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Nonlinear Model Reduction of Cable Slab Dynamics

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TECHNISCHE UNIVERSITEIT DELFT



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MASTER THESIS

Nonlinear Model Reduction of Cable Slab Dynamics

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Abstract

In high precision motion systems, rather flexible cable slabs provide electrical power and cooling liquid to moving stages. The positioning accuracy of free floating stages is defined by the disturbance action on it. So, it is of utmost importance to accurately quantify the disturbance forces of these cables. The cable slab is a highly flexible structure, hence the mathematical model of such a system must feature geometric nonlinearities due to finite displacements and rotations. The computational cost of such a model is often prohibitive, therefore model order reduction is needed to mitigate the cost of computation. However, the present reduction techniques are not adequate to model such a degree of nonlinearity.

This thesis presents an attempt to develop and apply a reduction technique that is applicable to the cable slab system. Firstly, a consistent theory of nonlinear model reduction is developed. The theory is combined with the concepts of the well known linear reduction technique, the Craig-Bampton component mode method to obtain the so called Generalized Craig-Bampton (GCB) method, applicable to geometrically nonlinear systems which is then used to reduce model of the cable slab.

The GCB technique allows the small amplitude internal dynamics of the cable slab to be linearized. This is exploited towards developing an efficient numerical integration procedure: a two stage offline-online scheme which allows for near real time simulation during the online stage. Interpolation is employed to map the solutions from offline to the online stage. The challenges in implementing such a procedure is addressed and resolved.

Finally, the GCB model is validated against experiment results of a cable slab setup. A good agreement with the experiments is shown.

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Nomenclature

Symbols

$\mathbf{u}$	discrete nodal coordinates	$\bar{\mathbf{E}}$	assumed strain field
$\mathbf{q}$	Generalized reduced coordinates	$\hat{\mathbf{S}}$	assumed stress filed
s	static mode	$\mathbf{N}$	displacement shape function
$\mathbf{x}$	eigenmode	$\mathbf{N}_E$	assumed strain shape function
$\hat{\mathbf{u}}_b$	applied boundary mode	$\mathbf{N}_S$	assumed stress shape function
$\mathbf{f}^{int}$	internal forces	W_0	internal stored energy
$\mathbf{f}^{ext}$	external forces	$\mathbf{b}$	body forces
$\mathbf{f}^d$	dissipation force	$\mathbf{t}$	traction forces
$\mathbf{M}$	mass matrix	V	volume of solid
$\mathbf{K}$	stiffness matrix	A	surface area
$\mathbf{C}$	damping matrix	E	Young's modulus
$\mathbf{v}$	displacement field	t	time
$\mathbf{E}$	Green-Lagrange strain field	$\mathbf{I}$	identity matrix
$\mathbf{B}$	element approximation of strain variation		
α	discretized assumed strains	μ	modal damping coefficient
β	discretized assumed stress	ζ	Generalized prescribed displacement
Γ	Parameterized manifold	ν	poisson's ratio
Φ	Tangent reduction basis	ρ	mass density
$\mathbf{X}$	Eigen mode basis	ϕ	general mode
Ω	Diagonal eigenfrequency matrix	ω	eigen frequency
η	Modal amplitude of eigenmode		
$(\bullet)_i$	internal node partition	$\dot{(\bullet)}$	time derivative
$(\bullet)_b$	boundary nodes partition	$(\bullet)^T$	matrix transpose
$(\bar{\bullet})$	static solution	$(\bullet)_e$	element partition
$(\bullet)^h$	finite element approximation of the field	$\Delta(\bullet)$	change in variable
$(\bullet)_{,y}$	partial derivative with respect to variable $\mathbf{y}$		

Abbreviations

MOR	Model Order Reduction
GCB	Generalized Craig Bampton
FEM	Finite Element Method
CR	Co-Rotational
SVD	Singular Value Decomposition
POD	Proper Orthogonal Decomposition
MD	Modal Derivatives (method)
dof	degree(s) of freedom

Introduction

1.1 Background

Precision stages (for e.g. wafer stages) are one the key components of modern mechatronic systems. It needs to be extremely fast and highly precise at the same time which is critical in keeping the overall productivity of these systems at an acceptable level. For this extreme criteria to be met the stages needs to be isolated from its surroundings so as to not induce any disturbances. However this cannot always be achieved since in some cases they require power, coolant, process gas etc, which is usually done via a cable slab. Since the cables are flexible, they induce dynamic disturbance forces at the contact points which can adversely affect the performance of the entire stage. Thus it is very crucial to be able to quantify these disturbance forces in order to compensate for it.

The main difficulty lies in characterizing the dynamics of the cable slab, which is nonlinear and therefore hard to describe using a simple model. Although accurate nonlinear modeling techniques exist today, considerable effort is still being put towards reducing the model. This is primarily due to the need for developing an efficient model to enable rapid prototyping of the cable slab subsystem. This is the current challenge in this area.

1.2 The Cable Slab System

A cable slab (see Figure 1.1) as the name suggests is flat bundle of wires or hoses which are rigidly bonded to each other. A cable slab system can consist of several cable slabs arranged in stacks along its thickness to form a single unit in which the slabs may or may not be rigidly connected to each other using external connectors at distinct locations. The cable slabs are rigidly constrained at its ends to two different stages which move relative to the other. The displacements of these stages is usually very large, hence the cable slab undergo finite deformations. The imposed stage displacements can also be fast. The accelerations can go up to the magnitude of 40g. The cable slab can be placed

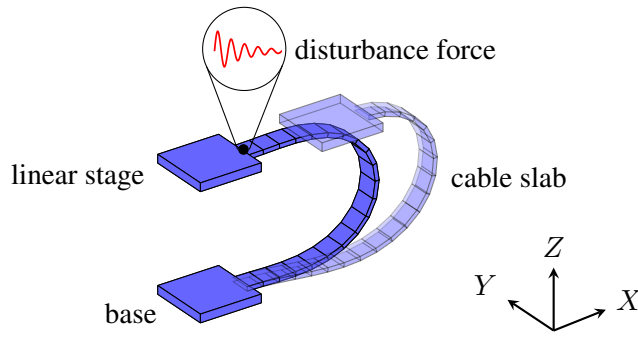


Figure 1.1: Depiction of a cable slab system in motion. The disturbance force is generated by the cable slab as a reaction to the imposed displacement of the linear stage.

either horizontally or vertically with respect to its width. The thickness of each slab is usually very small compared to its width and length.

Being a composite structure consisting of polymers and metal wires, the exact description of the material behavior of the cable slab is in general anisotropic and hard to determine. It usually has a low average modulus of Elasticity (order of 10^7Pa) to allow high flexibility. It also exhibits viscoelastic behavior.

Additional clamps may be placed on the sides of the cable slabs along its length without rigid connection to constrain the shape of the cable slab in order to improve its overall stiffness. Since the cable slab is not rigidly attached to it, there is a chance for the cable slab to impact the clamps during high frequent motions.

This Section is summarized in Table 1.1.

Geometric description	undergoes finite deformations
Material behavior	anisotropic, exhibits viscoelasticity
Applied boundary Conditions	imposed displacements on stages, contact constraint due to external clamps

Table 1.1: Brief description of the cable slab system

1.3 Motivation

Almost any complex structural problems can be effectively solved nowadays using the nonlinear finite element method. But nonlinear analysis is usually reserved until the final design phase of a project, to be used merely as a validation tool. This is primarily due to the following two reasons: a) Nonlinear analysis is computationally expensive making it time consuming for large design iterations and b) the results obtained from it are not intuitive in general and offer no physical insight or interpretation of the system under study. On the other hand linear analysis overcomes the above limitations and is usually the preferred tool for initial design phase. However in many cases linear models

are simply not applicable to the problem at hand due to the large influence of the system nonlinearities and a nonlinear model has to be resorted to. Therefore in order to alleviate the computational cost of nonlinear analysis in such a case, model reduction techniques are needed. Unfortunately, nonlinear model reduction is very challenging especially for geometrically nonlinearity and the techniques used today are still in their infancy.

The need for a fast, reliable and accurate nonlinear reduction technique for the simulation of the cable slab dynamics is the primary motivation of the work presented here.

1.4 Objectives and Scope

The objective of the work is to propose an accurate model reduction technique for a given nonlinear cable slab model and derive an efficient integration scheme for the reduced model capable of simulating the disturbance forces due to the stage motion. A very general or a complex model of the cable slab is not considered here. Instead, this work focuses solely on the geometric nonlinearity aspect of the cable slab. Any other influence is either neglected or approximated. Moreover, only special cases of cable slab excitations are considered here in order to further simplify the process.

An importance is also placed on the intuitiveness of the reduced model and its ability to yield physical interpretation of the results. Finally, experimental validation of the proposed reduced model and the assumptions is also another key goal of this work.

The scope of this work and the main assumptions made here can be summarized as follows.

1. Only geometric nonlinearities are considered. A linear isotropic material model is assumed. The nonlinear contact problem due to clamps is not considered.
2. A linear modal damping is assumed.
3. Only imposed stage displacement in the X direction is considered (refer Figure 1.1).
4. The effects of gravity is not modeled.
5. The amplitude of the dynamic oscillations of the cable slab about its static configuration is assumed to be small.

The assumption made in point 5 is one of the keys aspects of the work. It is pivotal towards developing a successful reduced model of the cable slab. This is elaborated in the Chapters 5 and 6.

1.5 Challenges

The geometric nonlinearity of the cable slab is the biggest challenge towards developing a successful reduced model of the cable slab. Standard projection based reduction techniques cannot be applied in this case. This applies even to methods that are used for nonlinear problems like the Proper Orthogonal Decomposition (POD) and Modal Derivatives (MD). The reason behind this lies in the assumption that the solution of the dynamic problem lies in a vector subspace of the configuration space. This approximation does not hold for structures undergoing finite deformations which cannot be represented simply as linear combination of a few basis vectors. Thus a completely different approach is required.

If an updated basis method is used where the reduction basis is regularly updated to describe the local dynamics at a given configuration, the challenge now becomes to efficiently and reliably derive these basis since they are usually computationally expensive to obtain.

The need for a model that offers good physical interpretation also puts some constraint on the range of techniques that can be used. It restricts the type of basis vectors that can be selected. Usually eigenmodes are selected since they give a good interpretation, but they do not evolve smoothly with the configuration and require preconditioning in order to implement them in an updated basis method.

Finally, experimental validation of the reduced model is also another challenge. The viscoelastic and damping properties of the cable slab are largely unknown and thus not accurately modeled. But these properties have a huge effect on the behavior of the cable slab which could adversely affect the response predicted by the model. Thus, it is very important to be able to distinguish the spurious effects due to these properties from the desired response while performing the experiment and to be able to quantify the uncertainty due to it.

Therefore the four main challenges that are addressed in this work can be summarized as follows.

1. Formulate a new theory of model reduction consistent with finite deformations.
2. Develop an efficient numerical scheme to integrate the reduced model.
3. Implement a conditioning algorithm that maintains the fidelity of the basis.
4. Identify spurious effects in the experimental results and attempt to compensate it or quantify the uncertainty due to it.

1.6 Thesis Outline

The contents of this thesis may be divided into 3 parts: theory, implementation and experimental validation. First, an extensive literature survey is carried out in Chapter 2 where various topics related to FEM modeling and model reduction is discussed. In Chapter 3, the finite element modeling of the cable slab is discussed in detail. In Chapter 4, the modal sensitivities of various important quantities is discussed. In Chapter 5, a new theory of model reduction consistent with finite deformations is introduced. This theory is then applied to develop a reduction technique known as Generalized Craig Bampton or GCB method which is an extension of the well known Craig Bampton technique.

In Chapter 6, a fast and reliable numerical integration scheme is discussed that implements the reduced GCB expressions. Finally, in Chapter 7, the reduced model is validated against the fully nonlinear model and experimental results from a test setup and the conclusions and recommendations are given in Chapter 8.

1.7 General nomenclature

The following are some of the notations that are followed in this thesis. All scalars are denoted using a normal font. Column vectors are indicated using bold symbols or bold lowercase alphabets. Matrices and tensors are denoted by bold uppercase alphabets. Dot product of tensors is indicated simply by placing the two quantities next to one another in an expression and sometimes by a placing a period in between. Double dot product is denoted by the symbol $(:)$.

Partial derivatives of any quantity $\mathbf{y}$ with respect to a quantity $\mathbf{x}$ is given as $\mathbf{y}_{,\mathbf{x}}$. Time derivatives are denoted by the symbol $(\dot{\bullet})$. Higher order derivatives are sometimes denoted by the superscript $(\bullet)^i$, where i denotes the order of the derivative.

Literature Survey

2.1 Nonlinear Finite Element Procedures

This section presents a study of the various Finite Element Method (FEM) formulations in existence with the key focus on effective and efficient modeling of the cable slab system. The objective is to do broad comparison of the existing modeling methods so that the reader who wishes to model cable slabs will be able to make an appropriate choice for the method. This survey is restricted to geometrically nonlinear analysis of beam and shell elements. Treatment of other nonlinearities are not considered here.

2.1.1 Degenerate versus Solid formulation

Any general purpose beam or shell formulation can be broadly classified as either degenerate or solid. A degenerate formulation means that the cross-sectional deformations are described analytically which are approximations of the actual deformation. Based on the order of the approximations degenerate formulations can be further classified as Classical or Zeroth Order Shear Deformation Theory (also known as Euler Bernoulli theory for beams [1] and Kirchoff-Love theory for shells and plates), First Order Shear Deformation Theory (also known as Timoshenko Beam theory for beams), Third and Higher Order Deformation Theory [2, 3]. Higher the order, better is the description of the true cross-sectional deformation.

The main advantage of using a degenerate formulation is that the cross-section need not be discretized and any complex cross section geometry can be explained easily. This however comes at the price of the a priori kinematic assumptions made which makes it less general purpose and sometimes more cumbersome to deal with in a nonlinear framework. The choice of the order of theory that should be selected generally depends on the thickness of the shell or beam. Thicker structures require a higher order description since shear force effects are more dominant in this case.

A solid formulation [4, 5] is the case where the cross-section is discretized as well. Due to the high aspect ratio of the elements, shear locking is the major concern of such a formulation. This can be dealt with by using mixed methods and assumed

strain interpolations (see next section). Solid formulations are robust since no a priori kinematic assumptions are needed to describe the cross-sectional deformation. This means that there is no restriction on the order of the bending strains and a fully non-linear formulation can be implemented (1D finite strain theories do exist but are not very popularly used; see Reissner [6]). Moreover rotations disappear from the formulation which makes integrations of 3D problems and coupling to other structures much simpler. The main disadvantage however is that the cross-section needs to be discretized which can dramatically increase the number of degrees of freedom of the system and the system needs to be re-discretized for every geometry change which can be a huge drawback for optimization problems.

2.1.2 Shell versus Beam Elements

To state it plainly, a beam is a special case of a shell element where the transverse deflections are a function of only one direction (along the beam axis) instead of two. The second direction is degenerated and the behavior is described analytically. This results in unique applications of each of the two structural elements. Since the cross-section of a beam is analytically described, complex geometries can be easily and flexibly modeled. Whereas for a shell element, only the thickness parameter can be varied. There are a few cases in which either a beam or a shell model can be applied such as thin beams (not to be confused with thin walled beams). In such a case there is only one essential difference between choosing a beam or a shell element which is the effect of lateral contraction due to Poisson's ratio which is accounted for in shells but not in beams. Lateral contraction may reduce the stiffness of the structure which otherwise might appear slightly stiffer. Other effects like warping and torsion bending coupling are also not generally accounted for in a beam but these can be neglected if the cross-section is rectangular and thin.

2.1.3 Shear and Membrane Locking Phenomena

In most FEM element beam and shell models a C_0 continuity is applied for to take advantage of isoparametric mapping. Although this might be enough to satisfy the convergence criteria the resulting formulation might contain spurious energy residuals. Membrane and shear locking is a case that occurs commonly in thin structures where a full integration of the total stiffness matrix leads to over-stiff solutions. Luo [7] gives an excellent explanation of the general locking phenomena in FEM formulations. According to the author low order interpolations (e.g. isoparametric interpolation) of the independent variables used in element formulations is the main reason for locking to occur. In general low order interpolations used in the case where the transverse displacements and rotations are coupled leads to shear locking. Similarly, when in-plane displacements are coupled with section rotations and a low order interpolation is used, membrane locking

will occur. The author uses a field consistent approach to derive higher order interpolation functions and the resulting matrices obtained are proved to be devoid of shear and membrane locking. The field consistent approach can also be used as a tool to determine whether a formulation is prone to locking or not.

Other more popular ways of preventing the locking phenomenon are reduced integration [8, 9], assumed strain interpolation/mixed interpolation method [2, 4, 10, 11], enhanced displacement method [12] and the mixed field method [2, 4, 13]. Reduced integration is the simplest and the least effective of all the methods. It avoids shear and membrane locking but introduces other issues. Bucelem and Bathe [11] briefly argue about the inefficacy of reduced integration methods. Uniform Reduced Integration and Selective Reduced Integration are two of its types out of which the former has more prominent numerical issue of spurious zero energy modes which can lead to rank deficiency in the global stiffness matrix. The later has the same issue but to a lesser degree. Another issue with the reduced integration techniques is that they are sensitive to geometric distortions.

In mixed field methods the stress and the strain field are taken as independent variables and the dependency on the constitutive law and the displacements is enforced using lagrange multipliers. The mixed method is very popular in dealing with incompressibility problems (Bathe [2]) and also prove to be very effective in dealing with locking phenomenon and high distortions. A complementary method of the mixed field formulation is the mixed interpolation method or in other cases assumed strain method. In these methods the strains are not directly derived from the displacement field but instead interpolated from an assumed strain field. These methods are also very robust in dealing with locking and severe element distortions.

2.1.4 Nonlinear FEM Solution Techniques

The three main nonlinear FEM solution techniques that are in use today are Total Lagrangian (TL), Updated Lagrangian (UL) and Co-rotational (CR) formulation. TL and UL are equivalent methods which ideally give the same results in the end regardless of the underlying theory used. Hence the choice between the two boils down to other heuristic needs like computational efficiency or numerical stability for a given problem. They are also the most widely used of the three methods.

Both Total and Updated Lagrangian have been very successful in dealing with nonlinear mechanical problems. The choice between the two depends on the nature of the problem and the requirement. According to the notes by Kouznetsova [14] the choice between TL and UL can depend on the constitutive law used. If the constitutive equations are formulated using tensors that are defined in the current configuration (e.g. the Cauchy stress tensor, Euler-Almansi strain tensor e.t.c) then the Updated Lagrange

formulation will have simpler expressions which are easy to compute, and same is the case for Total Lagrange formulation when the tensors are defined in the reference configuration. UL formulations can be used in the case where constitutive laws may be formulated in a rate form. Calculation of these rate-type quantities at time $t + dt$ clearly requires knowledge of the relevant tensors at the last iteration at time t . Implementation of these models is therefore easier if the updated Lagrange formulation is adopted.

The Corotational (CR) description is the most recent of the three and the least developed one. Felippa and Haugen [15] present a unified formulation of small-strain CR theory. The main a priori kinematic assumption made in this formulation is that the strains in the structure are small. The displacements and rotations experienced can be finite however. This is the main drawback of the formulation that has limited its widespread use. But it does have certain unique capabilities that make it attractive in some cases. One class of CR formulation is the Element Independent Corotational (EICR) description. Here the main idea is to separate the rigid body displacement of each element from its deformational displacements and process it individually and recombine to get the resultant displacement. The decomposition of the displacements acts like a pre-processing step which can be performed outside the standard element routines and thus is independent of the element type. This means that large rotation effects can be incorporated into linear FEM code without major alterations to the original script. This could save a lot of time and effort needed to upgrade an existing formulation valid only for the small rotation case. Another benefit of CR formulation is the cheap computation of the total stiffness matrix of the system compared to TL and UL.

Le *et. al.* [16] derives an efficient CR formulation for the dynamics of Bernoulli beam. Xu [17] derived it for the static case of Timoshenko Beam. Le *et. al.* [18] wrote another paper in which the authors point out the difficulty in implementing dynamics in 3D CR formulations. One reason is that is that the decomposition in rigid body and deformational parts leads to very complicated expressions for the dynamic terms. Another reason is that finite rotations in 3D are non commutative which means that a Newmark step cannot be directly applied. Therefore, it means that the method must be reformulated according to the parametrization of the finite rotations. Numerous techniques have been developed in this area [19–22].

Many CR Shell formulations also exist and have been proposed for static cases by Li *et. al.* [23, 24], Polat [25], Yang and Xia [26] to name a few and or dynamic problems by Satish[27] and Almeida and Awruch [28].

CR analysis is popularly used in modeling flexible multibody systems that couple rigid body motions with small elastic deformations. SPACAR [29] is an example of one such software. Such techniques also have a potential in model reduction. It allows for a linear description of the small amplitude dynamics in a system which is proven to be valuable for constructing efficient reduced models. One such application is described by

Boer [30]. Here, the author uses a 2 node linear super-element which is coupled using a component mode technique to obtain the assembled system. Euler parameters are used to parameterize the orientation of the CR frame. A SPACAR code is used to integrate the equations.

2.2 Nonlinear Modal Order Reduction Techniques

The idea of modal order reduction of nonlinear systems is desirable since they are computationally expensive to solve. But it is considerably more challenging compared to model reduction of linear systems. The techniques available can be classified based on the type and degree of nonlinearity present in the system.

Local nonlinearity

For systems with localized nonlinearities, a substructuring technique by Bathe [31] can be used where the linear degrees of freedom are condensed onto the nonlinear ones and equilibrium iterations are carried out on the reduced system. In the case where the nonlinear dofs cannot be easily separated, a technique suggested by Noor [32] can be used where a nonlinear node is identified by comparing the value of the higher order derivatives of the displacements with that of the first order at that node. If the magnitudes are comparable, then the node can be classified as a nonlinear node else it can be approximated as linear.

Moderate nonlinearity

For moderate geometrically nonlinear problems, two popular techniques exist: Proper Orthogonal Decomposition (POD) and Modal Derivatives (MD). A comparison of the two methods is given by Tiso and Rixen [33]).

A reduction basis is usually obtained from the linearized system of equations around a certain equilibrium configuration. Such a basis acts like a local linear approximation about the equilibrium point and can thus be termed as tangent reduction basis. MD utilizes higher order derivatives of the deformation modes in order to enrich the basis. This acts like a second (or higher) order approximation of the local behavior thereby extending the subspace to span nonlinear effects. The concept is based upon a method proposed by Noor [32]. The author developed an effective reduction basis to solve nonlinear static problems by using path derivatives of the static solution as the basis vectors. The path derivatives are obtained by successive differentiation of the internal force vector with a so called nonlinear path parameter.

This idea of path derivatives was later adopted by Idelsohn and Cardona [34] who applied it to an eigenmode basis and enriched it by including the derivatives of the modes

with respect to the modal amplitudes. To circumvent the singularity of the dynamic stiffness matrix in eigen value problems, the authors approximated the expressions by neglecting the inertial terms. Numerical experiments have shown that this approximation does not change the results. Slaats *et. al.* [35] later proved this analytically. This greatly simplifies the computation of the derivatives. The basis obtained using modal derivative method gave much better results compared to the regular tangent eigenmode basis. The authors later applied this concept to the reduction method proposed by Wilson *et. al.* [36] which uses ritz vectors rather than eigenmodes as a basis, [37].

Some important differences exist between eigenmodes and Ritz vectors which determine their applications. Eigen modes are mass orthogonal whereas Ritz vectors are not. Ritz vectors are computationally cheap to obtain compared to eigenmodes. Ritz vectors is better suited as basis when the loading of the structure is very complex. Eigen modes can be interpreted as the actual resonant mode shapes of the structures. In the case of Ritz vectors the first mode is the pure static response of the structure to a given load, the second mode is the static response to the inertial forces generated by the first mode, and in similar fashion the remaining modes are static responses of the inertia forces due to the $i - 1^{th}$ mode.

Further work has been done in this area by [33, 38, 39]. Tiso *et. al.* [40] presents a method to compute the modal derivatives in shells using a perturbation method. One drawback of the MD method could be that for a basis of n vectors, the possible addition of its derivatives is in the order of n^2 which could quickly cause the basis to become too large. In [38], Tiso addresses this and proposes a way to select only the optimal derivatives.

The other popularly used reduction method for moderately nonlinear problems is POD. It constructs a reduction basis by decomposing a discrete time series data of the particular process into a summation of its dominant modes. Unlike MD which is an analytical method, POD is a statistical technique that requires the knowledge of the solution of the system rather than the system itself. Chatterjee [41] gives an introduction to the POD technique. It utilizes the Singular Value Decomposition (SVD) procedure to decompose the solution. Provided with k solutions values at distinct time instances of an n dimensional system, an SVD procedure can be interpreted a minimization problem which seeks a m -dimensional subspace for which the mean square distance of the p solution points, from the subspace, is minimized. POD can robustly capture nonlinear modes that are not represented by tangent basis. Chatterjee illustrates this by applying it to a vibro-impact problem and extracts the impact modes of the system which are highly nonlinear.

SVD is strongly related to eigen value decomposition. Suppose A is a $n \times m$ matrix and an SVD is performed on it ($A = USV^T$), the resulting orthogonal matrices U ($n \times n$) and V ($m \times m$) obtained are the eigen vector basis of the matrices AA^T and $A^T A$

respectively and S contains the square roots of the eigenvalues of both the matrices. When A is symmetric and positive definite then its eigenvalues are also its singular values, and $U = V$.

In Getan *et. al.* [42] the authors compare the Proper Orthogonal Modes (POM) obtained using POD to the eigenmodes of a system. In general the POM bear no semblance to the eigenmodes. POM are orthogonal and eigenmodes are mass orthogonal. Only when the system is excited at its eigenfrequency the POM is forced to converge to the particular eigenmode. The only special case when both matrices are the same is when the Mass matrix of the system equals the identity matrix.

Although POD has been successfully applied in the field of nonlinear MOR, it is in general an expensive procedure as pointed out by Amsellem [43]. This is mainly because to obtain the reduced basis an entire time simulation has to be carried out for a large number of iterations. This can be very computationally inefficient if the system has some parametric dependencies that needs to be studied.

Finally, a third less popular method which can be thought of as a substitute of MD described by Boer [30]. Here the author constructs a reduction basis by combining the tangent basis obtained at several distinct equilibrium configurations and orthogonalizing it. Thus it allows the basis to capture the dynamics of a range of configurations.

High nonlinearity

The model reduction of structures undergoing finite deformations/displacements are the most challenging and the least developed of all methods. Such structures cannot be represented by the linear combination of a few basis vectors. Hence, using a constant basis fails in this case. Instead, an updated basis method can be applied where the reduction basis is updated regularly to adapt to the local dynamic behavior of the system at any given instant. One of the first attempts at this was made by Idelsohn and Cardona [34]. In addition to the high cost of computing the basis at every iteration, the authors demonstrate that updating the tangent reduction basis at every iteration introduces a truncation error which grows exponentially with every time step. This was due to incompatibility between the old and the new basis. Thus, a pure updation of the tangent modes is unpractical. The authors therefore propose the adding of modal derivatives to improve the quality of the basis and thereby reducing the frequency of required updates. They further propose an algorithm to mitigate the effects of the truncation error.

A consistent nonlinear reduction method suitable for flexible multibody structures is proposed by Br uls [44]. The author introduces the concept of Global Modal Parameterization (GMP) which is used as a kinematic description of the reduced model. Small strain are assumed to linearize the displacements due to dynamic elastic deformations.

The GMP is approximated using a metamodel constructed using piecewise functions. An equivalent technique with the same applications is proposed by Boer [30].

2.3 Derivatives of Eigenmodes

As described in section 2.2, modal derivatives have useful applications in nonlinear model reduction. Hence this section will focus on the different methods used to compute them. The simplest way to derive the eigenmode derivative would be using finite differences. But this is usually computationally expensive and its accuracy is limited by the proper selection of the step size. Iott 45 talks about how to decide the right step sizes for finite difference sensitivity analysis. A more numerically stable solution can be derived by referring to the governing expression for the derivatives. By differentiating the eigen value problem with respect to a given parameter, a linear expression for the eigenmode derivatives can be obtained. However this expression cannot be solved directly due to the singular nature of the eigen problem matrix (Or the dynamic stiffness matrix). Therefore numerous techniques have been developed over the past couple of years to deal with this issue and derive the modal derivatives. They can be broadly classified into 3 categories, Algebraic, Modal and Iterative methods. Each of them have their own unique advantages and disadvantages. This is summarized in Table 2.1.

2.3.1 Algebraic Method

Algebraic methods get around the singularity issue by explicitly manipulating the eigen problem matrix to remove the singularity. The most popular algebraic method was first proposed by Nelson [46]. The solution derived using Nelson's method is exact. A major advantage of this method is that only the knowledge of the eigenmode whose derivative is desired is needed and this saves a lot of numerical effort in computing unnecessary higher modes (which are required by other methods, see sec. 2.3.2). Nelson's method also preserves the banded form of the dynamic stiffness matrix unlike other algebraic methods which is an other advantage.

The main disadvantage of Nelson's method however is that the modified coefficient matrix has to be decomposed for every eigenmode, which makes is very computationally expensive if a large number of eigenmode derivatives are required. Fetterman [47] tries to address this issue by proposing reduced methods to solve Nelson's expression but does so with limited success. Another disadvantage is that this method is applicable only in the case of a system with distinct eigenvalues although modifications of this algorithm does exist to account for this (see sec 2.3.4). Another limitation can be that the method requires the eigenmodes be mass normalized and the derivatives are also computed such that the mass normality is preserved. Siddhi [48] proposes a modified Nelson's method to compute eigenmode derivatives for different eigenmode normalizations.

Algebraic methods to derive the second order derivative has been formulated by Friswell [49]. Extension to non-conservative systems has been carried out by Adhikari and Friswell [50] and Guedria *et.al.* [51].

2.3.2 Modal Method

The modal method was first proposed by Fox and Kapoor [52]. This method proposes the analytical solution of the eigenmode derivative as a linear combination of all the eigenmodes of the system. But this method already suffers the major drawback of having to compute all the eigenmodes which is computationally expensive. Hence usually a truncated eigenmode basis is chosen so as to give an approximate solution of the derivatives. The selection of the truncated basis is not straightforward and the convergence of the solution is inefficient as it can be controlled solely by the number of modes computed. Thus the truncated method proposed by Fox and Kapoor is usually not sufficient.

Fortunately, there have been several developments over the years to improve the accuracy of the truncated modal method without expensive procedures. Wang [53] proposed two methods, one explicit and another implicit. Both of these methods involve adding a correction term to the truncated series to partially account for the neglected eigenmodes. The concept used here is similar to the one used in the mode acceleration method. The implicit method is an extended version of the explicit method in which the correction term is further improved at a minor additional cost. Another work in this area was carried out by Liu *et.al.* [54, 55]. Here the authors added a convergent infinite series as a correction term. The correction term if summed up to infinity completely accounts for all the truncated modes. Thus this method can be thought of as an iterative method rather than modal. In most cases only the first few terms of this series need to be computed to obtain good accuracy. Liu's second method [54] is an extension of the first method [55] which again provides better accuracy for minor additional steps.

Yu *et.al.* [56] has done a comparison of the above mentioned methods with regards to its accuracy versus efficiency. According to the author's conclusion the second method of Liu [54] has the best convergence efficiency for a given accuracy followed by Wang's implicit method [53], Liu's first method [55], Wang's explicit method and finally regular truncated modal method.

Further work has been done by Adhikari and Friswell [57] in extending the modal method to the first-order and second-order derivatives of the eigen solutions of asymmetric damped systems. Modal methods like the algebraic methods require that the system has distinct eigen values. If this is not satisfied then other modified methods have to be used which is discussed in sec 2.3.4.

2.3.3 Iterative Method

When the number of eigenmode derivatives that need to be computed are large, then an iterative method can prove to be more advantages. As discussed in the previous section, Liu's method acts like an iterative method. A similar approach of successive correction of modal method was proposed by Zhang and Zerva [58], where they extend Wang's method [53] to an iterative procedure. The authors showed that the first iteration of their method gives the same correction term as Wang's explicit method. The authors further improved their method and developed an accelerated algorithm to give better convergence speed especially when the eigen frequencies are very closely spaced [59]. Alvin [60] applies a Positive Conjugate Projected Gradient (PCPG) algorithm to Zhang's method and thus achieves better convergence. The author also compares the cost of convergence with that of Nelson's method and concludes that the PCPG method is in general more efficient when the degrees of freedom of the system is higher than 1000. The author also compares the PCPG method with Zhang's method and concludes that the former is more superior compared to the later.

Other important iterative methods include the method proposed by Rudisill and Chu [61] and later improved by Tan [62, 63].

2.3.4 Eigenmode Derivatives of Systems with Repeated Eigenvalues

All of the above mentioned methods described above either fail or show poor convergence when repeated eigenvalues are present in the system. Therefore there are several independent and modified methods to deal with this issue. Friswell [64] proposes an extended version of Nelson method to tackle repeated eigen values and also shows that in general the eigenmode derivatives are discontinuous functions of two or more design parameters. Other algorithms have been developed to obtain the derivatives only for the special case of repeated eigenvalues by many authors including Juang *et.al.* [65], Bernard and Bronowiki [66], and Lim *et.al.* [67] to name a few.

2.4 Interpolation of Reduced Order Basis and other Matrices

This section discusses the work done by Amasalleem and Fahrat (*et. al.*) in the area of interpolation of reduced order basis [43, 68–70]. As it was discussed in the previous section a reduction basis usually represents only the local linear/nonlinear behavior of the system and it needs to be updated for large displacements (or other parameters). This is a serious drawback since the reduction basis are usually numerically expensive to compute. One way to overcome this limitation would be to interpolate the reduced basis instead. These basis usually have very important property which is orthogonality which

Method	Pros	Cons	When Efficient
Algebraic	• Solution is exact • Only eigenmodes whose derivative is required are needed	• Coefficient matrix needs to be decomposed for every mode • Can face problem with ill conditioning	The dof and number of modal derivatives required are small
Modal	• Depending on the problem the derivatives can be obtained very cheaply and accurately	• Poor Convergence • Usually knowledge of eigenmodes whose derivatives are not needed is required	The truncated modes do not strongly interact with the retained modes
Iterative	• Has good convergence rate • Only eigenmodes whose derivative is required are needed • Decomposed matrices can be reused	• Computationally expensive for small problems • Convergence rate usually suffers when eigen frequencies are close	The dof and the number of derivatives required are large

Table 2.1: Comparison of different methods to derive eigenmode derivatives

needs to be preserved. A general interpolation scheme does not guarantee this. Thus the authors propose a special interpolation scheme which preserves the orthogonality of the reduced basis upon interpolation. To differentiate it from standard interpolation techniques the proposed method will be termed as *Manifold* interpolation from hereon.

In [70], the authors derive the Manifold interpolation method which is based on the concept of differential geometry. To state it simply the reduced basis is first projected to a "flat" constraint free space where the orthogonality is always satisfied. Once the basis have been projected to this space it can be interpolated using any conventional interpolation technique and the resulting matrix is projected back to the original space to give the final interpolated basis. The authors then validate the results of the proposed method on a CFD-based aeroelastic problem of a fighter jet with the reduced basis obtained via POD for different Mac numbers. Excellent results were demonstrated for the interpolated basis.

In [69], the authors further extend the scope of the method towards matrices with other properties namely symmetric-positive definiteness (SPD) and mass orthogonality. This can be very useful in nonlinear structural dynamics which have stiffness and mass matrices which possess the SPD property and eigenmodes which possess the mass orthogonality property. However the algorithm proposed here for interpolating eigenmodes is not as straightforward compared to that of a purely orthonormal basis. This is mainly because when the method is applied to a truncated eigenmode basis with more than one mode the resulting basis is still valid but the shapes no longer correspond to the physical resonant modes (each column of the resulting basis is some linear combination of the

actual physical eigenmodes). This can be a drawback if the shape of each eigenmode also needs to be preserved. This can be avoided by interpolating the modes individually instead of collectively as a matrix. The authors validate the method for simple problems.

In [43], the authors propose a large scale implementation of the Manifold interpolation towards real-time CFD-based Aeroelastic computations. The idea is to generate a large database of reduced basis off-line which are randomly spaced in a given parameter space. During the on-line period the reduced basis at any other parameter set is interpolated using the Manifold interpolation scheme thus enabling real-time computations. In this work the authors use the Mac number and the angle of attack of a F-16 model as parameters for interpolation.

Finally in [68], the authors propose a two-step interpolation scheme which adds a pre-conditioning step prior to the Manifold interpolation. The proposed additional step transforms all the precomputed reduced basis into a consistent set of generalized coordinates. This greatly increases the robustness of the interpolation scheme especially for the case of mode veering (or mode crossing) phenomena where a normal Manifold interpolation would have failed.

FEM Model of the Cable Slab

3.1 Discrete governing equations

The discrete governing expressions of the cable slab dynamics can be written in general as:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{f}^{int}(\mathbf{u}) + \mathbf{f}^d(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{f}^{ext}(t) . \quad (3.1)$$

Here $\mathbf{M}$ represents the mass matrix of the system , $\mathbf{f}^{int}$ the internal force, $\mathbf{f}^d$ the dissipation forces and $\mathbf{f}^{ext}$ the external reaction/disturbance force due to the hinges connecting the slab to the linear stages. The derivation of these quantities is explained in the subsequent sections. The derivation of the dissipation forces specifically, is dealt at the end of the chapter.

As will be later shown in this work, it is convenient to partition the degrees of freedom and the governing equations into internal and boundary coordinates. The boundary degrees of freedom are those that connect the structure to the external world (the linear stage and the base as shown in Figure 1.1). The remaining are designated as internal degrees of freedom. The subscript $(\bullet)_b$ and $(\bullet)_i$ are used to denote the variables associated to the boundary and internal coordinates respectively. $\mathbf{u}$ can therefore be partitioned as:

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_b \end{bmatrix} . \quad (3.2)$$

Similarly, the governing expressions (Equation (3.1)) can also be re-written as follows:

$$\begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ib} \\ \mathbf{M}_{bi} & \mathbf{M}_{bb} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_i \\ \ddot{\mathbf{u}}_b \end{bmatrix} + \begin{bmatrix} \mathbf{f}_i^{int} \\ \mathbf{f}_b^{int} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_i^d \\ \mathbf{f}_b^d \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_b^{ext} \end{bmatrix} . \quad (3.3)$$

Here it is clearly seen that the external forces are applied only on the boundary nodes as reaction/disturbance forces due to the hinges. Also of interest is the static response of the cable slab system when the boundary degrees of freedom are imposed. Let $(\bar{\bullet})$ be used to indicate the solution terms of the static problem. Since $\mathbf{u}_b$ is imposed, $\mathbf{f}_b^{ext}$

becomes the unknown. The static equilibrium expression can be written as:

$$\begin{bmatrix} \mathbf{f}_i^{int}(\bar{\mathbf{u}}_i, \mathbf{u}_b) \\ \mathbf{f}_b^{int}(\bar{\mathbf{u}}_i, \mathbf{u}_b) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{f}}_b^{ext} \end{bmatrix}. \quad (3.4)$$

It follows from the above expression that $\bar{\mathbf{u}}_i$ and $\bar{\mathbf{f}}_b^{ext}$ are functions of $\mathbf{u}_b$.

3.2 Modeling elastic and inertial behavior

This section describes the derivation of the discrete elastic and inertial forces of the cable slab. The FEM formulation is based on the work done by Wagner [4]. An 8-node solid shell element with 3 translational degrees of freedom per node is used. The element formulation exhibits a superior in-plane bending behavior and is immune to membrane and shear locking making it ideal to model thin, highly flexible cable slabs. It is based on the Hu-Washizu three-field variational formulation where the displacements, assumed and assumed stress fields are taken as independent variables. A short improvised derivation of the governing FEM expressions from the variational principles assuming a linear constitutive model is presented here without going into details about the construction of the FEM approximations. For a detailed description of method, refer [4].

The Hu-Washizu functional is given as:

$$\Pi(\mathbf{v}, \hat{\mathbf{S}}, \bar{\mathbf{E}}) = \int_{\mathcal{B}_0} (W_0(\bar{\mathbf{E}}) + \hat{\mathbf{S}} : (\mathbf{E} - \bar{\mathbf{E}}) - \mathbf{v} \cdot \mathbf{b}) dV - \int_{\partial_\sigma \mathcal{B}_0} \mathbf{v} \cdot \mathbf{t} dA. \quad (3.5)$$

Here $\mathbf{v}$, $\hat{\mathbf{S}}$ and $\bar{\mathbf{E}}$ represent the displacement field and assumed stress and strain fields respectively. $\hat{\mathbf{S}}$ and $\bar{\mathbf{E}}$ are work conjugates in material description. W_0 is the stored elastic energy and is a function of $\bar{\mathbf{E}}$. $\mathbf{E}$ represents the Green-Lagrange strain tensor. $\mathbf{t}$ and $\mathbf{b}$ represent the traction and body forces respectively. The internal forces are defined on $\mathcal{B}_0$ which represents the body of volume V and tractions are defined on $\partial_\sigma \mathcal{B}_0$ which represents the surface of area A . Double contraction of tensors is indicated by the symbol $(:)$ and dot product by $(\cdot)$. $\hat{\mathbf{S}}$ and $\bar{\mathbf{E}}$ do not have a $\mathcal{C}_0$ continuity requirement and hence can be approximated by a discontinuous function which would allow it to be condensed on an element level leaving behind a pure displacement representation.

Let the superscript $(\bullet)^h$ indicate the finite element approximations of the respective fields. The double contractions in Equation (3.5) is replaced with matrix multiplication by representing the 2^{nd} order tensor fields in a column vector fashion (i.e. $\mathbf{A} = [A_{11} \ A_{22} \ A_{33} \ A_{23} \ A_{12} \ A_{13}]^T$). The Finite Element equations can be derived by substituting the FEM approximations of the fields and applying the principle of virtual

work with respect to the discretized degrees of freedom.

$$\int_{\mathcal{B}_0} (\delta \bar{\mathbf{E}}^{hT} \frac{\partial W_0(\bar{\mathbf{E}}^h)}{\partial \bar{\mathbf{E}}^h} + \delta \hat{\mathbf{S}}^{hT} (\mathbf{E}^h - \bar{\mathbf{E}}^h) + (\delta \mathbf{E}^{hT} - \delta \bar{\mathbf{E}}^{hT}) \hat{\mathbf{S}}^h - \delta \mathbf{v}^{hT} \mathbf{b}) dV - \int_{\partial_\sigma \mathcal{B}_0} \delta \mathbf{v}^{hT} \mathbf{t} dA = 0 \quad (3.6)$$

Let subscript $(\bullet)_e$ is used to represent the quantities at element level. $\mathbf{v}_e^h$, $\bar{\mathbf{E}}_e^h$ and $\hat{\mathbf{S}}_e^h$ are given follows:

$$\begin{aligned} \mathbf{v}_e^h &= \mathbf{N}(\mathbf{u}_e - \mathbf{u}_e^0) \\ \bar{\mathbf{E}}_e^h &= \mathbf{N}_E \boldsymbol{\alpha}_e \\ \hat{\mathbf{S}}_e^h &= \mathbf{N}_S \boldsymbol{\beta}_e \end{aligned} \quad (3.7)$$

Here, $\mathbf{N}$, $\mathbf{N}_E$ and $\mathbf{N}_S$ represent the respective shape functions and $\mathbf{u}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the corresponding discretized degrees of freedom. $\mathbf{u}$ is the nodal coordinates as denoted earlier and $\mathbf{u}^0$ is the initial undeformed configuration. The terms in Equation (3.6) can be expanded as follows:

$$\delta \mathbf{v}_e = \mathbf{N} \delta \mathbf{u}_e \quad (3.8a)$$

$$\delta \bar{\mathbf{E}}_e = \mathbf{N}_E \delta \boldsymbol{\alpha}_e \quad (3.8b)$$

$$\delta \hat{\mathbf{S}}_e = \mathbf{N}_S \delta \boldsymbol{\beta}_e \quad (3.8c)$$

$$\delta \mathbf{E}_e = \mathbf{B} \delta \mathbf{u}_e \quad (3.8d)$$

$$\frac{\partial W_0}{\partial \bar{\mathbf{E}}_e^h} = \mathbf{H} \bar{\mathbf{E}}_e^h \quad (3.8e)$$

Here $\mathbf{B}$ is the derivative of $\mathbf{E}_e^h$ with respect to $\mathbf{u}_e$. $\mathbf{H}$ is the linear elastic or Hookean constitutive matrix. Substituting Equation (3.8) in Equation (3.6) and equating the terms within the virtual displacements to zero gives the governing FEM expressions of each element:

$$\mathbf{f}_e^{int} = \mathbf{f}_e^{ext} \quad (3.9a)$$

$$\mathbf{a}_e = \mathbf{0} \quad (3.9b)$$

$$\mathbf{b}_e = \mathbf{0} \quad , \quad (3.9c)$$

where,

$$\mathbf{f}_e^{int} = \int_{\mathcal{B}_e} \mathbf{B}^T \hat{\mathbf{S}}_e^h dV_e \quad (3.10a)$$

$$\mathbf{f}_e^{ext} = \int_{\mathcal{B}_0} \mathbf{N}^T \mathbf{b} dV + \int_{\partial_\sigma \mathcal{B}_0} \mathbf{N}^T \mathbf{t} dA \quad (3.10b)$$

$$\mathbf{a}_e = \int_{\mathcal{B}_e} \mathbf{N}_E^T (\mathbf{H} \bar{\mathbf{E}}_e^h - \hat{\mathbf{S}}_e^h) dV_e \quad (3.10c)$$

$$\mathbf{b}_e = \int_{\mathcal{B}_e} \mathbf{N}_S^T (\mathbf{E}_e^h - \bar{\mathbf{E}}_e^h) dV_e \quad (3.10d)$$

Since $\bar{\mathbf{E}}_e^h$ and $\hat{\mathbf{S}}_e^h$ do not have any inter-element continuity requirement, they can be condensed on the element level when deriving the tangent stiffness matrix. In order to do this α_e and β_e are temporarily treated as dependent variables with respect to $\mathbf{u}_e$ in order to eliminate them from the expressions. The element tangent stiffness matrix is obtained by differentiating Equation (3.10a) with respect to $\mathbf{u}_e$:

$$\begin{aligned} \mathbf{K}_{Te} &= \int_{\mathcal{B}_e} (\mathbf{B}_{,u}^T \hat{\mathbf{S}}_e^h + \mathbf{B}^T \mathbf{N}_S \beta_{e,u}) dV_e \\ &= \mathbf{K}_e + \mathbf{L}_e^T \beta_{e,u} , \end{aligned} \quad (3.11)$$

where,

$$\mathbf{K}_e = \int_{\mathcal{B}_e} \mathbf{B}_{,u}^T \hat{\mathbf{S}}_e^h dV_e \quad \mathbf{L}_e = \int_{\mathcal{B}_e} \mathbf{N}_S^T \mathbf{B} dV_e .$$

$\beta_{e,u}$ is obtained by differentiating Equation (3.9b) and (3.9c) with respect to $\mathbf{u}_e$ and solving it:

$$\begin{aligned} \mathbf{a}_{e,u} &= \int_{\mathcal{B}_e} \mathbf{N}_E^T \mathbf{H} \mathbf{N}_E dV_e \alpha_{e,u} - \int_{\mathcal{B}_e} \mathbf{N}_E^T \mathbf{N}_S dV_e \beta_{e,u} = 0 \\ &= \mathbf{A}_e \alpha_{e,u} - \mathbf{Q}_e^T \beta_{e,u} = 0 \end{aligned} \quad (3.12)$$

$$\begin{aligned} \mathbf{b}_{e,u} &= \int_{\mathcal{B}_e} \mathbf{N}_S^T \mathbf{B} dV_e - \int_{\mathcal{B}_e} \mathbf{N}_S^T \mathbf{N}_E dV_e \alpha_{e,u} = 0 \\ &= \mathbf{L}_e - \mathbf{Q}_e \alpha_{e,u} = 0 , \end{aligned} \quad (3.13)$$

where,

$$\mathbf{A}_e = \int_{\mathcal{B}_e} \mathbf{N}_E^T \mathbf{H} \mathbf{N}_E dV_e \quad \mathbf{Q}_e = \int_{\mathcal{B}_e} \mathbf{N}_S^T \mathbf{N}_E dV_e$$

Thus,

$$\beta_{e,u} = (\mathbf{Q}_e \mathbf{A}_e^{-1} \mathbf{Q}_e^T)^{-1} \mathbf{L}_e = \mathbf{W}_e \mathbf{L}_e \quad (3.14)$$

$$\alpha_{e,u} = \mathbf{A}_e^{-1} \mathbf{Q}_e^T \beta_{e,u} , \quad (3.15)$$

where,

$$\mathbf{W}_e = \mathbf{Q}_e \mathbf{A}_e^{-1} \mathbf{Q}_e^T .$$

Substituting Equation (3.14) into Equation (3.11) gives the expression for the element tangent stiffness matrix:

$$\mathbf{K}_{Te} = \mathbf{K}_e + \mathbf{L}_e^T \mathbf{W}_e \mathbf{L}_e . \quad (3.16)$$

The element mass matrix is modeled by assuming that the cable slab is of uniform density and employing the expression for consistent mass matrix given as:

$$\mathbf{M}_e = \rho \int_{\mathcal{B}_e} \mathbf{N}_e^T \mathbf{N}_e dV_e . \quad (3.17)$$

Where ρ represents density of the cable slab. It is observed that the mass matrix is not a function of any of the independent coordinates. The system matrices are obtained by simply assembling the element matrices.

3.3 Linearization of the internal force problem

When applying the FEM model to solve nonlinear iterative problems, it is important to linearize the governing expressions. Linearizing Equation (3.9a) with respect to the independent variables gives:

$$\begin{bmatrix} \mathbf{f}_e^{int} - \mathbf{f}_e^{ext} \\ \mathbf{a}_e \\ \mathbf{b}_e \end{bmatrix} + \begin{bmatrix} \mathbf{K}_e & \mathbf{0} & \mathbf{L}_e^T \\ \mathbf{0} & \mathbf{A}_e & -\mathbf{Q}_e^T \\ \mathbf{L}_e & -\mathbf{Q}_e & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{u}_e \\ \Delta \boldsymbol{\alpha}_e \\ \Delta \boldsymbol{\beta}_e \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} . \quad (3.18)$$

Solving Equation (3.18) for $\Delta \boldsymbol{\alpha}_e$ and $\Delta \boldsymbol{\beta}_e$ gives:

$$\Delta \boldsymbol{\beta}_e = \mathbf{W}_e^{-1} (\mathbf{L}_e \Delta \mathbf{u}_e + \mathbf{Q}_e \mathbf{A}_e^{-1} \mathbf{a}_e + \mathbf{b}_e) \quad (3.19a)$$

$$\Delta \boldsymbol{\alpha}_e = \mathbf{A}_e^{-1} (\mathbf{Q}_e^T \Delta \boldsymbol{\beta}_e - \mathbf{a}_e) . \quad (3.19b)$$

Substituting Equation (3.19a) into the first row of Equation (3.18) gives:

$$\mathbf{K}_{Te} \Delta \mathbf{u} + \mathbf{f}_e = \mathbf{f}_e^{ext} , \quad (3.20)$$

where,

$$\mathbf{f}_e = \mathbf{f}_e^{int} - \mathbf{L}_e^T \mathbf{W}_e^{-1} (\mathbf{Q}_e \mathbf{A}_e^{-1} \mathbf{a}_e + \mathbf{b}_e) . \quad (3.21)$$

$\mathbf{K}_{Te}$ is given by Equation (3.16). $\mathbf{f}_e$ gives the iteration force. After every iteration, $\boldsymbol{\alpha}_e$ and $\boldsymbol{\beta}_e$ are updated using Equation (3.19).

3.4 Eigen modes of the cable slab

The eigenmodes shapes of the constrained cable slab give useful information about its internal dynamics. They are applied in the model reduction technique described in later chapters. Unique solutions for the eigenmodes are obtained by solving the following

eigen value problem and applying the mass normalization condition:

$$(\mathbf{K}_{ii} + \omega^2 \mathbf{M}_{ii}) \mathbf{x} = \mathbf{0} \quad (3.22)$$

$$\mathbf{x}^T \mathbf{M}_{ii} \mathbf{x} = \mathbf{I} \quad (3.23)$$

Here ω is the natural or eigen frequency of the eigenmode $\mathbf{x}$. k solutions of ω are obtained to the above problem where k is equal to the number of internal node degrees of freedom. 2 solutions, $\mathbf{x}_j$ and $-\mathbf{x}_j$ are obtained by substituting a particular solution, ω_j into Equation (3.22), thus $2k$ solutions of eigenmodes exist (provided that the eigen frequencies are distinct).

The first six eigenmodes of the cable slab with the boundary nodes constrained are shown in Figure A.1 and A.2 for the configuration when the upper stage is in the middle and for one in which it is at the extreme position respectively. The modes are classified as XZ and Y modes. XZ modes are those in which the displacements occur only in the XZ plane and Y modes are the ones in which the displacements are mostly in the Y direction (refer the aforementioned figures for axis definition). This distinction between the modes is important since each are uniquely excited by stage motions. Only XZ modes are excited for motions of the linear stages in X direction and only Y modes are excited for motions in Y direction. This is an important criteria for selecting the modes for model reduction. Since only motions of the stage in the X direction is considered in this work, the focus will be only on the XZ modes.

3.5 Modeling damping behavior

A rigorous modeling of dissipation or the damping forces in the cable slab is out of scope of the thesis. Instead an ad-hoc approach is used to approximate damping behavior. Viscous damping on a truncated set of eigenmodes are assumed. The damping coefficients are tuned such that they agree with the experiment. The nodal damping matrix $\mathbf{C}$ can be constructed from the assumed modal damping matrix $\mathbf{C}_d$ as follows. Suppose the nodal damping matrix of system was defined as follows:

$$\begin{bmatrix} \mathbf{X}_r^T \\ \mathbf{X}_t^T \end{bmatrix} \mathbf{C} \begin{bmatrix} \mathbf{X}_r & \mathbf{X}_t \end{bmatrix} = \begin{bmatrix} \mathbf{C}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (3.24)$$

Where $\mathbf{X}_r$ is a set of p retained eigenmodes and $\mathbf{X}_t$ is a the remaining set of $k - p$ truncated eigenmodes of the cable slab system. $\mathbf{C}_d$ is given as:

$$\mathbf{C}_d = \begin{bmatrix} 2\omega_1\mu_1 & & 0 \\ & \ddots & \\ 0 & & 2\omega_p\mu_p \end{bmatrix} \quad (3.25)$$

μ_i represents the modal damping coefficient of the i^{th} eigenmode. Using the above definition, the damping matrix can be derived as follows. Pre and post multiply Equation (3.25) by the full eigenmode set and apply the fundamental property of eigenmodes $\sum_{j=1}^k \mathbf{x}_j^T \mathbf{x}_j = \mathbf{M}_{ii}^{-1}$ to get:

$$\mathbf{M}_{ii}^{-1} \mathbf{C} \mathbf{M}_{ii}^{-1} = \begin{bmatrix} \mathbf{X}_r & \mathbf{X}_t \end{bmatrix} \begin{bmatrix} \mathbf{C}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{X}_r^T \\ \mathbf{X}_t^T \end{bmatrix}. \quad (3.26)$$

Hence $\mathbf{C}$ is given as:

$$\mathbf{C} = \mathbf{M}_{ii} \mathbf{X}_r \mathbf{C}_d \mathbf{X}_r^T \mathbf{M}_{ii}. \quad (3.27)$$

This expression has the advantage that only the knowledge of the retained eigenmodes are required and serves as convenient way to model the approximate damping behavior of the cable slab.

3.6 Model Updating

In order for the model to match experimental results it is required to perform a model updating procedure to tune the parameters of the model. The parameters of the model are as follows. A brick type element of is used here, hence the geometric parameters are simply the length L , width w and thickness a of the cable slab. Since a linear elastic isotropic material with constant density is assumed, the material parameters are described simply by the Young's modulus E , poisson's ratio ν and mass density ρ . The configuration parameters of the cable slab is given by distance the vector $\mathbf{d} = [d_x, d_z]$. Here d_z represents the distance of the stage from the base in the Z direction (refer Figure 1.1 for axis definition) also termed as the stage separation. And d_x represents the distance in the X direction also termed as the stroke displacement. These 2 parameters together determine the overall static configuration of the cable slab.

All the above discussed parameters give the minimum possible description of the full cable slab (not accounting for damping). However in reality the system is more complex with many independent design parameters than can be accounted for by the model. One of these aspects is the shape of the cross section of a cable slab, which can be complex but the model can only describe a rectangular. A finer discretization of the cross section may allow more complex shapes but will be computationally expensive. Thus a simplification of the cross-sectional effects is needed.

If only motions in the XZ plane is considered, the model properties can be hugely simplified. The thickness of cable slabs is usually very thin and if enough elements are used to discretize the system, the deformation of each element in the thickness direction can be approximated using simple Bernoulli beam bending. This implies that the elastic property of the entire slab can be represented using a single parameter (provided the

thickness is uniform), the flexural rigidity K_b of the cross-section given by $\frac{Ea^3w}{12}$. Any of the 3 parameters (E , w or a) can be tuned to the measured value of the flexural rigidity (the thickness should be kept small in general).

With the above assumptions even the mass matrix can also be easily tuned. The rotational inertia of the cross-section about Y axis is negligible and thus ρ can be tuned such that the total mass of the model equals that mass of the actual setup.

The poisson's ratio ν can be left un-tuned for motion in the XZ plane except when the length of the cable slab is very short. In such a case, lateral contractions along the width of the slab could play an important role in effectively lowering the stiffness of the slab. Hence, the value of ν becomes important here. Tuning of parameters for deformation in the Y direction is hard since the approximation of simple bending is no longer valid. In this case the parameters might have to be simultaneously optimized to match the experimental results. Otherwise, sophisticated tests may need to be performed in order estimate the parameters.

Modal sensitivities

This chapter discusses the derivation of the sensitivity expressions for the tangent stiffness matrix, the static response and the eigenmodes of the system with respect to a displacement mode. These quantities are applied during model order reduction introduced in later chapters. Suppose ϕ is a mode and q represents its modal amplitude defined as:

$$\mathbf{u}_{,q} = \phi . \quad (4.1)$$

Therefore the modal sensitivity of any quantity with respect to the mode ϕ can be obtained by differentiating it with respect to q .

4.1 Modal sensitivities of stiffness matrix

The modal sensitivity of the tangent stiffness is obtained by differentiating Equation (3.16) with respect to q . This is then assembled to give the system level derivative matrix:

$$\mathbf{K}_{Te,q} = \mathbf{K}_{e,q} + \mathbf{L}_{e,q}^T \mathbf{W}_e \mathbf{L}_e + \mathbf{L}_e^T \mathbf{W}_e \mathbf{L}_{e,q} , \quad (4.2)$$

where,

$$\mathbf{K}_{e,q} = \int_{\mathcal{B}_e} \mathbf{B}_{,u}^T \hat{\mathbf{S}}_{e,q}^h dV_e \quad (4.3)$$

$$\mathbf{L}_{e,q} = \int_{\mathcal{B}_e} \mathbf{N}_S^T \mathbf{B}_{,q} dV_e , \quad (4.4)$$

$\hat{\mathbf{S}}_{e,q}^h$ is function of $\beta_{e,q}$ which can be obtained by applying chain rule on Equation (3.14) using Equation (4.1) :

$$\beta_{e,q} = \mathbf{W}_e \mathbf{L}_e \phi_e . \quad (4.5)$$

Since $\mathbf{B}$ is linear with respect to $\mathbf{u}_e$, $\mathbf{B}_{,q}$ is simply obtained by substituting ϕ_e instead of $\mathbf{u}_e$.

Similarly, second order derivatives of the stiffness matrix with respect to the same mode can be obtained by differentiating Equation (4.2) with respect to q again:

$$\mathbf{K}_{T_{e,qq}} = \mathbf{K}_{e,qq} + \mathbf{L}_{e,q}^T \mathbf{W}_e \mathbf{L}_e + 2\mathbf{L}_{e,q}^T \mathbf{W}_e \mathbf{L}_{e,q} + \mathbf{L}_e^T \mathbf{W}_e \mathbf{L}_{e,qq} , \quad (4.6)$$

where,

$$\mathbf{K}_{e,qq} = \int_{\mathcal{B}_e} \mathbf{B}_{,u}^T \widehat{\mathbf{S}}_{e,qq}^h dV_e \quad (4.7)$$

$$\mathbf{L}_{e,qq} = \int_{\mathcal{B}_e} \mathbf{N}_S^T \mathbf{B}_{,qq} dV_e . \quad (4.8)$$

Again, $\widehat{\mathbf{S}}_{e,qq}^h$ is a function of $\beta_{e,qq}$ which is given as,

$$\beta_{e,qq} = \mathbf{W}_e \mathbf{L}_e \phi_{e,q} + \mathbf{W}_e \mathbf{L}_{e,q} \phi_e . \quad (4.9)$$

It is important to note here that the derivatives of β_e have to be computed before Equation (4.3) and (4.7) can be evaluated. This means that the Gauss integration iteration has to be carried out twice in which $\mathbf{K}_{e,q}$ or $\mathbf{K}_{e,qq}$ is computed in the second iteration.

4.2 Modal sensitivities of eigenmodes

A powerful algorithm for computing the eigenmode derivatives of general real matrices with non-repeated eigenvalues is given by Nelson [46]. It requires the knowledge of only the eigenmode that needs to be differentiated. The method assumes that the eigenmode is mass normalized.

The modal sensitivity of an eigenmode say, $\mathbf{x}_j$ with respect to an arbitrary mode ϕ can be obtained by differentiating the eigen value problem given by Equation (3.22) with respect to the corresponding modal amplitude of ϕ , q :

$$(\mathbf{K}_{ii,q} - \omega_{j,q}^2 \mathbf{M}_{ii}) \mathbf{x}_j + (\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii}) \mathbf{x}_{j,q} = \mathbf{0} . \quad (4.10)$$

The eigenvector sensitivity cannot be calculated directly from Equation (4.10) since it is singular. Instead it is proposed that the solution be written in the following form:

$$\mathbf{x}_{j,q} = \boldsymbol{\nu}_{jq} + c_{jq} \mathbf{x}_j , \quad (4.11)$$

for some vector $\boldsymbol{\nu}_{jq}$ and a scalar constant c_{jq} which can be calculated. Substituting the above Equation (4.11) into Equation (4.10) and simplifying yields:

$$(\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii}) \boldsymbol{\nu}_{jq} = \mathbf{F}_{jq} , \quad (4.12)$$

where the vector on the right hand side is given as:

$$\mathbf{F}_{jq} = -(\mathbf{K}_{ii,q} - \omega_{j,q}^2 \mathbf{M}_{ii}) \mathbf{x}_j . \quad (4.13)$$

$(\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii})$ on the left hand side of the equations is still singular which is removed by setting the p^{th} component of $\boldsymbol{\nu}_{jq}$ be set to zero, where p is the location at which $\mathbf{x}_j$ has the maximum absolute value. This is achieved by replacing the p^{th} row and column of $(\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii})$ to 0 except for the diagonal term, which is set to 1 and setting the corresponding term in $\mathbf{F}_{jq}$ to 0. The resulting partitioned form can be expressed as:

$$\begin{bmatrix} (\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii})_{11} & \mathbf{0} & (\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii})_{13} \\ \mathbf{0} & 1 & \mathbf{0} \\ (\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii})_{31} & \mathbf{0} & (\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii})_{33} \end{bmatrix} \boldsymbol{\nu}_{jq} = \begin{bmatrix} (\mathbf{F}_{jq})_1 \\ 0 \\ (\mathbf{F}_{jq})_3 \end{bmatrix} . \quad (4.14)$$

The new matrix formed is now non-singular. Once $\boldsymbol{\nu}_{jq}$ is calculated, the value of the scalar constant c_{jq} is determined by differentiating the mass normalization condition given by Equation (3.23) with respect to q as:

$$2\mathbf{x}_j^T \mathbf{M}_{ii} \mathbf{x}_{j,q} = 0 . \quad (4.15)$$

Substituting the expression for eigenmode sensitivity from Equation (4.11) in the above expression and using the mass normalization condition itself gives the solution for c_{jq} as:

$$c_{jq} = -\mathbf{x}_j^T \mathbf{M}_{ii} \boldsymbol{\nu}_{jq} . \quad (4.16)$$

Once c_{jq} and $\boldsymbol{\nu}_{jq}$ are known, they may be substitute back into Equation (4.11) to get the eigenmode sensitivity $\mathbf{x}_{j,q}$. The expressions for c_{jq} and $\boldsymbol{\nu}_{jq}$ require the knowledge of the corresponding modal sensitivity of the stiffness matrix, $\mathbf{K}_{ii,j}$ which was just derived in this section and $\omega_{j,q}^2$ which is given by the following expression:

$$\omega_{j,q}^2 = \mathbf{x}_j^T \mathbf{K}_{ii,q} \mathbf{x}_j . \quad (4.17)$$

Second order derivatives can be obtained in a similar fashion. Differentiating Equation (3.22) with respect to q twice one obtains:

$$(\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii}) \mathbf{x}_{j,qq} + 2(\mathbf{K}_{ii,q} - \omega_{j,q}^2 \mathbf{M}_{ii}) \mathbf{x}_{j,q} + (\mathbf{K}_{ii,qq} - \omega_{j,qq}^2 \mathbf{M}_{ii}) \mathbf{x}_j = \mathbf{0} . \quad (4.18)$$

Since the above expression is singular the solution is again proposed to be of the form given by Equation (4.11):

$$\mathbf{x}_{j,qq} = \boldsymbol{\nu}_{jq} + c_{jq} \mathbf{x}_j . \quad (4.19)$$

Substituting the above expression into Equation (4.18) and simplifying yields:

$$(\mathbf{K}_{ii} - \omega_j^2 \mathbf{M}_{ii}) \boldsymbol{\nu}_{jq} = \mathbf{F}_{jq} \quad , \quad (4.20)$$

where $\mathbf{F}_{jq}$ is given as:

$$\mathbf{F}_{jq} = -((\mathbf{K}_{ii,q} - \omega_{j,q}^2 \mathbf{M}_{ii}) \mathbf{x}_j + 2(\mathbf{K}_{ii,q} - \omega_{j,q}^2 \mathbf{M}_{ii}) \mathbf{x}_{j,q}) \quad . \quad (4.21)$$

$\boldsymbol{\nu}_{jq}$ is solved in the same manner as $\boldsymbol{\nu}_j$. c_{jq} can be obtained by solving the second derivative of the mass normalization equation twice with respect to q :

$$2\mathbf{x}_{j,q}^T \mathbf{M}_{ii} \mathbf{x}_{j,q} + 2\mathbf{x}_j^T \mathbf{M}_{ii} \mathbf{x}_{j,q} = 0 \quad . \quad (4.22)$$

Substituting the expression for eigenmode sensitivity from Equation (4.19) in the above expression and using the mass normalization condition itself gives the solution for c_{jq} as:

$$c_{jq} = -(\mathbf{x}_j^T \mathbf{M}_{ii} \boldsymbol{\nu}_{jq} + \mathbf{x}_{j,q}^T \mathbf{M}_{ii} \mathbf{x}_{j,q}) \quad . \quad (4.23)$$

Here again the derivation of $\mathbf{K}_{ii,q}$ is given previously in this section and $\omega_{j,q}$ is obtained as follows:

$$\omega_{j,q}^2 = \mathbf{x}_j^T \mathbf{K}_{ii,q} \mathbf{x}_j \quad . \quad (4.24)$$

4.3 Static perturbations of imposed displacement

Static perturbations are modes in which a substructure responds statically for small imposed displacements on the boundary nodes. Let $\hat{\mathbf{u}}_b$ be the imposed displacement mode and ζ the corresponding generalized model amplitude. If linear a displacement is applied $\hat{\mathbf{u}}_b$ is a constant else it is in general a function of ζ . From the definition we have the following relation:

$$\Delta \mathbf{u}_b = \hat{\mathbf{u}}_b \Delta \zeta \quad . \quad (4.25)$$

The static perturbations are given as the coefficients of the non zero order terms of ζ in the following Taylor expansion:

$$\bar{\mathbf{u}}_i(\mathbf{u}_b + \hat{\mathbf{u}}_b \Delta \zeta) = \bar{\mathbf{u}}_i(\mathbf{u}_b) + \bar{\mathbf{u}}_{i,\zeta}(\mathbf{u}_b) \Delta \zeta + \frac{1}{2} \bar{\mathbf{u}}_{i,\zeta\zeta}(\mathbf{u}_b) \Delta \zeta^2 + \mathcal{O}(\Delta \zeta^3) \quad . \quad (4.26)$$

The non zero order terms of ζ in the above expression give the static modes. The symbol $\mathbf{s}$ is used from here on instead of $\bar{\mathbf{u}}_{i,\zeta}(\mathbf{u}_b)$ and $\mathbf{s}_\zeta$ for $\bar{\mathbf{u}}_{i,\zeta\zeta}(\mathbf{u}_b)$ and so on. $\mathbf{s}$ can be obtained by differentiating the internal node partition of Equation (3.4) with respect to ζ :

$$\mathbf{f}_{i,\zeta}^{int}(\bar{\mathbf{u}}_i, \mathbf{u}_b) = \mathbf{K}_{ii}(\bar{\mathbf{u}}_i, \mathbf{u}_b) \mathbf{s} + \mathbf{K}_{ib}(\bar{\mathbf{u}}_i, \mathbf{u}_b) \hat{\mathbf{u}}_b = \mathbf{0} \quad . \quad (4.27)$$

Therefore $\mathbf{s}$ is given as:

$$\mathbf{s} = -\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \hat{\mathbf{u}}_b . \quad (4.28)$$

The second order static mode is obtained by differentiating Figure 4.27 twice with respect to ζ and solving for $\mathbf{s}_{,\zeta}$:

$$\mathbf{s}_{,\zeta} = -\mathbf{K}_{ii}^{-1} (\mathbf{K}_{ii,\zeta} \mathbf{s} + \mathbf{K}_{ib,\zeta} \hat{\mathbf{u}}_b) . \quad (4.29)$$

Higher order derivatives can be obtained from further differentiation.

Nonlinear Model Order Reduction

In this chapter a general theory of nonlinear model reduction is developed using the concept of a parameterized manifold. It is later applied to generalize the Craig Bampton method for large static displacements. The expressions are then further simplified for a more specific case applicable to the cable slab problem.

5.1 Model Reduction using Parameterized Manifolds

Let $\mathbf{\Gamma} : \mathbb{R}^m \mapsto \mathbb{R}^n \mid m \ll n$, describe the reduced kinematic description of the system dynamics in terms of a set of generalized coordinates $\mathbf{q} \in \mathbb{R}^m$. $\mathbf{u}$ can therefore be written as:

$$\mathbf{u} = \mathbf{\Gamma}(\mathbf{q}) \quad . \quad (5.1)$$

Mathematically speaking, $\mathbf{\Gamma}(\mathbf{q})$ is termed as a n dimensional parameterized manifold with respect to the parameter $\mathbf{q}$. A parametrized manifold is a topological space that locally behaves as an Euclidean space at every point. This is to say that the derivative of $\mathbf{\Gamma}$ with respect to $\mathbf{q}$ gives a set of basis vectors $\mathbf{\Phi} \in \mathbb{R}^{n \times m}$ that span the local linear dynamic behavior of the system. $\mathbf{\Phi}$ is termed as the tangent basis matrix. Therefore we have:

$$\mathbf{\Gamma}_{,q}|_{q=\tilde{q}} = \mathbf{\Phi}(\tilde{q}) \quad . \quad (5.2)$$

If the analytical expression of tangent basis is well defined for all $\mathbf{q}$, $\mathbf{\Gamma}(\mathbf{q})$ can be constructed by integrating the expression with respect to $\mathbf{q}$ and applying the boundary condition at the reference configuration:

$$\mathbf{\Gamma}(\mathbf{q}) = \int_0^{\mathbf{q}} \mathbf{\Phi}(\mathbf{q}) d\mathbf{q} \quad . \quad (5.3)$$

The expressions for velocities and accelerations are obtained by differentiating Equation (5.1) with respect to time:

$$\dot{\mathbf{u}} = \mathbf{\Gamma}_{,q}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{\Phi}(\mathbf{q})\dot{\mathbf{q}} \quad (5.4)$$

$$\ddot{\mathbf{u}} = \dot{\mathbf{\Phi}}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{\Phi}(\mathbf{q})\ddot{\mathbf{q}} \quad (5.5)$$

Since the tangent basis is a function of $\mathbf{q}$, it can be expressed via chain rule. Explicit dependency of $\mathbf{\Phi}$ on $\mathbf{q}$ is not shown from here on for convenience:

$$\dot{\mathbf{\Phi}} = \sum_{j=1}^m \mathbf{\Phi}_{,q_j} \dot{q}_j \quad (5.6)$$

The reduced governing expressions of the cable slab can now be obtained by substituting Equation (5.1), (5.4) and (5.5) into Equation (3.1) and applying the principle of virtual work with respect to $\mathbf{q}$ and equating the terms within the variation to zero:

$$\mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} \ddot{\mathbf{q}} + \mathbf{\Phi}^T \mathbf{M} \dot{\mathbf{\Phi}} \dot{\mathbf{q}} + \mathbf{\Phi}^T \mathbf{f}^{int}(\mathbf{\Gamma}(\mathbf{q})) + \mathbf{\Phi}^T \mathbf{f}^d(\mathbf{\Gamma}(\mathbf{q}), \mathbf{\Phi} \dot{\mathbf{q}}) = \mathbf{\Phi}^T \mathbf{f}^{ext}(t) \quad (5.7)$$

Since the reduced expressions contain spatial derivatives of $\mathbf{\Phi}$, a criteria for selecting the method to derive the tangent basis is that its derivative must be well defined at all points.

5.2 Generalized Craig Bampton Method

The general reduced expressions obtained in the previous section (Equation (5.7)) is developed further by formulating the exact expressions for the appropriate tangent basis. Here, the well known Craig-Bampton method is selected. It is chosen in particular because it offers an unique advantage in substructures (as will be discussed in detail subsequently in this work) where the internal vibrations remain small with respect to its overall motion. The system can thus be linearized with respect to the internal vibrations. This is the only assumption made during the derivation of the reduced expressions and is referred to as the *small internal vibration* assumption. This holds in general for most practical applications of the cable slab system and hence makes sense to proceed with it. Since the standard Craig-Bampton technique applies to linear systems and the method developed here extends the concept to geometric nonlinearities, it is termed here as *Generalized Craig Bampton* method or GCB for short.

According to concept of the Craig Bampton method, any linear substructure deformation can be decomposed into two type of modes, static and dynamic. A static mode is represented by the first order static perturbations to the imposed boundary mode given by Equation (4.28) and the dynamic mode is represented by the eigenmodes put

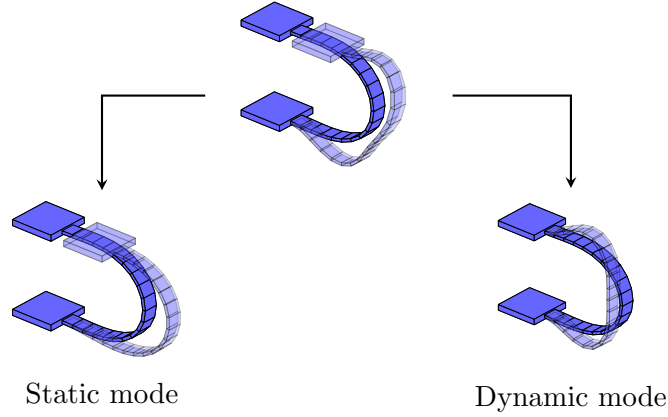


Figure 5.1: Illustration of the Craig Bampton Modes

forward by fixing the boundary nodes which are given by Equation (3.22) and (3.23). This concept is illustrated in Figure 5.1. Thus, the tangent basis can be expressed as:

$$\Phi = \begin{bmatrix} \Phi_i \\ \Phi_b \end{bmatrix} = \begin{bmatrix} \mathbf{X} & \mathbf{s} \\ \mathbf{0} & \hat{\mathbf{u}}_b \end{bmatrix}. \quad (5.8)$$

Here, $[\mathbf{X}^T \ \mathbf{0}^T]^T$ represents the dynamic mode where $\mathbf{X}$ is a matrix of p eigenmodes that have the highest participation with respect to $\hat{\mathbf{u}}_b$. $[\mathbf{s}^T \ \mathbf{u}_b^T]^T$ represents the static mode. In general, a complex imposed boundary mode can be applied but only a constant translational mode along the stroke direction as shown in Figure 5.1 is considered. As mentioned earlier, the generalized reduced coordinate vector associated with the static mode is given by ζ , and that associated with the dynamic mode is given by $\boldsymbol{\eta}$. If $\hat{\mathbf{u}}_b$ is a constant with unit values, then ζ gives the magnitude of the imposed displacements. The Craig Bampton parameterized manifold can be obtained by applying Equation (5.3). Since the dynamic displacements are assumed to be small, the dependency of the variables on $\boldsymbol{\eta}$ is dropped and are considered solely as a function of ζ . From definition, the integration of the static mode gives the static configuration, $\bar{\mathbf{u}}$ of the system defined by Equation (3.4). Hence:

$$\begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_b \end{bmatrix} = \int_0^\zeta \begin{bmatrix} \mathbf{s}(\zeta) \\ \hat{\mathbf{u}}_b \end{bmatrix} d\zeta + \int_0^\eta \begin{bmatrix} \mathbf{X}(\zeta) \\ \mathbf{0} \end{bmatrix} d\boldsymbol{\eta} \quad (5.9)$$

$$\approx \begin{bmatrix} \bar{\mathbf{u}}_b(\zeta) \\ \hat{\mathbf{u}}_b \zeta \end{bmatrix} + \begin{bmatrix} \mathbf{X}(\zeta) \boldsymbol{\eta} \\ \mathbf{0} \end{bmatrix}. \quad (5.10)$$

The derivative of the Craig-Bampton modes with respect to the generalized coordinates are given as follows:

$$\Phi_{,\eta} \approx \mathbf{0} \quad (5.11)$$

$$\Phi_{,\zeta} = \begin{bmatrix} \mathbf{X}_{,\zeta} & \mathbf{s}_{,\zeta} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} . \quad (5.12)$$

Here the derivatives with respect to η are neglected since it is small. $\mathbf{X}_{,\zeta}$ is eigenmode derivative with respect to ζ and is obtained using Nelson's method described in Section 4.2. $\mathbf{s}_{,\zeta}$ is given by Equation (4.29).

The internal force in the reduced balanced expression (Equation (5.7)) is linearized with respect to η by first applying Equation (5.9) into the internal force function and expanding it with respect to η using Taylor expansion and then applying Equation (3.4):

$$\mathbf{f}_i^{int}(\bar{\mathbf{u}}_i + \mathbf{X}\eta, \hat{\mathbf{u}}_b\zeta) \approx \mathbf{f}_i^{int}(\bar{\mathbf{u}}_i, \hat{\mathbf{u}}_b\zeta) + \mathbf{K}_{ii}\mathbf{X}\eta = \mathbf{K}_{ii}\mathbf{X}\eta \quad (5.13)$$

$$\mathbf{f}_b^{int}(\bar{\mathbf{u}}_i + \mathbf{X}\eta, \hat{\mathbf{u}}_b\zeta) \approx \mathbf{f}_b^{int}(\bar{\mathbf{u}}_i, \hat{\mathbf{u}}_b\zeta) + \mathbf{K}_{ib}\mathbf{X}\eta = \bar{\mathbf{f}}_b^{ext} + \mathbf{K}_{bi}\mathbf{X}\eta . \quad (5.14)$$

Thus, substituting Equation (5.8), (5.11), (5.13) and (5.9) into Equation (5.7) gives expression for the Generalized Craig Bampton (GCB) method (Equation (5.15)):

$$\begin{bmatrix} \mathbf{M}_{\eta\eta} & \mathbf{m}_{\eta\zeta} \\ \mathbf{m}_{\zeta\eta} & m_{\zeta\zeta} \end{bmatrix} \begin{bmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{\eta\eta} & \mathbf{g}_{\eta\zeta} \\ \mathbf{g}_{\zeta\eta} & g_{\zeta\zeta} \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \dot{\zeta} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_\eta^d \\ \mathbf{f}_\zeta^d \end{bmatrix} + \begin{bmatrix} \mathbf{\Omega}^2\eta \\ \bar{\mathbf{f}}_\zeta^{ext} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_\zeta^{ext} \end{bmatrix} , \quad (5.15)$$

where,

$$\mathbf{\Omega} = \begin{bmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_k \end{bmatrix} \quad (5.16a)$$

$$\mathbf{M}_{\eta\eta} = \mathbf{I} \quad (5.16b)$$

$$\mathbf{m}_{\eta\zeta} = \mathbf{X}^T \mathbf{M}_{ii} \mathbf{s} + \mathbf{X}^T \mathbf{M}_{ib} \hat{\mathbf{u}}_b \quad (5.16c)$$

$$\mathbf{m}_{\zeta\eta} = \mathbf{s}^T \mathbf{M}_{ii} \mathbf{X} + \hat{\mathbf{u}}_b^T \mathbf{M}_{ib} \mathbf{X} \quad (5.16d)$$

$$m_{\zeta\zeta} = \mathbf{s}^T \mathbf{M}_{ii} \mathbf{s} + 2\hat{\mathbf{u}}_b^T \mathbf{M}_{bi} \mathbf{s} + \hat{\mathbf{u}}_b^T \mathbf{M}_{bb} \hat{\mathbf{u}}_b \quad (5.16e)$$

$$\mathbf{G}_{\eta\eta} = \mathbf{X}^T \mathbf{M}_{ii} \mathbf{X}_{,\zeta} \quad (5.16f)$$

$$\mathbf{g}_{\eta\zeta} = \mathbf{X}^T \mathbf{M}_{ii} \mathbf{s}_{,\zeta} \quad (5.16g)$$

$$\mathbf{g}_{\zeta\eta} = \mathbf{s}^T \mathbf{M}_{ii} \mathbf{X}_{,\zeta} + \hat{\mathbf{u}}_b^T \mathbf{M}_{bi} \mathbf{X}_{,\zeta} \quad (5.16h)$$

$$g_{\zeta\zeta} = \mathbf{s}^T \mathbf{M}_{ii} \mathbf{s}_{,\zeta} + \hat{\mathbf{u}}_b^T \mathbf{M}_{bi} \mathbf{s}_{,\zeta} , \quad (5.16i)$$

and,

$$\mathbf{f}_\eta^d = \mathbf{X}^T \mathbf{f}_i^d \quad (5.17a)$$

$$f_\zeta^d = \mathbf{s}^T \mathbf{f}_i^d + \hat{\mathbf{u}}_b^T \mathbf{f}_i^d \quad (5.17b)$$

$$\bar{f}_\zeta^{ext} = \hat{\mathbf{u}}_b^T \bar{\mathbf{f}}_b^{ext} . \quad (5.17c)$$

Since ζ is imposed and hence a known function of time, the associated external applied force f_ζ^{ext} becomes a variable. The Equation (5.15) can thus be rearranged to highlight the independent variables by placing them on the left hand side of the expressions:

$$\ddot{\eta} + \Omega^2 \eta + \mathbf{G}_{\eta\eta} \dot{\eta} \dot{\zeta} + \mathbf{f}_\eta^d = -\mathbf{m}_{\eta\zeta} \ddot{\zeta} - \mathbf{g}_{\eta\zeta} \dot{\zeta}^2 \quad (5.18)$$

$$f_\zeta^{ext} = \bar{f}_\zeta^{ext} + \mathbf{m}_{\eta\zeta}^T \ddot{\eta} + m_{\zeta\zeta} \ddot{\zeta} + \mathbf{g}_{\zeta\eta}^T \dot{\eta} \dot{\zeta} + g_{\zeta\zeta} \dot{\zeta}^2 + f_\zeta^d . \quad (5.19)$$

The above expression gives the final reduced equation of the cable slab problem at hand. It is completely linear with respect to the unknowns. The coefficients are functions of ζ . f_ζ^{ext} gives the disturbance force of the cable slab in the direction of imposed motion, which needs to be determined.

Numerical Implementation

6.1 Introduction: A two stage integration scheme

This section describes the implementation of a near real time numerical scheme based on the GCB given by Equation (5.18) and (5.19). As discussed in the previous chapter, the coefficients of the reduced equations involve the computation of $\bar{\mathbf{u}}$, $\mathbf{s}$, $\mathbf{s}_{,\zeta}$, $\mathbf{X}$, $\mathbf{X}_{,\zeta}$, $\boldsymbol{\Omega}$ and $\bar{\mathbf{f}}^{ext}$ which are implicit functions of the imposed displacement, ζ . They are obtained by post-processing the static solution after nonlinear iterations at a given ζ and need to be updated at every time iteration. In addition, $\mathbf{X}$ and $\mathbf{X}_{,\zeta}$ are expensive to compute. Hence, the combined cost of the nonlinear static iterations and post-processing renders the online computation of these coefficients inefficient.

The alternative that circumvents this issue is to compute the coefficients at different, widely spaced stage positions that span the whole stage stroke and interpolate it at the intermediate instants. Since the static response is time independent, this can be done offline without prior knowledge of the input functions and can be later mapped to every time instant. The fact also allows the computed variables to be recycled for different inputs, thereby reducing the overall cost of simulation. Therefore, the reduced dynamic equations is solved in two separate stages, an *offline* and an *online* stage.

In the offline stage, the static response for the given imposed displacement is computed using a Newton Raphson solver with a fixed stepsize. Since each static load step is computationally expensive, optimal stepsize determination is crucial as it strongly affects number of steps needed and hence, the total computational time of the stage. This is discussed in more details in Section 6.4. The offline variables (coefficient values) are computed and stored in a database $\mathbf{D}^h$ at each ζ . The computed eigenmodes are susceptible to mode-crossing, veering and phase reversing phenomenon which produces a discontinuous response which cannot be interpolated accurately. Hence a preconditioning procedure is performed on the eigenmodes to obtain a smooth behavior. This is elaborated in Section 6.2.

In the online stage the dynamic equations are integrated using a linear Newmark scheme. The database $\mathbf{D}^h$ stored in the offline stage is invoked and interpolation is

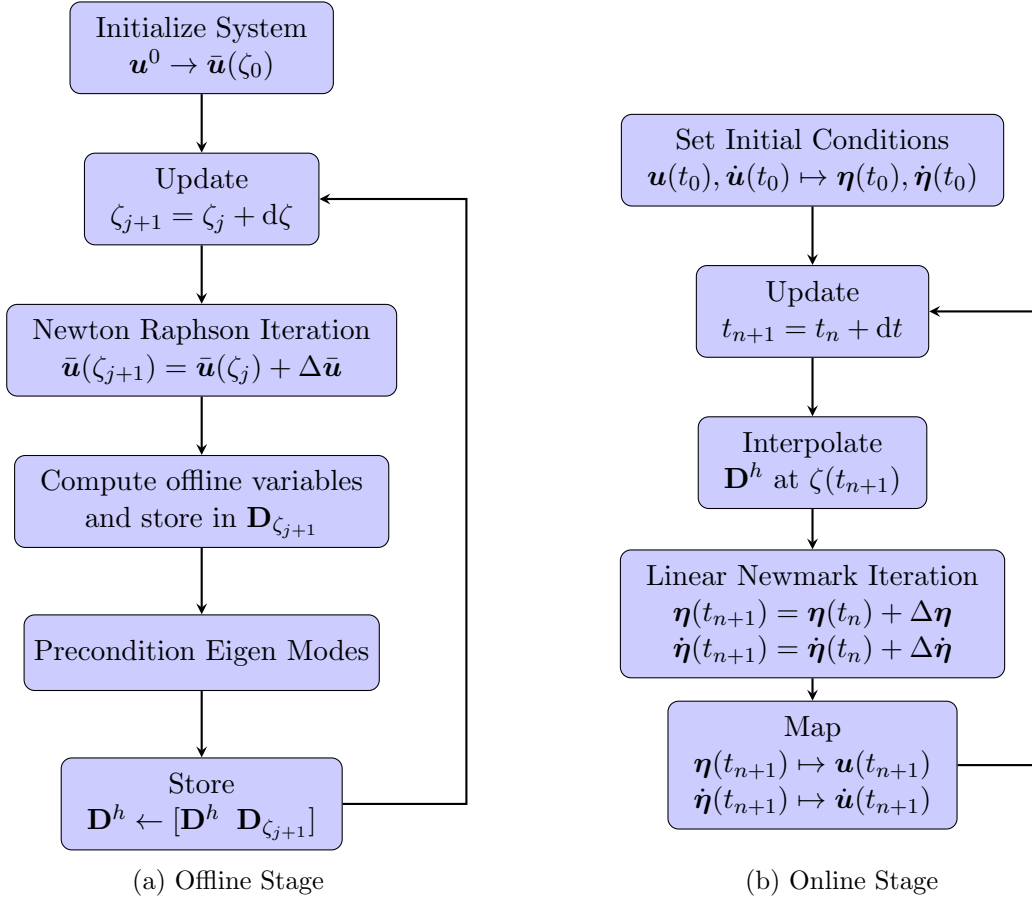


Figure 6.1: The two stage procedure for the time integration of reduced equations of the cable slab system. Here $\mathbf{u}^0$ represents the initial undeformed configuration of the FEM model. Here $\mathbf{D}_\zeta = \{\bar{\mathbf{u}}, \mathbf{s}, \mathbf{s}_{,\zeta}, \mathbf{X}, \mathbf{X}_{,\zeta}, \boldsymbol{\Omega}, \bar{\mathbf{f}}^{ext}\}$, is computed at a given value of ζ

used to update the system information at every instant. The details of the interpolation scheme is discussed in Section 6.3.

Figure 6.1 gives an outline of the different steps involved in the offline and online stages of the solution procedure.

6.2 Preconditioning of the Eigenmode

For the interpolation to work, a smooth function response is needed. This is usually not the case with eigenmodes. As discussed in section 3.4, two solutions of the eigenmodes exist for a given eigen frequency. This causes a phase switching phenomenon to occur where the mode switches rapidly between positive and negative values. Another issue with eigenmode response is mode veering which occurs when two eigenfrequency responses tend to cross over resulting in the switching of the responses of both the eigenmode and its eigenfrequency. These phenomenon are illustrated in Figure 7.1 and 7.2. Both these effects result in non-smooth behavior of the eigenmode and eigenfrequency

response. To circumvent this problem a preconditioning procedure given by is performed after the modes are computed. This procedure introduced here is adapted from the work done by [68].

The aim of the procedure is, for a given a set of two close-by eigenmodes $\mathbf{X}_j$ and $\mathbf{X}_{j+1}$, to construct a permutation matrix $\hat{\mathbf{Q}}$ that swaps and/or switches the sign of the columns of $\mathbf{X}_{j+1}$ such that it most closely resembles $\mathbf{X}_j$. This can be expressed as a minimization problem given as:

$$\min_{\hat{\mathbf{Q}}} \|\mathbf{X}_j - \mathbf{X}_{j+1}\hat{\mathbf{Q}}\|^2 . \quad (6.1)$$

Since the elements of $\hat{\mathbf{Q}}$ can take only discrete values of $\{-1, 0, 1\}$, Equation (6.1) becomes a discrete optimization problem which cannot be solved analytically. Hence problem is first made continuous by using a general orthonormal rotation matrix given by $\mathbf{Q}$ and substituting in Equation (6.1). An orthonormal matrix is selected since it preserves the subspace spanned by $\mathbf{X}_{j+1}$ upon multiplication. The solution to continuous problem is given in [68] which is as follows:

$$\mathbf{U}\mathbf{S}\mathbf{V}^T = \mathbf{X}_{j+1}^T \mathbf{M}_{ii} \mathbf{X}_j \text{ (SVD)} \quad (6.2)$$

$$\mathbf{Q} = \mathbf{U}\mathbf{V}^T . \quad (6.3)$$

Where $\mathbf{M}_{ii}$ is the mass matrix of the inner nodes. Although $\mathbf{Q}$ preserves the subspace of $\mathbf{X}_{j+1}$, the mode shapes are lost after multiplication. Since the mode shapes are of interest here, $\hat{\mathbf{Q}}$ must be determined. This is done by assuming that the eigenmode basis are sufficiently close to each so that $\mathbf{X}_{j+1}$ has rotated only slightly with respect to $\mathbf{X}_j$. $\hat{\mathbf{Q}}$ can then be obtained by rounding off the values of $\mathbf{Q}$ to the nearest integer. This operation is given as:

$$\hat{\mathbf{Q}} = \text{round}(\mathbf{Q}) \quad (6.4)$$

The new value of $\mathbf{X}_{j+1}$ is given by $\mathbf{X}_j \hat{\mathbf{Q}}$. In order to correct for mode veering in the eigenfrequencies, $\boldsymbol{\Omega}$ is multiplied by $\mathbf{Q}^*$, where $\mathbf{Q}^*$ is the absolute value of $\hat{\mathbf{Q}}$ given as:

$$\mathbf{Q}^* = |\hat{\mathbf{Q}}| . \quad (6.5)$$

The entire procedure is repeated sequentially on all pairs of eigenmode basis. This is given in Algorithm (1). The results after applying the procedure is given in Figure 7.1 and 7.2.

Algorithm 1 Preconditioning procedure

```

1: Input:  $\mathbf{X}_j$  and  $\mathbf{\Omega}_j$  where  $j = 1, 2..l$  and  $\mathbf{M}_{ii}$ 
2: Output:  $\tilde{\mathbf{X}}_j$  and  $\tilde{\mathbf{\Omega}}_j$  where  $j = 1, 2..l$ 
3: begin
4: for  $j = 1$  to  $l$  do
5:   if  $j = 1$  then
6:      $\tilde{\mathbf{X}}_j = \mathbf{X}_j$ 
7:      $\tilde{\mathbf{\Omega}}_j = \mathbf{\Omega}_j$ 
8:   else
9:      $\mathbf{P} = \mathbf{X}_{j-1}^T \mathbf{M}_{ii} \mathbf{X}_j$ 
10:     $\mathbf{P} = \mathbf{U}^T \mathbf{S} \mathbf{V}$  (SVD)
11:     $\mathbf{Q} = \mathbf{U}^T \mathbf{V}$ 
12:     $\hat{\mathbf{Q}} = \text{round}(\mathbf{Q})$ 
13:     $\hat{\mathbf{Q}}^* = |\hat{\mathbf{Q}}|$ 
14:     $\tilde{\mathbf{X}}_j = \mathbf{X}_j \hat{\mathbf{Q}}$ 
15:     $\tilde{\mathbf{\Omega}}_j = \mathbf{\Omega}_j \hat{\mathbf{Q}}^*$ 
16:   end if
17: end for
18: end

```

6.3 Interpolation of configuration quantities

Since certain quantities and its derivatives need to be interpolated like the eigenmodes ($\mathbf{X}$ and $\mathbf{X}_{,\zeta}$) and the static response ($\bar{\mathbf{u}}, \bar{\mathbf{s}}$ and $\bar{\mathbf{s}}_{,\zeta}$), it makes sense to implement an interpolation scheme that takes advantage of the available derivative information to improve its accuracy. To keep the interpolation scheme consistent for all quantities, up to the second order derivatives of each individual quantity is computed and a 5th order polynomial interpolation is constructed using the two adjacent points and its derivative information. Although this implies the need to compute additional derivatives of quantities that are not used in the dynamic equations, it still is numerically efficient

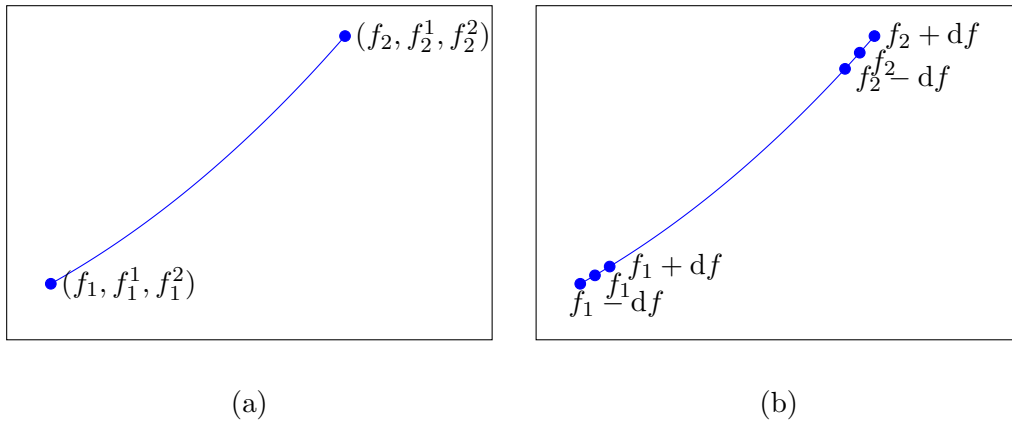


Figure 6.2: A polynomial interpolation using derivatives (a) is equivalent to one with points placed very close to each other (b)

since it maintains a consistent accuracy for each quantity which translates to better step size selection that ultimately reduces the total computational time by decreasing the number of required iterations.

Using derivatives to construct the polynomial also has an added benefit that it is stable and avoids the error cause by Runge phenomenon. This can be understood as follows. Consider a 5th order polynomial constructed using 2 function values, f_1 and f_2 and its first two derivatives evaluated at the each point, f'_1 , f'_2 , f''_1 and f''_2 . This is equivalent to a polynomial constructed using the same 2 function values f_1 and f_2 and another 4 function values $f_1 + dy$, $f_1 - dy$, $f_2 + dy$ and $f_2 - dy$ that are infinitesimally close to them as shown in Figure 6.2. The same can be said about higher order derivatives. Such a node distribution follows the general principle that avoids Runge error.

An alternative interpolation method is briefly explored here known as the manifold interpolation. However it is not implemented as it does not provide any significant improvement in the accuracy. A brief analysis of the accuracy of the method is given in Appendix B.

6.4 Step size selection

The step size of the static iteration is of utmost importance as it determines the total number of load steps required which in turn decides the overall computational speed of the offline stage. The choice of the stepsize is limited by the accuracy of the applied interpolation scheme. Hence for optimum step size selection it is important to quantify the accuracy of the interpolation scheme. This can be achieved by applying the Cauchy's error theorem which can be stated as follows:

Theorem. *Let $f(x)$ be a real $(n + 1)$ -times continuously differentiable function on the bounded interval $[a, b]$. For the interpolation polynomial $P_n(x)$ with $(n + 1)$ pairwise distinct nodes $x_0, \dots, x_n$ with $\min_i(x_i) = a$ and $\max_i(x_i) = b$ and values $y_i = f(x_i)$ we have for each $\bar{x} \in [a, b]$:*

$$\varepsilon(\bar{x}) = f(\bar{x}) - P_n(\bar{x}) = \frac{f^{n+1}(\xi)}{(n + 1)!} \prod_{i=0}^n (\bar{x} - x_i) . \quad (6.6)$$

Where $\xi \in [a, b]$ is independent of $\bar{x}$.

Here, $\varepsilon(\bar{x})$ represents the interpolation error at $\bar{x}$. The above expression applies to a scheme where only coordinate values are used for the interpolation and not its derivatives. Thus the equivalent polynomial representation of Figure 6.2 is applied again. Using this representation, the interpolation error of the 5th order polynomial

($n = 5$) between the points ζ_k and ζ_{k+1} is given as:

$$\varepsilon(\zeta) = \frac{f^6(\xi)}{6!} (\zeta - \zeta_k)^3 (\zeta - \zeta_{k+1})^3 . \quad (6.7)$$

To obtain the error norm in terms of the stepsize $\Delta\zeta = (\zeta_{k+1} - \zeta_k)$, $\zeta = (\zeta_{k+1} + \zeta_k)/2$ is substituted in Equation (6.7). After simplification one obtains:

$$\varepsilon = \frac{f^6(\xi)}{2^6 6!} \Delta\zeta^6 . \quad (6.8)$$

$f^6(\xi)$ is generally unknown and since only an error estimation is required, a finite difference approximation at an arbitrary value of ξ within the interpolation limits (usually at $\xi = (\zeta_{k+1} + \zeta_k)/2$) can be chosen. Thus, Equation (6.8) becomes an approximate expression:

$$\varepsilon \approx \frac{f^6(\xi)}{2^6 6!} \Delta\zeta^6 . \quad (6.9)$$

A relative error is obtained by dividing the RHS of Equation (6.8) by $f(\xi)$:

$$\varepsilon_{rel} \approx \frac{f^6(\xi)}{2^6 6! f(\xi)} \Delta\zeta^6 . \quad (6.10)$$

Finally, Equation (6.10) is solved for $\Delta\zeta$ by setting ε_{rel} as the desired tolerance ε_{tol} of the interpolation:

$$\Delta\zeta \approx 5.98 \left(\frac{f(\xi) \varepsilon_{tol}}{f^6(\xi)} \right)^{1/6} . \quad (6.11)$$

The choice of ε_{tol} is upto the user. The accuracy of quantities like eigenmodes is not as demanding as that for the tangent stiffness matrix for example. Moreover, since the accuracy of the interpolation scheme can never be higher than the accuracy of the quantities used for the interpolation itself, a general rule for setting ε_{tol} would be to keep it equal to or greater than the error tolerance of the offline solver ε_{sol} . Hence:

$$\varepsilon_{tol} \geq \varepsilon_{sol} . \quad (6.12)$$

The interpolation error of the first order derivative can be obtained by differentiating Equation (6.10) with respect to ζ :

$$\varepsilon_{rel}^{deriv} \approx \frac{f^6(\xi)}{2^6 5!} \Delta\zeta^5 . \quad (6.13)$$

The step size can also be determined using Equation (6.13) in case the accuracy on the derivatives have to be enforced.

Results

7.1 Output of the Preconditioning procedure

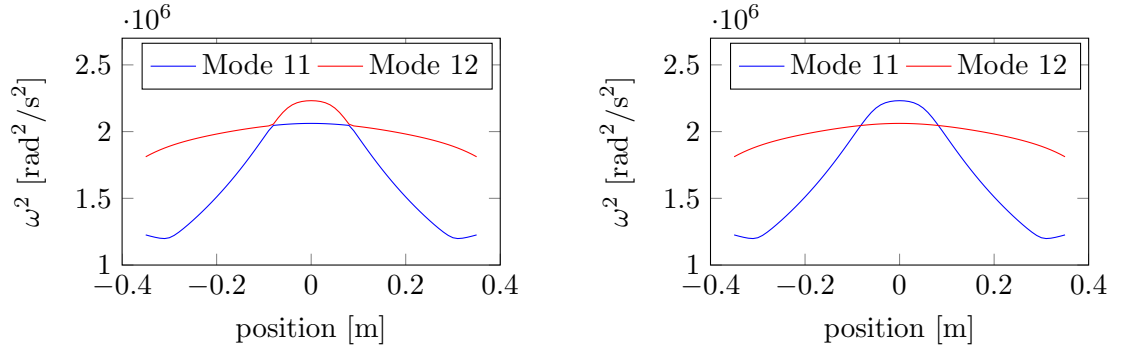


Figure 7.1: Mode veering phenomenon of mode 11 and 12 (left) and its corrected response (right)

The efficacy of the preconditioning procedure described in section 6.2 is tested by comparing the results of the eigen frequency and the eigenmode response. Figure 7.1 gives the plot of the eigen frequency response for mode 11 and 12 with respect to ζ before and after the preconditioning. A clear phenomenon of mode veering is observed in the first figure where the behavior of the two frequencies swap near the points where they

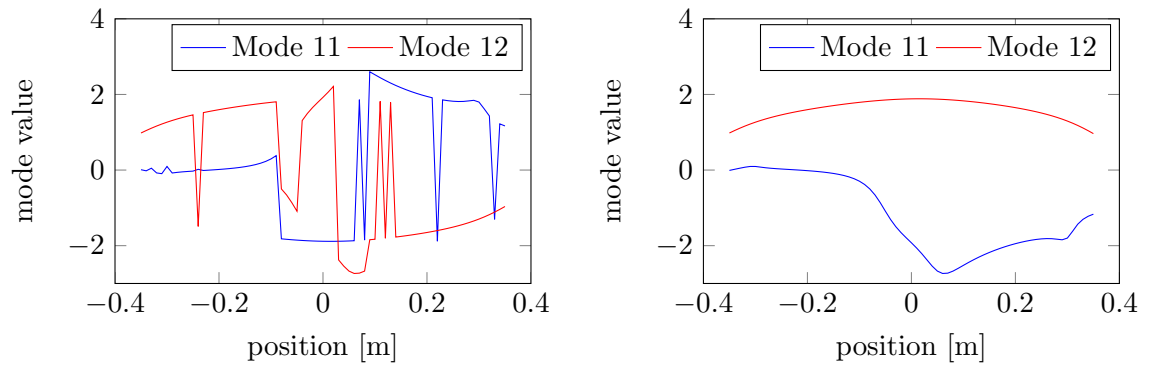


Figure 7.2: Eigen mode response of mode 11 and 12 without preconditioning exhibiting both mode veering and phase switching(left) and its response after correction (right)

coincide. This is due to the crossing of the eigenmodes at this point. This phenomenon is corrected after preconditioning the frequencies.

The eigenmode response of some arbitrary coordinate of the same two modes is now compared. The mode response before the preconditioning is highly irregular due to a combination of mode crossing and phase flipping. Preconditioning of the modes produces a smooth response curve devoid of these effects.

7.2 Model Verification

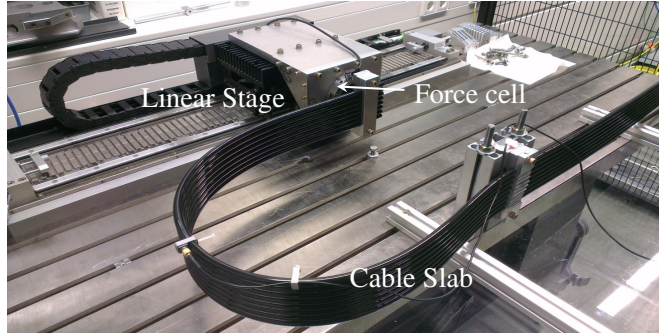


Figure 7.3: The experimental setup of a cable slab. Connection is made to the linear stage via a force cell that measures the disturbance forces generated by the cable slab

The GCB model is verified against the fully nonlinear model and experiment. A nonlinear implicit Newmark scheme is used to integrate the fully nonlinear model. For the following analysis, a stepsize of 10^{-2} [m] was selected for the offline stage to achieve a high interpolation accuracy of the order 10^{-6} to 10^{-8} in the interpolated quantities. The cable slab test setup is shown in figure 7.3. A force cell is mounted at the hinge of the slab and the stage in order to measure the reaction or the disturbance force. Only the force measurement in the stroke direction (X) is considered here. The length of the cable slab can be freely varied by adjusting the mounting.

The mass of the cable slab was tuned by weighing it and the stiffness was tuned by estimating the flexural rigidity K_b of the system. A rough estimation of K_b was made first and later tuned by measuring the first two eigen frequencies of the cable slab at a given length using experimental frequency response analysis and matching the eigen frequencies obtained from the model. The results of the analysis is shown in Table 7.1. It was seen that tuning the flexural rigidity of the slab predicted both the eigen frequencies quite well, thus validating the tuning technique. The experiment was repeated for a different length to again verify the tuning of the flexural rigidity. The tuned parameters is given in Table 7.2.

Length [m]	ζ [m]	Experiment		FEM	
		ω_1 [Hz]	ω_2 [Hz]	ω_1 [Hz]	ω_2 [Hz]
0.8	0.0	10.5	31.74	10.94	32.2
0.8	0.2	10.5	32.96	11.95	32.37
0.7	0.0	15.87	43.95	15.24	42.67
0.7	0.2	17.09	43.95	16.82	42.97

Table 7.1: Results of frequency response analysis comparing the first 2 eigenfrequencies obtained from the experiment and the model. The experimental result of the first row was used for tuning, and the subsequent rows for verifying the model.

Parameter	Value	Unit
stage separation (d_z)	0.3	m
linear mass density (m/L)	0.311	kg m ⁻¹
width* (w)	0.05	m
thickness* (a)	0.01	m
moment of inertia* (I_{yy})	4.17e-9	m ⁴
Young's modulus* (E)	154	MPa
Flexural rigidity (K_b)	0.641	N m ²

Table 7.2: Tuned parameters of the cable slab model. (*- These parameters may not physically correspond to that of the setup).

Before verifying the GCB model, the static response of the cable slab is measured and compared with the fully nonlinear model as a pre-verification of the FEM model and the actual setup.

7.2.1 Static response behavior

The following experiment was performed to test the static behavior of the cable slab. A setpoint with max acceleration of 0.1 m/s² and max velocity of 0.1 m/s is applied order to not excite the dynamics of the system. The stage is displaced from -0.1 to 0.1 [m] and then back to -0.1 [m]. The response is as shown in Figure 7.4. It is clear from the plot is clear that FEM model fails to predict the static behavior of the cable slab. This indicates that the predicted stresses in the slab is incorrect which is a consequence of the assumption of a linear elastic model. The cable slab exhibits creep behavior which allows it to relax the internal stresses in deformed configuration thus possibly explaining the failure of the model in predicting the static response. The FEM model is therefore not applicable towards predicting the static reaction forces at the hinges. Another effect not accounted in the model that is clearly visible from the force versus position plot of Figure 7.4 is the hysteresis caused due to visco-elasticity of the material of the cable slab. This effect is responsible for the slow drift of the reaction force over time when the system is at rest. The dynamics due to the visco-elasticity is very slow compared to the bandwidth of the controller hence can be neglected as a static effect.

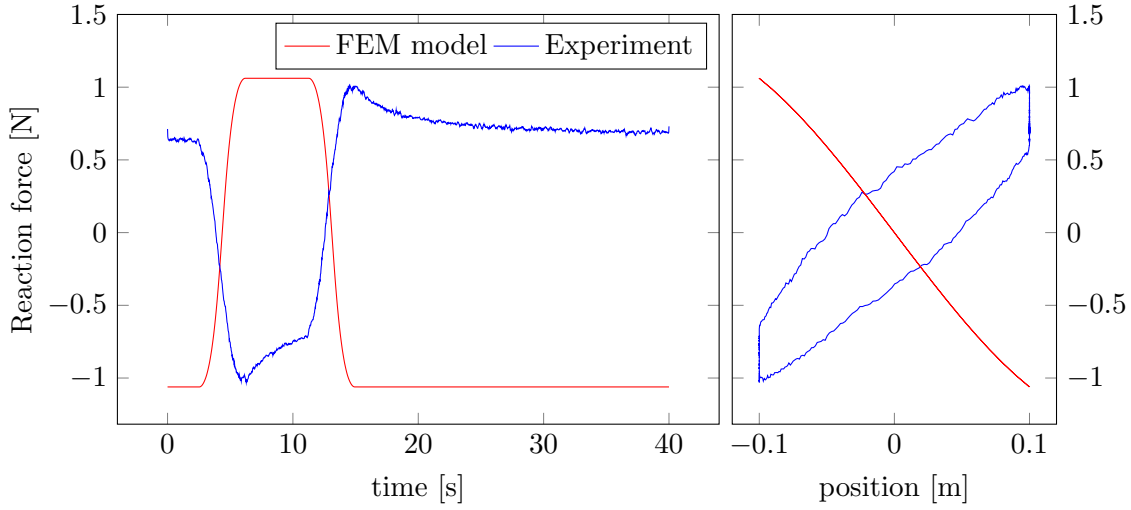


Figure 7.4: Comparison of the static response of the cable slab determined experimentally with that of the FEM model. The model completely fails to predict the response. A hysteresis due to viscoelastic effects is seen in the experimental setup.

Since the focus of this thesis is on the dynamic behavior of the cable, the prediction of the static forces is not important but the uncertainty introduced in the dynamic model due to the failure of predicting the stresses need to be accounted. From observing the cable slab setup it is seen that the configuration is more or less the same as that predicted by the model, hence the changes in the stress distribution due to creep does not affect the static configuration. The other quantity that is affected by the stresses is the tangent stiffness matrix. This effect can be understood as follows.

The tangent stiffness matrix can be additively decomposed into a material and geometric component denoted as $\mathbf{K}_{mat}$ and $\mathbf{K}_{geo}$ respectively:

$$\mathbf{K} = \mathbf{K}_{mat} + \mathbf{K}_{geo} . \quad (7.1)$$

The material stiffness matrix is a function of the stress distribution in the material and the geometric stiffness matrix is a function of the current configuration of the system. Since it is established that the configuration is not affected by the changes in stress distribution, the geometric stiffness matrix can be assumed to be unaffected by viscoelastic effects. The sensitivity of the static forces to the stress distribution can be understood by computing the tangent stiffness at the hinge in the direction of the force measurement (along ζ). This stiffness is given as:

$$k_{\zeta\zeta} = \hat{\mathbf{u}}_b^T (\mathbf{K}_{bb} - \mathbf{K}_{bi} \mathbf{K}_{ii} \mathbf{K}_{ib}) \hat{\mathbf{u}}_b . \quad (7.2)$$

The value of $k_{\zeta\zeta}$ is computed for the stiffness matrix with and without the material matrix contribution at $\zeta = 0$ [m] and $L = 0.9$ [m]. This is shown in Table 7.3. It is

	$\mathbf{K} = \mathbf{K}_{mat} + \mathbf{K}_{geo}$	$\mathbf{K} = \mathbf{K}_{geo}$
$k_{\zeta\zeta}$ [N/m]	-12.19	41.81

Table 7.3: Values of the tangent stiffness along ζ with and without the material stiffness contribution for $L=0.9$ [m] and $\zeta = 0$

seen that the tangent stiffness is highly sensitive to the stresses. Moreover, the sign of the stiffness has changed, further indicating that stress relaxation has occurred. It is

Mode Number	error% ω_j	MAC $\mathbf{x}_j$	error % g_j
1	5.7	1	0.0166
2	7.3	0.99	0.0682
3	4.7	0.99	0.9396
4	3.6	0.99	2.56

Table 7.4: Maximum uncertainty in the respective quantities due to unknown $\mathbf{K}_{mat}$

now left to see the influence of $\mathbf{K}_{mat}$ on the dynamics. The main quantities in the GCB expression determined by the stiffness matrix are the eigenmodes, eigen frequencies and the effective modal masses $g_j = \mathbf{x}_j^T \mathbf{M}_{ii} \mathbf{s}$. The uncertainty in these quantities for the first four XZ modes is shown in table 7.4. It is seen that the contribution of the stresses in the dynamics is low. Thus, it is valid to use the model the dynamic response.

7.2.2 Dynamic behavior

In order to study and verify the dynamic behavior of the cable slab, the length is first set to 0.9 [m] and an experiment is performed where the cable slab is excited by input prescribed motion shown in Figure 7.5. The force cell reading in the direction of the excitation is measured. The output is a combination of the cable slab disturbance force along with various other spurious contributions. Thus several data processing/filtering steps need to be carried out to extract the relevant dynamics and is briefly outlined here. Care is taken to account for the effects of the processing on the desired output and certain steps are repeated on the model itself to account for it. The following are the data processing steps taken.

The experiment is done twice, one with the cable slab and one without (figure 7.7a). The extra inertial force due to the mounting between the force cell and the cable slab is eliminated by subtracting the resulting output from the original one (Figure 7.7b). The experiment is repeated several times and the force cell output is averaged to eliminate stochastic noise. A low pass filtering is applied to further improve the signal to noise ratio (Figure 7.7c). Dynamics due to the cable slab can be clearly seen here after the setup comes to a halt after 0.4 [s]. However the constant velocity region still has poor signal to noise ratio due to the high bearing noises in the stages. Hence from now on only the region after 0.4 [s] is shown. From the filtered output, it is seen that the

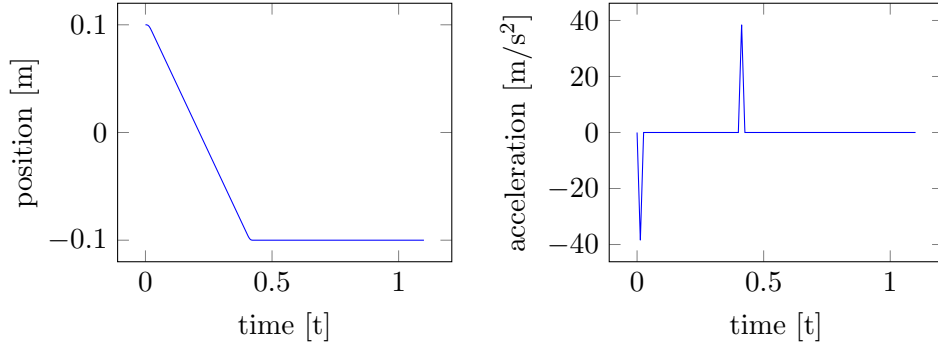


Figure 7.5: Imposed setpoint on the linear stage

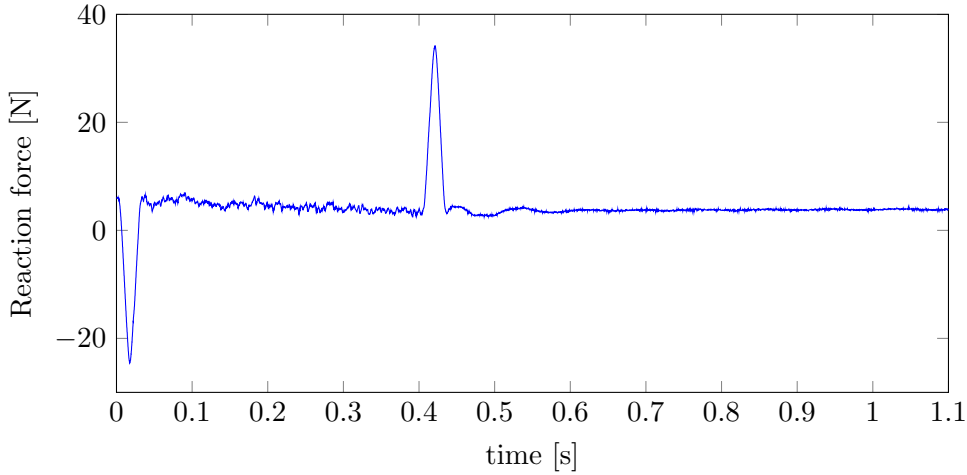


Figure 7.6: Output of the force cell in the direction of excitation for the given imposed setpoints

reaction force slowly drifts even after the cable slab has come to a halt. This is because the visco-elastic nature of the material of the slab. In order to compensate this drift, an exponential decay curve is constructed and tuned such that it matches the drift caused by visco-elasticity. This is shown in Figure 7.7d. With the drift corrected, the output data can finally be compared to numerical simulations.

The low pass filtering step performed on the experimental output is repeated on the simulation result in order to reproduce its effect of the truncation or phase shift on the dynamic response. The damping is varied till the model best agrees with the setup. The output of the tuned fully nonlinear model and the GCB model using the first 2 XZ modes is plotted in Figure 7.8. A good agreement is seen between the fully nonlinear model and the GCB model, verifying the method and a fair agreement is seen between the GCB model and the experimental results which verifies the method and the assumptions made. In order to verify the model to parameter changes, the experiment and the simulations are repeated with the length reduced to 0.8 [m]. No other parameter other than the damping is varied in the numerical model. The plot

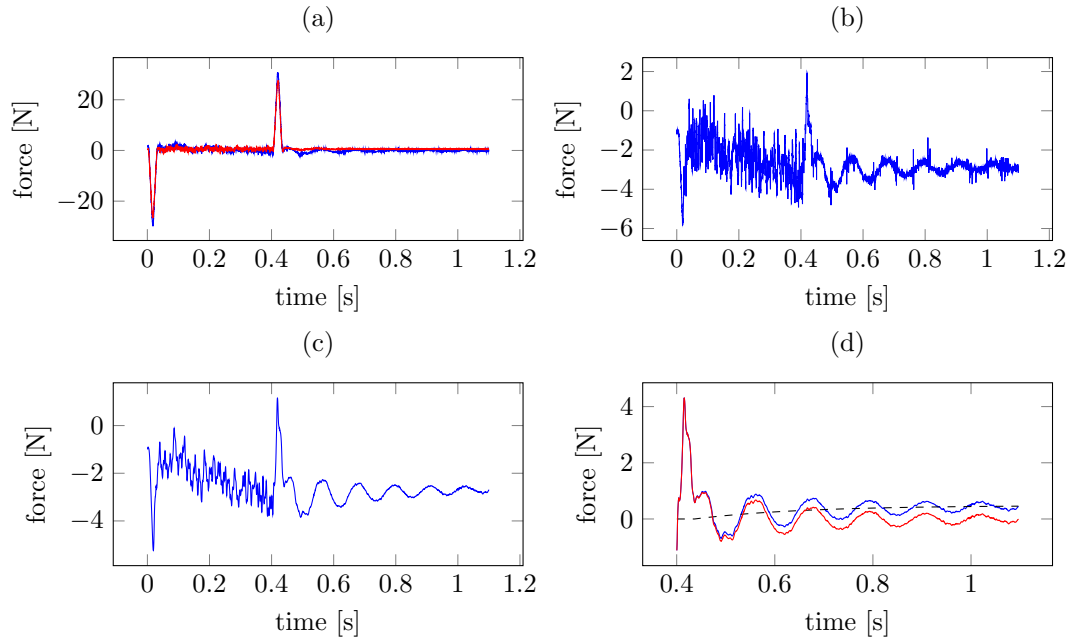


Figure 7.7: Data processing steps of the force cell output. (a) Force cell output with (blue) and without (red) cable slab is obtained. (b) The two outputs are subtracted to eliminate inertial contribution due to mounting. (c) The subtracted signal is averaged and filtered to remove noise. (d) Visco elastic drift is compensated.

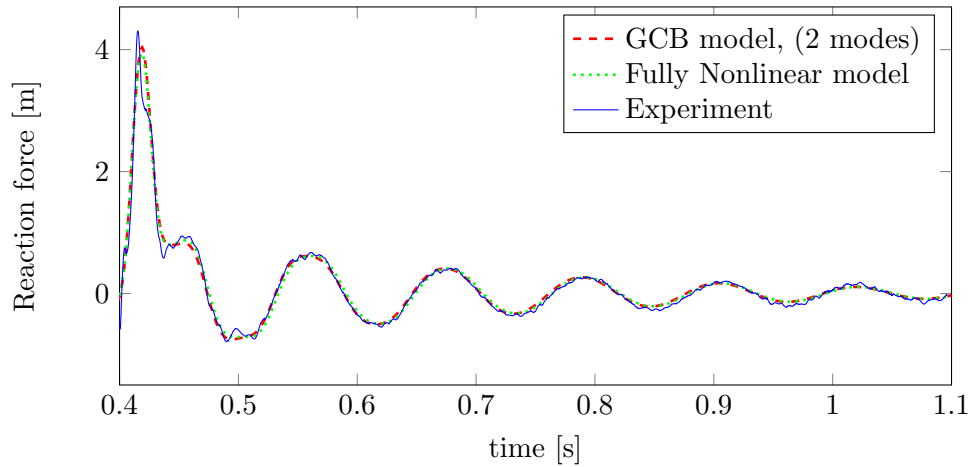


Figure 7.8: Comparison of the filtered experimental result with numerical simulation using the reduced GCB and the fully nonlinear FEM model for the given excitation and length equal to 0.9 [m]. The first 2 bending modes are used in the construction of the CB basis.

is shown in Figure 7.9. A good agreement is seen again between all the plots, further validating the model and the reduction technique used.

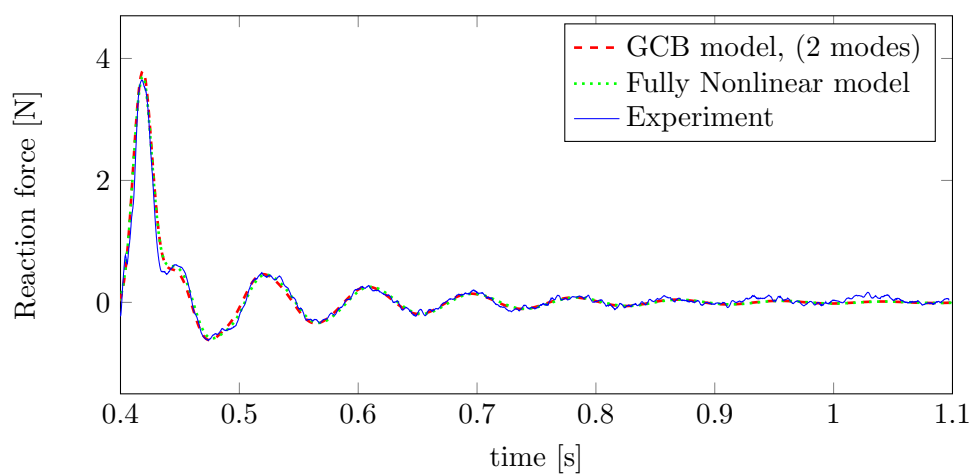


Figure 7.9: Comparison of the filtered experimental result with numerical simulation using the reduced GCB and the fully nonlinear FEM model for the given excitation and length equal to 0.8 [m]. The first 2 bending modes are used in the construction of the CB basis.

Conclusion and Recommendation

8.1 Conclusion

In this work, a general theory of model reduction is developed by replacing the idea of a subspace with that of a parameterized manifold. This allows geometrically nonlinear dynamic problems to be described using a small set of generalized coordinates. The idea that a parameterized manifold behaves locally as a subspace is used to extend the concept of a well known linear reduction technique, the Craig Bampton method to the so called Generalized Craig Bampton or the GCB method. This new technique could now be applied to describe substructures undergoing finite deformations due to imposed boundary modes.

The GCB technique is used to obtain the reduced equations of the cable slab. It allows the internal dynamics of the cable slab to be linearized using the assumption of small vibrations. However, the equations are still expensive to numerically integrate since the coefficients of the equations are functions of the imposed displacement which are computationally expensive to evaluate. Thus a two stage offline-online integration scheme is proposed where these coefficients are precomputed at discrete values of the imposed displacement and stored offline and then interpolated during the online dynamic integration stage. This eliminates the need to recompute the coefficient terms for multiple simulations which results in a huge gain in computational efficiency (about 100-1000 $\times$) over a fully nonlinear simulation. Furthermore, it is also seen that the combined computational speed of offline and online stage is still far higher than a fully nonlinear dynamic simulation (10-100 $\times$) because of the fact that the offline solver uses a much larger increment steps compared to a dynamic solver.

Finally, the methods described in the work is validated against an experimental cable slab setup. It is seen that the nonlinear model completely fails to predict the static response of the cable slab. It is concluded that this is due to the viscoelastic effects in the cable slab material that alters the static stresses in a cable slab which in turn affects the external reaction forces measured at the mounting. An estimation was made on the

effect of the altered stresses on the dynamics of the cable slab and it is that it has a very small influence on the dynamic coefficients, thus allowing the use of the model.

The FEM model was tuned to the given setup and a given setpoint was applied on the setup and the disturbance reaction force was measured and compared with numerical results. It was seen that the GCB model with the first two relevant modes matched almost exactly the result of the fully nonlinear model and the experiment thus validating the reduced model and the GCB method. To test the tuned model against parameter changes, the length of the cable slab was changed and the experiment was repeated and compared with simulation with only the corresponding parameter being modified. The results of the reduced model again predicted the dynamic behavior with very good accuracy, thus validating the model and the GCB method.

Thus the objectives set out in the introduction of this work have been successfully met. The Generalized Craig Bampton method is a fast and accurate technique to model the nonlinear behavior of the cable slab. It naturally extends the application of the Craig Bampton method which is an intuitive and well understood technique in the field of model reduction and thus gives a good interpretation of the system dynamics.

With the success of the GCB method in modeling the cable slab dynamics, further applications and modifications of the method can be envisioned. This is described in the next section.

8.2 Recommendations

This section outlines some of the drawbacks of the GCB method and proposes alternative techniques that can be implemented.

8.2.1 Use of Ritz vectors

In the GCB method, the dynamic component of the tangent basis is constructed using eigenmodes. They are the natural choice for describing dynamic motions since they provide a good approximation for a narrow bandwidth excitation and physical interpretation of the overall dynamics. But computationally they are very inefficient to implement for the following reasons.

- *They are expensive to compute.* Iterative procedures are required to compute the eigenmodes which involves the factorization of the stiffness matrix for every iteration. Moreover, some eigenmodes might have poor participation factor and this can be known only after computing it, thus adding to the computational effort. Even the eigenmode derivatives are equally if not more expensive to compute.
- *The methods used are prone to ill conditioning.* In geometrically nonlinear structures, the eigen frequencies evolve as the system deforms and inevitably cross each other at some point (especially for the higher eigen frequencies). The dynamic

stiffness matrix $(\mathbf{K} - \omega^2 \mathbf{M})$ is highly susceptible to ill conditioning around this region. This affects the computation of the eigenmodes and the derivatives. Ill conditioned problems mostly lead to incorrect results. Special algorithms need to be used in such a situation.

In order to overcome these limitations an alternative method to compute the basis is proposed based on the work done in [36]. These new set of vectors are known as Ritz vectors and are denoted by $\mathbf{R}$. They are obtained by solving the following recurrence relationship:

$$\begin{aligned} \mathbf{K}_{ii} \mathbf{r}_1 &= \mathbf{M}_{ii} \mathbf{s} \\ \mathbf{K}_{ii} \mathbf{r}_j &= \mathbf{M}_{ii} \mathbf{r}_{j-1}, \quad j = 2, \dots, k \end{aligned} \quad (8.1)$$

Where $\mathbf{s}$ is the static mode introduced in Chapter ???. If $\mathbf{D} = \mathbf{K}_{ii}^{-1} \mathbf{M}_{ii}$, $\mathbf{R}$ can be simply written as:

$$\mathbf{R} = \begin{bmatrix} \mathbf{D} \mathbf{s} & \mathbf{D}^2 \mathbf{s} & \dots & \mathbf{D}^k \mathbf{s} \end{bmatrix} \quad (8.2)$$

$\mathbf{R}$ is not orthogonal. Hence an orthogonalization procedure is performed using a projection matrix $\mathbf{P}$ to obtain $\tilde{\mathbf{R}}$:

$$\tilde{\mathbf{R}} = \mathbf{P} \mathbf{R}, \quad \tilde{\mathbf{R}}^T \tilde{\mathbf{R}} = \mathbf{I} \quad (8.3)$$

The Craig Bampton tangent basis can now be written as:

$$\Phi = \begin{bmatrix} \mathbf{s} & \tilde{\mathbf{R}} \\ \hat{\mathbf{u}}_b & \mathbf{0} \end{bmatrix} \quad (8.4)$$

The modal derivative of $\tilde{\mathbf{R}}$ with respect to ζ can be computed by direct differentiation of Equation (8.3):

$$\tilde{\mathbf{R}}_{,\zeta} = \mathbf{P}_{,\zeta} \mathbf{R} + \mathbf{P} \mathbf{R}_{,\zeta} \quad (8.5)$$

Where $\mathbf{R}_{,\zeta}$ is given by differentiating equation 8.1 with respect to ζ and solving it:

$$\begin{aligned} \mathbf{K}_{ii} \mathbf{r}_{1,\zeta} &= \mathbf{M}_{ii} \mathbf{s}_\zeta - \mathbf{K}_{ii,\zeta} \mathbf{r}_1 \\ \mathbf{K}_{ii} \mathbf{r}_{j,\zeta} &= \mathbf{M}_{ii} \mathbf{r}_{j-1,\zeta} - \mathbf{K}_{ii,\zeta} \mathbf{r}_j, \quad j = 2, \dots, k \end{aligned} \quad (8.6)$$

From the above expressions, it is seen that ritz vectors and its derivatives are fairly cheap to compute compared to eigenmodes. They are also not prone to ill conditioning. Ritz vectors span the subspace of only those modes that are excited by the applied forces/displacements. Hence only a few vectors need to be computed. The trade-off of this method is that the physical interpretation of the modes is lost upon orthogonalization.

8.2.2 GCB for multiple inputs

In cases where many imposed boundary conditions are applied (e.g./ in robotics), $\hat{\mathbf{u}}_b$ will span a multidimensional space. This would escalate the cost of the offline stage because of the need to compute the derivatives of the basis along each direction. Also, a multivariate interpolation needs to be implemented, which is cumbersome. An offline stage in such a case is therefore not recommended. In such a situation it is more efficient to compute the basis during the online stage along the instantaneously computed value of $\hat{\mathbf{u}}_b$.

Without the offline stage, a nonlinear dynamic solver is required. In this case the method proposed by Bathe [31] can be used to condense the linear equations before the nonlinear iterations thus effectively reducing the number of degrees of freedom of the system. A cheap way to approximate the modal basis can be constructed to avoid costly recomputations. Thus, prospects do exist even for this case.

Publications

Below are a list of selected abstracts based on the work done in report.

1. A. Sridhar, P. Tiso and T. Hardeman, Nonlinear model reduction of cable slab dynamics. International Conference on Noise and Vibration Engineering, September 2014, Leuven.
2. T. Hardeman, A. Sridhar, S. Boere and P. Tiso, Advanced modelling and simulation of non-linear cable slab dynamics in high precision systems. Dutch Society for Precision Engineering, 2014.

Eigenmodes of the cable slab

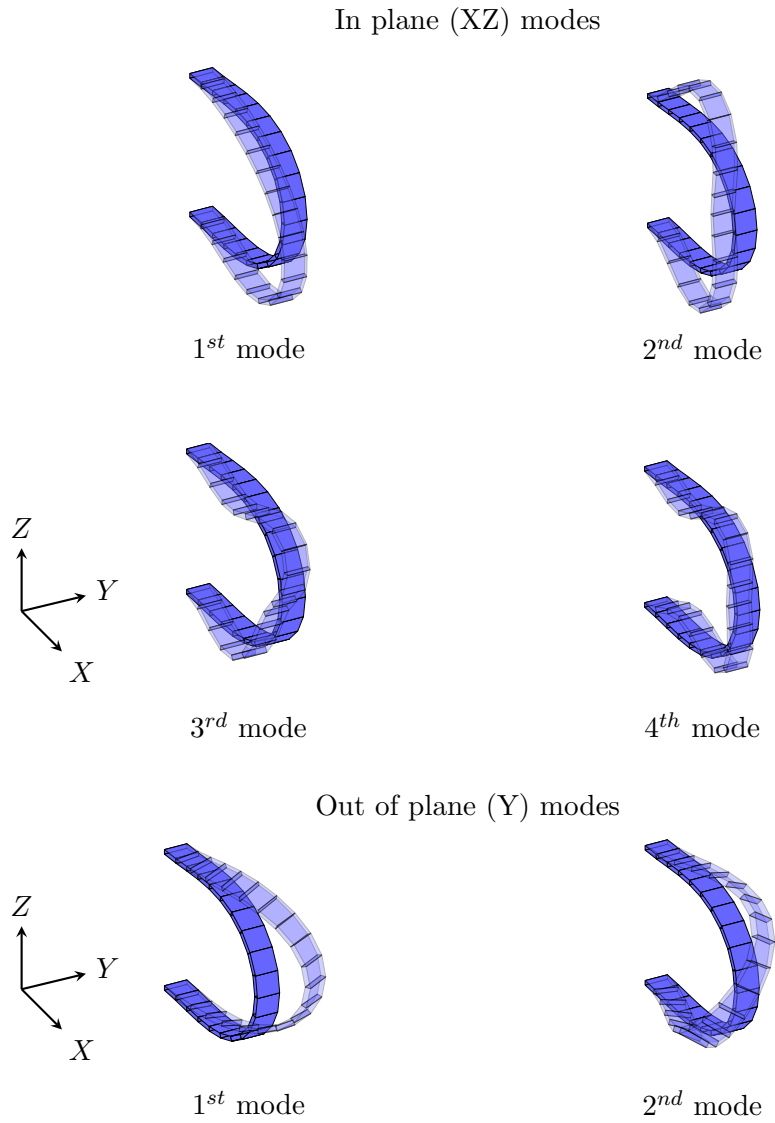


Figure A.1: The first 6 eigenmodes of the cable slab at mid stage configuration.

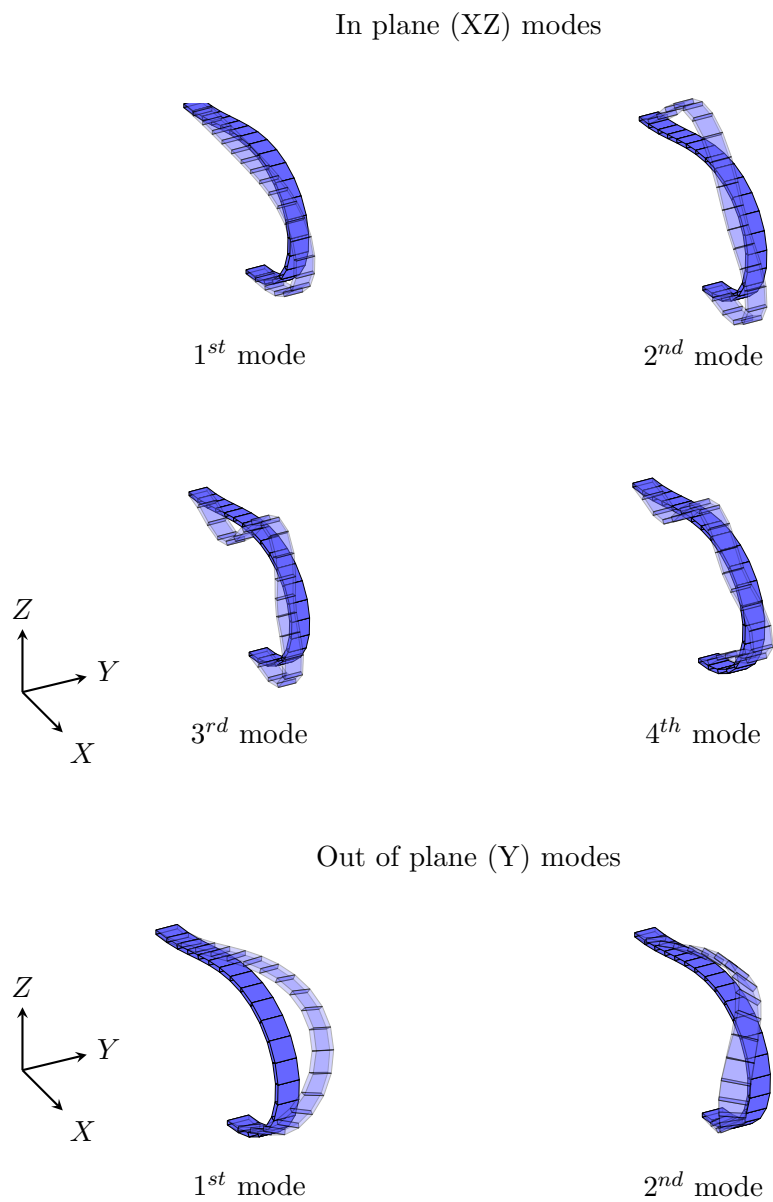


Figure A.2: The first 6 eigenmodes of the cable slab at extreme stage configuration.

Comparative analysis of Manifold interpolation

The method discussed here is based on the work done by Amasalleem [70] which is slightly modified for interpolating eigenmodes. The author introduces a new interpolation technique (termed as manifold interpolation here for convenience) suitable for interpolating reduced basis. The method has the property of preserving the orthogonality of the interpolated basis which is not guaranteed by direct interpolation. This section attempts to check whether this property has any benefit. In order to check the accuracy of the manifold interpolation it is compared with direct interpolation by checking the subspace spanned by the interpolated basis using both these methods. If the subspace spanned is the same in all cases, then the manifold interpolation can be concluded to have the same accuracy as that of direct interpolation.

Suppose two eigenmodes $\mathbf{X}_0$ and $\mathbf{X}_1$ computed at ζ_0 and ζ_1 respectively are to be interpolated. The manifold interpolation can be done only on orthonormal modes, hence the eigenmodes have to be orthonormalized to obtain Φ_0 and Φ_1 respectively:

$$\Phi_j = \text{orth}(\mathbf{X}_j) . \quad (\text{B.1})$$

Φ_1 is conditioned using $\mathbf{Q}$ as described in Section 6.2:

$$\Phi_1 \leftarrow \Phi_1 \mathbf{Q} . \quad (\text{B.2})$$

The manifold interpolation using Φ_0 and Φ_1 is given as:

$$(\mathbf{I} - \Phi_0 \Phi_0^T) \Phi_1 (\Phi_0^T \Phi_1)^{-1} = \mathbf{U} \mathbf{S} \mathbf{V}^T \quad (\text{SVD}) \quad (\text{B.3})$$

$$\Phi_{\mathcal{M}}(\zeta) = \Phi_0 \mathbf{V} \cos \left(\left(\frac{\zeta - \zeta_0}{\zeta_1 - \zeta_0} \right) \tan^{-1}(\mathbf{S}) \right) + \mathbf{U} \sin \left(\left(\frac{\zeta - \zeta_0}{\zeta_1 - \zeta_0} \right) \tan^{-1}(\mathbf{S}) \right) . \quad (\text{B.4})$$

Direct interpolation is given as:

$$\Phi_I(\zeta) = \left(\frac{\zeta - \zeta_1}{\zeta_1 - \zeta_0} \right) \Phi_0 + \left(\frac{\zeta - \zeta_0}{\zeta_1 - \zeta_0} \right) \Phi_1 . \quad (\text{B.5})$$

Φ_I is orthogonalized to obtain an orthogonal basis:

$$\Phi_I \leftarrow \text{orth}(\Phi_I) . \quad (\text{B.6})$$

In order to check whether $\Phi_{\mathcal{M}}$ and Φ_I span the same space, the following projection is first carried out on Φ_I :

$$\tilde{\Phi}_I = \Phi_{\mathcal{M}} \Phi_{\mathcal{M}}^T \Phi_I . \quad (\text{B.7})$$

The Modal Assurance Criteria (MAC) of both the basis is computed to see if they coincide. The interpolation interval $(\zeta_1 - \zeta_0)$ is varied from 0.01 to 0.7 [m] in steps of 0.01 [m]. In each case it is seen that the interpolated modes using both the methods coincide exactly. Thus a manifold interpolation is equivalent to direct interpolation combined with an orthogonalization procedure.

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