Short Papers

Best Linear Unbiased Estimation of the Fourier Coefficients of Periodic Signals

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Abstract—**Fourier coefficients of periodic signals are usually estimated by the discrete Fourier transform (DFT). Since DFT is equivalent to ordinary least squares, it might be expected to be best linear unbiased only if the errors in the observations are not covariant. Fortunately, for covariant errors, the DFT achieves this optimum property asymptotically.**

I. Introduction

The subject of this paper is the estimation of complex Fourier coefficients of periodic, multiharmonic signals from additively error-disturbed observations. For example, this problem is found in classical [1] and parametric [2]–[4] frequency response methods. In the latter methods, the parameters of the transfer function of a linear dynamic system are estimated from noise-disturbed responses to periodic, multiharmonic test signals. This is done by first estimating the complex Fourier coefficients of the harmonics of the response, and subsequently using the results to estimate the system parameters. So, to fully exploit the available observations, the variance of the estimator of the Fourier coefficients should be as small as possible.

If the discrete Fourier transform (DFT) is chosen as an estimator of the Fourier coefficients, this is equivalent to ordinary least-squares (OLS) estimation [5]. Generally, the OLS estimator is best linear unbiased (BLU) only if the errors in the observations are not covariant and have equal variance [6, p. 190]. Thus, if in the problem considered in this paper the errors meet these conditions, the DFT is BLU. If, in addition, the errors are normally distributed, OLS estimators are known to be maximum likelihood estimators and to attain the Cramér–Rao lower bound on the variance for any number of observations [6, p.413]. So, under these conditions, the DFT has the same optimal properties. On the other hand, if the errors are covariant, only the *weighted* least squares estimator with the inverse of the covariance matrix of the errors as a weighting matrix has the described optimal properties [6, p. 190]. Then the OLS estimator, and consequently the DFT, are no longer optimal.

Compared with the DFT, the weighted least-squares estimator has two important disadvantages. First, it requires knowledge of the covariance matrix of the errors. This knowledge is not always available. Second, the numerical computation of the weighted least-squares estimator is much more time consuming.

The purpose of this paper may now be described as follows. It will be shown that the covariance matrix of the OLS estimator of the Fourier coefficients, that is, the covariance matrix of the DFT, asymptotically approaches the covariance matrix of the weighted least-squares estimator. Therefore, for any probability density of the errors, the DFT is asymptotically BLU. In addition, for normally distributed errors, the DFT asymptotically attains the Cramér–Rao lower bound.

Earlier results concerning BLU and maximum likelihood estimation of Fourier coefficients are reported by Brillinger [7, p. 22] and by Grenander and Szegö [8, p. 204]. These concern the estimation of the complex amplitude of a monosinusoid. This paper addresses the joint estimation of the complex Fourier coefficients of a *multiharmonic* signal such as found in the above-mentioned applications. Grenander and Rosenblatt [9, p. 246] address trigonometric regression, that is, joint estimation of the complex amplitudes of a number of complex sinusoids of known but not necessarily harmonically related frequencies. The results of this paper agree with their results, but the specialization to harmonically related frequencies drastically simplifies the proofs and the results.

The outline of this paper is as follows. In Section II, the problem is formally stated. Section III deals with the asymptotic covariance matrices of OLS and BLU estimators. In Section IV, conclusions are drawn.

II. Least-Squares Estimation of Fourier Coefficients

Suppose that a number of JN complex observations w_n are available, described by

$$w_n = y_n + v_n \qquad n = 0, \cdots, JN - 1 \tag{1}$$

where the complex periodic sequence y_n is defined as

$$y_n = \sum_l c_l \exp(j2\pi k_l n/N) \tag{2}$$

with $l = 1, \cdots, L$ and $n = 0, \cdots, JN - 1$. The integers k_l, $l = 1, \cdots, L$ are the harmonic numbers and the c_l are the corresponding complex Fourier coefficients, respectively. Furthermore, in the expressions (1) and (2), the v_n, $n = 0, \cdots, JN - 1$, are a realization of a stationary stochastic process with expectation zero and covariance sequence $R_{vv}(m)$, $m = 0, \pm 1, \pm 2, \cdots$; the integer N is the period of y_n, and the integer J is the number of periods observed. Then (1) and (2) may be summarized as follows:

$$\boldsymbol{w} = \boldsymbol{X}\boldsymbol{c} + \boldsymbol{v} \tag{3}$$

where $\boldsymbol{w}$ and $\boldsymbol{v}$ are the $JN \times 1$ vectors of the observations w_n and the errors v_n, respectively, the $JN \times L$ matrix $\boldsymbol{X}$ is defined by its (p, q)th element $\exp(j2\pi k_q(p-1)/N)$, and $\boldsymbol{c}$ is the $L \times 1$ vector of c_l.

The OLS estimator of $\boldsymbol{c}$ in (3) is described by

$$\hat{\boldsymbol{c}} = (\boldsymbol{X}^H \boldsymbol{X})^{-1} \boldsymbol{X}^H \boldsymbol{w} \tag{4}$$

where the superscript H denotes complex conjugate transposition. Since differently numbered rows of $\boldsymbol{X}^H$ and columns of $\boldsymbol{X}$ are orthogonal, $\hat{\boldsymbol{c}}$ is equal to [5]

$$\hat{\boldsymbol{c}} = \frac{1}{JN} \boldsymbol{X}^H \boldsymbol{w}. \tag{5}$$

Thus, the elements of $\hat{\boldsymbol{c}}$ are described by

$$\hat{c}_l = \frac{1}{JN} \sum_n w_n \exp(-j2\pi k_l n/N) \tag{6}$$

Manuscript received May 11, 1992.
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IEEE Log Number 9206149.

and are, therefore, seen to be identical with the DFT of w_n, $n = 0, \cdots, JN - 1$. It also follows from (6) that $\hat{c}$ is unbiased for c since $E[w_n] = y_n$ because, by assumption, $E[v_n] = 0$ for all n. From this and from (5), it follows that the covariance matrix of $\hat{c}$ is equal to

$$\frac{1}{(JN)^2} X^H E[vv^H] X = \frac{1}{(JN)^2} X^H V X \tag{7}$$

where the $JN \times JN$ matrix V is the covariance matrix of the process v_n defined as the matrix whose (p, q)th element is $E[v_p v_q^*] = R_{vv}(p - q)$.

The properties of the OLS estimator are well known [6, p. 190]. It is BLU if the errors in the observations are not covariant. It is the maximum likelihood estimator if, in addition, the errors are normally distributed. Then its covariance matrix is equal to the Cramér–Rao lower bound for any number of observations [6, p. 413]. Thus, if in an experiment these conditions are met, the DFT being the OLS estimator is the best choice.

In principle, the OLS estimator loses these optimal properties if the errors are covariant. Then the weighted least-squares estimator with V^{-1} as a weighting matrix is known to be optimal in the above sense [6, p. 190]. So, if in the problem considered in this paper the stochastic process v_n is covariant, the BLU estimator of the Fourier coefficients is

$$\tilde{c} = (X^H V^{-1} X)^{-1} X^H V^{-1} w. \tag{8}$$

It is easily shown that the covariance matrix of this estimator is equal to

$$(X^H V^{-1} X)^{-1}. \tag{9}$$

It is clear that the BLU estimator (8) is computationally much more demanding than the OLS estimator. It requires the computation of the inverse of the $JN \times JN$ matrix V, the computation of the constants in the set of linear equations (8), and the subsequent solution of this set for $\tilde{c}$. The OLS estimator, on the other hand, only requires the evaluation of a number of L DFT's such as described by (6). This is why in the next section the covariance matrix (7) of the OLS estimator and the covariance matrix (9) of the BLU estimator will be compared.

III. Asymptotic Covariance Matrices

In this section, the asymptotic behavior of the covariance matrix (7) of the OLS estimator and the covariance matrix (9) of the BLU estimator will be analyzed. To simplify this analysis, use will be made of the fact that the covariance matrices of a general class of stochastic processes converge in norm to certain circulant matrices associated with them [10, p. 36], [11, p. 74]. Specifically, Brillinger [11, p. 74] shows that for square summable $R_{vv}(m)$, the matrix V converges in norm to the $JN \times JN$ circulant matrix U whose kth element of the first row is described by

$$R_{vv}(1 - k) + R_{vv}(1 - k + JN) \qquad k = 1, \cdots, JN \tag{10}$$

where $R_{vv}(JN)$ is taken equal to zero.

Generally, it can be shown that the eigenvalues of a circulant matrix are proportional to the DFT of the first row [11, p. 73]. Specifically, the eigenvalues of U are described by

$$\begin{aligned}\lambda_l &= \sum_{k=0}^{JN-1} \{R_{vv}(-k) + R_{vv}(-k + JN)\} \exp(-j2\pi lk/JN) \\ &= \sum_{m=-(JN-1)}^{JN-1} R_{vv}(-m) \exp(-j2\pi lm/JN)\end{aligned} \tag{11}$$

with $l = 0, \cdots, JN - 1$. It can also be shown [11, p. 73] that the eigenvectors of U are the columns of the $JN \times JN$ matrix F having $(JN)^{-1/2} \exp(-j2\pi(p - 1)(q - 1)/JN)$ as its (p, q)th element. Then it is easily shown that

$$U = F \Lambda F^H \tag{12}$$

where $\Lambda = \text{diag}(\lambda_0 \cdots \lambda_{JN-1})$. Hence, since F is unitary,

$$U^{-1} = F \Lambda^{-1} F^H. \tag{13}$$

Now, substituting this expression for V^{-1} in (9) yields

$$(X^H F \Lambda^{-1} F^H X)^{-1}. \tag{14}$$

By definition, the (p, q)th element of the $JN \times L$ matrix X is $\exp\{j2\pi k_q(p - 1)/N\} = \exp\{j2\pi(k_q J)(p - 1)/JN\}$. Therefore, the columns of X are a subset of those of F. Hence, the elements of $X^H F$ in (16) with indexes $(1, NJ - k_1 J + 1)$, $(2, NJ - k_2 J + 1)$, $\cdots$, $(L, NJ - k_L J + 1)$ are all equal to $(JN)^{1/2}$, and all other elements are equal to zero. Analogous considerations apply to the matrix $F^H X$. This leads to the conclusion that (14) is equal to

$$\frac{1}{JN} \text{diag}(\lambda(N - k_1)J, \cdots, \lambda(N - k_L)J). \tag{15}$$

On the other hand, the covariance matrix (7) of the ordinary least-squares estimator converges to

$$\frac{1}{(JN)^2} X^H F \Lambda F^H X = \frac{1}{JN} \text{diag}(\lambda(N - k_1)J, \cdots, \lambda(N - k_L)J). \tag{16}$$

From (15) and (16), it is concluded that the asymptotic covariance matrix of the OLS estimator, that is, the DFT estimator, and the asymptotic covariance matrix of the BLU estimator are equal. So, the DFT estimator asymptotically has the same optimal properties as the BLU estimator.

IV. Conclusions

It has been shown that the discrete Fourier transform as an estimator of Fourier coefficients of periodic signals from observations disturbed by covariant noise is asymptotically best linear unbiased. This also implies that the discrete Fourier transform asymptotically attains the Cramér–Rao lower bound on the variance if the errors are normally distributed.

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A Method to Measure the Aperture Field and Experimentally Determine the Near-Field Phase Center of a Horn Antenna

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Abstract—**A method whereby the aperture field distribution of a horn antenna can be measured is proposed. The method is based on the measurement of the scattered field of a small conducting sphere in the aperture. The position of the phase center of the aperture field is calculated using the measured phase distribution of the field in the aperture via a simple type of convolution. In principle, the method is suited to determine the field distribution of other types of apertures illuminated by an electromagnetic field.**

I. Introduction

Knowledge of the field distribution in the aperture of a horn antenna is essential when analyzing the characteristics of the antenna, especially if the far-field pattern or phase center of the far-zone radiated fields is important.

Assumptions regarding the amplitude distributions are often based on the dominant mode of the waveguide feeding the antenna. In the case of a rectangular horn, the dominant mode of the waveguide is TE_{10}, resulting in a cosine distribution in the E-plane and a constant distribution in the H-plane [1]. Quadratic models are used for the phase distribution in the aperture.

In this paper, a method is discussed whereby the amplitude and phase distribution of the field in the aperture of a horn antenna can be measured. As an example, results are presented for a rectangular X-band horn antenna. The results are then used to empirically determine the position of the phase center of the field in the aperture.

The phase center of the fields in the aperture of a horn antenna can be defined by considering the feed of the horn antenna as an equivalent point source within a certain angular region. This is valid only if equiphase surfaces of the aperture field are concentric spheres in this region. The phase center of these equiphase spheres is defined as the phase center for the radiating source. However, in nature, no true point source exists; therefore, a phase center is uniquely defined only in a particular plane and over a specific angular sector. Thus, a phase center does not result in a point in space, but rather in a line. It is therefore possible to define two phase centers for the field in the aperture of a horn antenna: one that corresponds to a line source in the E-plane, and one in the H-plane. The location of the aperture fields' phase center is required when using the UTD to calculate the far-zone radiated field, for example [2].

In principle, the method is suited to, or can be adopted to, determine the aperture field distribution of other types of apertures illuminated by an electromagnetic field, e.g., a parabolic antenna. The method is especially useful in cases where it is difficult to determine the aperture distribution analytically [3].

Manuscript received August 29, 1991; revised June 4, 1992.
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IEEE Log Number 9203661.

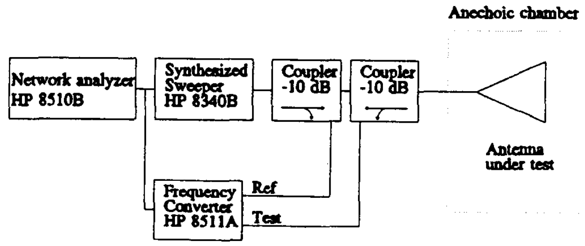

Fig. 1. The setup to measure the field in the aperture of a horn antenna.

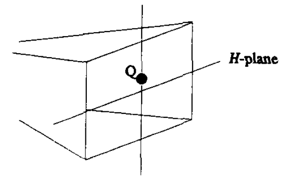

Fig. 2. A small conducting sphere at a position Q in the aperture of the horn antenna.

II. Measuring the Aperture Distribution

The measurement of the amplitude and phase of the field in the aperture is based on the measurement of the scattered field of a small conducting sphere that is introduced into the aperture for the purpose of the measurement. The measurements were performed in an anechoic chamber using an HP8510B vector network analyzer, as shown in Fig. 1. The powerful mathematical capabilities of the network analyzer were found to be very useful. Instead of an S-parameter test set, a frequency converter and two directional couplers were used to measure the S-parameters of the antenna under test in the anechoic chamber.

The measurement is performed by measuring the scattering parameter S_{11} with the network analyzer while the antenna under test is transmitting. This measurement of the amplitude and phase of the background is stored in the network analyzer's memory. A small conducting sphere is then positioned in the aperture of the horn at a known point Q, as shown in Fig. 2. Since one does not want the presence of the sphere in the aperture to distort the value of the field, a small sphere is desired. The scattering parameter S_{11} is measured again, this time with the sphere in the aperture. The background data are then subtracted with the vector math capability of the network analyzer. The result is then the scattered field (amplitude and phase) from the sphere at a known point Q in the aperture of the horn antenna. This field is, in effect, proportional to the field at Q in the aperture.

III. Example—X-Band Horn Antenna

The field in the aperture of a horn antenna with dimensions l_e = 272 mm, θ_e = 14.04°, l_h = 316 mm, and θ_h = 12.3° was measured. The parameters l_e, θ_e and l_h, θ_h are the flare length and half flare angle in the E- and H-plane, respectively.

Three conducting spheres with different radii (6.35, 4.75, and 3.85 mm diameter) were used to measure the field. One would ex-