

SPIN SWAP AND EXCHANGE COUPLING IN A QUANTUM DOT ARRAY

by

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1

INTRODUCTION

In 1965, Gordon E. Moore, one of the co-founders of Intel, noted that, in years prior, the transistors in computers became twice as small approximately every two years as a consequence of the constant technological innovation [1]. He also predicted that this trend would continue for at least ten more years. His prediction came true and it was given the name Moore's law. This law has proven true for more than 45 years. The computer industry even base their targets on it. However, the limit is about to be reached. Transistors have become so small, that reducing their size much further will yield quantum mechanical behaviour. A quantum computer makes use of quantum mechanical behaviour. Certain difficult problems that cannot be solved efficiently by classical computers can be solved by a quantum computer in polynomial time, like the problem of prime factorization [2], [3]. Furthermore, a quantum computer can simulate any finite physical system [4]. For instance, this can be used to determine the shape of a protein when the amino acid sequence is known, which will seriously benefit drug development [5].

Instead of using classical bits, a quantum computer uses qubits as units of information. In principle, any two-state system can be used as a qubit. The states are represented by $|0\rangle$ and $|1\rangle$. A qubit can be in a superposition of $|0\rangle$ and $|1\rangle$, as opposed to a classical bit, which is either 0 or 1. However, all qubits experience decoherence, information loss as a result of interactions with the environment. Reducing or preventing decoherence is one of the biggest challenges in quantum computing today.

A promising choice of qubit is an electron spin in a quantum dot. $|0\rangle$ and $|1\rangle$ represent spin down and spin up, respectively. Advantages of quantum dot systems include their efficient electrical control [6] and their potential for long coherence times [7], [8].

In this research, simulations were performed on systems of two, three and four linearly coupled quantum dots. In these simulations, the eigenenergies of the systems were determined and time evolutions were performed to get an understanding of the behaviour of electrons in small linear arrays of quantum dots. Next, the pulsing scheme as described by Hu and Das Sarma [9] that swaps two different spin states in the presence of Zeeman inhomogeneity, and the exchange coupling between two neighbouring electrons were reviewed.

2

THEORY

2.1. QUANTUM DOTS

A quantum dot is an artificially fabricated device in which one or more electrons can be contained [10], [11]. Quantum dots have a size of order 10 nm and can be made of semiconductor material. The behaviour of an electron is governed by quantum mechanics. Therefore, the energy levels it can assume are quantized. The energy and other properties, like spin, can be described by quantum numbers. The quantum number n represents the energy level, while s describes the spin number. An electron has spin number $s = \frac{1}{2}$.

A GaAs quantum dot mainly consists of Gallium Arsenide (GaAs) with a layer of Aluminium Gallium Arsenide (AlGaAs) on top of it. AlGaAs has a higher band gap than GaAs, resulting in the formation of a Two Dimensional Electron Gas (2DEG) at the GaAs/AlGaAs interface, typically 100 nm below the surface. Electron gas is formed because the difference in band gap causes the potential to drop below the Fermi Energy. The two dimensionality of the electron gas is caused by the narrowness of the GaAs/AlGaAs interface. Only the lowest standing wave fits into the interface. Finally, two gates are attached on top.

A quantum dot is capacitively attached to a reservoir and a gate. A double quantum dot is created by capacitively coupling two quantum dots. A lateral array of quantum dots can be created by repeating this process. In figure 2.1, a double quantum dot is depicted.

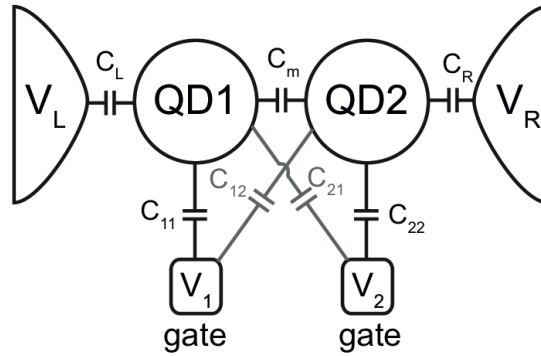


Figure 2.1: Model of a double quantum dot [12].

2.2. ENERGY

The Hamiltonian describes the energy of a system. A Hamiltonian is an operator, operating on a wave function. The eigenvalues of the Hamiltonian are the energies of the system [13]. All elements of the Hamiltonian are determined by an external potential interacting with the particle in the quantum dot. In this report, those elements are the Zeeman effect, exchange modelled by spin-spin interaction and hyperfine interaction.

2.2.1. ZEEMAN SPLITTING

Applying a uniform external magnetic field causes a splitting in the energy levels of degenerate spin states [14]. This is called the Zeeman effect. The Zeeman Hamiltonian is defined as

$$H_Z = \gamma \vec{B} \cdot \vec{S}. \quad (2.1)$$

In this expression, $\vec{B}$ denotes the external magnetic field, $\vec{S}$ is the spin operator and γ is the gyromagnetic ratio. γ is described as

$$\gamma = \frac{g\mu_B}{\hbar}, \quad (2.2)$$

in which g is the Landé g-factor, μ_B is the Bohr magneton and $\hbar$ is the reduced Planck constant. The g-factor depends on different parameters including the specifications of the quantum dot and is usually between -0.4 and -0.2 for GaAs [11]. The direction in which the magnetic field is applied, is typically taken to be the z-direction, reducing the Zeeman Hamiltonian to

$$H_Z = \gamma B S^z, \quad (2.3)$$

in which S^z is the electron spin in the z-direction. S^z has eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ for spin up ($\uparrow$) and spin down ($\downarrow$) respectively. $\uparrow$ is oriented in the +z-direction, while $\downarrow$ is oriented in the -z-direction. Because the g-factor is negative, $\downarrow$ has a higher energy than $\uparrow$. Compared to the system without Zeeman splitting, the energy of $\downarrow$ has increased by $-\frac{E_Z}{2}$, while the energy of $\uparrow$ has experienced a decrease by $-\frac{E_Z}{2}$, where E_Z is the Zeeman energy, which is described by

$$E_Z = \gamma \hbar B = g\mu_B B. \quad (2.4)$$

2.2.2. EXCHANGE INTERACTION

In a double quantum dot containing two electrons, an exchange interaction exists [15]. This interaction can be captured by spin-spin interaction and is defined as

$$H_C = J(\vec{S}_1 \cdot \vec{S}_2 - \frac{\hbar^2}{4}), \quad (2.5)$$

in which J is the exchange coupling and $\vec{S}_i$ is the spin operator working on electron i ($i \in \{1, 2\}$). J is dependent on the overlap of the wave functions of the two electrons. An increase in overlap yields an increase in J .

2.2.3. HYPERFINE INTERACTION

In a GaAs quantum dot, the host material contains many nuclear spins with which the electron spin interacts [11]. The Hamiltonian for this hyperfine interaction is given by

$$H_{HF} = \sum_{k=1}^N A_k \vec{I}_k \cdot \vec{S}, \quad (2.6)$$

in which A_k is the coupling strength between nucleus k and the electron and $\vec{I}_k$ is the spin operator for nucleus k . A_k is proportional to the square of the overlap between the nucleus and the electron wave function.

All nuclear spins combined can be treated as an apparent magnetic field B_n , yielding an alternative description of the effect of the nuclei on the electron spin. The nuclear field is known as the Overhauser field and acts on the electron spin like an external magnetic field:

$$\left(\sum_{k=1}^N A_k \vec{I}_k \right) \vec{S} = \gamma \vec{B}_n \vec{S}. \quad (2.7)$$

2.2.4. ANTICROSSING

Two eigenenergies of a Hamiltonian can only cross, i.e. become equal in value, under certain conditions. The Hamiltonian of a two-state system is

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}. \quad (2.8)$$

The eigenenergies of this Hamiltonian are

$$\lambda_1 = -\frac{H_{11} + H_{22}}{2} + \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2} \quad (2.9)$$

$$\lambda_2 = -\frac{H_{11} + H_{22}}{2} - \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}. \quad (2.10)$$

For these eigenenergies to be equal, the following conditions must hold:

$$H_{11} = H_{22}, \quad H_{12} = 0. \quad (2.11)$$

When these conditions are not met, the eigenenergies cannot cross. Therefore, an anticrossing can occur. For example, of the matrices

$$M_1 = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \quad M_2 = \begin{pmatrix} x & 1 \\ 1 & -x \end{pmatrix}, \quad (2.12)$$

only M_1 meets the conditions of 2.11 at $x = 0$. The left (right) side of figure 2.2 shows the eigenvalues of M_1 (M_2) as a function of x . The eigenvalues of M_2 anticross at $x = 0$.

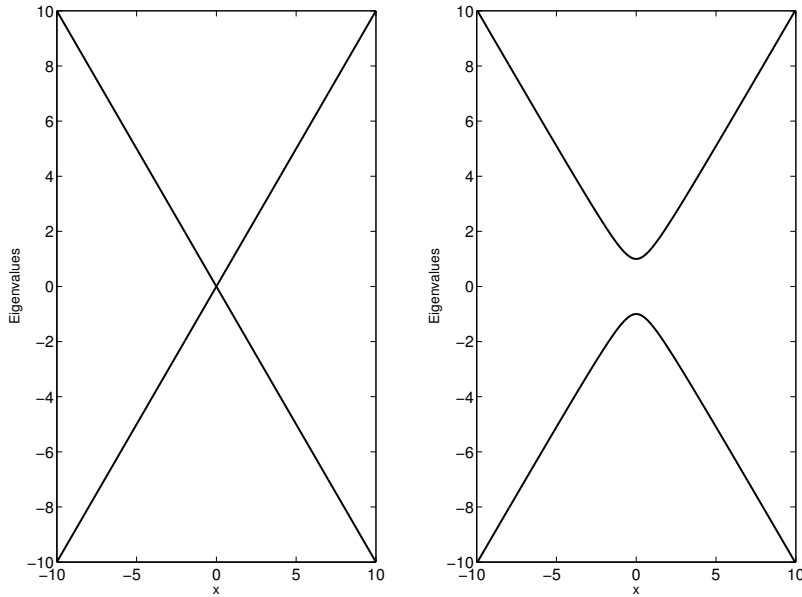


Figure 2.2: Eigenvalues of the matrices $M_1 = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ (on the left) and $M_2 = \begin{pmatrix} x & 1 \\ 1 & -x \end{pmatrix}$ (on the right) as a function of x . The off-diagonal elements in M_1 are 0, so the eigenvalues can cross. The off-diagonal elements in M_2 are non-zero, causing an anticrossing at $x = 0$.

The size of an anticrossing is equal to twice the coupling of two states, which is given by the off-diagonal elements in the Hamiltonian, and can therefore be used to determine the coupling of two states.

2.3. DOUBLE QUANTUM DOT

2.3.1. CHARGE STABILITY DIAGRAM

Consider two quantum dots, labelled 1 and 2 [11]. Their electrochemical potentials are controlled independently by gate voltages $V_{G,1}$ and $V_{G,2}$ respectively. When the dots are completely uncoupled, their electrochemical potentials are independent and therefore the number of electrons in both dots are too. In a charge stability diagram, the occupancy of the dots is plotted as a function of $V_{G,1}$ and $V_{G,2}$. In (x,y) x denotes the number of electrons in dot 1, while y is the number of electrons in dot 2. The charge stability diagram for the uncoupled case is shown in figure 2.3(a). When the capacitive coupling between the dots is nonzero, the electrostatic energy of one of the dots changes when an electron is added to the other dot. Furthermore, $V_{G,1}$ now influences the second dot as well as the first dot, and vice versa. The charge stability diagram in the case of capacitive coupling between the dots is shown in figure 2.3(b).

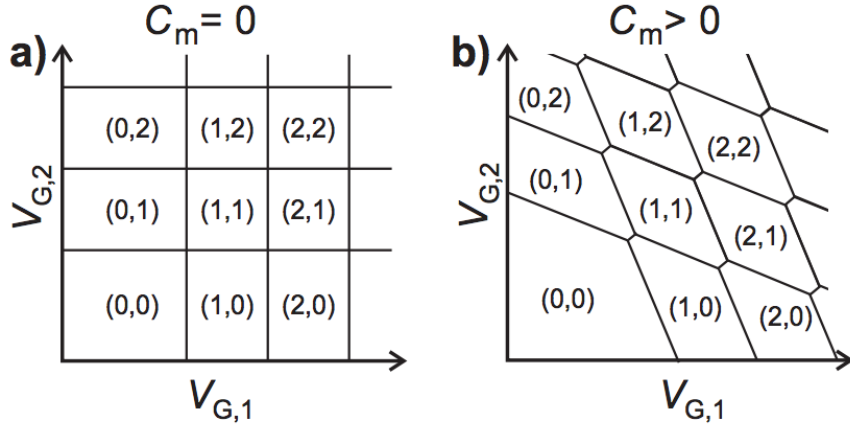


Figure 2.3: Charge stability diagrams for (a) uncoupled and (b) coupled double quantum dots [11].

2.3.2. DOUBLE QUANTUM DOT CONTAINING ONE ELECTRON

CHARGE TRANSITIONS

When two quantum dots are uncoupled, the eigenstates of the system are $|L\rangle$ and $|R\rangle$, in which $|L\rangle$ denotes an electron in the left dot, and $|R\rangle$ in the right dot [12], [16]. When tunnel coupling between the dots is introduced, the electron can tunnel between the dots, resulting in different eigenstates of the system. Using the standard basis

$$|L\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.13)$$

the Hamiltonian becomes

$$H = \begin{pmatrix} \frac{\epsilon}{2} & \tau \\ \tau & -\frac{\epsilon}{2} \end{pmatrix}, \quad (2.14)$$

in which the detuning ϵ is the energy difference between $|L\rangle$ and $|R\rangle$, and τ is the tunnel coupling constant. Detuning is defined along the line connecting $(0,1)$ and $(1,0)$ in the charge stability diagram, as shown in figure 2.4.

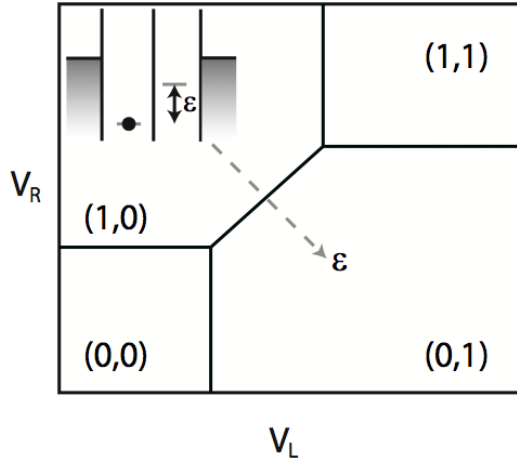


Figure 2.4: The detuning is defined along the axis connecting $(1,0)$ and $(0,1)$ [12].

SPIN

When an external magnetic field is applied to the system, the Zeeman Hamiltonian must be taken into account. With respect to the standard basis

$$|L \uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |L \downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |R \uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |R \downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (2.15)$$

the Hamiltonian becomes

$$H = \begin{pmatrix} \frac{\epsilon}{2} & 0 & \tau & 0 \\ 0 & \frac{\epsilon}{2} & 0 & \tau \\ \tau & 0 & -\frac{\epsilon}{2} & 0 \\ 0 & \tau & 0 & -\frac{\epsilon}{2} \end{pmatrix} + H_Z = \begin{pmatrix} \frac{\epsilon}{2} + \frac{\gamma_L \hbar B_0}{2} & 0 & \tau & 0 \\ 0 & \frac{\epsilon}{2} - \frac{\gamma_L \hbar B_0}{2} & 0 & \tau \\ \tau & 0 & -\frac{\epsilon}{2} + \frac{\gamma_R \hbar B_0}{2} & 0 \\ 0 & \tau & 0 & -\frac{\epsilon}{2} - \frac{\gamma_R \hbar B_0}{2} \end{pmatrix}, \quad (2.16)$$

in which the external magnetic field (applied in the z-direction) is denoted by B_0 .

DIFFERENT G-FACTORS

When the two dots have different g-factors, the Hamiltonian becomes

$$H = \begin{pmatrix} \frac{\epsilon}{2} + \frac{\gamma_L \hbar B_0}{2} & 0 & \tau & 0 \\ 0 & \frac{\epsilon}{2} - \frac{\gamma_L \hbar B_0}{2} & 0 & \tau \\ \tau & 0 & -\frac{\epsilon}{2} + \frac{\gamma_R \hbar B_0}{2} & 0 \\ 0 & \tau & 0 & -\frac{\epsilon}{2} - \frac{\gamma_R \hbar B_0}{2} \end{pmatrix}, \quad (2.17)$$

in which $\gamma_L = \frac{g_L \mu_B}{\hbar}$ and $\gamma_R = \frac{g_R \mu_B}{\hbar}$, with g_L and g_R denoting the g-factors of the left and right dot, respectively. ϵ is defined as the average of the energy difference between $|L \uparrow\rangle$ and $|R \uparrow\rangle$ and the energy difference between $|L \downarrow\rangle$ and $|R \downarrow\rangle$.

HYPERFINE INTERACTION

When the hyperfine interaction is added to the system, an extra term is added to the Hamiltonian:

$$H_{HF} = \gamma \vec{B}_n \cdot \vec{S}; \quad (2.18)$$

this expression can be decomposed into six different components:

$$H_{HF} = \gamma_L (B_{n,L}^x S_L^x + B_{n,L}^y S_L^y + B_{n,L}^z S_L^z) + \gamma_R (B_{n,R}^x S_R^x + B_{n,R}^y S_R^y + B_{n,R}^z S_R^z), \quad (2.19)$$

in which $B_{n,j}^i$ is the nuclear field in the i -direction in dot j and S_j^i is the spin in the i -direction in dot j , with $i \in \{x, y, z\}$ and $j \in \{L, R\}$.

S_x and S_y can be written in terms of the spin raising and lowering operators S^+ and S^- :

$$S_x = \frac{S^+ + S^-}{2}, \quad S_y = \frac{S^+ - S^-}{2i}, \quad (2.20)$$

yielding the following hyperfine Hamiltonian in matrix form:

$$H_{HF} = \begin{pmatrix} \gamma_L B_{n,z,L} & \gamma_L (B_{n,x,L} - i B_{n,y,L}) & 0 & 0 \\ \gamma_L (B_{n,x,L} + i B_{n,y,L}) & -\gamma_L B_{n,z,L} & 0 & 0 \\ 0 & 0 & \gamma_R B_{n,z,R} & \gamma_R (B_{n,x,R} - i B_{n,y,R}) \\ 0 & 0 & \gamma_R (B_{n,x,R} + i B_{n,y,R}) & -\gamma_R B_{n,z,R} \end{pmatrix}. \quad (2.21)$$

This Hamiltonian shows that $|L \uparrow\rangle$ and $|L \downarrow\rangle$ are coupled by the hyperfine interaction, as are $|R \uparrow\rangle$ and $|R \downarrow\rangle$

2.3.3. DOUBLE QUANTUM DOT CONTAINING TWO ELECTRONS

Neglecting the hyperfine interaction, the two terms contributing to the Hamiltonian in a quantum dot containing two electrons are the Zeeman effect and the exchange interaction:

$$H = H_Z + H_C = \gamma B_0 (S_1^z + S_2^z) + J \left(\vec{S}_1 \cdot \vec{S}_2 - \frac{\hbar^2}{4} \right). \quad (2.22)$$

By expanding the dot product, the following expression is obtained:

$$\vec{S}_1 \cdot \vec{S}_2 = S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z \quad (2.23)$$

As seen in equation (2.20), the S^x and S^y can be expressed in terms of the spin raising and lowering operators, yielding

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{S_1^+ S_2^- + S_1^- S_2^+}{2} + S_1^z + S_2^z. \quad (2.24)$$

Therefore, the total Hamiltonian is

$$\gamma B_0 (S_1^z + S_2^z) + J \left(\frac{S_1^+ S_2^- + S_1^- S_2^+}{2} + S_1^z + S_2^z - \frac{\hbar^2}{4} \right). \quad (2.25)$$

Using the standard basis

$$|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (2.26)$$

the Hamiltonian becomes

$$H = \hbar \begin{pmatrix} \gamma B_0 & 0 & 0 & 0 \\ 0 & -\frac{\hbar J}{2} & \frac{\hbar J}{2} & 0 \\ 0 & \frac{\hbar J}{2} & -\frac{\hbar J}{2} & 0 \\ 0 & 0 & 0 & -\gamma B_0 \end{pmatrix}. \quad (2.27)$$

EXCHANGE COUPLING IN A DOUBLE QUANTUM DOT

Taylor et al. [8] have found an expression for the exchange coupling J in a double quantum dot as a function of the detuning ϵ . ϵ is the energy difference between $|(1,1)S\rangle$ and $|(0,2)S\rangle$, in which S denotes the singlet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ and (x,y) is the charge state, with x the number of electrons in the left dot, and y the number of electrons in the right dot. When tunnel coupling between the two dots is enabled, and it is assumed that $|(2,0)S\rangle$ has a much higher energy than $|(1,1)S\rangle$ and $|(0,2)S\rangle$, the only charge transition occurs between $|(1,1)S\rangle$ and $|(0,2)S\rangle$. With respect to the basis

$$|(2,0)S\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |(1,1)S\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.28)$$

the Hamiltonian describing this charge transition is

$$H_{11-20} = \begin{pmatrix} -\epsilon & \tau \\ \tau & 0 \end{pmatrix}. \quad (2.29)$$

It can be shown that the eigenfunction corresponding to the lowest eigenenergy is equal to

$$|\tilde{S}\rangle = \cos\theta |(1,1)S\rangle + \sin\theta |(0,2)S\rangle \quad (2.30)$$

in which θ is a function of ϵ and τ :

$$\theta = \arctan \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}} \quad (2.31)$$

The energy of $\tilde{S}$ is

$$E_{\tilde{S}} = -\frac{\tau}{2} \tan \theta = -\frac{\tau^2}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}} = -\frac{\epsilon}{2} - \sqrt{\left(\frac{\epsilon}{2}\right)^2 + 4\tau^2}. \quad (2.32)$$

When the detuning is low, each dot contains one electron. So $\tilde{S} \approx (1,1)S$ and $E_{\tilde{S}} \approx E_{(1,1)S}$. Increasing the detuning yields an increase of $(0,2)S$ in $\tilde{S}$, thereby increasing the probability of tunnelling.

The exchange coupling J is equal to the energy difference between the T_0 (the triplet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$) and the singlet state. T_0 has energy zero, so the exchange coupling is equal to

$$J(\epsilon) = \frac{E_{T_0} - E_{\tilde{S}}}{\hbar^2} = -\frac{E_{\tilde{S}}}{\hbar^2} = \frac{1}{\hbar^2} \left(\frac{\epsilon}{2} + \sqrt{\left(\frac{\epsilon}{2}\right)^2 + 4\tau^2} \right). \quad (2.33)$$

However, at very high detuning, the triplet state hybridizes, so this expression for the exchange coupling is not valid in that regime.

DIFFERENT G-FACTORS

Considering the low detuning regime, in which $\tilde{S} \approx (1,1)S$, when the electrons in the two dots have different g-factors, the Hamiltonian becomes

$$H = B_0(\gamma_L S_1^z + \gamma_R S_2^z) + J \left(\frac{S_1^+ S_2^- + S_1^- S_2^+}{2} + S_1^z + S_2^z - \frac{\hbar^2}{4} \right), \quad (2.34)$$

yielding the following matrix:

$$H = \frac{\hbar}{2} \begin{pmatrix} (\gamma_L + \gamma_R)B_0 & 0 & 0 & 0 \\ 0 & (\gamma_L - \gamma_R)B_0 - \hbar J & \hbar J & 0 \\ 0 & \hbar J & (-\gamma_L + \gamma_R)B_0 - \hbar J & 0 \\ 0 & 0 & 0 & (-\gamma_L - \gamma_R)B_0 \end{pmatrix}. \quad (2.35)$$

2.4. TRIPLE QUANTUM DOT

Research on a triple quantum dot has been performed by Laird et al. [17]. They modelled the triple quantum dot as two double dots sharing one of the dots. The middle electron interacts with both the left and the right electron, but the left and the right electron do not interact with each other. Therefore, the Hamiltonian contains two terms that describe an exchange interaction:

$$H = J_{12} \left(\vec{S}_1 \cdot \vec{S}_2 - \frac{\hbar^2}{4} \right) + J_{23} \left(\vec{S}_2 \cdot \vec{S}_3 - \frac{\hbar^2}{4} \right) + \gamma B_0 (S_1^z + S_2^z + S_3^z), \quad (2.36)$$

in which J_{12} is the coupling between the first and second electron, and J_{23} between the second and third electron. Using the standard basis

$$\begin{aligned} |\uparrow\uparrow\uparrow\rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ |\downarrow\uparrow\uparrow\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |\downarrow\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \end{aligned} \quad (2.37)$$

the Hamiltonian becomes

$$H = \frac{\hbar}{2} \begin{pmatrix} 3\gamma B_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma B_0 - \hbar J_{23} & \hbar J_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & \hbar J_{23} & \gamma B_0 - \hbar J_{12} - \hbar J_{23} & 0 & \hbar J_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma B_0 - \hbar J_{12} & 0 & \hbar J_{12} & 0 & 0 \\ 0 & 0 & \hbar J_{12} & 0 & \gamma B_0 - \hbar J_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hbar J_{12} & 0 & -\gamma B_0 - \hbar J_{12} - \hbar J_{23} & \hbar J_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & \hbar J_{23} & -\gamma B_0 - \hbar J_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3\gamma B_0 \end{pmatrix}. \quad (2.38)$$

Here, the detuning ϵ is defined along the line connecting charge states (2,0,1), (1,1,1) and (1,0,2). This is depicted in figure 2.5. In (x,y,z) x is the charge occupation of the left dot, y of the middle dot and z of the right dot.

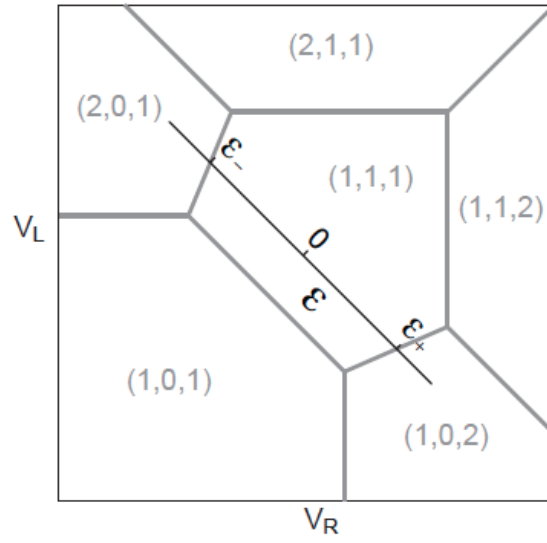


Figure 2.5: The detuning in a triple quantum dot is defined along the axis connecting (2,0,1) and (1,0,2) via (1,1,1) [17].

At $\epsilon = \epsilon_-$, the charge states (2,0,1) and (1,1,1) line up. At $\epsilon = \epsilon_+$, (1,1,1) and (1,0,2) line up. Then, the exchange couplings J_{12} and J_{23} are

$$J_{12}(\epsilon) = \frac{1}{\hbar^2} \left(\frac{\epsilon_- - \epsilon}{2} + \sqrt{(\epsilon_- - \epsilon)^2 + 4\tau_L^2} \right), \quad J_{23}(\epsilon) = \frac{1}{\hbar^2} \left(\frac{\epsilon - \epsilon_+}{2} + \sqrt{(\epsilon - \epsilon_+)^2 + 4\tau_R^2} \right), \quad (2.39)$$

in which τ_L is the tunnel coupling between the left and the middle dot and τ_R between the middle and the right dot.

2.5. QUADRUPLE QUANTUM DOT

The triple quantum dot Hamiltonian can be extended to the quadruple quantum dot, yielding

$$H = J_{12} \left(\vec{S}_1 \cdot \vec{S}_2 - \frac{\hbar^2}{4} \right) + J_{23} \left(\vec{S}_2 \cdot \vec{S}_3 - \frac{\hbar^2}{4} \right) + J_{34} \left(\vec{S}_3 \cdot \vec{S}_4 - \frac{\hbar^2}{4} \right) + \gamma B_0 (S_1^z + S_2^z + S_3^z + S_4^z). \quad (2.40)$$

3

TIME EVOLUTION AND SPIN SWAP

3.1. TIME EVOLUTION

Considering a time-independent Hamiltonian, the time-dependent Schrödinger equation is

$$H\Psi = i\hbar \frac{\partial \Psi}{\partial t}. \quad (3.1)$$

Solving for the wave function Ψ yields

$$\Psi(t) = e^{-\frac{iHt}{\hbar}} \Psi(0). \quad (3.2)$$

So by applying the unitary operator $e^{-\frac{iHt}{\hbar}}$ to the initial state of a system (the state at $t = 0$), the state at time t can be obtained.

3.2. OSCILLATION IN A TWO-STATE SYSTEM

In a two-level quantum mechanical system, two basis vectors can be defined, $|0\rangle$ and $|1\rangle$. All pure states of this system can be written as a linear combination of these two basis vectors, so the wavefunction of the initial state can be represented as

$$|\phi(0)\rangle = c_0|0\rangle + c_1|1\rangle, \quad (3.3)$$

in which c_0 and c_1 are complex numbers with $|c_0|^2 + |c_1|^2 = 1$. Then, the time-dependent wavefunction is

$$|\phi(t)\rangle = c_0|0\rangle e^{-\frac{iE_0t}{\hbar}} + c_1|1\rangle e^{-\frac{iE_1t}{\hbar}}, \quad (3.4)$$

in which E_0 and E_1 are the energies of $|0\rangle$ and $|1\rangle$ respectively.

The probability of this state returning to its initial state at time t is equal to $|\langle\phi(0)|\phi(t)\rangle|^2$. It can be shown that

$$|\langle\phi(0)|\phi(t)\rangle|^2 = 1 - 4|c_0^2 c_1^2| \sin^2\left(\frac{(E_1 - E_0)t}{2\hbar}\right). \quad (3.5)$$

$\sin^2 x$ is π -periodic, so the oscillation frequency is $\frac{E_1 - E_0}{2\pi\hbar}$.

3.3. BLOCH SPHERE

The Bloch sphere is a geometrical representation of a two-level quantum mechanical system. Given the two eigenvectors of a two-state quantum mechanical system, $|0\rangle$ and $|1\rangle$, all pure states of this system can be represented as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad (3.6)$$

in which α and β are complex numbers precluded $|\alpha|^2 + |\beta|^2 = 1$. Without loss of generality, ϕ can be rewritten as

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle, \quad (3.7)$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. The parameters ϕ and θ can be interpreted as spherical coordinates that define a point $(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ on the unit sphere. The Bloch sphere is useful for visualizing the time evolution of a two-state system. In figure 3.1, the Bloch sphere is depicted.

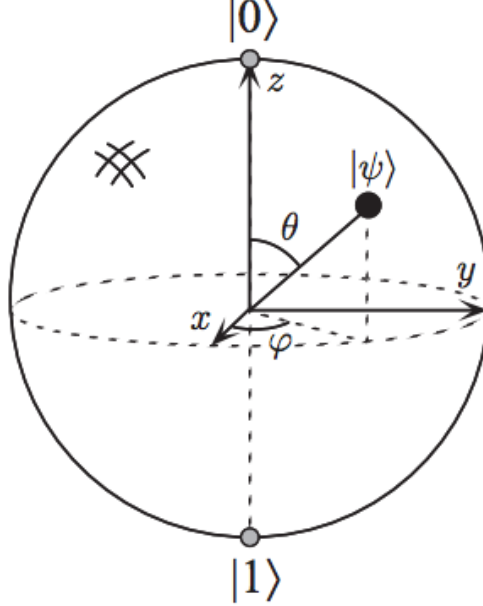


Figure 3.1: The Bloch Sphere: $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$ is depicted on the Bloch Sphere using ϕ as the azimuthal angle and θ as the polar angle. [18].

3.4. SPIN SWAP

Knowing how to execute a spin swap is important, because in a spin-based quantum computer, the swap gate is a useful tool that swaps two qubits.

Hu and Das Sarma [9] have designed a pulsing scheme that swaps two electron spins in a double quantum dot in the presence of Zeeman inhomogeneity, thereby making the change $|\uparrow\downarrow\rangle \rightarrow |\downarrow\uparrow\rangle$ or vice versa. They made two assumptions; one being that J can be turned on and off infinitely fast, the other that there is no decoherence in the system. Hyperfine interaction, for example, is known to cause decoherence.

As seen before, in the low detuning regime, the Hamiltonian describing a double quantum dot with different g-factors containing two electrons is

$$H = \frac{\hbar}{2} \begin{pmatrix} (\gamma_L + \gamma_R)B_0 & 0 & 0 & 0 \\ 0 & (\gamma_L - \gamma_R)B_0 - \hbar J & \hbar J & 0 \\ 0 & \hbar J & (-\gamma_L + \gamma_R)B_0 - \hbar J & 0 \\ 0 & 0 & 0 & (-\gamma_L - \gamma_R)B_0 \end{pmatrix}. \quad (3.8)$$

The off-diagonal elements indicate that $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are coupled, while $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ experience no coupling. Therefore, the two-state subspace spanned by $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ can be described by the effective Hamiltonian

$$H_{eff} = \frac{1}{2} \begin{pmatrix} \delta & \hbar^2 J \\ \hbar^2 J & -\delta \end{pmatrix} = \frac{1}{2} \hbar^2 J \sigma_x + \frac{1}{2} \delta \sigma_z = J_n \hat{n} \cdot \sigma \quad (3.9)$$

in which $\delta = \hbar(\gamma_L - \gamma_R)B_0$ is the difference in Zeeman energy between the two dots, σ_x and σ_z are the Pauli spin matrices in the x- and z-direction, $J_n = \frac{1}{2} \sqrt{\hbar^4 J^2 + \delta^2}$ and $\hat{n} = (\hbar^2 J, 0, \delta)/J_n$. Using this Hamiltonian, the swap operation can be explained geometrically. This effective Hamiltonian represents a rotation on the Bloch

sphere around the axis defined by $\hat{n}$. On this Bloch sphere, $|\uparrow\downarrow\rangle$ is located at the top of the sphere, while $|\downarrow\uparrow\rangle$ is located at the bottom. The rotation angle is determined by the duration and the strength of this Hamiltonian. When there is no Zeeman inhomogeneity, the rotation occurs around the x-axis. Therefore, a π rotation would result in a swap: $|\uparrow\downarrow\rangle \rightarrow |\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\rangle \rightarrow |\uparrow\downarrow\rangle$. A 2π rotation is shown in figure 3.2a.

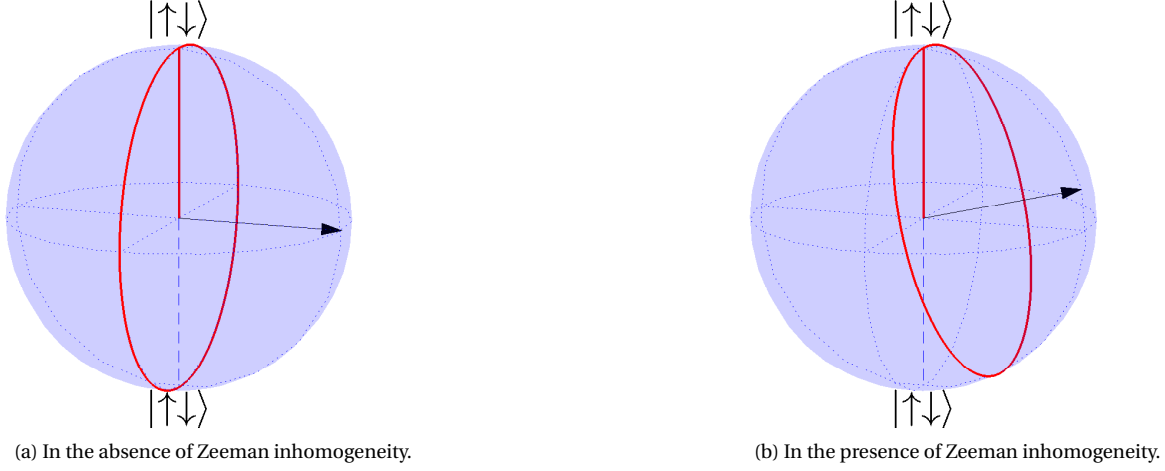


Figure 3.2: A 2π rotation of the initial state $|\uparrow\downarrow\rangle$, depicted on the Bloch Sphere. The black arrow indicates the rotational axis. When there is no Zeeman inhomogeneity, the time evolution occurs around the x-axis, so a π rotation will yield the final state $|\downarrow\uparrow\rangle$. In the presence of Zeeman inhomogeneity, the time evolution no longer takes place around the x-axis, so $|\uparrow\downarrow\rangle$ will never completely evolve to $|\downarrow\uparrow\rangle$.

When the dots have different g-factors, the rotational axis $\hat{n}$ veers away from the x-axis. When the initial state is $|\uparrow\downarrow\rangle$, the starting point is the north pole of the Bloch sphere. Because the rotation occurs around a different axis now, a π rotation will never yield a state situated at the south pole of the sphere, $|\downarrow\uparrow\rangle$. Instead, it will result in a superposition of these two states. A 2π rotation in the presence of Zeeman inhomogeneity is shown in figure 3.2b.

However, if the magnitude of the Zeeman inhomogeneity is known, the swap procedure can still be performed by adjusting the rotational axis. This, in turn, can be done by adjusting J . One way to perform a swap operation is by:

- First applying a square exchange pulse with magnitude J to execute a π rotation;
- Next turning J off so the system undergoes a π rotation around the z-axis, caused by the Zeeman inhomogeneity;
- Finally turning on a second exchange pulse with magnitude $J' = \frac{\delta^2}{\hbar^4 J}$, executing a π rotation.

The exchange pulse sequences for performing a swap in the absence and the presence of Zeeman inhomogeneity are depicted in figure 3.3. The frequency of the revolution around $\hat{n}$ is $\frac{\sqrt{\hbar^4 J^2 + \delta^2}}{2\pi\hbar}$. Therefore, the duration of a π pulse is $\frac{\pi\hbar}{\sqrt{\hbar^4 J^2 + \delta^2}}$. So the time it takes to perform the swap operation is

$$t_{\text{swap}} = \frac{\pi\hbar}{\sqrt{\hbar^4 J^2 + \delta^2}} + \frac{\pi\hbar}{|\delta|} + \frac{\pi\hbar}{\sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}}, \quad J, \delta > 0. \quad (3.10)$$

This pulse sequence will yield a swap of the two spins: $|\uparrow\downarrow\rangle \rightarrow |\downarrow\uparrow\rangle$.

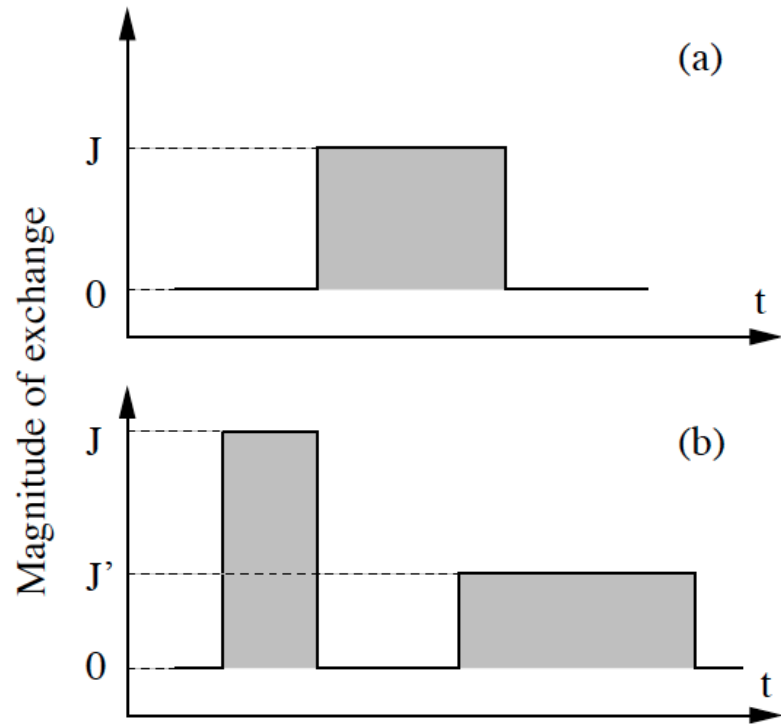


Figure 3.3: (a) In the absence of Zeeman inhomogeneity, a swap operation can be performed with a single exchange pulse. (b) In the presence of Zeeman inhomogeneity, a swap operation can be performed by applying two π pulses with magnitudes J and $J' = \frac{\delta^2}{\hbar^4 J}$ and a π pulse for free evolution in between [9].

4

RESULTS: EIGENENERGIES, TIME EVOLUTION AND SPIN SWAP

4.1. DOUBLE QUANTUM DOT

4.1.1. DOUBLE QUANTUM DOT CONTAINING ONE ELECTRON

The eigenstates and corresponding eigenenergies of Hamiltonian (2.14) are shown in table 4.1.

Eigenstate	Eigenenergy
$-\sin\theta L\rangle + \cos\theta R\rangle$	$\frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}$
$\cos\theta L\rangle + \sin\theta R\rangle$	$-\frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}$

Table 4.1: Eigenstates and corresponding eigenenergies of a double dot containing two electrons, neglecting the different spin states, with $\theta = \arctan \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}}$.

A plot of the eigenenergies as a function of the detuning ϵ , the energy difference between $|(1,0)\rangle$ and $|(0,1)\rangle$, is shown in figure 4.1. An anticrossing occurs at $\epsilon = 0$. The size of the anticrossing is 2τ , which is consistent with the tunnel coupling τ that couples the two states.

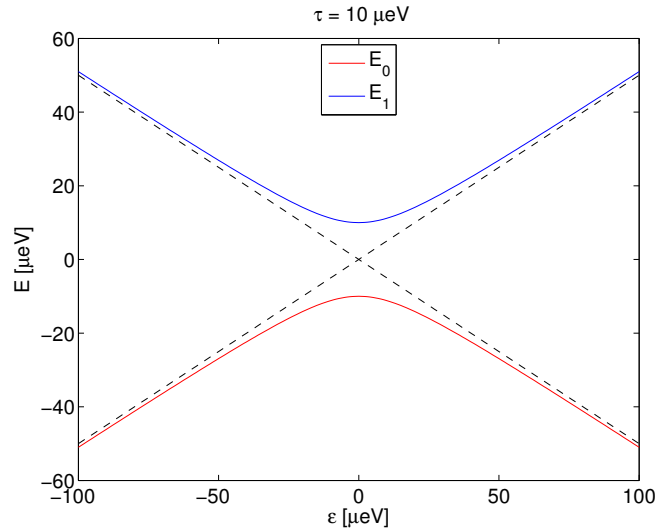


Figure 4.1: Eigenenergies of a double dot system containing one electron as a function of detuning ϵ . The dashed black lines represent the eigenenergies in the absence of tunnel coupling. An anticrossing occurs at $\epsilon = 0$.

When the Zeeman effect is taken into account, as in Hamiltonian (2.16), the eigenenergies are split by the Zeeman energy. This can be seen in table 4.2, in which the eigenstates and the corresponding eigenenergies are shown.

Eigenstate	Eigenenergy
$-\sin\theta L\uparrow\rangle + \cos\theta R\uparrow\rangle$	$\frac{\gamma\hbar B_0}{2} + \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}$
$\cos\theta L\uparrow\rangle + \sin\theta R\uparrow\rangle$	$\frac{\gamma\hbar B_0}{2} - \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}$
$-\sin\theta L\downarrow\rangle + \cos\theta R\downarrow\rangle$	$-\frac{\gamma\hbar B_0}{2} + \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}$
$\cos\theta L\downarrow\rangle + \sin\theta R\downarrow\rangle$	$-\frac{\gamma\hbar B_0}{2} - \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}$

Table 4.2: Eigenstates and corresponding eigenenergies of a double dot containing two electrons, in the presence of an external magnetic field, with $\theta = \arctan \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}}$. The Zeeman effect causes the different spin states to be separated by E_Z .

The Zeeman splitting can also be observed in figure 4.2. At $\epsilon = 0$, the spin up eigenstates, as well as the spin down eigenstate, experience an anticrossing with size 2τ , as expected.

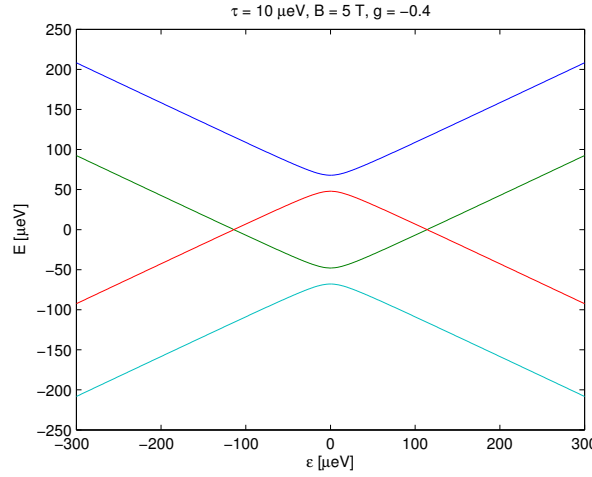


Figure 4.2: Eigenenergies of a double dot system containing one electron as a function of detuning. An external magnetic field is present, so Zeeman splitting occurs.

In each subplot of figure 4.3 one of the basis components $|L\uparrow\rangle$, $|L\downarrow\rangle$, $|R\uparrow\rangle$ and $|R\downarrow\rangle$ is plotted on top of the eigenenergies. The color bar shows that the lighter the line is, the larger the basis component. This plot shows that around $\epsilon = 0$, the eigenstates are mixed states, two consisting of $|L\uparrow\rangle$ and $|R\uparrow\rangle$ and two consisting of $|L\downarrow\rangle$ and $|R\downarrow\rangle$, while when $|\epsilon| \gg 0$, the eigenstates approach the basis components, due to the large energy difference between the left and the right dot.

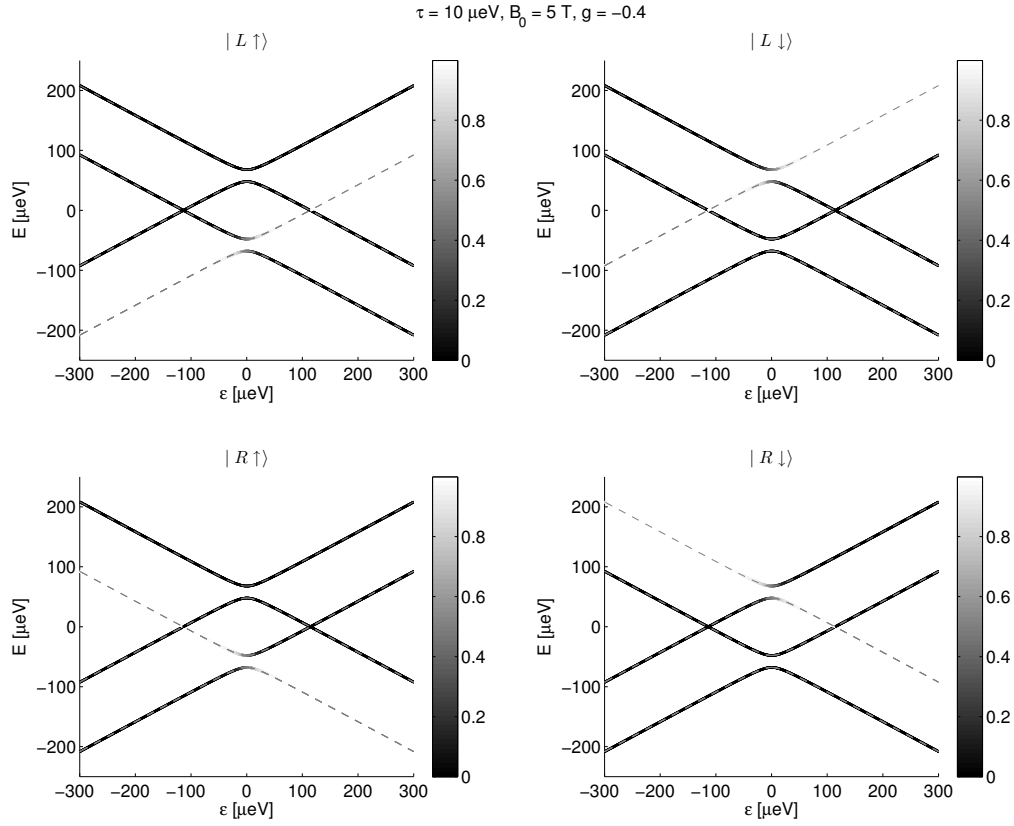


Figure 4.3: Eigenenergies of a double dot system containing one electron, as a function of detuning, in the presence of an external magnetic field. On top, the component of the basis vectors $|L\uparrow\rangle$, $|L\downarrow\rangle$, $|R\uparrow\rangle$ and $|R\downarrow\rangle$ in the corresponding eigenvectors is plotted. The lighter the line, the larger the basis component, so around $\epsilon = 0$, the eigenstates consist of either $|L\uparrow\rangle$ and $|R\uparrow\rangle$ or $|L\downarrow\rangle$ and $|R\downarrow\rangle$, while at large values of $|\epsilon|$, the eigenstates approach the basis states.

In Hamiltonian (2.17), different g-factors are taken into account. In figure 4.4, the corresponding eigenenergies are plotted as a function of the detuning ϵ . The different g-factors cause a difference in Zeeman energies in the dots. Therefore, there is a shift in the eigenenergies. Anticrossings occur when energy levels in the left and the right dot are equal. This happens when

$$\frac{\epsilon}{2} \pm \frac{\mu_B B_0 g_L}{2} = -\frac{\epsilon}{2} \pm \frac{\mu_B B_0 g_R}{2}. \quad (4.1)$$

Solving equation (4.1) for ϵ yields

$$\epsilon = \pm \frac{\mu_B B_0 (g_R - g_L)}{2}. \quad (4.2)$$

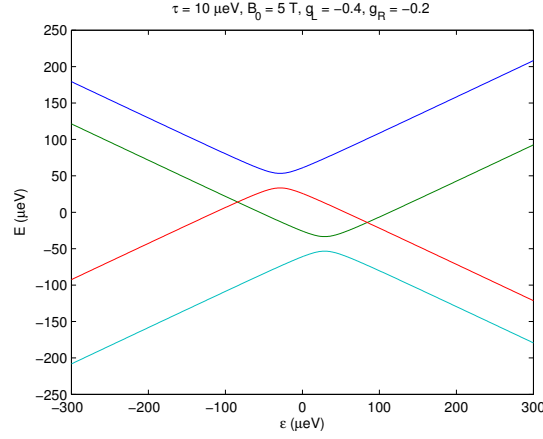


Figure 4.4: Eigenenergies as a function of detuning, in the presence of an external magnetic field. The two dots have different g-factors.

The energies of $|L \uparrow\rangle$ and $|R \uparrow\rangle$ align when $\epsilon = \frac{\mu_B B_0 (g_R - g_L)}{2} = 28.94 \mu\text{eV}$. The energies of $|L \downarrow\rangle$ and $|R \downarrow\rangle$ align when $\epsilon = \frac{\mu_B B_0 (g_L - g_R)}{2} = -28.94 \mu\text{eV}$.

Considering equal g-factors and the initial state $|L \uparrow\rangle$, when the system is pulsed into $\epsilon = 0$, an oscillation between $|L \uparrow\rangle$ and $|R \uparrow\rangle$ occurs.¹ This can be seen in the uppermost two plots in 4.5. At $\epsilon \neq 0$, the oscillation no longer has a peak-to-peak amplitude of 1, as can be seen in the two plots at the bottom of figure 4.5. $|L \uparrow\rangle$ will never completely evolve to $|R \uparrow\rangle$. The oscillation frequency is

$$f = \frac{\sqrt{\epsilon^2 + 4\tau^2}}{2\pi\hbar}, \quad (4.3)$$

while the oscillation peak-to-peak amplitude is

$$A_{pp} = \frac{4\tau^2}{\epsilon^2 + 4\tau^2}. \quad (4.4)$$

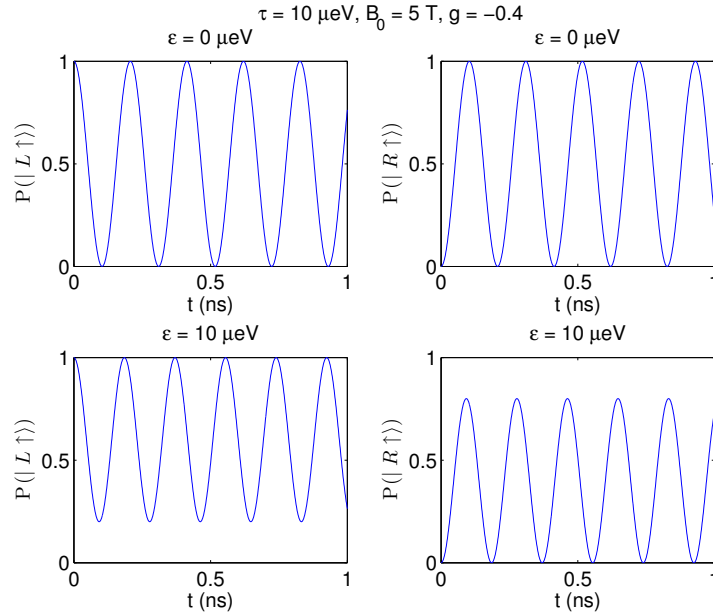


Figure 4.5: Time evolution of a system initially in the state $|L \uparrow\rangle$: probability of measuring $|L \uparrow\rangle$ and $|R \uparrow\rangle$ as a function of time. In the upper two plots, $\epsilon = 0$, yielding an oscillation between $|L \uparrow\rangle$ and $|R \uparrow\rangle$. In the lower two plots, $\epsilon = 10 \mu\text{eV} \neq 0$, so $|L \uparrow\rangle$ will never completely evolve to $|R \uparrow\rangle$.

¹When the initial state is $|L \downarrow\rangle$, an oscillation between $|L \downarrow\rangle$ and $|R \downarrow\rangle$ occurs.

Considering different g-factors and the initial state $|L \uparrow\rangle$, an oscillation between $|L \uparrow\rangle$ and $|R \uparrow\rangle$ occurs at the anticrossings. The oscillation between the spin up states occurs at $\epsilon_{\uparrow} = \frac{\mu_B B_0 (g_R - g_L)}{2}$, while the oscillation between the spin down states occurs at $\epsilon_{\downarrow} = \frac{\mu_B B_0 (g_L - g_R)}{2}$. The oscillation between $|L \uparrow\rangle$ and $|R \uparrow\rangle$ at the anticrossing $\epsilon = \epsilon_{\uparrow}$ can be seen in the uppermost plots in figure 4.6. The two plots at the bottom of this figure show the time evolution at a distance of $10 \mu\text{eV}$ of the anticrossing. In the case of an oscillation between the spin up states, when different g-factors are involved, the oscillation frequency and the peak-to-peak amplitude are ²

$$f = \frac{\sqrt{(\epsilon - \epsilon_{\uparrow})^2 + 4\tau^2}}{2\pi\hbar} \quad (4.5)$$

$$A_{pp} = \frac{4\tau^2}{(\epsilon - \epsilon_{\uparrow})^2 + 4\tau^2} \quad (4.6)$$

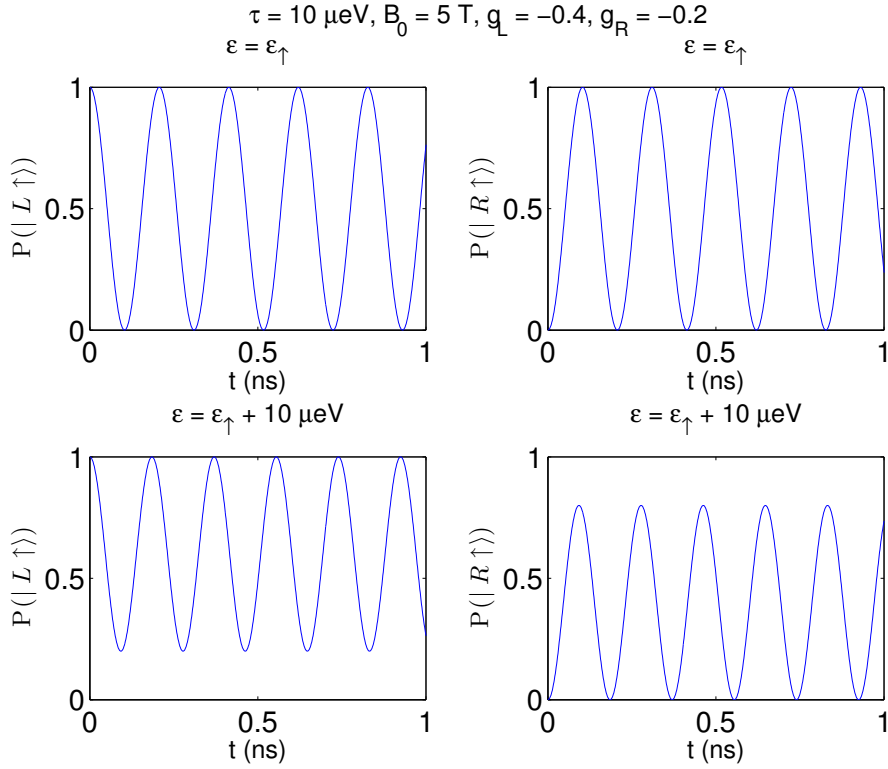


Figure 4.6: Time evolution of a system with different g-factors, initially in the state $|L \uparrow\rangle$: probability of measuring $|L \uparrow\rangle$ and $|R \uparrow\rangle$ as a function of time. In the upper two plots, the time evolution occurs at the anticrossing, $\epsilon = \epsilon_{\uparrow}$, causing an oscillation between $|L \uparrow\rangle$ and $|R \uparrow\rangle$. In the lower two plots, the time evolution occurs at a detuning $10 \mu\text{eV}$ from the anticrossing, so $|L \uparrow\rangle$ will never completely evolve to $|R \uparrow\rangle$.

In Hamiltonian (2.21), the hyperfine interaction is taken into account. A plot of the eigenenergies in the case of equal g-factors is shown in the left plot of figure 4.7. Zooming in on this plot reveals two new anticrossings. One is shown in the right plot of figure 4.7.

²In the case of an oscillation between the spin down states, $\epsilon_{\uparrow}$ must be replaced by $\epsilon_{\downarrow}$.

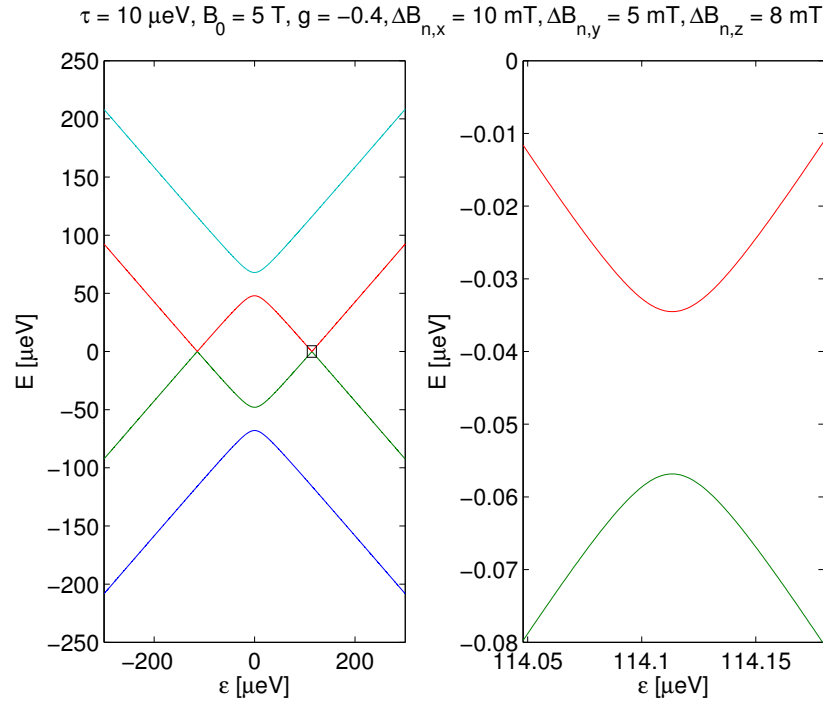


Figure 4.7: Eigenenergies of a double dot system containing one electron as a function of detuning, in the presence of an external magnetic field. Hyperfine interactions are taken into account, causing an anticrossing to occur. In the right plot, a small part of the left plot is magnified, revealing an anticrossing.

In the following, the dependency of the size of this anticrossing on the different parameters is studied. The results show the following relation between the anticrossing size and the parameters:

$$\text{Anticrossing size} \propto \tau \sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2}. \quad (4.7)$$

The anticrossing size as a function of $\Delta B_{n,x} = B_{n,x,L} - B_{n,x,R}$ for different values of $\Delta B_{n,y} = B_{n,y,L} - B_{n,y,R}$ is shown in figure 4.8. The plot shows the dependency of the anticrossing size on $\Delta B_{n,x}$ for different values of $\Delta B_{n,y}$. When the anticrossing size is plotted as a function of $\Delta B_{n,y}$ for different values of $\Delta B_{n,x}$, the results are similar.

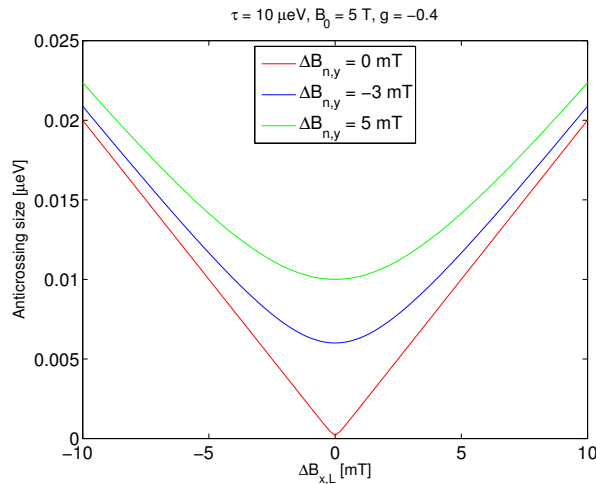


Figure 4.8: Size of the anticrossing induced by the hyperfine interaction as a function of $\Delta B_{n,x}$ for different values of $\Delta B_{n,y}$. The shapes of the curves suggest that the anticrossing size depends linearly on $\sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2}$.

In figure 4.9, the anticrossing size is plotted as a function of $\Delta B_{n,x}$ for different values of τ . The fit is

described by

$$c\sqrt{\Delta B_{N,x}^2 + \Delta B_{N,y}^2} \quad (4.8)$$

in which c is the fitting constant. The fits are indistinguishable from the data. The fitting constant depends linearly on τ , so the anticrossing size depends linearly on $\tau\sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2}$.

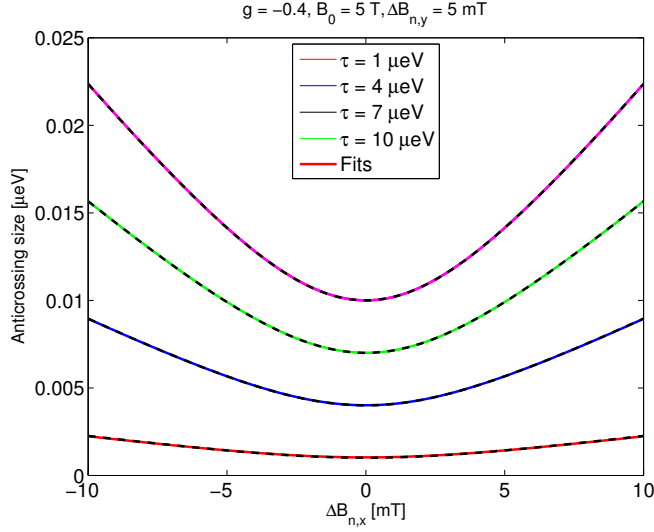


Figure 4.9: Size of the anticrossing that is induced by the hyperfine interaction as a function of $\Delta B_{N,x}$ for different values of τ . The curves have been fitted with $c\sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2}$. The fits are equal to the curves. The fitting constant c depends linearly on τ .

4.1.2. DOUBLE QUANTUM DOT CONTAINING TWO ELECTRONS

The eigenenergies and the corresponding eigenstates of Hamiltonian (2.27) with J as in (2.33) can be found in table 4.3. The eigenstates of a double dot containing two electrons are the three triplet states and the singlet state.

Name	Eigenstate	Eigenenergy	Total spin s	Spin number m_s
T_+	$ \uparrow\uparrow\rangle$	γB_0	1	1
T_0	$\frac{1}{\sqrt{2}}(\uparrow\downarrow\rangle + \downarrow\uparrow\rangle)$	0	1	0
T_-	$ \downarrow\downarrow\rangle$	$-\gamma B_0$	1	-1
S	$\frac{1}{\sqrt{2}}(\uparrow\downarrow\rangle - \downarrow\uparrow\rangle)$	$-\hbar^2 J$	0	0

Table 4.3: Eigenstates and corresponding eigenenergies of a double dot containing two electrons.

The triplet eigenenergies are constant as a function of ϵ , while the singlet eigenenergy depends on J , as expected, because $\hbar^2 J$ is the difference in energy between T_0 and S . A plot of the eigenenergies as a function of the detuning is shown in figure 4.10.

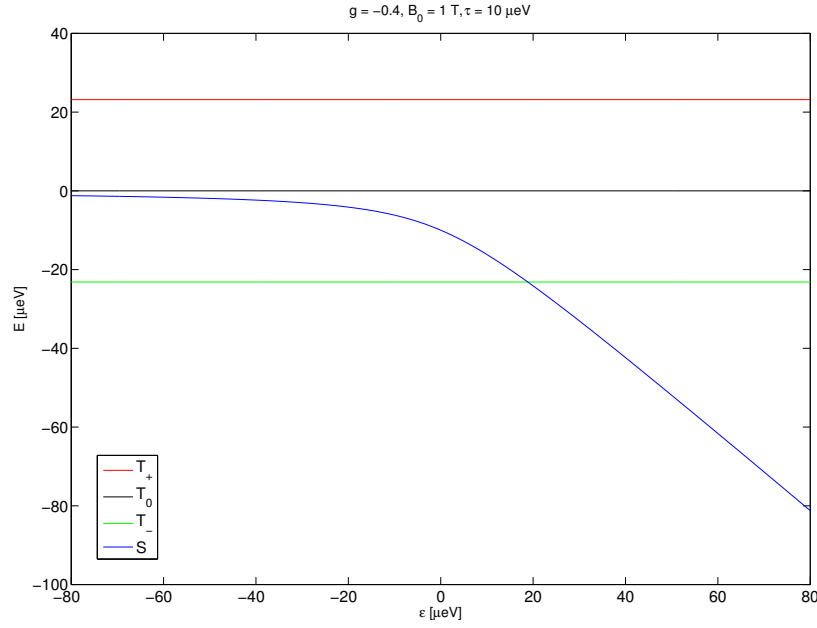
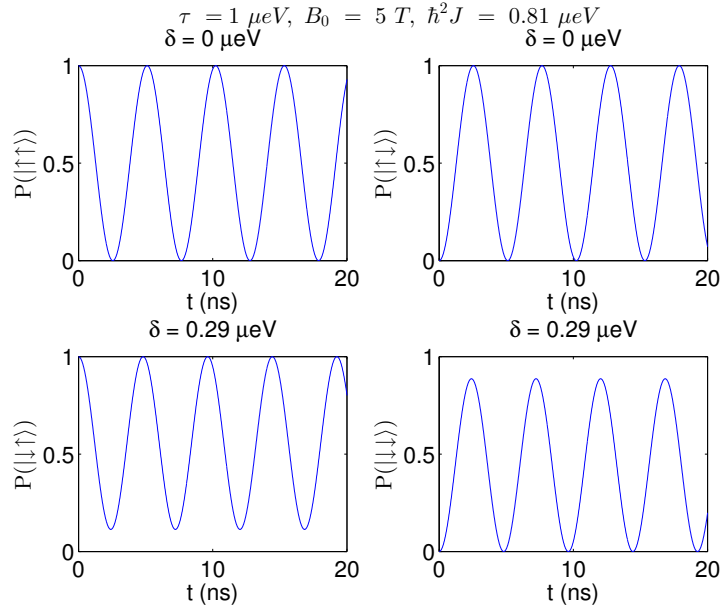


Figure 4.10: Eigenenergies of a double dot system as a function of detuning.

Considering equal g -factors and the initial state $|\uparrow\downarrow\rangle$, the state will oscillate between $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$, due to the exchange coupling J , that couples $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$. The probability of measuring $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ as a function of time is shown in the two uppermost plots of figure 4.11. When the two dots have different g -factors, and the initial state is $|\uparrow\downarrow\rangle$, the oscillation will never completely reach the state $|\downarrow\uparrow\rangle$, as is shown in the bottom two plots of figure 4.11.

Figure 4.11: Time evolution of the initial state $|\uparrow\downarrow\rangle$. In the upper two plots, there is no Zeeman inhomogeneity, so an oscillation between $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ occurs. In the two lower plots, a Zeeman inhomogeneity is present, so $|\uparrow\downarrow\rangle$ will never evolve to $|\downarrow\uparrow\rangle$.

Suppose the initial state of the two electron spins is $|\uparrow\downarrow\rangle$ and the goal is to achieve a spin swap, but there is a Zeeman inhomogeneity. As described in chapter 3, it is not possible to do this with a constant exchange coupling, but the procedure described by Hu and Das Sarma can be applied to reach $|\downarrow\uparrow\rangle$. The probabilities of measuring $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ as a function of time during this procedure have been plotted in figure 4.12.

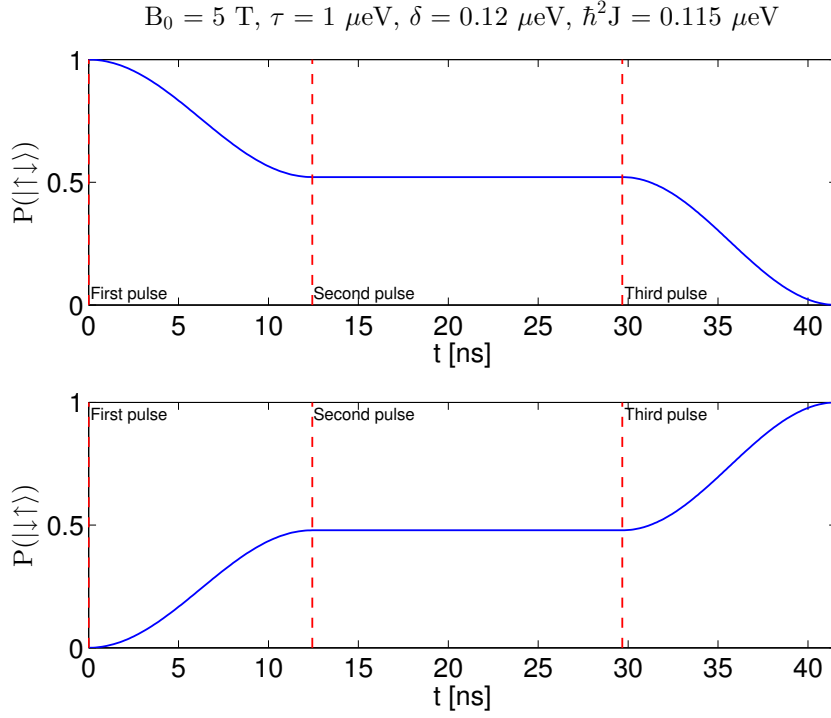


Figure 4.12: Time evolution of the initial state $|\uparrow\downarrow\rangle$ during the spin swap operation as described by Hu and Das Sarma. The first π pulse, in which the exchange coupling has magnitude J , is followed by a π pulse with exchange coupling 0. The third π pulse has magnitude $\frac{\delta^2}{\hbar^4 J}$. This plot shows that during this procedure, $|\uparrow\downarrow\rangle$ evolves to $|\downarrow\uparrow\rangle$.

In figure 4.13, the effect of the swapping procedure as described by Hu and Das Sarma is shown for different ratios of J and δ .

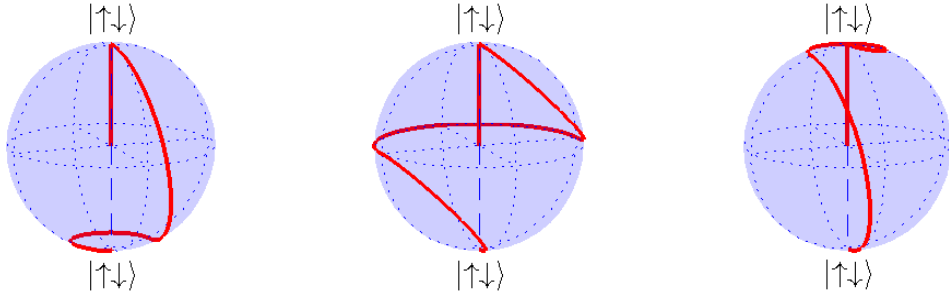


Figure 4.13: Time evolution of the state that is initialized as $|\uparrow\downarrow\rangle$ during the spin swap operation as described by Hu and Das Sarma. The operation is performed for different ratios of J and δ . $J = 5|\delta|$ in the leftmost, $J = |\delta|$ in the middle and $J = \frac{1}{5}|\delta|$ in the rightmost figure.

As mentioned in chapter 3, the duration of a π pulse is $\frac{\pi\hbar}{\sqrt{\hbar^4 J^2 + \delta^2}}$, so the total duration of the swapping procedure is given by

$$t_{\text{swap}} = \frac{\pi\hbar}{\sqrt{\hbar^4 J^2 + \delta^2}} + \frac{\pi\hbar}{|\delta|} + \frac{\pi\hbar}{\sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}}, \quad J, \delta > 0. \quad (4.9)$$

Using equation (4.9), it is possible to determine upper and lower bounds for the swapping time, depending on the magnitude of the Zeeman inhomogeneity δ . In appendix B, it is shown that

$$\frac{2\pi\hbar}{|\delta|} < t_{\text{swap}} \leq \frac{(\sqrt{2}+1)\pi\hbar}{|\delta|}. \quad (4.10)$$

The upper bound, $\frac{(\sqrt{2}+1)\pi\hbar}{|\delta|}$, is reached when $J = |\delta|$, while the lower bound, $\frac{2\pi\hbar}{|\delta|}$ is approached when $J \rightarrow 0$ and when $J \rightarrow \infty$. The swapping time as a function of the exchange coupling is plotted in figure 4.14.

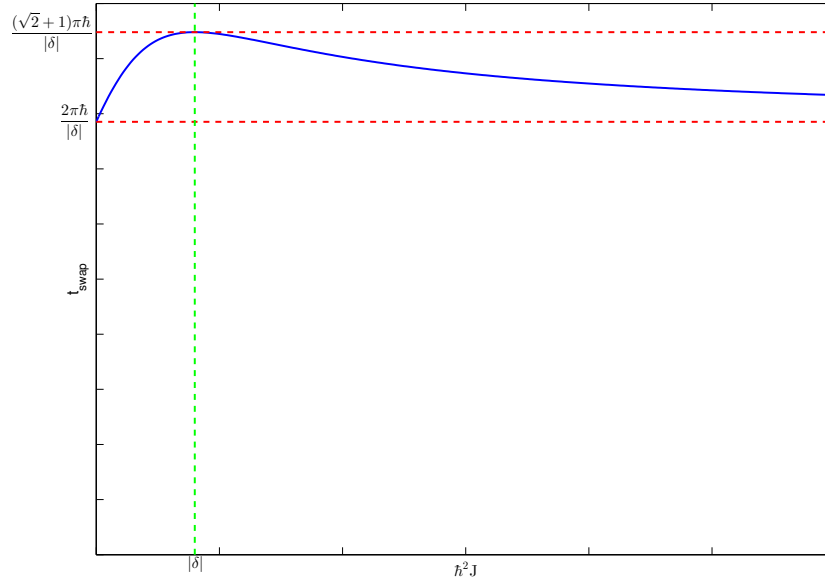
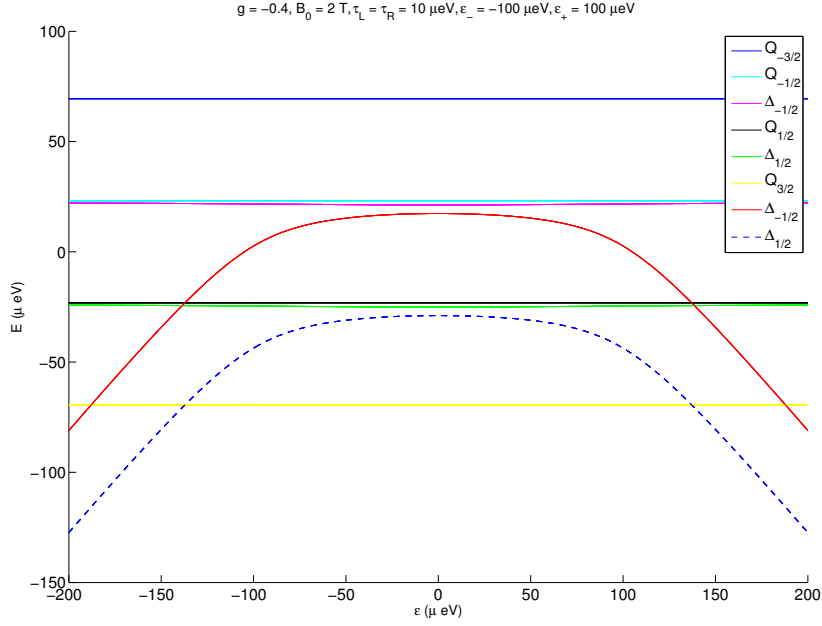


Figure 4.14: Duration of the swapping procedure as described by Hu and Das Sarma, as a function of the exchange coupling. The swap time is bounded from below by $\frac{2\pi\hbar}{|\delta|}$ and bounded from above by $\frac{(\sqrt{2}+1)\pi\hbar}{|\delta|}$. The upper bound is reached at $\hbar^2 J = |\delta|$, while the lower bound is approached when $\hbar^2 J \rightarrow 0$ and $\hbar^2 J \rightarrow \infty$.

In GaAs quantum dots, the decoherence time is in the order of 10 ns [19]. One of the DiVincenzo requirements of the implementation of quantum computation is that the gate operation times are much shorter than the decoherence time [20]. Previously, in measurements in GaAs quantum dots, a Zeeman inhomogeneity of approximately $0.1 \mu\text{eV}$ has been observed, when applying a magnetic field of 5 T. Using equation (4.10), the total required swap time is 37 ns, which is substantially larger than the coherence time of 10 ns in this material. Therefore, the Hu and Das Sarma pulsing scheme is not practically useful for obtaining a perfect spin swap in GaAs quantum dots. However, this pulsing scheme could be useful if a material or system with a longer coherence time were to be used.

4.2. TRIPLE QUANTUM DOT

Using Hamiltonian 2.36, the eigenenergies of the triple quantum dot can be calculated. In figure 4.15, the eigenenergies are shown as a function of the detuning.

Figure 4.15: Eigenenergies of a triple dot system as a function of ϵ .

The eigenstates are a quadruplet state, Q , and two doublet states, Δ and Δ' . The eigenstates and corresponding eigenenergies can be found in Laird et al.

When $\epsilon_+ - \epsilon_- \gg \tau_L, \tau_R$, initializing the system in the state $|\uparrow\downarrow\uparrow\rangle$ and letting it evolve freely at detuning level $\epsilon = \epsilon_+$ will yield oscillations not only between the initial state $|\uparrow\downarrow\uparrow\rangle$ and $|\uparrow\uparrow\downarrow\rangle$, but also between $|\uparrow\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$. However, since $J_{23} \gg J_{12}$ at $\epsilon = \epsilon_+$, the latter oscillation will be very slow compared to the former. In figure 4.16, the time evolution is shown for a relatively small time scale. A few oscillations between $|\uparrow\downarrow\uparrow\rangle$ and $|\uparrow\uparrow\downarrow\rangle$ occur, but the oscillation between $|\uparrow\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$ is negligible.

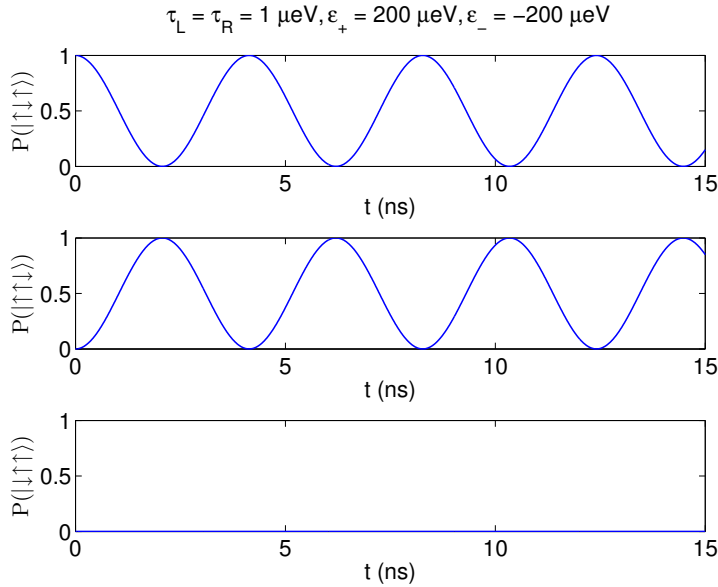


Figure 4.16: Time evolution of the three dot system initialized in the state $|\uparrow\downarrow\uparrow\rangle$ at $\epsilon = \epsilon_+$. The probabilities of measuring $|\uparrow\downarrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$ are plotted as a function of time. Here, a relatively small time scale is used, in which a few oscillations between $|\uparrow\downarrow\uparrow\rangle$ and $|\uparrow\uparrow\downarrow\rangle$ occur. The time scale is so small that the oscillation between $|\uparrow\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$ is negligible.

In figure 4.17, the time evolution is shown for a larger time scale, in which one oscillation between $|\uparrow\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$ occurs. The number of oscillations between $|\uparrow\downarrow\uparrow\rangle$ and $|\uparrow\uparrow\downarrow\rangle$ is so large that they are indistinguishable.

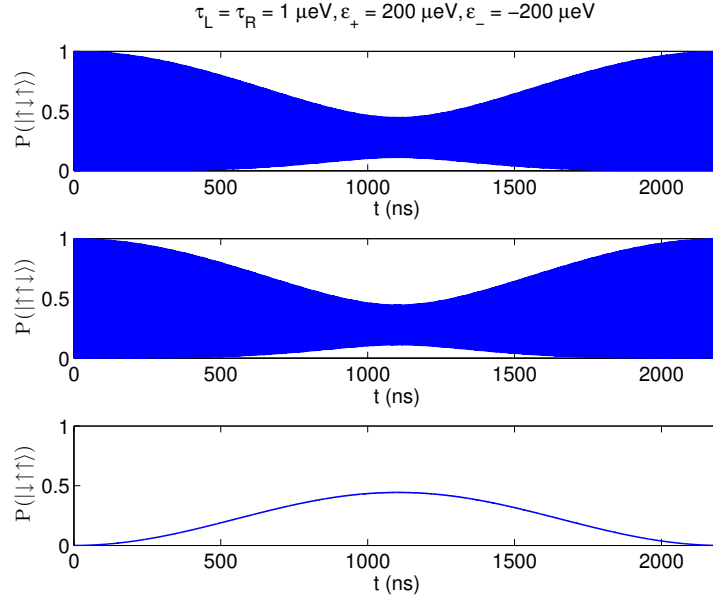


Figure 4.17: Time evolution of the three dot system initialized in the state $|\uparrow\uparrow\uparrow\rangle$ at $\epsilon = \epsilon_+$. The probabilities of measuring $|\uparrow\uparrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$ is plotted as a function of time. Here, a relatively large time scale is used, in which one oscillation between $|\uparrow\uparrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$ occurs. The time scale is so large that the different oscillations between $|\uparrow\uparrow\uparrow\rangle$ and $|\uparrow\uparrow\downarrow\rangle$ are indistinguishable. In this plot, coherence times are assumed to be very long.

If the initial state is $|\uparrow\uparrow\downarrow\rangle$ and the desired final state is $|\downarrow\uparrow\uparrow\rangle$, first the two rightmost spins must be swapped, followed by the two leftmost spins. This can be achieved by first pulsing to $\epsilon = \epsilon_+$ in order to obtain a high J_{12} and then pulsing to $\epsilon = \epsilon_-$ in order to obtain a high J_{23} . This is shown in figure 4.18. Both pulses should yield a π rotation. Therefore, the duration of the first pulse and the second pulse must be $\frac{\pi\hbar}{\hbar^2 J_{23}}$ and $\frac{\pi\hbar}{\hbar^2 J_{12}}$ respectively.

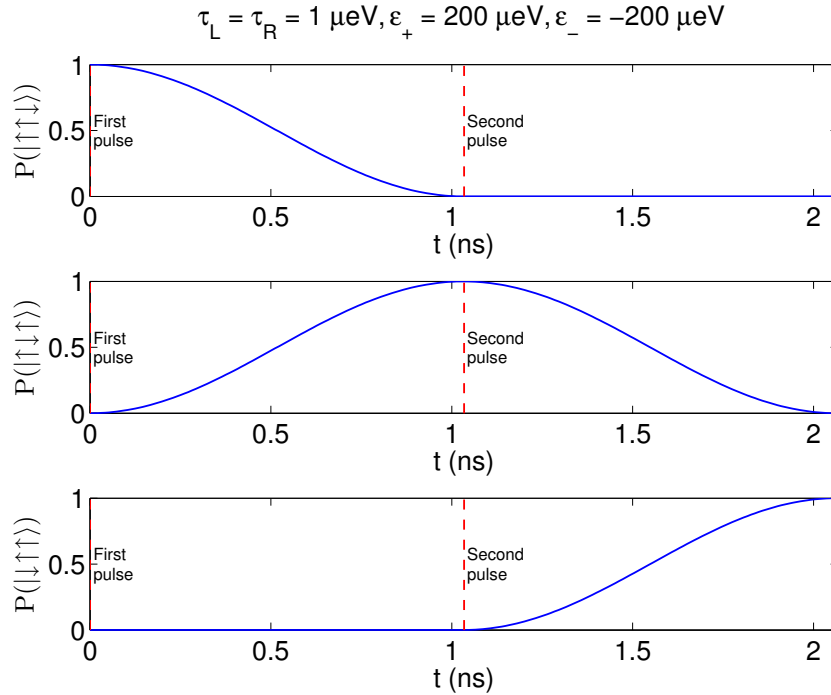


Figure 4.18: The initial state $|\uparrow\uparrow\downarrow\rangle$ will evolve to $|\downarrow\uparrow\uparrow\rangle$ when a π pulse is applied at $\epsilon = \epsilon_+$ first and a π pulse is applied at $\epsilon = \epsilon_-$ next. The probabilities of measuring $|\uparrow\uparrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$ are plotted as a function of time during this pulsing scheme.

In a triple dot, Zeeman inhomogeneities can be present as well. $\delta_{LM} = (\gamma_L - \gamma_M)\hbar B_0$ and $\delta_{MR} = (\gamma_M - \gamma_R)\hbar B_0$ are the Zeeman inhomogeneities between the two leftmost dots and the two rightmost dots, respec-

tively. Here, a pulsing scheme is given that swaps the leftmost and the rightmost spins when $g_L \neq g_R$ and $g_M \neq g_R$. Considering the initial state $|\uparrow\uparrow\downarrow\rangle$, the desired final state is $|\downarrow\uparrow\uparrow\rangle$. First, the two rightmost spins must be swapped. As long as $J_{12} \approx 0$, the swapping of the two leftmost spins will be negligible, so during the first three pulses the condition $J_{12} \approx 0$ must hold. First, a π pulse with magnitude $\tilde{J}_{23}$ is applied, followed by a π pulse with $J_{23} \approx 0$, followed by a π pulse with magnitude $J_{23} = \frac{\delta_{MR}^2}{\hbar^4 J_{23}}$. Then, the state will have evolved to $|\uparrow\uparrow\uparrow\rangle$. During the final three pulses, $J_{23} \approx 0$ must hold, so no swapping of the rightmost spins occurs. The final three pulses are a π pulse with magnitude $\tilde{J}_{23} \gg 0$, a π pulse with $J_{12} \approx 0$ and a π pulse with $J_{12} = \frac{\delta_{LM}^2}{\hbar^4 J_{12}}$, yielding the final state $|\downarrow\uparrow\uparrow\rangle$. All steps of this pulsing scheme can be found in table 4.4.

Pulse number	J_{12}	J_{23}
1	0	$\tilde{J}_{23} \gg 0$
2	0	0
3	0	$\frac{\delta_{MR}^2}{\hbar^4 J_{23}} \gg 0$
4	$\tilde{J}_{12} \gg 0$	0
5	0	0
6	$\frac{\delta_{LM}^2}{\hbar^4 J_{12}} \gg 0$	0

Table 4.4: Pulsing scheme that lets the initial state $|\uparrow\uparrow\downarrow\rangle$ evolve to $|\downarrow\uparrow\uparrow\rangle$ when Zeeman inhomogeneities are present.

The time evolution of the initial state $|\uparrow\uparrow\downarrow\rangle$ during the pulsing scheme described above is shown in figure 4.19.

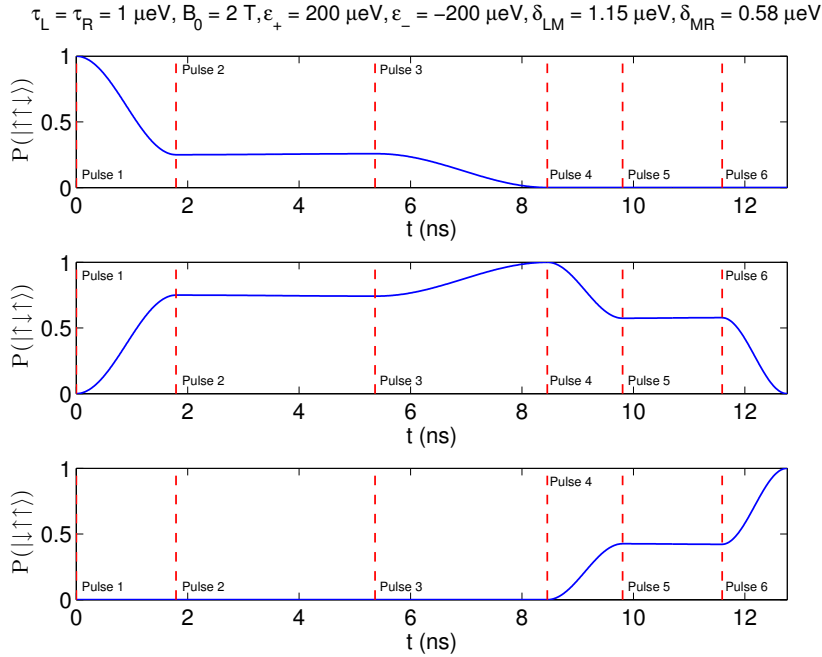


Figure 4.19: The initial state $|\uparrow\uparrow\downarrow\rangle$ will evolve to $|\downarrow\uparrow\uparrow\rangle$ when the six π pulses described in table 4.4 are applied. The probabilities of measuring $|\uparrow\uparrow\downarrow\rangle$, $|\uparrow\uparrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$ are plotted as a function of time during this pulsing scheme.

4.3. QUADRUPLE QUANTUM DOT

In a quadruple quantum dot, the spin swap $|\uparrow\uparrow\uparrow\downarrow\rangle \rightarrow |\downarrow\uparrow\uparrow\uparrow\rangle$ can be obtained by three π pulses in sequence: $J_{34} \gg J_{12}, J_{23}$ first, $J_{23} \gg J_{12}, J_{34}$ next and finally $J_{12} \gg J_{23}, J_{34}$. In figure 4.20, the probabilities of measuring $|\uparrow\uparrow\uparrow\downarrow\rangle$, $|\uparrow\uparrow\uparrow\uparrow\rangle$, $|\uparrow\downarrow\uparrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\uparrow\rangle$ is plotted as a function of time during this pulsing scheme.

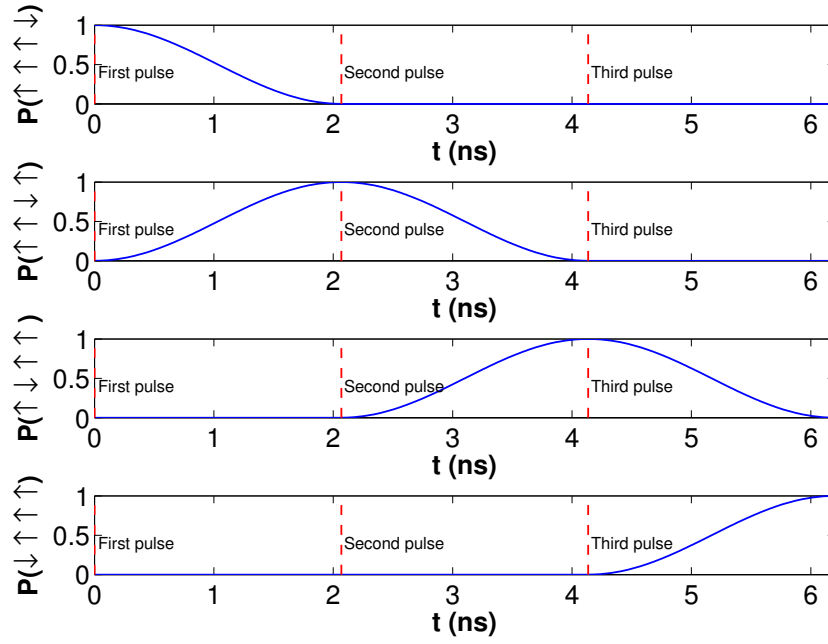


Figure 4.20: Four electrons initialized in the state $|\uparrow\uparrow\uparrow\downarrow\rangle$ can evolve to the state $|\downarrow\uparrow\uparrow\uparrow\rangle$ by applying three π pulses in sequence: $J_{34} \gg J_{12}, J_{23}$ first, $J_{23} \gg J_{12}, J_{34}$ next and finally $J_{12} \gg J_{23}, J_{34}$. Probabilities of measuring $|\uparrow\uparrow\uparrow\downarrow\rangle$, $|\uparrow\uparrow\downarrow\uparrow\rangle$, $|\uparrow\downarrow\uparrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\uparrow\rangle$ have been plotted as a function of time during this pulsing scheme. During the first pulse, $\hbar^2 J_{12} = 1 \mu\text{eV}$ and $\hbar^2 J_{23} = \hbar^2 J_{34} = 0$. During the second pulse, $\hbar^2 J_{23} = 1 \mu\text{eV}$ and $\hbar^2 J_{12} = \hbar^2 J_{34} = 0$. During the third pulse, $\hbar^2 J_{34} = 1 \mu\text{eV}$ and $\hbar^2 J_{12} = \hbar^2 J_{23} = 0$.

5

RESULTS: EXCHANGE COUPLING

Upon further inspection, Taylor et al.'s approach of getting an expression for the exchange coupling J yields interesting result. As seen before, neglecting $|(2,0)S\rangle$, the Hamiltonian describing the charge transition between $|(1,1)S\rangle$ and $|(0,2)S\rangle$ in a double dot with respect to the basis

$$|(0,2)S\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |(1,1)S\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (5.1)$$

is

$$H_{11-02} = \begin{pmatrix} -\epsilon & \tau \\ \tau & 0 \end{pmatrix}. \quad (5.2)$$

The eigenvalues are

$$\lambda = -\frac{\epsilon}{2} \pm \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}. \quad (5.3)$$

The ground state has the lowest eigenenergy, $\lambda_{gs} = -\frac{\epsilon}{2} - \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}$. The corresponding eigenvector can be found by solving $H_{11-02} - \lambda_{gs}I = 0$, yielding

$$|(1,1)S\rangle + \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}} |(0,2)S\rangle. \quad (5.4)$$

This eigenvector can be normalized using the identities $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$ and $\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$ with $x = \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}}$. Therefore, the ground state is the new singlet state as described by Taylor et al.:

$$|\tilde{S}\rangle = \cos\theta |(1,1)S\rangle + \sin\theta |(0,2)S\rangle \quad (5.5)$$

with $\theta = \arctan \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}}$, as Taylor et al. calculated. As seen in equation (2.33), $\hbar^2 J_{11-02} = \frac{\epsilon}{2} + \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2} = -\lambda_{gs}$. Using the equality $H|\tilde{S}\rangle = \lambda_{gs}|\tilde{S}\rangle$, the following set of equations is obtained:

$$\begin{pmatrix} -\epsilon \sin\theta + \tau \cos\theta \\ \tau \sin\theta \end{pmatrix} = -\hbar^2 J_{11-02} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \quad (5.6)$$

The bottom equation yields

$$\hbar^2 J_{11-02} = -\tau \frac{\sin\theta}{\cos\theta} = -\tau \frac{\langle(0,2)S|\tilde{S}\rangle}{\langle(1,1)S|\tilde{S}\rangle}. \quad (5.7)$$

So J_{11-02} is linearly dependent on the ratio of the magnitude of the components of $|(0,2)S\rangle$ and $|(1,1)S\rangle$ in $|\tilde{S}\rangle$.

The same can be applied to the system in which only $|(1,1)S\rangle$ and $|2,0)S\rangle$ are considered. With E_c the charging energy of the left quantum dot and ϵ the energy difference between $|(0,2)S\rangle$ and $|(1,1)S\rangle$, as before, $|(1,1)S\rangle$ and $|(2,0)S\rangle$ line up at $\epsilon = -E_c$. In this case,

$$\hbar^2 J_{11-20} = \frac{-\epsilon - E_c}{2} + \frac{1}{2} \sqrt{(-\epsilon - E_c)^2 + 4\tau^2} \quad (5.8)$$

and in terms of the ratio of the different components the expression is

$$\hbar^2 J_{11-20} = -\tau \frac{\langle (2,0)S | \tilde{S} \rangle}{\langle (1,1)S | \tilde{S} \rangle}. \quad (5.9)$$

Henceforth, J_{11-02} and J_{11-20} in terms of component ratios will be denoted as J_{11-02}^* and J_{11-20}^* respectively, while J_{11-02} and J_{11-20} will still be defined as in equation (2.33) and (5.8). When all three states are taken into consideration, the charge transition Hamiltonian with respect to the basis

$$|(2,0)S\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |(1,1)S\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |(0,2)S\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (5.10)$$

is

$$H = \begin{pmatrix} \epsilon + E_c & \tau & 0 \\ \tau & 0 & \tau \\ 0 & \tau & -\epsilon \end{pmatrix}. \quad (5.11)$$

Again $-\hbar^2 J$ is equal to the lowest eigenvalue of this Hamiltonian. However, the expressions for the eigenvalues of this Hamiltonian are very extensive. But, when $E_c \gg \tau$, J can be approximated by the sum of J_{11-02} and J_{11-20} :

$$\hbar^2 J \approx \hbar^2 (J_{11-02} + J_{11-20}) = -\frac{E_c}{2} + \frac{1}{2} \sqrt{\epsilon^2 + 4\tau^2} + \frac{1}{2} \sqrt{(-\epsilon - E_c)^2 + 4\tau^2}. \quad (5.12)$$

On the right side of figure 5.1, $\hbar^2 J$, $\hbar^2 (J_{11-02} + J_{11-20})$ have been plotted as a function of ϵ . It is clear that $\hbar^2 J$ can be approximated by $\hbar^2 (J_{11-02} + J_{11-20})$. On the left side of figure 5.1, $\hbar^2 J$, $\hbar^2 J_{11-02}$ and $\hbar^2 J_{11-20}$ have been plotted. This plot shows that $\hbar^2 J$ approaches $\hbar^2 J_{11-02}$ asymptotically for large ϵ , and $\hbar^2 J$ approaches $\hbar^2 J_{11-20}$ asymptotically for large $-\epsilon$, due to $\langle (2,0)S | \tilde{S} \rangle$ being negligibly small around the charge transition of $|(1,1)S\rangle$ and $|(0,2)S\rangle$ and $\langle (0,2)S | \tilde{S} \rangle$ is negligibly small around the charge transition of $|(1,1)S\rangle$ and $|(2,0)S\rangle$.

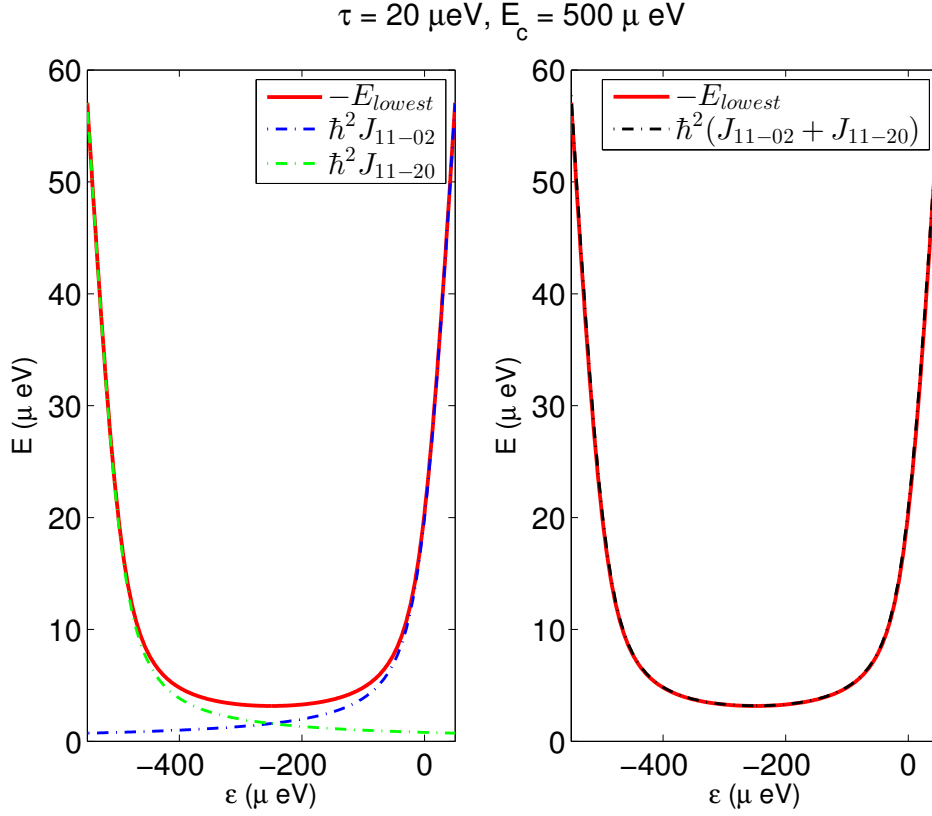


Figure 5.1: In red, the lowest eigenvalue of the charge state Hamiltonian (5.11) is plotted as a function of the detuning ϵ with $E_c = 25\tau$. The relation between the exchange coupling J and the lowest eigenvalue E_{lowest} is given by $\hbar^2 J = -E_{lowest}$. In the left figure, $\hbar^2 J_{11-02}$ and $\hbar^2 J_{11-20}$ have been plotted as dashed lines, showing that $\hbar^2 J$ approaches $\hbar^2 J_{11-02}$ asymptotically for large ϵ , and $\hbar^2 J$ approaches $\hbar^2 J_{11-20}$ asymptotically for large $-\epsilon$. In the right figure, $\hbar^2 J_{11-02} + \hbar^2 J_{11-20}$ is plotted as a dashed black line. The two lines are indistinguishable, showing that $\hbar^2 J$ can be approximated by $\hbar^2 (J_{11-02} + J_{11-20})$.

However, when the condition $E_c \gg \tau$ does not hold, equation (5.12) does not hold either. The right plot of figure 5.2, clearly shows that $\hbar^2 J$ and $\hbar^2 J_{11-02} + \hbar^2 J_{11-20}$ are not equal. For $\epsilon \gg 0$, J_{11-02} can still be used to approximate $\hbar^2 J$, and for $\epsilon \ll -E_c$, the approximation $\hbar^2 J \approx \hbar^2 J_{11-20}$ is valid.

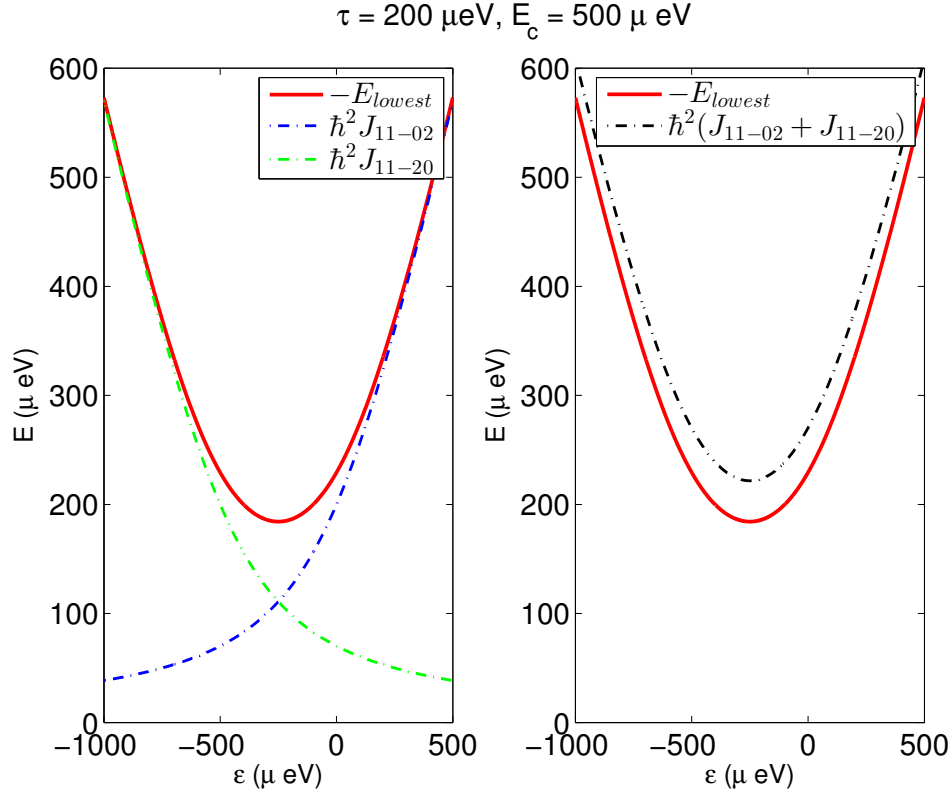


Figure 5.2: In red, the lowest eigenvalue of the charge state Hamiltonian (5.11) is plotted as a function of the detuning ϵ with $E_c = 2.5\tau$. The relation between the exchange coupling J and the lowest eigenvalue E_{lowest} is given by $\hbar^2 J = -E_{\text{lowest}}$. In the left figure, $\hbar^2 J_{11-02}$ and $\hbar^2 J_{11-20}$ have been plotted as dashed lines, showing that $\hbar^2 J$ approaches $\hbar^2 J_{11-02}$ asymptotically for large ϵ , and $\hbar^2 J$ approaches $\hbar^2 J_{11-20}$ asymptotically for large $-\epsilon$. In the right figure, $\hbar^2 J_{11-02} + \hbar^2 J_{11-20}$ is plotted as a dashed black line. In this case, the two lines are clearly different, showing that $\hbar^2 J$ cannot be approximated by $\hbar^2(J_{11-02} + J_{11-20})$.

In the right plot of figure 5.3, $\hbar^2 J$ and $\hbar^2(J_{11-02}^* + J_{11-20}^*)$ have been plotted as a function of ϵ . In the left plot, $\hbar^2 J_{11-02}^*$ and $\hbar^2 J_{11-20}^*$ have been plotted separately. The graphs of $\hbar^2 J$ and $\hbar^2(J_{11-02}^* + J_{11-20}^*)$ are indistinguishable, showing that

$$\hbar^2 J = \hbar^2(J_{11-02}^* + J_{11-20}^*). \quad (5.13)$$

Equation (5.13) was valid for all tested values of E_c and τ ¹. In all cases, relative errors had order of magnitude 10^{-16} , which indicates that the errors are purely numerical.

¹Tested values were in the range of a few μeV up until $10000 \mu\text{eV}$.

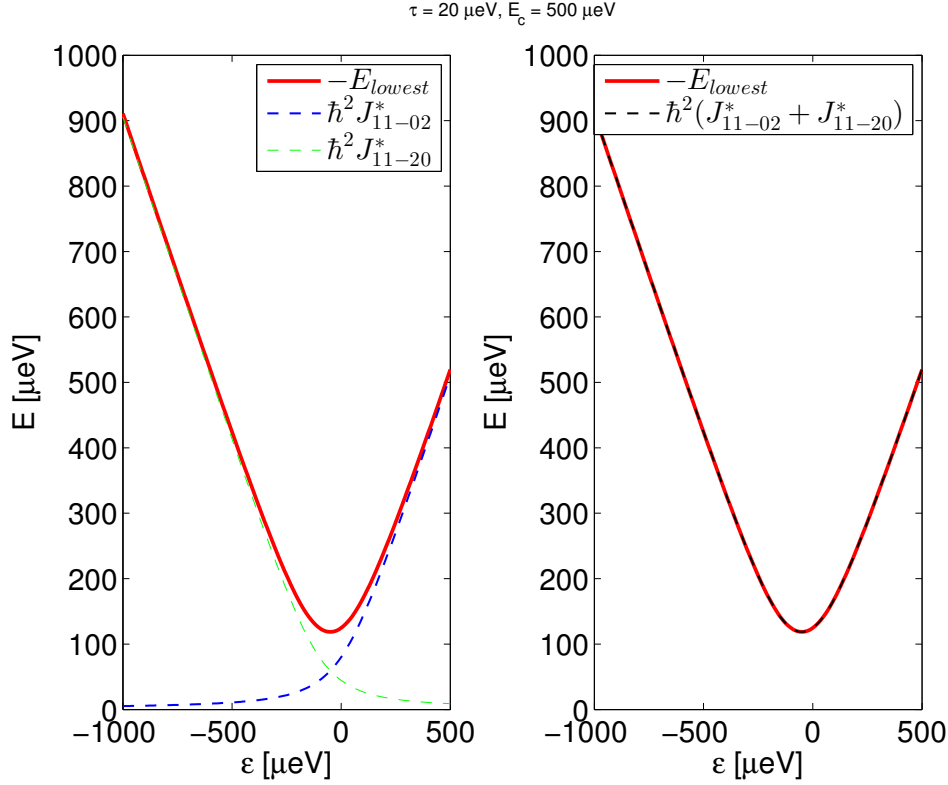


Figure 5.3: In red, the lowest eigenvalue of the charge state Hamiltonian (5.11) is plotted as a function of the detuning ϵ with $E_c = 2.5\tau$. In a double quantum dot, the relation between the exchange coupling J and the lowest eigenvalue E_{lowest} is given by $\hbar^2 J = -E_{lowest}$. In the left figure, $\hbar^2 J_{11-02}^*$ and $\hbar^2 J_{11-20}^*$ are plotted on top, showing that $\hbar^2 J$ approaches $\hbar^2 J_{11-02}^*$ asymptotically for large ϵ , and $\hbar^2 J$ approaches $\hbar^2 J_{11-20}^*$ asymptotically for large $-\epsilon$. In the right figure, $\hbar^2 J_{11-02}^* + \hbar^2 J_{11-20}^*$ is plotted on top. In this case, the two lines are indistinguishable. Relative errors are in the order of 10^{-16} , which indicates that the errors are numerical. So $\hbar^2 J = \hbar^2 (J_{11-02}^* + J_{11-20}^*)$.

As discussed in chapter 2, Laird et al. found expressions for the exchange coupling between the two electrons on the left (in dot 1 and 2) and the two electrons on the right (in dot 2 and 3) in a triple quantum dot by assuming that there was no interaction between non-neighbouring electrons. The expressions are

$$J_{12}(\epsilon) = \frac{1}{\hbar^2} \left(\frac{\epsilon_- - \epsilon}{2} + \sqrt{(\epsilon_- - \epsilon)^2 + 4\tau_L^2} \right), \quad J_{23}(\epsilon) = \frac{1}{\hbar^2} \left(\frac{\epsilon - \epsilon_+}{2} + \sqrt{(\epsilon - \epsilon_+)^2 + 4\tau_R^2} \right), \quad (5.14)$$

in which ϵ_+ denotes the detuning value at which (1,0,2) and (1,1,1) line up, while ϵ_- denotes the detuning value at which (2,0,1) and (1,1,1) line up.

In this three dot system, the charge state Hamiltonian with respect to the basis

$$|(2,0,1)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |(1,1,1)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |(1,0,2)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (5.15)$$

is

$$H = \begin{pmatrix} \epsilon - \epsilon_- & \tau_L & 0 \\ \tau_L & 0 & \tau_R \\ 0 & \tau_R & \epsilon_+ - \epsilon \end{pmatrix}, \quad (5.16)$$

in which τ_L and τ_R denote the tunnel couplings between the two dots on the left and the two dots on the right, respectively. Similar to the double dot, when $\epsilon_+ - \epsilon_- \gg \tau_L, \tau_R$, the expressions obtained by Laird et al. can be used to approximate the exchange coupling in the triple dot:

$$\hbar^2 J \approx \hbar^2 (J_{12} + J_{23}). \quad (5.17)$$

In the right plot of figure 5.4, $\hbar^2 J$ and $\hbar^2(J_{12} + J_{23})$ have been plotted, showing that $\hbar^2 J$ can be approximated by $\hbar^2(J_{12} + J_{23})$ when $\epsilon_+ - \epsilon_- \gg \tau_L, \tau_R$.

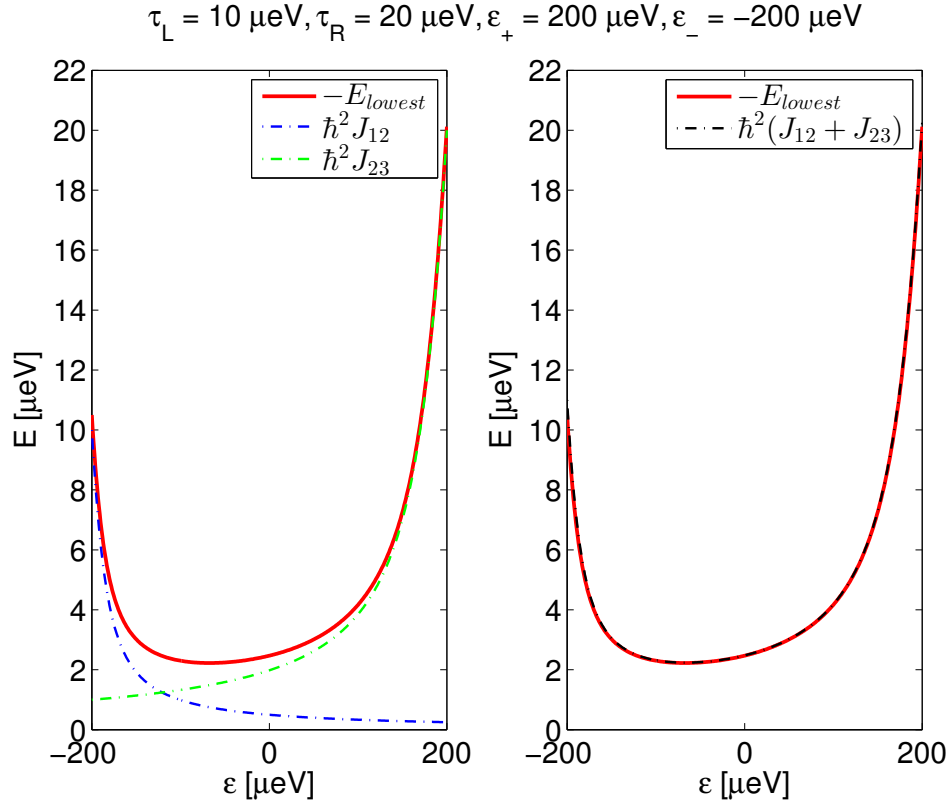


Figure 5.4: In red, the lowest eigenvalue of the charge state Hamiltonian (5.16) has been plotted as a function of the detuning ϵ with $\epsilon_+ - \epsilon_- = 40\tau_L = 20\tau_R$. The relation between the total exchange coupling J and the lowest eigenvalue E_{lowest} is given by $\hbar^2 J = -E_{lowest}$. In the left figure, $\hbar^2 J_{12}$ and $\hbar^2 J_{23}$ have been plotted as dashed lines. In the right figure, $\hbar^2(J_{12} + J_{23})$ is plotted as a black dashed line. In this case, the two lines are indistinguishable, showing that $\hbar^2 J$ can be described by $\hbar^2(J_{12} + J_{23})$.

However, when the condition $\epsilon_+ - \epsilon_- \gg \tau_L, \tau_R$ is not met, equation (5.17) is not valid anymore. Defining

$$J_{12}^* = -\tau_L \frac{\langle (2, 0, 1) | \psi \rangle}{\langle (1, 1, 1) | \psi \rangle}, \quad J_{23}^* = -\tau_R \frac{\langle (1, 0, 2) | \psi \rangle}{\langle (1, 1, 1) | \psi \rangle},$$

in which ψ as the eigenstate corresponding to the lowest eigenvalue of Hamiltonian (5.16), J can be written in terms of J_{12}^* and J_{23}^* :

$$\hbar^2 J = \hbar^2 (J_{12}^* + J_{23}^*). \quad (5.18)$$

In the right plot of figure 5.5, the left and right sides of equality in (5.18) are plotted and they are indistinguishable. The quality holds for all tested values².

²Tested values of $\tau_L, \tau_R, \epsilon_+, \epsilon_-$ were in the range of several μeV up until $10000 \mu\text{eV}$.

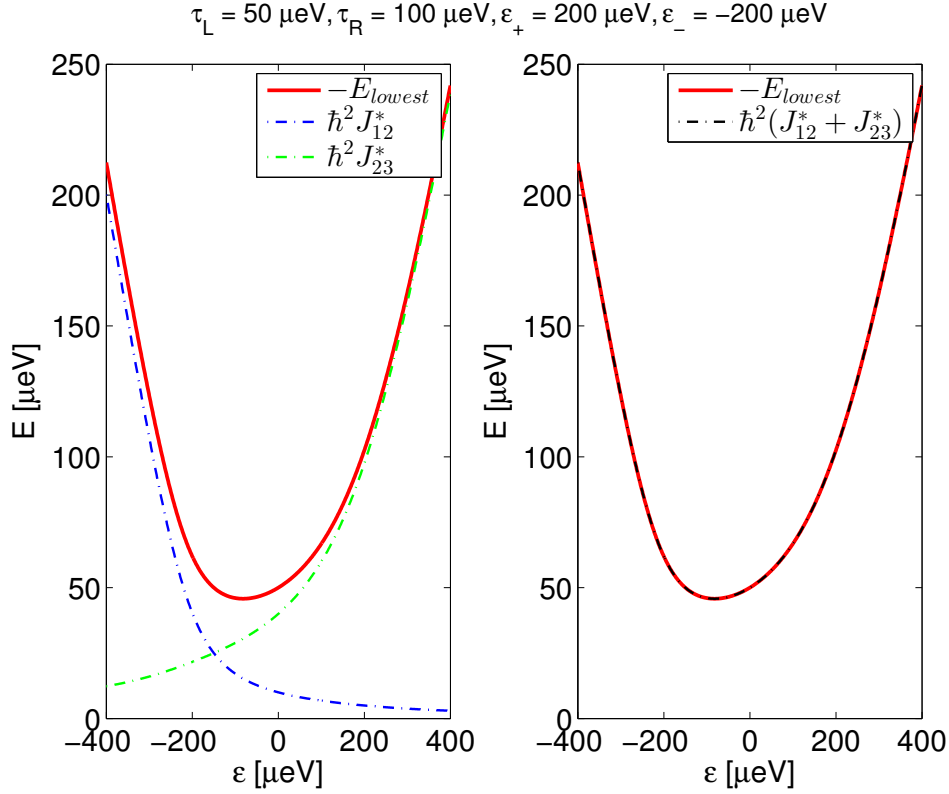


Figure 5.5: In red, the lowest eigenvalue of the charge state Hamiltonian (5.16) has been plotted as a function of the detuning ϵ with $\epsilon_+ - \epsilon_- = 4\tau_L = 2\tau_R$. The relation between the total exchange coupling J and the lowest eigenvalue E_{lowest} is given by $\hbar^2 J = -E_{lowest}$. In the left figure, $\hbar^2 J_{12}^*$ and $\hbar^2 J_{23}^*$ have been plotted as dashed lines. In the right figure, $\hbar^2 (J_{12}^* + J_{23}^*)$ has been plotted as a dashed black line. The two lines are indistinguishable. Relative errors are in the order of 10^{-15} , which indicates that the errors are purely numerical. So $\hbar^2 J = \hbar^2 (J_{12}^* + J_{23}^*)$.

The same procedure can be repeated for the quadruple dot. A charge state is denoted as (w, x, y, z) , in which w, x, y and z are the number of electrons in dot 1, 2, 3 and 4, respectively. Dot 1 is the leftmost dot and dot 4 is the rightmost dot. In this system, the basis

$$|(2, 0, 1, 1)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |(1, 1, 1, 1)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |(1, 0, 2, 1)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |(1, 1, 0, 2)\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (5.19)$$

with the Hamiltonian

$$H = \begin{pmatrix} -\epsilon_{12} + E_{c,12} & \tau_L & 0 & 0 \\ \tau_L & 0 & \tau_M & \tau_R \\ 0 & \tau_M & -\epsilon_{23} + E_{c,23} & 0 \\ 0 & \tau_R & 0 & -\epsilon_{34} + E_{c,34} \end{pmatrix}, \quad (5.20)$$

is considered. Possible couplings between non-neighbouring electrons are not taken into account. τ_L denotes the tunnel coupling between dot 1 and 2, τ_M the tunnel coupling between dot 2 and 3 and τ_R the tunnel coupling between dot 3 and 4, $E_{c,12}$ is the charging energy of the leftmost dot, $E_{c,23}$ is the charging energy of the third dot from the left, and $E_{c,34}$ is the charging energy of the rightmost dot. $\epsilon_{12} - E_{c,12}$ is the energy difference between $|(2, 0, 1, 1)\rangle$ and $|(1, 1, 1, 1)\rangle$, $\epsilon_{23} - E_{c,23}$ the energy difference between $|(1, 0, 2, 1)\rangle$, $|(1, 1, 1, 1)\rangle$ and $\epsilon_{34} - E_{c,34}$ the energy difference between $|(1, 1, 0, 2)\rangle$ and $|(1, 1, 1, 1)\rangle$. Denoting ϕ as the eigenstate corresponding to the lowest eigenvalue of Hamiltonian (5.20), J_{12}^* , J_{23}^* and J_{34}^* can be defined as

$$J_{12}^* = -\tau_L \frac{\langle (2, 0, 1, 1) | \phi \rangle}{\langle (1, 1, 1, 1) | \phi \rangle}, \quad J_{23}^* = -\tau_M \frac{\langle (1, 0, 2, 1) | \phi \rangle}{\langle (1, 1, 1, 1) | \phi \rangle}, \quad J_{34}^* = -\tau_R \frac{\langle (1, 1, 0, 2) | \phi \rangle}{\langle (1, 1, 1, 1) | \phi \rangle}. \quad (5.21)$$

As in the double and the triple dot, the equality

$$\hbar^2 J = \hbar^2 (J_{12}^* + J_{23}^* + J_{34}^*), \quad (5.22)$$

holds for all tested values of τ_L , τ_M , τ_R , $E_{c,12}$, $E_{c,23}$ and $E_{c,34}$ ³, because the relative errors had order of magnitude 10^{-15} , which indicates that the errors are purely numerical.

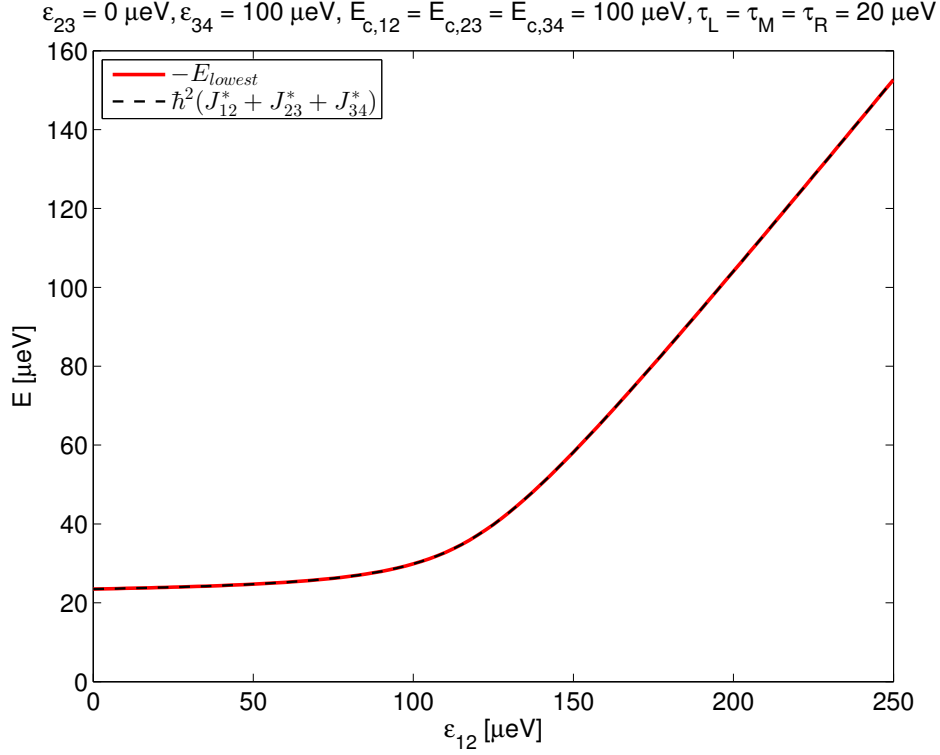


Figure 5.6: In red, the lowest eigenvalue of the charge state Hamiltonian (5.20) is plotted as a function of ϵ_{12} . The relation between the total exchange coupling J and the lowest eigenvalue E_{lowest} is given by $\hbar^2 J = -E_{lowest}$. In the same figure, $\hbar^2 (J_{12}^* + J_{23}^* + J_{34}^*)$ has been plotted as a dashed line. The lines are indistinguishable, the relative errors are in the order of magnitude 10^{-15} , which indicates that the errors are purely numerical. So $\hbar^2 J = \hbar^2 (J_{12}^* + J_{23}^* + J_{34}^*)$.

When the states $|(0, 2, 1, 1)\rangle$, $|(1, 2, 0, 1)\rangle$ and $|(1, 1, 2, 0)\rangle$ are taken into account as well, J_{12}^* , J_{23}^* and J_{34}^* have to be extended to

$$J_{12}^* = -\tau_L \left(\frac{\langle (2, 0, 1, 1) | \phi \rangle}{\langle (1, 1, 1, 1) | \phi \rangle} + \frac{\langle (0, 2, 1, 1) | \phi \rangle}{\langle (1, 1, 1, 1) | \phi \rangle} \right) \quad (5.23)$$

$$J_{23}^* = -\tau_M \left(\frac{\langle (1, 0, 2, 1) | \phi \rangle}{\langle (1, 1, 1, 1) | \phi \rangle} + \frac{\langle (1, 2, 0, 1) | \phi \rangle}{\langle (1, 1, 1, 1) | \phi \rangle} \right) \quad (5.24)$$

$$J_{34}^* = -\tau_R \left(\frac{\langle (1, 1, 0, 2) | \phi \rangle}{\langle (1, 1, 1, 1) | \phi \rangle} + \frac{\langle (1, 1, 2, 0) | \phi \rangle}{\langle (1, 1, 1, 1) | \phi \rangle} \right). \quad (5.25)$$

for equation (5.22) to be valid.

³Tested values ranged from a few μeV up until $10000 \mu\text{eV}$ for all parameters.

6

CONCLUSIONS

- The hyperfine interaction couples the up and down states in a double quantum dot containing one electron. The strength of the coupling depends linearly on $\tau\sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2}$
- The duration of the pulsing scheme described by Hu and Das Sarma that causes the initial state $|\uparrow\downarrow\rangle$ to evolve to $|\downarrow\uparrow\rangle$ in the presence of Zeeman inhomogeneity is bounded from below by $\frac{2\pi\hbar}{|\delta|}$.
- In a double quantum dot, the magnitude of the exchange coupling can be described by the equation $\hbar^2 J = -\frac{E_c}{2} + \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2} + \frac{1}{2}\sqrt{(-\epsilon - E_c)^2 + 4\tau^2}$ only when $E_c \gg \tau$.
- In a double quantum dot, the magnitude of the exchange coupling can always be described by $\hbar^2 J = -\tau \left(\frac{\langle(0,2)|S|\bar{S}\rangle}{\langle(1,1)|S|\bar{S}\rangle} + \frac{\langle(2,0)|S|\bar{S}\rangle}{\langle(1,1)|S|\bar{S}\rangle} \right)$.
- In a triple quantum dot, the magnitude of the total exchange coupling is $\hbar^2 J = -\tau_L \frac{\langle(2,0,1)|\psi\rangle}{\langle(1,1,1)|\psi\rangle} - \tau_R \frac{\langle(1,0,2)|\psi\rangle}{\langle(1,1,1)|\psi\rangle}$, in which ψ is the eigenstate corresponding to the lowest eigenvalue of the charge state Hamiltonian. The exchange coupling between electrons 1 and 2, and 2 and 3 can be described by $\hbar^2 J_{12}^* = -\tau_L \frac{\langle(2,0,1)|\psi\rangle}{\langle(1,1,1)|\psi\rangle}$ and $\hbar^2 J_{23}^* = -\tau_R \frac{\langle(1,0,2)|\psi\rangle}{\langle(1,1,1)|\psi\rangle}$, respectively.
- In a quadruple quantum dot, the magnitude of the total exchange coupling is $\hbar^2 J = \hbar^2 (J_{12}^* + J_{23}^* + J_{34}^*)$, in which $J_{12}^* = -\tau_L \left(\frac{\langle(2,0,1,1)|\phi\rangle}{\langle(1,1,1,1)|\phi\rangle} + \frac{\langle(0,2,1,1)|\phi\rangle}{\langle(1,1,1,1)|\phi\rangle} \right)$, $J_{23}^* = -\tau_M \left(\frac{\langle(1,0,2,1)|\phi\rangle}{\langle(1,1,1,1)|\phi\rangle} + \frac{\langle(1,0,2,1)|\phi\rangle}{\langle(1,1,1,1)|\phi\rangle} \right)$, $J_{34}^* = -\tau_R \left(\frac{\langle(1,1,0,2)|\phi\rangle}{\langle(1,1,1,1)|\phi\rangle} + \frac{\langle(1,1,2,0)|\phi\rangle}{\langle(1,1,1,1)|\phi\rangle} \right)$ denote the exchange couplings between electrons 1 and 2, 2 and 3, and 3 and 4, respectively.

7

OUTLOOK

7.1. FITTING THE EXCHANGE COUPLING IN A DOUBLE QUANTUM DOT

As found in chapter 5, the magnitude of the exchange coupling between the two electrons in a double quantum dot is a function of $\langle(2,0)S|\tilde{S}\rangle$, $\langle(0,2)S|\tilde{S}\rangle$ and $\langle(1,1)S|\tilde{S}\rangle$:

$$\hbar^2 J = \hbar^2 (J_{11-02} + J_{11-20}) = -\tau \left(\frac{\langle(0,2)S|\tilde{S}\rangle}{\langle(1,1)S|\tilde{S}\rangle} + \frac{\langle(2,0)S|\tilde{S}\rangle}{\langle(1,1)S|\tilde{S}\rangle} \right). \quad (7.1)$$

When $E_c \gg \tau$, $\hbar^2 J_{11-02}^*$ reduces to $\hbar^2 J_{11-02} = \frac{\epsilon}{2} + \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}$ and $\hbar^2 J_{11-20}^*$ reduces to $\hbar^2 J_{11-20} = \frac{-\epsilon - E_c}{2} + \frac{1}{2}\sqrt{(-\epsilon - E_c)^2 + 4\tau^2}$. However, when the condition $E_c \gg \tau$ is not met, J_{11-02} and J_{11-20} cannot be used to approximate J_{11-02}^* and J_{11-20}^* .

$\langle(2,0)S|\tilde{S}\rangle$, $\langle(0,2)S|\tilde{S}\rangle$ and $\langle(1,1)S|\tilde{S}\rangle$ have been plotted as a function of ϵ in figure 7.1. It can be noted that the plot of $|\langle(0,2)S|\tilde{S}\rangle|$ as a function of ϵ resembles a sigmoid function.

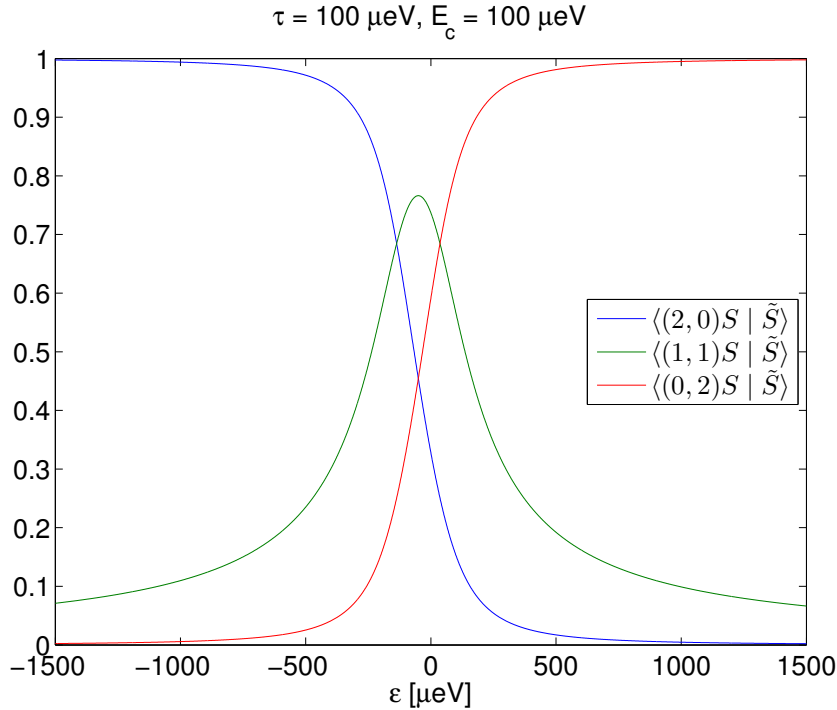


Figure 7.1: $\langle(2,0)S|\tilde{S}\rangle$, $\langle(0,2)S|\tilde{S}\rangle$ and $\langle(1,1)S|\tilde{S}\rangle$ have been plotted as a function of the detuning ϵ . It can be noted that $\langle(0,2)S|\tilde{S}\rangle$ resembles a sigmoid function.

$|\langle(0,2)S|\tilde{S}\rangle|$ has been fitted with the sigmoid function

$$f_{02}(\epsilon) = \frac{1}{2} + \frac{1}{2} \frac{\frac{\epsilon-b}{c}}{\sqrt{1 + \left(\frac{\epsilon-b}{c}\right)^2}}, \quad (7.2)$$

as shown in the upper left plot of figure 7.2. $|\langle(2,0)S|\tilde{S}\rangle|$ has been fitted with the function

$$f_{20}(\epsilon) = \frac{1}{2} - \frac{1}{2} \frac{\frac{\epsilon+b+E_c}{c}}{\sqrt{1 + \left(\frac{\epsilon+b+E_c}{c}\right)^2}}, \quad (7.3)$$

as shown in the upper right plot of figure 7.2. In the two lower plots of figure 7.2, the relative errors have been plotted as a function of ϵ .

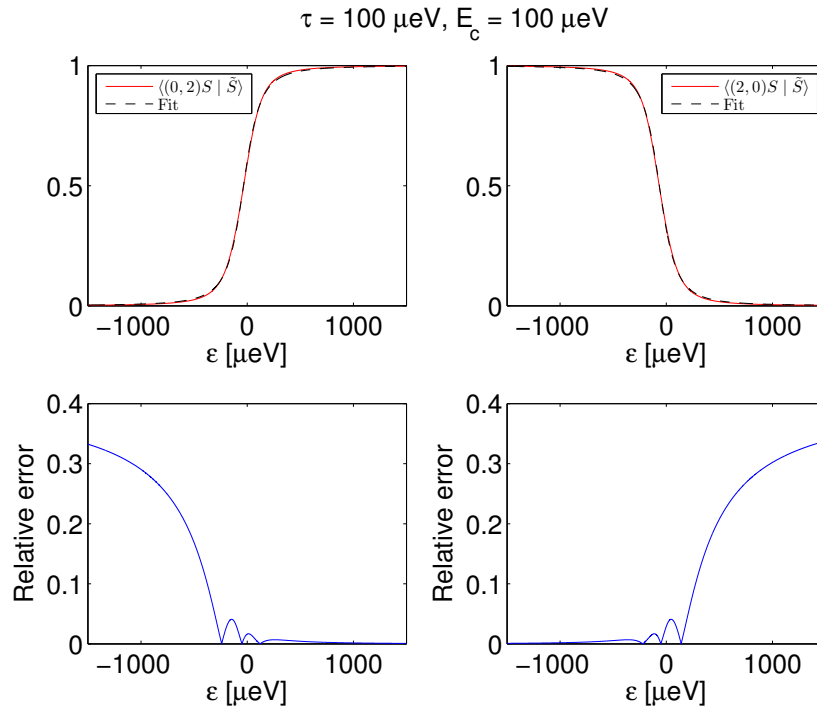


Figure 7.2: $|\langle(0,2)S|\tilde{S}\rangle|$ and $|\langle(2,0)S|\tilde{S}\rangle|$ have been plotted as a function of the detuning ϵ in the upper left and upper right plot, respectively. $|\langle(0,2)S|\tilde{S}\rangle|$ has been fitted with the sigmoid function $f_{02}(\epsilon)$, $|\langle(2,0)S|\tilde{S}\rangle|$ has been fitted with $f_{20}(\epsilon)$. The relative errors have been plotted in the two lower subfigures. The left plot shows that the relative errors in the left plot are below 0.05 when $\epsilon > -300 \mu\text{eV}$, but when ϵ decreases further, the fit $f_{02}(\epsilon)$ becomes less accurate. The right plot shows that $f_{20}(\epsilon)$ approximates $|\langle(2,0)S|\tilde{S}\rangle|$ with a relative error of less than 0.05 when $\epsilon < 200 \mu\text{eV}$, but when ϵ increases further, $f_{20}(\epsilon)$ is less accurate.

Because $|\tilde{S}\rangle$ is normalized, $|\langle(1,1)S|\tilde{S}\rangle| = \sqrt{1 - \langle(0,2)S|\tilde{S}\rangle^2 - \langle(2,0)S|\tilde{S}\rangle^2}$, so $\hbar^2 J$ can be fitted with $f(\epsilon) = \tau \left(\frac{f_{02}(\epsilon)}{\sqrt{1 - f_{20}(\epsilon)^2 - f_{02}(\epsilon)^2}} + \frac{f_{20}(\epsilon)}{\sqrt{1 - f_{20}(\epsilon)^2 - f_{02}(\epsilon)^2}} \right)$. Both $\hbar^2 J$ and $f(\epsilon)$ have been plotted in figure 7.3. $f(\epsilon)$ approximates $\hbar^2 J$ well for $-300 \mu\text{eV} < \epsilon < 200 \mu\text{eV}$, but the farther outside of this range, the more $f(\epsilon)$ deviates from $\hbar^2 J$, due to propagating errors in $f_{02}(\epsilon)$ and $f_{20}(\epsilon)$. However, outside of this interval, $\hbar^2 J$ approaches $\hbar^2(J_{11-02} + J_{11-20})$.

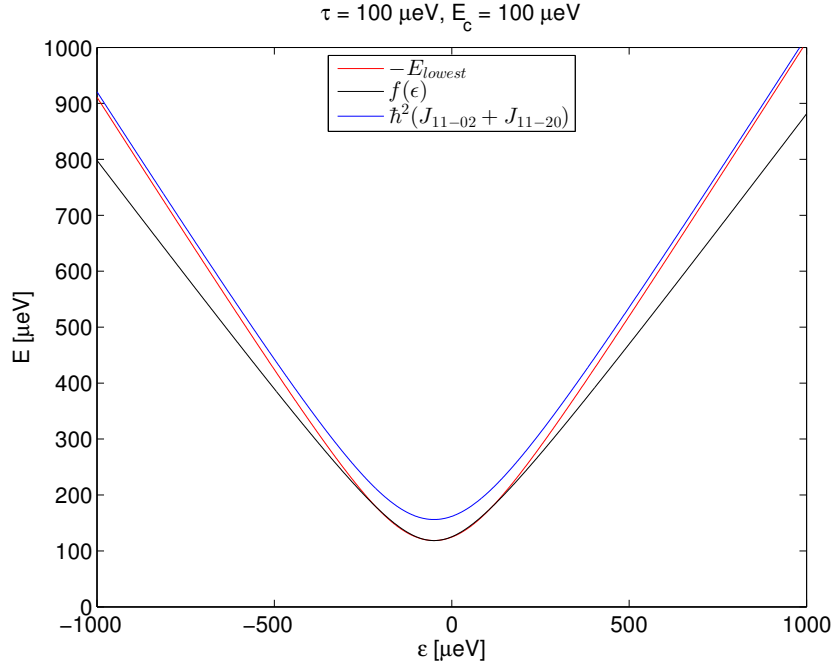


Figure 7.3: $\hbar^2 J = -E_{lowest}$ and $f(\epsilon)$ have been plotted as a function of ϵ . For $-300\mu\text{eV} < \epsilon < 200\mu\text{eV}$, $f(\epsilon)$, $\hbar^2 J$ and $f(\epsilon)$ are nearly equal. Relative errors are smaller than 0.02. However, the farther outside of this range, the more $f(\epsilon)$ deviates from $\hbar^2 J$, due to propagating errors in $f_{02}(\epsilon)$ and $f_{20}(\epsilon)$. Outside of the interval $-300\mu\text{eV} < \epsilon < 200\mu\text{eV}$, $\hbar^2 J$ approaches $\hbar^2(J_{11-02} + J_{11-20})$.

So, close to $\epsilon = 0$, $\hbar^2 J$ can be approximated by $f(\epsilon)$, and when $|\epsilon| \gg E_c$, $\hbar^2 J$ can be approximated by $\hbar^2(J_{11-02} + J_{11-20})$. However, in the intermediate regime, $\hbar^2 J$ cannot be approximated accurately by either of them.

The coefficients b and c in $f_{02}(\epsilon)$ and $f_{20}(\epsilon)$ have been determined for several different values of τ and E_c in the range of a few to 30000 μeV . The results can be found in appendix C. The values of b and c were fitted, yielding

$$b_{fit} = \left(0.02988 \left(\frac{\tau}{E_c} \right)^{-1.023} - 0.375 \right) E_c \quad (7.4)$$

$$c_{fit} = 0.2653 \tau^{-0.02465} E_c^{1.025} + 1.4142 \tau. \quad (7.5)$$

b_{fit} yields values of b within all 95% confidence intervals. However, c_{fit} yielded values within the 95% confidence intervals of only 58 of the 63 tested combinations of τ and E_c .

7.2. SUGGESTIONS FOR FURTHER RESEARCH

Further research could focus on finding an expression or a fit for $\hbar^2 J$ in a double quantum dot that is valid in all detuning regimes.

Secondly, in the derivation of the expressions that describe $\hbar^2 J$ in a triple and a quadruple quantum dot in chapter 5, the possibility of non-neighbouring electrons interacting with each other was not taken into account. It is unclear how valid this assumption is. It could be useful to find expressions describing $\hbar^2 J$ when coupling between non-neighbouring electrons is taken into account.

Furthermore, the effect of spin-orbit coupling and hyperfine interactions on the spin swapping procedure in a quantum dot array could be explored.

While the spin swap gate is useful in a spin-based quantum computer, the square-root-of-swap gate is even more important. $\sqrt{SWAP}$ is one of the building blocks of the cNOT operation, which is used for entangling operations. Further research could be dedicated to designing a pulsing scheme that carries out a $\sqrt{SWAP}$.

8

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A

STANDARD BASIS AND HAMILTONIAN OF A QUADRUPLE QUANTUM DOT

The standard basis is

$$\begin{aligned}
 |\uparrow\uparrow\uparrow\uparrow\rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\uparrow\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\uparrow\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\uparrow\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\downarrow\uparrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\downarrow\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\downarrow\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\downarrow\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
 |\downarrow\uparrow\uparrow\uparrow\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\uparrow\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\uparrow\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\uparrow\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\downarrow\uparrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\downarrow\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |\downarrow\downarrow\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, |\downarrow\downarrow\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
 \end{aligned}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_{I_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_{I_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{I_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{I_{12}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\gamma B_0 - h_{I_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h_{I_{12}} - h_{I_{34}} & h_{I_{34}} & -h_{I_{12}} - h_{I_{34}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_{I_{34}} & -h_{I_{12}} - h_{I_{34}} & h_{I_{34}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\gamma B_0 - \gamma I_{12} - h_{I_{23}} & -h_{I_{23}} & -2\gamma B_0 - h_{I_{23}} - h_{I_{34}} & h_{I_{34}} & -2\gamma B_0 - h_{I_{34}} & 0 \\ 0 & 0 & h_{I_{23}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{I_{23}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{I_{23}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -h_{I_{23}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h_{I_{23}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{I_{34}} & 0 & -4\gamma B_0 \end{pmatrix}$$

B

DERIVATION OF THE BOUNDS OF THE SWAPPING TIME

The total swapping time is given by

$$t_{\text{swap}} = \frac{\pi \hbar}{\sqrt{\hbar^4 J^2 + \delta^2}} + \frac{\pi \hbar}{|\delta|} + \frac{\pi \hbar}{\sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}}, \quad J, \delta > 0. \quad (\text{B.1})$$

We claim that

$$\frac{2\pi \hbar}{|\delta|} < t_{\text{swap}} \leq \frac{(\sqrt{2} + 1)\pi \hbar}{|\delta|}. \quad (\text{B.2})$$

Proof. Given a certain $\delta \neq 0$, the duration of the π rotation with the exchange coupling fixed to 0 is $\frac{\pi \hbar}{|\delta|}$, so only the left and right terms in equation (B.1) can influence the swapping time. Therefore, it suffices to show that

$$\frac{1}{|\delta|} < \frac{1}{\sqrt{\hbar^4 J^2 + \delta^2}} + \frac{1}{\sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}} \leq \frac{\sqrt{2}}{|\delta|} \quad (\text{B.3})$$

for all $J > 0$.

First, we will rewrite $\frac{1}{\sqrt{\hbar^4 J^2 + \delta^2}} + \frac{1}{\sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}}$:

$$\frac{1}{\sqrt{\hbar^4 J^2 + \delta^2}} + \frac{1}{\sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}} = \frac{\sqrt{\hbar^4 J^2 + \delta^2} + \sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}}{\sqrt{\hbar^4 J^2 + \delta^2} \sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}} = \frac{\sqrt{\hbar^4 J^2 + \delta^2} + \sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}}{|\delta| \sqrt{\hbar^4 J^2 + 2\delta^2 + \frac{\delta^4}{\hbar^4 J^2}}} = \frac{\sqrt{x} + \sqrt{y}}{|\delta| \sqrt{x+y}}, \quad (\text{B.4})$$

in which we have made the substitutions $x = \hbar^4 J^2 + \delta^2$ and $y = \frac{\delta^4}{\hbar^4 J^2} + \delta^2$. $x, y > 0$ because $J \neq 0$ and $\delta \neq 0$.

Now we will prove that $\frac{1}{\sqrt{\hbar^4 J^2 + \delta^2}} + \frac{1}{\sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}} > \frac{1}{|\delta|}$:

$$\frac{\sqrt{\hbar^4 J^2 + \delta^2} + \sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}}{|\delta| \sqrt{\hbar^4 J^2 + 2\delta^2 + \frac{\delta^4}{\hbar^4 J^2}}} = \frac{\sqrt{x} + \sqrt{y}}{|\delta| \sqrt{x+y}} \stackrel{(*)}{>} \frac{\sqrt{x+y}}{|\delta| \sqrt{x+y}} = \frac{1}{|\delta|},$$

in which (*) follows from the well known inequality $\sqrt{a} + \sqrt{b} > \sqrt{a+b}$ for all $a, b > 0$.

Finally, we will prove that $\frac{1}{\sqrt{\hbar^4 J^2 + \delta^2}} + \frac{1}{\sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}} \leq \frac{\sqrt{2}}{|\delta|}$:

$$\frac{\sqrt{\hbar^4 J^2 + \delta^2} + \sqrt{\frac{\delta^4}{\hbar^4 J^2} + \delta^2}}{|\delta| \sqrt{\hbar^4 J^2 + 2\delta^2 + \frac{\delta^4}{\hbar^4 J^2}}} = \frac{\sqrt{x} + \sqrt{y}}{|\delta| \sqrt{x+y}} \stackrel{(**)}{\leq} \frac{\sqrt{2}\sqrt{x+y}}{|\delta| \sqrt{x+y}} = \frac{\sqrt{2}}{|\delta|}. \quad (\text{B.5})$$

(**) follows from the inequality $\sqrt{a} + \sqrt{b} \leq \sqrt{2}\sqrt{a+b}$ for all $a, b > 0$. Below, a proof of this statement is given:

$$0 \leq (a - b)^2 \quad (\text{B.6})$$

$$\Leftrightarrow 0 \leq a^2 - 2ab + b^2 \quad (\text{B.7})$$

$$\Leftrightarrow 4ab \leq a^2 + b^2 + 2ab \quad (\text{B.8})$$

Taking the square root on both sides yields

$$2\sqrt{ab} \leq a + b \quad (\text{B.9})$$

$$\Leftrightarrow a + b + 2\sqrt{ab} \leq 2a + 2b \quad (\text{B.10})$$

$$\Leftrightarrow (\sqrt{a} + \sqrt{b})^2 \leq 2(a + b) \quad (\text{B.11})$$

Again, we take the square root on both sides of the inequality:

$$\sqrt{a} + \sqrt{b} \leq \sqrt{2}\sqrt{a+b}. \quad (\text{B.12})$$

We have shown that both sides of equation (B.3) hold, so equation (B.1) holds too. $\square$

Furthermore,

$$\lim_{J \rightarrow 0} t_{\text{swap}} = \lim_{J \rightarrow \infty} t_{\text{swap}} = \frac{2\pi\hbar}{|\delta|},$$

so the lower bound is approached when $J \rightarrow 0$ and when $J \rightarrow \infty$.

The upper bound is reached when $J = |\delta|$, because then $x = y$, and $\sqrt{x} + \sqrt{y} = 2\sqrt{x} = \sqrt{2}\sqrt{2x} = \sqrt{2}\sqrt{x+y}$.

C

FITTING THE EXCHANGE COUPLING IN A DOUBLE QUANTUM DOT: RESULTS

$\tau[\mu\text{eV}]$	$E_c[\mu\text{eV}]$	b	b_-	b_+	b_f	b_f within bounds?	c	c_-	c_+	c_f	c_f within bounds?
1600	800	-288.3	-288.8	-287.7	-288.2	Yes	2472	2470.5	2473.5	2471.85317	Yes
800	400	-144.1	-144.9	-143.3	-144.1185	Yes	1236	1234.5	1237.5	1235.90122	Yes
800	200	-73.55	-73.94	-73.16	-73.5529	Yes	1183	1181.5	1184.5	1182.73263	Yes
400	200	-72.06	-72.47	-71.66	-72.0593	Yes	617.9	617.15	618.65	617.93793	Yes
1600	400	-147.1	-147.3	-146.8	-147.1058	Yes	2366	2365.5	2367.5	2365.49020	No
600	100	-37.01	-37.2	-36.82	-37.0221	Yes	874.3	873.75	874.55	873.94470	Yes
400	100	-36.78	-36.97	-36.58	-36.7764	Yes	591.5	591.15	591.85	591.36009	Yes
1600	1600	-552.1	-553.2	-551	-552.1920	Yes	2688	2686.5	2690.5	2688.29752	Yes
400	0	0	$-5.907 \cdot 10^{-6}$	$1.172 \cdot 10^{-5}$	0.0000	Yes	565.7	565.65	565.75	565.68000	Yes
1600	0	0	-0.0006694	0.0006721	0.0000	Yes	2263	2262.5	2263.5	2262.72000	Yes
10000	10	-3.75	-3.756	-3.744	-3.7497	Yes	$1.414 \cdot 10^4$	$1.4135 \cdot 10^4$	$1.4145 \cdot 10^4$	14144.23943	Yes
20	10	-3.603	-3.624	-3.583	-3.6030	Yes	30.9	30.855	30.935	30.89416	Yes
6000	1000	-370.2	-370.8	-369.6	-370.2211	Yes	8743	8741.5	8744.5	8739.65197	No
1000	10	-3.747	-3.766	-3.729	-3.7473	Yes	1417	1416.5	1417.5	1416.57021	Yes
2000	10	-3.749	-3.765	-3.732	-3.7487	Yes	2831	2830.5	2831.5	2830.73006	Yes
100	100	-34.51	-34.72	-34.29	-34.5120	Yes	168	167.55	168.45	167.99280	Yes
1000	100	-37.21	-37.4	-37.02	-37.2166	Yes	1440	1439.5	1440.5	1439.30656	No
3200	3200	-1104	-1106	-1102	-1104.3840	Yes	5377	5372.5	5380.5	5376.80155	Yes
1600	1400	-488.5	-489.4	-487.5	-488.5092	Yes	2634	2631.5	2635.5	2633.85929	Yes
1600	1200	-423.3	-424.1	-422.5	-423.2853	Yes	2579	2577.5	2581.5	2579.61579	Yes
1600	1000	-356.6	-357.2	-355.9	-356.5258	Yes	2525	2523.5	2526.5	2525.59888	Yes
1600	600	-218.4	-218.8	-218.1	-218.4270	Yes	2419	2417.5	2419.5	2418.44586	Yes
1600	200	-74.28	-74.4	-74.15	-74.2879	Yes	2314	2313.5	2314.5	2313.22233	No
800	100	-37.14	-37.2	-37.08	-37.1439	Yes	1157	1156.5	1157.5	1156.60504	Yes
800	50	-18.66	-18.69	-18.63	-18.6624	Yes	1144	1143.5	1144.5	1143.76567	Yes
800	150	-55.43	-55.52	-55.34	-55.4414	Yes	1170	1169.5	1170.5	1169.61336	Yes
800	600	-211.6	-212	-211.2	-211.6427	Yes	1290	1288.5	1290.5	1289.76946	Yes
400	150	-54.61	-54.7	-54.51	-54.6067	Yes	604.6	604.45	604.85	604.59258	Yes
400	300	-105.8	-106	-105.6	-105.8213	Yes	644.8	644.45	645.15	644.86552	Yes
1000	500	-180.2	-180.5	-179.8	-180.1481	Yes	1545	1543.5	1545.5	1544.88673	Yes
1000	800	-281	-281.5	-280.4	-280.9747	Yes	1626	1624.5	1626.5	1625.77020	Yes
1000	300	-109.8	-110	-109.7	-109.8842	Yes	1492	1491.5	1492.5	1491.61704	Yes
10000	1000	-372.1	-372.7	-371.5	-372.1661	Yes	$1.440 \cdot 10^4$	$1.435 \cdot 10^4$	$1.445 \cdot 10^4$	14393.26804	Yes

Table C.1: Values of the fitting constants b and c in $f_{02}(e)$ and $f_{20}(e)$ (equations (7.2) and (7.3)). b_- and b_+ represent the lower and upper bounds of the 95% confidence intervals of b . c_- and c_+ denote the lower and upper bounds of the 95% confidence intervals of c . b_f and c_f are the values given by the functions b_{fit} and c_{fit} in equations (7.4) and (7.5).

$\tau[\mu\text{eV}]$	$E_c[\mu\text{eV}]$	b	b_-	b_+	b_f	b_f within bounds?	c	c_-	c_+	c_f	c_f within bounds?
10000	5000	-1802	-1805	-1799	-1801.4815	Yes	$1.545 \cdot 10^4$	$1.5435 \cdot 10^4$	$1.5455 \cdot 10^4$	15449.92097	Yes
10000	3000	-1099	-1101	-1097	-1098.8425	Yes	$1.492 \cdot 10^4$	$1.4915 \cdot 10^4$	$1.4925 \cdot 10^4$	14916.79451	Yes
10000	8000	-2810	-2815	-2804	-2809.7469	Yes	$1.626 \cdot 10^4$	$1.6245 \cdot 10^4$	$1.6265 \cdot 10^4$	16259.40769	Yes
6000	0	0	-0.00162	0.001621	0.0000	Yes	8485	8484.5	8485.5	8485.20000	Yes
6000	2000	-730.6	-731.8	-729.3	-730.5770	Yes	9004	9001.5	9007.5	9002.99943	Yes
6000	3000	-1081	-1083	-1079	-1080.8889	Yes	9269	9265.5	9273.5	9269.81229	Yes
6000	4000	-1421	-1424	-1419	-1421.0596	Yes	9538	9532.5	9543.5	9538.90081	Yes
1000	2000	-627.7	-629.2	-626.1	-628.5593	Yes	1955	1951.5	1958.5	1955.38157	Yes
1600	3200	-1004	-1007	-1002	-1005.6949	Yes	3128	3123.5	3132.5	3128.75296	Yes
1000	0	0	$-8.88 \cdot 10^{-7}$	$1.155 \cdot 10^{-6}$	0.0000	Yes	1414	1413.5	1414.5	1414.20000	Yes
1000	800	-281	-281.5	-280.4	-280.9747	Yes	1626	1624.5	1627.5	1625.77020	Yes
800	1600	-502.1	-503.4	-500.9	-502.8474	Yes	1564	1561.5	1566.5	1564.27144	Yes
800	800	-276	-276.6	-275.5	-276.0960	Yes	1344	1342.5	1345.5	1344.09714	Yes
400	800	-251.1	-251.7	-250.4	-251.4237	Yes	782	780.85	783.45	782.08321	Yes
400	400	-138	-138.3	-137.7	-138.0480	Yes	672	671.55	672.55	672.02277	Yes
6000	12000	-3765	-3775	-3756	-3771.3558	Yes	$1.173 \cdot 10^4$	$1.1715 \cdot 10^4$	$1.1755 \cdot 10^4$	11734.32634	Yes
6000	500	-186.2	-186.5	-185.9	-186.3242	Yes	8613	8612.5	8614.5	8610.24032	No
6000	5000	-1751	-1754	-1748	-1751.0210	Yes	9810	9803.5	9815.5	9809.69425	Yes
400	600	-197.7	-198.1	-197.2	-197.8560	Yes	727.1	726.35	727.95	726.81931	Yes
6000	9000	-2965	-2972	-2958	-2967.8406	Yes	$1.091 \cdot 10^4$	$1.0895 \cdot 10^4$	$1.0925 \cdot 10^4$	10904.58171	Yes
1000	1500	-494.2	-495.3	-493.1	-494.6401	Yes	1818	1815.5	1820.5	1817.17749	Yes
1600	2400	-790.7	-792.5	-789	-791.4242	Yes	2908	2904.5	2911.5	2907.59006	Yes
10000	15000	-4941	-4952	-4930	-4946.4010	Yes	$1.818 \cdot 10^4$	$1.8155 \cdot 10^4$	$1.825 \cdot 10^4$	18175.02385	Yes
800	1200	-395.4	-396.2	-394.5	-395.7121	Yes	1454	1452.5	1456.5	1453.71682	Yes
6000	6000	-2070	-2074	-2066	-2070.7200	Yes	$1.008 \cdot 10^4$	$1.0075 \cdot 10^4$	$1.0095 \cdot 10^4$	10081.85415	Yes
800	0	0	$-4.493 \cdot 10^{-7}$	$5.024 \cdot 10^{-7}$	0.0000	Yes	1131	1130.5	1131.5	1131.36000	Yes
800	300	-109.2	-109.4	-109	-109.2135	Yes	1209	1208.5	1210.5	1209.20404	Yes
10000	20000	-6275	-6291	-6259	-6285.5930	Yes	$1.956 \cdot 10^4$	$1.9525 \cdot 10^4$	$1.9585 \cdot 10^4$	19558.17883	Yes
10000	10000	-3450	-3457	-3444	-3451.2000	Yes	$1.68 \cdot 10^4$	$1.6785 \cdot 10^4$	$1.6815 \cdot 10^4$	16803.56606	Yes
10000	500	-186.8	-187.1	-186.5	-186.8027	Yes	$1.427 \cdot 10^4$	$1.4265 \cdot 10^4$	$1.4275 \cdot 10^4$	14265.47570	Yes

Table C.2: Values of the fitting constants b and c in $f_{02}(e)$ and $f_{30}(e)$ (equations (7.2) and (7.3)). b_- and b_+ represent the lower and upper bounds of the 95% confidence intervals of b . c_- and c_+ denote the lower and upper bounds of the 95% confidence intervals of c . b_f and c_f are the values given by the functions b_{fit} and c_{fit} in equations (7.4) and (7.5).

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