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Inversion of the Multidimensional Marchenko Equation

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SUMMARY

Focusing functions are defined as wavefields that focus at a specified location in a heterogeneous subsurface. These functions can be directly related to Green's functions and hence they can be used for seismic imaging of complete wavefields, including not only primary reflections but all orders of internal multiples. Recently, it has been shown that focusing functions can be retrieved from single-sided reflection data and an initial operator (which can be computed in a smooth background velocity model of the subsurface) by iterative substitution of the multidimensional Marchenko equation. In this work, we show that the Marchenko equation can also be inverted directly for the focusing functions. Although this approach is computationally more expensive than iterative substitution, additional constraints can easily be imposed. Such a flexibility might be beneficial in specific cases, for instance when the recorded data are incomplete or when additional measurements (e.g. from downhole receivers) are available.

Introduction

The pioneering work of Broggini and Snieder (2012) revealed that Green's functions in an unknown acoustic medium can be retrieved from a single-sided reflection response by an iterative scheme that is derived from the 1D Marchenko equation. This idea was extended by Wapenaar et al. (2014) for the retrieval of Green's functions in multidimensional media. The retrieved Green's functions find valuable applications in modern seismic imaging schemes, where not only primary reflections but also internal multiples are correctly migrated (Slob et al., 2014; Broggini et al., 2014). Extensions of the scheme have been presented for elastic media (da Costa Filho et al., 2014; Wapenaar and Slob, 2014) and it has been shown how free-surface multiples can also be included (Singh et al., 2014). In all these cases, the multidimensional Marchenko equation is solved by iterative substitution. In this work, we will show that this equation can also be directly inverted.

Theory

To invert the multidimensional Marchenko equation, that was derived by Wapenaar et al. (2014), it is useful to cast the relevant representations in a discrete framework of matrix-vector multiplications. In this framework, that is discussed extensively by Van der Neut et al. (2015), Green's functions in the subsurface are represented as vectors, in which all relevant traces are concatenated in the time-space domain. Vector $\mathbf{g}^-$ contains the upgoing Green's functions with multiple sources located at the surface and a receiver positioned at a specified focal point. Vector $\mathbf{g}^{+\star}$ contains the equivalent downgoing Green's function, but its traces are time-reversed, which is indicated by the superscript *. In the representations of Wapenaar et al. (2014), these Green's functions are related to so-called focusing functions that, when injected into the subsurface from the acquisition level at the Earth's surface, focus at the focal point. We distinguish the downgoing part of the focusing function $\mathbf{f}_1^+$ and the upgoing part $\mathbf{f}_1^-$. The downgoing part can be partitioned as $\mathbf{f}_1^+ = \mathbf{f}_{1d}^+ + \mathbf{f}_{1m}^+$, where $\mathbf{f}_{1d}^+$ is related to the direct wavefield and $\mathbf{f}_{1m}^+$ is a scattering coda. In Marchenko imaging, it is assumed that $\mathbf{f}_{1d}^+$ is known. This wavefield is generally estimated by modeling the direct wavefield in a macro velocity model and reversing it in time (Broggini et al., 2014). The focusing functions and Green's functions are related through two representations that can be concatenated as

$$\begin{pmatrix} -\mathbf{g}^- \\ \mathbf{g}^{+\star} \end{pmatrix} = \left[\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{R} \\ \mathbf{R}^\star & \mathbf{0} \end{pmatrix} \right] \begin{pmatrix} \mathbf{f}_1^- \\ \mathbf{f}_{1d}^+ + \mathbf{f}_{1m}^+ \end{pmatrix}. \tag{1}$$

In this equation, matrices $\mathbf{I}$ and $\mathbf{0}$ are the identity matrix and a matrix with zeros, respectively. Matrix $\mathbf{R}$ applies multidimensional convolution with the reflection response, while $\mathbf{R}^\star$ applies multidimensional crosscorrelation. Since equation 1 has four unknowns ($\mathbf{g}^-$, $\mathbf{g}^{+\star}$, $\mathbf{f}_1^-$ and $\mathbf{f}_{1m}^+$), it does not have a unique solution. Fortunately, the Green's functions and focusing functions map differently in the time-space domain and therefore they can be separated by a window function. We introduce this window as a matrix Θ that removes all data after the direct wavefield, including the direct wavefield itself. This window is symmetric in time, such that all data before the time-reversed direct wavefield is also removed. Because of causality, the Green's functions are assumed to be zero before the direct wavefield. Hence they vanish when Θ is applied, i.e. $\Theta\mathbf{g}^- = \mathbf{0}$ and $\Theta\mathbf{g}^{+\star} = \mathbf{0}$. The direct part of the focusing function is also removed by the window: $\Theta\mathbf{f}_{1d}^+ = \mathbf{0}$. The coda and the upgoing part, however, contain data only before the direct wavefield (and after the time-reversed direct wavefield) (Wapenaar et al., 2014; Van der Neut et al., 2015). Hence they are preserved by the window function: $\Theta\mathbf{f}_{1m}^+ = \mathbf{f}_{1m}^+$ and $\Theta\mathbf{f}_1^- = \mathbf{f}_1^-$. Based on these observations, we can apply $\begin{pmatrix} \Theta & 0 \\ 0 & \Theta \end{pmatrix}$ to equation 1 and rewrite the result as

$$\begin{pmatrix} \mathbf{0} & \Theta\mathbf{R} \\ \Theta\mathbf{R}^\star & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{f}_{1d}^+ \end{pmatrix} = \left[\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \Theta\mathbf{R} \\ \Theta\mathbf{R}^\star & \mathbf{0} \end{pmatrix} \right] \begin{pmatrix} \mathbf{f}_1^- \\ \mathbf{f}_{1m}^+ \end{pmatrix}. \tag{2}$$

Equation 2 can be recognized as a (discretized) Fredholm integral equation of the second kind ($\mathbf{y} = [\mathbf{I} - \mathbf{A}]\,\mathbf{x}$), which can be expanded as a Neumann series ($\mathbf{x} = \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{y}$ given that $|\mathbf{A}^k \mathbf{y}|_2 \to 0$ as $k \to \infty$), yielding

$$\begin{pmatrix} \mathbf{f}_1^- \\ \mathbf{f}_{1m}^+ \end{pmatrix} = \sum_{k=1}^{\infty} \begin{pmatrix} \mathbf{0} & \Theta\mathbf{R} \\ \Theta\mathbf{R}^\star & \mathbf{0} \end{pmatrix}^k \begin{pmatrix} \mathbf{0} \\ \mathbf{f}_{1d}^+ \end{pmatrix}. \tag{3}$$

Equation 3 is a concise representation of the iterative scheme that was proposed by Slob et al. (2014) and several others. Alternatively, we can invert equation 2 directly, e.g. by least-squares inversion. From practical experience, we have learned that straightforward inversion of this equation yields unphysical solutions, containing data after the direct wave (and before the time-reversed direct wave). To avoid these solutions, we replace $\begin{pmatrix} \mathbf{f}_1^- \\ \mathbf{f}_{1m}^+ \end{pmatrix}$ in this equation by $\begin{pmatrix} \Theta & 0 \\ 0 & \Theta \end{pmatrix}\begin{pmatrix} \mathbf{f}_1^- \\ \mathbf{f}_{1m}^+ \end{pmatrix}$, leading to

$$\begin{pmatrix} \mathbf{0} & \Theta\mathbf{R} \\ \Theta\mathbf{R}^\star & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} \\ \mathbf{f}_{1d}^+ \end{pmatrix} = \left[\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \Theta\mathbf{R} \\ \Theta\mathbf{R}^\star & \mathbf{0} \end{pmatrix}\right]\begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \Theta \end{pmatrix}\begin{pmatrix} \mathbf{f}_1^- \\ \mathbf{f}_{1m}^+ \end{pmatrix}. \tag{4}$$

In the following example, we will show that this equation can be successfully inverted for the unknowns $\mathbf{f}_1^-$ and $\mathbf{f}_{1m}^+$. Having retrieved these focusing functions, the Green's functions can be computed with equation 1.

Example

We illustrate Marchenko redatuming by inversion with a numerical example. In Figure 1a, we show a velocity model. We place 301 sources and 301 receivers on a fixed grid at the surface with 10m spacing (laid out between $x = -1500m$ and $x = +1500m$). A focal point is selected at $(x,z) = (0m, 1100m)$. We compute the wavefield from the surface to the focal point in the true model and remove the scattered coda. The direct part of the focusing function is constructed by reversing this truncated wavefield in time. Our aim is to reconstruct the scattering coda of the Green's function, by first retrieving the focusing function and subsequently applying equation 1. We investigate two options to achieve this. First, the focusing functions are retrieved by iterative substitution as in equation 2. Second, they are retrieved by direct inversion of equation 4, for which we make use of the least-squares algorithm LSQR that was introduced by Pauge and Saunders (1982). In Figures 1b and 1c, we show the retrieved down- and upgoing parts of the focusing functions, using both methods. A good match is found, confirming that the Marchenko equation can indeed be directly inverted, using the framework that we derived. We compute the up- and downgoing Green's functions with equation 1 and show them in Figures 2a and 2b. To confirm these results, the down- and upgoing parts are added and compared with the exact Green's function that was obtained by placing a source at the focal point, see Figure 2c. The responses match well, demonstrating that we indeed have managed to retrieve the scattering coda of the Green's function from the reflection response at the surface.

Discussion

Now that the Marchenko equation can be solved by inversion, additional constraints can easily be imposed. For example, it is possible to include the recordings from a Vertical Seismic Profiling (VSP) experiment as an additional equation. In this way, the recorded Green's function from a surface location to a downhole receiver can be combined with surface seismic data to predict the wavefield from another surface location to that same receiver. In this way, VSP coverage can be effectively extended or interpolated by utilizing additional surface seismic data.

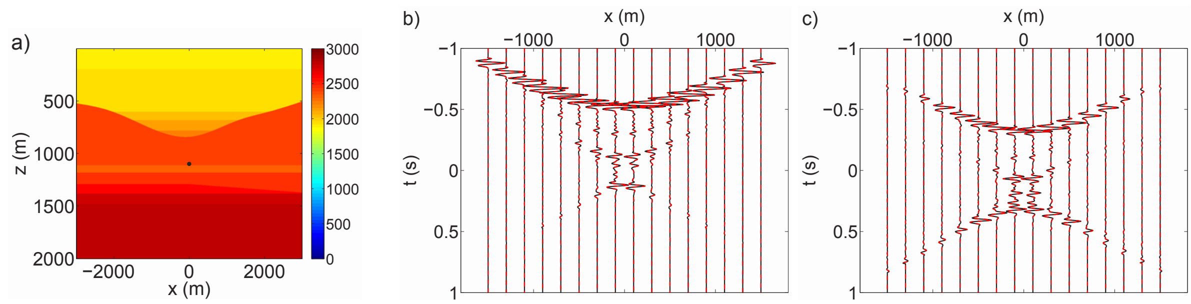

Figure 1 *a) Velocity model (in m/s) of the subsurface with the focal point indicated by the black dot. b) Retrieved downgoing focusing function* $\mathbf{f}_1^+ = \mathbf{f}_{1d}^+ + \mathbf{f}_{1m}^+$ *obtained by inversion (in red) and iterative substitution (in black). c) Same for the upgoing focusing function* $\mathbf{f}_1^-$.

Note that for the inversion of equation 4, we do not necessarily require complete data, as we do for iterative substitution (Van der Neut et al., 2015). Even if a system of equations is underdetermined, irregularly sampled and corrupted with noise, satisfying solutions can sometimes still be obtained by posing additional constraints on the solution (Aravkin et al., 2014). With an inversion-based Marchenko methodology, we might benefit from similar advantages.

Singh et al. (2014) have shown how free-surface multiples can be included in the multidimensional Marchenko equation. From their forward model, the following representation can be derived, using a derivation akin to that of equation 4:

$$\begin{pmatrix} \mathbf{0} & \Theta\overline{\mathbf{R}} \\ \Theta\overline{\mathbf{R}}^\star & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{f}_{1d}^+ \end{pmatrix} = \left[\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} - \begin{pmatrix} -r\Theta\overline{\mathbf{R}} & \Theta\overline{\mathbf{R}} \\ \Theta\overline{\mathbf{R}}^\star & -r\Theta\overline{\mathbf{R}}^\star \end{pmatrix} \right] \begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \Theta \end{pmatrix} \begin{pmatrix} \mathbf{f}_1^- \\ \mathbf{f}_{1m}^+ \end{pmatrix}. \quad (5)$$

Here, $\overline{\mathbf{R}}$ and $\overline{\mathbf{R}}^\star$ are matrices that apply multidimensional convolution and crosscorrelation with the data with free-surface multiples (indicated by the bar) and r is the reflectivity of the surface. This equation can also be inverted for the desired focusing functions. In both equations 4 and 5, the source signature should be deconvolved from the data prior to inversion. Alternatively, we can include this signature in the representation. Let $\mathbf{Q}$ and $\mathbf{Q}^\star$ be matrices that apply convolution and crosscorrelation with the source signature. Now, $\overline{\mathbf{P}} = \mathbf{Q}\overline{\mathbf{R}}$ and $\overline{\mathbf{P}}^\star = \mathbf{Q}^\star\overline{\mathbf{R}}^\star$ can be defined as matrices that apply multidimensional convolution and crosscorrelation with the recorded data (including source signature and free-surface multiples). After applying $\begin{pmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{Q}^\star \end{pmatrix}$ to equation 5, we find the following inverse problem:

$$\begin{pmatrix} \mathbf{0} & \Theta\overline{\mathbf{P}} \\ \Theta\overline{\mathbf{P}}^\star & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{f}_{1d}^+ \end{pmatrix} = \left[\begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^\star \end{pmatrix} - \begin{pmatrix} -r\Theta\overline{\mathbf{P}} & \Theta\overline{\mathbf{P}} \\ \Theta\overline{\mathbf{P}}^\star & -r\Theta\overline{\mathbf{P}}^\star \end{pmatrix} \right] \begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \Theta \end{pmatrix} \begin{pmatrix} \mathbf{f}_1^- \\ \mathbf{f}_{1m}^+ \end{pmatrix}. \quad (6)$$

The structure of this equation bears much similarity to the forward model for free-surface multiple elimination (Lin and Herrmann, 2013). The solutions of both problems rely heavily on the source signature, which is generally not sufficiently known. However, it has been shown for the free-surface demultiple problem that the source signature can be estimated during inversion, e.g. by alternating optimization (Lin and Herrmann, 2013). Building on the mathematical similarity between both problems, we are currently investigating whether the focusing function and the (unknown) source signature can be estimated simultaneously by applying a similar type of algorithm for the inversion of equation 6.

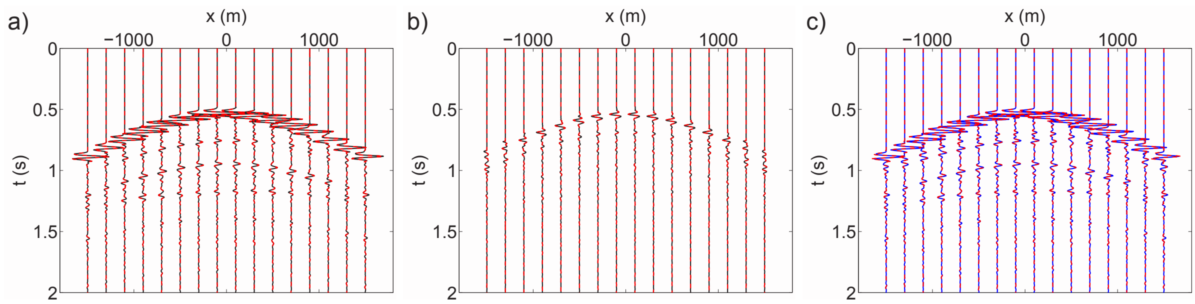

Figure 2 *a) Retrieved downgoing Green's function* $\mathbf{g}^+$ *(after time-reversal of* $\mathbf{g}^{+\star}$*) obtained by inversion (in red) and iterative substitution (in black). Same for the upgoing Green's function* $\mathbf{g}^-$*. c) Green's function with a source at the focal point obtained by finite-difference modeling (in blue) versus the retrieved Green's function by inversion (obtained by adding* $\mathbf{g}^+$ *and* $\mathbf{g}^-$*).*

Conclusion

We have demonstrated that the multidimensional Marchenko equation can be solved by inversion. If no additional constraints are imposed, this yields a solution that is similar to the solution of an iterative scheme that has recently been developed. Although retrieving Green's functions by inversion is computationally expensive, this strategy allows for imposing additional constraints and it does not necessarily require complete reflection data.

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