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***Abstract*— In this article, we study the conditions to be satisfied by a discrete-time linear system to ensure output controllability using sparse control inputs. A set of necessary and sufficient conditions can be directly obtained by extending the Kalman rank test for output controllability. However, the verification of these conditions is computationally heavy due to their combinatorial nature. Therefore, we derive noncombinatorial conditions for output sparse controllability that can be verified with polynomial time complexity. Our results also provide bounds on the minimum sparsity level required to ensure output controllability of the system. This additional insight is useful for designing sparse control input that drives the system to any desired output.**

***Index Terms*—Controllability, discrete-time system, general linear systems, Kalman rank test, linear dynamical systems, minimal input, optimal sparse control, output controllability, sparsity, time-varying support.**

I. Introduction

With the widespread acceptance and use of networked control systems, various new challenging theoretical issues have emerged in control theory. One such problem is the analysis of a network system with *sparse control inputs*. In particular, the controllability of systems under the sparsity constraints on the input is a relatively new concept [1], [2]. This article characterizes *output controllability* of a linear system with sparse control, i.e., the input applied at every time instant has a few nonzero entries compared to its dimension.

A. Practical Context and Examples

Constraining the inputs to be sparse is often necessary to select a small subset of the available sensors or actuators at each time instant, due to bandwidth, energy, or physical constraints. The sparse control inputs arise in several areas, such as multiagent systems [3]; optimal actuator placement [4], [5]; nodes selection [6], [7]; opinion dynamics [8]; environmental monitoring systems [1], [9]; and robotics [10], to name a few.

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In the following, we discuss two examples of systems, wherein the input is sparse and its support is time varying.

1) Networked Control Systems: Networked control systems, where the controller and plant communicate over a network, are often constrained by energy and bandwidth. In energy-constrained networks, energy-aware scheduling of actuators can help to extend the battery life of the nodes [11]. In this case, choosing the same support for a longer time drains the battery of the selected set of nodes. Therefore, using a different set of nodes at each time instant can result in a longer network lifetime. Further, in networked control systems, the control inputs are required to meet the bandwidth constraints imposed by the links over which they are exchanged [12], [13]. The sparse vectors admit compressed representations, and consequently, using sparse inputs helps to reduce the bandwidth requirements [14]–[16]. Also, restricting the control inputs to a fixed support may severely limit the set of admissible inputs to the system. On the other hand, using different supports provides much greater flexibility without significantly increasing the communication requirements. Thus, this approach combines the benefits of the other two methods.

2) Social Networks: A social network is often modeled using a graph, whose vertices represent the individuals in the network, and the edges represent the social connection between the individuals. A popular model for the evolution of network opinion is the DeGroot model that uses a linear dynamical system [17], [18]. Here, the system state is denoted by a vector containing the opinion of each individual in the network, and the state matrix is the adjacency matrix of the graph. Further, it is assumed that an external agent, such as an election candidate, a paid blogger, or a marketing agent, desires to drive the network opinion to a particular state by influencing only a few people on the network [19]–[21]. For example, consider an election candidate visiting the voters as part of a political campaign [8]. At each time instant, the candidate can only visit and influence a small group of people. Also, for better campaigning, the candidate does not visit the same set of people at each time. As a result, the support of the sparse control input (due to the candidate) also varies with time. Moreover, the goal of the candidate is not to influence all the voters, but it is enough if the candidate can influence the majority of the voters. Hence, the candidate designs the campaign strategy such that a subset of the network opinion can be driven to the desired value. This problem can be solved via the analysis of the output controllability of the network opinion.

B. Related Literature

Before we present our model and results, we provide a brief review of the existing literature on sparse control.

1) Structural Controllability of Networks: The characterization of controllability of networks by using a few nodes is a well-studied problem [22]–[30]. However, these papers focused on structural controllability or strongly structural controllability. On the contrary, we deal with output controllability by assuming the knowledge of the system matrices. Also, in such problems, sparsity refers to the number of driver nodes and not the number of nonzero entries in the control input.

2) Minimal Controllability Problem: The minimal controllability refers to the problem of selecting a small set of input variables so that the system is controllable using the selected set [31]–[33]. This model is similar to ours except that the support of the control inputs does not change with time. Here, the support refers to the indices of the nonzero entries of the input. However, the time-varying support model is more flexible and offers better control over the system while incurring similar cost (in terms of energy and bandwidth) as that of the time-invariant support model [1]. Therefore, we analyze the controllability of a linear system with inputs having supports that change with time.

3) Time-Varying Actuator Scheduling: The time-varying actuator scheduling problem deals with the control of linear systems using sparse inputs with a time-varying support model [6], [7], [34]–[40]. However, previous studies on this problem mainly focused on the design of sparse control inputs and the optimal actuator scheduling (choosing the support of the control inputs at every time instant) problems. Such problems were formulated as optimization problems with an ℓ_0-norm constraint on the input. The ℓ_0-norm-based problems are NP-hard and, thus, they were solved using ℓ_p-norm-based relaxations ($0 < p \leq 1$) or greedy algorithms. While these studies attempted to devise approximation algorithms to design the control inputs, our focus in this article is to gain new fundamental insights into the conditions for output controllability of a system using sparse inputs that follow the time-varying support model.

4) Sparse Controllability: Sparse controllability defined in [1] refers to the controllability of a linear system when the inputs are sparse, and their supports are time-varying. Joseph and Murthy [1] derived the necessary and sufficient conditions for sparse controllability that are noncombinatorial. In particular, they established that any controllable system is sparse controllable if and only if the sparsity level exceeds the nullity of the state matrix.[1] This work was also extended to controllability using non-negative sparse control inputs [2]. However, Joseph and Murthy [1] only dealt with state controllability. A similar algebraic characterization of output sparse controllability is not straightforward. This is because the results on sparse controllability in [1] are based on the Popov–Belevitch–Hautus (PBH) test [41] for controllability. However, an analogous PBH test for output sparse controllability is not available in the literature. Consequently, the proof technique used in [1] is not applicable for output sparse controllability.

[1]The precise statements of the results are presented in Section III-E.

In a nutshell, in this article, we derive the conditions for output sparse controllability of a linear system using the fundamental tools from linear algebra and matrix theory.

C. Our Contributions

The rest of this article is organized as follows. We present a discrete-time linear time-invariant dynamical system with sparse control inputs and a time-varying support model in Section II. We then show that the direct extension of the Kalman-type rank test for output sparse controllability leads to a verification procedure with exponential time complexity. In Section III, we show that any linear system that is output controllable is also output sparse controllable if and only if the sparsity level exceeds a certain bound which we present in Theorem 1. Hence, our result also provides the minimum sparsity level that ensures output controllability. In addition, we present several implications and insights from our result and compare it with the existing results on controllability and sparsity in Sections III-A–III-E. We discuss the design of sparse control inputs that drive the system output to a desired value in Section III-F. Finally, Section IV concludes this article.

Notations: In the sequel, boldface lowercase letters denote vectors, boldface uppercase letters denote matrices, and calligraphic letters denote sets. The ith column of the matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}_i$ while the submatrix of $\boldsymbol{A}$ formed by the columns indexed by the set $\mathcal{A}$ is denoted by $\boldsymbol{A}_{\mathcal{A}}$. The symbols $(\cdot)^{\mathsf{T}}$, $\mathsf{Rank}\{\cdot\}$, $(\cdot)^{-1}$, $(\cdot)^{\dagger}$, and $\mathcal{CS}\{\cdot\}$ denote the transpose, rank, inverse, pseudoinverse, and column space of a matrix, respectively. Also, the cardinality of a set is denoted using $|\cdot|$, and the ceiling function is denoted using $\lceil\cdot\rceil$. Further, the notations $\boldsymbol{I}$ and $\boldsymbol{0}$ represent the identity matrix and the zero matrix (or vector), respectively. Finally, we use $\mathbb{R}$ to denote the set of real numbers and $\mathbb{C}$ for the set of complex numbers.

II. Output Sparse Controllability

We consider the discrete-time linear dynamical system described by the triple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ in which the state and output evolve as follows:

$$\boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{x}_{k-1} + \boldsymbol{B}\boldsymbol{u}_k \text{ and } \boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k. \tag{1}$$

Here, $\boldsymbol{x}_k \in \mathbb{R}^N$ denotes the state vector, $\boldsymbol{u}_k \in \mathbb{R}^m$ denotes the control input vector, and $\boldsymbol{y}_k \in \mathbb{R}^n$ denotes the output vector at time k. Also, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are the state matrix, input matrix, and output matrix of the system, respectively. We assume that the control vectors are constrained to be s-sparse, i.e., at most s entries of $\boldsymbol{u}_k$ are nonzero, for all values of k. Under this sparsity constraint on the input, we revisit the classical output controllability problem. To be specific, our goal is to check if it is possible to drive the output to any final state $\boldsymbol{y}_f \in \mathbb{R}^n$, starting from any initial state $\boldsymbol{x}_0 \in \mathbb{R}^N$, using s-sparse control inputs within a finite time. This notion of controllability is referred to as *output s-sparse controllability*, henceforth.

Using (1), the output at any time $K > 0$ is

$$\boldsymbol{y}_K = \boldsymbol{C}\sum_{k=1}^{K}\boldsymbol{A}^{K-k}\boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{C}\boldsymbol{A}^K\boldsymbol{x}_0. \tag{2}$$

So the system is output s-sparse controllable if and only if there exists an integer $0 < K < \infty$ such that

$$\bigcup_{\substack{\{\mathcal{S}_k \subseteq \{1,2,\ldots,m\}:\\ |\mathcal{S}_k| \leq s, 1\leq k\leq K\}}} \mathcal{CS}\left\{\boldsymbol{C}\left[\boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} \quad \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2}\ldots\boldsymbol{B}_{\mathcal{S}_K}\right]\right\} = \mathbb{R}^n. \tag{3}$$

However, a vector space over an infinite field cannot be a finite union of proper subspaces [42]. Then, from (3), output s-sparse controllability holds if and only if there exist an integer $N < K < \infty$ and index sets $\{\mathcal{S}_i, |\mathcal{S}_i| \leq s\}_{i=1}^K$ such that

$$\mathcal{CS}\left\{\left[\boldsymbol{C}\boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} \quad \boldsymbol{C}\boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} \quad \ldots \quad \boldsymbol{C}\boldsymbol{B}_{\mathcal{S}_K}\right]\right\} = \mathbb{R}^n. \tag{4}$$

The direct evaluation of the condition (4) requires computation of the column spaces of $\binom{N}{s}^K$ matrices of size $n \times Ks$. Thus, the verification of the condition is computationally expensive. Motivated by this, in Section III, we present some noncombinatorial conditions that help to test output sparse controllability.

III. Necessary and Sufficient Conditions

The results of this section are based on the controllability matrix $\boldsymbol{W}$ and a new metric R_i as defined in the following:

$$\boldsymbol{W} \triangleq \left[\boldsymbol{A}^{N-1}\boldsymbol{B} \quad \boldsymbol{A}^{N-2}\boldsymbol{B} \quad \ldots \quad \boldsymbol{B}\right] \in \mathbb{R}^{N\times Nm} \tag{5}$$

$$R_i \triangleq \text{Rank}\left\{\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{W}\right\} - \text{Rank}\left\{\boldsymbol{C}\boldsymbol{A}^{i+1}\boldsymbol{W}\right\} \tag{6}$$

where $i \geq 0$ is an integer. The main result of this section is as follows.

Theorem 1: Consider the discrete-time linear dynamical system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ defined in (1) whose controllability matrix $\boldsymbol{W}$ is given by (5). Then, for any integer $0 < s \leq m$, a set of necessary conditions for output s-sparse controllability is

$$\text{Rank}\left\{\boldsymbol{C}\boldsymbol{W}\right\} = n \text{ and } \max_{0\leq i\leq N-1} \frac{\sum_{j=0}^{i} R_j}{i+1} \leq s, \tag{7}$$

and a set of sufficient conditions is

$$\text{Rank}\left\{\boldsymbol{C}\boldsymbol{W}\right\} = n \text{ and } \max_{0\leq i\leq N-1} R_i \leq s. \tag{8}$$

Here, R_i is as defined in (6).

Proof: See Appendix I. ■

In the following sections, we discuss the geometric intuition and insights from Theorem 1.

A. Geometric Intuition

The rank condition in (7) and (8) is straightforward from the Kalman rank test for (nonsparse) output controllability (see Theorem A). The bounds on the sparsity in Theorem 1 can be intuitively explained as follows.

1) Necessary Condition: From (4), the system is s-sparse output controllable if and only if the columns of the matrix given in (4) span $\mathbb{R}^n$. Thus, the last $(i+1)s$ columns of the matrix in (4) span the subspace orthogonal to the column space $\mathcal{U}_i$ of the remaining $Ks - (i+1)s$ columns, namely, the column space given by

$$\mathcal{U}_i = \mathcal{CS}\left\{\left[\boldsymbol{C}\boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} \quad \ldots \quad \boldsymbol{C}\boldsymbol{A}^{i+1}\boldsymbol{B}_{\mathcal{S}_{K-i-1}}\right]\right\} \tag{9}$$

$$\subseteq \mathcal{CS}\left\{\boldsymbol{C}\boldsymbol{A}^{i+1}\boldsymbol{W}\right\}. \tag{10}$$

Thus, (4) holds *only if* the last $(i+1)s$ columns of the matrix in (4) span the left null space of $\boldsymbol{C}\boldsymbol{A}^{i+1}\boldsymbol{W}$. So, we arrive at

$$(i+1)s \geq n - \text{Rank}\left\{\boldsymbol{C}\boldsymbol{A}^{i+1}\boldsymbol{W}\right\} = \sum_{j=0}^{i} R_i. \tag{11}$$

The above relation leads to the bound on sparsity given in the necessary condition (7). The bound is not sufficient because it only considers the possibility of spanning the smaller subset of the left null space of $\boldsymbol{C}\boldsymbol{A}^{i+1}\boldsymbol{W}$, which is a subset of the subspace orthogonal to $\mathcal{U}_i$ (see Example 1).

2) Sufficient Condition: We first observe that the column space of $\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{W}$ contains that of $\boldsymbol{C}\boldsymbol{A}^{i+1}\boldsymbol{W}$. Let $\mathcal{V}_i$ be the column space of $\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{W}$, which is orthogonal to that of $\boldsymbol{C}\boldsymbol{A}^{i+1}\boldsymbol{W}$. Consequently, when $\boldsymbol{x}_0 = \boldsymbol{0}$, $\mathcal{V}_i$ contains the subspace of output vectors that can be reached at the time instant i but not at time $i+1$. Also, the dimension of $\mathcal{V}_i$ is R_i. One case where the set of all possible output vectors forms $\mathbb{R}^n$ starting from $\boldsymbol{x}_0 = \boldsymbol{0}$ is when the column space of $\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{B}_{\mathcal{S}_{K-i}} \in \mathbb{R}^{n\times s}$ contains the subspace $\mathcal{V}_i$, for each value for i. This case leads to

$$R_i \leq \text{Rank}\left\{\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{B}_{\mathcal{S}_{K-i}}\right\} \leq s. \tag{12}$$

The above relation results in the bound on sparsity given in (8) of Theorem 1. However, the condition is not necessary because it considers only the possibility of spanning the larger set $\mathcal{V}_i$ (see Example 2).

Refer to Appendix I for a rigorous proof of the above intuitive explanation. We illustrate our idea using the following examples.

Example 1: Consider the system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ in (1) with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} 1 & 1\\ 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1 \end{bmatrix} \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}. \tag{13}$$

For this system, we have $\text{Rank}\{\boldsymbol{C}\boldsymbol{W}\} = 3 = n$

$$\max_{0\leq i\leq N-1} R_i = 2 \text{ and } \max_{0\leq i\leq N-1} \frac{\sum_{j=0}^{i} R_j}{i+1} = 1. \tag{14}$$

Therefore, when $s = 1$, the system satisfies the necessary condition, but it does not satisfy the sufficient condition. Using the brute force verification of output sparse controllability [given by (4)], we see that the system is not output $1-$sparse controllable. Thus, this example shows that the necessary conditions of Theorem 1 are not always sufficient for output sparse controllability.

Example 2: Consider the system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ in (1) with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^{\mathsf{T}}. \quad (15)$$

For this system, we have $\text{Rank}\{\boldsymbol{CW}\} = 2 = n$

$$\max_{0 \le i \le N-1} R_i = 2 \text{ and } \max_{0 \le i \le N-1} \frac{\sum_{j=0}^{i} R_j}{i+1} = 1. \quad (16)$$

Therefore, when $s = 1$, the system satisfies the necessary condition, but it does not satisfy the sufficient condition. However, the system defined by $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C})$ is output sparse controllable, where $\boldsymbol{B}_2 \in \mathbb{R}^4$ is the second column of $\boldsymbol{B}$. Thus, this example shows that the sufficient conditions of Theorem 1 are not always necessary.

B. Simultaneous Necessity and Sufficiency

Both necessary and sufficient conditions in Theorem 1 become identical under some mild assumptions on the system, which we present in the following corollary. If the two sets of conditions in Theorem 1 are not identical and the sparsity level of the system lies between the necessary bound and the sufficient bound given by Theorem 1, our sparse-output controllability test fails (as presented in Examples 1 and 2). Then, we use the brute force test given by (4) to check the output controllability using s-sparse control inputs.

Corollary 1: Consider the discrete-time linear dynamical system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ defined in (1), whose controllability matrix $\boldsymbol{W}$ is given by (5). The necessary conditions (7) are also sufficient for s-sparse output controllability if $\max_{0 \le i \le N-1} R_i = R_0$, where R_i is as defined in (6). Also, in this case, (7) and (8) reduce to

$$\text{Rank}\{\boldsymbol{CW}\} = n \text{ and } s \ge n - \text{Rank}\{\boldsymbol{CAW}\}. \quad (17)$$

Proof: When $\max_{0 \le i \le N-1} R_i = R_0$, the relation (8) reduces to

$$\text{Rank}\{\boldsymbol{CW}\} = n \text{ and } s \ge R_0 = n - \text{Rank}\{\boldsymbol{CAW}\}. \quad (18)$$

Since (7) also implies that $s \ge R_0$, and the necessary conditions are less stringent than the sufficiency conditions, we conclude that both (7) and (8) reduce to (17). ∎

We note that the assumption in Corollary 1 is satisfied by a large class of matrices, for which the output controllability test in Theorem 1 never fails for any value of s. For example, suppose that $\text{Rank}\{\boldsymbol{A}\} = \text{Rank}\{\boldsymbol{A}^2\}$. In this case, the row space and the column space of $\boldsymbol{A}^i$ are the same as those of $\boldsymbol{A}$, for $i \ge 1$. As a result, we obtain

$$\text{Rank}\{\boldsymbol{CAW}\} = \text{Rank}\{\boldsymbol{CA}^i\boldsymbol{W}\}. \quad (19)$$

Consequently, from (6), we get $R_i = 0 \le R_0$ for all values of $i \ge 1$. Hence, $\max_{0 \le i \le N-1} R_i = R_0$, and by Corollary 1, the necessary and sufficient conditions of Theorem 1 reduce to (17). Here, the condition $\text{Rank}\{\boldsymbol{A}\} = \text{Rank}\{\boldsymbol{A}^2\}$ implies that the algebraic and geometric multiplicities of the eigenvalue 0 of $\boldsymbol{A}$ are the same [43, ch. 3]. This condition is satisfied by the families of matrices, such as the diagonalizable matrices and the matrices with rank greater than or equal to $N - 1$. In such cases, using the inherent special structure of the matrix $\boldsymbol{A}$, we can verify if $\max_{0 \le i \le N-1} R_i = R_0$ holds without specifying $\boldsymbol{B}$ and $\boldsymbol{C}$.

The above observation is particularly useful to analyze the network opinion of social networks such as Facebook (see Section I-A2). Such a network is modeled using an undirected graph, and therefore, the state matrix $\boldsymbol{A}$ of the corresponding linear dynamical system is the symmetric adjacency matrix of the graph [8]. Since symmetric matrices are always diagonalizable, (17) gives the necessary and sufficient conditions in this case.

C. Computational Complexity

The computational complexity to verify all the conditions of Theorem 1 depends on the complexity to compute the rank of matrices $\boldsymbol{CA}^i\boldsymbol{W}$, for $i = 0, 1, \ldots, N$. So, unlike the verification of the combinatorial condition (4), the verification of the conditions in Theorem 1 possesses polynomial time complexity (in N and n), and the complexity is independent of s. Moreover, the complexity of the verification test can further be reduced by using a simpler condition which does not involve the computation of $\{R_i\}_{i=0}^{N-1}$ as presented in the following.

Corollary 2: Consider the discrete-time linear dynamical system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ in (1) whose controllability matrix $\boldsymbol{W}$ is given by (5). The system is output s-sparse controllable for any $s > 0$ if

$$\text{Rank}\{\boldsymbol{CW}\} = n \text{ and } s \ge N - \text{Rank}\{\boldsymbol{A}\}. \quad (20)$$

Proof: See Appendix III. ∎

Clearly, the relaxed bound on s given in (20) is easy to calculate. So, if the system satisfies the bound in Corollary 2, we can avoid the more computationally heavy conditions of Theorem 1. Also, from the proof of the result, we notice that (20) in Corollary 2 can also be replaced with a more stringent condition

$$s \ge \text{Rank}\{\boldsymbol{W}\} - \text{Rank}\{\boldsymbol{AW}\} \quad (21)$$

which follows from the proof of Corollary 2 (see (99) in Appendix III).

Further, Corollary 2 implies that if a linear system is reversible, i.e., $\boldsymbol{A}$ is nonsingular, then (output) controllability implies and is implied by (output) s-sparse controllability, for any $1 \le s \le m$. This property also holds for sparse controllability [1].

D. Additional Insights

Some interesting observations from Theorem 1 are as follows:

1) Bound on Necessary Sparsity: If $\text{Rank}\{\boldsymbol{CW}\} = n$, the system is s-sparse output controllable when $s = m$. Thus, $s = m$ satisfies the necessary condition (7) of Theorem 1, and we arrive at

$$m \ge \max_{0 \le i \le N-1} \frac{\sum_{j=0}^{i} R_j}{i+1}. \quad (22)$$

This condition holds for any output (nonsparse) controllable systems.

2) Time-Invariant System: We note that the sparse inputs having a time-invariant support are special cases of the sparse inputs without any additional constraints. Hence, (7) is necessary for output controllability using sparse inputs with time-invariant support model.

3) Noncanonical Basis: If a system is output controllable using control inputs which are s-sparse in the canonical basis, it is output controllable using inputs that admit s-sparse representations under any other basis $\Phi \in \mathbb{R}^{m\times m}$. This is because the change of basis is equivalent to replacing $\boldsymbol{B}$ with $\boldsymbol{B}\Phi$ which does not change the condition in (4). Sparse controllability also possesses a similar property [1].

E. Comparison With Existing Results

In this section, we compare Theorem 1 with the existing results on controllability and sparsity.

1) Output Controllability Without Constraints: The classical result for output controllability is as follows.

Theorem A ([44]): Consider the linear dynamical system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ defined in (1) whose controllability matrix $\boldsymbol{W}$ is given by (5). The system is output controllable if and only if $\text{Rank}\{\boldsymbol{CW}\} = n$.

From (70) and (74) in Appendix I, we see that when $\text{Rank}\{\boldsymbol{CW}\} = n$, we have

$$R_i = \text{Rank}\left\{\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right\} \tag{23}$$

$$\leq \text{Rank}\left\{\left[\boldsymbol{I} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\boldsymbol{CA}^i\boldsymbol{B}\right\} \leq \text{Rank}\left\{\boldsymbol{B}\right\} \tag{24}$$

$$\leq m \tag{25}$$

where $\tilde{\boldsymbol{C}}_{(i)}$ is defined in (39) using the Kalman decomposition given by (43)–(45). Therefore, if we remove the sparsity constraint, i.e., when $s = m$, the sparsity bound in (8) holds true, and as a result, Theorem 1 coincides with Theorem A.

2) Controllability With Sparse Inputs: The next result gives the necessary and sufficient conditions for controllability with sparse control inputs.

Theorem B ([1, Th. 1]): Consider the linear dynamical system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ defined in (1) whose controllability matrix $\boldsymbol{W}$ is given by (5). The system is controllable using s-sparse inputs if and only if the following conditions hold:

$$\text{Rank}\left\{\left[\lambda\boldsymbol{I} - \boldsymbol{A} \quad \boldsymbol{B}\right]\right\} = N \leq \text{Rank}\left\{\boldsymbol{A}\right\} + s \quad \forall\lambda \in \mathbb{C}. \tag{26}$$

The connections between Theorems 1 and B are as follows.

1) When $\boldsymbol{C} = \boldsymbol{I}$, the notion of output sparse controllability and sparse controllability are the same. If we substitute $\boldsymbol{C} = \boldsymbol{I}$ in Theorem 1, the rank conditions of (7) and (8) are equivalent to $\text{Rank}\{\boldsymbol{W}\} = N$. Also, using the arguments presented in the proof of Corollary 2 (see (99) in Appendix III), for all $0 \leq i \leq N$

$$R_i \leq N - \text{Rank}\left\{\boldsymbol{A}\right\} \tag{27}$$

$$= \text{Rank}\left\{\boldsymbol{W}\right\} - \text{Rank}\left\{\boldsymbol{AW}\right\} = R_0 \tag{28}$$

which follows when $\text{Rank}\{\boldsymbol{W}\} = N$. Hence, by Corollary 1, the system is output s-sparse controllable if and only if (17) holds which is equivalent to

$$\text{Rank}\left\{\boldsymbol{W}\right\} = N \text{ and } s \geq N - \text{Rank}\left\{\boldsymbol{A}\right\}. \tag{29}$$

However, the condition $\text{Rank}\{\boldsymbol{W}\} = N$ is equivalent to the rank condition in (26) due to the equivalence of the PBH test [41] and Kalman rank test for controllability [45]. In other words, when $\boldsymbol{C} = \boldsymbol{I}$, Theorem 1 reduces to Theorem B.

2) The proof of Theorem B given in [1] is based on the PBH test for controllability, whereas our proof of Theorem 1 is based on the fundamental results in linear algebra. Therefore, the proof of Theorem 1 provides an alternate method to establish Theorem B.

3) Comparing Corollary 2 and Theorem B, we conclude that when the sparsity $s \geq N - \text{Rank}\{\boldsymbol{A}\}$, the system is output s-sparse controllable if it is output controllable (i.e., $\boldsymbol{CW}$ is full row rank); and the system is s-sparse controllable if it is controllable (i.e., $\boldsymbol{W}$ is full row rank).

3) Necessary Conditions for Output Sparse Controllability: We next present a known set of necessary conditions for output s-sparse controllability.

Theorem C ([1, Corollary 1]): Consider the linear dynamical system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ defined in (1) whose controllability matrix $\boldsymbol{W}$ is given by (5). The system is output controllable using s-sparse vectors only if the following conditions hold:

$$\text{Rank}\left\{\boldsymbol{C}\left[\lambda\boldsymbol{I} - \boldsymbol{A} \quad \boldsymbol{B}\right]\right\} = n \quad \forall\lambda \in \mathbb{C} \tag{30}$$

$$\text{Rank}\left\{\boldsymbol{CA}\right\} \geq n - s. \tag{31}$$

Our necessary conditions in Theorem 1 are stronger than those in Theorem C. To verify this claim, suppose that (30) does not hold, i.e., there exist $\lambda \in \mathbb{C}$ and $\boldsymbol{z} \in \mathbb{R}^n$ such that $\boldsymbol{z}^\mathsf{T}\boldsymbol{CA} = \lambda\boldsymbol{z}^\mathsf{T}\boldsymbol{C}$ and $\boldsymbol{z}^\mathsf{T}\boldsymbol{CB} = \boldsymbol{0}$. In this case, we obtain $\boldsymbol{z}^\mathsf{T}\boldsymbol{CW} = \boldsymbol{0}$, which implies that the rank condition of (7) does not hold. Thus, (30) is necessary for (7) to hold. Also, the necessary condition (7) of Theorem 1 implies that if the system in (1) is output s-sparse controllable,

$$R_0 = n - \text{Rank}\left\{\boldsymbol{CAW}\right\} \geq n - \text{Rank}\left\{\boldsymbol{CA}\right\}. \tag{32}$$

As a consequence, (31) is necessary for the sparsity bound in (7) to hold. Hence, we conclude that Theorem 1 is stronger than Theorem C.

F. Design of Sparse Control Inputs

Theorem 1 focuses on the existence of a set of control inputs that ensures output controllability while satisfying the sparsity constraints. However, another problem related to output controllability is the design of this set of sparse vectors. The problem can be cast as a sparse signal recovery problem using (2), where we solve for the unknown sparse vectors $\{\boldsymbol{u}_k\}_{k=1}^K$ [46], [47].

We first note from [1, Corollary 2] that to drive the system from any given initial state $\boldsymbol{x}_0 \in \mathbb{R}^N$ to any final output $\boldsymbol{y}_f \in \mathbb{R}^n$, we need at most n control inputs ($K = n$). Thus, from (2), the design of control inputs reduces to solving for

$\tilde{\boldsymbol{u}} = \begin{bmatrix}\boldsymbol{u}_1^\mathsf{T} & \boldsymbol{u}_2^\mathsf{T} & \dots & \boldsymbol{u}_n^\mathsf{T}\end{bmatrix}^\mathsf{T} \in \mathbb{R}^{nm}$ using

$$\boldsymbol{y}_f - \boldsymbol{C}\boldsymbol{A}^n\boldsymbol{x}_0 = \boldsymbol{C}\begin{bmatrix}\boldsymbol{A}^{n-1}\boldsymbol{B} & \boldsymbol{A}^{n-2}\boldsymbol{B} & \dots & \boldsymbol{B}\end{bmatrix}\tilde{\boldsymbol{u}}. \quad (33)$$

Here, the unknown vector $\tilde{\boldsymbol{u}}$ is formed by concatenating n vectors which are s-sparse. This signal structure is known as piecewise sparsity. Hence, (33) can be efficiently solved (in polynomial time) using piecewise sparse recovery algorithms, such as the piecewise orthogonal matching pursuit [9], [48].

IV. Conclusion

In this article, we derived a set of necessary and sufficient conditions under which a discrete-time linear system is output sparse controllable. Our results apply to any general linear system and do not impose any restrictions on the system matrices. Both necessary and sufficient conditions included a rank condition on the output controllability matrix and a lower bound on the sparsity bound. We also derived the conditions under which both sets of conditions became identical and showed that the results on output controllability (without any constraints) and controllability (with and without sparsity constraints on the inputs) can be derived as a special case of our result. An important direction for future work is to derive the conditions that are jointly necessary and sufficient for output sparse controllability. Studying output sparse controllability under other constraints on the systems, such as bounded energy and nonnegativity, are also avenues for future work.

Appendix I
Proof of Theorem1

The key idea of the proof is to use the Kalman decomposition [49, Sec. 6.4] to prove the necessity of (7) and sufficiency of (8). The proof also relies on the following results from linear algebra.

Proposition 1 ([50]): For any matrix $\boldsymbol{A}$ and any orthogonal matrix $\boldsymbol{Q}$ of compatible dimension, we have $(\boldsymbol{Q}\boldsymbol{A})^\dagger = \boldsymbol{A}^\dagger\boldsymbol{Q}^\mathsf{T}$.

Proposition 2: Any matrices $\boldsymbol{A}$ and $\boldsymbol{W}$ of compatible dimensions satisfy

$$\mathsf{Rank}\left\{\boldsymbol{A}\boldsymbol{W}\right\} = \mathsf{Rank}\left\{\boldsymbol{A}\boldsymbol{W}\boldsymbol{W}^\dagger\right\}. \quad (34)$$

Proof: The result follows because

$$\mathsf{Rank}\left\{\boldsymbol{A}\boldsymbol{W}\right\} \geq \mathsf{Rank}\left\{\boldsymbol{A}\boldsymbol{W}\boldsymbol{W}^\dagger\right\} \geq \mathsf{Rank}\left\{\boldsymbol{A}\boldsymbol{W}\boldsymbol{W}^\dagger\boldsymbol{W}\right\}$$
$$= \mathsf{Rank}\left\{\boldsymbol{A}\boldsymbol{W}\right\}. \quad (35)$$

■

Proposition 3: For a given nonzero square matrix $\boldsymbol{A} \in \mathbb{R}^{N\times N}$, let $R = \mathsf{Rank}\{\boldsymbol{A}^N\}$. Then, for any given integers $N \leq p \leq q$, there exist real numbers $\{\alpha_i\}_{i=1}^R$ such that

$$\boldsymbol{A}^p = \boldsymbol{A}^q\sum_{i=1}^R \alpha_i\boldsymbol{A}^i. \quad (36)$$

Proof: See Appendix II. ■

Next, we prove the desired result using the above lemmas. We first note that the necessity of the rank condition in (7) is straightforward from Theorem A. Therefore, to prove (7), it is enough to show that when the system is s-sparse output controllable, the lower bound in (7) holds. The proof for the necessity of sparsity bound in (7) and sufficiency of (8) is presented next. At a high level, the proof has the following steps.

1) We first use the Kalman decomposition to construct two matrices $\tilde{\boldsymbol{C}} \in \mathbb{R}^{n\times r}$ and $\tilde{\boldsymbol{A}} \in \mathbb{R}^{r\times r}$ with $r \triangleq \mathsf{Rank}\{\boldsymbol{W}\}$ such that
$$R_i = \mathsf{Rank}\left\{\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\right\} - \mathsf{Rank}\left\{\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}\right\},\ i \geq 0. \quad (37)$$
2) Using (37), we show that when the system is output s-sparse controllable, (7) of Theorem 1 holds. This proof implies that (7) is necessary for output s-sparse controllability.
3) To prove the sufficiency of (8), we first prove that when (8) holds, for any vector $\boldsymbol{y} \in \mathbb{R}^n$, there exists an s-sparse vector $\boldsymbol{u} \in \mathbb{R}^m$ satisfying
$$\left[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\boldsymbol{y} = \left[\boldsymbol{I} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{B}\boldsymbol{u} \quad (38)$$
where we define $\tilde{\boldsymbol{C}}_{(i)} \in \mathbb{R}^{n\times n}$ as
$$\tilde{\boldsymbol{C}}_{(i)} = \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\left(\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\right)^\dagger. \quad (39)$$
4) When (8) holds, we prove that for any vector $\boldsymbol{y} \in \mathbb{R}^n$, there exist s-sparse vectors $\{\boldsymbol{u}_k \in \mathbb{R}^m\}_{k=1}^r$ such that
$$\boldsymbol{y} = \sum_{k=1}^r \boldsymbol{C}\boldsymbol{A}^{k-1}\boldsymbol{B}\boldsymbol{u}_k + \tilde{\boldsymbol{C}}_{(r)}\left[\boldsymbol{y} - \sum_{k=1}^r \boldsymbol{C}\boldsymbol{A}^{k-1}\boldsymbol{B}\boldsymbol{u}_k\right]. \quad (40)$$
5) Finally, when (8) holds, we also show that there exist an integer $0 < K < \infty$ and s-sparse vectors $\{\boldsymbol{u}_k \in \mathbb{R}^m\}_{k=r+1}^K$ such that
$$\tilde{\boldsymbol{C}}_{(r)}\left[\boldsymbol{y} - \sum_{k=1}^r \boldsymbol{C}\boldsymbol{A}^{k-1}\boldsymbol{B}\boldsymbol{u}_k\right] = \sum_{k=r+1}^K \boldsymbol{C}\boldsymbol{A}^{k-1}\boldsymbol{B}\boldsymbol{u}_k. \quad (41)$$
Combining Steps 4) and 5), we establish the sufficiency of (8).

In the reminder of this section, we provide the details of each step.

A. Equivalent Definition of R_i in (6)

By the Kalman decomposition [49, Sec. 6.4], there exists an orthogonal matrix $\boldsymbol{Q}$ such that

$$\boldsymbol{Q} = \begin{bmatrix}\tilde{\boldsymbol{Q}} \in \mathbb{R}^{N\times r} & \boldsymbol{R} \in \mathbb{R}^{N\times N-r}\end{bmatrix} \in \mathbb{R}^{N\times N} \quad (42)$$

$$\boldsymbol{W} = \begin{bmatrix}\tilde{\boldsymbol{Q}} & \boldsymbol{R}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{W}} \in \mathbb{R}^{r\times Nm} \\ \boldsymbol{0} \in \mathbb{R}^{N-r\times Nm}\end{bmatrix} \quad (43)$$

$$\boldsymbol{A} = \begin{bmatrix}\tilde{\boldsymbol{Q}} & \boldsymbol{R}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{A}} \in \mathbb{R}^{r\times r} & \boldsymbol{A}_{(1)} \\ \boldsymbol{0} \in \mathbb{R}^{N-r\times r} & \boldsymbol{A}_{(2)}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{Q}} & \boldsymbol{R}\end{bmatrix}^{-1} \quad (44)$$

$$\boldsymbol{B} = \begin{bmatrix}\tilde{\boldsymbol{Q}} & \boldsymbol{R}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{B}} \in \mathbb{R}^{r\times m} \\ \boldsymbol{0} \in \mathbb{R}^{N-r\times m}\end{bmatrix} \quad (45)$$

where $r = \text{Rank}\{\boldsymbol{W}\} = \text{Rank}\{\tilde{\boldsymbol{W}}\}$. Then, for any integer $i \geq 0$, it is easy to see that

$$\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{W}\boldsymbol{W}^\dagger = \boldsymbol{C}\boldsymbol{A}^i\boldsymbol{Q}\begin{bmatrix}\tilde{\boldsymbol{W}}\\ \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{W}}^\dagger & \boldsymbol{0}\end{bmatrix}\boldsymbol{Q}^\mathsf{T} \quad (46)$$

$$= \boldsymbol{C}\begin{bmatrix}\tilde{\boldsymbol{Q}} & \boldsymbol{R}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{A}} & \boldsymbol{A}_{(1)}\\ \boldsymbol{0} & \boldsymbol{A}_{(2)}\end{bmatrix}^i\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{Q}^\mathsf{T} \quad (47)$$

$$= \boldsymbol{C}\begin{bmatrix}\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{A}}^i & \boldsymbol{0}\end{bmatrix}\boldsymbol{Q}^\mathsf{T} \quad (48)$$

where to get (46), we use (43) and Proposition 1. Also, (47) follows from (44) and the fact that $\tilde{\boldsymbol{W}}$ is a full row rank matrix. Consequently, we conclude that

$$\text{Rank}\left\{\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{W}\right\} = \text{Rank}\left\{\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{W}\boldsymbol{W}^\dagger\right\} = \text{Rank}\left\{\boldsymbol{C}\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{A}}^i\right\} \quad (49)$$

where we also use Proposition 2. Thus, we establish (37) by defining

$$\tilde{\boldsymbol{C}} \triangleq \boldsymbol{C}\tilde{\boldsymbol{Q}} \in \mathbb{R}^{n\times r} \quad (50)$$

and Step 1) is completed.

B. Necessity of Sparsity Bound in (20)

Using (44), (45), and (50), we rewrite (4) as

$$\mathcal{CS}\left\{\tilde{\boldsymbol{C}}\begin{bmatrix}\tilde{\boldsymbol{A}}^{K-1}\tilde{\boldsymbol{B}}_{\mathcal{S}_1} & \tilde{\boldsymbol{A}}^{K-2}\tilde{\boldsymbol{B}}_{\mathcal{S}_2} & \ldots & \tilde{\boldsymbol{B}}_{\mathcal{S}_K}\end{bmatrix}\right\} = \mathbb{R}^n. \quad (51)$$

Here, for every integer $0 \leq i \leq N-1$, the first $(K-i-1)s$ columns of the matrix in (51) belong to $\mathcal{CS}\{\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}\}$. As a consequence, the last $(i+1)s$ columns of the matrix span the null space of $\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}$. Since the dimension of the null space of $\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}$ is $n - \text{Rank}\{\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}\}$, we deduce that

$$(i+1)s \geq n - \text{Rank}\left\{\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}\right\} = \sum_{j=0}^{i} R_j \quad (52)$$

where we use (37). Therefore, we obtain that when (4) holds, the bound on s given by (7) is satisfied. Thus, we complete Step 2).

C. Characterizing $\mathcal{CS}\{\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\}$

To prove Step 3), we first show that

$$\mathcal{CS}\left\{\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right\} \subseteq \mathcal{CS}\left\{\left[\boldsymbol{I} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{B}\right\}. \quad (53)$$

Then, it is enough to prove that

$$\text{Rank}\left\{\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right\} \leq s \quad (54)$$

which implies that at most s columns of $[\boldsymbol{I} - \tilde{\boldsymbol{C}}_{(i+1)}]\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{B}$ are needed to span the column space of $\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}$. Thus, the goal of this step is to establish (53) and (54).

To prove (53), we start with two observations. First, from (5) and using the fact that $r = \text{Rank}\{\boldsymbol{W}\}$ from Step 1), we have

$$r = \text{Rank}\left\{\begin{bmatrix}\boldsymbol{A}^{N-1}\boldsymbol{B} & \boldsymbol{A}^{N-2}\boldsymbol{B} & \ldots & \boldsymbol{B}\end{bmatrix}\right\} \quad (55)$$

$$= \text{Rank}\left\{\begin{bmatrix}\tilde{\boldsymbol{A}}^{N-1}\tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}}^{N-2}\tilde{\boldsymbol{B}} & \ldots & \tilde{\boldsymbol{B}}\end{bmatrix}\right\} \quad (56)$$

which follows because $\boldsymbol{A}^{N-1}\boldsymbol{B} = \boldsymbol{Q}[(\tilde{\boldsymbol{A}}^{N-1}\tilde{\boldsymbol{B}})^\mathsf{T} \ \ \boldsymbol{0}]^\mathsf{T}$. Therefore, the system defined by $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}})$ is controllable by the classical Kalman rank test for (nonsparse) controllability [49]. So, for any $\boldsymbol{z} \in \mathbb{R}^r$, there exist (nonsparse) vectors $\{\boldsymbol{v}_k \in \mathbb{R}^m\}_{k=1}^r$ such that

$$\boldsymbol{z} = \sum_{k=1}^{r}\tilde{\boldsymbol{A}}^{k-1}\tilde{\boldsymbol{B}}\boldsymbol{v}_k. \quad (57)$$

The second observation is as follows. From the definition of $\tilde{\boldsymbol{C}}_{(i)}$ given by (39), when $k > i$

$$\left[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^k$$

$$= \left[\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\left(\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\right)^\dagger - \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}\left(\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}\right)^\dagger\right]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}\tilde{\boldsymbol{A}}^{k-(i+1)} \quad (58)$$

$$= \left[\left(\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\right)\tilde{\boldsymbol{A}} - \left(\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^{i+1}\right)\right]\tilde{\boldsymbol{A}}^{k-(i+1)} = \boldsymbol{0}. \quad (59)$$

Here, we use the fact that by the definition of pseudoinverse, for any matrix $\boldsymbol{A}$, we have $\boldsymbol{A}\boldsymbol{A}^\dagger\boldsymbol{A} = \boldsymbol{A}$. Similarly, we can simplify the same expression for the case when $k = i$ to obtain

$$\left[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^k = \begin{cases}\left[\boldsymbol{I} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i, & \text{if } k = i\\ \boldsymbol{0}, & \text{if } k > i\end{cases}. \quad (60)$$

Combining the above two observations, we premultiply (57) with $[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i$ and simplify using (60) to show that for any $\boldsymbol{z} \in \mathbb{R}^r$, there exists $\boldsymbol{v}_1 \in \mathbb{R}^m$ such that

$$\left[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\boldsymbol{z} = \left[\boldsymbol{I} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\tilde{\boldsymbol{B}}\boldsymbol{v}_1. \quad (61)$$

Here, we note that all the terms in the summation of (57) except the term corresponding to $k = 1$ vanish when premultiplied with $[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i$. The relation (61) leads to the following:

$$\mathcal{CS}\left\{\left[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\right\} \subseteq \mathcal{CS}\left\{\left[\boldsymbol{I} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{B}\right\} \quad (62)$$

which follows because $\boldsymbol{C}\boldsymbol{A}^i\boldsymbol{B} = \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\tilde{\boldsymbol{B}}$ from (44), (45), and (50). Next, we show that

$$\mathcal{CS}\left\{\left[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\right\} = \mathcal{CS}\left\{\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right\} \quad (63)$$

to arrive at (53). To this end, we have

$$\text{Rank}\left\{\left[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}_{(i)}\right\}$$

$$\leq \text{Rank}\left\{\left[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\right\} \quad (64)$$

because $\tilde{\boldsymbol{C}}_{(i)} = (\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i)(\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i)^\dagger$. Here, from the definition of the matrix $\tilde{\boldsymbol{C}}_{(i)}$ given by (39) and symmetry of $\tilde{\boldsymbol{C}}_{(i)}$ and $\tilde{\boldsymbol{C}}_{(i+1)}$, we derive

$$\left[\tilde{\boldsymbol{C}}_{(i)} - \tilde{\boldsymbol{C}}_{(i+1)}\right]\tilde{\boldsymbol{C}}_{(i)}$$

$$= \tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\left(\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\right)^\dagger\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\left(\tilde{\boldsymbol{C}}\tilde{\boldsymbol{A}}^i\right)^\dagger - \tilde{\boldsymbol{C}}_{(i+1)}^\mathsf{T}\tilde{\boldsymbol{C}}_{(i)}^\mathsf{T} \quad (65)$$

$$= \tilde{\boldsymbol{C}}_{(i)} - \left[\tilde{\boldsymbol{C}}_{(i)}\tilde{\boldsymbol{C}}_{(i+1)}\right]^\mathsf{T} \quad (66)$$

$$= \tilde{C}_{(i)} - \left[\tilde{C}\tilde{A}^i\left(\tilde{C}\tilde{A}^i\right)^{\dagger}\tilde{C}\tilde{A}^{i+1}\left(\tilde{C}\tilde{A}^{i+1}\right)^{\dagger}\right]^{\mathsf{T}} \quad (67)$$

$$= \tilde{C}_{(i)} - \left[\tilde{C}_{(i+1)}\right]^{\mathsf{T}} = \tilde{C}_{(i)} - \tilde{C}_{(i+1)}. \quad (68)$$

Thus, from (64), we get

$$\text{Rank}\left\{\tilde{C}_{(i)} - \tilde{C}_{(i+1)}\right\} \leq \text{Rank}\left\{\left[\tilde{C}_{(i)} - \tilde{C}_{(i+1)}\right] CA^i\right\}. \quad (69)$$

However, we also know that

$$\mathcal{CS}\left\{\tilde{C}_{(i)} - \tilde{C}_{(i+1)}\right\} \supseteq \mathcal{CS}\left\{\left[\tilde{C}_{(i)} - \tilde{C}_{(i+1)}\right]\tilde{C}\tilde{A}^i\right\} \quad (70)$$

and as a result, we derive (63), which leads to (53).

Next, we complete Step 3) by establishing (54). For this, we note from (68) that

$$\text{Rank}\left\{\tilde{C}_{(i)} - \tilde{C}_{(i+1)}\right\} = \text{Rank}\left\{\left[\tilde{C}_{(i)} - \tilde{C}_{(i+1)}\right]\tilde{C}_{(i)}\right\} \quad (71)$$

$$= \text{Rank}\left\{\left[I - \tilde{C}_{(i+1)}\right]\tilde{C}_{(i)}\right\} \quad (72)$$

$$= \text{Rank}\left\{\tilde{C}_{(i)}\right\} - \text{Rank}\left\{\tilde{C}_{(i+1)}\right\} \quad (73)$$

$$= R_i \leq s \quad (74)$$

where (72) follows from (60) with $k = i$. Also, (73) is because $\mathcal{CS}\{\tilde{C}_{(i)}\} \supseteq \mathcal{CS}\{\tilde{C}_{(i+1)}\}$, and $I - \tilde{C}_{(i+1)}$ is the projection onto the subspace orthogonal to $\mathcal{CS}\{\tilde{C}_{(i+1)}\}$. Finally, (74) follows from (37) and assumption (8).

D. Sparse Representation of the Null Space of $\tilde{C}\tilde{A}^r$

We prove a more general result: for any vector $y \in \mathbb{R}^n$ and integer $1 \leq i \leq r$, there exist s-sparse vectors $\{\tilde{u}_k \in \mathbb{R}^m\}_{k=1}^i$ such that

$$y = \sum_{k=1}^{i} CA^{k-1}B\tilde{u}_k + \tilde{C}_{(i)}\left[y - \sum_{k=1}^{i} CA^{k-1}B\tilde{u}_k\right]. \quad (75)$$

We prove (75) using mathematical induction, and for this, we first verify this result for $i = 1$. Using (38) of Step 3), for any given $y \in \mathbb{R}^n$, there exists an s-sparse vector $\tilde{u}_1 \in \mathbb{R}^m$ such that

$$\left[\tilde{C}_{(0)} - \tilde{C}_{(1)}\right] y = \left[I - \tilde{C}_{(1)}\right] CB\tilde{u}_1. \quad (76)$$

However, we observe from (43) that $\mathcal{CS}\{W\} = \mathcal{CS}\{\tilde{Q}\}$, and this observation combined with (8) leads to the following:

$$\text{Rank}\left\{\tilde{C}\right\} = \text{Rank}\left\{C\tilde{Q}\right\} = \text{Rank}\left\{CW\right\} = n. \quad (77)$$

As a result, we get

$$\tilde{C}_{(0)} = \tilde{C}\tilde{C}^{\dagger} = I. \quad (78)$$

Therefore, (76) yields that for any $y \in \mathbb{R}^n$, there exists an s-sparse vector $\tilde{u}_1 \in \mathbb{R}^m$ such that

$$y = CB\tilde{u}_1 + \tilde{C}_{(1)}\left(y - CB\tilde{u}_1\right). \quad (79)$$

Consequently, (75) holds for $i = 1$.

By inductive hypothesis, we assume that (75) holds for some integer $1 \leq i < r$. So, we again apply (38) by substituting y as $\tilde{C}_{(i)}[y - \sum_{k=1}^{i} CA^{k-1}B\tilde{u}_k]$ to deduce that there exists an s-sparse vector $\tilde{u}_{i+1} \in \mathbb{R}^m$ such that

$$\left[\tilde{C}_{(i)} - \tilde{C}_{(i+1)}\right]\tilde{C}_{(i)}\left[y - \sum_{k=1}^{i} CA^{k-1}B\tilde{u}_k\right]$$
$$= \left[I - \tilde{C}_{(i+1)}\right] CA^iBu_{i+1}. \quad (80)$$

Combining (80) and (68), we deduce that

$$\tilde{C}_{(i)}\left[y - \sum_{k=1}^{i} CA^{k-1}B\tilde{u}_k\right]$$
$$= CA^iBu_{i+1} + \tilde{C}_{(i+1)}\left[y - \sum_{k=1}^{i+1} CA^{k-1}B\tilde{u}_k\right] \quad (81)$$

where the term $\tilde{C}_{i+1}CA^iBu_{i+1}$ is absorbed into the summation term on the right-hand side. Adding (81) and the inductive hypothesis (75), we get

$$y = \sum_{k=1}^{i+1} CA^{k-1}B\tilde{u}_k + \tilde{C}_{(i+1)}\left[y - \sum_{k=1}^{i+1} CA^{k-1}B\tilde{u}_k\right]. \quad (82)$$

In conclusion, we obtain that the desired result (75) holds for $i + 1$, and thus, the relation (75) is proved. Finally, we choose $i = r$ in (75).

E. Sparse Representation of the Column Space of $\tilde{C}\tilde{A}^r$

Using (57), for any $z \in \mathbb{R}^r$, there exist (nonsparse) vectors $\{v_k \in \mathbb{R}^m\}_{k=1}^r$ such that

$$\tilde{C}\tilde{A}^r z = \tilde{C}\sum_{k=1}^{r} \tilde{A}^{r+k-1}\tilde{B}v_k. \quad (83)$$

Here, $v_k \in \mathbb{R}^m$ can be represented as $v_k = \sum_{j=1}^{\lceil m/s\rceil} u_j^{(k)}$, where $\{u_j^{(k)} \in \mathbb{R}^m\}_{j,k}$ are all s-sparse vectors. Therefore

$$\tilde{C}\tilde{A}^r z = \tilde{C}\sum_{k=1}^{r}\sum_{j=1}^{\lceil m/s\rceil} \tilde{A}^{r+k-1}\tilde{B}u_j^{(k)}. \quad (84)$$

However, from Proposition 3, there exists a set of real numbers $\{\alpha_i{}^{(j,k)} \in \mathbb{R}\}_{i=1}^{\text{Rank}\{\tilde{A}^r\}}$ such that

$$\tilde{A}^{r+k-1}\tilde{B}u_j^{(k)} = \tilde{A}^{r+q_{k,j}} \sum_{i=1}^{\text{Rank}\{\tilde{A}^r\}} \alpha_i{}^{(j,k)}\tilde{A}^i\tilde{B}u_j^{(k)} \quad (85)$$

where we define the quantity $q_{k,j} \geq k - 1$ as

$$q_{k,j} \triangleq \left[(k-1)\left\lceil\frac{m}{s}\right\rceil + (j-1)\right]\text{Rank}\left\{\tilde{A}^r\right\}. \quad (86)$$

Substituting (85) into (84), we deduce that

$$\tilde{C}\tilde{A}^r z = \tilde{C}\tilde{A}^r \sum_{k=1}^{r} \sum_{j=1}^{\lceil m/s \rceil} \sum_{i=1}^{\text{Rank}\{\tilde{A}^r\}} \tilde{A}^{q_{k,j}+i}\tilde{B}\left(\alpha_i^{(j,k)} u_j^{(k)}\right) \tag{87}$$

$$= CA^r \sum_{k=1}^{r} \sum_{j=1}^{\lceil m/s \rceil} \sum_{i=1}^{\text{Rank}\{\tilde{A}^r\}} A^{q_{k,j}+i} B\left(\alpha_i^{(j,k)} u_j^{(k)}\right) \tag{88}$$

which follows from (44), (45), and (50). Here, the powers of A in each term of the summation are distinct, and $\alpha_i^{(j,k)}$ $u_j^{(k)}$ is s-sparse for all values of i, j, and k. Consequently, for any vector $z \in \mathbb{R}^r$, there exists an integer $0 < K = r + r\lceil m/s \rceil$ $\text{Rank}\{\tilde{A}^r\} < \infty$ and s-sparse vectors $\{u_k \in \mathbb{R}^m\}_{k=r+1}^{K}$ such that

$$\tilde{C}\tilde{A}^r z = \sum_{k=r+1}^{K} CA^{k-1}Bu_k. \tag{89}$$

Finally, we choose

$$z = \left(\tilde{C}\tilde{A}^r\right)^\dagger \left[y - \sum_{k=1}^{r} CA^{k-1}Bu_k\right] \in \mathbb{R}^r \tag{90}$$

in (89) to complete Step 5).

Appendix II
Proof of Proposition 3

To prove the result, we consider the real Jordan canonical form [43] of A

$$A = P^{-1}\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} P \tag{91}$$

where $P \in \mathbb{R}^{N\times N}$ is an invertible matrix. Also, the square matrices J and N are formed by the Jordan blocks of A corresponding to the nonzero and zero eigenvalues of A, respectively. In other words, J is an invertible matrix and N is a nilpotent matrix, i.e., $N^N = 0$. Consequently, the desired result (36) is equivalent to

$$P^{-1}\begin{bmatrix} J^p & 0 \\ 0 & 0 \end{bmatrix} P = P^{-1}\begin{bmatrix} J^q & 0 \\ 0 & 0 \end{bmatrix}\left(\sum_{i=1}^{R}\alpha_i \begin{bmatrix} J^i & 0 \\ 0 & N^i \end{bmatrix} P\right) \tag{92}$$

$$= P^{-1}\begin{bmatrix} J^q \sum_{i=1}^{R}\alpha_i J^i & 0 \\ 0 & 0 \end{bmatrix} P \tag{93}$$

which follows because $p, q \geq N$. Hence, to prove Proposition 3, it suffices to show that for any $N \leq p \leq q$, there exist real numbers $\{\alpha_i\}_{i=1}^{R}$ such that

$$J^{p-q} = \sum_{i=1}^{R}\alpha_i J^i. \tag{94}$$

For this, we first note that

$$R = \text{Rank}\left\{A^N\right\} = \text{Rank}\left\{J^N\right\} = \text{Rank}\left\{J\right\} \tag{95}$$

which is due to the invertibility of J. Consequently, by Cayley–Hamilton theorem, we know that the characteristic polynomial of J has degree at most R. Hence, for any integer $p - q$, the relation (94) holds.

Appendix III
Proof of Corollary 2

From Step 1) of the proof of Theorem 1 given in Appendix I,

$$R_i = \text{Rank}\left\{\tilde{C}\tilde{A}^i\right\} - \text{Rank}\left\{\tilde{C}\tilde{A}^{i+1}\right\} \tag{96}$$

where $\tilde{A} \in \mathbb{R}^{r\times r}$ and $\tilde{C} \in \mathbb{R}^{N\times r}$ are as defined in (44) and (50), respectively, and $r = \text{Rank}\{W\}$. Using the Sylvester rank inequality [43, Sec. 0.4.5], we deduce that

$$R_i \leq r - \text{Rank}\left\{\tilde{A}\right\} = \text{Rank}\left\{W\right\} - \text{Rank}\left\{\tilde{A}\right\}. \tag{97}$$

Here, we simplify the second term as follows:

$$\text{Rank}\left\{\tilde{A}\right\} = \text{Rank}\left\{\begin{bmatrix} \tilde{A} & A_{(1)} \\ 0 & A_{(2)} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

$$= \text{Rank}\left\{Q\begin{bmatrix} \tilde{A} & A_{(1)} \\ 0 & A_{(2)} \end{bmatrix} Q^{-1}Q\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}\right\}$$

$$= \text{Rank}\left\{AWW^\dagger\right\} = \text{Rank}\left\{AW\right\} \tag{98}$$

where $A_{(1)}$ and $A_{(2)}$ are defined in (44) and Q is defined in (42). Also, (98) follows from the arguments similar to those in (46)–(48) and Proposition 2. Substituting (98) into (97), we obtain

$$R_i \leq \text{Rank}\left\{W\right\} - \text{Rank}\left\{AW\right\} \leq N - \text{Rank}\left\{A\right\} \tag{99}$$

where we use the Sylvester rank inequality [43, Sec. 0.4.5].

Hence, using the condition in Corollary 2, we arrive at $s \geq \max_{0\leq i\leq N-1} R_i$. This relation implies that the sufficient condition (8) of Theorem 1 holds, and the desired result follows.

References

[1] G. Joseph and C. R. Murthy, "Controllability of linear dynamical systems under input sparsity constraints," *IEEE Trans. Autom. Control*, vol. 66, no. 2, pp. 924–931, Feb. 2021.

[2] G. Joseph, "Controllability of a linear system with nonnegative sparse controls," *IEEE Trans. Autom. Control*, vol. 67, no. 1, pp. 468–473, Jan. 2022.

[3] M. Caponigro, M. Fornasier, B. Piccoli, and E. Trélat, "Sparse stabilization and control of alignment models," *Math. Models Methods Appl. Sci.*, vol. 25, no. 3, pp. 521–564, Mar. 2015.

[4] G. Stadler, "Elliptic optimal control problems with L^1−control cost and applications for the placement of control devices," *Comput. Optim. Appl.*, vol. 44, no. 2, pp. 159–181, Nov. 2009.

[5] P. V. Chanekar, N. Chopra, and S. Azarm, "Optimal actuator placement for linear systems with limited number of actuators," in *Proc. Amer. Control Conf.*, May 2017, pp. 334–339.

[6] T. Ikeda and K. Kashima, "On sparse optimal control for general linear systems," *IEEE Trans. Autom. Control*, vol. 64, no. 5, pp. 2077–2083, May 2019.

[7] T. Ikeda and K. Kashima, "Sparsity-constrained controllability maximization with application to time-varying control node selection," *IEEE Control Syst. Lett.*, vol. 2, no. 3, pp. 321–326, Jul. 2018.

[8] G. Joseph, B. Nettasinghe, V. Krishnamurthy, and P. Varshney, "Controllability of network opinion in Erdos-Renyi graphs using sparse control inputs," *SIAM J. Control Optim.*, vol. 59, no. 3, pp. 2321–2345, Jun. 2021.

[9] C. Sriram, G. Joseph, and C. R. Murthy, "Control of linear dynamical systems using sparse inputs," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process.*, May 2020, pp. 5765–5769.
[10] G. Vossen and H. Maurer, "On L^1-minimization in optimal control and applications to robotics," *Optim. Control Appl. Methods*, vol. 27, no. 6, pp. 301–321, Nov. 2006.
[11] M. Siami, A. Olshevsky, and A. Jadbabaie, "Deterministic and randomized actuator scheduling with guaranteed performance bounds," *IEEE Trans. Autom. Control*, vol. 66, no. 4, pp. 1686–1701, Apr. 2021.
[12] M. Nagahara and D. E. Quevedo, "Sparse representations for packetized predictive networked control," *IFAC-PapersOnLine*, vol. 44, no. 1, pp. 84–89, Jan. 2011.
[13] Z. Li, Y. Xu, H. Huang, and S. Misra, "Sparse control and compressed sensing in networked switched systems," *IET Control Theory Appl.*, vol. 10, no. 9, pp. 1078–1087, Jun. 2016.
[14] D. L. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1289–1306, Apr. 2006.
[15] E. J. Candes, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 489–509, Feb. 2006.
[16] R. G. Baraniuk, "Compressive sensing [lecture notes]," *IEEE Signal Process. Mag.*, vol. 24, no. 4, pp. 118–121, Jul. 2007.
[17] M. H. DeGroot, "Reaching a consensus," *J. Amer. Statist. Assoc.*, vol. 69, no. 345, pp. 118–121, Mar. 1974.
[18] B. Golub and M. O. Jackson, "How homophily affects the speed of learning and best-response dynamics," *Quart. J. Econ.*, vol. 127, no. 3, pp. 1287–1338, Aug. 2012.
[19] M. Nabi-Abdolyousefi, "Social control and optimal marketing," in *Controllability, Identification, and Randomness in Distributed Systems*. Berlin, Germany: Springer, 2014, pp. 137–146.
[20] K. Shu, H. R. Bernard, and H. Liu, "Studying fake news via network analysis: Detection and mitigation," in *Emerging Research Challenges and Opportunities in Computational Social Network Analysis and Mining*. Berlin, Germany: Springer, 2019, pp. 43–65.
[21] M. Cremonini and F. Casamassima, "Controllability of social networks and the strategic use of random information," *Comput. Soc. Netw.*, vol. 4, no. 10, pp. 1–22, Dec. 2017.
[22] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, no. 7346, pp. 167–173, May 2011.
[23] M. Egerstedt, S. Martini, M. Cao, K. Camlibel, and A. Bicchi, "Interacting with networks: How does structure relate to controllability in single-leader, consensus networks?," *IEEE Control Syst. Mag.*, vol. 32, no. 4, pp. 66–73, Aug. 2012.
[24] A. Chapman and M. Mesbahi, "On strong structural controllability of networked systems: A constrained matching approach," in *Proc. Amer. Control Conf.*, 2013, pp. 6126–6131.
[25] A. Chapman, M. Nabi-Abdolyousefi, and M. Mesbahi, "Controllability and observability of network-of-networks via Cartesian products," *IEEE Trans. Autom. Control*, vol. 59, no. 10, pp. 2668–2679, Oct. 2014.
[26] S. Pequito, S. Kar, and A. P. Aguiar, "A framework for structural input/output and control configuration selection in large-scale systems," *IEEE Trans. Autom. Control*, vol. 61, no. 2, pp. 303–318, Feb. 2016.
[27] A. Chapman and M. Mesbahi, "State controllability, output controllability and stabilizability of networks: A symmetry perspective," in *Proc. IEEE 54th Conf. Decis. Control*, 2015, pp. 4776–4781.
[28] S. S. Mousavi, M. Haeri, and M. Mesbahi, "On the structural and strong structural controllability of undirected networks," *IEEE Trans. Autom. Control*, vol. 63, no. 7, pp. 2234–2241, Jul. 2018.
[29] S. S. Mousavi, M. Haeri, and M. Mesbahi, "Strong structural controllability of networks under time-invariant and time-varying topological perturbations," *IEEE Trans. Autom. Control*, vol. 66, no. 3, pp. 1375–1382, Mar. 2021.
[30] C. Commault, J. van der Woude, and P. Frasca, "Functional target controllability of networks: Structural properties and efficient algorithms," *IEEE Trans. Netw. Sci. Eng*, vol. 7, no. 3, pp. 1521–1530, Jul./Sep. 2020.
[31] Z. Liu et al., "Minimal input selection for robust control," in *Proc. IEEE 56th Annu. Conf. Decis. Control*, 2017, pp. 2659–2966.
[32] V. Tzoumas, M. A. Rahimian, G. J. Pappas, and A. Jadbabaie, "Minimal actuator placement with bounds on control effort," *IEEE Trans. Control Netw. Syst.*, vol. 3, no. 1, pp. 67–78, Mar. 2016.
[33] A. Olshevsky, "Minimal controllability problems," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 3, pp. 249–258, Sep. 2014.
[34] F. Dörfler, M. R. Jovanović, M. Chertkov, and F. Bullo, "Sparsity-promoting optimal wide-area control of power networks," *IEEE Trans. Power Syst.*, vol. 29, no. 5, pp. 2281–2291, Sep. 2014.
[35] K. Ito and K. Kunisch, "Optimal control with $L^p(\Omega), p \in (0,1)$, control cost," *SIAM J. Control Optim.*, vol. 52, no. 2, pp. 1251–1275, 2014.
[36] D. Chatterjee, M. Nagahara, D. E. Quevedo, and K. M. Rao, "Characterization of maximum hands-off control," *Syst. Control.*, vol. 94, pp. 31–36, Aug. 2016.
[37] D. Kalise, K. Kunisch, and Z. Rao, "Infinite horizon sparse optimal control," *J. Optim. Theory Appl.*, vol. 172, no. 2, pp. 481–517, Feb. 2017.
[38] Z. Rao, "A dynamic programming approach for L^0 optimal control design," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 2886–2891, Jul. 2017.
[39] T. Zhou, "Minimal inputs/outputs for a networked system," *IEEE Control Syst. Lett.*, vol. 1, no. 2, pp. 298–303, Oct. 2017.
[40] T. Ikeda and M. Nagahara, "Maximum hands-off control without normality assumption," in *Proc. Amer. Control Conf.*, 2016, pp. 209–214.
[41] M. L. J. Hautus, "Stabilization controllability and observability of linear autonomous systems," *Indagationes Mathematicae*, vol. 73, pp. 448–455, Jan. 1970.
[42] P. L. Clark, "Covering numbers in linear algebra," *Amer. Math. Monthly*, vol. 119, no. 1, pp. 65–67, Jan. 2012.
[43] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 2012.
[44] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. New York, NY, USA: Springer, 2013.
[45] R. Kalman, "On the general theory of control systems," *IRE Trans. Autom. Control*, vol. 4, no. 3, pp. 110–110, Dec. 1959.
[46] S. Sefati, N. J. Cowan, and R. Vidal, "Linear systems with sparse inputs: Observability and input recovery," in *Proc. Amer. Control Conf.*, Jul. 2015, pp. 5251–5257.
[47] M. Kafashan, A. Nandi, and S. Ching, "Relating observability and compressed sensing of time-varying signals in recurrent linear networks," *Neural Netw.*, vol. 83, pp. 11–20, Nov. 2016.
[48] K. Li, C. R. Rojas, T. Yang, H. Hjalmarsson, K. H. Johansson, and S. Cong, "Piecewise sparse signal recovery via piecewise orthogonal matching pursuit," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process.*, May 2016, pp. 4608–4612.
[49] C.-T. Chen, *Linear System Theory and Design*. London, U.K.: Oxford Univ. Press, 1998.
[50] D. S. Djordjević and N. Č. Dinčić, "Reverse order law for the Moore–Penrose inverse," *J. Math. Anal. Appl.*, vol. 361, no. 1, pp. 252–261, Jan. 2010.

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