Commutator estimates for normal operators in factors with applications to derivations

Ber, Aleksei F.; Borst, Matthijs J.; Sukochev, Fedor A.

DOI
10.7900/jot.2023apr21.2442

Publication date
2025

Document Version
Accepted author manuscript

Published in
Journal of Operator Theory

Citation (APA)
Ber, A. F., Borst, M. J., & Sukochev, F. A. (2025). Commutator estimates for normal operators in factors with applications to derivations. *Journal of Operator Theory*, *93*(1), 251-289. https://doi.org/10.7900/jot.2023apr21.2442

COMMUTATOR ESTIMATES FOR NORMAL OPERATORS IN FACTORS WITH APPLICATIONS TO DERIVATIONS

ALEKSEI F. BER, MATTHIJS J. BORST, AND FEDOR A. SUKOCHEV

ABSTRACT. For a normal measurable operator a affiliated with a von Neumann factor $\mathcal{M}$ we show:
If $\mathcal{M}$ is infinite, then there is $\lambda_0 \in \mathbb{C}$ so that for $\varepsilon > 0$ there are $u_\varepsilon = u_\varepsilon^*$, $v_\varepsilon \in \mathcal{U}(\mathcal{M})$ with

$$v_\varepsilon |[a, u_\varepsilon]| v_\varepsilon^* \geq (1-\varepsilon)(|a - \lambda_0 \mathbf{1}| + u_\varepsilon |a - \lambda_0 \mathbf{1}| u_\varepsilon).$$

If $\mathcal{M}$ is finite, then there is $\lambda_0 \in \mathbb{C}$ and $u, v \in \mathcal{U}(\mathcal{M})$ so that

$$v|[a,u]|v^* \geq \frac{\sqrt{3}}{2}(|a - \lambda_0 \mathbf{1}| + u|a - \lambda_0 \mathbf{1}|u^*).$$

These bounds are optimal for infinite factors, II_1-factors and some I_n-factors. Furthermore, for finite factors applying $\|\cdot\|_1$-norms to the inequality provides estimates on the norm of the inner derivation $\delta_a : \mathcal{M} \to L_1(\mathcal{M}, \tau)$ associated to a. While by [3, Theorem 1.1] it is known for finite factors and self-adjoint $a \in L_1(\mathcal{M}, \tau)$ that $\|\delta_a\|_{\mathcal{M} \to L_1(\mathcal{M},\tau)} = 2 \min_{z \in \mathbb{C}} \|a - z\|_1$, we present concrete examples of finite factors $\mathcal{M}$ and normal operators $a \in \mathcal{M}$ for which this fails.

1. INTRODUCTION

Derivations are linear maps δ that satisfy the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$. They play an essential role in the theory of Lie algebras, Cohomology, the study of Semi-groups and in Quantum Physics, see [17, 19, 24]. A classical result on derivations is due to Stampfli [25] which asserts that for $a \in B(H)$, a bounded operator on a Hilbert space H, the derivation $\delta_a : B(H) \to B(H)$ defined by the commutator $\delta_a(x) = [a, x] = ax - xa$ has operator norm $\|\delta_a\| = 2 \inf_{z \in \mathbb{C}} \|a - z\mathbf{1}\|$. Through the work of [13, 18, 27], the result of Stampfli has been extended to derivations on arbitrary von Neumann algebras $\mathcal{M}$ (see also [21] for more in this direction). More precisely, the result of Zsidó [27, Corollary] asserts that for $\mathcal{M}$ a von Neumann algebra and $a \in \mathcal{M}$, the derivation $\delta_a : \mathcal{M} \to \mathcal{M}$ associated to a satisfies the distance formula:

$$\|\delta_a\|_{\mathcal{M} \to \mathcal{M}} = 2 \min_{z \in Z(\mathcal{M})} \|a - z\|, \tag{1}$$

where $Z(\mathcal{M})$ denotes the center of $\mathcal{M}$.

Our research aims to obtain results similar to (1) for derivations that map $\mathcal{M}$ into the predual $\mathcal{M}_*$. Indeed, the predual $\mathcal{M}_*$ is a $\mathcal{M}$-bimodule (see Section 7) and therefore it is possible to consider derivations $\delta : \mathcal{M} \to \mathcal{M}_*$. Important work on such derivations was done in [2, 7, 14] and particularly the result of [14, Theorem 4.1] showed that all these derivations are inner (i.e. of the form $\delta = \delta_a$ for some $a \in \mathcal{M}_*$, defined by $\delta_a(x) = ax - xa$). These studies arose after Connes proved in [8] that all amenable C*-algebras are necessarily nuclear. Haagerup proved in [14] that the reverse implication is also true.

In [3] the norms of these derivations were studied and results analogouos to (1) were found in certain cases: for $\mathcal{M}$ properly infinite it was shown that some form of formula (1) holds true and for $\mathcal{M}$ finite the same was proved under the condition that a is self-adjoint. The proofs of these results were based on improvements of the operator estimates obtained in [4, 5], see below:

Theorem 1.1. *[4, Theorem 1] Let $\mathcal{M}$ be a factor and let $a = a^* \in S(\mathcal{M})$ (here $S(\mathcal{M})$ is the algebra of measurable operators attached to $\mathcal{M}$).*

(1) If $\mathcal{M}$ is a finite factor or else a purely infinite σ-finite factor, then there exists $\lambda_0 \in \mathbb{R}$ and $u_0 = u_0^* \in \mathcal{U}(\mathcal{M})$, such that

$$|[a, u_0]| = u_0 |a - \lambda_0 \mathbf{1}| u_0 + |a - \lambda_0 \mathbf{1}|$$

where $\mathcal{U}(\mathcal{M})$ is the group of all unitary operators in $\mathcal{M}$;

Date: April 24, 2023.
2010 *Mathematics Subject Classification.* 47B47, 46L10.
Key words and phrases. von Neumann factors, operator inequalities, derivations, commutators.

(2) there exists $\lambda_0 \in \mathbb{R}$ so that for any $\varepsilon > 0$ there exists $u_\varepsilon = u_\varepsilon^* \in \mathcal{U}(\mathcal{M})$ such that

$$|[a, u_\varepsilon]| \geq (1-\varepsilon)|a - \lambda_0 \mathbf{1}|. \tag{2}$$

This theorem was extended to arbitrary von Neumann algebras in [5] with the replacement of $\lambda_0\mathbf{1}$ by an element from the center. In [3, Theorem B.1] inequality (2) was extended to:

$$|[a, u_\varepsilon]| \geq (1-\varepsilon)(|a - \lambda_0\mathbf{1}| + u_\varepsilon|a - \lambda_0\mathbf{1}|u_\varepsilon). \tag{3}$$

The question arises: is such an inequality as (3) true for arbitrary $a \in S(\mathcal{M})$? More precisely, are there such $\lambda_0 \in \mathbb{C}$, $u, v, w \in \mathcal{U}(\mathcal{M})$ and a constant $C > 0$ such that

$$|[a, u]| \geq C(v|a - \lambda_0\mathbf{1}|v^* + w|a - \lambda_0\mathbf{1}|w^*) \tag{4}$$

holds true? In this paper, we give an answer to this question in the case when a is a normal operator (see Theorems 5.6, 6.4). It turns out that if $\mathcal{M}$ is an infinite factor, then the constant C can be chosen arbitrarily close to 1, just as in the case of self-adjoint a. However, in the case when $\mathcal{M}$ is a finite factor, the situation changes. For II_1-factors the optimal constant C turns out to be equal to $\frac{\sqrt{3}}{2}$ and for I_n-factors appropriate upper and lower bounds on the optimal constant are given by $\Lambda_n \leq C \leq \frac{1}{2}\widetilde{\Lambda}_n$ (see (10) and (11) for definitions of these constants and (12) for estimates). We summarize above results in the following theorem.

Theorem 1.2 (see Theorems 5.6, 6.4)**.** *Let $\mathcal{M}$ be a factor and let $a \in S(\mathcal{M})$ be normal. Then there is a $\lambda_0 \in \mathbb{C}$ and unitaries $u, v, w \in \mathcal{U}(\mathcal{M})$ such that*

$$|[a, u]| \geq C\,(v|a - \lambda_0\mathbf{1}|v^* + w|a - \lambda_0\mathbf{1}|w^*) \tag{5}$$

for some constant $C > 0$ independent of a. Moreover

(1) when $\mathcal{M}$ is a I_n-factor, $n < \infty$, the optimal constant satisfies $\Lambda_n \leq C \leq \frac{1}{2}\widetilde{\Lambda}_n$.
(2) when $\mathcal{M}$ is a II_1-factor, the optimal constant is $C = \frac{\sqrt{3}}{2}$.
(3) when $\mathcal{M}$ is an infinite factor, we can choose C arbitrarily close to 1.

This theorem can be applied to obtain norm estimates for derivations $\delta : \mathcal{M} \to \mathcal{M}_*$ and extend results of [3]. Specifically, we consider the case that $\mathcal{M}$ is finite, and τ is a faithful normal tracial state on $\mathcal{M}$. In this case $\mathcal{M}_*$ is isomorphic to $L_1(\mathcal{M}, \tau)$ (see e.g. [26, Lemma 2.12 and Theorem 2.13]). As an application of inequality (3), it was proved in [3, Theorem 1.1] that, for $a = a^* \in L_1(\mathcal{M}, \tau)$, we have

$$\|\delta_a\|_{\mathcal{M} \to L_1(\mathcal{M},\tau)} = 2 \min_{z \in Z(S(\mathcal{M}))} \|a - z\|_1 \tag{6}$$

(here $Z(S(\mathcal{M}))$ denotes the center of $S(\mathcal{M})$) and that the minimum is attained at a self-adjoint element $c_a = c_a^* \in L_1(\mathcal{M}, \tau) \cap Z(S(\mathcal{M}))$. In the present paper, using Theorem 1.2, we show that for a finite factor $\mathcal{M}$ and for an arbitrary normal measurable $a \in L_1(\mathcal{M}, \tau)$, the estimate

$$\sqrt{3} \min_{z \in \mathbb{C}} \|a - z\|_1 \leq \|\delta_a\|_{\mathcal{M} \to L_1(\mathcal{M},\tau)} \leq 2 \min_{z \in \mathbb{C}} \|a - z\|_1 \tag{7}$$

holds (see Theorem 7.3). In Section 7 we show that the estimates given in (7) are sharp. In particular, in Theorem 7.3 we demonstrate that for any finite II_1-factor $\mathcal{M}$ there exists a normal $a \in \mathcal{M}$ such that the derivation δ_a is non-zero and satisfies $\|\delta_a\|_{\mathcal{M} \to L_1(\mathcal{M},\tau)} = \sqrt{3}\min_{z\in\mathbb{C}} \|a - z\|_1$, whereas it follows from Theorem 6.4 and [3, Theorem 3.1] that for any infinite factor $\mathcal{M}$ formula (6) holds for an arbitrary normal $a \in L_1(\mathcal{M}, \tau)$.

Finally, we remark that (7) is in fact an estimate for the L_1-diameter of the unitary orbit $\mathcal{O}(a) = \{uau^* : u \in \mathcal{U}(\mathcal{M})\}$ of a as $\mathrm{Diam}_{L_1(\mathcal{M},\tau)}(\mathcal{O}(a)) = \|\delta_a\|_{\mathcal{M} \to L_1(\mathcal{M},\tau)}$, see end of Section 7.

1.1. **Structure and overview.** In Section 2 we introduce standard terminology, recall the definitions of (locally) measurable operators and prove Proposition 2.1 and Theorem 2.2 that extend some results to locally measurable operators. In Section 3 we introduce the constants Λ_n and $\widetilde{\Lambda}_n$ for $n \in \mathbb{N} \cup \{\infty\}$ that will be used throughout the paper. In Section 4 our main result is Theorem 4.3, which is closely related to the constants Λ_n and to the operator inequality (4). In Section 5 we use this result to obtain Theorem 5.6 which establishes the operator inequality of Theorem 1.2 for normal elements in finite factors. In Section 6 we obtain the inequality of Theorem 1.2 for normal locally measurable operators affiliated with an infinite factor, see Theorem 6.4. In Section 7 we apply our results to obtain the estimate (7) for the norm of derivations $\delta_a : \mathcal{M} \to L_1(\mathcal{M}, \tau)$ for normal $a \in L_1(\mathcal{M}, \tau)$, and we show the given bounds are optimal in some cases. In the Appendix we prove two technical results regarding the constants Λ_n and $\widetilde{\Lambda}_n$. In particular, Theorem A.1 determines the exact value of Λ_n for $n \neq 4$.

2. Preliminaries

We establish notation on von Neumann algebras and (locally) measurable operators (for a thorough discussion of these topics we refer to [10, 12]). Furthermore, we prove two results, Proposition 2.1 and Theorem 2.2, which generalize a known result (a type of triangle inequality for operators) to locally measurable operators.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H with unit $\mathbf{1}$. We let $\mathcal{U}(\mathcal{M})$ be the group of unitaries in $\mathcal{M}$, let $\mathcal{P}(\mathcal{M})$ be the lattice of projections in $\mathcal{M}$ and let $Z(\mathcal{M})$ be the center of $\mathcal{M}$.

Recall that two projections $e, f \in \mathcal{M}$ are called *Murray-von Neumann equivalent* (denoted by $e \sim f$) if there exists an element $u \in \mathcal{M}$ such that $u^*u = e$ and $uu^* = f$. A projection $p \in \mathcal{M}$ is called *finite*, if the conditions $q \leq p$ and $q \sim p$ imply that $q = p$.

Let $x : \mathrm{dom}(x) \to H$ be a densely defined closed linear operator (the domain $\mathrm{dom}(x)$ of x is a linear subspace in H). Then x is said to be *affiliated* with $\mathcal{M}$ if $yx \subset xy$ for all y from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. A linear operator x affiliated with $\mathcal{M}$ is called *measurable* with respect to $\mathcal{M}$ if $\chi_{(\lambda,\infty)}(|x|)$ is a finite projection for some $\lambda > 0$. Here $\chi_{(\lambda,\infty)}(|x|)$ is the spectral projection of $|x|$ corresponding to the interval $(\lambda, +\infty)$. We denote the set of all measurable operators by $S(\mathcal{M})$. Clearly, $\mathcal{M}$ is a subset of $S(\mathcal{M})$. It is clear that if $\mathcal{M}$ is a factor of type I or III then $S(\mathcal{M}) = \mathcal{M}$.

Let $x, y \in S(\mathcal{M})$. It is well known that $x + y$ and xy are densely-defined and preclosed operators [10]. We define the *strong sum* respectively the *strong product* of x and y as the closures of these operators, which we simply also denote by $x+y$ and xy respectively. When $S(\mathcal{M})$ is equipped with the operation of strong sum, operation of strong product, and the $*$-operation, it becomes a unital $*$-algebra over $\mathbb{C}$. It is clear that $\mathcal{M}$ is a $*$-subalgebra of $S(\mathcal{M})$. Moreover, in the case that $\mathcal{M}$ is finite, every operator affiliated with $\mathcal{M}$ becomes measurable. In particular, the set of all affiliated operators then forms a $*$-algebra, which coincides with $S(\mathcal{M})$. Following [19, 20], in the case when the von Neumann algebra $\mathcal{M}$ is finite, we refer to the algebra $S(\mathcal{M})$ as the Murray-von Neumann algebra associated with $\mathcal{M}$.

Let $\mathcal{M}$ be semi-finite and let τ be a faithful normal semi-finite trace on $\mathcal{M}$. A linear operator x affiliated with $\mathcal{M}$ is called *τ-measurable* with respect to $\mathcal{M}$ if $\tau(\chi_{(\lambda,\infty)}(|x|)) < \infty$ for some $\lambda > 0$. We denote the set of all τ-measurable operators by $S(\mathcal{M}, \tau)$. The set $S(\mathcal{M}, \tau)$ is a $*$-subalgebra of $S(\mathcal{M})$ that contains $\mathcal{M}$. Consider the topology t_τ of convergence in measure or *measure topology* on $S(\mathcal{M}, \tau)$, which is defined by the following neighborhoods of zero:

$$N(\varepsilon, \delta) = \{x \in S(\mathcal{M}, \tau) : \exists\, e \in \mathcal{P}(\mathcal{M}),\ \tau(\mathbf{1} - e) \leq \delta,\ xe \in \mathcal{M},\ \|xe\|_{\mathcal{M}} \leq \varepsilon\},$$

where ε, δ are positive numbers. The algebra $S(\mathcal{M}, \tau)$ equipped with the measure topology is a topological $*$-algebra and F-space [10].

A linear operator x affiliated with $\mathcal{M}$ is called *locally measurable* with respect to $\mathcal{M}$ if there exist increasing central projections (p_n) in $\mathcal{P}(Z(\mathcal{M}))$ converging strongly to $\mathbf{1}$, and such that $xp_n \in S(\mathcal{M})$. The set $LS(\mathcal{M})$ of locally measurable operators forms a $*$-algebra with respect to the operations of a strong sum and a strong product. It is clear that if $\mathcal{M}$ is a factor then $LS(\mathcal{M}) = S(\mathcal{M})$.

Let $x \in LS(\mathcal{M})$. Denote by $\mathbf{l}(x)$ - the left carrier of x, by $\mathbf{r}(x)$ - the right carrier of x and $\mathbf{s}(x) = \mathbf{l}(x) \vee \mathbf{r}(x)$. If $x = u|x|$ is the polar decomposition of x, then $\mathbf{l}(x) = uu^*$ and $\mathbf{r}(x) = u^*u$. We denote $\Re(x) = \frac{x+x^*}{2}$ and $\Im(x) = \frac{x-x^*}{2i}$ for respectively the real and imaginary part of x. For a self-adjoint $x \in LS(\mathcal{M})$ we denote by x_+ (respectively, x_-) its positive (respectively negative) part, defined by $x_+ = \frac{x+|x|}{2}$ (respectively, $x_- = -\frac{x-|x|}{2}$). We note that x_- and x_+ are orthogonal, that is $x_-x_+ = 0$.

We require Theorem 2.2 which states a triangle inequality for operators $x \in LS(\mathcal{M})$. The statement is similar to [1, Theorem 2.2] where for operators $x \in \mathcal{M}$ the result was shown with partial isometries instead of isometries (see also [12, Lemma 4.3] and [15, Lemma 4.15]). To prove Theorem 2.2, we will need the following statement which is similar to [1, Proposition 2.1]. Here, $v \in \mathcal{M}$ is called an *isometry* if $v^*v = \mathbf{1}$.

Proposition 2.1. *For each $x \in LS(\mathcal{M})$ there is an isometry $v \in \mathcal{M}$ such that $\Re(x)_+ \leq v|x|v^*$.*

Proof. Let $p = \mathbf{s}(\Re(x)_+)$, $a = p(x + |x|)$. Then clearly $\mathbf{l}(a) \leq p$. We show $p = \mathbf{l}(a)$. Put $r = p - \mathbf{l}(a)$ so that $0 = ra = rar = rxr + r|x|r$. Taking the real part of this equation gives $0 = r\Re(x)r + r|x|r$. Since $r \leq p$ we have $r\Re(x)_-r = 0$ and therefore $r\Re(x)r = r\Re(x)_+r$. Then $0 = r\Re(x)r + r|x|r = r\Re(x)_+r + r|x|r$ and hence $r\Re(x)_+r = 0$. Then as $(\Re(x)_+^{\frac{1}{2}}r)^*(\Re(x)_+^{\frac{1}{2}}r) = r\Re(x)r = 0$, we obtain $\Re(x)_+^{\frac{1}{2}}r = 0$ and hence $\Re(x)_+r = 0$. Therefore, $\Re(x)_+(\mathbf{1} - r) = \Re(x)_+$ which shows $(\mathbf{1} - r) \geq \mathbf{s}(\Re(x)_+) = p$ and we conclude $r = 0$, i.e. $p = \mathbf{l}(a)$.

Let $a = w|a|$ be the polar decomposition of a. Then $ww^* = p$. Put $q = w^*w$ and $s = (\mathbf{1} - q) \wedge p$. We show $s = 0$. Indeed $as = aqs = 0$, thus $s(x + |x|)s = sas = 0$ and taking the real part of this equation gives $s\Re(x)s + s|x|s = 0$. As $s \leq p$ we have $s\Re(x)_- s = 0$ so that $s\Re(x)s = s\Re(x)_+ s$. Again, by the same arguments as before, this implies $s\Re(x)_+ s = 0$ and subsequently $(\mathbf{1}-s) \geq p$. Thus $s \leq (\mathbf{1}-p) \wedge p = 0$.

Let $(\mathbf{1}-p)(\mathbf{1}-q) = w_0|(\mathbf{1}-p)(\mathbf{1}-q)|$ be the polar decomposition of $(\mathbf{1}-p)(\mathbf{1}-q)$. Then $w_0 w_0^* \leq \mathbf{1}-p$ and $w_0^* w_0 \leq \mathbf{1}-q$. Moreover, if $t = \mathbf{1}-q-w_0^*w_0 = \mathbf{1}-q-\mathbf{r}((\mathbf{1}-p)(\mathbf{1}-q))$ then we see $(\mathbf{1}-q)t = t$ and

$$(\mathbf{1}-p)t = ((\mathbf{1}-p)(\mathbf{1}-q))t = 0 \Rightarrow t \leq p \Rightarrow t \leq s = 0.$$

Thus we obtain the equality $w_0^* w_0 = \mathbf{1}-q$ and obtain that $v = w + w_0$ is an isometry in $\mathcal{M}$.

The inequality $\Re(x)_+ \leq v|x|v^*$ is proved in the same way as in the proof of [1, Proposition 2.1] (the monotonicity of the square root function follows from [10, Corollary 2.2.28]). □

The proof of Theorem 2.2 is exactly the same as the proof of [1, Theorem 2.2], but instead of [1, Proposition 2.1] we use Proposition 2.1 above. We include the proof for completeness.

Theorem 2.2. *For any $x, y \in LS(\mathcal{M})$ there are isometries $v, w \in \mathcal{M}$ such that*

$$|x+y| \leq v|x|v^* + w|y|w^*.$$

Proof. We write the polar decomposition $x + y = u|x+y|$. Then

$$|x+y| = \frac{1}{2}(u^*(x+y) + (x+y)^*u) = \Re(u^*x) + \Re(u^*y) \tag{8}$$

Furthermore, $|u^*x| = (x^*u^*ux)^{\frac{1}{2}} \leq \|u\|(x^*x)^{\frac{1}{2}} \leq |x|$ and similarly $|u^*y| \leq |y|$. Now apply Proposition 2.1 to u^*x and to u^*y to obtain isometries $v, w \in \mathcal{M}$ so that

$$|x+y| = \Re(u^*x) + \Re(u^*y) \leq v|u^*x|v^* + w|u^*y|w^* \leq v|x|v^* + w|y|w^* \tag{9}$$

□

3. Constants Λ_n and $\widetilde{\Lambda}_n$

For $n \in \mathbb{N}$ we denote by (Ω_n, μ_n) the set $\{1, 2, \ldots, n\}$ equipped with the normalized counting measure, and by $(\Omega_\infty, \mu_\infty)$ we denote the interval $[0,1]$ equipped with Lebesgue measure. We will moreover write $S(\Omega_n)$ for the set of complex measurable functions on Ω_n, which is simply the collection of all n-tuples of complex numbers. We write Aut_n for the automorphism group of (Ω_n, μ_n), $n \in \mathbb{N} \cup \{\infty\}$, where automorphism is defined as follows:

Definition 3.1. *Let (X_1, μ_1) and (X_2, μ_2) be measure spaces. We will say that a map T is an isomorphism between X_1 and X_2 if T is a measurable bijective map $T : N_1 \to N_2$ between two sets $N_1 \subseteq X_1$ and $N_2 \subseteq X_2$ of full measure, and such that moreover T^{-1} is also measurable, and $\mu_1 \circ T^{-1} = \mu_2$. Whenever $(X_1, \mu_1) = (X_2, \mu_2)$ we will call T an automorphism.*

Let $n \in \mathbb{N} \cup \{\infty\}$. We now introduce two constant Λ_n and $\widetilde{\Lambda}_n$ as follows. Let $g \in S(\Omega_n)$, $T \in \mathrm{Aut}_n$, $z \in \mathbb{C}$, and put

$$\Lambda(g, T, z) = \operatorname{ess\,inf} \frac{|g - g \circ T|}{|g - z| + |g \circ T - z|},$$

where we assume $\frac{0}{0} = 1$. By the triangle inequality we have $|g - g \circ T| \leq |g - z| + |g \circ T - z|$ which shows $\Lambda(g, T, z) \leq 1$ for all g, T, z. We put

$$\Lambda(g) = \sup\{\Lambda(g, T, z) : \ T \in \mathrm{Aut}_n, z \in \mathbb{C}\}$$

and define Λ_n by

$$\Lambda_n = \inf_{g \in S(\Omega_n)} \Lambda(g). \tag{10}$$

For $n > 1$ we define $\widetilde{\Lambda}_n$ by setting

$$\widetilde{\Lambda}_n = \begin{cases} 2 & \text{if } n = 2, \ n = 4 \\ \sqrt{3} & \text{if } n = 3k, \\ \frac{2\sqrt{3}}{\sqrt{\frac{3k-3}{3k+1} + \frac{3k+3}{3k+1}}} & \text{if } n = 3k+1, \ n \neq 4 \\ \frac{2\sqrt{3}}{\sqrt{\frac{3k+6}{3k+2} + \frac{3k}{3k+2}}} & \text{if } n = 3k+2, \\ \sqrt{3} & \text{if } n = \infty. \end{cases} \tag{11}$$

In the Appendix we will prove two results on the constants Λ_n and $\widetilde{\Lambda}_n$. In Theorem A.1 we will precisely determine Λ_n for all values except for $n = 4$. It turns out that

$$\Lambda_1 = \Lambda_2 = 1, \quad \text{and} \quad \frac{\sqrt{3}}{2} \leq \Lambda_4 \leq 1, \quad \text{and} \quad \Lambda_n = \frac{\sqrt{3}}{2} \text{ for } n \notin \{1, 2, 4\}. \tag{12}$$

We observe that this implies that $2\Lambda_n \leq \widetilde{\Lambda}_n$ for $n > 1$ with equality when $n \equiv 0 \mod 3$ or $n = \infty$ and that moreover $\lim_{n\to\infty} 2\Lambda_n = \sqrt{3} = \lim\limits_{n\to\infty} \widetilde{\Lambda}_n$.

We denote the diameter of a set $A \subseteq \mathbb{C}$ by $\mathrm{Diam}(A) := \sup_{z,w\in A} |z - w|$. In Lemma A.2 we will show for $n > 1$ that there exists $g \in L_\infty(\Omega_n)$ with $\mathrm{Diam}(g(\Omega_n)) = 1$ and $\widetilde{\Lambda}_n = \sup_{z\in\mathbb{C}} \frac{1}{\|g-z\|_1}$, which will be used throughout the text.

4. Technical result

This section is devoted to the proof of Theorem 4.3, which is closely connected to the operator inequality (4) and to the constants Λ_n. To fully state the result we first give the following definition:

Definition 4.1. *Let $z \in \mathbb{C}$, $0 \leq \alpha \leq \pi$. The sets $A, B \subset \mathbb{C}$ will be called (z, α)-**conjugate** if there are two lines in $\mathbb{C}$ that intersect at the point z at an angle α, such that the sets A and B lie in opposite closed corners with the vertex z and the magnitude α (see Fig. 1)*

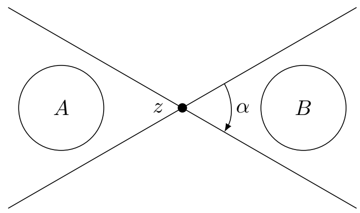

FIGURE 1. Two (z, α)-conjugate sets A and B are depicted.

Remark 4.2. Let the sets A, B be (z, α)-conjugate, $a \in A$, $b \in B$. It is easy to see that

$$|a - b| \geq (|a - z| + |b - z|) \cos \frac{\alpha}{2}.$$

Indeed, it is enough to consider the projections of points a, b on the bisector of the angle α.

Theorem 4.3. *Let $g \in S(\Omega_n)$, $n \in \mathbb{N} \cup \{\infty\}$. Then there exists a $z_0 \in \mathbb{C}$ and an automorphism T of Ω_n such that*

$$|g \circ T - g| \geq \frac{\sqrt{3}}{2}\big(|g - z_0| + |g \circ T - z_0|\big). \tag{13}$$

i.e.

$$\Lambda(g) \geq \frac{\sqrt{3}}{2}. \tag{14}$$

Moreover, the set Ω_n can be partitioned into disjoint measurable sets as follows:

(i) if n is even or $n = \infty$ then there is a partition $\{X_1\} \cup \{X_2^{m,i} : 1 \leq m, 1 \leq i \leq 2\}$ so that $g(X_1) \subset \{z_0\}$, $\mu_n(X_2^{m,1}) = \mu_n(X_2^{m,2})$ and the sets $g(X_2^{m,1})$, $g(X_2^{m,2})$ are $(z_0, \frac{\pi}{3})$-conjugate for $m = 1, 2, \ldots$; Moreover, denoting $X_2 = \Omega_n \setminus X_1$ we have that $T^k|_{X_k} = \mathrm{Id}_{X_k}$ for $k = 1, 2$.

(ii) if n is odd then there is a partition X_1, X_2, X_3, X_5, so that $T^k|_{X_k} = Id_{X_k}$, $k = 1, 2, 3, 5$.

If $n < \infty$ then there exists $z_0 \in \mathbb{C}$ and $T \in Aut_n$ so that

$$\Lambda(g, T, z_0) = \Lambda(g). \tag{15}$$

The above theorem relates to the operator inequality (4) through functional calculus. This is best visible in the case of finite-dimensional factors, see Theorem 5.1. Furthermore, we note that Theorem 4.3 provides a lower bound on the constants Λ_n. Indeed, given $g \in S(\Omega_n)$ the obtained z_0, T are such that $\Lambda(g, T, z_0) \geq \frac{\sqrt{3}}{2}$. Hence $\Lambda_n \geq \frac{\sqrt{3}}{2}$ for all $n \in \mathbb{N} \cup \{\infty\}$. In the Appendix, Theorem A.1, it is proved that in fact $\Lambda_n = \frac{\sqrt{3}}{2}$ for $n = 3$ and $n \geq 5$. This means that, for these values of n, the constant $\frac{\sqrt{3}}{2}$ in the above theorem is best possible (i.e. maximal so that for all $g \in S(\Omega_n)$ there exist z_0, T satisfying (13)).

The proof of Theorem 4.3 is somewhat technical and requires two lemmas, Lemma 4.4 and Lemma 4.5 We give a sketch of the proof. Given a measurable function $g : \Omega_n \to \mathbb{C}$ we first use Lemma 4.5 to locate a point $z_0 \in \mathbb{C}$, and divide the plane into 6 components by drawing 3 lines intersecting in z_0 making angles of $\frac{2\pi}{6}$. The way we do this is such that the measure of the inverse image of g of opposing components is equal. We can then construct an automorphism T by just mapping the inverse image of g of each component to the inverse image of its opposing component. For all $\omega \in \Omega$, we then obtain the estimate $\angle g(w), z_0, g(T(w)) \geq \frac{2\pi}{3}$ for the angle. Lemma 4.4 will then imply that (13) holds true. In the actual proof of Theorem 4.3 some difficulties arise with the boundaries of the components, and particularly for the case that we are dealing with the measure space Ω_n with n odd. Because of this reason, it is necessary to consider multiple cases in the proof.

The following lemma gives for complex numbers z_0, z_1, z_2 a sufficient condition for

$$|z_1 - z_2| \geq \frac{\sqrt{3}}{2}(|z_1 - z_0| + |z_2 - z_0|) \tag{16}$$

to hold, namely when the angle satisfies $\angle z_1 z_0 z_2 \geq \frac{2\pi}{3}$. Equation (16) can also be described geometrically as saying that the point z_0 lies in the ellipse with foci z_1 and z_2 and eccentricity $\frac{\sqrt{3}}{2}$.

Lemma 4.4. *Let $z_0, z_1, z_2 \in \mathbb{C}$ be points in the plane, and consider the triangle $\triangle z_0 z_1 z_2$. Denote $a = |z_1 - z_2|$, $b = |z_1 - z_0|$, $c = |z_2 - z_0|$, and $\alpha = \angle z_1 z_0 z_2$. If $\alpha \geq \frac{2\pi}{3}$ then*

$$a \geq \frac{\sqrt{3}}{2}(b + c).$$

Proof. According to the cosine theorem we have

$$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$

Since $\cos \alpha \leq -\frac{1}{2}$ and $b^2 + c^2 \geq 2bc$ we obtain

$$4a^2 \geq 4(b^2 + c^2 + bc) \geq 3b^2 + 3c^2 + 6bc = 3(b + c)^2$$

which shows the result. □

The following lemma is used, for a given function $g \in S(\Omega_n)$, to choose the point $z_0 \in \mathbb{C}$ adequately such that (13) holds for some automorphism T that we will later determine. The point $z_0 \in \mathbb{C}$ should be thought of as the center (or rather *a* center) of the image of g. In Lemma 4.5 we have identified $\mathbb{C}$ with $\mathbb{R}^2$ and the point $z_0 \in \mathbb{C}$ is represented as a vector $\mathbf{z}_0 \in \mathbb{R}^2$. This vector $\mathbf{z}_0$ is chosen together with three affine hyperplanes (i.e. lines) through $\mathbf{z}_0$ that are represented by unit vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ orthogonal to those affine hyperplanes. The unit vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ moreover make angles $\angle \mathbf{v}_i 0 \mathbf{v}_j$ for $i \neq j$ of $\frac{2\pi}{3}$ (this means that the affine hyperplances intersect at angles of $\frac{2\pi}{6}$). To each of the affine hyperplanes correspond two closed halfspaces. The lemma tells us that $\mathbf{z}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ can be chosen in such a way that the inverse image of g of each of these closed halfspaces has measure larger or equal to $\frac{1}{2}$. This explains why we think of $\mathbf{z}_0$ as a center of the image of g. Namely, for all three affine hyperplanes it must hold that an equal portion of the domain is mapped to each side (or possibly on the affine hyperplane). However, we remark that such a 'center point' $\mathbf{z}_0$ with the above properties does not need to be unique.

Lemma 4.5. *Let (Ω, μ) be a probability space and let g be a measurable $\mathbb{R}^2$-valued function. Then, there exists a point $\mathbf{z}_0 \in \mathbb{R}^2$, unit vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ with angles $\angle \mathbf{v}_1 0 \mathbf{v}_2 = \angle \mathbf{v}_2 0 \mathbf{v}_3 = \angle \mathbf{v}_3 0 \mathbf{v}_1 = \frac{2\pi}{3}$ so that for $i = 1, 2, 3$, denoting $a_i := \langle \mathbf{z}_0, \mathbf{v}_i \rangle$, we have*

$$m_i^L := \mu\Big(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}_i \rangle \leq a_i\}\Big) \geq \frac{1}{2}, \qquad m_i^R := \mu\Big(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}_i \rangle \geq a_i\}\Big) \geq \frac{1}{2}.$$

For $i = 1, 2, 3$ we point out that $m_i^L + m_i^R = 1$ holds if and only if $\mu(\{\omega \in \Omega : \langle g(w), \mathbf{v}_i \rangle = a_i\}) = 0$.

Proof. We first prove the result for the case that g is bounded. Denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and for $t \in \mathbb{T}$ set $\mathbf{v}(t) = (\cos(t), \sin(t))$ and define

$$\Omega(t, r) = \{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \leq r\}, \; r \in \mathbb{R},$$

$$A(t) = \left\{ r \in \mathbb{R} : \frac{1}{2} \leq \mu(\Omega(t, r)) \right\},$$

$$a(t) = \inf A(t).$$

If $r_n \downarrow a(t)$ and $\frac{1}{2} \leq \mu(\Omega(t, r_n))$ then $\Omega(t, r_1) \supset \Omega(t, r_2) \supset \dots$ and $\Omega(t, a(t)) = \bigcap_n \Omega(t, r_n)$. Hence,

$$\frac{1}{2} \leq \mu(\Omega(t, a(t))). \tag{17}$$

If $r_n \uparrow a(t)$ then $\frac{1}{2} \geq \mu(\Omega(t, r_n))$ and $\Omega(t, r_1) \subset \Omega(t, r_2) \subset \dots$ and $\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle < a(t)\} = \bigcup_n \Omega(t, r_n)$. Hence,

$$\mu\Big(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle < a(t)\}\Big) \leq \frac{1}{2} \leq \mu\Big(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle \leq a(t)\}\Big) \tag{18}$$

and therefore

$$\mu\Big(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle \geq a(t)\}\Big) \geq \frac{1}{2}. \tag{19}$$

We note that it follows from the definition of a that

$$a(t+\pi) = -\sup\Big\{r \in \mathbb{R} : \frac{1}{2} \leq \mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle \geq r\})\Big\} \tag{20}$$

since

$$\Omega(t+\pi, r) = \{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle \geq -r\}, \ r \in \mathbb{R}.$$

Hence, we obtain by (19), (20) and by properties of the supremum that $a(t) \leq -a(t+\pi)$ for all $t \in \mathbb{T}$ since $a(t) \in \Big\{r \in \mathbb{R} : \frac{1}{2} \leq \mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle \geq r\})\Big\}$. Moreover, in the second inequality of (18), replacing t by $t+\pi$ we obtain

$$\frac{1}{2} \leq \mu\Big(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle \geq -a(t+\pi)\}\Big). \tag{21}$$

Hence, for any $t \in \mathbb{T}$, and any $b \in [a(t), -a(t+\pi)]$ we obtain using (18) and (21) that

$$\frac{1}{2} \leq \mu\Big(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle \leq b\}\Big), \qquad \frac{1}{2} \leq \mu\Big(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t)\rangle \geq b\}\Big). \tag{22}$$

We show that the function a is continuous. Indeed, let $\varepsilon > 0$, and choose $\delta > 0$ such that $\|\mathbf{v}(t) - \mathbf{v}(s)\|_2 < \varepsilon$ for all $t, s \in \mathbb{T}$ with $\mathrm{dist}(s,t) < \delta$. Now, fix $t, s \in \mathbb{T}$ with $\mathrm{dist}(t,s) < \delta$. Then for $\omega \in \Omega$ we have

$$|\langle g(\omega), \mathbf{v}(t)\rangle - \langle g(\omega), \mathbf{v}(s)\rangle| \leq \|g\|_\infty \|\mathbf{v}(t) - \mathbf{v}(s)\|_2 < \varepsilon \|g\|_\infty.$$

But this means for $r \in \mathbb{R}$ that

$$\{\omega \in \Omega : \ \langle g(\omega), \mathbf{v}(t)\rangle \leq r\} \subseteq \{\omega \in \Omega : \ \langle g(\omega), \mathbf{v}(s)\rangle \leq r + \varepsilon\|g\|_\infty\}.$$

This implies in particular that

$$\frac{1}{2} \leq \mu\Big(\{\omega : \ \langle g(\omega), \mathbf{v}(t)\rangle \leq a(t)\}\Big) \leq \mu\Big(\{\omega : \ \langle g(\omega), \mathbf{v}(s)\rangle \leq a(t) + \varepsilon\|g\|_\infty\}\Big)$$

so that $a(s) \leq a(t) + \varepsilon\|g\|_\infty$. By symmetry of s and t we obtain similarly $a(t) \leq a(s) + \varepsilon\|g\|_\infty$, which implies $|a(t) - a(s)| < \varepsilon\|g\|_\infty$ and shows the continuity of a.

Now, for $t \in \mathbb{T}$ and $b \in \mathbb{R}$ consider the line

$$L(t,b) = \{\mathbf{w} \in \mathbb{R}^2 : \ \langle \mathbf{w}, \mathbf{v}(t)\rangle = b\} = b\mathbf{v}(t) + \mathbb{R}\mathbf{v}(t + \frac{\pi}{2}).$$

For $s \neq t \mod \pi$, the lines $L(s)$ and $L(t)$ intersect at a unique point $\mathbf{w}(L(s,b), L(t,c))$. In particular there is a $r \in \mathbb{R}$ such that

$$\mathbf{w}(L(s,b), L(t,c)) = b\mathbf{v}(s) + r\mathbf{v}(s + \frac{\pi}{2}).$$

Therefore $c := \langle \mathbf{w}(L(s,b), L(t,c)), \mathbf{v}(t)\rangle = b\langle \mathbf{v}(s), \mathbf{v}(t)\rangle + r\langle \mathbf{v}(s+\frac{\pi}{2}), \mathbf{v}(t)\rangle$ so that $r = \frac{c - b\langle \mathbf{v}(s), \mathbf{v}(t)\rangle}{\langle \mathbf{v}(s+\frac{\pi}{2}), \mathbf{v}(t)\rangle}$ and thus

$$\mathbf{w}(L(s,b), L(t,c)) = b\mathbf{v}(s) + \frac{c - b\langle \mathbf{v}(s), \mathbf{v}(t)\rangle}{\langle \mathbf{v}(s+\frac{\pi}{2}), \mathbf{v}(t)\rangle}\mathbf{v}(s + \frac{\pi}{2}).$$

Let $t \in \mathbb{T}$. We are interested in finding values $b_1, b_2, b_3 \in \mathbb{R}$ such that the lines $L(t - \frac{2\pi}{3}, b_1)$, $L(t + \frac{2\pi}{3}, b_2)$ and $L(t, b_3)$ intersect at a single point. This is to say that the intersection point $\mathbf{w}(L(t - \frac{2\pi}{3}, b_1), L(t + \frac{2\pi}{3}, b_2))$ must lie on the line $L(t, b_3)$. From this we obtain the expression for b_3, namely:

$$\begin{aligned}
b_3 &:= \langle \mathbf{w}(L(t - \frac{2\pi}{3}, b_1), L(t + \frac{2\pi}{3}, b_2)), \mathbf{v}(t) \rangle \\
&= b_1 \langle \mathbf{v}(t - \frac{2\pi}{3}), \mathbf{v}(t) \rangle + \frac{b_2 - b_1 \langle \mathbf{v}(t - \frac{2\pi}{3}), \mathbf{v}(t + \frac{2\pi}{3}) \rangle}{\langle \mathbf{v}(t - \frac{\pi}{6}), \mathbf{v}(t + \frac{2\pi}{3}) \rangle} \langle \mathbf{v}(t - \frac{\pi}{6}), \mathbf{v}(t) \rangle \\
&= b_1 \cos(\frac{2\pi}{3}) + \frac{b_2 - b_1 \cos(\frac{4\pi}{3})}{\cos(\frac{5\pi}{6})} \cos(\frac{\pi}{6}) \\
&= b_1 \cos(\frac{2\pi}{3}) - \left(b_2 - b_1 \cos(\frac{4\pi}{3}) \right) \\
&= -b_1 - b_2.
\end{aligned}$$

This shows that the lines $L(t - \frac{2\pi}{3}, b_1), L(t + \frac{2\pi}{3}, b_2)$ and $L(t, b_3)$ intersect precisely when $b_1 + b_2 + b_3 = 0$.

Define $c : \mathbb{T} \to \mathbb{R}$ as $c(t) = a(t - \frac{2\pi}{3}) + a(t) + a(t + \frac{2\pi}{3})$, which is a continuous function. Now, we note that, similar to a, we have $c(t) \leq -c(t + \pi)$ for all t, so that $\int_{\mathbb{T}} c(t)dt \leq -\int_{\mathbb{T}} c(t + \pi)dt = -\int_{\mathbb{T}} c(t)dt$, and hence that $\int_{\mathbb{T}} c(t)dt \leq 0$. We can thus find a t_1 such that $c(t_1) \leq 0$. If also $0 \leq -c(t_1 + \pi)$ then we set $t_0 := t_1$. If instead $-c(t_1 + \pi) < 0$, we set $t_2 := t_1 + \pi$ and obtain $-c(t_2 + \pi) = -c(t_1) \geq c(t_1 + \pi) > 0$. By the intermediate value theorem, we then find a $t_0 \in \mathbb{T}$ such that $-c(t_0 + \pi) = 0$. Then $c(t_0) \leq -c(t_0 + \pi) = 0$.

In both cases, we found $t_0 \in \mathbb{T}$ with $c(t_0) \leq 0 \leq -c(t_0 + \pi)$. Now, as moreover $a(t) \leq -a(t + \pi)$ for all $t \in \mathbb{T}$, we can determine

$$\begin{aligned}
b_1 &\in [a(t_0 - \frac{2\pi}{3}), -a(t_0 + \frac{\pi}{3})], \\
b_2 &\in [a(t_0 + \frac{2\pi}{3}), -a(t_0 - \frac{\pi}{3})], \\
b_3 &\in [a(t_0), -a(t_0 + \pi)]
\end{aligned}$$

such that $b_1 + b_2 + b_3 = 0$. Indeed, this is possible as the sum of the left-endpoints of the intervals equals $c(t_0)$, whereas the sum of the right-endpoints of the intervals equals $-c(t_0 + \pi)$. We now set $\mathbf{v}_1 := \mathbf{v}(t_0 - \frac{2\pi}{3})$, $\mathbf{v}_2 := \mathbf{v}(t_0 + \frac{2\pi}{3})$ and $\mathbf{v}_3 := \mathbf{v}(t_0)$ and let $\mathbf{z}_0$ be the unique intersection point of the lines $L(t_0 - \frac{2\pi}{3}, b_1)$, $L(t_0 + \frac{2\pi}{3}, b_2)$ and $L(t_0, b_3)$. As $\mathbf{z}_0$ lies on each of the three lines, we obtain that $a_i := \langle \mathbf{z}_0, \mathbf{v}_i \rangle = b_i$ for $i = 1, 2, 3$. By the choice of the b_i's in the intervals, it (see (22)) now follows that the properties of the lemma are fulfilled. The last line of the lemma follows from the fact that $m_i^L + m_i^R = \mu(\Omega) + \mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}_i \rangle = a_i\})$.

The result for unbounded g follows by the following reduction to the case of bounded functions. For $j \in \mathbb{N}$ let $\Omega_j \subseteq \Omega$ be a measurable subset for which $g\chi_{\Omega_j}$ is bounded and with $\Omega_j \uparrow \Omega$. Denote $\mu_j := \frac{1}{\mu(\Omega_j)}\mu$ and $g_j := g|_{\Omega_j} \in L_\infty(\Omega_j, \mu_j)$. Applying the result of the lemma to g_j, we find $\mathbf{z}_{0,j}$ and $\mathbf{v}_{i,j}$ and $a_{i,j} = \langle \mathbf{z}_{0,j}, \mathbf{v}_{i,j} \rangle$ with the stated properties. The sequence $\mathbf{z}_{0,j}$ must be bounded. Indeed, otherwise there is an $i \in \{1, 2, 3\}$ such that for a subsequence of $(a_{i,j})_{j \geq 1}$ we have $a_{i,j} \to +\infty$. However, this would contradict $\frac{1}{2} \leq \mu_j \Big(\{\omega \in \Omega_j : \ \langle g_j(\omega), \mathbf{v}_{i,j} \rangle \geq a_{i,j}\} \Big)$. Thus, by boundedness of the sequences $(\mathbf{z}_{0,j})_{j \geq 1}$ and $(\mathbf{v}_{i,j})_{j \geq 1}$, we have for some strictly increasing sequence $(j_k)_{k \geq 1}$ in $\mathbb{N}$, that the limits $\mathbf{z}_0 := \lim_{k \to \infty} \mathbf{z}_{0,j_k}$ and $\mathbf{v}_i := \lim_{k \to \infty} \mathbf{v}_{i,j_k}$ exist. Setting $a_i := \langle \mathbf{z}_0, \mathbf{v}_i \rangle$ we also have $a_i = \lim_{k \to \infty} a_{i,j_k}$. Using (reversed) Fatou's lemma, we now obtain for $i = 1, 2, 3$ that

$$\begin{aligned}
\mu\Big(\{\omega \in \Omega : \ \langle g(\omega), \mathbf{v}_i \rangle \leq a_i\} \Big) &\geq \mu\Big(\bigcap_{K=1}^{\infty} \bigcup_{k \geq K} \{\omega \in \Omega_{j_k} : \ \langle g(\omega), \mathbf{v}_{i,j_k} \rangle \leq a_{i,j_k}\} \Big) \\
&\geq \limsup_{k \to \infty} \mu\Big(\{\omega \in \Omega_{j_k} : \ \langle g(\omega), \mathbf{v}_{i,j_k} \rangle \leq a_{i,j_k}\} \Big) \\
&\geq \limsup_{k \to \infty} \mu_{j_k}\Big(\{\omega \in \Omega_{j_k} : \ \langle g_{j_k}(\omega), \mathbf{v}_{i,j_k} \rangle \leq a_{i,j_k}\} \Big) \\
&\geq \frac{1}{2}.
\end{aligned}$$

In the same way $\mu\bigg(\{\omega \in \Omega | \langle g(\omega), \mathbf{v}_i \rangle \geq a_i\}\bigg) \geq \frac{1}{2}$ can be shown. The last line of the lemma follows as before. This proves the lemma.

□

We are now fully equipped to prove Theorem 4.3.

Proof of Theorem 4.3. By identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can apply Lemma 4.5, to obtain $\mathbf{z}_0$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and a_1, a_2, a_3 which we will use to prove the result. Without loss of generality we can moreover assume that $\mathbf{v}_1$,$\mathbf{v}_2$ and $\mathbf{v}_3$ are orientated counter-clockwise. In the proof, we distinguish cases, depending on n. We prove the result separately for the cases: (1) for n even, or $n = \infty$ and (2) for n odd,

(1). *n is even, or $n = \infty$.*

First, suppose that $n \in \mathbb{N}$ is even. Then, by the choice of the point $\mathbf{z}_0$ and of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (see Lemma 4.5) and the fact that n is even, we can for $j = 1, 2, 3$ create partitions $\{I_j^+, I_j^-\}$ of Ω_n such that $\mu_n(I_j^+) = \frac{\mu_n(\Omega_n)}{2} = \mu_n(I_j^-)$ and such that $\langle g(\omega), \mathbf{v}_j \rangle \leq a_j$ whenever $\omega \in I_j^-$ and $\langle g(\omega), \mathbf{v}_j \rangle \geq a_j$ whenever $\omega \in I_j^+$. If instead $n = \infty$ then the same is true, because of the fact that μ_n is atomless in that case. We can now define the sets

$$\begin{aligned} P_1^+ &= I_1^+ \cap I_2^- \cap I_3^-, \quad & P_1^- &= I_1^- \cap I_2^+ \cap I_3^+, \\ P_2^+ &= I_1^- \cap I_2^+ \cap I_3^-, \quad & P_2^- &= I_1^+ \cap I_2^- \cap I_3^+, \\ P_3^+ &= I_1^- \cap I_2^- \cap I_3^+, \quad & P_3^+ &= I_1^+ \cap I_2^+ \cap I_3^-, \\ P_4^+ &= I_1^+ \cap I_2^+ \cap I_3^+, \quad & P_4^- &= I_1^- \cap I_2^- \cap I_3^- \end{aligned}$$

that partition Ω_n.

We show that $g(P_4^+ \cup P_4^-) \subseteq \{\mathbf{z}_0\}$. We have that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0$ and therefore $a_1 + a_2 + a_3 = 0$. For $\omega \in I_1^+ \cap I_2^+ \cap I_3^+$ we have $\langle g(\omega), \mathbf{v}_i \rangle \geq a_i, i = 1, 2, 3$, and $\sum_{i=1}^3 \langle g(\omega), \mathbf{v}_i \rangle = 0$. Hence, $\langle g(\omega), \mathbf{v}_i \rangle = a_i, i = 1, 2, 3$. But this means precisely that $g(\omega) = \mathbf{z}_0$. Similarly $g(P_4^-) \subseteq \{\mathbf{z}_0\}$. For benefit of the reader, we have visualized the partition sets in Fig. 2.

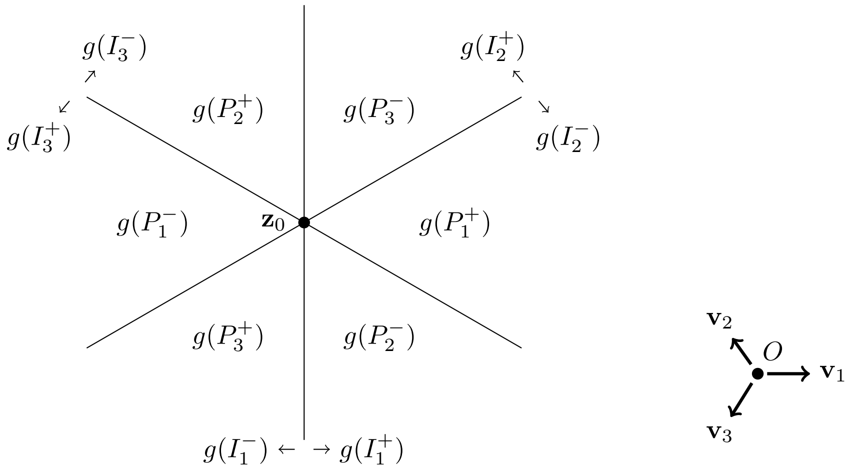

FIGURE 2. The partition sets are visualized for a universal example (any example is like this, except for shifting $\mathbf{z}_0$ and rotating the lines). The 3 lines intersect in a single point $\mathbf{z}_0$. For every line, the set Ω_n is partitioned in two sets I_i^+ and I_i^-, so that $g(I_i^+)$ and $g(I_i^-)$ lie only on one side of this line. The partition sets $P_j^{\pm}$ are then such that $g(P_j^{\pm})$ lies in one connected component (or its boundary). The sets $g(P_4^+)$ and $g(P_4^-)$ are not visualized. For these we must have $g(P_4^+ \cup P_4^-) \subseteq \{\mathbf{z}_0\}$.

We have

$$\mu_n(P_1^+ \cup P_2^- \cup P_3^- \cup P_4^+) = \mu_n(I_1^+) = \mu_n(I_1^-) = \mu_n(P_1^- \cup P_2^+ \cup P_3^+ \cup P_4^-), \tag{23}$$

$$\mu_n(P_1^- \cup P_2^+ \cup P_3^- \cup P_4^+) = \mu_n(I_2^+) = \mu_n(I_2^-) = \mu_n(P_1^+ \cup P_2^- \cup P_3^+ \cup P_4^-), \tag{24}$$

$$\mu_n(P_1^- \cup P_2^- \cup P_3^+ \cup P_4^+) = \mu_n(I_3^+) = \mu_n(I_3^-) = \mu_n(P_1^+ \cup P_2^+ \cup P_3^- \cup P_4^-), \tag{25}$$

(23)+(24):

$$\mu_n(P_3^-)+\mu_n(P_4^+)=\mu_n(P_3^+)+\mu_n(P_4^-),$$

(23)+(25):

$$\mu_n(P_2^-)+\mu_n(P_4^+)=\mu_n(P_2^+)+\mu_n(P_4^-),$$

(24)+(25):

$$\mu_n(P_1^-)+\mu_n(P_4^+)=\mu_n(P_1^+)+\mu_n(P_4^-).$$

We thus obtain that $t:=\mu_n(P_j^+)-\mu_n(P_j^-)$ is independent of $j=1,2,3,4$.

Let us assume that $t\geq 0$ so that $\mu_n(P_j^+)\geq t$. Choose $A_j\subseteq P_j^+$ with $\mu_n(A_j)=t$. We denote $X_1=(P_4^+\cup P_4^-)\setminus A_4$ and

$$X_2^{1,1}=P_1^+\setminus A_1,\quad X_2^{1,2}=P_1^-,\ X_2^{2,1}=P_2^+\setminus A_2,\ X_2^{2,2}=P_2^-,\ X_2^{3,1}=P_3^+\setminus A_3,\ X_2^{3,2}=P_3^-.$$

First, suppose that $n\in\mathbb{N}$ is even. Then $A_j=\{a_{j,1},\dots,a_{j,l}\},\ j=1,2,3,4,\ l=tn$. Fix $k=1,\dots,l$. In each triple $(a_{1,k},a_{2,k},a_{3,k})$ there will be such $i,j\in\{1,2,3\}$ that $\frac{2\pi}{3}\leq\angle g(a_{i,k}),\mathbf{z}_0,g(a_{j,k})\leq\pi$ (see Fig. 2). Let $\{q\}=\{1,2,3\}\setminus\{i,j\}$. Then $\{g(a_{i,k})\}$ and $\{g(a_{j,k})\}$, and also $\{g(a_{q,k})\}$ and $\{g(a_{4,k})\}$ form pairs of $(\mathbf{z}_0,\frac{\pi}{3})$-conjugate sets. We put $X_2^{2k+2,1}=\{a_{i,k}\},\ X_2^{2k+2,2}=\{a_{j,k}\},\ X_2^{2k+3,1}=\{a_{q,k}\},\ X_2^{2k+3,2}=\{a_{4,k}\}$ and $X_2^{m,1}=X_2^{m,2}=\emptyset$ for $m\geq 2l+4$.

We assume now that $n=\infty$. Let $\Sigma_j=\{Y_j^1,Y_j^2,\dots\}$ be a maximal system of pairwise disjoint measurable subsets of $A_j,\ j=1,2,3,4$, such that $\mu_\infty(Y_1^k)=\mu_\infty(Y_2^k)=\mu_\infty(Y_3^k)=\mu_\infty(Y_4^k)>0$ and the four $(g(Y_1^k),g(Y_2^k),g(Y_3^k),g(Y_4^k))$ is divided into two pairs of $(\mathbf{z}_0,\frac{\pi}{3})$-conjugate sets for $k=1,2,\dots$.

Put $B_j=A_j\setminus\bigcup_k Y_j^k$. Then $\mu_\infty(B_1)=\mu_\infty(B_2)=\mu_\infty(B_3)=\mu_\infty(B_4)=t_0$. Suppose that $t_0>0$. If the sets $g(B_1),g(B_2),g(B_3)$ are located on three rays emanating from $\mathbf{z}_0$ and forming angles $\frac{2\pi}{3}$ then $g(B_1),g(B_2)$ are $(\mathbf{z}_0,\frac{\pi}{3})$-conjugate sets and the same for $g(B_3),g(B_4)$. This contradicts the maximality of the above set systems Σ_j.

Otherwise, there will be such $b_1\in B_i,\ b_2\in B_j,\ i,j\in\{1,2,3\},\ i\neq j$, that $\angle g(b_1),\mathbf{z}_0,g(b_2)>\frac{2\pi}{3}$ and $g(b_1),g(b_2)$ are essential values of $g|_{B_i\cup B_j}$. Then there will be such neighborhoods V_1 and V_2 of the points $g(b_1)$ and $g(b_2)$, respectively, that V_1,V_2 are $(\mathbf{z}_0,\frac{\pi}{3})$-conjugate sets. Therefore there exist sets $Y_1\subset B_i,\ Y_2\subset B_j$ so that $\mu_\infty(Y_1)=\mu_\infty(Y_2)>0$ and $g(Y_k)\subset V_k,\ k=1,2$. Hence, $g(Y_1),g(Y_2)$ are $(\mathbf{z}_0,\frac{\pi}{3})$-conjugate sets. Let $\{q\}=\{1,2,3\}\setminus\{i,j\}$. There exists $Y_3\subset B_q,\ Y_4\subset B_4,\ \mu_\infty(Y_3)=\mu_\infty(Y_4)=\mu_\infty(Y_1)$. It is clear that $g(Y_3),g(Y_4)$ are $(\mathbf{z}_0,\frac{\pi}{3})$-conjugate sets. The presence of sets Y_1,Y_2,Y_3,Y_4 contradicts the maximality of the above systems Σ_j.

The contradiction obtained in both cases shows $t_0=0$. Therefore the system $\{X_1\}\cup\{X_2^{m,i}:1\leq m\leq 3,1\leq i\leq 2\}$ can be completed using $\Sigma_j,\ j=1,2,3,4$.

It remains to define T so that $T_{X_1}=Id_{X_1},\ T(X_2^{m,1})=X_2^{m,2},\ T(X_2^{m,2})=X_2^{m,1}$ for $m=1,2,\dots$ and such that $T^2=\mathrm{Id}_{\Omega_n}$. Then the inequality (13) follows from the Lemma 4.4.

The case that $t\leq 0$ is similar, by changing the roles of P_j^+ and P_j^-.

(2). *n is odd.*

We can for $i=1,2,3$ instead build partitions $\{I_i^+,\{\omega_i\},I_j^-\}$ of Ω_n with $\mu_n(I_i^+)=\mu_n(I_i^-)$ and such that $\langle g(\omega),\mathbf{v}_i\rangle\leq a_i$ whenever $\omega\in I_i^-$ and $\langle g(\omega),\mathbf{v}_i\rangle\geq a_i$ whenever $\omega\in I^+$ and $\langle g(\omega_i),\mathbf{v}_i\rangle=a_i$. Indeed, such ω_i exist because $|\{\omega\in\Omega_n:\ \langle g(\omega),\mathbf{v}_i\rangle\leq a_i\}|,|\{\omega\in\Omega_n:\ \langle g(\omega),\mathbf{v}_i\rangle\geq a_i\}|\geq\frac{n+1}{2}$ and therefore $\{\omega\in\Omega_n:\ \langle g(\omega),\mathbf{v}_i\rangle\leq a_i\}\cap\{\omega\in\Omega_n:\ \langle g(\omega),\mathbf{v}_i\rangle\geq a_i\}\neq\emptyset$. Denote $Y_0=\{\omega_1,\omega_2,\omega_3\}$.

Now, suppose that $\mathbf{z}_0\in g(\Omega_n)$. Then we could have chosen $\omega_1=\omega_2=\omega_3$ all equal and such that $g(\omega_i)=\mathbf{z}_0$. Then $|Y_0|=1$ and the sets $\{I_i^+,I_i^-\}$ are all partitions of $\Omega_n\setminus Y_0$ similar to (1), and we can build the corresponding automorphism T of $\Omega_n\setminus Y_0$. This completes the proof for that case by setting $T(\omega_1)=\omega_1$.

We can thus assume that $\mathbf{z}_0\notin g(\Omega_n)$ so that in particular $g(\omega_i)\neq g(\omega_j)$ for $i\neq j$ and $|Y_0|=3$. Now suppose first that $\mathbf{z}_0\in\mathrm{Conv}(g(Y_0))$. For all $i\in\{1,2,3\}$ we then have that $Y_0\cap I_i^+$ and $Y_0\cap I_i^-$ both consist of 1 element. Hence, $\mu_n(I_i^+\setminus Y_0)=\mu_n(I_i^-\setminus Y_0)$ and the partitions $\{I_i^+\setminus Y_0,I_i^-\setminus Y_0\}$ of $\Omega_n\setminus Y_0$ satisfy the same properties as (1). We thus obtain a measure preserving automorphism T of $\Omega_n\setminus Y_0$ with the same properties. Now we can set $T(\omega_1)=\omega_2,\ T(\omega_2)=\omega_3$ and $T(\omega_3)=\omega_1$, so that $\angle g(\omega_i),\mathbf{z}_0,g(T(\omega_i))=\frac{2\pi}{3}$. This finishes the proof by Lemma 4.4

Now suppose that $\mathbf{z}_0\notin\mathrm{Conv}(g(Y_0))$. Then it can be seen geometrically (for intuition see Fig. 3), that there is a unique choice of (distinct) indices $i_1,i_2,i_3\in\{1,2,3\}$ such that

$$\{\omega_{i_1}\}=Y_0\cap I_{i_2}^-\quad\{\omega_{i_3}\}=Y_0\cap I_{i_2}^+. \tag{26}$$

Now, suppose that $\omega_{i_1}\notin I_{i_3}^+$. Then as $\omega_{i_1}\neq\omega_{i_3}$ we get $\omega_{i_1}\in I_{i_3}^-$. But then as $\omega_{i_1}\in\{\omega_{i_1}\}\cap I_{i_2}^-\cap I_{i_3}^-$ we would get $g(\omega_{i_1})=\mathbf{z}_0$ by the same argument as why $g(P_4^+\cup P_4^-)\subseteq\{\mathbf{z}_0\}$ in (1). However, $\mathbf{z}_0\notin g(\Omega_n)$

by our assumption so this cannot be the case. We conclude that we must have $\omega_{i_1} \in I_{i_3}^+$. By a same argument we find that we must have $\omega_{i_3} \in I_{i_1}^-$ (Indeed, otherwise $\omega_{i_3} \in I_{i_1}^+$ so that $\omega_{i_3} \in I_{i_1}^+ \cap I_{i_2}^+ \cap \{\omega_{i_3}\}$, which would imply $g(\omega_{i_3}) = \mathbf{z}_0$, which gives a contradiction). Furthermore, we claim that $\omega_{i_2} \in I_{i_3}^+$. Indeed, if $\omega_{i_2} \in I_{i_3}^-$ then we could rearrange the indexes as $i_1' = i_2$, $i_2' = i_3$ and $i_3' = i_1$, so that we get $\{w_{i_1'}\} = \{w_{i_2}\} = Y_0 \cap I_{i_3}^- = Y_0 \cap I_{i_2'}^-$ and $\{w_{i_3'}\} = \{w_{i_1}\} = Y_0 \cap I_{i_3}^+ = Y_0 \cap I_{i_2'}^+$. This contradicts the uniqueness of the choice i_1, i_2, i_3 satisfying (26). We conclude that indeed $\omega_{i_2} \in I_{i_3}^+$. By the same argument we find $\omega_{i_2} \in I_{i_1}^-$ (Indeed, if $\omega_{i_2} \in I_{i_1}^+$ we could take the rearrangement $i_1' = i_3$, $i_2' = i_1$ and $i_3' = i_2$ to obtain $\{w_{i_1'}\} = \{w_{i_3}\} = Y_0 \cap I_{i_1}^- = Y_0 \cap I_{i_2'}^-$ and $\{w_{i_3'}\} = \{w_{i_2}\} = Y_0 \cap I_{i_1}^+ = Y_0 \cap I_{i_2'}^+$, which contradicts the uniqueness). For clarity we summarize the results:

$$\{\omega_{i_1}\} = Y_0 \cap I_{i_2}^- \quad \{\omega_{i_3}\} = Y_0 \cap I_{i_2}^+,$$
$$\{\omega_{i_2}, \omega_{i_3}\} = Y_0 \cap I_{i_1}^- \quad \{\omega_{i_1}, \omega_{i_2}\} = Y_0 \cap I_{i_3}^+.$$

We now obtain

$$\mu_n(I_{i_1}^+ \cap I_{i_2}^-) + \mu_n(I_{i_1}^+ \cap I_{i_2}^+) = \mu_n(I_{i_1}^+ \setminus \{\omega_{i_2}\}) = \mu_n(I_{i_1}^+), \tag{27}$$
$$\mu_n(I_{i_1}^+ \cap I_{i_2}^-) + \mu_n(I_{i_1}^- \cap I_{i_2}^-) = \mu_n(I_{i_2}^- \setminus \{\omega_{i_1}\}) = \mu_n(I_{i_2}^-) - \frac{1}{n}, \tag{28}$$
$$\mu_n(I_{i_2}^+ \cap I_{i_3}^-) + \mu_n(I_{i_2}^- \cap I_{i_3}^-) = \mu_n(I_{i_3}^- \setminus \{\omega_{i_2}\}) = \mu_n(I_3^-), \tag{29}$$
$$\mu_n(I_{i_2}^+ \cap I_{i_3}^-) + \mu_n(I_{i_2}^+ \cap I_{i_3}^+) = \mu_n(I_{i_2}^+ \setminus \{\omega_{i_3}\}) = \mu_n(I_{i_2}^+) - \frac{1}{n}. \tag{30}$$

Hence, by (27) + (28) we obtain $\mu_n(I_{i_1}^+ \cap I_{i_2}^+) = \frac{1}{n} + \mu_n(I_{i_1}^- \cap I_{i_2}^-)$ and by summing up (29) with (30) we obtain $\mu_n(I_{i_2}^- \cap I_{i_3}^-) = \frac{1}{n} + \mu_n(I_{i_3}^+ \cap I_{i_2}^+)$. We conclude the existences of $\omega_4 \in I_{i_1}^+ \cap I_{i_2}^+$ and $\omega_5 \in I_{i_2}^- \cap I_{i_3}^-$. Now, for the sets $P_4^+ := I_{i_1}^+ \cap I_{i_2}^+ \cap I_{i_3}^+$ and $P_4^- := I_{i_1}^- \cap I_{i_2}^- \cap I_{i_3}^-$ we have that $g(P_4^+ \cup P_4^-) \subseteq \{\mathbf{z}_0\}$ (same as in (1)), and hence $P_4^+ \cup P_4^- = \emptyset$ as $\mathbf{z}_0 \notin g(\Omega_n)$ by assumption. This means that $\omega_4 \notin I_{i_3}^+$ and $\omega_5 \notin I_{i_1}^-$. Also, as $\omega_{i_3} \in I_{i_1}^-$ and $\omega_{i_1} \in I_{i_3}^+$ we get that $\omega_4 \neq \omega_{i_3}$ and $\omega_5 \neq \omega_{i_1}$. As $\{I_i^+, \{\omega_i\}, I_i^-\}$ are partitions of Ω_n, we conclude that $\omega_4 \in I_{i_1}^+ \cap I_{i_2}^+ \cap I_{i_3}^-$ and $\omega_5 \in I_{i_1}^+ \cap I_{i_2}^- \cap I_{i_3}^-$ Denote $Y_1 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, so that by the above we have $|Y_1| = 5$ and moreover:

$$Y_1 \cap I_{i_1}^- = \{\omega_{i_2}, \omega_{i_3}\}, \qquad Y_1 \cap I_{i_2}^- = \{\omega_{i_1}, \omega_5\}, \qquad Y_1 \cap I_{i_3}^- = \{\omega_4, \omega_5\},$$
$$Y_1 \cap I_{i_1}^+ = \{\omega_4, \omega_5\}, \qquad Y_1 \cap I_{i_2}^+ = \{\omega_{i_3}, \omega_4\}, \qquad Y_1 \cap I_{i_3}^+ = \{\omega_{i_1}, \omega_{i_2}\}.$$

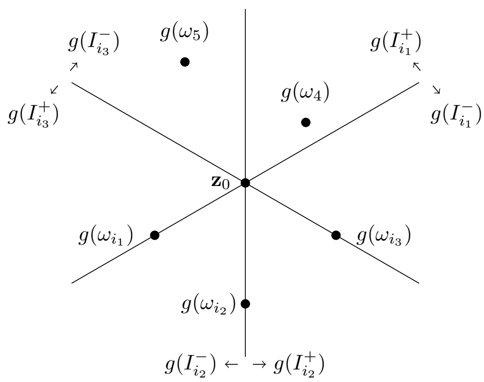

FIGURE 3. The 5 points are depicted for an example.

Now, as all these sets have size 2, we must have that

$$\mu_n(I_i^+ \setminus Y_1) = \mu_n(I_i^+) - \frac{2}{n} = \mu_n(I_i^-) - \frac{2}{n} = \mu_n(I_i^- \setminus Y_1).$$

This means that the partitions $\{I_i^+ \setminus Y_1, I_i^- \setminus Y_1\}$ of $\Omega_n \setminus Y_1$ satisfy the same properties as in (1). We can therefore find a transformation T of $\Omega_n \setminus Y_1$ with the same properties. We can now define T on Y_1 by setting $T(\omega_{i_1}) = \omega_4$, $T(\omega_4) = \omega_{i_2}$, $T(\omega_{i_2}) = \omega_5$, $T(\omega_5) = \omega_{i_3}$ and $T(\omega_{i_3}) = \omega_{i_1}$. Then $\angle g(\omega_i), \mathbf{z}_0, g(T(\omega_i)) \geq \frac{2\pi}{3}$ for all i. Appealing to Lemma 4.4 this impies $|g(w) - g(T(w))| \leq \frac{\sqrt{3}}{2}(|g(w) - \mathbf{z}_0| + |g(T(w) - \mathbf{z}_0|)$ for all $\omega \in \Omega$, which shows that (13) holds true. The inequality (14) follows from it. Furthermore, in each of the considered cases it clear how to split Ω_n into the parts X_1, X_2, X_3, X_5 (note that by construction we have $T^k(\omega) = \omega$ for some $k \in \{1,2,3,5\}$ for $\omega \in \Omega$). We prove the final statement.

Let $n < \infty$ and let (T_m), (z_m) be sequences such that $0 < \Lambda(g, T_m, z_m) \uparrow \Lambda(g)$. Then

$$\Lambda(g, T_m, z_m)^{-1}|g(\omega) - g(T_m(\omega))| \geq |g(\omega) - z_m| + |g(T_m(\omega)) - z_m| \geq |g(\omega) - z_m|$$

for any $\omega \in \Omega_n$. Let $\omega_0 \in \Omega_n$. Then

$$|g(\omega_0) - z_m| \leq \Lambda(g, T_m, z_m)^{-1} \operatorname{Diam}(g(\Omega_n)) \leq \Lambda(g, T_1, z_1)^{-1} \operatorname{Diam}(g(\Omega_n)).$$

Since $card(Aut_n) = n! < \infty$ and since $\{z \in \mathbb{C} : \ |g(\omega_0) - z| \leq \Lambda(g, T_1, z_1)^{-1} \operatorname{Diam}(g(\Omega_n))\}$ is compact, there exists a sequence (m_k) and a $z_0 \in \mathbb{C}$ such that

$$z_0 = \lim_k z_{m_k}, \ T_0 := T_{m_1} = T_{m_2} = \cdots = T_{m_k} = \dots \ .$$

Then $\Lambda(g, T_0, z_0) = \Lambda(g)$. □

5. Commutator estimates for normal operators in finite factors

The main result of this section, Theorem 5.6 below, establishes the commutator inequality (4) for normal element $a \in S(\mathcal{M})$, where $\mathcal{M}$ is a finite factor, and provides upper and lower bounds on the optimal constant $C_{\mathcal{M}}$. This yields a version of [3, Theorem 1.1], suitable for normal elements. We consider the case of I_n-factors ($n < \infty$) in Theorem 5.1) and the case of II_1-factors in Theorem 5.4 and show that the commutator inequality holds for the constant $\frac{\sqrt{3}}{2}$. The proof for II_1-factors requires two additional results, Theorem 5.2 and Lemma 5.3. Furthermore, in order to prove the upper bounds in Theorem 5.6 we provide Proposition 5.5.

Theorem 5.1. *Let $\mathcal{M} = B(\mathcal{H})$ be an I_n-factor for $n \in \mathbb{N}$. For an arbitrary normal operator $a \in \mathcal{M}$ there is a unitary $u \in \mathcal{U}(\mathcal{M})$ and a $z_0 \in \mathbb{C}$ such that*

$$|[a,u]| \geq \frac{\sqrt{3}}{2}(|a - z_0\mathbf{1}| + u|a - z_0\mathbf{1}|u^*). \tag{31}$$

Moreover, u can be chosen so that

- *when n is even there are projections p_1, p_2 such that $p_1 + p_2 = \mathbf{1}$*
- *when n is odd there are projections p_1, p_2, p_3, p_5 such that $p_1 + p_2 + p_3 + p_5 = \mathbf{1}$*

so that

$$p_k u = u p_k, \ u^k p_k = p_k, \ k = 1, 2, 3, 5.$$

If $a \in \mathcal{M}$ is such that its spectrum $\sigma(a)$ lies on a straight line, then we can obtain true equality:

$$|[a,u]| = |a - z_0\mathbf{1}| + u|a - z_0\mathbf{1}|u, \ \text{for some } u^* = u \in \mathcal{U}(\mathcal{M}), z_0 \in \mathbb{C}. \tag{32}$$

We remark that when $n = 1, 2$ every normal $a \in \mathcal{M}$ satisfies this extra condition.

Proof. Since a is a normal element on an n-dimensional Hilbert space, it follows from the spectral mapping theorem that there is a unitary $U : \mathcal{H} \to L_2(\Omega_n)$ such that $a = U^* M_g U$, where M_g is the multiplication operator on $L_2(\Omega_n)$ for some $g \in L_\infty(\Omega_n)$. Applying Theorem 4.3 to g, we find a transformation T and a $z_0 \in \mathbb{C}$ such that

$$|g \circ T - g| \geq \frac{\sqrt{3}}{2}(|g - z_0| + |g \circ T - z_0|) \tag{33}$$

together with the given partition of Ω_n consisting of the sets X_1, X_2 (when n is even) or X_1, X_2, X_3, X_5 (when n is odd) and that satisfy $T^k|_{X_k} = \mathrm{Id}_{X_k}$. Now let u_T be the Koopman operator on $L_2(\Omega_n)$ corresponding to T, i.e. $u_T f = f \circ T$. Denote $u = U^* u_T U$. Then

$$\begin{aligned}
|[a,u]| &= |u(u^* a u - a)| \\
&= |u a u^* - a| \\
&= U^*|u_T M_g u_T^* - M_g|U \\
&= U^*|M_{g\circ T} - M_g|U \\
&= U^* M_{|g\circ T - g|} U \\
&\geq \Lambda_n \left(U^*|M_g - z_0|U + U^*|M_{g\circ T} - z_0|U\right) \\
&= \Lambda_n \left(U^*|M_g - z_0|U + U^* u_T|M_g - z_0|u_T^* U\right). \\
&= \Lambda_n \left(|a - z_0| + u|a - z_0|u^*\right)
\end{aligned}$$

We now define the projections by setting $p_k = U^* \chi_{X_k} U$ which clearly satisfy the statements.

If $\sigma(a)$ lies on a straight line, then there exist scalars $\alpha, \beta \in \mathbb{C}$, $|\alpha| = 1$, such that $a_1 := \alpha(a - \beta\mathbf{1}) \in \mathcal{M}$ is self-adjoint. It follows from Theorem 1.1 that there exist $z_0 \in \mathbb{R}$ and $u = u^* \in \mathcal{U}(\mathcal{M})$ that

$$|[a, u]| = |[a_1, u]| = |a_1 - z_0\mathbf{1}| + u|a_1 - z_0\mathbf{1}|u = |a - (\beta + z_0\alpha^{-1})\mathbf{1}| + u|a - (\beta + z_0\alpha^{-1})\mathbf{1}|u.$$

□

We need the following result, which for a diffuse semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ and a normal measurable $a \in S(\mathcal{M})$ establishes an injective $*$-homomorphism F between $S[0,1]$ and $S(\mathcal{M})$ which preserves measure and is such that a lies in the image of F. Special cases of the result which follows for positive bounded elements of $\mathcal{M}$ and positive elements of $L_1(M, \tau)$ can be found in [11, Lemma 9] and in [9, Lemma 4.1] respectively.

Theorem 5.2. *Let $\mathcal{M}$ be a diffuse (i.e. atomless) von Neumann algebra with a faithful normal tracial state τ, let $a \in S(\mathcal{M})$ be a normal operator. There exists such an injective $*$-homomorphism $F : S[0,1] \to S(\mathcal{M})$ such that $a \in \mathrm{Image}(F)$ and $m(A) = \tau(F(\chi_A))$ for any measurable subset $A \subset [0,1]$ (here m is the Lebesgue measure on $[0,1]$).*

Proof. Let e be a spectral measure of the operator a defined on the σ-algebra $\mathcal{B}(\sigma(a))$ of Borel subsets in $\sigma(a)$. Then $\tau(e(\cdot))$ yields a probability measure on $\mathcal{B}(\sigma(a))$. By the spectral theorem (see [22, Theorem 13.33]), we have

$$a = \int_{\sigma(a)} \lambda de(\lambda).$$

Let X_0 be a set of eigenvalues a. It is clear that $X_0 \subset \sigma(a)$ and $card(X_0) \leq \aleph_0$. Indeed, if $t \in X_0$ then $e(\{t\}) \neq 0$ and $\sum_{t \in X_0} \tau(e(\{t\})) = \tau(e(X_0)) \leq 1$. Let $t \in X_0$. Since $\mathcal{M}$ is diffuse, it follows that in $\mathcal{M}$ there is such a chain of projections $f_s^t \uparrow_s e(\{t\})$ such that $\tau(f_s^t) = s$, $s \in Y_t := [0, \tau(e(\{t\}))]$.

Denote by f_t the spectral measure by $\mathcal{B}(Y_t)$ given by the equality

$$f_t((s_1, s_2)) = f_{s_2}^t - f_{s_1}^t.$$

We have $\tau(f_t(A)) = m(A)$ for any $a \in \mathcal{B}(Y_t)$. Let us now set

$$X = (\sigma(a) \setminus X_0) \sqcup \bigsqcup_{t \in X_0} Y_t.$$

On $\mathcal{B}(X)$, we define a spectral measure g such that

$$g|_{\mathcal{B}(\sigma(a)\setminus X_0)} = e|_{\mathcal{B}(\sigma(a)\setminus X_0)},\ g|_{\mathcal{B}(Y_t)} = f_t,\ t \in X_0,$$

and a scalar measure

$$\mu_X(A) = \tau(e(A \cap (\sigma(a) \setminus X_0))) + \sum_{t \in X_0} \mu(A \cap Y_t).$$

It follows that $(X, \mathcal{B}(X), \mu_X)$ is a Lebesgue space with an atomless probability measure. Hence, it is isomorphic to the segment $[0,1]$ equipped with Lebesgue measure m, see e.g. [6, Theorem 9.5.1].

A linear mapping $F : S(X, \mathcal{B}(X), \mu_X) \to S(\mathcal{M})$ is defined by

$$F(\varphi) = \int_X \varphi(x) dg(x)$$

for any $\varphi \in S(X, \mathcal{B}(X), \mu_X)$ (see [10, Definition 1.5.6]). We remark that $F(\chi_A) = g(A)$ for measurable $A \subseteq X$ and that $\mu_X(A) = \tau(F(\chi_A))$. Furthermore $F(\chi_A\chi_B) = F(\chi_{A\cap B}) = g(A \cap B) = g(A)g(B) = F(\chi_A)F(\chi_B)$ for measurable sets $A, B \subseteq X$. Therefore, as F is continuous with respect to the topologies of convergence in measure in $S(X, \mathcal{B}(X), \mu_X)$ and $S(\mathcal{M}, \tau)$ and since simple functions in $S(X, \mathcal{B}(X), \mu_X)$ are dense with respect to the measure topology, it follows that $F(\varphi\psi) = F(\varphi)F(\psi)$ for all $\varphi, \psi \in S(X, \mathcal{B}(X), \mu_X)$. Moreover, $F(\overline{\varphi}) = \int_X \overline{\varphi(x)} dg(x) = F(\varphi)^*$ so we find that F is a $*$-homomorphism. Now, suppose $\varphi \in S(X, \mathcal{B}(X), \mu_X)$ is such that $F(\varphi) = 0$ and $B \subseteq X$ is such that $\varphi(x) \neq 0$ for a.e. $x \in B$. Then $g(B) = F(\chi_B) = F(\frac{1}{\varphi}\chi_B)F(\varphi) = 0$, thus $\mu_X(B) = \tau(g(B)) = 0$. This shows that F is injective.

Finally, let us define the function f by setting $f(t) = t$ for $t \in \mathcal{B}(\sigma(a) \setminus X_0)$ or $t \in Y_t$. Then $f \in S(X, \mathcal{B}(X), \mu_X)$ and $F(f) = a$. □

Lemma 5.3. *Let $\mathcal{M}$ be a finite von Neumann algebra, let $a, b \in S(\mathcal{M})$ be normal operators, $z_0 \in \mathbb{C}$, $0 \leq \alpha < \pi$ and let $\sigma(a)$, $\sigma(b)$ be (z_0, α)-conjugate sets. Then*

$$v|a - b|v^* \geq (|a - z_0\mathbf{1}| + |b - z_0\mathbf{1}|)\cos\frac{\alpha}{2} \tag{34}$$

for some $v \in \mathcal{U}(\mathcal{M})$.

Proof. Since $\sigma(a)$ and $\sigma(b)$ are (z_0,α)-conjugate, the shifted sets $\sigma(a)-z_0$ and $\sigma(b)-z_0$ are $(0,\alpha)$-conjugate. We can then obtain a pair of lines as in Fig. 1, intersecting at the origin with an angle α. By rotating the complex plane around the origin we can assure that these lines are symmetric with respect to the real axis. This is to say that there exists a function $f(z)=c(z-z_0)$ with $|c|=1$ so that

$$f(\sigma(a))\subset\{z:\ -\frac{\alpha}{2}\le Arg(z)\le\frac{\alpha}{2}\},\ f(\sigma(b))\subset\{z:\ \pi-\frac{\alpha}{2}\le Arg(z)\le\pi+\frac{\alpha}{2}\}.$$

Let $a_1=f(a),\ b_1=f(b)$. We have

$$|a_1|\cos\frac{\alpha}{2}\le\Re a_1,\ |b_1|\cos\frac{\alpha}{2}\le-\Re b_1.$$

Therefore

$$(|a-z_0\mathbf{1}|+|b-z_0\mathbf{1}|)\cos\frac{\alpha}{2}=(|a_1|+|b_1|)\cos\frac{\alpha}{2}\le\Re a_1-\Re b_1=\Re(a_1-b_1)\le\Re(a_1-b_1)_+ \tag{35}$$

By Proposition 2.1, we obtain

$$\Re(a_1-b_1)_+\le v|a_1-b_1|v^*=v|a-b|v^*. \tag{36}$$

for some $v\in\mathcal{M}$ with $v^*v=\mathbf{1}$. Since $\mathbf{1}$ is a finite projection it follows that $vv^*=\mathbf{1}$, i.e. $v\in\mathcal{U}(\mathcal{M})$. Combining (35) and (36) establishes (34) □

We now prove a version of Theorem 5.1 for II$_1$-factors. Equation (37) below is slightly different from (31) as it involves a second unitary $w\in\mathcal{U}(\mathcal{M})$.

Theorem 5.4. *Let $\mathcal{M}$ be a factor of type II_1, $a\in S(\mathcal{M})$ be normal. Then there exists a $z_0\in\mathbb{C}$, $u=u^*\in\mathcal{U}(\mathcal{M})$ and $w\in\mathcal{U}(\mathcal{M})$ so that*

$$w|[a,u]|w^*\ge\frac{\sqrt{3}}{2}\cdot(|a-z_0\mathbf{1}|+u|a-z_0\mathbf{1}|u). \tag{37}$$

If $\sigma(a)$ lies on a straight line then

$$|[a,u]|=|a-z_0\mathbf{1}|+u|a-z_0\mathbf{1}|u. \tag{38}$$

Proof. Let τ be a faithful normal tracial state on $\mathcal{M}$ and let $F:S[0,1]\to S(\mathcal{M})$ be an injective $*$-homomorphism from Theorem 5.2 satisfying $a\in\text{Image}(F)$. Let $g=F^{-1}(a)$.

It follows from Theorem 4.3 that there exists z_0 such that $[0,1]$ can be divided into disjoint measurable parts $\{X_1\}\cup\{X_2^{m,i}:m\ge1,1\le i\le2\}$ so that $g(X_1)\subset\{z_0\}$, $\mu(X_2^{m,1})=\mu(X_2^{m,2})$ and the sets $g(X_2^{m,1})$, $g(X_2^{m,2})$ are $(z_0,\frac{\pi}{3})$-conjugate for $m=1,2,\dots$ (where μ is the Lebesgue measure on $[0,1]$).

Let $e=F(\chi_{X_1})$, $p_m=F(\chi_{X_2^{m,1}})$, $q_m=F(\chi_{X_2^{m,2}})$, $m=1,2,\dots$. Then $p_m\sim q_m$, $m=1,2,\dots$, since $\tau(p_m)=\mu(X_2^{m,1})=\mu(X_2^{m,2})=\tau(q_m)$. Besides $e+\sum_{m\ge1}(p_m+q_m)=\mathbf{1}$. Hence, there exists such $u=u^*\in\mathcal{U}(\mathcal{M})$ that

$$ue=e,\ up_m=q_mu,\ m=1,2,\dots.$$

Note also that $p_mu=uq_m$ since u self-adjoint. It is clear that

$$|[a,u]|e=|[a-z_0\mathbf{1},u]|e=0=(|a-z_0\mathbf{1}|+u|a-z_0\mathbf{1}|u)e.$$

For any $m=1,2,\dots$ $\sigma(ap_m)$ coincides with the set A_m of essential values of the function $g|_{X_2^{m,1}}$ and $\sigma(uaup_m)=\sigma(aq_m)$ coincides with the set B_m of essential values of the function $g|_{X_2^{m,2}}$ (here the operators ap_m and $uaup_m$ are considered as elements of the algebra $p_m\mathcal{M}p_m$). The sets A_m and B_m are $(z_0,\frac{\pi}{3})$-conjugate sets. It follows from the Lemma 5.3 that

$$v_m|a-uau|v_m^*p_m=v_m|a-uau|p_mv_m^*\ge\frac{\sqrt{3}}{2}\cdot(|a-z_0\mathbf{1}|+u|a-z_0\mathbf{1}|u)p_m \tag{39}$$

for some $v_m\in\mathcal{U}(p_m\mathcal{M}p_m)$.

Applying the automorphism $u\cdot u$ to (39), and noting that $u|a-uau|u=|a-uau|$, we obtain

$$(uv_mu)|a-uau|(uv_mu)^*q_m\ge\frac{\sqrt{3}}{2}\cdot(|a-z_0\mathbf{1}|+u|a-z_0\mathbf{1}|u)q_m. \tag{40}$$

To complete the proof, it remains to define

$$w=e+\sum_{n=1}^{\infty}(v_n+uv_nu)$$

which is a unitary (the series converges in the strong operator topology) (note here that $uv_mu \in \mathcal{U}(q_m\mathcal{M}q_m)$). We observe that

$$w|[a,u]|w^*p_n = w|[a,u]|p_nv_n^*p_n = wp_n|[a,u]|v_n^*p_n = v_n|a-uau|v_n^*p_n \tag{41}$$

and similarly, $w|[a,u]|w^*q_n = (uv_nu)|a-uau|(uv_nu)^*q_n$ and $w|[a,u]|w^*e = |[a,u]|e = 0$. Summing up the inequalities (39) and (40) in the measure topology we arrive at

$$\begin{aligned} w|[a,u]|w^* &= w|[a,u]|w^*e + \sum_{n=1}^{\infty} w|[a,u]|w^*(p_n+q_n) \\ &= \sum_{n=1}^{\infty} v_n|a-uau|v_n^*p_n + (uv_nu)|a-uau|(uv_nu)^*q_n \\ &\geq \sum_{n=1}^{\infty} \frac{\sqrt{3}}{2}(|a-z_0\mathbf{1}| + u|a-z_0\mathbf{1}|u)(p_n+q_n) \\ &= \frac{\sqrt{3}}{2}(|a-z_0\mathbf{1}| + u|a-z_0\mathbf{1}|u) \end{aligned}$$

which proves (37). Regarding the proof of equality (38), see the end of the proof of the Theorem 5.1. □

We have now established in Theorem 5.1 and Theorem 5.4 that for finite factors the commutator estimate (4) holds with the constant $\frac{\sqrt{3}}{2}$. However, this may not be the best constant for which, for all normal $a \in \mathcal{M}$, the inequality holds. We will now establish upper bounds on the best possible constant and we will in particular show that $\frac{\sqrt{3}}{2}$ is in fact the best possible constant when $\mathcal{M}$ is a II_1-factor or a I_n-factor ($n < \infty$) with $n \equiv 0 \mod 3$, . To do this we need the following proposition, which is partly motivated by the proof of [16, Theorem 1]. Here, for a given algebra $\mathcal{A}$ we denote by $\mathrm{Mat}_n(\mathcal{A})$ the set of all $n \times n$ matrices with entries in $\mathcal{A}$.

Proposition 5.5. *Let $\mathcal{N}$ be a finite factor with a faithful normal tracial state $\tau_{\mathcal{N}}$, $\mathbb{M}_n = \mathrm{Mat}_n(\mathbb{C})$, $n \in \mathbb{N}$, $\mathcal{M} = \mathbb{M}_n \otimes \mathcal{N} \cong \mathrm{Mat}_n(\mathcal{N})$, $\tau_{\mathcal{M}} = \frac{1}{n}Tr \otimes \tau_{\mathcal{N}}$ be a tracial state on $\mathcal{M}$. Denote $\mathcal{U}_n^{per} \subseteq \mathbb{M}_n$ for the group of permutation matrices and $\mathbb{D}_n \subseteq \mathbb{M}_n$ for the set of diagonal matrices.*

If $a \in \mathbb{D}_n \otimes \mathbf{1}_{\mathcal{N}}$ then

$$\sup_{u \in \mathcal{U}(\mathcal{M})} \|a - u^*au\|_2 = \max_{u \in \mathcal{U}_n^{per} \otimes \mathbf{1}_{\mathcal{N}}} \|a - u^*au\|_2$$

(The isomorphism (identification) of $\mathrm{Mat}_n(\mathcal{N}) \to \mathbb{M}_n \otimes \mathcal{N}$ is given by the mapping $(a_{ij})_{i,j=1}^n \to \sum_{i,j=1}^n \alpha_{ij} \otimes a_{ij}$ where α_{ij} are matrix units of $\mathbb{M}_n$.)

Proof. Write $a = \mathrm{Diag}(a_i)_{i=1}^n \otimes \mathbf{1}_{\mathcal{N}}$ with $a_i \in \mathbb{C}$ and let $u = (u_{ij})_{i,j=1}^n \in \mathcal{U}(\mathcal{M})$, $u_{ij} \in \mathcal{N}$, $i,j = 1,\dots,n$. We note that

$$\|a - uau^*\|_2^2 = \tau_{\mathcal{M}}((a-uau^*)(a^*-ua^*u)) = 2\tau_{\mathcal{M}}(|a|^2) - 2\Re(\tau_{\mathcal{M}}(aua^*u^*)).$$

We are interested in finding a unitary element $u \in \mathcal{M}$ for which the scalar

$$R(u) := -\Re(\tau_{\mathcal{M}}(aua^*u^*)) = -\frac{1}{n}\sum_{i,j}\Re(\tau_{\mathcal{N}}(a_iu_{ij}\overline{a_j}u_{ij}^*)) = -\frac{1}{n}\sum_{i,j}\Re(a_i\overline{a_j})\tau_{\mathcal{N}}(u_{ij}u_{ij}^*)$$

attains its maximum. For convenience, let $(d_{ij}) \in \mathbb{M}_n$ be the matrix given by $d_{ij} = -\frac{1}{n}\Re(a_i\overline{a_j})$, so that $R(u) = \sum_{ij} d_{ij}\tau_{\mathcal{N}}(u_{ij}u_{ij}^*)$. Denote $\mathcal{W}_n = \{(\tau_{\mathcal{N}}(v_{ij}v_{ij}^*))_{ij} \in \mathbb{M}_n : v = (v_{ij}) \in \mathcal{U}(\mathrm{Mat}_n(\mathcal{N}))\}$. We observe for $w = (\tau_{\mathcal{N}}(v_{ij}v_{ij}^*))_{ij} \in \mathcal{W}_n$ and every j such that $1 \leq j \leq n$, we have $\sum_i w_{ij} = \tau_{\mathcal{N}}(\sum_i v_{ij}v_{ij}^*) = \tau_{\mathcal{N}}(\mathbf{1}_{\mathcal{N}}) = 1$. Similarly, for every $1 \leq i \leq n$ we have that $\sum_j w_{ij} = \tau_{\mathcal{N}}(\sum_j v_{ij}v_{ij}^*) = \tau_{\mathcal{N}}(\mathbf{1}_{\mathcal{N}}) = 1$. Furthermore, as $v_{ij}v_{ij}^* \geq 0$ in $\mathcal{N}$, it is clear that $w_{ij} \geq 0$ for all i,j. Now, denote by $\mathcal{X}_n$ the set of all elements $x = (x_{ij}) \in \mathbb{M}_n$ satisfying

$$\forall j : \sum_i x_{ij} = 1, \quad \forall i : \sum_j x_{ij} = 1, \quad \forall i,j : x_{ij} \geq 0$$

so that $\mathcal{W}_n \subseteq \mathcal{X}_n$. Considering $\mathcal{X}_n$ as a subset of $\mathbb{R}^{n^2}$, we see that $\mathcal{X}_n$ defines a closed convex polytope. By [16, Lemma], the vertices of $\mathcal{X}_n$ are the permutation matrices. Hence the maximum of the linear form

$(x_{ij}) \to \sum_{ij} d_{ij}x_{ij}$ on $\mathcal{X}_n$ is attained for some permutation matrix $\widetilde{u} = (\widetilde{u}_{ij}) \in \mathcal{U}_n^{per}$. As $\widetilde{u} \in \mathcal{U}_n^{per} \subseteq \mathrm{Mat}_n(\mathcal{N})$ we have that $\tau_{\mathcal{N}}(\widetilde{u}_{ij}\widetilde{u}_{ij}^*) = \widetilde{u}_{ij}$ and so

$$R(\widetilde{u}) = \sum_{i,j} d_{ij}\tau_{\mathcal{N}}(\widetilde{u}_{ij}\widetilde{u}_{ij}^*) = \sum_{i,j} d_{ij}\widetilde{u}_{ij} = \max_{x\in\mathcal{X}_n}\sum_{i,j} d_{ij}x_{ij} \geq \sup_{w\in\mathcal{W}_n}\sum_{i,j} d_{ij}w_{ij} = \sup_{u\in\mathcal{U}(\mathcal{M})} R(u).$$

Thus, $\sup_{u\in\mathcal{U}(\mathcal{M})} \|a - u^*au\|_2 \leq \|a - (\widetilde{u}\otimes \mathbf{1}_{\mathcal{N}})^*a(\widetilde{u}\otimes\mathbf{1}_{\mathcal{N}})\|_2$ and the claim follows. □

Combining Theorem 5.1 and Theorem 5.4, we estimate the maximal constant $C_{\mathcal{M}}$ that satisfies the commutator estimate (4) for finite factors $\mathcal{M}$ in Theorem 5.6 below. For the definitions of the constants Λ_n and $\widetilde{\Lambda_n}$ we refer to (10) and (11) and for the exact values of Λ_n we refer to Theorem A.1.

Theorem 5.6. *Let $\mathcal{M}$ be a finite factor with $\mathcal{M} \neq \mathbb{C}$. Then there is a constant $C > 0$ with the property that:*

(∗) *For any normal $a \in S(\mathcal{M})$ there exists a complex number $z_0 \in \mathbb{C}$ and unitaries $u, v, w \in \mathcal{U}(\mathcal{M})$ such that*

$$|[a, u]| \geq C(v|a - z_0\mathbf{1}|v^* + w|a - z_0\mathbf{1}|w^*). \tag{42}$$

Moreover, a maximal constant $C_{\mathcal{M}}$ with this property exists and it satisfies $\Lambda_n \leq C_{\mathcal{M}} \leq \frac{1}{2}\widetilde{\Lambda}_n$ when $\mathcal{M}$ is a I_n-factor ($1 < n < \infty$), and $C_{\mathcal{M}}$ equals $\frac{1}{2}\sqrt{3}$ when $\mathcal{M}$ is a II_1-factor.

Proof. Combining Theorem 5.1 and Theorem 5.4 we obtain for any finite factor that the constant $C = \frac{1}{2}\sqrt{3}$ is admissable for (∗). By Theorem A.1 we have that $\Lambda_n = \frac{1}{2}\sqrt{3}$ when $n = 3$ or $5 \leq n \leq \infty$. Let $n < \infty$. To see that $C = \Lambda_n$ is admissible for all n we note that by Theorem 4.3 we have for $g \in S(\Omega_n)$ that there exist $z_0 \in \mathbb{C}$, $T \in \mathrm{Aut}_n$ such that $\Lambda(g, T, z_0) = \Lambda_n(g) \geq \Lambda_n$, which means

$$|g\circ T - g| \geq \Lambda_n(|g - z_0| + |g\circ T - z_0|). \tag{43}$$

Repeating the proof of Theorem 5.1, replacing (33) with (43), we obtain that $C = \Lambda_n$ is also an admissible constant for (∗). We will later see that the maximal admissible constant $C_{\mathcal{M}}$ actually exists. First we prove upper bounds on constants C satisfying (∗) for $\mathcal{M}$. Let τ be a tracial state on $\mathcal{M}$.

Let $\mathcal{M}$ be a I_n-factor with $1 < n < \infty$. Let $g \in S(\Omega_n)$ be the the function from Lemma A.2 and let $a = \mathrm{Diag}(g(1), ...g(n)) \in \mathcal{M}$. Let $z_0 \in \mathbb{C}$, $u, v, w \in \mathcal{U}(\mathcal{M})$ such that (∗) is satisfied for a with constant C. It follows from Proposition 5.5 ($\mathcal{N} = \mathbb{C}$) that

$$\|[a, u]\|_1 \leq \|[a, u]\|_2 = \|a - u^*au\|_2 \leq \max_{u_0\in\mathcal{U}_n^{per}} \|a - u_0^*au_0\|_2 \leq \mathrm{Diam}(\sigma(a)).$$

Hence,

$$2C\|a - z_0\mathbf{1}\|_1 = C\|v|a - z_0\mathbf{1}|v^* + w|a - z_0\mathbf{1}|w^*\|_1 \leq \|[a, u]\|_1 \leq \mathrm{Diam}(\sigma(a)).$$

Now, choosing g as in the assertion of Lemma A.2 we obtain

$$1 \geq \mathrm{Diam}(\sigma(a)) \geq 2C\|a - z_0\mathbf{1}\|_1 \geq 2C\|g - z_0\|_1 \geq 2C\widetilde{\Lambda}_n^{-1}.$$

Hence, $C \leq \frac{1}{2}\widetilde{\Lambda}_n$.

Let $\mathcal{M}$ be of type II_1. Then $\mathcal{M} \cong \mathrm{Mat}_3(\mathbb{C}) \otimes \mathcal{N}$ for some II_1-factor $\mathcal{N}$. Let the function $g \in S(\Omega_3)$ be as in Lemma A.2 and let $a_1 = \mathrm{Diag}(g(1), g(2), g(3)) \in \mathrm{Mat}_3(\mathbb{C})$ and $a = a_1 \otimes \mathbf{1}_{\mathcal{N}} \in \mathcal{M}$. Let $z_0 \in \mathbb{C}$, $u, v, w \in \mathcal{U}(\mathcal{M})$ be such that (∗) holds for a with constant C. We have

$$\|[a, u]\|_1 \leq \|[a, u]\|_2 = \|a - u^*au\|_2 \leq \max_{u_0\in\mathcal{U}_3^{per}\otimes\mathbf{1}_{\mathcal{N}}} \|a - u_0^*au_0\|_2 = \max_{u_0\in\mathcal{U}_3^{per}} \|a_1 - u_0^*a_1u_0\|_2 \leq \mathrm{Diam}(\sigma(a_1)).$$

Hence,

$$2C\|a_1 - z_0\mathbf{1}\|_1 = 2C\|a - z_0\mathbf{1}\|_1 = C\|v|a - z_0\mathbf{1}|v^* + w|a - z_0\mathbf{1}|w^*\|_1 \leq \|[a, u]\|_1 \leq \mathrm{Diam}(\sigma(a_1)) \leq 1.$$

It follows from Lemma A.2 that

$$1 \geq \mathrm{Diam}(\sigma(a_1)) \geq 2C\|g - z_0\|_1 \geq 2C\widetilde{\Lambda}_3^{-1}.$$

Hence, $C \leq \frac{1}{2}\widetilde{\Lambda}_3 = \frac{\sqrt{3}}{2}$. For $\mathcal{M}$ a II_1-factor, this shows that in fact $C_{\mathcal{M}}$ exists and that $C_{\mathcal{M}} = \frac{1}{2}\sqrt{3}$.

We now show that the maximal constant $C_{\mathcal{M}}$ also exists when $\mathcal{M}$ is a I_n-factor ($1 < n < \infty$). Let $(C_i)_{i\geq 1}$ be an increasing sequence of positive constants admissible for (∗) and set $C = \sup C_i \leq \frac{1}{2}\widetilde{\Lambda}_n$. For a normal $a \in \mathcal{M}$ there exists corresponding $u_i \in \mathcal{U}(\mathcal{M})$ and $z_{0,i} \in \mathbb{C}$ such that the equation (43) holds with the constant C_i. Now by

$$2\|a\|_1 \geq \|[a, u_i]\|_1 \geq 2C_i\|a - z_{0,i}\mathbf{1}\|_1 \geq 2C_i(|z_{0,i}| - \|a\|_1)$$

we obtain $|z_{0,i}| \leq \frac{1+C_i}{C_i}\|a\|_1 \leq \frac{1+C_1}{C_1}\|a\|_1$. Therefore, as the sequences $(u_i)_i$ and $(z_{0,i})_i$ are bounded and as $\mathcal{M}$ is finite-dimensional, we can assume these sequences converge in norm to some $u \in \mathcal{U}(\mathcal{M})$ and some $z_0 \in \mathbb{C}$ (otherwise restrict to a subsequence). Now the elements $d_i := |[a,u_i]| - C_i(|a - z_{0,i}\mathbf{1}| + u_i|a - z_{0,i}\mathbf{1}|u_i^*)$ are all positive and converge to $d = |[a,u]| - C(|a-z_0\mathbf{1}| + u|a-z_0\mathbf{1}|u^*)$. As the cone of positive elements in $\mathcal{M}$ is closed in the norm, we obtain $d \geq 0$. This shows that $|[a,u]| \geq C(|a-z_0\mathbf{1}|+u|a-z_0\mathbf{1}|u^*)$ holds, and therefore C is admissible for $(*)$ as well. Hence, the supremum of all admissible constants (which is finite), is again admissible, and this shows that $C_{\mathcal{M}}$ exists. It now follows that $\Lambda_n \leq C_{\mathcal{M}} \leq \frac{1}{2}\widetilde{\Lambda}_n$ ☐

6. Commutator estimates for normal operators in infinite factors

We shall now obtain the commutator estimate (4) for normal elements in an infinite factor. We show in Theorem 6.4 that for such factors the constant C in this estimate can be chosen arbitrary close to 1. For infinite factors, this extends the result of [3, Theorem B.1] to normal elements. The proof of Theorem 6.4 extensively uses the geometry of projections. Before we start its proof, we state and prove three short technical lemmas.

Lemma 6.1. *Let $\mathcal{M}$ be an infinite factor and p be a infinite projection from $\mathcal{M}$. If $p_1, \dots, p_n \in \mathcal{P}(\mathcal{M})$ are pairwise commuting and $p_1, \dots, p_n \prec p$, then $p_1 \vee \dots \vee p_n \prec p$. (We understand the symbol "$\prec$" as "$\precsim$".)*

Proof. Let $q_1 = p_1$ and $q_{k+1} = p_{k+1}(\mathbf{1} - q_1 - \dots q_k)$ for $k = 1, \dots, n-1$. Then $q_i q_j = 0$ for $i \neq j$, $q_k \prec p$ for $k = 1, \dots, n$ and $p_1 \vee \dots \vee p_n = q_1 + \dots + q_n \prec p$ (see [4, Lemma 2 (ii)]). ☐

Lemma 6.2. *Let $\mathcal{M}$ be a factor, a be a normal operator from $S(\mathcal{M})$, $p, q \in \mathcal{P}(\mathcal{M})$, $q \preceq p$. Suppose that one of the following conditions holds:*

(i). q is finite and there exists a sequence of finite projections (p_n) in $\mathcal{M}$ such that $p_n \uparrow p$ and $[a, p_n] = 0$ for all $n \in \mathbb{N}$;

(ii). q is an infinite projection and $[a, p] = 0$.

Then there exists a projection $q_1 \in \mathcal{M}$ such that $q_1 \sim q$, $[a, q_1] = 0$ and such that $q_1 \leq p$.

Proof. The proof follows along the lines of [4, Lemma 3] and is therefore omitted. ☐

Lemma 6.3. *Let $\mathcal{M}$ be a von Neumann algebra, $a, b \in LS(\mathcal{M})$, $\alpha_1, \alpha_2 > 0$, and*

$$|a| \geq \alpha_1 \mathbf{1}, \ , \ 2\alpha_2 < \alpha_1, \ \alpha_2 \mathbf{1} \geq |b|.$$

Then there exists $v \in \mathcal{U}(\mathcal{M})$ such that

$$v|a-b|v^* \geq (1 - \frac{2\alpha_2}{\alpha_1})|a| + |b|.$$

Proof. Let $a, b \in LS(\mathcal{M})$, $\alpha_1, \alpha_2 > 0$ satisfy the assumption of the lemma. By Theorem 2.2, we have that

$$|a| \leq v|a-b|v^* + w|b|w^*$$

for some $v, w \in \mathcal{M}$ with $v^*v = w^*w = \mathbf{1}$. Then

$$\begin{aligned} v|a-b|v^* &\geq |a| - w|b|w^* \geq |a| - \alpha_2 ww^* \geq |a| - \alpha_2 \mathbf{1} \\ &\geq |a| + |b| - 2\alpha_2\mathbf{1} \geq |a| + |b| - \frac{2\alpha_2}{\alpha_1}|a| = (1 - \frac{2\alpha_2}{\alpha_1})|a| + |b|. \end{aligned}$$

Since $v|a-b|v^* \geq (1 - \frac{2\alpha_2}{\alpha_1})|a| \geq (\alpha_1 - 2\alpha_2)\mathbf{1}$, it follows

$$0 = (\mathbf{1} - vv^*)v|a-b|v^*(\mathbf{1} - vv^*) \geq (\alpha_1 - 2\alpha_2)(\mathbf{1} - vv^*) \geq 0.$$

Therefore, we have $\mathbf{1} - vv^* = 0$, i.e. $v \in \mathcal{U}(\mathcal{M})$. ☐

Theorem 6.4. *Let $\mathcal{M}$ be an infinite factor, and let $a \in S(\mathcal{M})$ be normal. There is a $\lambda_0 \in \mathbb{C}$ such that for any $\varepsilon > 0$ there exist $u_\varepsilon = u_\varepsilon^* \in \mathcal{U}(\mathcal{M})$, $w_\varepsilon \in \mathcal{U}(\mathcal{M})$ so that*

$$w_\varepsilon|[a, u_\varepsilon]|w_\varepsilon^* \geq (1-\varepsilon)(|a - \lambda_0 \mathbf{1}| + u_\varepsilon|a - \lambda_0\mathbf{1}|u_\varepsilon). \tag{44}$$

Proof. Let $e(\cdot)$ be the spectral measure of a on $\mathbb{C}$, in particular, $e(X) = \chi_X(a)$ for any $X \in \mathcal{B}(\mathbb{C})$. Since $a \in S(\mathcal{M})$ there exists a $R > 0$ so that $e(X_R)$ is a finite projection, where $X_R = \{\lambda \in \mathbb{C} : \ |\lambda| > R\}$. Then $Y_R := \mathbb{C} \setminus X_R$ is compact and it follows from Lemma 6.1 that $e(Y_R) \sim \mathbf{1}$. A point $\lambda \in \mathbb{C}$ will be called a *point of densification* for a if $e(V) \sim \mathbf{1}$ for any neighborhood V of a point λ. Denote by A the set of all points of densification for a.

We claim that $A \neq \emptyset$. To see that the claim holds it is sufficient to show there exists a system of nested sets $B_n = [\alpha_n, \alpha_n + \frac{5R}{2^n}) \times [\beta_n, \beta_n + \frac{5R}{2^n})$, with $e(B_n) \sim \mathbf{1}$. We put $\alpha_1 = \beta_1 = -R$ so that clearly $Y_R \subset B_1$ and therefore $e(B_1) \sim \mathbf{1}$. Now suppose $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ are already constructed so that $e(B_1) \sim \cdots \sim e(B_n) \sim \mathbf{1}$. We can divide the rectangle B_n into 4 smaller rectangles by

$$B_n = \bigcup_{k,l=0}^{1} [\alpha_n + k \cdot \frac{5R}{2^{n+1}}, \alpha_n + (k+1) \cdot \frac{5R}{2^{n+1}}) \times [\beta_n + l \cdot \frac{5R}{2^{n+1}}, \beta_n + (l+1) \cdot \frac{5R}{2^{n+1}}).$$

It follows from Lemma 6.1 that one of the sets from this union can be taken for B_{n+1} (which then defines $\alpha_{n+1}, \beta_{n+1}$). This completes the induction. The point $\lambda := (\sup_n \alpha_n) + (\sup_n \beta_n)i$ is a point of densification for a since any neighbourhood V of λ contains a set B_n for some n. Therefore $A \neq \emptyset$.

We show that A is closed. Indeed, if λ is a limit point of A and V is a neighborhood of λ, then V is also a neighborhood of some point from A. Hence $e(V) \sim \mathbf{1}$. This shows $\lambda \in A$. Thus A is closed. Obviously, $A \subset Y_R$. Therefore, A is a nonempty compact subset in $\mathbb{C}$.

Let us consider three cases covering the full picture.

- 1. *There is a point $\lambda_0 \in \mathbb{C}$ such that $e(\{\lambda_0\}) \sim \mathbf{1}$.* Then $e(\mathbb{C} \setminus \{\lambda_0\}) \preceq e(\{\lambda_0\})$ and therefore there is a $v \in \mathcal{M}$ with $v^*v = e(\mathbb{C} \setminus \{\lambda_0\})$ and $vv^* \leq e(\{\lambda_0\})$. Let 's put $u = v + v^* + (e(\{\lambda_0\}) - vv^*)$. Then $u = u^* \in \mathcal{U}(\mathcal{M})$. Since

$$\begin{aligned}(a - \lambda_0\mathbf{1})u(a - \lambda_0\mathbf{1})^* &= (a - \lambda_0\mathbf{1})u(\mathbf{1} - e(\{\lambda_0\}))(a - \lambda_0\mathbf{1})^* \\ &= (a - \lambda_0\mathbf{1})v(\mathbf{1} - vv^*)(a - \lambda_0\mathbf{1})^* \\ &= (a - \lambda_0\mathbf{1})e(\{\lambda_0\})u(a - \lambda_0\mathbf{1})^* = 0\end{aligned}$$

  and, similarly,

$$(a - \lambda_0\mathbf{1})^*u(a - \lambda_0\mathbf{1}) = 0$$

  then

$$|[a, u]| = |(a - \lambda_0\mathbf{1}) - u(a - \lambda_0\mathbf{1})u| = |a - \lambda_0\mathbf{1}| + u|a - \lambda_0\mathbf{1}|u$$

  which shows the result for this case with $w_\varepsilon = \mathbf{1}$.

In the following two cases, the scalar $\lambda_0 \in \mathbb{C}$ will be found and for a fixed number $\varepsilon > 0$ a sequence of pairs of projectors $\{(p_n, q_n)\}_{n \geq 1}$ of $\mathcal{M}$ will be constructed together with a sequence (γ_n) of positive numbers converging to zero satisfying the following conditions:

(i). $p_n q_m = 0,\ p_n p_m = \delta_{nm} p_n,\ q_n q_m = \delta_{nm} q_n,\ [a, p_n] = [a, q_n] = 0,\ p_n \sim q_n$ for all n, m;
(ii). $q_n \leq e(W_n),\ p_n \leq e(V_n)$ for all $n \geq 1$;
(iii). $\bigvee_{n\geq 0} p_n \vee \bigvee_{n \geq 0} q_n = \mathbf{1} - e(\{\lambda_0\})$,

where $V_n := \{\lambda : |\lambda - \lambda_0| > \gamma_n\}$ and $W_n := \{\lambda : |\lambda - \lambda_0| < \frac{\varepsilon}{2}\gamma_n\}$.

- 2. *The set A has a limit point λ_0.* We can assume that $\varepsilon < \frac{1}{2}$. We inductively construct the sequences of positive numbers (γ_n) (and hence the sets V_n, W_n), numbers (λ_n) from A, and sets

$$U_n = \{\lambda : \ |\lambda - \lambda_{2n}| < \gamma_{n+1}\} \tag{45}$$

  in such a way that $U_n \subseteq W_n \cap V_{n+1}$ and the set $V_{n+1} \setminus \bigcup_{k=1}^n (U_k \cup V_k))$ is a neighborhood of the point λ_{2n+1}. First, let $\lambda_1 \in A \setminus \{\lambda_0\}$ and put $\gamma_1 = \frac{|\lambda_1 - \lambda_0|}{2}$. Then V_1 is a neighborhood of the point λ_1. Next, in the set W_1 there will be different points λ_2, λ_3 from $A \setminus \{\lambda_0\}$. Put $\gamma_2 = \frac{1}{2}\min\{|\lambda_3 - \lambda_0|, |\lambda_2 - \lambda_3|, \frac{\varepsilon}{2}\gamma_1 - |\lambda_2 - \lambda_0|, |\lambda_2 - \lambda_0|\}$ and note that $\gamma_2 < \frac{1}{2}|\lambda_3 - \lambda_0| \leq \frac{\gamma_1}{4}$. Note also that the set $V_2 \setminus (V_1 \cup U_1)$ is a neighborhood of the point λ_3 and that $U_1 \subseteq W_1 \cap V_2$. We continue this process by induction. Let these sequences be constructed for the indices $1, \ldots, n$. Then in the set W_n there will be different points $\lambda_{2n}, \lambda_{2n+1}$ from $A \setminus \{\lambda_0\}$. Put $\gamma_{n+1} = \frac{1}{2}\min\{|\lambda_{2n+1} - \lambda_0|, |\lambda_{2n} - \lambda_{2n+1}|, \frac{\varepsilon}{2}\gamma_n - |\lambda_{2n} - \lambda_0|, |\lambda_{2n} - \lambda_0|\}$. Then $\gamma_{n+1} < \frac{\gamma_n}{4}$ and $V_{n+1} \setminus \bigcup_{k=1}^n (U_k \cup V_k))$ is a neighborhood of the point λ_{2n+1}, and $U_n \subseteq W_n \cap V_{n+1}$. Thus, the above sequences are constructed. We remark that for $n < m$ we have $U_n \cap U_m \subseteq W_m \cap V_{n+1} = \emptyset$

  Put $p_1 = e(V_1),\ q_1 = e(U_1);\ q_n = e(U_n),\ p_n = e(V_n \setminus \bigcup_{k=1}^{n-1}(U_k \cup V_k))$ for $n > 1$. Then we have by the construction that $p_1, q_1, p_2, q_2, ...$ are pairwise orthogonal and $p_n \sim \mathbf{1} \sim q_n$ for any n. Now since $V_n = (V_n \setminus \bigcup_{k=1}^{n-1}(U_k \cup V_k))) \cup \bigcup_{k=1}^{n-1}(U_k \cup V_k))$ and $\bigcup_{n=1}^\infty V_n = \mathbb{C} \setminus \{\lambda_0\}$ we find $\bigvee_{n\geq 0} p_n \vee \bigvee_{n\geq 0} q_n = \mathbf{1} - e(\{\lambda_0\})$.
- 3. *The set A is finite and $e(\{\lambda\}) \prec \mathbf{1}$ for any $\lambda \in A$.* We can by assumption write $A = \{\lambda_0, \ldots, \lambda_m\}$ for some $m \geq 0$ (note A is non-empty). When $|A| = 1$ put $r = 1$ and when $|A| > 1$ let r be the minimum distance between points in A. Consider the sets $V(t) = \mathbb{C} \setminus \bigcup_{k=0}^m \{\lambda : |\lambda - \lambda_k| < t\}$ for $0 < t < \frac{r}{2}$. It is clear that $V(t) \uparrow \mathbb{C} \setminus A$ at $t \downarrow 0$.

We show that $e(V(t)) \prec \mathbf{1}$ for $0 < t < \frac{r}{2}$. Indeed, for any point $z \in V(t) \setminus X_R$ there is a neighborhood U_z of z with $e(U_z) \prec \mathbf{1}$. Now as the set $V(t) \setminus X_R$ is compact we can let $\{U_{z_1}, \ldots, U_{z_l}\}$ be a finite subcover for $V(t) \setminus X_R$. Then $\{X_R, U_{z_1}, \ldots, U_{z_l}\}$ is the coverage of the set $V(t)$. It follows from Lemma 6.1 that $e(V(t)) \prec \mathbf{1}$.

There are now two possible cases:

3.1. All projections $e(V(t))$, $t > 0$, are finite. In this case, put $\gamma_1 = \frac{r}{3}$.

3.2. There is a $0 < t_0 < \frac{r}{3}$ so that the projection $e(V(t_0))$ is infinite. In this case put $\gamma_1 = t_0$.

Set $\gamma_n = \frac{\gamma_1}{2^{n-1}}$, $n > 1$ (and hence V_n, W_n are defined as well); We set $p_1 := e(V(\gamma_1) \cup (A \setminus \{\lambda_0\})) \leq e(V_1)$. It follows from Lemma 6.1 that $p_1 \prec \mathbf{1}$ and $p := e(W_1) \sim \mathbf{1}$. If we put $q = p_1$, then for p, q the conditions Lemma 6.2 are met: condition (ii) is met if q is an infinite projection, and condition (i) is met in case 3.1 if q is an finite projection (in this case, the set W_1 is covered by the system $V(t)$, $t > 0$). Therefore, there is a projection $q_1 \leq e(W_1)$ such that $q_1 \sim p_1$ and $[a, q_1] = 0$. Now, suppose the projections $p_1, q_1, \ldots, p_n, q_n \prec \mathbf{1}$ are constructed. We build projections p_{n+1}, q_{n+1}. We put $p_{n+1} = e(V(\gamma_{n+1})) \cdot (\mathbf{1} - \sum_{k=1}^{n}(p_k + q_k))$. Then $p_{n+1} \prec \mathbf{1}$ since $p_{n+1} \leq e(V(\gamma_{n+1}))$. Furthermore, since $e(W_n) \sim \mathbf{1}$ and $p_1, q_1, \ldots, p_n, q_n \prec \mathbf{1}$ we find $e(W_n) \cdot (\mathbf{1} - \sum_{k=1}^{n}(p_k + q_k)) \sim \mathbf{1}$. Again using Lemma 6.2, we find such a projection $q_{n+1} \sim p_{n+1}$ that $q_{n+1} \leq e(W_n) \cdot (\mathbf{1} - \sum_{k=1}^{n}(p_k + q_k))$ and $[a, q_{n+1}] = 0$ (two cases are considered again: p_{n+1} is a infinite projection; p_{n+1} is a finite projection and the condition 3.1 is met). As $p_{n+1} + \sum_{k=1}^{n}(p_k + q_k) \geq e(V(\gamma_{n+1}))$ and $p_1 \geq e(A \setminus \{\lambda_0\})$ we conclude $\sum_{k=1}^{\infty}(p_k + q_k) = \mathbf{1} - e(\{\lambda_0\})$. Therefore, the projections $p_1, q_1, p_2, q_2, \ldots$ satisfy the conditions (i)-(iii).

In the cases (2) and (3) we can now find partial isometries $v_n \in \mathcal{M}$ so that $v_n^* v_n = p_n$, $v_n v_n^* = q_n$, for $n = 1, 2, \ldots$. We put $u_\varepsilon = e(\{\lambda_0\}) + \sum_{n=1}^{\infty}(v_n + v_n^*)$. Then $u_\varepsilon = u_\varepsilon^* \in \mathcal{U}(\mathcal{M})$, $u_\varepsilon e(\{\lambda_0\}) = e(\{\lambda_0\})$ and $u_\varepsilon p_n = q_n u_\varepsilon$ for all n. We have

$$|a - \lambda_0 \mathbf{1}| p_n \geq \gamma_n p_n, \ |a - \lambda_0 \mathbf{1}| q_n \leq \frac{\varepsilon}{2} \gamma_n q_n, \ \forall n. \tag{46}$$

Therefore

$$|u_\varepsilon a u_\varepsilon - \lambda_0 \mathbf{1}| p_n = u_\varepsilon |a - \lambda_0 \mathbf{1}| q_n u_\varepsilon \leq \frac{\varepsilon}{2} \gamma_n u_\varepsilon q_n u_\varepsilon = \frac{\varepsilon}{2} \gamma_n p_n, \ \forall n. \tag{47}$$

Since $[a, p_n] = [u_\varepsilon a u_\varepsilon, p_n] = 0$ then

$$|a - u_\varepsilon a u_\varepsilon| p_n = |(a - \lambda_0 \mathbf{1}) p_n - (u_\varepsilon a u_\varepsilon - \lambda_0 \mathbf{1}) p_n|. \tag{48}$$

It follows from Lemma 6.3 that

$$w_n |a - u_\varepsilon a u_\varepsilon| p_n w_n^* \geq ((1 - \varepsilon)|a - \lambda_0 \mathbf{1}| + |u_\varepsilon a u_\varepsilon - \lambda_0 \mathbf{1}|) p_n$$

for some $w_n \in \mathcal{U}(p_n \mathcal{M} p_n)$.

Therefore

$$w_n |a - u_\varepsilon a u_\varepsilon| w_n^* p_n \geq ((1 - \varepsilon)(|a - \lambda_0 \mathbf{1}| + |u_\varepsilon a u_\varepsilon - \lambda_0 \mathbf{1}|)) p_n. \tag{49}$$

Applying the automorphism $u_\varepsilon \cdot u_\varepsilon$ to (49), and noting that $u_\varepsilon |a - u_\varepsilon a u_\varepsilon| u_\varepsilon = |a - u_\varepsilon a u_\varepsilon|$, we obtain

$$(u_\varepsilon w_n u_\varepsilon)|a - u_\varepsilon a u_\varepsilon|(u_\varepsilon w_n u_\varepsilon)^* q_n \geq ((1 - \varepsilon)(|a - \lambda_0 \mathbf{1}| + |u_\varepsilon a u_\varepsilon - \lambda_0 \mathbf{1}|)) q_n. \tag{50}$$

Recall that $S(\mathcal{M}) = \mathcal{M}$ if $\mathcal{M}$ has type I or III. In this case, we denote by t the strong operator topology in $\mathcal{M}$. If the factor $\mathcal{M}$ is of type II then $\mathcal{S}(\mathcal{M}) = \mathcal{S}(\mathcal{M}, \tau)$ for any faithful semi-finite normal trace τ on $\mathcal{M}$. In this case we let t stand for the measure topology t_τ (this topology is defined in Section 2, the need to use this topology is due to the fact that a can be an unbounded operator).

To complete the proof, it remains to set

$$w_\varepsilon = e(\{\lambda_0\}) + \sum_{n=1}^{\infty}(w_n + u_\varepsilon w_n u_\varepsilon)$$

(the series converges in the strong operator topology) and sum up the inequalities (49) and (50) in the topology t. □

7. Estimates for inner derivations associated to normal elements

In this section we apply the operator estimates from Theorem 5.1 and Theorem 5.6 to extend the result of [3, Theorem 1.1] and estimate the norm of inner derivations $\delta_a : \mathcal{M} \to L_1(\mathcal{M}, \tau)$ in the case when $\mathcal{M}$ a finite factor with faithful normal trace τ and $a \in L_1(\mathcal{M}, \tau)$ is normal.

We establish some notation first. Let $\mathcal{M}$ be a von Neumann algebra with predual $\mathcal{M}_*$. The Banach space $\mathcal{M}_*$ can be embedded into its double dual $(\mathcal{M}_*)^{**} = \mathcal{M}^*$. In this way we identify $\mathcal{M}_*$ with the space of ultra weakly continuous linear functionals on $\mathcal{M}$. The predual $\mathcal{M}_*$ is a Banach $\mathcal{M}$-bimodule with the bimodule actions given by:

$$(a \cdot \omega)(x) = \omega(xa),\ \ (\omega \cdot a)(x) = \omega(ax),\ \ a, x \in \mathcal{M},\ \ \omega \in \mathcal{M}_*. \tag{51}$$

If there is a faithful normal semi-finite trace τ on $\mathcal{M}$, then the Banach $\mathcal{M}$-bimodule $\mathcal{M}_*$ is isomorphic to $L_1(\mathcal{M}, \tau)$ (see e.g. [26, Lemma 2.12 and Theorem 2.13]).

A linear operator $\delta : \mathcal{M} \to \mathcal{M}_*$ is called a *derivation* if

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in \mathcal{M}$. For each $a \in \mathcal{M}_*$ a derivation $\delta_a : \mathcal{M} \to \mathcal{M}_*$ can be defined by the equality

$$\delta_a(x) = [a, x] = ax - xa$$

(using the $\mathcal{M}$-bimodule structure as defined in (51)). Such derivations are called *inner*. In fact it holds true that any derivation $\delta : \mathcal{M} \to \mathcal{M}_*$ is inner. Moreover, there exists $a \in \mathcal{M}_*$ so that $\delta = \delta_a$ and $\|a\|_{\mathcal{M}_*} \le \|\delta\|_{\mathcal{M}\to\mathcal{M}_*}$ see [14, Theorem 4.1] and [2, Corollary C]. We are interested in describing the norm of the derivations $\delta_a : \mathcal{M} \to \mathcal{M}_*$ for $a \in \mathcal{M}_*$. Is it true that a distance formula similar to (1) holds true? This question has been fully settled in [3, Theorem 3.1] for infinite factors. Moreover, in [3] the following theorem was proved:

Theorem 7.1. *[3, Theorem 1.1] If $\mathcal{M}$ is a von Neumann algebra with a faithful normal finite trace τ and $a = a^* \in L_1(\mathcal{M}, \tau)$, then there exists $c_a = c_a^* \in L_1(\mathcal{M}, \tau) \cap Z(S(\mathcal{M}))$ such that*

$$\|\delta_a\|_{\mathcal{M}\to L_1(\mathcal{M},\tau)} = 2\,\|a - c_a\|_1 = 2 \min_{z\in Z(S(\mathcal{M}))} \|a - z\|_1 \tag{52}$$

where $Z(S(\mathcal{M}))$ stands for the center of the algebra of all measurable operators affiliated with $\mathcal{M}$

We focus on the case that $\mathcal{M}$ is finite. For brevity, we will denote the norm $\|\cdot\|_{\mathcal{M}\to L_1(\mathcal{M},\tau)}$ by $\|\cdot\|_{\infty,1}$. For general $a \in L_1(\mathcal{M}, \tau)$ we do not know the relationship between $\|\delta_a\|_{\infty,1}$ and $\inf\{\|a - z\|_1 : z \in Z(S(\mathcal{M}))$. In Theorem 7.3, we shall give upper and lower estimates of this relation in the case when $\mathcal{M}$ is a finite factor and a is a normal operator. We will see a substantial difference with the case of inner derivations associated to self-adjoint elements. First we state Theorem 7.2 which is related and is used in the proof of Theorem 7.3. Recall that when $n \equiv 0 \pmod 3$ or $n = \infty$ we have $2\Lambda_n = \sqrt{3} = \widetilde{\Lambda}_n$ and that in addition,

$$\lim_{n\to\infty} \widetilde{\Lambda}_n = \sqrt{3},$$

and

$$2\Lambda_n = \sqrt{3} \text{ for } n = 3, \text{ or } n \ge 5.$$

For convenience, we define for a finite factor $\mathcal{M}$ the value

$$n(\mathcal{M}) = \begin{cases} n & \mathcal{M} \text{ is a } \mathrm{I}_n\text{-factor} \\ \infty & \mathcal{M} \text{ is a } \mathrm{II}_1\text{-factor} \end{cases} \tag{53}$$

Theorem 7.2. *Let $\mathcal{M}$ be a finite factor with a faithful tracial state τ. Assume $\mathcal{M} \neq \mathbb{C}$. Then*

1. *For every derivation $\delta_a : \mathcal{M} \to L_1(\mathcal{M}, \tau)$ with $a \in \mathcal{M}$ normal, there is a normal $b \in \mathcal{M}$ such that $\delta_a = \delta_b$ and $\|\delta_b\|_{\infty,1} \ge 2\Lambda_{n(\mathcal{M})}\|b\|_1$.*
2. *There exists a normal $a \in \mathcal{M}$ for which the derivation $\delta_a : \mathcal{M} \to L_1(\mathcal{M}, \tau)$ is non-zero and such that for every $b \in \mathcal{M}$ with $\delta_a = \delta_b$ we have $\|\delta_b\|_{\infty,1} \le \widetilde{\Lambda}_{n(\mathcal{M})}\|b\|_1$.*

Proof. (1) Let $a \in \mathcal{M}$ be normal. By Theorem 5.6 there exist $u, w \in \mathcal{U}(\mathcal{M})$, $z_0 \in \mathbb{C}$ satisfying the commutator estimate (42), hence $\|\delta_a\|_{\infty,1} \ge \|\delta_a(u)\|_1 \ge 2\Lambda_{n(\mathcal{M})}\|a - z_0\mathbf{1}\|_1$. This shows the result since $b := a - z\mathbf{1}$ is normal and $\delta = \delta_a = \delta_b$.

(2) Let $\mathcal{M}$ be a finite factor. When $\mathcal{M}$ is a I_n-factor, we set $m := n$ and we can write $\mathcal{M} = \mathrm{Mat}_m(\mathcal{N})$, with $\mathcal{N} = \mathbb{C}$. When $\mathcal{M}$ is a II_1-factor we set $m = 3$ and we can write $\mathcal{M} = \mathrm{Mat}_m(\mathcal{N})$ for some II_1-factor $\mathcal{N}$. We now let $g \in L_\infty(\Omega_m)$ be non-constant and let a be the diagonal matrix $a = \mathrm{Diag}(g(1), \ldots, g(m)) \otimes \mathbf{1}_\mathcal{N} \in \mathrm{Mat}_m(\mathbb{C}) \otimes \mathcal{N} = \mathcal{M}$. Then $\delta_a : \mathcal{M} \to L_1(\mathcal{M}, \tau)$ is a non-zero derivation. To estimate $\|\delta_a\|_{\infty,1}$ we recall that the Russo–Dye Theorem, [23, Theorem 1], asserts for a unital C*-algebra that the closed unit ball equals the closed convex hull of all the unitaries. Now, for $x \in \mathrm{Conv}(\mathcal{U}(\mathcal{M}))$ we can write $x = \sum_{i=1}^N c_i u_i$ with $N \in \mathbb{N}$, $u_i \in \mathcal{U}(\mathcal{M})$ and $c_i \ge 0$ with $\sum_{i=1}^N c_i = 1$. Then clearly $\|\delta_a(x)\|_1 \le \sum_{i=1}^N c_i\|\delta_a(u_i)\|_1 \le \max_{1\le i\le N} \|\delta(u_i)\|_1 \le \sup_{u\in\mathcal{U}(\mathcal{M})} \|\delta_a(u)\|_1$. By continuity of

δ_a this inequality holds for all x in the closed convex hull as well. By the Russo-Dye Theorem this shows that

$$\|\delta_a\|_{\infty,1} = \sup_{x\in\mathcal{M},\|x\|\leq 1} \|\delta_a(x)\|_1 = \sup_{x\in\overline{\mathrm{Conv}(\mathcal{U}(\mathcal{M}))}} \|\delta_a(x)\|_1 = \sup_{u\in\mathcal{U}(\mathcal{M})} \|\delta_a(u)\|_1. \tag{54}$$

Using this and Proposition 5.5 we find

$$\begin{aligned}
\|\delta_a\|_{\infty,1} &= \sup_{u\in\mathcal{U}(\mathcal{M})} \|\delta_a(u)\|_1 \\
&= \sup_{u\in\mathcal{U}(\mathrm{Mat}_m(\mathcal{N}))} \|u^*[a,u]\|_1 \\
&= \sup_{u\in\mathcal{U}(\mathrm{Mat}_m(\mathcal{N}))} \|u^*au - a\|_1 \\
&\leq \sup_{u\in\mathcal{U}(\mathrm{Mat}_m(\mathcal{N}))} \|u^*au - a\|_2 \\
&= \sup_{u\in\mathcal{U}_m^{per}\otimes\mathbf{1}_\mathcal{N}} \|u^*au - a\|_2 \\
&= \sup_{\substack{T:\Omega_m\to\Omega_m \\ \text{permutation}}} \|g\circ T - g\|_2.
\end{aligned}$$

The last step follows from the fact that, for $u \in \mathcal{U}_m^{per}\otimes\mathbf{1}_\mathcal{N}$, we have $u^*au = \mathrm{Diag}(g\circ T(1),\ldots,g\circ T(n))\otimes\mathbf{1}_\mathcal{N}$ for some permutation T. By Lemma A.2 we can fix a g so that $\mathrm{Diam}(g(\Omega_m)) = 1 \leq \widetilde{\Lambda}_m \inf_{z\in\mathbb{C}} \|g - z\|_1$ (note that such g is non-constant). Take any $b \in \mathcal{M}$ with $\delta_a = \delta_b$. Then $a - b$ lies in the center of $\mathcal{M}$, so $a - b = z_0\mathbf{1}$ for some $z_0 \in \mathbb{C}$. Hence, $\|b\|_1 = \|a - z_0\mathbf{1}\|_1 = \|g - z_0\|_1$ so that $\|\delta\|_{\infty,1} \leq \mathrm{Diam}(g(\Omega_m)) \leq \widetilde{\Lambda}_m\|b\|_1$. The result now follows. Indeed, when $\mathcal{M}$ is a I_n-factor, we obtained $\|\delta\|_{\infty,1} \leq \widetilde{\Lambda}_{n(\mathcal{M})}\|b\|_1$ and when $\mathcal{M}$ is a II_1-factor we obtained $\|\delta\|_{\infty,1} \leq \widetilde{\Lambda}_3\|b\|_1 = \widetilde{\Lambda}_\infty\|b\|_1 = \widetilde{\Lambda}_{n(\mathcal{M})}\|b\|_1$. □

The following theorem shows that for (most) finite factors the distance formula from (52) does not hold for arbitrary normal $a \in L_1(\mathcal{M},\tau)$, which shows a crucial difference with the classical result of Stampfli and its generalisations describing the norm of derivations $\delta_a : \mathcal{M} \to \mathcal{M}$, as for these derivations the distance formula (1) holds for all $a \in \mathcal{M}$. While the distance formula does not hold true, we are able to obtain constant bounds on the ratio $\frac{\|\delta_a\|_{\infty,1}}{\min_{z\in\mathbb{C}}\|a-z\mathbf{1}\|_1}$. In the case of II_1-factors and I_n-factors ($1 < n < \infty$) with $n \equiv 0 \mod 3$ these constants can not be improved.

Theorem 7.3. *Let $\mathcal{M}$ be a finite factor with a faithful tracial state τ and let $a \in L_1(\mathcal{M},\tau) \setminus Z(\mathcal{M})$ be normal and measurable. Then the derivation $\delta_a : \mathcal{M} \to L_1(\mathcal{M},\tau)$ satisfies:*

$$2\Lambda_{n(\mathcal{M})} \leq \frac{\|\delta_a\|_{\infty,1}}{\min_{z\in\mathbb{C}}\|a - z\mathbf{1}\|_1} \leq 2. \tag{55}$$

Moreover, when $\mathcal{M} \neq \mathbb{C}$ there exist non-zero derivations δ_a, δ_b corresponding to normal $a, b \in \mathcal{M}$ such that $\|\delta_a\|_{\infty,1} \leq \widetilde{\Lambda}_{n(\mathcal{M})} \min_{z\in\mathbb{C}} \|a - z\mathbf{1}\|_1$ and $\|\delta_b\|_{\infty,1} = 2\min_{z\in\mathbb{C}} \|b - z\mathbf{1}\|_1$. We remark that

1. *When $n(\mathcal{M}) \notin \{1,2,4\}$ then the distance formula of (52) does not extend to arbitrary normal measurable $a \in L_1(\mathcal{M},\tau) \setminus Z(\mathcal{M})$, since $\widetilde{\Lambda}_{n(\mathcal{M})} < 2$ in these cases.*
2. *When $\mathcal{M}$ is a II_1-factor or a I_n-factor with $n \equiv 0 \mod 3$ then the constant bounds given in (55) can not be improved as in these cases $2\Lambda_{n(\mathcal{M})} = \sqrt{3} = \widetilde{\Lambda}_{n(\mathcal{M})}$.*

Proof. Let $a \in L_1(\mathcal{M},\tau) \setminus Z(\mathcal{M})$ be normal and measurable. By Theorem 5.6 there exist $u, w \in \mathcal{U}(\mathcal{M})$, $z_0 \in \mathbb{C}$ satisfying (42) so that $\|\delta_a\|_{\infty,1} \geq \|\delta_a(u)\|_1 \geq 2\Lambda_{n(\mathcal{M})}\|a - z_0\mathbf{1}\|_1$, from which the first inequality follows. The second inequality follows from the fact that $\|\delta_a(x)\|_1 = \|(a - z\mathbf{1})x - x(a - z\mathbf{1})\|_1 \leq 2\|a - z\mathbf{1}\|_1\|x\|$ holds for any $x \in \mathcal{M}$, $z \in \mathbb{C}$.

For the next statement, we note by (52) that $\|\delta_b\|_{\infty,1} = 2\inf_{z\in\mathbb{C}} \|b - z\mathbf{1}\|_1$ holds for any self-adjoint $b \in \mathcal{M}$, and that when $\mathcal{M} \neq \mathbb{C}$ we can choose b so that moreover $b \notin Z(\mathcal{M})$, ensuring that δ_b is non-zero. Moreover, by Theorem 7.2(2) we obtain a normal $a \in \mathcal{M}$ such that δ_a is a non-zero derivation with $\|\delta_a\|_{\infty,1} \leq \widetilde{\Lambda}_{n(\mathcal{M})}\|a - z\mathbf{1}\|_1$ for every $z \in \mathbb{C}$ since $\delta_a = \delta_{a-z\mathbf{1}}$. Thus $\|\delta_a\|_{\infty,1} \leq \widetilde{\Lambda}_{n(\mathcal{M})} \min_{z\in\mathbb{C}} \|a - z\mathbf{1}\|_1$ (it is clear the minimum exists). The last two remarks follow directly. □

We remark that the above argument actually yields an estimate on the L_1-diameter of the unitary orbit $\mathcal{O}(a) = \{uau^* : u \in \mathcal{U}(\mathcal{M})\}$ of a. Indeed, as we already showed in (54), we obtain by the Russo-Dye Theorem [23, Theorem 1] that $\|\delta_a\|_{\infty,1} = \sup_{u\in\mathcal{U}(\mathcal{M})} \|\delta_a(u)\|_1$. Therefore

$$\mathrm{Diam}_{L_1(\mathcal{M},\tau)}(\mathcal{O}(a)) = \sup_{u\in\mathcal{U}(\mathcal{M})} \|a - uau^*\|_1 = \sup_{u\in\mathcal{U}(\mathcal{M})} \|\delta_a(u)\|_1 = \|\delta_a\|_{\infty,1}.$$

Appendix A.

We prove two technical results concerning the constants Λ_n and $\widetilde{\Lambda}_n$. In Theorem A.1 we will for $n \neq 4$ determine the exact value of Λ_n with the help of Theorem 4.3. In Lemma A.2 we prove the main property of the constants $\widetilde{\Lambda}_n$ that we used in the paper.

Theorem A.1. *We have $\Lambda_1 = \Lambda_2 = 1$, $\frac{\sqrt{3}}{2} \leq \Lambda_4 \leq 1$ and $\Lambda_n = \frac{\sqrt{3}}{2}$ for any $n \notin \{1, 2, 4\}$.*
Moreover, for $n \neq 4$ there exists a $g \in L_\infty(\Omega_n)$, $T \in \mathrm{Aut}_n$, $z \in \mathbb{C}$ such that that $\Lambda(g, T, z) = \Lambda(g) = \Lambda_n$.

Proof. If $n = 1$ then $\Lambda(g, \mathrm{Id}, g(1)) = 1$ for all $g \in \mathcal{S}(\Omega_n)$ since we agreed to count $\frac{0}{0} = 1$. Hence, $\Lambda_1 = 1$. If $n = 2$ then $\Lambda(g, T, \frac{g(1)+g(2)}{2}) = 1$ for all $g \in \mathcal{S}(\Omega_n)$ where $T(1) = 2$. Hence, $\Lambda_2 = 1$. It follows from Theorem 4.3 that $\Lambda_n \geq \frac{\sqrt{3}}{2}$ for all $n \geq 3$. It only remains to show that this is in fact an equality whenever $n = 3$ or $n \geq 5$, which we shall do now. For the given values of n, we can find a partition $\{A_1, A_2, A_3\}$ of Ω_n such that $\frac{1}{5} \leq \frac{\mu_n(A_j)}{\mu_n(\Omega_n)} \leq \frac{2}{5}$ for $j = 1, 2, 3$. Now, denote $w_j := e^{\frac{2\pi i j}{3}}$ for $j = 1, 2, 3$ and construct the function $g = \sum_{j=1}^{3} w_j \chi_{A_j} \in L_\infty(\Omega_n)$. We will show that $\Lambda(g) \leq \frac{\sqrt{3}}{2}$.

Suppose $\Lambda(g) > \frac{\sqrt{3}}{2}$. Then there exists $T \in \mathrm{Aut}_n$, $z_0 \in \mathbb{C}$ and $\lambda > \frac{\sqrt{3}}{2}$ so that
$$|g(T(\omega)) - g(\omega)| \geq \lambda(|g(T(\omega)) - z_0| + |g(\omega) - z_0|)$$
a.e..

We note that for $k \neq l$ we have
$$|w_k - w_l| = \sqrt{3}.$$
Denote $B_{k,j} = A_k \cap T^{-1}(A_j)$ so that $B_{k,j} \subseteq A_k$ and $T(B_{k,j}) \subseteq A_j$. Moreover, since $\{A_1, A_2, A_3\}$ is a partition of Ω_n, we have for $l = 1, 2, 3$ that
$$A_l = B_{l,1} \cup B_{l,2} \cup B_{l,3} \quad T^{-1}(A_l) = B_{1,l} \cup B_{2,l} \cup B_{3,l}. \tag{56}$$
We note that if $\mu_n(B_{k,j} \cup B_{j,k}) > 0$ we must by the assumption have that
$$|w_k - w_j| \geq \lambda(|w_k - z_0| + |w_j - z_0|).$$
This is to say that z_0 lies within the ellipse with foci w_k and w_j and eccentricity λ.

Now suppose $\mu_n(B_{k,k}) > 0$ for some k. Then $z_0 = w_k$ and for $l, j \neq k$ we have
$$|w_l - w_j| \leq \sqrt{3} < 2\lambda < 2\lambda\sqrt{3} = \lambda(|w_l - w_k| + |w_j - w_k|) = \lambda(|w_l - z_0| + |w_j - z_0|)$$
and hence $\mu_n(B_{l,j}) = 0$. However, (56) then implies for $j \neq k$ that
$$\mu_n(A_j) = \mu_n(B_{j,1}) + \mu_n(B_{j,2}) + \mu_n(B_{j,3}) = \mu_n(B_{j,k}).$$
Therefore, using this and (56) we obtain
$$\begin{aligned} 2\mu_n(A_k) &= \mu_n(A_k) + (\mu_n(B_{1,k}) + \mu_n(B_{2,k}) + \mu_n(B_{3,k})) \\ &= \mu_n(A_k) + \left(\sum_{\substack{1 \leq l \leq 3 \\ l \neq k}} \mu_n(B_{l,k}) \right) + \mu_n(B_{k,k}) \\ &= \mu_n(A_k) + \left(\sum_{\substack{1 \leq l \leq 3 \\ l \neq k}} \mu_n(A_l) \right) + \mu_n(B_{k,k}) \\ &= \mu_n(B_{k,k}) + \mu_n(A_1) + \mu_n(A_2) + \mu_n(A_3) \\ &= \mu_n(\Omega_n) + \mu_n(B_{k,k}) > \mu_n(\Omega_n). \end{aligned}$$
Hence $\frac{\mu_n(A_k)}{\mu_n(\Omega_n)} > \frac{1}{2}$, which is a contradiction with the choice of the partition.

We conclude that $\mu_n(B_{k,k}) = 0$ for $k = 1, 2, 3$. Now suppose that for some $1 \leq l, j \leq 3$ with $l \neq j$ we have $\mu_n(B_{l,j} \cup B_{j,l}) = 0$. Let $k \in \{1, 2, 3\}$ such that $k \neq l, j$. Then we obtain $\mu_n(A_l) = \mu_n(B_{l,l}) + \mu_n(B_{j,l}) + \mu_n(B_{k,l}) = \mu_n(B_{k,l})$ and $\mu_n(A_j) = \mu_n(B_{l,j}) + \mu_n(B_{j,j}) + \mu_n(B_{k,j}) = \mu_n(B_{k,j})$. We thus have
$$\begin{aligned} 2\mu_n(A_k) &= \mu_n(A_k) + \mu_n(B_{k,l}) + \mu_n(B_{k,j}) + \mu_n(B_{k,k}) \\ &= \mu_n(A_k) + \mu_n(A_l) + \mu_n(A_j) = \mu_n(\Omega_n) \end{aligned}$$
and thus $\frac{\mu_n(A_k)}{\mu_n(\Omega_n)} = \frac{1}{2}$. This contradicts the choice of the partition sets.

Hence, $\mu_n(B_{l,j} \cup B_{j,l}) > 0$ for all l, j with $l \neq j$. This means that the point z_0 lies in all three ellipses (i.e. for $l \neq j$ the point z_0 has to lie inside the ellipse with foci w_l and w_j and eccentricity λ). We obtain that for $\lambda = \frac{\sqrt{3}}{2}$ the only point in the intersection of the three ellipses is 0, and that for $\lambda > \frac{\sqrt{3}}{2}$ the intersection is empty (see Fig. 4)

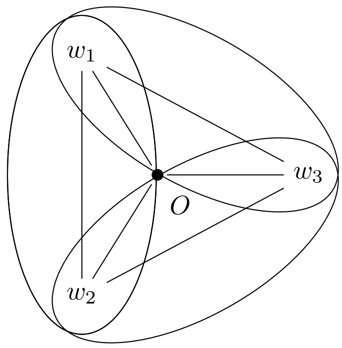

FIGURE 4. The image of the simple function g consists of the three points w_1, w_2 and w_3. The three ellipses with foci w_l and w_j (for l and j different) and eccentricity $\lambda = \frac{\sqrt{3}}{2}$ are drawn. The only point that lies in all three the ellipses is the point $z_0 := 0$.

Hence, $\Lambda(g) \leq \frac{\sqrt{3}}{2}$. Therefore $\Lambda_n = \frac{\sqrt{3}}{2}$. □

Lemma A.2. *Let $1 < n \leq \infty$. Then there is a $g \in L_\infty(\Omega_n)$ with Diam$(g(\Omega_n)) = 1$ and so that $\widetilde{\Lambda}_n = \sup_{z\in\mathbb{C}} \frac{1}{\|g-z\|_1}$.*

Proof. The result for $n = 2$ follows directly by taking $g = \chi_{\{1\}}$.

Thus, suppose $n \geq 3$. We can build a partition $\{A_1, A_2, A_3\}$ of Ω_n so that:

- If $n = 3k,\ k \in \mathbb{N}$, or $n = \infty$, then $\mu_n(A_1) = \mu_n(A_2) = \mu_n(A_3) = \frac{1}{3}$.
- If $n = 3k+1,\ k \in \mathbb{N}$, then $\mu_n(A_1) = \mu_n(A_2) = \frac{k}{n},\ \mu_n(A_3) = \frac{k+1}{n}$.
- If $n = 3k+2,\ k \in \mathbb{N}$, then $\mu_n(A_1) = \mu_n(A_2) = \frac{k+1}{n},\ \mu_n(A_3) = \frac{k}{n}$.

For convenience let us denote

$$a = \mu_n(A_1) = \mu_n(A_2),\ b = \mu_n(A_3),\ w_k = e^{\frac{2\pi k i}{3}},\ k = 0, 1, 2.$$

Define $g_0 \in L_\infty(\Omega_n, \mu_n)$ by

$$g_0 = \chi_{A_1} w_1 + \chi_{A_2} w_2 + \chi_{A_3} w_0.$$

Since $\mu_n(A_1) = \mu_n(A_2)$, it is clear that the minimum of $\mathbb{C} \ni z \mapsto \|g_0 - z\|_1$ is attained for real-valued z, and moreover that $-\frac{1}{2} \leq z \leq 1$. When $n = 4$, it is clear from the triangle inequality that the minimum is attained at the point $t_0 = 1$ and we have $\|g_0 - t_0\|_1 = \frac{\sqrt{3}}{2}$. Now assume $n \neq 4$ so that the ratio $\frac{b}{a}$ satisfies $\frac{b}{a} < \sqrt{3}$ (the ratio $\frac{b}{a}$ is maximal for $n = 7$ in which case we have $\frac{b}{a} = \frac{\frac{3}{7}}{\frac{2}{7}} = \frac{3}{2} < \sqrt{3}$). Hence $\sqrt{3}a - b > 0$. We have for $t \in [-\frac{1}{2}, 1]$ that

$$\|g_0 - t\|_1 = 2a|w_1 - t| + b(1 - t).$$

Then

$$\frac{d}{dt}\|g_0 - t\|_1 = 2a\frac{t + \frac{1}{2}}{|w_1 - t|} - b.$$

As $\frac{d}{dt}\|g_0 - t\|_1$ is negative when evaluated at $-\frac{1}{2}$ and positive when evaluated at 1 (as $\sqrt{3}a - b > 0$), the minimum of $\|g_0 - t\|_1$ must be assumed at a point $t_0 \in [-\frac{1}{2}, 1]$ satisfying

$$b|w_1 - t_0| = 2a(t_0 + \frac{1}{2}).$$

Then

$$b^2((t_0 + \frac{1}{2})^2 + \frac{3}{4}) = 4a^2(t_0 + \frac{1}{2})^2$$

and

$$(t_0 + \frac{1}{2})^2 = \frac{3b^2}{4(4a^2 - b^2)} = \frac{3b^2}{4(2a - b)}$$

since $2a+b=1$. Therefore

$$(t_0+\frac{1}{2})^2+\frac{3}{4}=\frac{3b^2}{4(2a-b)}+\frac{3}{4}=\frac{3a^2}{(2a-b)}$$

and

$$\begin{aligned}\|g_0-t_0\|_1 &= 2a|t_0-w_1|+b(1-t_0)\\ &= 2\frac{\sqrt{3}a^2}{\sqrt{2a-b}}+b-b(\frac{\sqrt{3}b}{2\sqrt{2a-b}}-\frac{1}{2})\\ &= \frac{\sqrt{3}\sqrt{2a-b}}{2}+\frac{3b}{2}\\ &= \frac{\sqrt{3-6b}}{2}+\frac{3b}{2}.\end{aligned}$$

- For $n=3k$ or $n=\infty$ we have $\mu_n(A_3)=\frac{1}{3}$ and find $\|g_0-t_0\|_1=1$.
- For $n=3k+1$ ($n\neq 4$) we have $\mu_n(A_3)=\frac{k+1}{3k+1}$ and find

$$\|g_0-t_0\|_1=\frac{1}{2}\sqrt{\frac{3k-3}{3k+1}}+\frac{1}{2}\cdot\frac{3k+3}{3k+1}.$$

- For $n=3k+2$ we have $\mu_n(A_3)=\frac{k}{3k+2}$ and find

$$\|g_0-t_0\|_1=\frac{1}{2}\sqrt{\frac{3k+6}{3k+2}}+\frac{1}{2}\cdot\frac{3k}{3k+2}.$$

Now, take $g=\frac{1}{\sqrt{3}}g_0$ so that $\mathrm{Diam}(g(\Omega_n))=1$. Then

$$\sup_{z\in\mathbb{C}}\frac{1}{\|g-z\|_1}=\sup_{z\in\mathbb{C}}\frac{\sqrt{3}}{\|g_0-z\|_1}=\frac{\sqrt{3}}{\|g_0-t_0\|_1}=\tilde{\Lambda}_n.$$

□

Acknowledgements

We like to thank Jinghao Huang and Thomas Scheckter for providing feedback that helped improve the exposition.

References

[1] C.A. Akemann, T. Andersen, G.K. Pedersen, *Triangle inequalities in operator algebras*, Linear and Multilinear Algebra, vol. 11, 2 (1982), 167–178.

[2] U. Bader, T. Gelander, N. Monod,*A fixed point theorem for L_1 spaces*, Invent. Math. 189 (2012), 143–148.

[3] A. Ber, J. Huang, F. Sukochev, *Norms of skew-adjoint derivations with values in the predual of a semifinite von Neumann algebra*, Put into print.

[4] A. Ber, F. Sukochev, *Commutator estimates in W^*-factors,* Trans. Amer. Math. Soc. 364(10) (2012), 5571–5587.

[5] A.F. Ber, F.A. Sukochev, *Commutator estimates in W^*-algebras*, J. Funct. Anal., 262 (2012), 537–568.

[6] V.I. Bogachev, *Measure Theory, volume II*, Springer-Verlag, (2007)

[7] J. Bunce, W. Paschke, *Derivations on a C^*-algebra and its double dual*, J. Funct. Anal. 37 (1980), 235–247.

[8] A. Connes *On the cohomology of operator algebras*, J. Funct. Anal 28(2) (1978) 248–253.

[9] V.I. Chilin, F.A. Sukochev, *Weak convergence in non-commutative symmetric spaces* J. Operator Theory 31 (1994), no. 1, 35–65.

[10] P.G. Dodds, B. de Pagter, F.A. Sukochev, *Non-Commutative Integration*, Springer Nature, 2022.

[11] K. Dykema, F. Sukochev, D. Zanin, *A decomposition theorem in II_1 -factors* J. Reine Angew. Math. 708 (2015), 97–114.

[12] T. Fack, H. Kosaki, *Generalized s-numbers of τ-measurable operators* Pacific J. Math. 123 (1986), no. 2, 269–300.

[13] P. Gajendragadkar, *Norm of a derivation on a von Neumann algebra*, Transactions of the American Mathematical Society, 170 (1972), 165–170

[14] U. Haagerup, *All nuclear C^*-algebras are amenable,* Invent. Math. 74 (1983), 305–319.

[15] F. Hiai *Lectures on Selected Topics in von Neumann Algebras* EMS. Press. (2021)

[16] A.J. Hoffman, H.W. Wielandt, *The variation of the spectrum of a normal matrix,* Selected papers of Alan J. Hoffman: With commentary, (1952), 118-120

[17] J. Huang, F. Sukochev *Derivations with values into bimodules into τ-measurable operators*, Submitted manuscript.

[18] R.V. Kadison, E.C. Lance, J.R. Ringrose, *Derivations and automorphisms of operator algebras. II*, J. Funct. Anal., 1(2) (1967), 204–221

[19] R.V. Kadison, Z. Liu, *A note on derivations of Murray–von Neumann algebras*, PNAS, vol. 111, no. 6, (2014) 2087–2093.

[20] R.V. Kadison, Z. Liu, A. Thom, *A note on commutators in algebras of unbounded operators*, arXiv:1901.10711.

[21] B. Magajna *The Haagerup norm on the tensor product of operator modules*, J. Funct. Anal., 129(2) (1995), 325–348

[22] W. Rudin, *Functional analysis*, International Editions 1991.
[23] B. Russo, H.A. Dye, *A note on unitary operators in* C^**-algebras* Duke Math. J, 33 (1966), 413–416
[24] A Sinclair, R. Smith *Hochschild cohomology of von Neumann algebras*, vol 203, Cambridge University Press (1995).
[25] J.G. Stampfli, *The norm of a derivation*, Pacific P. Math. 33(3) (1970), 737-747
[26] M. Takesaki, *Theory of operator algebras. II.* Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer–Verlag, Berlin, 2003.
[27] L. Zsidó, *The norm of a derivation in a* W^**-algebra*, Proceedings of the American Mathematical Society, 38(1) (1973), 147–150

Department of Mathematics, National University of Uzbekistan, Vuzgorodok, 100174, Tashkent, Uzbekistan, *E-mail :* ber@ucd.uz

Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA, Delft, The Netherlands, *E-mail :* m.j.borst@tudelft.nl

School of Mathematics and Statistics, University of New South Wales, Kensington, 2052, NSW, Australia *E-mail :* f.sukochev@unsw.edu.au