$

where κ is such that E has a κ-norm. □

Corollary 4.18. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and suppose that $\varepsilon_+, \varepsilon_- > 0$. Let $\boldsymbol{p} \in (0,\infty]^{\ell}$ satisfy $\varepsilon_+ > \operatorname{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$. Given $(s_{n,k})_{(n,k)} \in y^{\boldsymbol{A}}(E)$, set $g_n = \sum_{k\in\mathbb{Z}^d} s_{n,k}\chi^{\boldsymbol{A}}_{n,k}$. Then $\sum_{n=0}^{\infty}|g_n|$ in $L_0(S; L_{\boldsymbol{p},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$ and the series $\sum_{n=0}^{\infty} g_n$ converges almost everywhere, and in $L_0(S; L_{\boldsymbol{p},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$ (when $\boldsymbol{p} \in (0,\infty)^{\ell}$).*

Proof. This follows from (33), see [24, Corollary 1.2.5] for more details. □

Lemma 4.19. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$, $b \in (0,\infty)$ and $\lambda \in (\varepsilon_-, \infty)$. Define the sublinear operator*

$$
T_\lambda : L_0(S)^{\mathbb{N}\times\mathbb{Z}^d} \longrightarrow L_0(S;[0,\infty])^{\mathbb{N}\times\mathbb{Z}^d}, \qquad (s_{n,k})_{(n,k)} \mapsto (t_{n,k})_{(n,k)},
$$

by

$$
t_{n,k} := \sum_{m,l} 2^{\lambda(n-m)}|s_{m,l}|,
$$

the sum being taken over all indices $(m,l) \in \mathbb{N}\times\mathbb{Z}^d$ such that $Q^{\boldsymbol{A}}_{m,l}(b) \supset Q^{\boldsymbol{A}}_{n,k}$ and $m < n$. Then T_λ restricts to a bounded sublinear operator from $y^{\boldsymbol{A}}(E)$ to $y^{\boldsymbol{A}}(E)$.

Proof. This can be proved in the same way as [24, Lemma 1.2.6]. □

Lemma 4.20. *Let $\boldsymbol{r} \in (0,1]^{\ell}$ and $\rho \in (0,1)$ satisfy $\rho < \boldsymbol{r}_{\min}$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}^d$ satisfying*

$$
0 \leq \gamma_n(x) \lesssim (1 + 2^n\rho_{\boldsymbol{A}}(x))^{-\operatorname{tr}(\boldsymbol{A}^{\oplus})/\rho}.
$$

If $(s_{n,k})_{(n,k)} \in L_0(S)^{\mathbb{N}\times\mathbb{Z}^d}$, $g_n = \sum_{k\in\mathbb{Z}^d} s_{n,k}\chi^{\boldsymbol{A}}_{n,k}$ *and* $h_n = \sum_{k\in\mathbb{Z}^d} |s_{n,k}|\, \gamma_n(\,\cdot\, - \boldsymbol{A}_{2^{-n}}k)$, *then*

$$h_n \lesssim M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_n), \qquad n \in \mathbb{N}.$$

Proof. We may of course without loss of generality assume that $\boldsymbol{r} = (r, \dots, r)$ with $r \in (0, 1]$. Now the statement can be established as in [24, Lemma 1.2.7]. □

Lemma 4.21. *Let* $M \in \mathbb{N}$, $\lambda \in (0,\infty)$ *and* $\Phi \in C^M(\mathbb{R}^d; X)$ *be such that*

$$(1+\rho_{\boldsymbol{A}}(x))^{\lambda}\|D^{\beta}\Phi(x)\|_X \lesssim 1, \qquad x \in \mathbb{R}^d, |\beta| \leq M,$$

and let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ *be such that* $\Psi \perp \mathcal{P}^d_{M-1}$. *Set* $\Psi_t := t^{-\mathrm{tr}(\boldsymbol{A}^{\oplus})}\Psi(\boldsymbol{A}_{t^{-1}}\,\cdot\,)$ *for* $t \in (0,\infty)$. *Then, given* $\varepsilon \in (0, \lambda^{\boldsymbol{A}}_{\min})$,

$$\|\Phi * \Psi_t\,(x)\|_X \lesssim_{\varepsilon} \frac{t^{(\lambda^{\boldsymbol{A}}_{\min}-\varepsilon)M}}{(1+\rho_{\boldsymbol{A}}(x))^{\lambda}}, \qquad x \in \mathbb{R}^d, t \in (0,1].$$

Proof. As Ψ is a Schwartz function, there in particular exists $C \in (0,\infty)$ such that

$$|\Psi(x)| \leq C(1+\rho_{\boldsymbol{A}}(x))^{-\lambda}(1+|x|)^{-(d+M+1)}, \qquad x \in \mathbb{R}^d.$$

The desired inequality can now be obtained as in [24, Lemma 1.2.8]. □

Lemmas 4.22 and 4.23 are the corresponding versions of Lemmas 4.17 and 4.19, respectively, for $\widetilde{y}^{\boldsymbol{A}}(E;X)$ instead of $y^{\boldsymbol{A}}(E;X)$.

Lemma 4.22. *Let* $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$, $b \in (0,\infty)$ *and suppose that* $\varepsilon_+, \varepsilon_- > 0$. *Let* $\boldsymbol{p} \in (0,\infty]^{\ell}$ *satisfy* $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$. *Define the sublinear operator*

$$T^{\boldsymbol{A}}_{\boldsymbol{p}} : L_0(S)^{X^*\times\mathbb{N}\times\mathbb{Z}^d} \longrightarrow L_0(S;[0,\infty])^{X^*\times\mathbb{N}\times\mathbb{Z}^d}, \qquad (s_{x^*,n,k})_{(x^*,n,k)} \mapsto (t_{x^*,n,k})_{(x^*,n,k)},$$

by

$$t_{x^*,n,k} := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\Big\|\sum_{m,l}|s_{x^*,m,l}|\chi^{\boldsymbol{A}}_{m,l}\Big\|_{L_{\boldsymbol{p},\mathpzc{d}}}$$

and the sum is taken over all indices $(m,l) \in \mathbb{N}\times\mathbb{Z}^d$ *such that* $Q^{\boldsymbol{A}}_{m,l} \subset Q^{\boldsymbol{A}}_{n,k}(b)$ *and* $m \geq n$. *Then* $T^{\boldsymbol{A}}_{\boldsymbol{p}}$ *restricts to a bounded sublinear operator on* $\widetilde{y}^{\boldsymbol{A}}(E)$.

Proof. Let $\delta \in (0,\infty)$ be as in the proof of Lemma 4.17. Let $(s_{n,k})_{(x^*,n,k)} \in \widetilde{y}^{\boldsymbol{A}}(E)$ and $(t_{x^*,n,k})_{(n,k)} = T^{\boldsymbol{A}}_{\boldsymbol{p}}[(s_{x^*,n,k})_{(x^*,n,k)}] \in L_0(S;[0,\infty])^{X^*\times\mathbb{N}\times\mathbb{Z}^d}$. Define

$$g_{x^*,m} := \sum_{l\in\mathbb{Z}^d} s_{x^*,m,l}\chi^{\boldsymbol{A}}_{m,l} \in L_0(S), \qquad m \in \mathbb{N}.$$

Then $(g_{x^*,m})_{(x^*,m)} \in \mathscr{F}_{\mathrm{M}}(X^*;E)$ with $\|(g_{x^*,m})_{(x^*,m)}\|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} = \|(s_{x^*,n,k})_{(x^*,n,k)}\|_{\widetilde{y}^{\boldsymbol{A}}(E)}$. So there exists $(g_m)_m \in E_+$ with $\|(g_m)_m\| \leq 2\|(s_{x^*,n,k})_{(x^*,n,k)}\|_{\widetilde{y}^{\boldsymbol{A}}(E)}$ such that $|g_{x^*,m}| \leq \|x^*\|g_m$. By (32) from the proof of Lemma 4.17,

$$\begin{aligned} t_{x^*,n,k}\chi^{\boldsymbol{A}}_{n,k} &\lesssim_{b,\boldsymbol{A},\boldsymbol{r}} \sum_{m=n}^{\infty} 2^{(m-n)((\varepsilon_+-\delta))}M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_{x^*,m}) \\ &\leq \|x^*\|\sum_{m=n}^{\infty} 2^{(m-n)((\varepsilon_+-\delta))}M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_m). \end{aligned}$$

As (33) in proof of Lemma 4.17, we find that $(t_{x^*,n,k})_{(x^*,n,k)} \in \widetilde{y}^{\boldsymbol{A}}(E;X)$ with

$$\|(t_{x^*,n,k})_{(x^*,n,k)}\|_{\widetilde{y}^{\boldsymbol{A}}(E;X)} \lesssim \|(g_m)_m\| \leq 2\|(s_{x^*,n,k})_{(x^*,n,k)}\|_{\widetilde{y}^{\boldsymbol{A}}(E)}. \quad \square$$

Lemma 4.23. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$, $b \in (0,\infty)$ and $\lambda \in (\varepsilon_-, \infty)$. Define the sublinear operator*

$$T_\lambda : L_0(S)^{X^* \times \mathbb{N} \times \mathbb{Z}^d} \longrightarrow L_0(S;[0,\infty])^{X^* \times \mathbb{N} \times \mathbb{Z}^d}, \qquad (s_{x,^*,n,k})_{(x^*,n,k)} \mapsto (t_{x^*,n,k})_{(x^*,n,k)},$$

by

$$t_{x^*,n,k} := \sum_{m,l} 2^{\lambda(n-m)} |s_{x^*,m,l}|,$$

the sum being taken over all indices $(m,l) \in \mathbb{N} \times \mathbb{Z}^d$ such that $Q^{\boldsymbol{A}}_{m,l}(b) \supset Q^{\boldsymbol{A}}_{n,k}$ and $m < n$. Then T_λ restricts to a bounded sublinear operator on $\widetilde{y}^{\boldsymbol{A}}(E;X)$.

Proof. This can be proved in the same way as [24, Lemma 1.2.6]. $\square$

Lemma 4.24. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{1}, (S, \mathscr{A}, \mu))$ and let $k \in L_{1,\mathrm{c}}(\mathbb{R}^d)$ fulfill the Tauberian condition*

$$|\hat{k}(\xi)| > 0, \qquad \xi \in \mathbb{R}^d, \frac{\epsilon}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\epsilon,$$

for some $\epsilon \in (0,\infty)$. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\operatorname{supp}\hat{\psi} \subset \{\xi : \epsilon \leq \rho_{\boldsymbol{A}}(\xi) \leq B\}$ for some $B \in (\epsilon,\infty)$. Define $(k_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ by $k_n := 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})} k(\boldsymbol{A}_{2^n}\,\cdot)$ and $\psi_n := 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})}\psi(\boldsymbol{A}_{2^n}\,\cdot)$. Then

$$\|(\psi_n * f_n)_n\|_{E(X)} \lesssim \|(k_n * f_n)_n\|_{E(X)}, \qquad f \in L_0(S; L_{1,\mathrm{loc}}(\mathbb{R}^d;X)).$$

Proof. Pick $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\eta \subset B^{\boldsymbol{A}}(0,2\epsilon)$ and $\eta(\xi) = 1$ for $\rho_{\boldsymbol{A}}(\xi) \leq \frac{3\epsilon}{2}$. Define $m \in \mathcal{S}(\mathbb{R}^d)$ by $m(\xi) := [\eta(\xi) - \eta(\boldsymbol{A}_2\xi)]\hat{k}(\xi)^{-1}$ if $\frac{\epsilon}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\epsilon$ and $m(\xi) := 0$ otherwise; note that this gives a well-defined Schwartz function on $\mathbb{R}^d$ because $\eta - \eta(\boldsymbol{A}_2\,\cdot)$ is a smooth function supported in the set $\{\xi : \frac{\epsilon}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\epsilon\}$ on which the function $\hat{k} \in C^\infty_{L_\infty}(\mathbb{R}^d)$ does not vanish. Define $(m_n)_{n\in\mathbb{N}}$ by $m_n := m(\boldsymbol{A}_{2^{-n}}\,\cdot)$. Then, by construction,

$$\sum_{l=n}^{n+N} m_l \hat{k}_l(\xi) = \eta(\boldsymbol{A}_{2^{-(n+N)}}\xi) - \eta(\boldsymbol{A}_{2^{-n+1}}\xi) = 1$$

for $2^n\epsilon \leq \rho_{\boldsymbol{A}}(\xi) \leq 2^{n+N-1}3\epsilon$, $n \in \mathbb{N}$, $N \in \mathbb{N}$. Since $\operatorname{supp}\hat{\psi}_n \subset \{\xi : 2^n\epsilon \leq \rho_{\boldsymbol{A}}(\xi) < 2^n B\}$ for every $n \in \mathbb{N}$, there thus exists $N \in \mathbb{N}$ such that $\sum_{l=n}^{n+N} m_l\hat{k}_l \equiv 1$ on $\operatorname{supp}\hat{\varphi}_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we consequently have

$$\psi_n = \psi_n * \left(\sum_{l=n}^{n+N} \check{m}_l * k_l\right) = \sum_{l=n}^{n+N} \psi_n * \check{m}_l * k_l = \sum_{l=0}^{N} \psi_n * \check{m}_{n+l} * k_{n+l}.$$

As $\psi, m \in \mathcal{S}(\mathbb{R}^d)$, we obtain the pointwise estimate

$$\|\psi_n * f\|_X \leq \sum_{l=0}^{N} \|\psi_n * \check{m}_{n+l} * k_{n+l} * f\|_X \lesssim \sum_{l=0}^{N} M^{\boldsymbol{A}}(M^{\boldsymbol{A}}(\|k_{n+l} * f\|_X)).$$

It follows that

$$\begin{aligned}\|(\psi_n * f)_n\|_{E(X)} &\lesssim \sum_{l=0}^{N} \left\| \left(M^{A}(M^{A}(\|k_{n+l} * f\|_X))\right)_n \right\|_E \\ &\lesssim \sum_{l=0}^{N} \|(k_{n+l} * f)_n\|_{E(X)} \lesssim \sum_{l=0}^{N} 2^{-\varepsilon + l} \|(k_n * f)_n\|_{E(X)} \\ &\lesssim \|(k_n * f)_n\|_{E(X)}. \quad \square\end{aligned}$$

4.4. *Proofs of the results in Section 4.2*

Proof of Theorem 4.4. (i) $\Rightarrow$ (ii): Fix $\omega \in C_c^\infty((-1,2)^d)$ with the property that

$$\sum_{k \in \mathbb{Z}^d} \omega(x-k) = 1, \qquad x \in \mathbb{R}^d.$$

Let $(f_n)_n$ be as in Definition 3.11 with $\|(f_n)_n\|_{E(X)} \leq 2\|f\|_{Y L^{A}(E;X)}$. For each $(n,k) \in \mathbb{N} \times \mathbb{Z}^d$, we put

$$\tilde{a}_{n,k} := \omega(\boldsymbol{A}_{2^n}(\,\cdot\, - \boldsymbol{A}_{2^{-n}} k)) f_n, \qquad s_{n,k} := \|\tilde{a}_{n,k}(\boldsymbol{A}_{2^{-n}}\,\cdot\,)\|_{C_b^M(\mathbb{R}^d;X)},$$

and

$$a_{n,k} := \frac{\tilde{a}_{n,k}}{s_{n,k}} 1_{\{s_{n,k} \neq 0\}}.$$

Note that

$$\begin{aligned}|s_{n,k}| &= \|\tilde{a}_{n,k}(\boldsymbol{A}_{2^{-n}}\,\cdot\,)\|_{C_b^M(\mathbb{R}^d;X)} = \|\omega(\,\cdot\, - k) f_n(\boldsymbol{A}_{2^{-n}}\,\cdot\,)\|_{C_b^M(\mathbb{R}^d;X)} \\ &\lesssim \|\omega(\,\cdot\, - k)\|_{C_b^M(\mathbb{R}^d)} \|f_n(\boldsymbol{A}_{2^{-n}}\,\cdot\,)\|_{C_b^M([-1,2]^d + k;X)} \\ &\lesssim \sup_{|\alpha| \leq M} \sup_{y \in [-1,2]^d + k} \|D^\alpha [f_n(\boldsymbol{A}_{2^{-n}}\,\cdot\,)](y)\|_X\end{aligned}$$

Given $x \in Q^{\boldsymbol{A}}_{n,k}$ and $\tilde{x} = \boldsymbol{A}_{2^n} x \in [0,1)^d + k$, for $y \in [-1,2]^d + k$ we can write $y = \tilde{x} + z$ with

$$z = y - \tilde{x} = (y-k) - (\tilde{x} - k) \in [-1,2]^d - [0,1)^d, \quad \text{so, in particular, } \rho_{\boldsymbol{A}}(z) \leq C_d.$$

Combining the above and subsequently applying Lemma A.1 to $f_n(\boldsymbol{A}_{2^{-n}}\,\cdot\,)$, whose spectrum satisfies $\operatorname{supp} \mathcal{F}[f_n(\boldsymbol{A}_{2^{-n}}\,\cdot\,)] \subset B^{\boldsymbol{A}}(0,2)$, we find

$$\begin{aligned}|s_{n,k}| &\lesssim \sup_{|\alpha| \leq M} \sup_{\rho_{\boldsymbol{A}}(z) \leq C_d} \|D^\alpha [f_n(\boldsymbol{A}_{2^{-n}}\,\cdot\,)](\tilde{x} + z)\|_X \\ &\lesssim M^{\boldsymbol{A}}_{\boldsymbol{r}}[\|f_n(\boldsymbol{A}_{2^{-n}}\,\cdot\,)\|_X](\boldsymbol{A}_{2^n} x) = M^{\boldsymbol{A}}_{\boldsymbol{r}}(\|f_n\|_X)(x)\end{aligned}$$

for $x \in Q^{\boldsymbol{A}}_{n,k}$. Therefore, $(s_{n,k})_{(n,k)} \in y^{\boldsymbol{A}}(E)$ with

$$\|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)} \lesssim \left\|(M^{\boldsymbol{A}}_{\boldsymbol{r}}(\|f_n\|_X))_n\right\|_E \lesssim \|(f_n)_n\|_{E(X)} \leq 2\|f\|_{Y L^{\boldsymbol{A}}(E;X)}.$$

Finally, the convergence (25) follows from Corollary 4.18 and the observation that

$$f = \sum_{n=0}^\infty f_n = \sum_{n=0}^\infty \sum_{k \in \mathbb{Z}^d} s_{n,k} a_{n,k} \quad \text{in} \quad L_0(S; L_{\boldsymbol{r},d,\mathrm{loc}}(\mathbb{R}^d; X)).$$

(ii) $\Rightarrow$ (i): Set $g_n := \sum_{k\in\mathbb{Z}^d} |s_{n,k}|\chi^{A}_{n,k}$ for $n \in \mathbb{N}$. For $n \in \mathbb{Z}_{<0}$, set $f_n := 0$ and $g_n := 0$. Pick $\kappa \in (0,1]$ such that E has a κ-norm. Pick $\varepsilon \in (0, \lambda^{A}_{\min})$ such that $(\lambda^{A}_{\min} - \varepsilon)M > \varepsilon_-$. Pick $\lambda \in (0,\infty)$ such that $\mathrm{tr}(A^{\oplus})/\lambda < \boldsymbol{r}_{\min} \wedge 1$. Pick $\psi = (\psi_n)_{n\in\mathbb{N}} \in \Phi^{A}(\mathbb{R}^d)$ such that

$$\operatorname{supp}\hat{\psi}_0 \subset B^{A}(0,2), \qquad \operatorname{supp}\hat{\psi}_n \subset B^{A}(0,2^{n+1}) \setminus B^{A}(0,2^{n-1}), \quad n \geq 1,$$

and set $\Psi_n := 2^{n\mathrm{tr}(A^{\oplus})}\psi_0(A_{2^n}\,\cdot)$ for each $n \in \mathbb{N}$. Note that

$$a_{n,k} * \Psi_n = [b_{n,k} * \Psi](A_{2^n}\,\cdot\, - k)$$

and

$$a_{n,k} * \psi_m = [b_{n,k} * \psi_{m-n}](A_{2^n}\,\cdot\, - k), \qquad n < m.$$

An application of Lemma 4.21 thus yields that

$$\|a_{n,k} * \Psi_n(x)\|_X \lesssim \frac{1}{(1 + 2^n\rho_A(x - A_{2^{-n}}k))^{\lambda}} \tag{34}$$

and

$$\|a_{n,k} * \psi_m(x)\|_X \lesssim \frac{2^{-(m-n)(\lambda^{A}_{\min}-\varepsilon)M}}{(1 + 2^n\rho_A(x - A_{2^{-n}}k))^{\lambda}}, \qquad n < m. \tag{35}$$

Now put

$$\tilde{a}_{n,k,m} := \begin{cases} a_{n,k} * \Psi_n, & n = m, \\ a_{n,k} * \psi_m, & n < m. \end{cases}$$

Let $L_{\mathrm{M}}(\mathbb{R}^d;X)$ denote the Fréchet space of all equivalence classes of strongly measurable X-valued functions on $\mathbb{R}^d$ that are of polynomial growth; this space can for instance be described as

$$L_{\mathrm{M}}(\mathbb{R}^d;X) := \left\{ f \in L_0(\mathbb{R}^d;X) : \forall \phi \in \mathcal{S}(\mathbb{R}^d), \phi f \in L_\infty(\mathbb{R}^d;X) \right\}.$$

Using Lemma 4.11 together with the support condition of the $a_{n,k}$ and $\|a_{n,k}\|_{L_\infty(\mathbb{R}^d;X)} \leq 1$, it can be shown that the series $\sum_{k\in\mathbb{Z}^d} s_{n,k}a_{n,k}$ converges in $L_0(S; L_{\mathrm{M}}(\mathbb{R}^d;X))$. Since $L_{\mathrm{M}}(\mathbb{R}^d;X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d;X)$ and convolution gives rise to a separately continuous bilinear mapping $\mathcal{S} \times \mathcal{S}' \to \mathscr{O}_{\mathrm{M}}$, it follows that

$$f_{n,m} := \sum_{k\in\mathbb{Z}^d} s_{n,k}\tilde{a}_{n,k,m} = \Big(\sum_{k\in\mathbb{Z}^d} s_{n,k}a_{n,k}\Big) * \begin{cases} \Psi_n, & n = m, \\ \psi_m, & n < m, \end{cases} \quad \text{in} \quad L_0(S; \mathscr{O}_{\mathrm{M}}(\mathbb{R}^d;X)) \tag{36}$$

for each $n, m \in \mathbb{N}$ with $m \geq n$.

It will be convenient to define

$$f^{+}_{n,m} := \sum_{k\in\mathbb{Z}^d} |s_{n,k}|\,\|\tilde{a}_{n,k,m}\|_X, \qquad n, m \in \mathbb{N}, m \geq n.$$

By a combination of (34), (35) and Lemma 4.20,

$$f^{+}_{m-l,m} \lesssim 2^{-l(\lambda^{A}_{\min}-\varepsilon)M} M^{A}_{\boldsymbol{r}}(g_{m-l}), \qquad m, l \in \mathbb{N}, m \geq l.$$

From this it follows that

$$\begin{aligned} \|(f^{+}_{m-l,m})_{m\geq l}\|_{E(\mathbb{N}_{\geq l})} &\lesssim 2^{-l(\lambda^{A}_{\min}-\varepsilon)M}\|(M^{A}_{\boldsymbol{r}}(g_{m-l}))_{m\geq l}\|_{E(\mathbb{N}_{\geq l})} \\ &= 2^{-l(\lambda^{A}_{\min}-\varepsilon)M}\left\|(S_-)^l(M^{A}_{\boldsymbol{r}}(g_m))_{m\in\mathbb{N}}\right\|_E \end{aligned}$$

$$\begin{aligned}&\lesssim 2^{-l((\lambda^{\boldsymbol{A}}_{\min}-\varepsilon)M-\varepsilon_-)}\|(g_m)_{m\in\mathbb{N}}\|_E\\&= 2^{-l((\lambda^{\boldsymbol{A}}_{\min}-\varepsilon)M-\varepsilon_-)}\|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)}.\end{aligned}\tag{37}$$

Therefore, by Lemma 3.8 and the assumption $(\lambda^{\boldsymbol{A}}_{\min}-\varepsilon)M > \varepsilon_-$,

$$\sum_{l=0}^{\infty}\sum_{m=l}^{\infty} f^+_{m-l,m} = \sum_{l=0}^{\infty}\sum_{m=l}^{\infty}\sum_{k\in\mathbb{Z}^d} |s_{m-l,k}|\,\|\tilde{a}_{m-l,k,m}\|_X$$

belongs to $E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}] \hookrightarrow L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d))$. By Lebesgue domination this implies that $\sum_{l=0}^{\infty}\sum_{m=l}^{\infty}\sum_{k\in\mathbb{Z}^d} s_{m-l,k}\tilde{a}_{m-l,k,m}$ converges unconditionally in the space $L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X))$. In particular,

$$\sum_{l=0}^{\infty}\sum_{m=l}^{\infty}\sum_{k\in\mathbb{Z}^d} s_{m-l,k}\tilde{a}_{m-l,k,m} = \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}^d} s_{n,k}\sum_{m=n}^{\infty}\tilde{a}_{n,k,m} \quad \text{in} \quad L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)).$$

Since

$$a_{n,k} = \lim_{N\to\infty} \Psi_N * a_{n,k} = \lim_{N\to\infty}\sum_{m=n}^{N}\tilde{a}_{n,k,m} \quad \text{in} \quad L_0(S; L_1(\mathbb{R}^d; X)),$$

and since f has the representation (25), it follows that

$$f = \sum_{l=0}^{\infty}\sum_{m=l}^{\infty}\sum_{k\in\mathbb{Z}^d} s_{m-l,k}\tilde{a}_{m-l,k,m} \quad \text{in} \quad L_0(S; L_{\boldsymbol{r}\wedge\boldsymbol{p},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)).$$

Combining the latter with (36), we find

$$f = \sum_{l=0}^{\infty}\sum_{m=l}^{\infty} f_{m-l,m} \quad \text{in} \quad L_0(S; L_{\boldsymbol{r}\wedge\boldsymbol{p},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)).\tag{38}$$

Note that

$$\|(f_{m-l,m})_{m\geq l}\|_{E(\mathbb{N}_{\geq l};X)} \lesssim 2^{-l((\lambda^{\boldsymbol{A}}_{\min}-\varepsilon)M-\varepsilon_-)}\|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)}$$

by (37). Since

$$\operatorname{supp}\hat{f}_{m-l,m} \subset \begin{cases}\operatorname{supp}\hat{\Psi}_m, & l=0,\\ \operatorname{supp}\hat{\psi}_m, & l\geq 1,\end{cases} \subset \bar{B}^{\boldsymbol{A}}(0,2^{m+1}), \qquad m\geq l,$$

it follows that (see Remark 3.14)

$$F_l := \sum_{m=l}^{\infty} f_{m-l,m} \qquad \text{in} \quad L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)),$$

defines an element of $YL^{\boldsymbol{A}}(E;X)$ with

$$\|F_l\|_{YL^{\boldsymbol{A}}(E;X)} \lesssim 2^{-l((\lambda^{\boldsymbol{A}}_{\min}-\varepsilon)M-\varepsilon_-)}\|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)}.$$

As $(\lambda^{\boldsymbol{A}}_{\min}-\varepsilon)M > \varepsilon_-$, we find that $F := \sum_{l=0} F_l \in YL^{\boldsymbol{A}}(E;X)$ with

$$\|F\|_{YL^{\boldsymbol{A}}(E;X)} \lesssim \|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)}.$$

But $f = F$ in view of (38) and $YL^{\boldsymbol{A}}(E;X) \hookrightarrow L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X))$ (see Remark 3.14), yielding the desired result.

(ii) ⇒ (iii): We will write down the proof in such a way that the proof of Proposition 4.10 only requires a slight modification. Combining the estimate corresponding to (ii) ⇒ (i) with $YL^{\boldsymbol{A}}(E;X) \hookrightarrow E_0(X)$ (see (9)), we find

$$\|f\|_{E_0(X)} \lesssim \|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)}.$$

So let us focus on the remaining part of the required inequality. To this end, fix $c \in \mathbb{R}$ and choose $R \in [1,\infty)$ such that

$$\rho_{\boldsymbol{A}}(tz) \leq R\rho_{\boldsymbol{A}}(z), \qquad z \in \mathbb{R}^d, t \in [0,|c|+M].$$

Put

$$d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto L_{cz}\Delta^M_z f \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n});X)}, \qquad n \in \mathbb{N}.$$

Now let f has a representation as in (ii) and write $h_n := \sum_{k\in\mathbb{Z}^d} s_{n,k}a_{n,k}$. Then

$$\begin{aligned} d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,n}(f)(x) &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto \sum_{m=0}^{n-1} \|L_{cz}\Delta^M_z h_m(x)\|_X \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &\quad + 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto \sum_{m=n}^{\infty} \|L_{cz}\Delta^M_z h_m(x)\|_X \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))}. \end{aligned} \tag{39}$$

We use the identity

$$L_{cz}\Delta^M_z h_m(x) = \sum_{l=0}^{M}(-1)^{M-l}\binom{M}{l} h_j(x+(c+l)z)$$

to estimate the second term in (39) as follows

$$\begin{aligned} &2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto \sum_{m=n}^{\infty} \|L_{cz}\Delta^M_z h_m(x)\|_X \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &\lesssim \sum_{l=0}^{M} 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto \sum_{m=n}^{\infty} \|h_m(x+(c+l)z)\|_X \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| \sum_{m=n}^{\infty} \|h_m\|_X \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(x,R2^{-n}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| \sum_{m=n}^{\infty} \sum_{k\in\mathbb{Z}^d} \|s_{m,k}\|_X 1_{Q^{\boldsymbol{A}}_{m,k}(3)} \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(x,R2^{-n}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| \sum_{m,l} \|s_{m,l}\|_X 1_{Q^{\boldsymbol{A}}_{m,l}(3)} \right\|_{L_{\boldsymbol{p},\mathscr{d}}}, \end{aligned}$$

where the last sum is taken over all (m,l) such that $Q^{\boldsymbol{A}}_{m,l}(3)$ intersects $(B^{\boldsymbol{A}}(x,R2^{-n}))$ and $m \geq n$. From this it follows that

$$\begin{aligned} &2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto \sum_{m=n}^{\infty} \|L_{cz}\Delta^M_z h_m\|_X \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &\lesssim \sum_{k\in\mathbb{Z}^d} 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| \sum_{m,l} \|s_{m,l}\|_X 1_{Q^{\boldsymbol{A}}_{m,l}(3)} \right\|_{L_{\boldsymbol{p},\mathscr{d}}} 1_{Q^{\boldsymbol{A}}_{n,k}(3R)}, \end{aligned} \tag{40}$$

where the sum is taken over all (m,l) such that $Q^{\boldsymbol{A}}_{m,l}(3) \subset Q^{\boldsymbol{A}}_{n,k}(3R)$ and $m \geq n$.

In order to estimate the first term in (39), note that

$$\Delta_z^M h_m(x) = \int_{[0,1]^M} D^M h_m(x + (t_1 + \cdots + t_M)z)(z, \ldots, z)\, d(t_1, \ldots, t_M)$$

and thus that

$$\begin{aligned}\|\Delta_z^M h_m(x)\|_X &\leq \sup_{t \in [0,M]} \|D^M h_m(x + tz)(z, \ldots, z)\|_X \\ &= \sup_{t \in [0,M]} \left\| D^M [h_m \circ \boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m} x + t\boldsymbol{A}_{2^m} z)(\boldsymbol{A}_{2^m} z, \ldots, \boldsymbol{A}_{2^m} z) \right\|_X \\ &\lesssim \sup_{t \in [0,M]} \sup_{|\alpha| \leq M} \left\| D^\alpha [h_m \circ \boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m} x + t\boldsymbol{A}_{2^m} z) \right\|_X |\boldsymbol{A}_{2^m} z|^M,\end{aligned}$$

from which it follows that

$$\begin{aligned}\|L_{cz} \Delta_z^M h_m(x)\|_X &\lesssim \sup_{t \in [0,M]} \sup_{|\alpha| \leq M} \left\| D^\alpha [h_m \circ \boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m} x + (c+t)\boldsymbol{A}_{2^m} z) \right\|_X |\boldsymbol{A}_{2^m} z|^M \\ &\leq \sup_{y \in B^{\boldsymbol{A}}(0, R\rho_{\boldsymbol{A}}(z))} \sup_{|\alpha| \leq M} \left\| D^\alpha [h_m \circ \boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m} [x + y]) \right\|_X |\boldsymbol{A}_{2^m} z|^M.\end{aligned}$$

Given $\varepsilon \in (0, \lambda_{\min}^{\boldsymbol{A}})$, for $m \in \{0, \ldots, n-1\}$ and $z \in B^{\boldsymbol{A}}(0, 2^{-n})$ this gives

$$\begin{aligned}\|L_{cz} \Delta_z^M h_m(x)\|_X &\lesssim_\varepsilon \sup_{y \in B^{\boldsymbol{A}}(0, R2^{-n})} \sup_{|\alpha| \leq M} \left\| D^\alpha [h_m \circ \boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m} [x + y]) \right\|_X \rho_{\boldsymbol{A}}(\boldsymbol{A}_{2^m} z)^{(\lambda_{\min}^{\boldsymbol{A}} - \varepsilon)M} \\ &\lesssim \sup_{y \in B^{\boldsymbol{A}}(0, R2^{-n})} \sup_{|\alpha| \leq M} \left\| D^\alpha [h_m \circ \boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m} [x + y]) \right\|_X 2^{(\lambda_{\min}^{\boldsymbol{A}} - \varepsilon)M(m-n)}.\end{aligned}$$

Since

$$\begin{aligned}\left\| D^\alpha [h_m \circ \boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m} [x + y]) \right\|_X &\leq \sum_{l \in \mathbb{Z}^d} \|s_{m,l}\|_X 1_{[-1,2]^d + l}(\boldsymbol{A}_{2^m} [x + y]) \\ &\leq \sum_{l \in \mathbb{Z}^d} \|s_{m,l}\|_X 1_{Q_{m,l}^{\boldsymbol{A}}(3)}(x + y),\end{aligned}$$

it follows that

$$\begin{aligned}&2^{n \operatorname{tr}(\boldsymbol{A}) \cdot \boldsymbol{p}^{-1}} \left\| z \mapsto \sum_{m=0}^{n-1} \|L_{cz} \Delta_z^M h_m(x)\|_X \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &\quad\lesssim_\varepsilon \sum_{m=0}^{n-1} \sup_{z \in B^{\boldsymbol{A}}(0,2^{-n})} \|L_{cz} \Delta_z^M h_m(x)\|_X 2^{(\lambda_{\min}^{\boldsymbol{A}} - \varepsilon)M(m-n)} \\ &\quad\lesssim \sum_{m,l} 2^{(\lambda_{\min}^{\boldsymbol{A}} - \varepsilon)M(m-n)} \|s_{m,l}\|_X,\end{aligned}$$

where the last sum is taken over all (m, l) such that $Q_{m,l}^{\boldsymbol{A}}(3)$ intersects $B^{\boldsymbol{A}}(x, R2^{-n})$ and $m < n$. From this it follows that

$$2^{n \operatorname{tr}(\boldsymbol{A}) \cdot \boldsymbol{p}^{-1}} \left\| z \mapsto \sum_{m=0}^{n-1} \|L_{cz} \Delta_z^M h_m(x)\|_X \right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \lesssim \sum_{m,l} \|s_{m,l}\|_X, \tag{41}$$

where the last sum is taken over all (m, l) such that $Q_{m,l}^{\boldsymbol{A}}(3R) \supset Q_{n,k}^{\boldsymbol{A}}(3)$ and $m < n$.

A combination of (39), (40), Lemma 4.17, (41) and Lemma 4.19 give the desired result. □

Proof of Theorem 4.6. The chain of implications (I) $\Leftrightarrow$ (II) $\Rightarrow$ (III) with corresponding estimates for $f \in L_0(S; L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d; X))$ can be obtained in the same way as Theorem 4.4 with some natural modifications; in particular, Lemmas 4.17 and 4.19 need to be replaced with Lemmas 4.22 and 4.23, respectively. Furthermore, (II) $\Rightarrow$ (IV) can be done in the same way as [24, Theorem 1.1.14], similarly to the implication (II) $\Rightarrow$ (III) (see the proof of (ii) $\Rightarrow$ (iii) in Theorem 4.4).

Fix $q \in (0, \infty)$ with $q \leq \boldsymbol{r}_{\min} \wedge \boldsymbol{p}_{\min}(III)^*_q$ and let (IV)$^*_q$ be the statements (III) and (IV), respectively, in which $\boldsymbol{p}$ gets replaced by $\boldsymbol{q} := (q, \dots, q) \in (0, \infty)^\ell$. Then, clearly, (III) $\Rightarrow$ (III)$^*_q$ and (IV) $\Rightarrow$ (IV)$^*_q$.

To finish this proof, it suffices to establish the implication (V) $\Rightarrow$ (IV)$^*_q$ for $f \in L_0(S; L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d; X))$ and the implications (III)$^*_q$ $\Rightarrow$ (V) and (IV)$^*_q$ $\Rightarrow$ (II) for f of the form $f = \sum_{i \in I} 1_{S_i} \otimes f^{[i]}$ with $(S_i)_{i \in I} \subset \mathscr{A}$ a countable family of mutually disjoint sets and $(f^{[i]})_{i \in I} \in L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d; X)$.

(V) $\Rightarrow$ (IV)$^*_q$: For this implication we just observe that, for $x \in Q^{\boldsymbol{A}}_{n,k}$ and $n \geq 1$,

$$\mathcal{E}^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f)(x) \lesssim \overline{\mathcal{E}}_M(\langle f, x^* \rangle, Q^{\boldsymbol{A}}_{n,k}(3), L_q) \lesssim M^{\boldsymbol{A}}_{\boldsymbol{q}}(g_{x^*,n})(x) \leq M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_{x^*,n})(x).$$

(III)$^*_q$ $\Rightarrow$ (V) *for f of the form $f = \sum_{i \in I} 1_{S_i} \otimes f^{[i]}$ with $(S_i)_{i \in I} \subset \mathscr{A}$ a countable family of mutually disjoint sets and $(f^{[i]})_{i \in I} \in L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d; X)$:* By Lemma 4.12, for each $i \in I$ and $(x^*, n, k) \in X^* \times \mathbb{N}_{\geq 1} \times \mathbb{Z}^d$ there exists a $\pi^{[i]}_{x^*,n,k} \in \mathcal{P}^d_{M-1}$ such that

$$\left| \langle f^{[i]}, x^* \rangle - \pi^{[i]}_{x^*,n,k} \right| 1_{Q^{\boldsymbol{A}}_{n,k}(3)} \lesssim d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f^{[i]}) + \left(\fint_{Q^{\boldsymbol{A}}_{n,k}(6)} d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f^{[i]})(y)^q \, dy \right)^{1/q}.$$

Defining $\pi_{x^*,n,k} \in L_0(S; \mathcal{P}^d_{M-1})$ by $\pi_{x^*,n,k} := \sum_{i \in I} 1_{S_i} \otimes \pi^{[i]}_{x^*,n,k}$, we obtain

$$\left| \langle f, x^* \rangle - \pi_{x^*,n,k} \right| 1_{Q^{\boldsymbol{A}}_{n,k}(3)} \lesssim d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f) + M^{\boldsymbol{A}}_{\boldsymbol{q}}(d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f)) \leq 2M^{\boldsymbol{A}}_{\boldsymbol{r}}(d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f)).$$

Since

$$\#\left\{ k \in \mathbb{Z}^d : x \in Q^{\boldsymbol{A}}_{n,k}(3) \right\} \lesssim 1, \qquad x \in \mathbb{R}^d, n \in \mathbb{N},$$

it follows that

$$\begin{aligned} &\left\| \{g_{x^*,n}\}_{(x^*,n) \in X^* \times \mathbb{N}_{\geq 1}} \right\|_{\mathscr{F}_{\mathrm{M}}(X^*; E(\mathbb{N}_1))} \\ &\qquad \lesssim \left\| \{M^{\boldsymbol{A}}_{\boldsymbol{r}}[d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)]\}_{(x^*,n) \in X^* \times \mathbb{N}_{\geq 1}} \right\|_{\mathscr{F}_{\mathrm{M}}(X^*; E(\mathbb{N}_1))} \\ &\qquad \lesssim \left\| \{d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)\}_{(x^*,n) \in X^* \times \mathbb{N}_{\geq 1}} \right\|_{\mathscr{F}_{\mathrm{M}}(X^*; E(\mathbb{N}_1))}. \end{aligned}$$

(IV)$^*_q$ $\Rightarrow$ (II) *for f of the form $f = \sum_{i \in I} 1_{S_i} \otimes f^{[i]}$ with $(S_i)_{i \in I} \subset \mathscr{A}$ a countable family of mutually disjoint sets and $(f^{[i]})_{i \in I} \in L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d; X)$:* Let $\omega \in C^\infty_c([-1, 2]^d)$ be such that

$$\sum_{k \in \mathbb{Z}^d} \omega(x - k) = 1, \qquad x \in \mathbb{R}^d,$$

and put $\omega_{n,k} := \omega(\boldsymbol{A}_{2^n} \cdot -k)$ and $Q^\omega_{n,k} := \boldsymbol{A}_{2^{-n}}([-1, 2]^d + k)$ for $(n, k) \in \mathbb{N} \times \mathbb{Z}^d$; so $\mathrm{supp}\,(\omega_{n,k}) \subset Q^\omega_{n,k}$. Define

$$I_{n,k} := \left\{ l \in \mathbb{Z}^d : Q^\omega_{n,k} \cap Q^\omega_{n-1,l} \neq \varnothing \right\}, \qquad (n, k) \in \mathbb{N}_1 \times \mathbb{Z}^d.$$

Then $\#I_{n,k} \lesssim 1$ and there exists $b \in (1, \infty)$ such that

$$Q^\omega_{n,k} \subset Q^{\boldsymbol{A}}_{n,k}(b) \cap Q^{\boldsymbol{A}}_{n-1,l}(b), \qquad l \in I_{n,k}, (n, k) \in \mathbb{N}_1 \times \mathbb{Z}^d. \tag{42}$$

Furthermore, there exists $n_0 \in \mathbb{N}_1$ such that

$$Q^{\boldsymbol{A}}_{n,k}(b) \cup Q^{\boldsymbol{A}}_{n-1,l}(b) \subset B^{\boldsymbol{A}}(x, 2^{-(n-n_0)}), \qquad x \in Q^{\omega}_{n,k}, (n,k) \in \mathbb{N} \times \mathbb{Z}^d. \tag{43}$$

For each $i \in I$, let us pick $(\pi^{[i]}_{x^*,n,k})_{(x^*,n,k) \in X^* \times \mathbb{N} \times \mathbb{Z}^d} \subset \mathcal{P}^d_{M-1}$ with the property that

$$\| \langle f^{[i]}, x^* \rangle - \pi_{x^*,n,k} \|_{L_q(Q^{\boldsymbol{A}}_{n,k}(b))} \leq 2\mathcal{E}_M(\langle f^{[i]}, x^* \rangle, Q^{\boldsymbol{A}}_{n,k}(b), L_q) \tag{44}$$

and put $\pi_{x^*,n,k} := \sum_{i \in I} 1_{S_i} \otimes \pi^{[i]}_{x^*,n,k} \in L_0(S; \mathcal{P}^d_{M-1})$. Define

$$u_{x^*,n,k} := \begin{cases} \omega_{n,k} \sum_{l \in \mathbb{Z}^d} \omega_{n-1,l}[\pi_{x^*,n,k} - \pi_{x^*,n-1,l}], & n > n_0, \\ \omega_{n,k}\pi_{x^*,n,k}, & n = n_0, \\ 0, & n < n_0. \end{cases}$$

Let $x^* \in X^*$ and $(n,k) \in \mathbb{N}_{\geq n_0+1} \times \mathbb{Z}^d$. Let $l \in I_{n,k}$. For $x \in Q^{\omega}_{n,k}$ we can estimate

$$\begin{aligned} \| \pi_{x^*,n,k} - \pi_{x^*,n-1,l} \|_{L_q(Q^{\omega}_{n,k})} &\overset{(42)}{\lesssim} \| \langle f, x^* \rangle - \pi_{x^*,n,k} \|_{L_q(Q^{\boldsymbol{A}}_{n,k}(b))} \\ &\quad + \| \langle f, x^* \rangle - \pi_{x^*,n-1,l} \|_{L_q(Q^{\boldsymbol{A}}_{n-1,l}(b))} \\ &\overset{(43),(44)}{\leq} 4\mathcal{E}_M(\langle f, x^* \rangle, B^{\boldsymbol{A}}(x, 2^{-(n-n_0)}), L_q), \end{aligned}$$

implying

$$\begin{aligned} &\| (\pi_{x^*,n,k} - \pi_{x^*,n-1,l})(\boldsymbol{A}_{2^{-n}} \cdot + k) \|_{C^M_b([-1,2]^M)} \\ &\qquad \lesssim 2^{n \operatorname{tr}(\boldsymbol{A}^{\oplus})/q} \mathcal{E}_M(\langle f, x^* \rangle, B^{\boldsymbol{A}}(x, 2^{-(n-n_0)}), L_q) \end{aligned}$$

in view of Corollary 4.15. Since $\# I_{n,k} \lesssim 1$, it follows that

$$\begin{aligned} \| u_{x^*,n,k}(\boldsymbol{A}_{2^{-n}} \cdot + k) \|_{C^M_b([-1,2]^M)} &\lesssim \overline{\mathcal{E}}_M(\langle f, x^* \rangle, B^{\boldsymbol{A}}(x, 2^{-(n-n_0)}), L_q) \\ &= \mathcal{E}^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n-n_0}(f)(x), \qquad x \in Q^{\omega}_{n,k}. \end{aligned} \tag{45}$$

For $n = n_0$ we similarly have

$$\begin{aligned} \| u_{x^*,n_0,k}(\boldsymbol{A}_{2^{-n_0}} \cdot + k) \|_{C^M_b([-1,2]^M)} &\lesssim \| \langle f, x^* \rangle \|_{L_{\boldsymbol{q},d}(B^{\boldsymbol{A}}(x,1))} \\ &\lesssim \| x^* \| \, M^{\boldsymbol{A}}_{\boldsymbol{q}}(\| f \|_X)(x) \\ &\leq \| x^* \| \, M^{\boldsymbol{A}}_{\boldsymbol{r}}(\| f \|_X)(x), \qquad x \in Q^{\omega}_{n_0,k}. \end{aligned} \tag{46}$$

Define $s_{x^*,n,k} := \| u_{x^*,n,k}(\boldsymbol{A}_{2^{-n}} \cdot + k) \|_{C^M_b([-1,2]^M)}$,

$$a_{x^*,n,k} := \begin{cases} \frac{u_{x^*,n,k}}{s_{x^*,n,k}}, & s_{x^*,n,k} \neq 0, \\ 0, & s_{x^*,n,k} = 0, \end{cases}$$

and $b_{x^*,n,k} := u_{x^*,n,k}(\boldsymbol{A}_{2^{-n}} \cdot + k)$. Then $b_{x^*,n,k} \in C^M_c([-1,2]^d)$ with $\| b_{x^*,n,k} \|_{C^M_b} \leq 1$ and $(s_{x^*,n,k})_{(x^*,n,k)} \in \widetilde{\mathfrak{y}}^{\boldsymbol{A}}(E; X)$ with

$$\begin{aligned} \| (s_{x^*n,k})_{(x^*,n,k)} \|_{\widetilde{\mathfrak{y}}^{\boldsymbol{A}}(E;X)} &\overset{(45),(46)}{\lesssim} \| M^{\boldsymbol{A}}_{\boldsymbol{r}}(\| f \|_X) \|_{E_0} \\ &\quad + \left\| \{ \mathcal{E}^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n-n_0}(f) \}_{(x^*,n) \in X^* \times \mathbb{N}_{\geq n_0}} \right\|_{\mathscr{F}_{\mathrm{M}}(X^*; E(\mathbb{N}_{\geq n_0+1}))} \\ &\lesssim \| f \|_{E_0(X)} + 2^{\varepsilon - n_0} \left\| \{ \mathcal{E}^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f) \}_{(x^*,n)} \right\|_{\mathscr{F}_{\mathrm{M}}(X^*; E(\mathbb{N}_{\geq 1}))}. \end{aligned}$$

Note that, for $n \geq n_0 + 1$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} s_{x^*,n,k} a_{x^*,n,k} &= \sum_{k\in\mathbb{Z}^d} u_{x^*,n,k} \\ &= \sum_{k\in\mathbb{Z}^d} \pi_{x^*,n,k}\omega_{x^*,n,k} \sum_{l\in\mathbb{Z}^d} \omega_{n-1,l} - \sum_{k\in\mathbb{Z}^d} \omega_{n,k} \sum_{l\in\mathbb{Z}^d} \pi_{x^*,n-1,l}\omega_{n-1,l} \\ &= \sum_{k\in\mathbb{Z}^d} \pi_{x^*,n,k}\omega_{n,k} - \sum_{l\in\mathbb{Z}^d} \pi_{x^*,n-1,l}\omega_{n-1,l}.\end{aligned}$$

In combination with Lemma 4.16 and an alternating sum argument, this implies that

$$\langle f, x^* \rangle = \sum_{n=0}^{\infty} \sum_{k\in\mathbb{Z}^d} s_{x^*,n,k} a_{x^*,n,k} \qquad \text{in } L_0(S; L_{q,\mathrm{loc}}(\mathbb{R}^d)).$$

The required convergence finally follows from this with an argument as in (the last part of) the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 4.4. $\square$

Proof of Corollary 4.7. This is an immediate consequence of Theorems 3.22, 4.4, 4.6 and the observation that

$$\|(d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f))_{(x^*,n)}\|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} \leq \|(d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_{n\geq 1}\|_{E(\mathbb{N}_1)}. \quad \square$$

Proof of Theorem 4.8. The estimates

$$\|f\|_{Y^{\boldsymbol{A}}(E;X)} \eqsim \|f\|_{YL^{\boldsymbol{A}}(E;X)} \eqsim \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$$

follow from Theorem 3.22. Combining the inclusion

$$YL^{\boldsymbol{A}}(E;X) \overset{(9)}{\hookrightarrow} E_0(X)$$

with the estimate corresponding to the implication (i)$\Rightarrow$(iii) in Theorem 4.4 gives

$$\|f\|_{E_0(X)} + \|(d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_{n\geq 1}\|_{E(\mathbb{N}_1;X)} \lesssim \|f\|_{YL^{\boldsymbol{A}}(E;X)}.$$

As it clearly holds that

$$\|I^{\boldsymbol{A}}_{M,n}(f)\|_X \leq d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f), \qquad n \in \mathbb{N},$$

it remains to be shown that

$$\|f\|_{Y^{\boldsymbol{A}}(E;X)} \lesssim \|f\|_{E_0(X)} + \|(I^{\boldsymbol{A}}_{M,n}(f))_{n\geq 1}\|_{E(\mathbb{N}_1;X)}. \tag{47}$$

Put $K := 1_{B^{\boldsymbol{A}}}(0,1)$ and $K^{\Delta^M} := \sum_{l=0}^{M-1}(-1)^l\binom{M}{l}\tilde{K}_{[M-l]^{-1}}$, where $\tilde{K}_t := t^d K(-t\,\cdot)$ for $t \in (0,\infty)$. Furthermore, put

$$K^{\boldsymbol{A}}_M(t,f) := t^{-\mathrm{tr}(\boldsymbol{A}^{\oplus})} K^{\Delta^M}(\boldsymbol{A}_{t^{-1}}\,\cdot) * f + (-1)^M \hat{K}(0) f, \qquad t \in (0,\infty).$$

Note that

$$I^{\boldsymbol{A}}_{M,n}(f) = K^{\boldsymbol{A}}_M(2^{-n}, f), \qquad n \in \mathbb{N}. \tag{48}$$

As $\widehat{K^{\Delta^M}}(0) = \sum_{l=0}^{M-1}(-1)^l\binom{M}{l}\hat{K}(0) = (-1)^{M+1}\hat{K}(0) \neq 0$, we can pick $\epsilon, c \in (0,\infty)$ such that K^{Δ^m} fulfills the Tauberian condition

$$|\mathscr{F} K^{\Delta^m}(\xi)| \geq c, \qquad \xi \in \mathbb{R}^d, \frac{\epsilon}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\epsilon.$$

So there exists $N \in \mathbb{N}$ such that $k := 2^{N\mathrm{tr}(A^{\oplus})}K^{\Delta^m}(\boldsymbol{A}_{2^N}\,\cdot) - K^{\Delta^m} \in L_{1,\mathrm{c}}(\mathbb{R}^d)$ satisfies

$$|\hat{k}(\xi)| \geq \frac{c}{2} > 0, \qquad \xi \in \mathbb{R}^d, \frac{\delta}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\delta,$$

for $\delta := 2^N\epsilon > 0$. Let $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ be such that $\operatorname{supp}\hat{\varphi}_1 \subset \{\xi : 2\epsilon \leq \rho_{\boldsymbol{A}}(\xi)\}$ (see Definition 3.18). Let $(k_n)_{n\in\mathbb{N}}$ be defined by $k_n := 2^{n\mathrm{tr}(A^{\oplus})}k(\boldsymbol{A}_{2^n}\,\cdot)$. Then, by construction,

$$k_n * f = K_M^{\boldsymbol{A}}(2^{-(n+N)}, f) - K_M^{\boldsymbol{A}}(2^{-n}, f) \overset{(48)}{=} I_{M,n+N}^{\boldsymbol{A}}(f) - I_{M,n}^{\boldsymbol{A}}(f), \qquad n \in \mathbb{N}.$$

An application of Lemma 4.24 thus yields that

$$\begin{aligned} \|(\varphi_n * f)_{n\geq 1}\|_{E(\mathbb{N}_1;X)} &\lesssim \|(k_n * f)_{n\geq 1}\|_{E(\mathbb{N}_1;X)} \\ &\lesssim \|(I_{M,n+N}^{\boldsymbol{A}}(f))_{n\geq 1}\|_{E(\mathbb{N}_1;X)} + \|(I_{M,n}^{\boldsymbol{A}}(f))_{n\geq 1}\|_{E(\mathbb{N}_1;X)} \\ &\lesssim (2^{-\varepsilon_+ N} + 1)\|(I_{M,n}^{\boldsymbol{A}}(f))_{n\geq 1}\|_{E(\mathbb{N}_1;X)}. \end{aligned} \tag{49}$$

As $\|\varphi_0 * f\|_X \lesssim M^{\boldsymbol{A}}(\|f\|_X)$, it furthermore holds that

$$\|\varphi_0 * f\|_{E_0(X)} \lesssim \|f\|_{E_0(X)}. \tag{50}$$

A combination of Proposition 3.19, (49) and (50) finally gives (47). □

Proof of Proposition 4.10. Using the estimate corresponding to the implication (i) ⇒ (ii) in Theorem 4.4, the first estimate can be obtained as in the proof of the implication (ii) ⇒ (iii) in Theorem 4.4. The second estimate can be obtained similarly, replacing Theorem 4.4 by Theorem 4.6. □

5. An intersection representation

In this section we come to the main results of this paper, namely, intersection representations. In particular, these include Theorem 1.1 from the introduction of this paper as a special case. Before we can state the results, we need to introduce some notation.

Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ with $\varepsilon_+, \varepsilon_- > 0$. Let J be a nonempty subset of $\{1, \ldots, \ell\}$, say $J = \{j_1, \ldots, j_k\}$ with $1 \leq j_1 \leq \cdots \leq j_k \leq \ell$. Put $\mathpzc{d}_J = (\mathpzc{d}_{j_1}, \ldots, \mathpzc{d}_{j_k})$, $d_J := |\mathpzc{d}_J|_1$ $\boldsymbol{A}_J := (A_{j_1}, \ldots, A_{j_k})$, $\boldsymbol{r}_J := (r_{j_1}, \ldots, r_{j_k})$ and

$$(S_J, \mathscr{A}_J, \mu_J) := (\mathbb{R}^{d-d_J}, \mathcal{B}(\mathbb{R}^{d-d_J}), \lambda^{d-d_J}) \otimes (S, \mathscr{A}, \mu)$$

Furthermore, define $E_{[\mathpzc{d};J]}$ as the quasi-Banach space E viewed as quasi-Banach function space on the measure space $\mathbb{R}^{d_J} \times \mathbb{N} \times S_J$. Then

$$E_{[\mathpzc{d};J]} \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}_J, \boldsymbol{r}_J, (S_J, \mathscr{A}_J, \mu_J))$$

By Remark 3.14,

$$\widetilde{YL}^{\boldsymbol{A}}(E; X) \hookrightarrow E_{\otimes}^{\boldsymbol{A}}(B_{\boldsymbol{A}}^{1,w_{\boldsymbol{A},\boldsymbol{r}}}(X)) \hookrightarrow L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)).$$

In the same way,

$$\widetilde{YL}^{\boldsymbol{A}_J}(E_{[\mathpzc{d};J]}; X) \hookrightarrow E_{\otimes}^{\boldsymbol{A}}(B_{\boldsymbol{A}}^{1,w_{\boldsymbol{A},\boldsymbol{r}}}(X)) \hookrightarrow L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)),$$

In particular, it makes sense to compare $\widetilde{YL}^{\boldsymbol{A}_J}(E_{[\mathpzc{d};J]}; X)$ with $\widetilde{YL}^{\boldsymbol{A}}(E; X)$.

Theorem 5.1. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ with $\varepsilon_+, \varepsilon_- > 0$. Let $\{J_1, \ldots, J_L\}$ be a partition of $\{1, \ldots, \ell\}$.*

(i) There is the estimate

$$\|f\|_{\widetilde{YL}^{A_{J_l}}(E_{[d;J_l]};X)} \leq \|f\|_{\widetilde{YL}^{A}(E;X)}, \qquad l \in \{1,\dots,L\},$$

for all $f \in L_0(S; L_{r,d,\mathrm{loc}}(\mathbb{R}^d; X))$.

(ii) There is the estimate

$$\|f\|_{\widetilde{YL}^{A}(E;X)} \lesssim \sum_{l=1}^{L} \|f\|_{\widetilde{YL}^{A_{J_l}}(E_{[d;J_l]};X)}$$

for all $f \in L_0(S; L_{r,d,\mathrm{loc}}(\mathbb{R}^d; X))$ *of the form* $f = \sum_{i\in I} 1_{S_i} \otimes f^{[i]}$ *with* $(S_i)_{i\in I} \subset \mathscr{A}$ *a countable family of mutually disjoint sets and* $(f^{[i]})_{i\in I} \in L_{r,d,\mathrm{loc}}(\mathbb{R}^d; X)$.

In particular, in case $(S, \mathscr{A}, \mu)$ *is atomic,*

$$\widetilde{YL}^{A}(E;X) = \bigcap_{l=1}^{L} \widetilde{YL}^{A_{J_l}}(E_{[d;J_l]};X)$$

with an equivalence of quasi-norms.

Remark 5.2. The analogous estimate in Theorem 5.1(i) for $YL^{A}(E;X)$ holds as well, with a slightly modified proof that actually is a little bit easier. However, we are not able to obtain a version of Theorem 5.1(ii) for $YL^{A}(E;X)$ due to the unavailability of the crucial implication (iii) ⇒ (i) (plus a corresponding estimate of the involved quasi-norm) in Theorem 4.4, see Remark 4.5.

Proof of Theorem 5.1. Let us start with (i). Fix $l \in \{1,\dots,L\}$ and write $J := J_l$. Let $f \in \widetilde{YL}^{A}(E;X)$. Let $\epsilon > 0$. Choose $(g_n)_n$ and $(f_{x^*,n})_{(x^*,n)}$ as in Definition 3.12 with $\|(g_n)_n\|_E \leq (1+\epsilon)\|f\|_{\widetilde{YL}^{A}(E;X)}$. As $f_{x^*,n} \in L_0(S; \mathcal{S}'(\mathbb{R}^d))$ with $\operatorname{supp} \hat{f}_{x^*,n} \subset B^{A}(0, 2^{n+1})$, we can naturally view $f_{x^*,n}$ as an element of $L_0(S_J; \mathcal{S}'(\mathbb{R}^{d-d_J}))$ with $\operatorname{supp} \hat{f}_{x^*,n} \subset B^{A_J}(0, 2^{n+1})$. Since

$$L_0(S; L_{r,d,\mathrm{loc}}(\mathbb{R}^d)) \hookrightarrow L_0(S_J; L_{r_J,d_J,\mathrm{loc}}(\mathbb{R}^{d_J})),$$

it follows that $f \in \widetilde{YL}^{A_J}(E_{[d;J]};X)$ with

$$\|f\|_{\widetilde{YL}^{A_J}(E_{[d;J]};X)} \lesssim \|(g_n)_n\|_{E_{[d;J]}} = \|(g_n)_n\|_E \leq (1+\epsilon)\|f\|_{\widetilde{YL}^{A}(E;X)}.$$

Let us next treat (ii). We may without loss of generality assume that $L = \ell$ and that $J_l = \{l\}$ for each $l \in \{1,\dots,\ell\}$. We will write $E_{[d;j]} = E_{[d;\{j\}]}$.

Let $f \in \bigcap_{j=1}^{\ell} \widetilde{YL}^{A_j}(E_{[d;j]};X)$ be of the form $f = \sum_{i\in I} 1_{S_i} \otimes f^{[i]}$ with $(S_i)_{i\in I} \subset \mathscr{A}$ a countable family of mutually disjoint sets and $(f^{[i]})_{i\in I} \in L_{r,d,\mathrm{loc}}(\mathbb{R}^d; X)$. In order to establish the desired inequality, we will combine the estimate corresponding to the implication (III) ⇒ (I) from Theorem 4.6 for the space $\widetilde{YL}^{A}(E;X)$ with the estimates from Proposition 4.10 for each of the spaces $\widetilde{YL}^{A_j}(E_{[d;j]};X)$. To this end, pick $M \in \mathbb{N}$ with $M\lambda^{A}_{\min} > \varepsilon_-$. Now, let us define $(g_{x^*,n})_{(x^*,n)\in X^*\times\mathbb{N}}$ and $(g_{c,x^*,n,j})_{(x^*,n)\in X^*\times\mathbb{N}}$, with $j \in \{1,\dots,\ell\}$ and $c \in \mathbb{R}$, by

$$g_{x^*,n} := \begin{cases} d^{A,r}_{0,x^*,0}(f), & n = 0, \\ d^{A,r}_{\ell M,x^*,n}(f), & n \geq 1, \end{cases}$$

and

$$g_{c,x^*,n,j} := \begin{cases} d^{[d;j],A_j,r_j}_{0,x^*,0}(f), & n=0, \\ d^{[d;j],A_j,r_j}_{M,c,x^*,n}(f), & n \geq 1, \end{cases}$$

where the notation is as in Theorem 4.6 and Proposition 4.10.

For $n=0$ we have

$$\begin{aligned} g_{x^*,0} = d^{\boldsymbol{A},\boldsymbol{r}}_{0,x^*,0}(f) &\lesssim \left[\bigcirc_{i=2}^{\ell} M^{[d;i],A_i}_{r_i}\right]\left(d^{[d;1],A_1,r_1}_{0,x^*,0}(f)\right) \\ &\leq M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[d^{[d;1],A_1,r_1}_{0,x^*,0}(f)\right] = M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[g_{c,x^*,0,1}\right], \qquad c \in \mathbb{R}, \end{aligned} \tag{51}$$

where $\bigcirc_{i=2}^{\ell} M^{[d;i],A_i}_{r_i}$ stands for the composition $M^{[d;\ell],A_\ell}_{r_\ell} \circ \ldots \circ M^{[d;2],A_2}_{r_2}$.

Now let $n \geq 1$. We will use the following elementary fact (cf. [57, 4.16]): there exist $C \in (0,\infty)$, $K \in \mathbb{N}$ and $\{c^{[k]}_j\}_{j=1,\ldots,\ell;k=0,\ldots,K} \subset \mathbb{R}$ such that

$$|\Delta^{\ell M}_z h(x)| \leq C \sum_{k=0}^{K}\sum_{j=1}^{\ell} \left|\Delta^M_{\iota_{[d;j]}z_j} h(x + \sum_{i=1}^{\ell} c^{[k]}_i \iota_{[d;i]}z_i)\right|$$

for all $h \in L_0(\mathbb{R}^d)$. Applying this pointwise in S to $\langle f, x^*\rangle$, we find that

$$\begin{aligned} g_{x^*,n} &= d^{\boldsymbol{A},\boldsymbol{r}}_{\ell M,x^*,n}(f) = 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}}\left\| z \mapsto \Delta^{\ell M}_z \langle f, x^*\rangle \right\|_{L_{\boldsymbol{r},d}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &\lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell} 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}}\left\| z \mapsto \left[\prod_{i=1}^{\ell} L_{c^{[k]}_i \iota_{[d;i]}z_i}\right]\Delta^M_{\iota_{[d;j]}z_j}\langle f, x^*\rangle\right\|_{L_{\boldsymbol{r},d}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &\lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell} 2^{n\mathrm{tr}(A_j)/r_j}\left[\bigcirc_{i\neq j} M^{[d;i],A_i}_{r_i}\right]\left[\left\| z_j \mapsto L_{c^{[k]}_j \iota_{[d;j]}z_j}\Delta^M_{\iota_{[d;j]}z_j}\langle f, x^*\rangle\right\|_{L_{r_j}(B^{A_j}(0,2^{-n}))}\right] \\ &\leq \sum_{k=0}^{K}\sum_{j=1}^{\ell} M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[2^{n\mathrm{tr}(A_j)/r_j}\left\| z_j \mapsto L_{c^{[k]}_j \iota_{[d;j]}z_j}\Delta^M_{\iota_{[d;j]}z_j}\langle f, x^*\rangle\right\|_{L_{r_j}(B^{A_j}(0,2^{-n}))}\right] \\ &= \sum_{k=0}^{K}\sum_{j=1}^{\ell} M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[d^{[d;j],A_j,r_j}_{M,c^{[k]}_j,x^*,n}(f)\right] = \sum_{k=0}^{K}\sum_{j=1}^{\ell} M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[g_{c^{[k]}_j,x^*,n,j}\right]. \end{aligned} \tag{52}$$

A combination of (51) and (52) gives

$$g_{x^*,n} \lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell} M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[d^{[d;j],A_j,r_j}_{M,c^{[k]}_j,x^*,n}(f)\right] = \sum_{k=0}^{K}\sum_{j=1}^{\ell} M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[g_{c^{[k]}_j,x^*,n,j}\right]$$

for all $(x^*,n) \in X^* \times \mathbb{N}$. Therefore,

$$\begin{aligned} \left\|\{g_{x^*,n}\}_{(x^*,n)}\right\|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} &\lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell} \left\|\left\{M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[g_{c^{[k]}_j,x^*,n,j}\right]\right\}_{(x^*,n)}\right\|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} \\ &\lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell} \left\|\{g_{c^{[k]}_j,x^*,n,j}\}_{(x^*,n)}\right\|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} \\ &= \sum_{k=0}^{K}\sum_{j=1}^{\ell} \left\|\{g_{c^{[k]}_j,x^*,n,j}\}_{(x^*,n)}\right\|_{\mathscr{F}_{\mathrm{M}}(X^*;E_{[d;j]})}. \end{aligned}$$

The desired result now follows from a combination of Theorem 4.6 and Proposition 4.10. □

As an immediate corollary to Theorems 3.22 and 5.1 we have:

Corollary 5.3. *Let* $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ *with* $\varepsilon_+, \varepsilon_- > 0$ *and* $(S, \mathscr{A}, \mu)$ *atomic. Let* $\{J_1, \dots, J_L\}$ *be a partition of* $\{1, \dots, \ell\}$. *If* $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{1})_+$, *where* $\mathrm{tr}(\boldsymbol{A}) = (\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_\ell))$, *then*

$$Y^{\boldsymbol{A}}(E; X) = YL^{\boldsymbol{A}}(E; X) = \widetilde{YL}^{\boldsymbol{A}}(E; X) = \bigcap_{l=1}^{L} \widetilde{YL}^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]}; X)$$
$$= \bigcap_{l=1}^{L} YL^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]}; X) = \bigcap_{l=1}^{L} Y^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]}; X)$$

with an equivalence of quasi-norms.

In the case that $\boldsymbol{r} = 1$, the above intersection representation simplifies a bit thanks to the corresponding simplification in the crucial estimate involving differences, also see Remark ??. In particular, we can drop the assumption of $(S, \mathscr{A}, \mu)$ being atomic.

Theorem 5.4. *Let* $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{1}, (S, \mathscr{A}, \mu))$ *with* $\varepsilon_+, \varepsilon_- > 0$. *Let* $\{J_1, \dots, J_L\}$ *be a partition of* $\{1, \dots, \ell\}$. *Then*

$$Y^{\boldsymbol{A}}(E; X) = YL^{\boldsymbol{A}}(E; X) = \widetilde{YL}^{\boldsymbol{A}}(E; X) = \bigcap_{l=1}^{L} \widetilde{YL}^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]}; X)$$
$$= \bigcap_{l=1}^{L} YL^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]}; X) = \bigcap_{l=1}^{L} Y^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]}; X)$$

with an equivalence of quasi-norms.

Proof. In view of Theorem 3.22, this can be proved in exactly the same way as Theorem 5.1, using Theorem 4.8 instead of Theorem 4.6. □

Remark 5.5. In light of Example 3.20, the intersection representation

$$Y^{\boldsymbol{A}}(E; X) = \bigcap_{l=1}^{L} Y^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]}; X) \tag{53}$$

from Corollary 5.3 and Theorem 5.4 extends the well-known Fubini property for the classical Lizorkin–Triebel spaces $F^s_{p,q}(\mathbb{R}^d)$ (see [57, Section 4] and the references given therein). It also covers Theorem 1.1 and thereby (1), the intersection representation from [16, Proposition 3.23]. The intersection representation [33, Proposition 5.2.38] for anisotropic weighted mixed-norm Lizorkin–Triebel is a special case as well. Furthermore, it suggests an operator sum theorem for generalized Lizorkin–Triebel spaces in the sense of [32].

Example 5.6. Let us state the intersection representation (53) from Corollary 5.3 and Theorem 5.4 for some concrete choices of E (see Examples 3.5 and 3.20) for the case that $\ell = 2$ with partition $\{\{1\}, \{2\}\}$ of $\{1, 2\}$.

(I) Let $\boldsymbol{p} \in (0, \infty)^2$, $q \in (0, \infty]$, $\boldsymbol{w} \in A_\infty(\mathbb{R}^{d_1}, A_1) \times A_\infty(\mathbb{R}^{d_2}, A_2)$ and $s \in \mathbb{R}$. Pick $\boldsymbol{r} \in (0, \infty)^2$ such that $r_1 < p_1 \wedge q$, $r_2 < p_1 \wedge p_2 \wedge q$ and $\boldsymbol{w} \in A_{p_1/r_1}(\mathbb{R}^{d_1}, A_1) \times A_{p_2/r_2}(\mathbb{R}^{d_2}, A_2)$.

If $s > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \mathbf{1})_+$, then

$$F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X) = \mathbb{F}^{s,A_2}_{p_2,q}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1); X) \cap L_{p_2}(\mathbb{R}^{d_2}, w_2; F^{s,A_1}_{p_1,q}(\mathbb{R}^{d_1}, w_1; X)).$$

(II) Let $\boldsymbol{p} \in (0,\infty)^2$, $q \in (0,\infty]$, $\boldsymbol{w} \in A_\infty(\mathbb{R}^{d_1}, A_1) \times A_\infty(\mathbb{R}^{d_2}, A_2)$ and $s \in \mathbb{R}$. Pick $\boldsymbol{r} \in (0,\infty)^2$ such that $r_1 < p_1$, $r_2 < p_1 \wedge p_2 \wedge q$ and $\boldsymbol{w} \in A_{p_1/r_1}(\mathbb{R}^{d_1}, A_1) \times A_{p_2/r_2}(\mathbb{R}^{d_2}, A_2)$. If $s > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \mathbf{1})_+$, then

$$Y^{\boldsymbol{A}}\left(L_{p_2}(\mathbb{R}^{d_2}, w_2)[[\ell^s_q(\mathbb{N})]L_{p_1}(\mathbb{R}^{d_1}, w_1)]; X\right) = F^{s,A_2}_{p_2,q}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1; X)) \cap L_{p_2}(\mathbb{R}^{d_2}, w_2; B^{s,A_1}_{p_1,q}(\mathbb{R}^{d_1}, w_1; X)).$$

To finish this section, let us finally state the Fubini property variants of the two examples from Example 5.6 (cf. Remark 5.5).

Example 5.7. Taking $\boldsymbol{p} = (p,q)$ in (I) and (II) of Example 5.6, an application of Fubini's theorem yields the following.

(I) Let $p, q \in (0,\infty)$, $\boldsymbol{w} \in A_\infty(\mathbb{R}^{d_1}, A_1) \times A_\infty(\mathbb{R}^{d_2}, A_2)$ and $s \in \mathbb{R}$. Pick $\boldsymbol{r} \in (0,\infty)^2$ such that $r_1, r_2 < p \wedge q$ and $\boldsymbol{w} \in A_{p/r_1}(\mathbb{R}^{d_1}, A_1) \times A_{q/r_2}(\mathbb{R}^{d_2}, A_2)$. If $s > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \mathbf{1})_+$, then

$$F^{s,\boldsymbol{A}}_{(p,q),p}(\mathbb{R}^d, \boldsymbol{w}; X) = F^{s,A_2}_{q,p}(\mathbb{R}^{d_2}, w_2; L_p(\mathbb{R}^{d_1}, w_1; X)) \cap L_q(\mathbb{R}^{d_2}, w_2; B^{s,A_1}_{p,p}(\mathbb{R}^{d_1}, w_1; X)).$$

(II) Let $p \in (0,\infty)$, $q \in (0,\infty]$, $\boldsymbol{w} \in A_\infty(\mathbb{R}^{d_1}, A_1) \times A_\infty(\mathbb{R}^{d_2}, A_2)$ and $s \in \mathbb{R}$. Pick $\boldsymbol{r} \in (0,\infty)^2$ such that $r_1 < p$, $r_2 < p \wedge q$ and $\boldsymbol{w} \in A_{p/r_1}(\mathbb{R}^{d_1}, A_1) \times A_{q/r_2}(\mathbb{R}^{d_2}, A_2)$. If $s > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \mathbf{1})_+$, then

$$B^{s,\boldsymbol{A}}_{(p,q),q}(\mathbb{R}^d, \boldsymbol{w}; X) = B^{s,A_2}_{q,q}(\mathbb{R}^{d_2}, w_2; L_p(\mathbb{R}^{d_1}, w_1; X)) \cap L_q(\mathbb{R}^{d_2}, w_2; B^{s,A_1}_{p,q}(\mathbb{R}^{d_1}, w_1; X)).$$

In applications to parabolic partial differential equations, one uses anisotropies of the form $\boldsymbol{A} = (a_1 I_{d_1}, a_2 I_{d_2})$ with $a_1 = 2m$, $a_2 = 1$, $d_1 \in \{n-1, n\}$ and $d_2 = 1$, where $2m$ is the order of the elliptic operator under consideration and n is the dimension of the spatial domain (see e.g. [35,36]). So let us for convenience of reference state Examples 5.6 and 5.7 for such anisotropies.

In view of Example 3.29 and the fact that $A_p(\mathbb{R}^n, \lambda A) = A_p(\mathbb{R}^n, A)$ for every $\lambda \in (0,\infty)$, the following two examples are obtained as special cases of Examples 5.6 and 5.7.

Example 5.8. Let $\boldsymbol{d} \in (\mathbb{N}_1)^2$ and $\boldsymbol{a} \in (0,\infty)^2$.

(I) Let $\boldsymbol{p} \in (0,\infty)^2$, $q \in (0,\infty]$, $\boldsymbol{w} \in A_\infty(\mathbb{R}^{d_1}) \times A_\infty(\mathbb{R}^{d_2})$ and $s \in \mathbb{R}$. Pick $\boldsymbol{r} \in (0,\infty)^2$ such that $r_1 < p_1 \wedge q$, $r_2 < p_1 \wedge p_2 \wedge q$ and $\boldsymbol{w} \in A_{p_1/r_1}(\mathbb{R}^{d_1}) \times A_{p_2/r_2}(\mathbb{R}^{d_2})$. If $s > a_1 d_1 (r_1^{-1} - 1)_+ + a_2 d_2 (r_2^{-1} - 1)_+$, then

$$F^{s,(\boldsymbol{a};\boldsymbol{d})}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X) = \mathbb{F}^{s/a_2}_{p_2,q}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1); X) \cap L_{p_2}(\mathbb{R}^{d_2}, w_2; F^{s/a_1}_{p_1,q}(\mathbb{R}^{d_1}, w_1; X)).$$

(II) Let $\boldsymbol{p} \in (0,\infty)^2$, $q \in (0,\infty]$, $\boldsymbol{w} \in A_\infty(\mathbb{R}^{d_1}) \times A_\infty(\mathbb{R}^{d_2})$ and $s \in \mathbb{R}$. Pick $\boldsymbol{r} \in (0,\infty)^2$ such that $r_1 < p_1$, $r_2 < p_1 \wedge p_2 \wedge q$ and $\boldsymbol{w} \in A_{p_1/r_1}(\mathbb{R}^{d_1}) \times A_{p_2/r_2}(\mathbb{R}^{d_2})$. If

$s > a_1 d_1(r_1^{-1} - 1)_+ + a_2 d_2(r_2^{-1} - 1)_+$, then

$$Y^{(\boldsymbol{a};\mathpzc{d})}\left(L_{p_2}(\mathbb{R}^{d_2}, w_2)[[\ell^s_q(\mathbb{N})]L_{p_1}(\mathbb{R}^{d_1}, w_1)]; X\right) = F^{s/a_2}_{p_2,q}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1; X)) \cap L_{p_2}(\mathbb{R}^{d_2}, w_2; B^{s/a_1}_{p_1,q}(\mathbb{R}^{d_1}, w_1; X)).$$

Example 5.9. Let $\mathpzc{d} \in (\mathbb{N}_1)^2$ and $\boldsymbol{a} \in (0, \infty)^2$.

(I) Let $p, q \in (0, \infty)$, $\boldsymbol{w} \in A_\infty(\mathbb{R}^{d_1}) \times A_\infty(\mathbb{R}^{d_2})$ and $s \in \mathbb{R}$. Pick $\boldsymbol{r} \in (0, \infty)^2$ such that $r_1, r_2 < p \wedge q$ and $\boldsymbol{w} \in A_{p/r_1}(\mathbb{R}^{d_1}) \times A_{q/r_2}(\mathbb{R}^{d_2})$. If $s > a_1 d_1(r_1^{-1} - 1)_+ + a_2 d_2(r_2^{-1} - 1)_+$, then

$$F^{s,(\boldsymbol{a};\mathpzc{d})}_{(p,q),p}(\mathbb{R}^d, \boldsymbol{w}; X) = F^{s/a_2}_{q,p}(\mathbb{R}^{d_2}, w_2; L_p(\mathbb{R}^{d_1}, w_1; X)) \cap L_q(\mathbb{R}^{d_2}, w_2; B^{s/a_1}_{p,p}(\mathbb{R}^{d_1}, w_1; X)).$$

(II) Let $p \in (0, \infty)$, $q \in (0, \infty]$, $\boldsymbol{w} \in A_\infty(\mathbb{R}^{d_1}) \times A_\infty(\mathbb{R}^{d_2})$ and $s \in \mathbb{R}$. Pick $\boldsymbol{r} \in (0, \infty)^2$ such that $r_1 < p$, $r_2 < p \wedge q$ and $\boldsymbol{w} \in A_{p/r_1}(\mathbb{R}^{d_1}) \times A_{q/r_2}(\mathbb{R}^{d_2})$. If $s > a_1 d_1(r_1^{-1} - 1)_+ + a_2 d_2(r_2^{-1} - 1)_+$, then

$$B^{s,(\boldsymbol{a};\mathpzc{d})}_{(p,q),q}(\mathbb{R}^d, \boldsymbol{w}; X) = B^{s/a_2}_{q,q}(\mathbb{R}^{d_2}, w_2; L_p(\mathbb{R}^{d_1}, w_1; X)) \cap L_q(\mathbb{R}^{d_2}, w_2; B^{s/a_1}_{p,q}(\mathbb{R}^{d_1}, w_1; X)).$$

Combining Example 5.8(I) together with a randomized Littlewood–Paley decomposition for UMD Banach space-valued Bessel potential spaces and type and cotype considerations (we refer the reader to [26] for the notions of type and cotype), we find the following embedding.

Example 5.10. Let X be a UMD Banach space with type $\rho_0 \in [1, 2]$ and cotype $\rho_1 \in [2, \infty]$. Let $\mathpzc{d} \in (\mathbb{N}_1)^2$, $\boldsymbol{a} \in (0, \infty)^2$, $\boldsymbol{p} \in (1, \infty)^2$, $q \in [\rho_0, \rho_1]$, $\boldsymbol{w} \in A_\infty(\mathbb{R}^{d_1}) \times A_{p_2}(\mathbb{R}^{d_2})$ and $s \in \mathbb{R}$. Pick $r \in (0, \infty)$ such that $r < p_1 \wedge q$ and $w_1 \in A_{p_1/r}(\mathbb{R}^{d_1})$. If $s > a_1 d_1(r^{-1} - 1)_+$, then

$$F^{s,(\boldsymbol{a};\mathpzc{d})}_{\boldsymbol{p},\rho_0}(\mathbb{R}^d, \boldsymbol{w}; X) \hookrightarrow H^{s/a_2}_{p_2}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1; X)) \cap L_{p_2}(\mathbb{R}^{d_2}, w_2; F^{s/a_1}_{p_1,q}(\mathbb{R}^{d_1}, w_1; X)) \hookrightarrow F^{s,(\boldsymbol{a};\mathpzc{d})}_{\boldsymbol{p},\rho_1}(\mathbb{R}^d, \boldsymbol{w}; X).$$

Proof. By [41, Proposition 3.2] and the fact that $L_{p_1}(\mathbb{R}^{d_1}, w_1; X)$ is a UMD Banach space (see e.g. [25, Proposition 4.2.15]),

$$H^{s/a_2}_{p_2}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1; X)) = F^{s/a_2}_{p_2,\mathrm{rad}}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1; X)). \tag{54}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\epsilon_k)_{k\in\mathbb{N}}$ a Rademacher sequence on $(\Omega, \mathcal{F}, \mathbb{P})$. The space $\mathrm{Rad}_p(\mathbb{N}; X)$, where $p \in [1, \infty)$, is defined as the Banach space of sequences $(x_k)_{k\in\mathbb{N}}$ for which there is convergence of $\sum_{k=0}^\infty \epsilon_k x_k$ in $L_p(\Omega; X)$, endowed with the norm

$$\|(x_k)_{k\in\mathbb{N}}\|_{\mathrm{Rad}_p(\mathbb{N};X)} := \|\sum_{k=0}^\infty \epsilon_k x_k\|_{L_p(\Omega;X)} = \sup_{K\geq 0} \|\sum_{k=0}^K \epsilon_k x_k\|_{L_p(\Omega;X)}.$$

As a consequence of the Kahane–Khintchine inequalities (see e.g. [26, Proposition 6.3.1]), $\mathrm{Rad}_p(\mathbb{N}; X) = \mathrm{Rad}_{\tilde{p}}(\mathbb{N}; X)$ with an equivalence of norms for any $p, \tilde{p} \in [1, \infty)$. We put $\mathrm{Rad}(\mathbb{N}; X) = \mathrm{Rad}_2(\mathbb{N}; X)$.

With the just introduced notation, the type and cotype assumptions on X can be reformulated as

$$\ell_{\rho_0}(\mathbb{N}; X) \hookrightarrow \mathrm{Rad}(\mathbb{N}; X) \hookrightarrow \ell_{\rho_1}(\mathbb{N}; X).$$

Combining this with the identity

$$\mathrm{Rad}(\mathbb{N}; L_{p_1}(\mathbb{R}^{d_1}, w_1; X)) = L_{p_1}(\mathbb{R}^{d_1}, w_1; \mathrm{Rad}(\mathbb{N}; X))$$

obtained from Fubine's theorem and the Kahane–Khintchine inequalities, we find

$$L_{p_1}(\mathbb{R}^{d_1}, w_1; \ell_{\rho_0}(\mathbb{N}; X)) \hookrightarrow \mathrm{Rad}(\mathbb{N}; L_{p_1}(\mathbb{R}^{d_1}, w_1; X)) \hookrightarrow L_{p_1}(\mathbb{R}^{d_1}, w_1; \ell_{\rho_1}(\mathbb{N}; X)). \quad (55)$$

A combination of (54) and (55) yields

$$\begin{aligned}\mathbb{F}^{s/a_2}_{p_2,\rho_1}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1); X) &\hookrightarrow H^{s/a_2}_{p_2}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1; X)) \\ &\hookrightarrow \mathbb{F}^{s/a_2}_{p_2,\rho_2}(\mathbb{R}^{d_2}, w_2; L_{p_1}(\mathbb{R}^{d_1}, w_1); X).\end{aligned}$$

The desired result now follows from Example 5.8(I) and 'monotonicity' of Lizorkin–Triebel spaces in the microscopic parameter. □

6. Duality

Definition 6.1. Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. We define $Y^{\boldsymbol{A}}(E; X^*, \sigma(X^*, X))$ as the space of all $f \in \mathcal{S}'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X)))$ which have a representation

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X)))$$

with $(f_n)_n \subset \mathcal{S}'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X)))$ satisfying the spectrum condition

$$\begin{aligned}&\mathrm{supp}\, \hat{f}_0 \subset \bar{B}^{\boldsymbol{A}}(0, 2) \\ &\mathrm{supp}\, \hat{f}_n \subset \bar{B}^{\boldsymbol{A}}(0, 2^{n+1}) \setminus B^{\boldsymbol{A}}(0, 2^{n-1}), \qquad n \in \mathbb{N},\end{aligned}$$

and $(f_n)_n \in E(X)$. We equip $Y^{\boldsymbol{A}}(E; X^*, \sigma(X^*, X))$ with the quasinorm

$$\|f\|_{Y^{\boldsymbol{A}}(E;X^*,\sigma(X^*,X))} := \inf \|(f_n)\|_{E(X^*,\sigma(X^*,X))},$$

where the infimum is taken over all representations as above.

Similarly to Proposition 3.19 we have the following Littlewood–Paley decomposition description for $Y^{\boldsymbol{A}}(E; X^*, \sigma(X^*, X))$:

Proposition 6.2. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. Let $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ with associated sequence of convolution operators $(S_n)_{n\in\mathbb{N}}$. Then*

$$\begin{aligned}&Y^{\boldsymbol{A}}(E; X^*, \sigma(X^*, X)) \\ &\quad = \left\{ f \in \mathcal{S}'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X))) : (S_n f)_{n\in\mathbb{N}} \in E(X^*, \sigma(X^*, X)) \right\}\end{aligned}$$

with

$$\|f\|_{Y^{\boldsymbol{A}}(E;X^*,\sigma(X^*,X))} \eqsim \|(S_n f)_{n\in\mathbb{N}}\|_{E(X^*,\sigma(X^*,X))}. \quad (56)$$

Using the description from the above proposition it is easy to see that

$$Y^{\boldsymbol{A}}(E; X^*) = Y^{\boldsymbol{A}}(E; X^*, \sigma(X^*, X)) \cap \mathcal{S}'(\mathbb{R}^d; L_0(S; X)).$$

with an equivalence of quasinorms.

Theorem 6.3. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ be a Banach function space with an order continuous norm and a weak order unit such that $E^\times \in \mathcal{S}(-\varepsilon_-, -\varepsilon_+, \boldsymbol{A}, \mathbf{1}, (S, \mathscr{A}, \mu))$. Assume that there exists a Banach function space F on S with an order continuous norm and a weak order unit such that $\mathcal{S}(\mathbb{R}^d; F(X)) \overset{d}{\hookrightarrow} Y^{\boldsymbol{A}}(E; X)$. Viewing*

$$\begin{aligned}[Y^{\boldsymbol{A}}(E; X)]^* \hookrightarrow \mathcal{S}'(\mathbb{R}^d; [F(X)]^*) &= \mathcal{S}'(\mathbb{R}^d; F^\times(X^*, \sigma(X^*, X))) \\ &\hookrightarrow \mathcal{S}'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X)))\end{aligned}$$

via the natural pairing, we have

$$[Y^{\boldsymbol{A}}(E; X)]^* = Y^{\boldsymbol{A}}(E^\times; X^*, \sigma(X^*, X)).$$

Consequently, if X^* has the Radon–Nikodým property with respect to μ, then

$$Y^{\boldsymbol{A}}(E^\times; X^*) = [Y^{\boldsymbol{A}}(E; X)]^* \hookrightarrow \mathcal{S}'(\mathbb{R}^d; F^\times(X^*)) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; L_0(S; X^*)).$$

Example 6.4. Let us consider the notation introduced in Example 3.20. For a weight vector $\boldsymbol{w}$ and $\boldsymbol{p} \in (1, \infty)^\ell$ we define the $\boldsymbol{p}$-dual weight of $\boldsymbol{w}$ by $\boldsymbol{w}'_{\boldsymbol{p}} := (w_1^{-\frac{1}{p_1-1}}, \dots, w_\ell^{-\frac{1}{p_\ell-1}})$ and we write $\boldsymbol{p}'$ for the Hölder conjugate vector of $\boldsymbol{p}$.

(i) Let $\boldsymbol{p} \in (1, \infty)^\ell$, $q \in [1, \infty)$, $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j}(\mathbb{R}^{d_j}, A_j)$ and $s \in \mathbb{R}$. Then

$$[F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)]^* = F^{-s,\boldsymbol{A}}_{\boldsymbol{p}',q'}(\mathbb{R}^d, \boldsymbol{w}'_{\boldsymbol{p}}; X^*).$$

(ii) Let $\boldsymbol{p} \in (1, \infty)^\ell$, $q \in [1, \infty)$, $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j}(\mathbb{R}^{d_j}, A_j)$ and $s \in \mathbb{R}$. Then

$$[B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)]^* = B^{-s,\boldsymbol{A}}_{\boldsymbol{p}',q'}(\mathbb{R}^d, \boldsymbol{w}'_{\boldsymbol{p}}; X^*).$$

(iii) Let F be a UMD Banach function space, $\boldsymbol{p} \in (1, \infty)^\ell$, $q \in [1, \infty)$, $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j}(\mathbb{R}^{d_j}, A_j)$ and $s \in \mathbb{R}$. If X^* has the Radon–Nikodým property with respect to μ, then

$$[\mathbb{F}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X)]^* = \mathbb{F}^{-s,\boldsymbol{A}}_{\boldsymbol{p}',q'}(\mathbb{R}^d, \boldsymbol{w}'_{\boldsymbol{p}}; F^\times; X^*).$$

Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \mathbf{1}, (S, \mathscr{A}, \mu))$ be a Banach function space. By Remark 3.10 we then have

$$E_i \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{1, w_{\boldsymbol{A},\mathbf{1}}}_{\boldsymbol{A}}) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}[B^{1, w_{\boldsymbol{A},\mathbf{1}}}_{\boldsymbol{A}}],$$

from which it follows that

$$\begin{aligned}E_i(X^*, \sigma(X^*, X)) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}[B^{1, w_{\boldsymbol{A},\mathbf{1}}}_{\boldsymbol{A}}](X^*, \sigma(X^*, X)) &\hookrightarrow \mathcal{S}'(\mathbb{R}^d; E^{\boldsymbol{A}}_{\otimes}(X^*, \sigma(X^*, X))) \\ &\hookrightarrow \mathcal{S}'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X))).\end{aligned}$$

Lemma 6.5. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \mathbf{1}, (S, \mathscr{A}, \mu))$ be a Banach function space and let Z be a Banach space with $Z \hookrightarrow L_0(S; X^*, \sigma(X^*, X))$. Let $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ with associated sequence of convolution operators $(S_n)_{n\in\mathbb{N}}$ be such that*

$$\begin{cases} \operatorname{supp} \hat{\varphi}_0 & \subset \bar{B}^{\boldsymbol{A}}(0, 2) \\ \operatorname{supp} \hat{\varphi}_n & \subset \bar{B}^{\boldsymbol{A}}(0, 2^{n+1}) \setminus B^{\boldsymbol{A}}(0, 2^{n-1}), \qquad n \in \mathbb{N}. \end{cases} \tag{57}$$

Then

$$\begin{aligned} &Y^{A}(E; X^{*}, \sigma(X^{*}, X)) \cap \mathcal{S}'(\mathbb{R}^{d}; Z) \\ &\quad = \left\{ f \in \mathcal{S}'(\mathbb{R}^{d}; Z) : \exists (f_k)_k \in E(X^{*}, \sigma(X^{*}, X)),\, f = \sum_{k=0}^{\infty} S_k f_k \text{ in } \mathcal{S}'(\mathbb{R}^{d}; Z) \right\} \end{aligned}$$

with

$$\| f \|_{Y^{A}(E; X^{*}, \sigma(X^{*}, X))} \eqsim \inf \| (f_k)_k \|_{E(X^{*}, \sigma(X^{*}, X))}.$$

Proof. Given $f \in Y^{A}(E; X^{*}, \sigma(X^{*}, X)) \cap \mathcal{S}'(\mathbb{R}^{d}; Z)$, let $f_k := T_k f$, where $T_k := S_{k-1} + S_k + S_{k+1}$. Then $S_k f_k = S_k f$, so $f = \sum_{k=0}^{\infty} S_k f_k$ in $\mathcal{S}'(\mathbb{R}^{d}; Z)$. From

$$|\langle f_k, x \rangle| = |T_k \langle S_k f, x \rangle| \lesssim M^{A} \langle S_k f, x \rangle \leq M^{A} \vartheta(S_k f), \qquad x \in B_X,$$

it follows that $\vartheta(f_k) \lesssim M^{A} \vartheta(S_k f)$. Using that M^{A} is bounded on E, we find

$$\| (f_k)_k \|_{E(X^{*}, \sigma(X^{*}, X))} \lesssim \| (S_k f)_k \|_{E(X^{*}, \sigma(X^{*}, X))} \overset{(56)}{\eqsim} \| f \|_{Y^{A}(E; X^{*}, \sigma(X^{*}, X))}.$$

For the converse, let $f = \sum_{k=0}^{\infty} S_k f_k$ in $\mathcal{S}'(\mathbb{R}^{d}; Z)$ with $(f_k)_k \in E(X^{*}, \sigma(X^{*}, X))$. Then

$$|\langle S_k f_k, x \rangle| = |S_k \langle f_k, x \rangle| \lesssim M^{A} \langle f_k, x \rangle \leq M^{A} \vartheta(f_k), \qquad x \in B_X,$$

so that $\vartheta(S_k f) \lesssim M^{A} \vartheta(f_k)$. In view of

$$f = \sum_{k=0}^{\infty} S_k f_k \qquad \mathcal{S}'(\mathbb{R}^{d}; Z) \hookrightarrow \mathcal{S}'(\mathbb{R}^{d}; L_0(S; X^{*}, \sigma(X^{*}, X))),$$

(57) and the boundedness of M^{A} on E, it follows that $f \in Y^{A}(E; X^{*}, \sigma(X^{*}, X))$ with

$$\| f \|_{Y^{A}(E; X^{*}, \sigma(X^{*}, X))} \lesssim \| (S_k f_k)_k \|_{E(X^{*}, \sigma(X^{*}, X))} \lesssim \| (f_k)_k \|_{E(X^{*}, \sigma(X^{*}, X))}. \qquad \square$$

Proof of Theorem 6.3. By assumption and Proposition 3.26,

$$\mathcal{S}(\mathbb{R}^{d}; F(X)) \hookrightarrow Y^{A}(E; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^{d}; E^{A}_{\otimes}(X)),$$

from which it follows that $F \hookrightarrow E^{A}_{\otimes}$, implying in turn that $[E^{A}_{\otimes}]^{\times} \hookrightarrow F^{\times}$. On the other hand it holds that $[E^{\times}]^{A}_{\otimes} \hookrightarrow [E^{A}_{\otimes}]^{\times}$. Therefore, $[E^{\times}]^{A}_{\otimes} \hookrightarrow F^{\times}$. By (a variant of) Proposition 3.26 we thus obtain

$$Y^{A}(E^{\times}; X^{*}, \sigma(X^{*}, X)) \hookrightarrow \mathcal{S}'(\mathbb{R}^{d}; [E^{\times}]^{A}_{\otimes}(X^{*}, \sigma(X^{*}, X))) \hookrightarrow \mathcal{S}'(\mathbb{R}^{d}; F^{\times}(X^{*}, \sigma(X^{*}, X))). \tag{58}$$

So we can use Lemma 6.5 with $Z = F^{\times}(X^{*}, \sigma(X^{*}, X))$ to describe $Y^{A}(E^{\times}; X^{*}, \sigma(X^{*}, X))$.

Let $(S_k)_{k \in \mathbb{N}}$ be as in Lemma 6.5 and equip $Y^{A}(E; X)$ with the corresponding equivalent norm from Proposition 3.19. Then

$$\iota : Y^{A}(E; X) \longrightarrow E(X),\; f \mapsto (S_k f)_k$$

defines an isometric linear mapping. By order continuity of E and F, there are the natural identifications

$$[E(X)]^{*} = E^{\times}(X^{*}, \sigma(X^{*}, X)) \qquad \text{and} \qquad [F(X)]^{*} = F^{\times}(X^{*}, \sigma(X^{*}, X)).$$

As $\mathcal{S}(\mathbb{R}^d; F(X)) \overset{d}{\hookrightarrow} Y^{\boldsymbol{A}}(E; X)$, we may thus view

$$[Y^{\boldsymbol{A}}(E; X)]^* \hookrightarrow \mathcal{S}'(\mathbb{R}^d; [F(X)]^*) = \mathcal{S}'(\mathbb{R}^d; F^{\times}(X^*, \sigma(X^*, X))).$$

Denoting the adjoint of ι by j, we thus obtain the following commutative diagram:

$$\begin{array}{ccc} E^{\times}(X^*, \sigma(X^*, X)) & \xrightarrow{\;T\;} & \mathcal{S}'(\mathbb{R}^d; F^{\times}(X^*, \sigma(X^*, X))) \\ \downarrow & \searrow^{j} & \big\uparrow \\ E^{\times}(X^*, \sigma(X^*, X))/\ker j & \underset{\simeq}{\xrightarrow{\;\tilde{j}\;}} & [Y^{\boldsymbol{A}}(E; X)]^* \end{array}.$$

Here T is explicitly given by

$$T(f_k)_k = \sum_{k=0}^{\infty} S_k f_k \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d; F^{\times}(X^*, \sigma(X^*, X))),$$

which can be seen by testing against $\phi \in \mathcal{S}(\mathbb{R}^d; F(X))$:

$$\langle T(f_k)_k, \phi \rangle = \langle (f_k)_k, \iota\phi \rangle = \langle (f_k)_k, (S_k\phi)_k \rangle = \sum_{k=0}^{\infty} \langle f_k, S_k\phi \rangle = \sum_{k=0}^{\infty} \langle S_k f_k, \phi \rangle.$$

The desired result follows by an application of Lemma 6.5 with $Z = F^{\times}(X^*, \sigma(X^*, X))$ (recall (58)). $\square$

7. A sum representation

In this section we combine the intersection representation for $Y^{\boldsymbol{A}}(E; X)$ from Theorem 5.4 and the duality result Theorem 6.3 with the following fact on duality for intersection spaces: given an interpolation couple of Banach spaces (Y, Z) for which $Y \cap Z$ is dense in both Y and Z, it holds that (X^*, Y^*) is an interpolation couple of Banach space and

$$[Y \cap Z]^* = Y^* + Z^*, \qquad [X + Y]^* = X^* \cap Y^*, \tag{59}$$

hold isometrically under the natural identifications (see [31, Theorem I.3.1]).

We let the notation be as in Section 5.

Corollary 7.1. *Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{1}, (S, \mathscr{A}, \mu))$ be a Banach function space such that $E^{\times}$ has an order continuous norm and a weak order unit and $E^{\times} \in \mathcal{S}(-\varepsilon_-, -\varepsilon_+, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ with $\varepsilon_+, \varepsilon_- < 0$. Suppose that X is reflexive. Let F Banach function space on S with an order continuous norm such that $\mathcal{S}(\mathbb{R}^d; F(X)) \overset{d}{\hookrightarrow} Y^{\boldsymbol{A}}(E^{\times}; X)$. Let $\{J_1, \dots, J_L\}$ be a partition of $\{1, \dots, \ell\}$ and, for each $l \in \{1, \dots, L\}$, let F_l be a Banach function space on S_{J_l} with an order continuous norm and a weak order unit such that*

$$\mathcal{S}(\mathbb{R}^d; F(X)) \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^{d-d_{J_l}}; F_l(X)) \overset{d}{\hookrightarrow} Y^{\boldsymbol{A}_{J_l}}(E^{\times}_{[d;J_l]}; X).$$

Then

$$Y^{\boldsymbol{A}}(E; X) = \sum_{l=1}^{L} Y^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]}; X)$$

with an equivalence of norms.

Proof. As E has the Fatou property, $E = (E^{\times})^{\times}$. The desired result thus follows from a combination of Theorems 5.4, 6.3, (59), the reflexivity of X and the fact that the Radon–Nikodým property is implied by reflexivity (see [26, Theorem 1.3.21]). □

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Appendix. Some maximal function inequalities

Suppose that $\mathbb{R}^d$ is $\mathpzc{d}$-decomposed with $\mathpzc{d} \in (\mathbb{N}_1)^{\ell}$ and let $\boldsymbol{A} = (A_1, \dots, A_{\ell})$ be a $\mathpzc{d}$-anisotropy.

Lemma A.1 (*Anisotropic Peetre's inequality*)**.** *Let X be a Banach space, $\boldsymbol{r} \in (0,\infty)^{\ell}$, $K \subset \mathbb{R}^d$ a compact set and $N \in \mathbb{N}$. For all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq N$ and $f \in \mathcal{S}'(\mathbb{R}^d; X)$ with $\operatorname{supp}(\hat{f}) \subset K$, there is the pointwise estimate*

$$\sup_{z \in \mathbb{R}^d} \frac{\|D^{\alpha} f(x+z)\|_X}{\prod_{j=1}^{\ell}(1+\rho_{A_j}(z_j))^{\operatorname{tr}(A_j)/r_j}} \lesssim \sup_{z \in \mathbb{R}^d} \frac{\|f(x+z)\|_X}{\prod_{j=1}^{\ell}(1+\rho_{A_j}(z_j))^{\operatorname{tr}(A_j)/r_j}} \lesssim \left[M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f\|_X)\right](x), \qquad x \in \mathbb{R}^d.$$

Proof. This can be obtained by combining the proof of [28, Proposition 3.11] for the case $\mathpzc{d} = \mathbf{1}$ with the proof of [10, Lemma 3.4] for the case $\ell = 1$. Although it get quite technical, we have decided to not provide the details. □

For $f \in \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X)$, $\boldsymbol{r} \in (0,\infty)^{\ell}$, $\boldsymbol{R} \in (0,\infty)^{\ell}$ we define the *maximal function of Peetre–Fefferman–Stein type* $f^*(\boldsymbol{A}, \boldsymbol{r}, \boldsymbol{R};\, \cdot\,)$ by

$$f^*(\boldsymbol{A}, \boldsymbol{r}, \boldsymbol{R}; x) := \sup_{z \in \mathbb{R}^d} \frac{\|f(x+z)\|_X}{\prod_{j=1}^{\ell}(1+R_j\rho_{A_j}(z_j))^{\operatorname{tr}(A_j)/r_j}}.$$

Corollary A.2. *Let X be a Banach space and $\boldsymbol{r} \in (0,\infty)^{\ell}$. For all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ and $\boldsymbol{R} \in (0,\infty)^{\ell}$ with $\operatorname{supp}(\hat{f}) \subset B^{\boldsymbol{A}}(0, \boldsymbol{R})$, there is the pointwise estimate*

$$f^*(\boldsymbol{A}, \boldsymbol{r}, \boldsymbol{R}; x) \lesssim_{\boldsymbol{A},\boldsymbol{r}} \left[M_{\boldsymbol{r}}^{\boldsymbol{A}}(\|f\|_X)\right](x), \qquad x \in \mathbb{R}^d.$$

Proof. By a dilation argument it suffices to consider the case $\boldsymbol{R} = \mathbf{1}$, which is contained in Lemma A.1. □

Lemma A.3. *Let X and Y be Banach spaces. For all $(M_n)_{n\in\mathbb{N}} \subset \mathcal{F}L^1(\mathbb{R}^d; \mathcal{B}(X,Y))$, $(\boldsymbol{R}^{(n)})_{n\in\mathbb{N}} \subset (0,\infty)^{\ell}$, $\boldsymbol{\lambda} \in (0,\infty)^{\ell}$, $c \in [1,\infty)$ and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X)$, there is the pointwise estimate*

$$\begin{aligned} &\left\|[\mathcal{F}(M_n\hat{f}_n)](x)\right\|_Y \\ &\quad\lesssim c^{\sum_{j=1}^{\ell}\lambda_j} \sup_{k\in\mathbb{N}} \int_{\mathbb{R}^d} \|\check{M}_n(\boldsymbol{A}_{\boldsymbol{R}^{(n)}}y)\|_{\mathcal{B}(X,Y)} \prod_{j=1}^{\ell}(1+\rho_{A_j}(y_j))^{\lambda_j}\,dy \\ &\qquad\cdot \sup_{z\in\mathbb{R}^d} \frac{\|f_n(x+z)\|_X}{\prod_{j=1}^{\ell}(1+cR_j^{(n)}\rho_{A_j}(y_j))^{\lambda_j}}. \end{aligned}$$

Proof. This can be shown as the pointwise estimate in the proof of [33, Proposition 3.4.8], which was in turn based on [39, Proposition 2.4]. □

The following proposition is an extension of [28, Proposition 3.13] to our setting, which is in turn a version of the pointwise estimate of pseudo-differential operators due to Marschall [38]. In order to state it, we first need to introduce the anisotropic mixed-norm homogeneous Besov space $\dot{B}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d;Z)$.

Let Z be a Banach space, $\boldsymbol{p} \in (1,\infty)^{\ell}$, $q \in (0,\infty]$ and $s \in \mathbb{R}$. Fix $(\phi_k)_{k\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^d)$ that satisfies $\hat{\phi}_k = \hat{\psi}(\boldsymbol{A}_{2^{-k}}\,\cdot) - \hat{\psi}(\boldsymbol{A}_{2^{-(k+)}}\,\cdot)$ for some $\psi \in \mathcal{F}C^{\infty}_c(\mathbb{R}^d)$ with $\hat{\psi} \equiv 1$ on a neighborhood of 0. Then $\dot{B}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d;Z)$ is defined as the space of all $f \in [\mathcal{S}'/\mathcal{P}](\mathbb{R}^d;Z)$ for which

$$\|f\|_{\dot{B}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d;Z)} := \left\|(2^{sk}\phi_k * f)_{k\in\mathbb{Z}}\right\|_{\ell_q(\mathbb{Z})[L_{\boldsymbol{p},d}(\mathbb{R}^d)](Z)} < \infty.$$

Proposition A.4. *Let X and Y be Banach spaces and $\boldsymbol{r} \in (0,1]^{\ell}$. Put $\tau := \boldsymbol{r}_{\min} \in (0,1]$. For all $b \in \mathcal{S}(\mathbb{R}^d;\mathcal{B}(X,Y))$, $u \in \mathcal{S}'(\mathbb{R}^d;X)$, $c \in (0,\infty)$ and $R \in [1,\infty)$ with $\operatorname{supp}(b) \subset B^{\boldsymbol{A}}(0,c)$ and $\operatorname{supp}(\hat{u}) \subset B^{\boldsymbol{A}}(0,cR)$, there is the pointwise estimate*

$$\|b(D)u(x)\|_Y \lesssim_{\boldsymbol{A},\boldsymbol{r}} (cR)^{\operatorname{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})}\|b\|_{\dot{B}^{\operatorname{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1},\boldsymbol{A}}_{1,\tau}(\mathbb{R}^d;\mathcal{B}(X,Y))}\left[M^{\boldsymbol{A}}_{\boldsymbol{r}}(\|u\|_X)\right](x)$$

for each $x \in \mathbb{R}^d$.

In the proof of Proposition A.4 we will use the following lemma.

Lemma A.5. *Let X be a Banach space and $\boldsymbol{p},\boldsymbol{q} \in (0,\infty)^{\ell}$ with $\boldsymbol{p} \leq \boldsymbol{q}$. For every $f \in \mathcal{S}'(\mathbb{R}^d;X)$ and $\boldsymbol{R} \in (0,\infty)^{\ell}$ with $\operatorname{supp}(\hat{f}) \subset B^{\boldsymbol{A}}(0,\boldsymbol{R})$,*

$$\|f\|_{L_{\boldsymbol{q},d}(\mathbb{R}^d;X)} \lesssim_{\boldsymbol{p},\boldsymbol{q},d} \prod_{j=1}^{\ell} R_j^{\operatorname{tr}(A_j)(\frac{1}{p_j}-\frac{1}{q_j})}\|f\|_{L_{\boldsymbol{p},d}(\mathbb{R}^d;X)}$$

Proof. By a scaling argument we may restrict ourselves to the case $\boldsymbol{R} = \boldsymbol{1}$. Now pick $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{\phi} \equiv 1$ on $B^{\boldsymbol{A}}(0,\boldsymbol{1})$. Then $f = \phi * f$ and the desired inequality follows from an iterated use of Young's inequality for convolutions. □

Proof of Proposition A.4. It holds that

$$b(D)u(x) = \int_{\mathbb{R}^d}\check{b}(y)u(x-y)\,dy, \qquad x \in \mathbb{R}^d.$$

For fixed $x \in \mathbb{R}^d$, by the quasi-triangle inequality for $\rho_{\boldsymbol{A}}$ (with constant $c_{\boldsymbol{A}}$),

$$\operatorname{supp}(\mathcal{F}[y \mapsto \check{b}(y)u(x-y)]) \subset B_{\boldsymbol{A}}(0,c) + B_{\boldsymbol{A}}(0,cR) \subset B_{\boldsymbol{A}}(0,c_{\boldsymbol{A}}(R+1)c).$$

Therefore,

$$\begin{aligned}\|b(D)u(x)\|_Y &\leq \|y \mapsto \check{b}(y)u(x-y)\|_{L_1(\mathbb{R}^d)}\\ &\lesssim (c_{\boldsymbol{A}}(R+1)c)^{\sum_{j=1}^{\ell}\operatorname{tr}(A_j)(\frac{1}{r_j}-1)}\|y \mapsto \check{b}(y)u(x-y)\|_{L_{\boldsymbol{r},d}(\mathbb{R}^d)}\\ &\lesssim (Rc)^{\sum_{j=1}^{\ell}\operatorname{tr}(A_j)(\frac{1}{r_j}-1)}\|y \mapsto \check{b}(y)u(x-y)\|_{L_{\boldsymbol{r},d}(\mathbb{R}^d)}, \end{aligned} \tag{60}$$

where we used Lemma A.5 for the second estimate.

Let $(\phi_k)_{k\in\mathbb{Z}}$ be as in the definition of the anisotropic homogeneous Besov space $\dot{B}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}$ as given preceding the proposition. Then $\sum_{k=-\infty}^{\infty}\hat{\phi}_k(-\,\cdot\,)=1$ on $\mathbb{R}^d\setminus\{0\}$, so that

$$\|\check{b}\,u(x-\,\cdot\,)\|_{L_{\boldsymbol{r},d}(\mathbb{R}^d)} \leq \left(\sum_{k\in\mathbb{Z}}\|\hat{\phi}_k(-\,\cdot\,)\,\check{b}\,u(x-\,\cdot\,)\|^{\tau}_{L_{\boldsymbol{r},d}(\mathbb{R}^d)}\right)^{1/\tau}. \tag{61}$$

Since

$$\begin{aligned}\sup_{y\in\mathbb{R}^d}\|\hat{\phi}_k(-y)\check{b}(y)\|_{\mathcal{B}(X,Y)} &\leq \|\mathcal{F}^{-1}[\hat{\phi}_k(-\,\cdot\,)\,\check{b}]\|_{L_1(\mathbb{R}^d;\mathcal{B}(X,Y))}\\ &= (2\pi)^{-d}\|\mathcal{F}^{-1}[\hat{\phi}_k\hat{b}]\|_{L_1(\mathbb{R}^d;\mathcal{B}(X,Y))}\end{aligned}$$

and $\operatorname{supp}(\hat{\phi}_k)\subset B^{\boldsymbol{A}}(0,2^{k+1})$, it follows from a combination of (60) and (61) that

$$\begin{aligned}\|b(D)u(x)\|_Y &\lesssim (Rc)^{\sum_{j=1}^{\ell}\operatorname{tr}(A_j)(\frac{1}{r_j}-1)}\left(\sum_{k\in\mathbb{Z}}\|\hat{\phi}_k(-\,\cdot\,)\,\check{b}\,u(x-\,\cdot\,)\|^{\tau}_{L_{\boldsymbol{r},d}(\mathbb{R}^d)}\right)^{1/\tau}\\ &\lesssim (Rc)^{\sum_{j=1}^{\ell}\operatorname{tr}(A_j)(\frac{1}{r_j}-1)}\left(\sum_{k\in\mathbb{Z}}\left[2^{k\sum_{j=1}^{\ell}\operatorname{tr}(A_j)\frac{1}{r_j}}\|\mathcal{F}^{-1}[\hat{\phi}_k\hat{b}]\|_{L_1}\right]^{\tau}\right)^{1/\tau}\\ &\qquad \sup_{k\in\mathbb{Z}}2^{-(k+1)\operatorname{tr}(A_j)\frac{1}{r_j}}\|1_{B^{\boldsymbol{A}}(0,2^{k+1})}u(x-\,\cdot\,)\|_{L_{\boldsymbol{r},d}(\mathbb{R}^d)}\\ &\leq (Rc)^{\sum_{j=1}^{\ell}\operatorname{tr}(A_j)(\frac{1}{r_j}-1)}\|b\|_{\dot{B}^{\sum_{j=1}^{\ell}\operatorname{tr}(A_j)\frac{1}{r_j},\boldsymbol{A}}_{1,\tau}(\mathbb{R}^d;\mathcal{B}(X,Y))}\left[M^{\boldsymbol{A}}_{\boldsymbol{r}}(\|u\|_X)\right](x). \quad \square\end{aligned}$$

Corollary A.6. *Let X and Y be Banach spaces, $\boldsymbol{r}\in(0,1]^{\ell}$ and $\psi\in C^{\infty}_c(\mathbb{R}^d;\mathcal{B}(X,Y))$. Put $\psi_k:=\psi(\boldsymbol{A}_{2^{-k}}\,\cdot\,)$ for each $k\in\mathbb{N}$. Then, for all $(f_k)_{k\in\mathbb{N}}\subset\mathcal{S}'(\mathbb{R}^d;X)$ with $\operatorname{supp}\hat{f}_k\subset B^{\boldsymbol{A}}(0,r2^k)$ for some $r\in[1,\infty)$, there is the pointwise estimate*

$$\|\psi_k(D)f_k(x)\|_Y \lesssim r^{\operatorname{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})}\left[M^{\boldsymbol{A}}_{\boldsymbol{r}}(\|f_k\|_X)\right](x), \qquad x\in\mathbb{R}^d.$$

Proof. Set $\sigma=\operatorname{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}=\sum_{j=1}^{\ell}\operatorname{tr}(A_j)\frac{1}{r_j}$. Let $c\in[1,\infty)$ be such that $\operatorname{supp}(\psi)\subset B^{\boldsymbol{A}}(0,c)$. Applying Proposition A.4 to $b=\psi_k$, $u=f_k$ and $R=r2^k$, we find that

$$\|\psi_k(D)f_k(x)\|_Y \lesssim (cr2^k)^{\operatorname{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})}\|\psi_k\|_{\dot{B}^{\sigma,\boldsymbol{A}}_{1,\tau}(\mathbb{R}^d)}\left[M^{\boldsymbol{A}}_{\boldsymbol{r}}(\|f_k\|_X)\right](x).$$

Observing that

$$\|\psi_k\|_{\dot{B}^{\sigma,\boldsymbol{A}}_{1,\tau}(\mathbb{R}^d)} = 2^{-k\operatorname{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})}\|\psi\|_{\dot{B}^{\sigma,\boldsymbol{A}}_{1,\tau}(\mathbb{R}^d)},$$

we obtain the desired estimate. $\square$

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