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On the complete bounds of L_p-Schur multipliers

Martijn Caspers and Guillermo Wildschut

Abstract. We study the class $\mathcal{M}_p$ of Schur multipliers on the Schatten-von Neumann class $\mathcal{S}_p$ with $1 \leq p \leq \infty$ as well as the class of completely bounded Schur multipliers $\mathcal{M}_p^{cb}$. We first show that for $2 \leq p < q \leq \infty$ there exists $m \in \mathcal{M}_p^{cb}$ with $m \notin \mathcal{M}_q$, so in particular the following inclusions that follow from interpolation are strict: $\mathcal{M}_q \subsetneq \mathcal{M}_p$ and $\mathcal{M}_q^{cb} \subsetneq \mathcal{M}_p^{cb}$. In the remainder of the paper we collect computational evidence that for $p \neq 1, 2, \infty$ we have $\mathcal{M}_p = \mathcal{M}_p^{cb}$, moreover with equality of bounds and complete bounds. This would suggest that a conjecture raised by Pisier (Astérisque 247:vi+131, 1998) is false.

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1. Introduction. The Schur product of matrices is given by the entry-wise product. For $m \in M_n(\mathbb{C})$ the linear map

$$M_m : M_n(\mathbb{C}) \to M_n(\mathbb{C}) : x \mapsto M_m(x) := (m_{i,j}x_{i,j})_{i,j},$$

is then called a Schur multiplier.

Schur multipliers appear in several different contexts. They are widely applied in harmonic analysis because of their close connection with Fourier multipliers and transference techniques, see, e.g., [4,6,16]. In operator theory Schur multipliers of divided differences occur naturally in problems involving commutators of operators, see, e.g., [20] and references given there. Further, recently new applications of transference techniques have been found in approximation properties of Lie groups [8,9,15]. In each of these applications crucial new results were obtained on the (complete) bounds of Schur multipliers.

The boundedness properties of M_m depend on the norm imposed on $M_n(\mathbb{C})$. If $M_n(\mathbb{C})$ is equipped with the operator norm, the bounds of M_m can be described by Grothendieck's characterization [19, Theorem 5.1]. In particular

the bounds and complete bounds of a Schur multiplier agree and in an infinite dimensional setting we see that every bounded multiplier is in particular automatically completely bounded. If $M_n(\mathbb{C})$ is equipped with the Schatten $\mathcal{S}_p$-norm, then finding bounds, or even optimal bounds, of M_m becomes very complicated as there is no such charaterization as Grothendieck's available.

In the current paper we show two things. Let $\mathcal{M}_p$ (resp. $\mathcal{M}_p^{cb}$) be the collection of symbols m that are bounded (resp. completely bounded) Schur multipliers of the Schatten-von Neumann classes $\mathcal{S}_p$ associated with an infinite dimensional Hilbert space. We refer to Section 2 for exact definitions. Through complex interpolation we have that $\mathcal{M}_q \subseteq \mathcal{M}_p$ in case $2 \leq p < q \leq \infty$. We show that this inclusion is strict; in fact we get a slightly stronger result in particular yielding the parallel result on the complete bounds as well. This extends the results by Harcharras [14, Theorem 5.1] which proves this for even p and it settles the question of strict inclusions (the problem was also stated in [11, p. 51]).

Secondly, we study the question whether $\mathcal{M}_p$ and $\mathcal{M}_p^{cb}$ are equal for $1 < p \neq 2 < \infty$. In fact, the following conjecture is stated in [17]:

Conjecture 1.1 (Conjecture 8.1.12 in [17]) For every $1 < p \neq 2 < \infty$ we have that $\mathcal{M}_p \neq \mathcal{M}_p^{cb}$.

If we replace $\mathcal{M}_p$ and $\mathcal{M}_p^{cb}$ by the class of, respectively, the bounded and completely bounded *Fourier* multipliers on a locally compact abelian group, then Conjecture 1.1 is true as is proven in [17] in case of the torus and in [1] for arbitrary locally compact abelian groups. Pisier's argument relies on lacunary sets in $\mathbb{Z}$ (for the bounds) and transference to Schur multipliers and unconditionality of the matrix units as a basis for $\mathcal{S}_p$ (for failure of the complete bounds). From this perspective it is very reasonable to state Conjecture 1.1.

In [15] it was proved that for continuous Schur multipliers on $\mathcal{B}(L_2(\mathbb{R}))$ we have $\mathcal{M}_p = \mathcal{M}_p^{cb}$ with *equal* bounds and complete bounds as operators on $\mathcal{S}_p(L_2(\mathbb{R}))$. The continuity is essential in their proof and leaves open Conjecture 1.1. It deserves to be noted that Lafforgue and De la Salle find several other fundamental properties of Schur multipliers in the same paper [15].

In the current paper we approximate the norms of Schur multipliers by computer algorithms; they suggest that $\mathcal{M}_p = \mathcal{M}_p^{cb}$ with equality of norms and complete norms (just as in the case $p = \infty$). We show that this is true in case of the triangular truncation (Corollary 4.2).

2. Preliminaries.

2.1. Schatten classes $\mathcal{S}_p$. Let H be a Hilbert space and let $\mathcal{B}(H)$ be the space of bounded operators on H. For $1 \leq p < \infty$ we let $\mathcal{S}_p = \mathcal{S}_p(H)$ be the space of operators $x \in \mathcal{B}(H)$ such that

$$\|x\|_p := \mathrm{Tr}(|x|^p)^{\frac{1}{p}} < \infty.$$

The assignment $\| \ \|_p$ defines a norm on $\mathcal{S}_p$ which turns it into a Banach space and which is moreover an ideal in $\mathcal{B}(H)$. We set $\mathcal{S}_\infty$ for the C*-algebra of compact operators with operator norm $\| \ \|_\infty$. In case $H = \mathbb{C}^n$ we write $\mathcal{S}_p^n$

for $\mathcal{S}_p = \mathcal{S}_p(\mathbb{C}^n)$. Fixing an orthonormal basis $f_j, j \in \mathbb{N}_{\geq 1}$, we have that we may identify $\mathcal{S}_p^n$ (completely) isometrically as a subspace of $\mathcal{S}_p$ by mapping the matrix unit $e_{i,j} \in \mathcal{S}_p^n$ to the matrix unit $e_{f_i,f_j} \in \mathcal{S}_p$ given by $e_{f_i,f_j} f_k = \langle f_k, f_j \rangle f_i$. Let P_n be the projection onto the span of $f_1, \ldots, f_n$. Then this map is an isometric isomorphism $\mathcal{S}_p^n \simeq P_n \mathcal{S}_p P_n$. Moreover, under this isomorphsim $\cup_n \mathcal{S}_p^n$ is dense in $\mathcal{S}_p$. In case $1 \leq p \leq q \leq \infty$ we have $\mathcal{S}_p \subseteq \mathcal{S}_q$ and the inclusion is (completely) contractive. This in particular turns $(\mathcal{S}_p, \mathcal{S}_q)$ into a compatible couple of Banach spaces and for any $p \leq r \leq q$ we have that $\mathcal{S}_r$ is a complex interpolation space between $(\mathcal{S}_p, \mathcal{S}_q)$, see [3,18]. Any tensor product $\mathcal{S}_p^n \otimes \mathcal{S}_p$ will be understood as a L_p-tensor product, i.e. the p-norm closure as a subspace of $\mathcal{S}_p(\mathbb{C}^n \otimes H)$.

2.2. Operator space structure. For the theory of operator spaces we refer to [12], [18]; we shall only need a result of Pisier on completely bounded maps on Schatten classes which we recall here. In [17] Pisier shows that $\mathcal{S}_p$ have a natural operator space structure as interpolation spaces between $\mathcal{S}_1$ and $\mathcal{S}_\infty$. In [17] it was proved that a linear map $M : \mathcal{S}_p \to \mathcal{S}_p$ is completely bounded iff for every $s \in \mathbb{N}$ the amplification

$$\mathrm{id}_s \otimes M : \mathcal{S}_p^s \otimes \mathcal{S}_p \to \mathcal{S}_p^s \otimes \mathcal{S}_p$$

is bounded with bound uniform in s. Moreover,

$$\|M : \mathcal{S}_p \to \mathcal{S}_p\|_{\mathcal{CB}(\mathcal{S}_p)} = \sup_{s \in \mathbb{N}} \|(\mathrm{id}_s \otimes M) : \mathcal{S}_p^s \otimes \mathcal{S}_p \to \mathcal{S}_p^s \otimes \mathcal{S}_p\|_{\mathcal{B}(\mathcal{S}_p^s \otimes \mathcal{S}_p)}. \tag{2.1}$$

The reader may take (2.1) as a definition, other properties (besides interpolation) of the operator space structure of $\mathcal{S}_p$ shall not be used in this text.

2.3. Schur multipliers. A symbol is a function $m : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$. We call m an L_p-Schur multiplier if there exists a map $M_m : \mathcal{S}_p \to \mathcal{S}_p$ determined by

$$M_m : \mathcal{S}_p^n \to \mathcal{S}_p^n : (x_{i,j})_{i,j} \mapsto (m(i,j) x_{i,j})_{i,j}.$$

Here we view again $\mathcal{S}_p^n$ as a subspace of $\mathcal{S}_p$ by fixing a basis. From the closed graph theorem, as M_m is presumed to be defined on all of $\mathcal{S}_p$, the map M_m is automatically bounded. The space of all L_p-Fourier multipliers will be denoted by $\mathcal{M}_p$ which carries the operator norm $\|\cdot\|_{\mathcal{M}_p}$ of $\mathcal{B}(\mathcal{S}_p)$. This turns $\mathcal{M}_p$ into a Banach space. We denote $\mathcal{M}_p^{cb}$ for the subset of $m \in \mathcal{M}_p$ such that $M_m : \mathcal{S}_p \to \mathcal{S}_p$ is completely bounded. We equip $\mathcal{M}_p^{cb}$ with the completely bounded norm $\|\cdot\|_{\mathcal{M}_p^{cb}}$ as completely bounded maps on $\mathcal{S}_p$. With slight abuse of terminology we shall refer to both the symbol m as well as the map M_m as a Schur multiplier and usually write $\|M_m\|_{\mathcal{M}_p}$ for $\|m\|_{\mathcal{M}_p}$ (and similarly for the completely bounded norms). Obviously $\mathcal{M}_p^{cb} \subseteq \mathcal{M}_p$. The question whether this inclusion is strict remains open, see Conjecture 1.1.

3. Strict inclusions of the set of Schur multipliers. Here we prove that for $2 \leq p < q \leq \infty$ there exists a symbol $m : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ that is a completely bounded L_p-Schur multiplier but which fails to be a bounded L_q-Schur multiplier. The following lemma is based on [7, Lemma 1]. For a finitely supported meausre μ on the torus $\mathbb{T}$ we write $\mu*$ for the convolution operator $L_p(\mathbb{T}) \to L_p(\mathbb{T}) : f \mapsto \mu * f$. We let $\|\mu\|$ be the norm of the measure.

Lemma 3.1. *Let $2 \leq p \leq \infty$. There exists a finitely supported measure $\mu_n, n \in \mathbb{N}$, on the torus $\mathbb{T}$ such that*

$$2^{n/p} \leq \|\mu_n * \|_{\mathcal{B}(L_p(\mathbb{T}))} \leq \|\mu_n * \|_{\mathcal{CB}(L_p(\mathbb{T}))} \leq \sqrt{2}\, 2^{n/p}. \tag{3.1}$$

Proof. Let $s_n = e^{\frac{\pi i}{2^n}}$. Set $\mu_0 = \nu_0 = \delta_1$, the Dirac delta measure in $1 \in \mathbb{T} \subseteq \mathbb{C}$. Then define inductively

$$\mu_{n+1} = \mu_n + s_n * \nu_n, \qquad \nu_{n+1} = \mu_n - s_n * \nu_n,$$

and note that the supports of μ_n and $s_n * \nu_n$ have empty intersection.

We claim that for every $f \in C_c(\mathbb{T}, \mathcal{S}_2^m), m \in \mathbb{N}$, we have

$$\|\nu_n * f\|_2^2 + \|\mu_n * f\|_2^2 = 2^{n+1}\|f\|_2^2. \tag{3.2}$$

Indeed, this is clear for $n = 0$ and further by the parallellogram law

$$\begin{aligned}\|\nu_{n+1} * f\|_2^2 + \|\mu_{n+1} * f\|_2^2 &= 2(\|\nu_n * f\|_2^2 + \|s_n * \mu_n * f\|_2^2) \\ &= 2(\|\nu_n * f\|_2^2 + \|\mu_n * f\|_2^2).\end{aligned}$$

Then (3.2) follows by induction. From (3.2) we obtain that

$$\|\mu_n * f\|_2^2 \leq \|\mu_n * f\|_2^2 + \|\nu_n * f\|_2^2 = 2^{n+1}\|f\|_2^2.$$

So that $\|\mu_n * \|_{\mathcal{CB}(L_2(\mathbb{T}))} \leq 2^{(n+1)/2}$. Also $\|\mu_n * \|_{\mathcal{CB}(L_1(\mathbb{T}))} \leq \|\mu_n\| = 2^n$ and by duality also $\|\mu_n * \|_{\mathcal{CB}(L_\infty(\mathbb{T}))} \leq 2^n$. By complex interpolation therefore $\|\mu_n * \|_{\mathcal{CB}(L_p(\mathbb{T}))} \leq \sqrt{2}\, 2^{n/p}$. This proves the upperbound in (3.1).

For the lower bounds let $f \in C_c(\mathbb{T})$ be a function with small support close to $1 \in \mathbb{T}$. If the support is small enough, then $\mu_n * f$ consists of 2^n disjointly supported translates of f so that $\|\mu_n * f\|_p = 2^{n/p}\|f\|_p$. This yields the lower bound. □

The following theorem shows in particular that the class of L_p-Schur multipliers can be distinguished from the L_q-Schur multipliers for $2 \leq p < q \leq \infty$.

Theorem 3.2. *Let $2 \leq p < q \leq \infty$ or $1 \leq q < p \leq 2$. There exists a symbol $m : \mathbb{Z}^2 \to \mathbb{C}$ such that $m \in \mathcal{M}_p^{cb}$ but $m \notin \mathcal{M}_q$.*

Proof. We first treat the case $2 \leq p < q < \infty$. Let μ_n be the finitely supported measure on $\mathbb{T}$ of Lemma 3.1. Let $m_n : \mathbb{Z} \to \mathbb{C}$ be its Fourier transform given by

$$m_n(k) = \sum_{\theta \in \mathrm{supp}(\mu_n)} \mu_n(\theta) e^{ik\theta}.$$

Then set $\widetilde{m}_n : \mathbb{Z}^2 \to \mathbb{C}$ by $\widetilde{m}_n(k, l) = m_n(k - l)$. By [16, Theorem 1.2] or [6, Theorem 4.2 and Corollary 5.3] we have

$$\|\mu_n * \|_{\mathcal{CB}(L_p(\mathbb{T}))} = \|M_{\widetilde{m}_n}\|_{\mathcal{CB}(\mathcal{S}_p)}. \tag{3.3}$$

We amplify $\widetilde{m}_n$ by defining $\widetilde{m}_n^{cb} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C} : (k,l) = (k_1,k_2,l_1,l_2) \mapsto \widetilde{m}(k_1,l_1)$. Then,

$$\|M_{\widetilde{m}_n^{cb}}\|_{\mathcal{B}(\mathcal{S}_p)} = \|M_{\widetilde{m}_n^{cb}}\|_{\mathcal{CB}(\mathcal{S}_p)} = \|M_{\widetilde{m}_n}\|_{\mathcal{CB}(\mathcal{S}_p)}. \tag{3.4}$$

Combining (3.3) and (3.4) with the estimates obtained in Lemma 3.1 we find that

$$2^{n/p} \leq \|M_{\widetilde{m}_n^{cb}}\|_{\mathcal{B}(\mathcal{S}_p)}, \qquad \text{and} \qquad \|M_{\widetilde{m}_n^{cb}}\|_{\mathcal{CB}(\mathcal{S}_q)} \leq \sqrt{2}\, 2^{n/q}. \tag{3.5}$$

From these two estimates we are able to prove the theorem as follows.

Suppose that the theorem is false, so that we have an inclusion map $i : \mathcal{M}_p^{cb} \to \mathcal{M}_q$. By the closed graph theorem this inclusion is continuous. Indeed, if $k_j \in \mathcal{M}_p^{cb}$ is a net in symbols such that $k_j \to 0$ in $\mathcal{M}_p^{cb}$ and such that k_j converges to k in $\mathcal{M}_q$, then for every matrix $x \in \mathcal{S}_p^n \subseteq \mathcal{S}_p$ we find that $M_{k_j}(x) \to 0$ in $\mathcal{S}_p^n$ and hence also in the norm of $\mathcal{S}_q^n$. This shows that for $x \in \mathcal{S}_q^n \subseteq \mathcal{S}_q$ we have $M_k(x) = \lim_j M_{k_j}(x) = 0$. But by density of $\cup_n \mathcal{S}_q^n$ in $\mathcal{S}_q$ we get that $M_k(x) = 0$. Hence the graph of i is closed indeed.

However, the estimates (3.5) show that

$$\frac{\|M_{\widetilde{m}_n^{cb}}\|_{\mathcal{B}(\mathcal{S}_p)}}{\|M_{\widetilde{m}_n^{cb}}\|_{\mathcal{CB}(\mathcal{S}_q)}} \geq 2^{\frac{n}{p}-\frac{n}{q}-\frac{1}{2}},$$

which converges to infinity if $n \to \infty$. This contradicts that $i : \mathcal{M}_p^{cb} \to \mathcal{M}_q$ is bounded.

Now, if $1 < q < p \leq 2$, then the statement follows from duality as $M_m^* = M_{m^\vee}$, where $m^\vee(k,l) = \overline{m(k,l)}$ and duality preserves the (complete) bounds of linear maps. In case $q = 1$ or $q = \infty$ the counter example is given by triangular truncation, see [10]. □

In particular we get the weaker statements that give non-inclusions of bounded and completely bounded multipliers.

Corollary 3.3. *Let either $2 \leq p < q \leq \infty$ or $1 \leq q < p \leq 2$. We have that $\mathcal{M}_p \subsetneq \mathcal{M}_q$ and $\mathcal{M}_p^{cb} \subsetneq \mathcal{M}_q^{cb}$.*

We may in fact improve on this theorem in the following way.

Corollary 3.4. *Let $2 \leq p < \infty$. There exists a symbol $m \in \mathcal{M}_p^{cb}$ such that for any $q > p$ we have have $m \notin \mathcal{M}_q$.*

Proof. Let $q_n > p$ be a decreasing sequence with $q_n \searrow p$. Let $m_n \in \mathcal{M}_p^{cb}$ with $\|m_n\|_{\mathcal{M}_p^{cb}} = 1$ be such that $m_n \notin \mathcal{M}_{q_n}$. We copy-paste part of the symbols m_n to diagonal blocks of a new symbol m as follows. Let $k_n \in \mathbb{N}$ be such that there exist $x_n \in \mathcal{S}_{q_n}^{k_n}$ with $\|x_n\|_q = 1$ and $\|M_{m_n}(x_n)\|_{q_n} > n$. Let $m_n' : [-k_n,k_n] \times [-k_n,k_n] \to \mathbb{C}$ be the restriction of m_n to a discrete interval. As $\|M_{m_n'}(x_n)\|_q \geq n$ we see that $\|M_{m_n'}\| \geq n$. Then let $m : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be the block symbol given by

$$m = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots \\ 0 & m_1' & 0 & 0 & \dots \\ \vdots & 0 & m_2' & 0 & \dots \\ \vdots & 0 & 0 & m_3' & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We find that $M_m = M_{m_1} \oplus M_{m_2} \oplus M_{m_2} \oplus \dots$. So that $\|M_m\|_{\mathcal{M}_p^{cb}} = \sup_k \|M_{m_k}\|_{\mathcal{M}_p^{cb}} \leq 1$. And similarly,

$$\|M_m\|_{\mathcal{M}_q^{cb}} = \sup_k \|M_{m_k}\|_{\mathcal{M}_q^{cb}} \geq \sup_{k, q_k \leq q} \|M_{m_k}\|_{\mathcal{M}_q^{cb}} = \infty.$$

□

Remark 3.5. The proof of Corollary 3.4 gives in fact a stronger result. It shows that for any $p_0 > 2$ there is a symbol m such that $M_m : \mathcal{S}_p \to \mathcal{S}_p$ is completely contractive if $p \in [2, p_0]$ and unbounded if $p \in (p_0, \infty)$.

4. Reduction of the variables. Let $n, s \in \mathbb{N}$ and consider $\mathcal{S}_p^n$. Let $e_{i,i}$ be the diagonal matrix unites of $M_n(\mathbb{C})$ and consider the subgroup of $M_s(\mathbb{C}) \otimes M_n(\mathbb{C})$ given by all diagonal unitaries $U_1 \otimes e_{1,1} + \dots + U_n \otimes e_{n,n}$ with $U_i \in U(s) \subseteq M_s(\mathbb{C})$ the unitary group. We denote this group by $\oplus_{i=1}^n U(s)$. Naturally $\oplus_{i=1}^n U(s)$ acts isometrically on $\mathcal{S}_p^s \otimes \mathcal{S}_p^n$ by left and right multiplications.

Proposition 4.1. *Let $m \in \mathcal{M}_p^n$ and let $s \in \mathbb{N}$. Consider the set of maximum points C_m^s consisting of all $x \in \mathcal{S}_p^s \otimes \mathcal{S}_p^n$ for which $\|x\|_p = 1$ and such that $\|(id_s \otimes M_m)(x)\|_p = \|id_s \otimes M_m\|_{\mathcal{M}_p}$. Then C_m^s is invariant for the left and right action of $\oplus_{i=1}^n U(s)$. In particular, it follows that there exists an $x \in C_m^s$ such that for every $1 \leq i \leq n$ we have that $x_{i,i} := (id_s \otimes \langle \cdot\ e_i, e_i \rangle)(x) \in \mathcal{S}_p^s$ is a diagonal matrix with non-negative eigenvalues.*

Proof. The first statement is a consequence of the fact that the Schur multiplier $(\mathrm{id}_s \otimes M_m)$ commutes with the isometric action of $\oplus_{i=1}^n U(s)$. Therefore, for $U \in \oplus_{i=1}^n U(s)$ we have $\|Ux\|_p = \|x\|_p$ and $\|(\mathrm{id}_s \otimes M_m)(Ux)\|_p = \|U(\mathrm{id}_s \otimes M_m)(x)\|_p = \|(\mathrm{id}_s \otimes M_m)(x)\|_p = \|\mathrm{id}_s \otimes M_m\|_{\mathcal{M}_p}$.

The second statement follows from the polar decomposition. Indeed, take $x \in C_m^s$. We claim first that we may assume that $x_{i,i}$ is a positive semi-definite matrix. For each $1 \leq i \leq n$ consider the polar decomposition $x_{i,i} = v_i |x_{i,i}|$ where v_i is a partial isometry with $\ker(v_i)^\perp = \mathrm{ran}(|x_{i,i}|)$. By dimension considerations we may extend v_i to a unitary $u_i \in U(s)$ that agrees with v_i on $\mathrm{ran}(|x_{i,i}|)$ so that still $x_{i,i} = u_i |x_{i,i}|$. Then put $u = \oplus_{i=1}^n u_i \in \oplus_{i=1}^n U(s)$. Then $u^* x \in C_m^s$ by the previous paragraph and $(u^* x)_{i,i} = |x_{i,i}|$ is positive semi-definite. Let $w_i \in U(s)$ be such that $w_i |x_{i,i}| w_i^*$ is a diagonal matrix, say d_i, with entries ≥ 0. Then put $w = \oplus_{i=1}^n w_i \in \oplus_{i=1}^n U(s)$. We find that $w^* u^* x w \in C_m^s$ and further $(w^* u^* x w)_{i,i} = d_i$. □

Proposition 4.1 can be used to significaty speed up our computations in Section 5.1.

As a side remark we obtain the following corollary that shows that the bounds and complete bounds of an infinite dimensional triangular truncation agree. This result was already recorded in (the discussion before) [16, Proposition 6.3]. In Corollary 4.2 we have that $\|h\|_{\mathcal{M}_p}$ is finite by the stronger result from [10, Theorem 1.4].

Corollary 4.2. *Let $h : \mathbb{Z}^2 \to \mathbb{C}$ be the symbol of triangular truncation given by $h(i,j) = \delta_{\geq 0}(i-j)$. Then for every $1 < p < \infty$ we have $\|h\|_{\mathcal{M}_p} = \|h\|_{\mathcal{M}_p^{cb}}$.*

Proof. We use the notation of Proposition 4.1. Let $\pi_s : \mathcal{S}_p^s \otimes \mathcal{S}_p \to \mathcal{S}_p : e_{i,j} \otimes e_{k,l} \mapsto e_{sk+i,sl+j}$ be the isometric isomorphism that re-indexes matrix units. Let $x \in C_h^s, s \in \mathbb{N}_{\geq 2}$. By Proposition 4.1 we may assume that each $x_{i,i} = (\mathrm{id}_s \otimes \langle \cdot e_i, e_i \rangle)(x) \in \mathcal{S}_p^s, i \in \mathbb{Z}$ is a diagonal matrix. Then $(\mathrm{id}_s \otimes M_h)(x) = M_h(\pi_s(x))$ and therefore,

$$\|(\mathrm{id}_s \otimes M_h)(x)\|_p = \|M_h(\pi_s(x))\|_p \leq \|M_h\|_{\mathcal{M}_p} \|\pi_s(x)\|_p = \|M_h\|_{\mathcal{M}_p} \|x\|_p.$$

□

5. Approximation. In this section we argue that if Conjecture 1.1 is true, then we should be able to find computer based evidence for it, which we make precise in the following way. Consider the following three statements:

1. For every $1 < p \neq 2 < \infty$ there exists a bounded Schur multiplier that is not completely bounded.
2. For every $1 < p \neq 2 < \infty$ there exists a completely bounded Schur multiplier $m \in \mathcal{M}_p^{cb}$ such that $\|M_m\|_{\mathcal{M}_p^{cb}} \neq \|M_m\|_{\mathcal{M}_p}$.
3. For every $1 < p \neq 2 < \infty$ and every $m \in \mathcal{M}_p$ we have $m \in \mathcal{M}_p^{cb}$ and moreover $\|M_m\|_{\mathcal{M}_p^{cb}} = \|M_m\|_{\mathcal{M}_p}$.

Statement 1 is Pisier's Conjecture 1.1. 2 is weaker than 1, and 3 is just the negative of 2. If 2 is already true, then it is possible to show this by sampling dense sets of Schur multipliers on finite dimensional Schatten classes and by approximating their norms with finite sets. The problem however is that it is not clear how much computations and computational power is needed in order to obtain a symbol m that witnesses statement 2 above. We state a quantitative statement in this direction in the next proposition.

Proposition 5.1. *Let $2 \leq p < \infty$. Fix $n \in \mathbb{N}$ and let $\varepsilon > 0, \delta > 0$. Let A_ε be the set of all symbols $m : \{1,\dots,n\}^2 \to \varepsilon\mathbb{Z} \cap [-1,1]$ and let A be the set of all symbols $m : \{1,\dots,n\}^2 \to [-1,1]$. Let $B_\delta \subseteq \mathcal{S}_p^n$ be the set of all $x \in \mathcal{S}_p^n$ with $\Re(x_{i,j}) \in \delta\mathbb{Z} \cap [-1,1]$ and $\Im(x_{i,j}) \in \delta\mathbb{Z} \cap [-1,1]$. For any symbol $m \in A$ we have that for $\delta < (\sqrt{2}n^{1+\frac{1}{p}})^{-1}$,*

$$\Delta_m := \|M_m\|_{\mathcal{M}_p} - \sup_{y \in B_\delta} \frac{\|M_m(y)\|_p}{\|y\|_p} \leq \|M_m\|_{\mathcal{M}_p} \frac{2\delta\sqrt{2}n^{1+\frac{1}{p}}}{1 - \delta\sqrt{2}n^{1+\frac{1}{p}}}. \tag{5.1}$$

Further, for every $m \in A_\varepsilon$ we have,

$$\inf_{m' \in A_\varepsilon} \|M_m - M_{m'}\| \leq n\varepsilon. \tag{5.2}$$

Proof. Take $x \in \mathcal{S}_p^n$ be such that $\|x\|_p = 1$ and $\|M_m\|_{\mathcal{M}_p} = \|M_m(x)\|_p$. Let $x^\delta \in B_\delta$ be such that for each coefficient at entry i,j we have $|x_{i,j} - x^\delta_{i,j}| < \delta\sqrt{2}$. Let x_i and x_i^δ be the i-th off-diagional of x and of x^δ, respectively. That is, $x_i(k,l) = x(k,l)$ if $k - l = i \mod n$ and $x_i(k,l) = 0$ otherwise. By the triangle inequality,

$$\|x - x^\delta\|_p \leq \sum_{i=1}^n \|x_i - x_i^\delta\|_p \leq n(\delta\sqrt{2}n^{\frac{1}{p}}).$$

Further $\|x^\delta\|_p \leq 1 + \|x - x^\delta\|_p \leq 1 + \delta\sqrt{2}n^{1+\frac{1}{p}}$ and similarly $\|x^\delta\|_p \geq 1 - \delta\sqrt{2}n^{1+\frac{1}{p}}$. We have

$$\begin{aligned}\|M_m(x^\delta)\|_p &\geq \|M_m(x)\|_p - \|M_m(x^\delta - x)\|_p \geq \|M_m\|_{\mathcal{M}_p} \\ &\quad -\|M_m\|_{\mathcal{M}_p}\|x - x^\delta\|_p.\end{aligned}$$

So combining these estimates yields

$$\begin{aligned}\|M_m\|_{\mathcal{M}_p} - \frac{\|M_m(x^\delta)\|_p}{\|x^\delta\|_p} &= \frac{\|M_m\|_{\mathcal{M}_p}\|x^\delta\|_p - \|M_m(x^\delta)\|_p}{\|x^\delta\|_p} \\ &\leq \frac{\|M_m\|_{\mathcal{M}_p}\|x^\delta\|_p - \|M_m\|_{\mathcal{M}_p} + \|M_m\|_{\mathcal{M}_p}\|x - x^\delta\|_p}{\|x^\delta\|_p} \\ &\leq \|M_m\|_{\mathcal{M}_p}\frac{1 + \delta\sqrt{2}n^{1+\frac{1}{p}} - 1 + \delta\sqrt{2}n^{1+\frac{1}{p}}}{1 - \delta\sqrt{2}n^{1+\frac{1}{p}}} = \|M_m\|_{\mathcal{M}_p}\frac{2\delta\sqrt{2}n^{1+\frac{1}{p}}}{1 - \delta\sqrt{2}n^{1+\frac{1}{p}}}.\end{aligned}$$

This proves (5.1). For (5) take $m \in A$. Let $m^\varepsilon \in A_\varepsilon$ be a symbol such that for each coefficient at entry i,j we have $|m_{i,j} - m^\varepsilon_{i,j}| \leq \varepsilon$. Let m_i^ε be the i-th off-diagional of the symbol m^ε; that is, $m_i^\varepsilon(k,l) = m^\varepsilon(k,l)$ if $k - l = i \mod n$ and $m_i^\varepsilon(k,l) = 0$ otherwise. Similarly, let m_i be the i-th off-diagonal of m. We find that

$$\|M_m - M_{m^\varepsilon}\|_{\mathcal{M}_p} \leq \sum_{i=1}^n \|M_{m_i} - M_{m_i^\varepsilon}\|_{\mathcal{M}_p} \leq n\varepsilon.$$

□

Proposition 5.1 shows that we can approximate the norms of Schur multipliers on $\mathcal{S}_p^n$. Naturally also the norms of each of the individual matrix amplifications $\mathrm{id}_s \otimes M_m$, $s \in \mathbb{N}$ can be approximated by viewing them as Schur multipliers on $\mathcal{S}_p^{sn}$. Note that shows that we may limit ourselves to Schur multipliers taking values in discrete intervals, i.e. with symbol in A_ε. If Statement 2 above would be true, then by approximation we would be able to find counter examples for every $1 < p \neq 2 < \infty$. However, our computer simulations exhibit the behaviour of the converse statement 3.

5.1. Approximation with gradient descent methods. We have used the Broyden-Fletcher-Goldfarb-Shanno algorithm (BFGS algorithm, see [2]), which

TABLE 1. Approximations of the symbol m of (5.3)

s	Approximation of $\|M_m^{(s)}\|_{\mathcal{B}(\mathcal{S}_4^{3^s})}$
1	3.0491549804518234
2	3.0491549802194102
3	3.0491549798442240
4	3.0491549798208277
5	3.0491549794012864

TABLE 2. n = dimension of the symbol, N = number of random sample multipliers m, second and third column = approximation of the 2nd and 3rd amplification of m

n	N	$\max_m(\|M_m^{(2)}\|_{\mathcal{B}(\mathcal{S}_4^{2n})} - \|M_m\|_{\mathcal{B}(\mathcal{S}_4^n)})$	$\max_m(\|M_m^{(3)}\|_{\mathcal{B}(\mathcal{S}_4^{3n})}) - \|M_m\|_{\mathcal{B}(\mathcal{S}_4^n)}))$
2	500	0.000000000000000	$-8.881784197001252 \cdot 10^{-16}$
3	100	$-3.841578721797134 \cdot 10^{-9}$	$-3.735681430860893 \cdot 10^{-9}$
4	20	$-2.227329432002989 \cdot 10^{-12}$	$-5.260509805538049 \cdot 10^{-10}$

is a gradient descent algorithm to find local minima/maxima of a function. We apply it here to find local maxima of

$$f_m(x) = \frac{\|M_m(x)\|_p^p}{\|x\|_p^p} = \frac{\operatorname{Tr}\left((M_m(x)^* M_m(x))^{p/2}\right)}{\operatorname{Tr}\left((x^*x)^{p/2}\right)}.$$

In case $p \in 2\mathbb{N}_{\geq 1}$, so that the $p/2$-powers are integer powers, this expression is faster to compute as it avoids determining eigenvalues of x^*x and $M_m(x)^*M_m(x)$. The precise algorithm is available on [5] and it makes use of the reductions in Section 4. Note that the sample sets A_ε in Proposition 5.1 scale exponentially with the dimension and therefore we are bound to use faster algorithms that only allow us to compute local maxima.

5.2. Approximation for a fixed Schur multiplier. In order to illustrate our larger computations we start with the example (fixed) Schur multiplier,

$$m = \begin{pmatrix} 1 & 2 & 2 \\ -2 & 1 & 3 \\ 0 & 2 & -2 \end{pmatrix}. \tag{5.3}$$

The following table shows the approximation of the norm of

$$M_m^{(s)} := \mathrm{id}_{\mathcal{S}_p^s} \otimes M_m = M_{m^{(s)}}, \qquad \text{with symbol} \qquad m^{(s)}(i,j) = \left(\left[\frac{i}{s}\right], \left[\frac{j}{s}\right]\right),$$

where $[r]$ is the largest integer k with $k \leq r$ (Table 1).

5.3. Approximations for random Schur multipliers. Next we sample random symbols m of Schur multipliers. In the next table N is the number of random samples $m \in M_n(\mathbb{C})$ and to each of these we approximate its norm in essentially the same way as we did to the single example of Section 5.2. We then take the maximum over all samples m over the difference of the norms (Table 2).

Table 3. n = dimension of the symbol, N = number of random sample multipliers m, remaining columns = approximation over the random sample set of $\max_m(\|M_m^{(3)}\|_p - \|M_m\|_p)$

n	N	$p=3$	$p=3.5$
2	100	$-1.965094753586527 \cdot 10^{-13}$	$-2.632338791386246 \cdot 10^{-13}$
3	10	$-2.616529215515584 \cdot 10^{-10}$	$-2.0181634141636096 \cdot 10^{-10}$

n	N	$p=4.5$	$p=5$
2	100	$-1.9551027463649007 \cdot 10^{-13}$	$-3.397282455352979 \cdot 10^{-13}$
3	10	$-3.2847524700230224 \cdot 10^{-10}$	$-5.515231604746873 \cdot 10^{-9}$

Note that the values in this table are negative because it is harder to approximate $\|M_m^{(2)}\|_{\mathcal{B}(\mathcal{S}_4^{2n})}$ than $\|M_m\|_{\mathcal{B}(\mathcal{S}_4^n)}$. Therefore if it would be true that $\|M_m^{(2)}\|_{\mathcal{B}(\mathcal{S}_4^{2n})} = \|M_m\|_{\mathcal{B}(\mathcal{S}_4^n)}$, then the approximation of $\|M_m^{(2)}\|_{\mathcal{B}(\mathcal{S}_4^{2n})}$ is smaller than the approximation of $\|M_m\|_{\mathcal{B}(\mathcal{S}_4^n)}$.

5.4. Different values of $2 \leq p < \infty$. For arbitrary p we may still approximate the norm by the same algorithm except that $\|x\|_p$ is computed by determining the eigenvalues $\lambda_1, \ldots, \lambda_n$ of x^*x so that then $\|x\|_p^p = \sum_{i=1}^n |\lambda_i|^{p/2}$. Though that this is computationally more involved we can still carry out our approximations which are displayed in the following figure (Table 3).

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