Can we improve the triangle inequality in L^p?

by

Veronique Bogouslavskii

to obtain the Bachelor degree of Applied Mathematics at the Delft University of Technology.

To be defended on July 16th, 2026 at 05:00 PM.

Student number:	6040551	
Project duration:	27.02.2026–17.07.2026	
Thesis committee:	Prof. Dr. Ir. M.C. (Mark) Veraar	TU Delft, supervisor
	Dr. D. (David) de Laat	TU Delft

TU Delft

Lay summary

This thesis is about adding two mathematical functions together. If we know how large each function is separately, we can estimate how large their sum will be. However, this estimate becomes more accurate if it is also known how much the two functions overlap. If two functions are large in the same places, their sum can be larger than if they are large in different places.

In this thesis, I study how this overlap affects the best possible estimates for the size of the sum. The problem can be represented using three quantities: the size of the first function, the size of the second function, and the amount of overlap between them. These three quantities can be studied using a geometric picture. In this picture, two special surfaces, called F_p and G_p, describe the best possible upper and lower estimates.

I also show that these estimates are optimal. This means that they cannot be improved if we only use the three quantities mentioned above. Finally, I study what happens when the power in the estimate is changed from p to another exponent q.

This research contributes to a better understanding of how overlap between functions influences optimal L^p-inequalities. The reduction to finite-dimensional optimization provides additional insight into the structure of extremizers and the role of the overlap parameter.

Abstract

This thesis studies sharp inequalities for sums of two functions in L^p. The classical triangle inequality gives a general upper bound for $\|f+g\|_p$, but it does not use information about how much the two functions overlap. The aim of this thesis is to obtain sharper bounds by also taking into account the quantity $\|fg\|_{p/2}^{p/2}$. Together with $\|f\|_p^p$ and $\|g\|_p^p$, this overlap term gives an associated triple

$$(x, y, z) = \left(\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}\right),$$

which lies in a three-dimensional admissible cone.

The main part of this thesis is based on the envelope method developed by Ivanisvili and Mooney, together with related work on sharpened triangle inequalities in L^p [1, 2, 3, 4, 5]. The problem of estimating $\|f+g\|_p^p$ is translated into a geometric problem on the admissible cone. Two envelope functions, denoted by F_p and G_p, are constructed from the boundary data of the cone. Depending on the value of $p > 0$, these envelopes give the sharp upper and lower bounds for $\|f+g\|_p^p$. The thesis explains how these envelopes are constructed, how their concavity and convexity are checked, and why they give optimal inequalities. It also discusses equality and extremal cases by studying the affine regions of the envelopes.

In the final part of the thesis, the p-power problem is extended to q-powers. Instead of estimating only $\|f+g\|_p^p$, we study inequalities for

$$\|f+g\|_p^q,$$

where $q > 2$. Using the sharp p-power envelope, the infinite-dimensional problem is reduced to a finite-dimensional optimization problem. The behavior of this optimization depends on the exponent q/p. For $2 < q \leq p$, the problem reduces to a proportional family of functions. For $q \geq p$, the optimization is controlled by the balanced case $x = y$. This shows that the envelope method remains useful for q-powers, but that an additional optimization step is needed once the final power is changed.

Beyond the obtained inequalities, the results illustrate how envelope methods can reduce functional optimization problems to finite-dimensional optimization problems. This provides additional insight into the structure of sharp inequalities and the role of overlap between functions.

Keywords: L^p-spaces, triangle inequality, convex and concave envelopes, power-type extensions.

Contents

1

Introduction

"A mathematician, like a painter or a poet, is a maker of patterns."

— G. H. Hardy

The triangle inequality is one of the most fundamental inequalities in analysis. In L^p, the triangle inequality states that for all $p \in [1, \infty]$ and $f, g \in L^p$,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

This inequality is one of the reasons why L^p-spaces are normed spaces for $p \geq 1$. The triangle inequality gives the sharpest upper bound that is valid for all functions. However, for specific pairs of functions it may greatly overestimate. In particular, it does not distinguish between functions that strongly overlap and functions with disjoint supports. For example, the inequality treats the case where f and g are supported on the same set in the same way as the case where their supports are disjoint. In many situations this is too rough, since the size of $\|f + g\|_p$ depends strongly on the interaction between f and g.

This becomes particularly clear in the case $p = 2$. If $f, g \in L^2$ are real-valued functions, then expanding the square gives

$$\|f + g\|_2^2 = \|f\|_2^2 + 2 \int fg \, d\mu + \|g\|_2^2.$$

If f and g are non-negative, then this can be written as

$$\|f + g\|_2^2 = \|f\|_2^2 + 2\|fg\|_1 + \|g\|_2^2.$$

More generally, by taking absolute values, we obtain the estimate

$$\|f + g\|_2^2 \leq \|f\|_2^2 + 2\|fg\|_1 + \|g\|_2^2.$$

The term $\|fg\|_1$ measures the overlap between f and g. If f and g have disjoint supports, then $fg = 0$ almost everywhere and the estimate reduces to

$$\|f + g\|_2^2 = \|f\|_2^2 + \|g\|_2^2.$$

On the other hand, if $f = g$, then the overlap is maximal and the inequality becomes an equality.

Thus, in L^2, for nonnegative functions, the size of $\|f + g\|_2$ is completely controlled by three quantities:

$$\|f\|_2, \qquad \|g\|_2, \qquad \|fg\|_1.$$

The subsequent natural question is whether an analogous refinement exists in L^p-spaces for $p \neq 2$.

The usual convexity estimate gives

$$\|f+g\|_p^p \le 2^{p-1}\big(\|f\|_p^p+\|g\|_p^p\big), \qquad p\ge 1.$$

This follows from the convexity of the function $t\mapsto |t|^p$. However, this upper bound again ignores the overlap between f and g. While it is sharp when f and g are equal, it is far from optimal when f and g have disjoint supports. Therefore, the aim is to replace this convexity estimate by a sharper inequality that includes an overlap term.

For $p>0$, the natural extension of the L^2-overlap term $\|fg\|_1$ is:

$$\|fg\|_{p/2}.$$

We seek to find a sharp estimate for $\|f+g\|_p^p$ depending only on

$$\|f\|_p, \qquad \|g\|_p \qquad \text{and} \quad \|fg\|_{p/2}.$$

The restriction to these three quantities is important. It means that the estimate should not use any additional information about the functions, such as the shape of their supports or higher-order interaction terms. The problem is therefore not merely to find a bound, but to find the best possible bound under this restriction.

This restricted formulation leads naturally to the question of whether the classical triangle inequality can be sharpened in a way that depends only on the relative overlap between the two functions. Carbery raised this question in 2009, and it has since been studied by Carlen, Frank, Ivanisvili, Lieb, and others. [1, 2, 3]. They found sharpened triangle inequalities in L^p in which the upper and lower bound depend on the relative overlap between f and g. As an example, for $p\in(0,1]\cup[2,\infty)$, and any non-negative functions f,g one obtains an upper bound of the form

$$\|f+g\|_p^p \le \left(\left(\frac{1+\sqrt{1-\Gamma_p^2}}{2}\right)^{1/p}+\left(\frac{1-\sqrt{1-\Gamma_p^2}}{2}\right)^{1/p}\right)^p (\|f\|_p^p+\|g\|_p^p). \tag{1.1}$$

with

$$\Gamma_p := \frac{2\|fg\|_{p/2}^{p/2}}{\|f\|_p^p+\|g\|_p^p}.$$

Take

$$C_p := \frac{\min\{\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}\}}{\|fg\|_{p/2}^{p/2}}.$$

Then for any $p\in(1,2)$ and any nonnegative f,g on any measure space we have

$$\|f+g\|_p^p \le \|f\|_p^p+\|g\|_p^p+\big((C_p^{-1/p}+C_p^{1/p})^p - C_p^{-1}-C_p\big)\|fg\|_{p/2}^{p/2}. \tag{1.2}$$

For other ranges of p, the direction of the inequality is reversed. This result already shows that the classical triangle inequality can be sharpened by including only the overlap term $\|fg\|_{p/2}$. It also shows that the exponent of the overlap term is optimal.

The main focus of this thesis is the work of Ivanisvili and Mooney, *Sharpening the Triangle Inequality: Envelopes Between L^2 and L^p Spaces* [4]. Their work gives a more precise and more geometric answer to the same problem. Instead of proving only one explicit inequality, they construct envelope functions that give the sharpest possible bounds depending on the three quantities

$$x=\|f\|_p^p, \qquad y=\|g\|_p^p, \qquad z=\|fg\|_{p/2}^{p/2}.$$

In this way, the problem is transformed into a geometric problem in the variables (x,y,z), where the relevant domain is the cone

$$\Omega=\{(x,y,z): x\ge 0,\ y\ge 0,\ 0\le z\le \sqrt{xy}\}.$$

The inequality is then obtained by studying the convex or concave envelope of a suitable one-homogeneous function on this cone.

The term “envelope” can be understood as follows. Suppose we start with a function that describes the desired estimate on simple or extreme configurations. This function may not itself have the convexity or concavity needed to apply Jensen’s inequality. By taking its convex or concave envelope, one constructs the largest convex function below it, or the smallest concave function above it, depending on the range of p. After integration, the envelope gives the optimal inequality.

The advantage of this method is that it gives the best possible bound among all inequalities depending only on $\|f\|_p$, $\|g\|_p$, and $\|fg\|_{p/2}$. Furthermore, the envelope method clarifies the equality cases. Equality in the final inequality can occur only when the pointwise triples

$$\left(f^p, g^p, (fg)^{p/2}\right)$$

lie almost everywhere in a region where the envelope is affine. Ivanisvili and Mooney show that these affine regions correspond to a small number of special configurations. In related endpoint or boundary cases, disjoint support also appears as an extremal situation. Thus the geometry of the envelope directly explains the structure of the equality cases.

The aim of this thesis is to give an explanation of these sharpened triangle inequalities. We begin by recalling the preliminaries on L^p-spaces, Minkowski’s inequality, convexity, and Jensen’s inequality in Chapter 2. In Chapter 3, we study the two-function problem for positive exponents $p > 0$. Following Ivanisvili and Mooney, we introduce the three quantities

$$x = \|f\|_p^p, \qquad y = \|g\|_p^p, \qquad z = \|fg\|_{p/2}^{p/2}.$$

These quantities form an admissible cone, and the problem of estimating $\|f + g\|_p^p$ is reduced to an envelope problem on this cone.

The negative range requires a separate framework involving negative moments and extended-valued functions, and is therefore outside the scope of this thesis.

We construct two envelope functions, denoted by F_p and G_p, and explain how their concavity and convexity properties determine the sharp upper and lower bounds. We also discuss why these bounds are optimal and how the affine regions of the envelopes lead to extremal and equality cases.

In the final chapter, we consider an extension to q-powers. Instead of only estimating $\|f+g\|_p^p$, we study inequalities for $\|f + g\|_p^q$ with $q > 2$. The sharp p-power envelope F_p is still the starting point, but the change from the power p to the power q introduces an additional finite-dimensional optimization problem. This optimization depends on the value of q/p, and naturally splits into the two ranges $2 < q \leq p$ and $q \geq p$.

Throughout the thesis, the main emphasis is that the estimates depend only on

$$\|f\|_p^p, \qquad \|g\|_p^p, \qquad \|fg\|_{p/2}^{p/2}.$$

No further information about the functions, such as the precise shape of their supports, is used. In this sense, the results can be viewed as L^p-extensions of the exact L^2-identity, where the size of the sum is determined by the two individual sizes and their overlap.

2

Preliminaries

Throughout this chapter, $(S, \mathcal{A}, \mu)$ denotes a measure space. Unless stated otherwise, all functions are assumed to be measurable and real-valued.

This chapter collects the background material needed in the rest of the thesis. We first recall the classical L^p setting and the triangle inequality. We then explain why the usual triangle inequality is not always optimal, and why an overlap term naturally appears. Finally, we introduce Jensen's inequality, one-homogeneous functions, the cone Ω, and convex and concave envelopes.

2.1. L^p-spaces

We use the standard definition of L^p as presented in the TU Delft lecture notes on real analysis [6]:
Definition 2.1 (L^p)**.** Let $p \in [1, \infty)$,we define

$$L^p(S) := \left\{ f : S \to \mathbb{R} : f \text{ is measurable and } \int_S |f|^p \, d\mu < \infty \right\}.$$

For $f \in L^p(S)$, the L^p-norm is defined by

$$\|f\|_p := \left(\int_S |f|^p \, d\mu \right)^{1/p}.$$

Equivalently,

$$\|f\|_p^p = \int_S |f|^p \, d\mu.$$

The quantity $\|f\|_p$ measures the size of a function in an averaged sense. In this thesis, we often work with the p-th power of the norm,

$$\|f\|_p^p,$$

because the sharpened inequalities are naturally written in terms of these quantities.

Definition 2.2 (Normed space)**.** Let V be a vector space over $\mathbb{R}$. A function

$$\| \cdot \| : V \to [0, \infty)$$

is called a norm if, for all $x, y \in V$ and all $\lambda \in \mathbb{R}$, it satisfies

$$\|x\| = 0 \quad \Longleftrightarrow \quad x = 0,$$

$$\|\lambda x\| = |\lambda| \, \|x\|,$$

and

$$\|x + y\| \leq \|x\| + \|y\|.$$

A vector space equipped with a norm is called a normed space.

For $p \geq 1$, the space $L^p(S)$, equipped with the norm

$$\|f\|_p = \left(\int_S |f|^p \, d\mu \right)^{1/p},$$

is a normed space. The triangle inequality for this norm is precisely Minkowski's inequality, which is stated in the next section.

Remark (The range $0 < p < 1$). For $0 < p < 1$, the quantity

$$\|f\|_p = \left(\int_S |f|^p \, d\mu \right)^{1/p}$$

is not a norm, since the triangle inequality does not hold. However, the quantity

$$\|f\|_p^p = \int_S |f|^p \, d\mu$$

is still well defined. Since the envelope inequalities are naturally formulated in terms of the p-th powers, we also include the range $0 < p < 1$.

2.2. Minkowski's inequality

The most fundamental inequality for L^p spaces is Minkowski's inequality.

Proposition 2.1 (Minkowski's inequality). *Let $p \in [1, \infty)$. For all real-valued nonnegative functions $f, g \in L^p(S)$, we have $f + g \in L^p(S)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Minkowski's inequality is the triangle inequality in L^p. It says that the size of the sum of two functions is at most the sum of their sizes.

In this thesis, we study refinements of this inequality. More precisely, we want to understand when one can obtain a sharper estimate for $\|f + g\|_p^p$ by using information about how much f and g overlap.

2.3. A basic convexity estimate

Definition 2.3 (Convex function). Let $C \subseteq \mathbb{R}^n$ be a convex set. A function

$$\Phi : C \to \mathbb{R}$$

is called convex if for all $x, y \in C$ and all $\lambda \in [0, 1]$,

$$\Phi(\lambda x + (1 - \lambda)y) \leq \lambda\Phi(x) + (1 - \lambda)\Phi(y).$$

For $p \geq 1$, the function $x \mapsto |x|^p$ is convex. Hence, for all $a, b \in \mathbb{R}$,

$$\left| \frac{a + b}{2} \right|^p \leq \frac{|a|^p + |b|^p}{2}.$$

Multiplying by 2^p, we get

$$|a + b|^p \leq 2^{p-1} \left(|a|^p + |b|^p \right).$$

Applying this pointwise to $a = f(s)$ and $b = g(s)$, and integrating over S, gives

$$\|f + g\|_p^p \leq 2^{p-1} \left(\|f\|_p^p + \|g\|_p^p \right). \tag{2.1}$$

This estimate is useful, but it does not use any information about the relation between f and g. For example, it does not distinguish between the case where f and g are equal and the case where their supports are disjoint.

2.4. Sharpness

Definition 2.4 (Sharp inequality)**.** An inequality is called sharp if its constant or bound cannot be improved in general.

The constant 2^{p-1} in (2.1) is sharp in general. Indeed, take $f = g$. Then

$$\|f+g\|_p^p = \|2f\|_p^p = 2^p\|f\|_p^p.$$

The right-hand side of (2.1) becomes

$$2^{p-1}\left(\|f\|_p^p + \|f\|_p^p\right) = 2^p\|f\|_p^p.$$

Thus equality holds. Therefore, the constant 2^{p-1} cannot be replaced by a smaller constant that works for all functions f and g.

However, although the constant is sharp in general, it is not always optimal for specific pairs of functions. This is the starting point of the sharpened triangle inequalities studied in this thesis.

2.5. Disjoint supports and overlap

Definition 2.5 (Support and disjoint support)**.** Let $f : S \to \mathbb{R}$ be measurable. The support of f is the set

$$\operatorname{supp}(f) := \{s \in S : f(s) \neq 0\}.$$

We say that f and g have disjoint supports if

$$f(s)g(s) = 0$$

for almost every $s \in S$.

If f and g have disjoint supports, then for $p > 0$,

$$|f+g|^p = |f|^p + |g|^p$$

almost everywhere. Therefore,

$$\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p.$$

This is much sharper than the general estimate (2.1).

This motivates the use of an overlap term. For nonnegative functions, a natural overlap quantity is

$$\|fg\|_{p/2}^{p/2} = \int_S (fg)^{p/2}\, d\mu.$$

This quantity is zero when f and g have disjoint supports, and it becomes larger when f and g are simultaneously large on the same part of the measure space.

2.6. Hölder's inequality and Cauchy-Schwarz

We will use Hölder's inequality and its special case, the Cauchy–Schwarz inequality, several times.

Proposition 2.2 (Hölder's inequality)**.** *Let $p, q \in (1, \infty)$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $f \in L^p(S)$ and $g \in L^q(S)$, then $fg \in L^1(S)$ and

$$\|fg\|_1 \leq \|f\|_p\|g\|_q.$$

Proposition 2.3 (Cauchy–Schwarz inequality)**.** *Let $u, v \in L^2(S)$. Then $uv \in L^1(S)$ and*

$$\int_S |uv|\, d\mu \leq \left(\int_S |u|^2\, d\mu\right)^{1/2} \left(\int_S |v|^2\, d\mu\right)^{1/2}. \tag{2.2}$$

The Cauchy–Schwarz inequality is the special case of Hölder's inequality where both exponents are equal to 2. Indeed,

$$\frac{1}{2} + \frac{1}{2} = 1.$$

We will mainly use Cauchy–Schwarz to control the overlap term. Taking

$$u = |f|^{p/2}, \qquad v = |g|^{p/2}$$

in Proposition 2.3, we obtain

$$\int_S |f|^{p/2}|g|^{p/2}\,d\mu \le \left(\int_S |f|^p\,d\mu\right)^{1/2}\left(\int_S |g|^p\,d\mu\right)^{1/2}.$$

Therefore,

$$\|fg\|_{p/2}^{p/2} \le \left(\|f\|_p^p\|g\|_p^p\right)^{1/2}. \tag{2.3}$$

This inequality explains why the triple

$$\left(\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}\right)$$

always lies in the cone Ω introduced below.

Proposition 2.4 (Arithmetic-geometric mean inequality)**.** *For all $a, b \ge 0$,*

$$\sqrt{ab} \le \frac{a+b}{2}.$$

Equivalently,

$$2\sqrt{ab} \le a + b.$$

Equality holds if and only if $a = b$.

Proof. Since $(\sqrt{a} - \sqrt{b})^2 \ge 0$, we have $a + b - 2\sqrt{ab} \ge 0$. Therefore $2\sqrt{ab} \le a + b$. Equality holds precisely when $\sqrt{a} = \sqrt{b}$, or equivalently when $a = b$. ☐

Combining the Cauchy–Schwarz inequality with the arithmetic-geometric mean inequality gives a normalized measure of the overlap between f and g. This motivates the parameter introduced in the next section.

2.7. The overlap parameter

For $p > 0$, define the normalized overlap parameter

$$\Gamma_p := \frac{2\|fg\|_{p/2}^{p/2}}{\|f\|_p^p + \|g\|_p^p}. \tag{2.4}$$

By the Cauchy–Schwarz estimate (2.3) and Proposition 2.4,

$$2\|fg\|_{p/2}^{p/2} \le 2\sqrt{\|f\|_p^p\|g\|_p^p} \le \|f\|_p^p + \|g\|_p^p.$$

Thus

$$0 \le \Gamma_p \le 1.$$

The value of Γ_p measures the relative overlap between f and g. If f and g have disjoint supports, then $\Gamma_p = 0$. If equality holds both in Cauchy–Schwarz and in the arithmetic-geometric mean inequality, then $\Gamma_p = 1$.

2.8. Concavity, midpoint concavity and Jensen's inequality

Definition 2.6 (Concave function)**.** Let $C \subseteq \mathbb{R}^n$ be a convex set. A function

$$\Phi : C \to \mathbb{R}$$

is called concave if for all $x, y \in C$ and all $\lambda \in [0,1]$,

$$\Phi(\lambda x + (1-\lambda)y) \geq \lambda\Phi(x) + (1-\lambda)\Phi(y).$$

Definition 2.7 (Midpoint concavity and midpoint convexity)**.** Let $C \subset \mathbb{R}^n$ be a convex set, and let $F : C \to \mathbb{R}$. We say that F is midpoint concave if

$$F\left(\frac{x+y}{2}\right) \geq \frac{F(x)+F(y)}{2} \qquad \text{for all } x, y \in C.$$

Similarly, we say that F is midpoint convex if

$$F\left(\frac{x+y}{2}\right) \leq \frac{F(x)+F(y)}{2} \qquad \text{for all } x, y \in C.$$

Definition 2.8 (Locally bounded function)**.** Let $C \subseteq \mathbb{R}^n$, and let $F : C \to \mathbb{R}$. We say that F is locally bounded on C if, for every $x \in C$, there exist a neighbourhood U of x and a constant $M > 0$ such that

$$|F(y)| \leq M \qquad \text{for all } y \in U \cap C.$$

In other words, F is bounded in a sufficiently small neighbourhood of every point of its domain.

Lemma 2.1 (Midpoint concavity implies concavity)**.** *Let $C \subset \mathbb{R}^n$ be a convex set, and let $F : C \to \mathbb{R}$ be locally bounded. If F is midpoint concave, then F is concave on C. That is, for all $x, y \in C$ and all $\lambda \in [0,1]$,*

$$F(\lambda x + (1-\lambda)y) \geq \lambda F(x) + (1-\lambda)F(y).$$

Similarly, if F is locally bounded and midpoint convex, then F is convex on C.

Proof. We prove the midpoint concave case. The midpoint convex case is analogous, with all inequalities reversed.

First, midpoint concavity implies the concavity inequality for dyadic coefficients. That is, for every

$$\lambda = \frac{k}{2^m}, \qquad k = 0, \dots, 2^m,$$

repeated application of midpoint concavity gives

$$F(\lambda x + (1-\lambda)y) \geq \lambda F(x) + (1-\lambda)F(y).$$

Thus the desired inequality holds for all dyadic $\lambda \in [0,1]$.

Since the dyadic numbers are dense in $[0,1]$, it remains to pass from dyadic values of λ to arbitrary values of λ. A standard result states that a locally bounded midpoint concave function on a convex subset of $\mathbb{R}^n$ is continuous on the relative interior of its domain. Therefore, by continuity, the inequality extends from dyadic λ to all $\lambda \in [0,1]$. Hence F is concave.

The midpoint convex case follows in the same way, with the inequalities reversed. □

Proposition 2.5 (Jensen's inequality)**.** *Let $(S, \mathcal{A}, \nu)$ be a probability space, and let $X : S \to \mathbb{R}^n$ be integrable. If $\Phi : \mathbb{R}^n \to \mathbb{R}$ is convex, then*

$$\Phi\left(\int_S X\, d\nu\right) \leq \int_S \Phi(X)\, d\nu.$$

If Φ is concave, then the inequality reverses:

$$\Phi\left(\int_S X\, d\nu\right) \geq \int_S \Phi(X)\, d\nu.$$

Jensen's inequality is one of the main tools in the proof of the envelope inequalities. The rough idea is that, at every point $s \in S$, one constructs a point

$$\left(\frac{f(s)^p}{(f(s)+g(s))^p}, \frac{g(s)^p}{(f(s)+g(s))^p}, \frac{(f(s)g(s))^{p/2}}{(f(s)+g(s))^p}\right)$$

on the boundary of Ω. Then Jensen's inequality is applied to a convex or concave envelope function.

Equality in Jensen's inequality is closely related to linearity. If the envelope is affine on a certain region and the pointwise triples lie in that region, then equality may occur.

2.9. Convex and concave envelopes

The main inequalities in this thesis are obtained by extending known boundary values to the interior of a convex domain. For this purpose, we use convex and concave envelopes.

Definition 2.9 (Majorant and minorant)**.** Let $C \subseteq \mathbb{R}^n$, let $B \subseteq C$, and let

$$\varphi : B \to \mathbb{R}.$$

A function $H : C \to \mathbb{R}$ is called a majorant of φ if $H(x) \geq \varphi(x)$ for all $x \in B$.

Similarly, H is called a minorant of φ if $H(x) \leq \varphi(x)$ for all $x \in B$.

Definition 2.10 (Concave and convex envelopes)**.** Let $C \subseteq \mathbb{R}^n$ be convex, let $B \subseteq C$, and let

$$\varphi : B \to \mathbb{R}.$$

The concave envelope of φ on C is the smallest concave majorant of φ. Thus, it is a concave function $H_{\mathrm{conc}} : C \to \mathbb{R}$ such that

$$H_{\mathrm{conc}}(x) \geq \varphi(x) \qquad \text{for all } x \in B,$$

and whenever $H : C \to \mathbb{R}$ is another concave majorant of φ, we have

$$H_{\mathrm{conc}}(x) \leq H(x) \qquad \text{for all } x \in C.$$

The convex envelope of φ on C is the largest convex minorant of φ. Thus, it is a convex function $H_{\mathrm{conv}} : C \to \mathbb{R}$ such that

$$H_{\mathrm{conv}}(x) \leq \varphi(x) \qquad \text{for all } x \in B,$$

and whenever $H : C \to \mathbb{R}$ is another convex minorant of φ, we have

$$H_{\mathrm{conv}}(x) \geq H(x) \qquad \text{for all } x \in C.$$

The concave envelope is therefore the lowest concave function lying above the given data, while the convex envelope is the highest convex function lying below the data.

In Chapter 3, the set C will be the admissible cone Ω, the set B will be its boundary, and φ_p will describe the known boundary values. The concave envelope will give an upper bound for $\|f+g\|_p^p$, while the convex envelope will give a lower bound.

2.10. Affine regions

Definition 2.11 (Affine function)**.** A function $A : \mathbb{R}^n \to \mathbb{R}$ is called affine if it can be written as

$$A(x) = \ell(x) + c,$$

where $\ell : \mathbb{R}^n \to \mathbb{R}$ is linear and $c \in \mathbb{R}$ is constant.

A function may not be affine everywhere, but it may be affine on certain subsets of its domain. Such subsets are called affine regions or linearity regions.

In the article, equality cases are connected to the affine regions of the envelope functions. For example, if the pointwise triples

$$(f^p, g^p, (fg)^{p/2})$$

lie almost everywhere in a region where the envelope is affine, then the Jensen step can be an equality.

2.11. One-homogeneous functions

Definition 2.12 (One-homogeneous function)**.** Let $C \subseteq \mathbb{R}^n$ be a cone. A function

$$H : C \to \mathbb{R}$$

is called one-homogeneous if for every $t > 0$ and every $x \in C$,

$$H(tx) = tH(x).$$

In this thesis, the relevant functions depend on triples

$$(x, y, z) = \left(\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2} \right).$$

These quantities all scale in the same way. Indeed, if we replace f and g by $t^{1/p}f$ and $t^{1/p}g$, then

$$\|t^{1/p}f\|_p^p = t\|f\|_p^p, \qquad \|t^{1/p}g\|_p^p = t\|g\|_p^p,$$

and

$$\|(t^{1/p}f)(t^{1/p}g)\|_{p/2}^{p/2} = t\|fg\|_{p/2}^{p/2}.$$

Therefore the triple (x, y, z) is multiplied by t. This is why one-homogeneous functions naturally appear.

Lemma 2.2 (Reduction to a slice)**.** *Let $C \subseteq \mathbb{R}^n$ be a cone and let $H : C \to \mathbb{R}$ be one-homogeneous. Suppose that every nonzero point $x \in C$ can be written as*

$$x = \lambda s,$$

where $\lambda > 0$ and s lies in a fixed cross-section $S \subset C$. Then H is completely determined by its values on S.

Proof. By one-homogeneity,

$$H(x) = H(\lambda s) = \lambda H(s).$$

Thus, once H is known on S, it is known on all of C. □

2.12. Bonami's two-point inequality

Besides Jensen's inequality, another classical inequality appearing in the background is Bonami's two-point inequality. We do not use it as the main tool in the envelope proof, but it helps place one of the scalar inequalities in context.

Proposition 2.6 (Bonami's two-point inequality)**.** *Let $1 < p \le q < \infty$. Then, for real numbers y, u, one has*

$$\left(\frac{\left| y + u\sqrt{\frac{p-1}{q-1}} \right|^q + \left| y - u\sqrt{\frac{p-1}{q-1}} \right|^q}{2} \right)^{1/q} \le \left(\frac{|y+u|^p + |y-u|^p}{2} \right)^{1/p}.$$

Proof. A proof of Bonami's two-point inequality can be found in *Analysis in Banach Spaces, Volume I* by Hytönen, van Neerven, Veraar, and Weis [7]. □

This inequality compares two different L^p-type averages on a two-point space. In the article, a related two-point inequality appears after rewriting the sharpened triangle inequality in terms of

$$q = \frac{1}{p}, \qquad x = \sqrt{1 - \Gamma^2}.$$

3

The two-function case

3.1. Motivation and statement of the problem

Throughout this chapter, we assume that $p > 0$ and that f and g are non-negative measurable functions for which the relevant p-moments are finite.

We study sharp estimates for the quantity

$$\|f + g\|_p^p.$$

The goal is to understand how large or small this quantity can be if we only know

$$\|f\|_p^p, \qquad \|g\|_p^p, \qquad \|fg\|_{p/2}^{p/2}.$$

The last term measures the overlap between the two functions. The main idea is to translate this analytic problem into a geometric problem on the admissible cone.

3.2. The admissible cone

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f, g \geq 0$ be measurable.

Definition 3.1 (Associated triple)**.** Let $p > 0$, and let f and g be non-negative measurable functions satisfying

$$\int_X f^p \, d\mu < \infty, \qquad \int_X g^p \, d\mu < \infty.$$

We define their associated triple by

$$x := \int_X f^p \, d\mu, \qquad y := \int_X g^p \, d\mu, \qquad z := \int_X (fg)^{p/2} \, d\mu.$$

For convenience, we also write

$$x = \|f\|_p^p, \qquad y = \|g\|_p^p, \qquad z = \|fg\|_{p/2}^{p/2}.$$

Definition 3.2 (Admissible cone)**.** We define the admissible cone by

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ 0 \leq z \leq \sqrt{xy}\}.$$

A triple $(x, y, z) \in \Omega$ is called admissible.

Lemma 3.1. *Every associated triple belongs to the admissible cone Ω.*

Proof. Clearly, $x = \|f\|_p^p \geq 0, y = \|g\|_p^p \geq 0, z = \|fg\|_{p/2}^{p/2} \geq 0$. Moreover, by the Cauchy–Schwarz inequality in Proposition 2.3 applied to $f^{p/2}$ and $g^{p/2}$, we get $z = \int f^{p/2} g^{p/2} \, d\mu \leq \left(\int f^p \, d\mu\right)^{1/2} \left(\int g^p \, d\mu\right)^{1/2} = \sqrt{xy}$. Hence $x \geq 0, y \geq 0, 0 \leq z \leq \sqrt{xy}$, so $(x, y, z) \in \Omega$. □

3.3. Boundary

Definition 3.3 (Boundary data)**.** Let $p > 0$. We define the boundary data $\varphi_p : \partial\Omega \to \mathbb{R}$ by

$$\varphi_p(x, y, z) = \begin{cases} x + y, & z = 0, \\ \left(x^{1/p} + y^{1/p}\right)^p, & z = \sqrt{xy}. \end{cases}$$

The boundary data are determined by the two extreme cases of overlap between f and g.

First we consider the lower boundary $z = 0$ for $p > 0$. Since $z = \int f^{p/2} g^{p/2}\, d\mu$, and $f, g \geq 0$, this implies $f^{p/2} g^{p/2} = 0$ almost everywhere. Equivalently, $fg = 0$ almost everywhere. Thus f and g have disjoint support, as discussed in Section 2.5. At each point, at most one of the two functions is nonzero, and therefore $(f + g)^p = f^p + g^p$ almost everywhere. Hence $\|f + g\|_p^p = \int (f + g)^p\, d\mu = \int f^p\, d\mu + \int g^p\, d\mu = x + y$.

Second, consider the upper boundary $z = \sqrt{xy}$. This is the equality case in the Cauchy–Schwarz inequality applied to $f^{p/2}$ and $g^{p/2}$. Hence these two functions are proportional. Therefore there exists a constant $\lambda \geq 0$ such that $f = \lambda g$ almost everywhere on their common support.

It follows that $x = \|f\|_p^p = \lambda^p \|g\|_p^p = \lambda^p y$. Moreover,

$$\begin{aligned} \|f + g\|_p^p &= \int_X (\lambda g + g)^p\, d\mu \\ &= (\lambda + 1)^p \int_X g^p\, d\mu \\ &= (\lambda + 1)^p y. \end{aligned}$$

Since $x^{1/p} = \lambda y^{1/p}$, we obtain

$$\|f + g\|_p^p = \left(x^{1/p} + y^{1/p}\right)^p.$$

Thus the boundary values of $\|f + g\|_p^p$ are known on the relevant boundary pieces of $\partial\Omega$. The main problem is to extend these values to the interior of the cone Ω. Ivanisvili and Mooney solved this by computing the concave envelope and convex envelopes of φ_p, defined in Definition 2.10. The concave envelope gives the upper bound, while the convex envelope gives the lower bound [1, 2].

3.4. Reduction to an envelope problem

The sharp estimate for $\|f + g\|_p^p$ follows from an envelope problem on the cone Ω.

Lemma 3.2. *Let $p > 0$, and assume that $H \in C(\Omega)$ is a concave, one-homogeneous function, in the sense of Definition 2.12 on Ω with $H|_{\partial\Omega} = \varphi_p$. Then*

$$\|f + g\|_p^p \leq H(\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}).$$

If H is convex, the inequality reverses.

Proof. Consider the pointwise triple $\left(\frac{f^p}{(f+g)^p}, \frac{g^p}{(f+g)^p}, \frac{(fg)^{p/2}}{(f+g)^p}\right)$, defined on the set where $f + g > 0$. This triple lies on the boundary of Ω, because pointwise we have $\frac{(fg)^{p/2}}{(f+g)^p} = \sqrt{\frac{f^p}{(f+g)^p} \frac{g^p}{(f+g)^p}}$. Since H agrees with the boundary data, we get pointwise

$$H\left(\frac{f^p}{(f+g)^p}, \frac{g^p}{(f+g)^p}, \frac{(fg)^{p/2}}{(f+g)^p}\right) = 1.$$

Now define a probability measure on $X' := \{t \in X : f(t) + g(t) > 0\}$. by $d\nu = \frac{(f+g)^p}{\|f+g\|_p^p}\, d\mu$. Integrating the identity above with respect to ν, we get

$$1 = \int_{X'} H\left(\frac{f^p}{(f+g)^p}, \frac{g^p}{(f+g)^p}, \frac{(fg)^{p/2}}{(f+g)^p}\right) d\nu.$$

Since H is concave, Jensen's inequality, Proposition 2.5, gives

$$1 \leq H\left(\int_{X'} \frac{f^p}{(f+g)^p}\, d\nu, \int_{X'} \frac{g^p}{(f+g)^p}\, d\nu, \int_{X'} \frac{(fg)^{p/2}}{(f+g)^p}\, d\nu\right).$$

Substituting the definition of $d\nu$, the first integral becomes $\int_{X'} \frac{f^p}{(f+g)^p}\, d\nu = \frac{1}{\|f+g\|_p^p} \int_{X'} f^p\, d\mu$. Since on $X \setminus X'$ we have $f+g=0$, and $f, g \geq 0$, it follows that $f=0$ and $g=0$ there. Hence the integrals of f^p, g^p, and $(fg)^{p/2}$ vanish on $X \setminus X'$. Therefore we may extend the integrals from X' to the whole space X. Thus

$$\int_{X'} \frac{f^p}{(f+g)^p}\, d\nu = \frac{\|f\|_p^p}{\|f+g\|_p^p}.$$

The same calculation applies to g and fg. Therefore Jensen's inequality gives $1 \leq H\left(\frac{\|f\|_p^p}{\|f+g\|_p^p}, \frac{\|g\|_p^p}{\|f+g\|_p^p}, \frac{\|fg\|_{p/2}^{p/2}}{\|f+g\|_p^p}\right)$.

Finally, by one-homogeneity of H, $1 \leq \frac{1}{\|f+g\|_p^p} H\left(\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}\right)$, which is equivalent to

$$\|f+g\|_p^p \leq H\left(\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}\right).$$

Thus every concave one-homogeneous extension of the boundary data gives an upper bound.

If H is convex instead of concave, Jensen's inequality gives the reverse inequality. Hence every convex one-homogeneous extension of the boundary data gives a lower bound. ☐

Thus the problem of estimating $\|f+g\|_p^p$ reduces to computing the concave and convex envelopes of φ_p.

3.5. The first envelope: the horizontal chord construction

We now construct the first envelope, denoted by F_p. We will later show that this function gives the upper bound in the range

$$p \in (0,1] \cup [2,\infty),$$

and the lower bound in the range

$$p \in (1,2).$$

First we use one-homogeneity to reduce the three-dimensional cone to a two-dimensional slice. On this slice the domain becomes an upper half-disc and the envelope is constructed by filling the domain with horizontal chords by extending the boundary values.

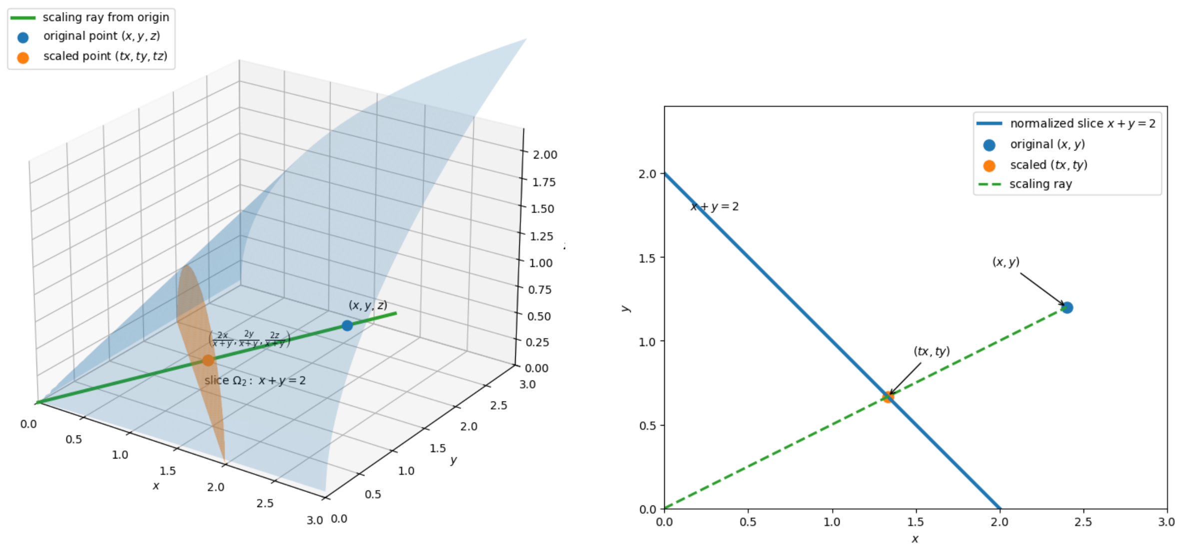
3.5.1. Derivation of the formula for F_p

We first introduce a lemma that reduces the problem to a normalized slice.

Lemma 3.3 (Reduction to the normalized slice)**.** *Let $H : \Omega \to \mathbb{R}$ be one-homogeneous. Then H is completely determined by its values on the slice $\Omega_2 := \{(x,y,z) \in \Omega : x+y=2\}$.*

Proof. Let $(x,y,z) \in \Omega$ with $x+y>0$, and define $t = \frac{2}{x+y}$. Then $tx+ty = t(x+y) = 2$, so the point $(tx,ty,tz) = \left(\frac{2x}{x+y}, \frac{2y}{x+y}, \frac{2z}{x+y}\right)$ lies in the normalized slice Ω_2. By one-homogeneity, $H(x,y,z) = \frac{1}{t}H(tx,ty,tz) = \frac{x+y}{2} H\left(\frac{2x}{x+y}, \frac{2y}{x+y}, \frac{2z}{x+y}\right)$. Thus knowing H on Ω_2 determines H on all of Ω. ☐

The scaling argument in Lemma 3.3 is illustrated in Figure 3.1. Here we see that every point can be scaled until it reaches the orange slice.



(a) Scaling a point in Ω to the slice $x + y = 2$.

(b) The same scaling shown in the (x, y)-plane.

Figure 3.1: Reduction to the normalized slice $\Omega_2 = \{(x, y, z) \in \Omega : x + y = 2\}$.

Assume that the function F_p is one-homogeneous. By Lemma 3.3, it is enough to work on the slice $x + y = 2$.

On this slice, the two variables x and y are not independent anymore: once x is chosen, y is determined by $y = 2 - x$. It is therefore convenient to introduce a single parameter s measuring the difference between x and y:

$$s := \frac{x - y}{2}.$$

It follows that $x = 1 + s, y = 1 - s$, where $s \in [-1, 1]$. The admissibility condition $0 \leq z \leq \sqrt{xy}$ then becomes $0 \leq z \leq \sqrt{1 - s^2}$. Thus, in the (s, z)-plane, the normalized cone is the upper half-disc:

$$D = \{(s, z) : -1 \leq s \leq 1,\ 0 \leq z \leq \sqrt{1 - s^2}\}.$$

Figure 3.2 visualizes this change of coordinates from the normalized slice Ω_2 to the half-disc D.

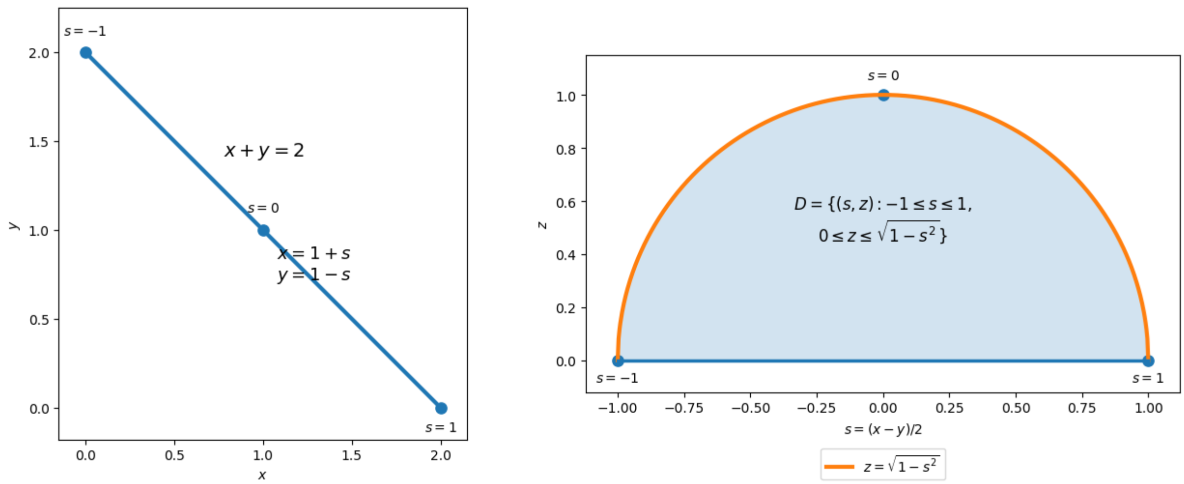


Figure 3.2: The normalized slice $\Omega_2 = \{(x, y, z) \in \Omega : x + y = 2\}$ becomes the upper half-disc D after the change of variables $x = 1 + s, y = 1 - s$.

On the upper boundary of this half-disc, we have $z = \sqrt{1-s^2}$. The boundary data are then

$$\varphi_p(1+s, 1-s, \sqrt{1-s^2}) = \left((1+s)^{1/p} + (1-s)^{1/p}\right)^p.$$

For convenience, define $\phi(s) := \left((1+s)^{1/p} + (1-s)^{1/p}\right)^p$. Notice that ϕ is symmetric: $\phi(s) = \phi(-s)$.

We construct F_p by connecting boundary points with the same height z. Fix some $z \in [0,1]$. The horizontal line at height z intersects the upper boundary at the two points

$$s = \sqrt{1-z^2} \qquad \text{and} \qquad s = -\sqrt{1-z^2}.$$

Since $\phi(\sqrt{1-z^2}) = \phi(-\sqrt{1-z^2})$, the linear interpolation along this horizontal chord is constant.

This horizontal chord construction is visualized in Figure 3.3.

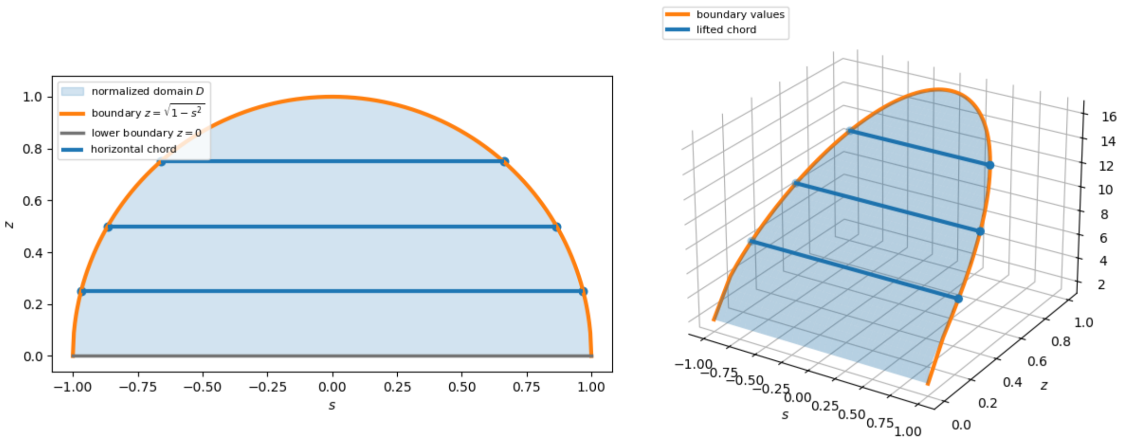


Figure 3.3: Construction of the candidate envelope F_p. The horizontal chords fill the normalized domain D, and lifting these chords according to the boundary data produces the surface F_p.

This gives the candidate function on the normalized slice:

$$F_p(1+s, 1-s, z) = \left[(1+\sqrt{1-z^2})^{1/p} + (1-\sqrt{1-z^2})^{1/p}\right]^p.$$

Thus, on the slice $x + y = 2$, the value of F_p depends only on z, not on s.

To return to a general point $(x, y, z) \in \Omega$, we use one-homogeneity. Define the normalized overlap parameter

$$w := \frac{2z}{x+y}.$$

Then $0 \le w \le 1$, because $2z \le 2\sqrt{xy} \le x + y$. Replacing z in the normalized formula by w, and multiplying by the homogeneity factor $(x + y)/2$, gives

$$F_p(x,y,z) = \frac{x+y}{2}\left[(1+\sqrt{1-w^2})^{1/p} + (1-\sqrt{1-w^2})^{1/p}\right]^p, \qquad w = \frac{2z}{x+y}. \tag{3.1}$$

Notice that this function is also one-homogeneous. Geometrically, it is obtained by extending the boundary values along horizontal chords of the normalized half-disc.

3.5.2. Concavity/convexity check

It remains to check in which range of p the function F_p is concave or convex. For this purpose, we first prove the following lemma.

Lemma 3.4 (Concavity and convexity on a slice). *Let* $\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : x \ge 0,\ y \ge 0,\ 0 \le z \le \sqrt{xy}\right\}$, *and let* $H : \Omega \to \mathbb{R}$ *be one-homogeneous. Then* H *is concave on* Ω *if and only if its restriction to the slice* $\Omega_2 := \{(x,y,z) \in \Omega : x+y = 2\}$ *is concave. Similarly,* H *is convex on* Ω *if and only if its restriction to* Ω_2 *is convex.*

Proof. We prove the concave case. The convex case is analogous. One direction is immediate: if H is concave on Ω, then its restriction to any convex subset, in particular Ω_2, is concave. Conversely, assume that H is concave on Ω_2. Take two points $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ arbitrary in Ω, with $x_1+y_1 > 0$ and $x_2+y_2 > 0$. Define $a = x_1+y_1$ and $b = x_2+y_2$. Then $\tilde{u} := \frac{2}{a}u, \tilde{v} := \frac{2}{b}v$ both lie in the normalized slice Ω_2. For $\lambda \in [0,1]$, set $w = \lambda u+(1-\lambda)v$. Then $w = \frac{\lambda a+(1-\lambda)b}{2}\left(\frac{\lambda a}{\lambda a+(1-\lambda)b}\tilde{u} + \frac{(1-\lambda)b}{\lambda a+(1-\lambda)b}\tilde{v}\right)$. The expression inside the parentheses is a convex combination of $\tilde{u}$ and $\tilde{v}$, hence it lies in Ω_2. Since

$$w = \frac{\lambda a + (1-\lambda)b}{2}\left(\frac{\lambda a}{\lambda a + (1-\lambda)b}\widetilde{u} + \frac{(1-\lambda)b}{\lambda a + (1-\lambda)b}\widetilde{v}\right),$$

one-homogeneity gives

$$H(w) = \frac{\lambda a + (1-\lambda)b}{2} H\left(\frac{\lambda a}{\lambda a + (1-\lambda)b}\widetilde{u} + \frac{(1-\lambda)b}{\lambda a + (1-\lambda)b}\widetilde{v}\right).$$

The two coefficients inside H are non-negative and sum to 1. Since $\widetilde{u}, \widetilde{v} \in \Omega_2$ and H is concave on Ω_2, we obtain

$$H(w) \ge \frac{\lambda a + (1-\lambda)b}{2}\left(\frac{\lambda a}{\lambda a + (1-\lambda)b} H(\widetilde{u}) + \frac{(1-\lambda)b}{\lambda a + (1-\lambda)b} H(\widetilde{v})\right).$$

Using one-homogeneity again,

$$H(\widetilde{u}) = H\left(\frac{2}{a}u\right) = \frac{2}{a}H(u), \qquad H(\widetilde{v}) = H\left(\frac{2}{b}v\right) = \frac{2}{b}H(v).$$

Therefore,

$$\begin{aligned} H(w) &\ge \frac{\lambda a + (1-\lambda)b}{2}\left(\frac{\lambda a}{\lambda a + (1-\lambda)b}\frac{2}{a}H(u) + \frac{(1-\lambda)b}{\lambda a + (1-\lambda)b}\frac{2}{b}H(v)\right) \\ &= \lambda H(u) + (1-\lambda)H(v). \end{aligned}$$

Therefore H is concave on Ω. □

This is illustrated in Figure 3.4.

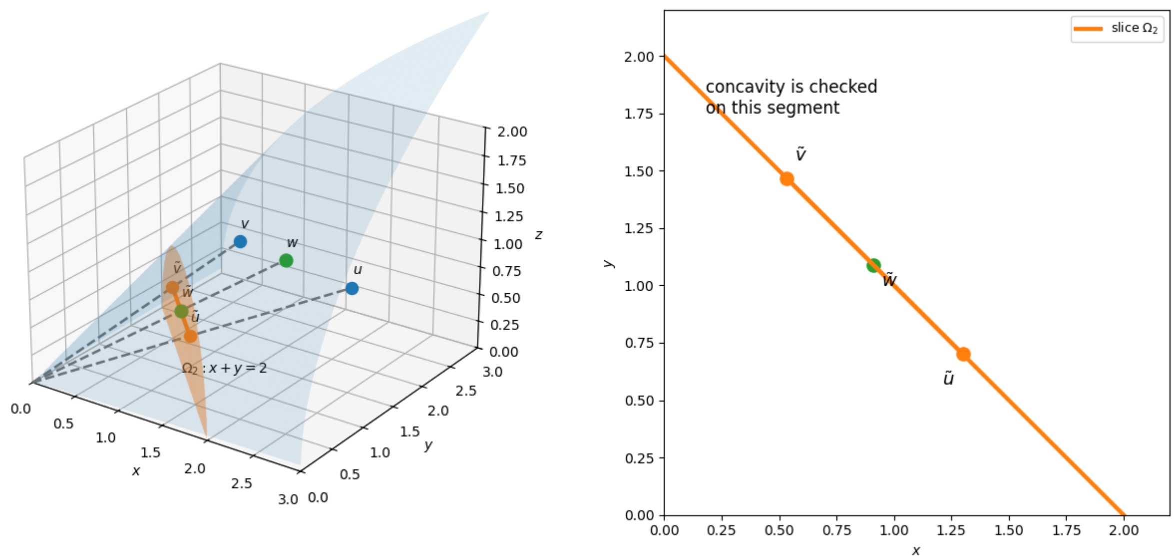


Figure 3.4: Reduction of concavity to the normalized slice D. Points u, v, and $w = \lambda u + (1-\lambda)v$ in Ω are scaled to $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ on the slice $x + y = 2$. Concavity only needs to be checked on the segment between $\tilde{u}$ and $\tilde{v}$.

Since F_p is one-homogeneous, it is enough to check whether F_p is concave on the slice $x + y = 2$ because of Lemma 3.4. On this slice, F_p is constant along horizontal chords, so we only need to study the one-variable function $F_p(1+s, 1-s, t) =$

$$u(t) := \left[(1+\sqrt{1-t^2})^{1/p} + (1-\sqrt{1-t^2})^{1/p}\right]^p, \qquad 0 \le t \le 1.$$

The computations in Appendix A show that

$$u''(t) \le 0 \qquad \text{for} \qquad p \in (0,1] \cup [2,\infty),$$

and

$$u''(t) \ge 0 \qquad \text{for} \qquad p \in (1,2).$$

Therefore F_p is concave in the first range and convex in the second range. This proves the desired concavity or convexity of F_p.

Consequently, F_p is a concave one-homogeneous extension of the boundary data for $p \in (0,1] \cup [2,\infty)$, and therefore gives an upper bound in that range. For $p \in (1,2)$, it is convex and gives a lower bound instead.

3.5.3. Envelope check

We have now constructed the function F_p, verified that it agrees with the boundary data φ_p, and checked its concavity and convexity properties. It remains to prove that F_p is indeed an envelope, and not only a concave or convex extension of the boundary data.

Lemma 3.5 (Extension of the envelope property from the slice)**.** *Let $\Omega_2 := \Omega \cap \{x + y = 2\}$. Let $H : \Omega \to \mathbb{R}$ be one-homogeneous and assume that $H = \varphi_p$ on $\partial\Omega$.*

Suppose that the restriction $H|_{\Omega_2}$ is the concave envelope of the boundary data $\varphi_p|_{\partial\Omega_2}$ on Ω_2. Then H is the smallest one-homogeneous concave majorant, in the sense of Definition 2.9 on Ω of the boundary data on $\partial\Omega$.

Similarly, if $H|_{\Omega_2}$ is the convex envelope of $\varphi_p|_{\partial\Omega_2}$ on Ω_2, then H is the largest one-homogeneous convex minorant on Ω of the boundary data on $\partial\Omega$.

Proof. We prove the concave case. The convex case follows analogously with all inequalities reversed. Assume the following: H is an one-homogeneous function, $H|_{\Omega_2}$ is the concave envelope on Ω_2 and that $H = \varphi_p$ on $\partial\Omega$ holds. By definition of $H|_{\Omega_2}$, it is concave on Ω_2. By Lemma 3.4 we get that H is concave on the full cone Ω. Hence H is itself a one-homogeneous concave majorant of the boundary data.

It remains to show that H is the smallest majorant. Let $L : \Omega \to \mathbb{R}$ be an arbitrary one-homogeneous concave function satisfying $L \ge \varphi_p$ on $\partial\Omega$. Then $L|_{\Omega_2}$ is a concave majorant of the boundary data on Ω_2, because $L|_{\partial\Omega_2} \ge \varphi_p|_{\partial\Omega_2}$. Since $H|_{\Omega_2}$ is the concave envelope on Ω_2, it is the smallest concave majorant of the boundary data on Ω_2. Therefore,

$$H \le L \text{ on } \Omega_2.$$

Now take any $(x, y, z) \in \Omega$ with $x + y > 0$. Normalize this point to the slice Ω_2 by setting $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \left(\frac{2x}{x+y}, \frac{2y}{x+y}, \frac{2z}{x+y}\right)$. Then $(\tilde{x}, \tilde{y}, \tilde{z}) \in \Omega_2$, and $(x, y, z) = \frac{x+y}{2}(\tilde{x}, \tilde{y}, \tilde{z})$. Using the one-homogeneity of H and L, we obtain

$$H(x, y, z) = \frac{x+y}{2} H(\tilde{x}, \tilde{y}, \tilde{z}) \le \frac{x+y}{2} L(\tilde{x}, \tilde{y}, \tilde{z}) = L(x, y, z).$$

At the origin, the same inequality holds because one-homogeneity gives $H(0, 0, 0) = L(0, 0, 0) = 0$. Thus $H \le L$ on all of Ω. Since L was an arbitrary one-homogeneous concave majorant of the boundary data, H is the smallest such majorant on Ω.

The convex statement is proved in the same way, replacing concave majorants by convex minorants and reversing the inequalities. □

So when the envelope property has been established on Ω_2, the corresponding one-homogeneous envelope property follows on the full cone Ω. But Lemma 3.5 assumes that the envelope property holds on Ω_2, therefore it remains to verify this property for F_p on $\Omega_2 = \Omega \cap \{x+y=2\}$.

Recall that every point in Ω_2 can be written as:

$$(1+s, 1-s, z), \qquad -1 \leq s \leq 1, \qquad 0 \leq z \leq \sqrt{1-s^2}.$$

Fix $z \in [0,1]$. The horizontal chord of Ω_2 at height z has endpoints

$$a_z = \left(1-\sqrt{1-z^2}, 1+\sqrt{1-z^2}, z\right) \qquad \text{and} \qquad b_z = \left(1+\sqrt{1-z^2}, 1-\sqrt{1-z^2}, z\right).$$

These endpoints lie on the upper boundary of D, because for both points we have $z = \sqrt{xy}$. Moreover, every point $u = (1+s, 1-s, z) \in \Omega_2$ lies on exactly one such horizontal chord. Hence there exists $\lambda \in [0,1]$ such that

$$u = \lambda a_z + (1-\lambda) b_z.$$

On the slice Ω_2, the formula for F_p depends only on the coordinate z. Therefore F_p is constant, and hence affine, on every horizontal chord, see Definition 2.11. Since F_p agrees with the boundary data on the upper boundary, we have

$$F_p(a_z) = \varphi_p(a_z) \qquad \text{and} \qquad F_p(b_z) = \varphi_p(b_z).$$

It follows that

$$F_p(u) = \lambda F_p(a_z) + (1-\lambda) F_p(b_z) = \lambda \varphi_p(a_z) + (1-\lambda)\varphi_p(b_z).$$

We first consider the concave case. Let L be any concave majorant of the boundary data on Ω_2, that is,

$$L \geq \varphi_p \qquad \text{on } \partial\Omega_2.$$

Since $u = \lambda a_z + (1-\lambda) b_z$, concavity of L gives $L(u) \geq \lambda L(a_z) + (1-\lambda) L(b_z)$. Using $L \geq \varphi_p$ on the boundary, we obtain

$$L(u) \geq \lambda \varphi_p(a_z) + (1-\lambda)\varphi_p(b_z) = F_p(u).$$

Thus every concave majorant of the boundary data on Ω_2 lies above F_p. Therefore, whenever F_p is concave, it is the smallest concave majorant of the boundary data on Ω_2, and hence the concave envelope on Ω_2.

The convex case is analogous. If L is any convex minorant of the boundary data on Ω_2, so that

$$L \leq \varphi_p \qquad \text{on } \partial\Omega_2,$$

then convexity gives

$$L(u) \leq \lambda L(a_z) + (1-\lambda) L(b_z).$$

Using $L \leq \varphi_p$ on the boundary, we get $L(u) \leq \lambda \varphi_p(a_z) + (1-\lambda)\varphi_p(b_z) = F_p(u)$. Hence every convex minorant of the boundary data on Ω_2 lies below F_p. Therefore, whenever F_p is convex, it is the largest convex minorant of the boundary data on Ω_2, and hence the convex envelope on Ω_2.

By Lemma 3.5, these envelope properties extend from Ω_2 to the full cone Ω. Consequently,

$$F_p = \overline{H}_p \qquad \text{for } p \in (0,1] \cup [2,\infty),$$

and

$$F_p = \underline{H}_p \qquad \text{for } p \in (-\infty,0) \cup (1,2).$$

3.6. The second envelope: the corner construction

The second envelope, denoted by G_p, is needed because F_p gives the sharp upper bound only when it is concave; in the complementary range $p \in (1,2)$, F_p becomes convex and therefore gives a lower bound instead.

F_p was obtained by using horizontal chords in the normalized half-disc, G_p is constructed differently, but still with line segments. The line segments come from the lower boundary and the corner regions of the admissible cone. Therefore, we refer to this construction as the corner construction.

It will later be shown, that G_p will give the concave envelope when

$$p \in (1,2),$$

and the convex envelope when

$$p \in (0,1] \cup [2,\infty).$$

3.6.1. Derivation of the formula for G_p

As before, we first work on the normalized slice $x+y=2$. We write $x = 1+s$ and $y = 1-s$, so that the normalized cone becomes the upper half-disc

$$D = \{(s,z) : -1 \le s \le 1,\ 0 \le z \le \sqrt{1-s^2}\}.$$

Define

$$b_p(s,z) := G_p(1+s, 1-s, z).$$

The goal is to derive a formula for b_p on D, and then extend it to the full cone by one-homogeneity.

We divide the half-disc D into two types of regions: the central region and the two corner regions. These are shown in Figure 3.5.

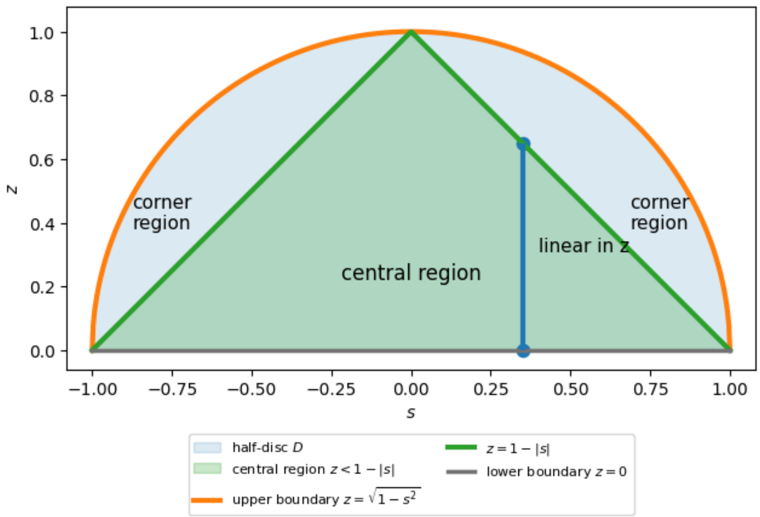


Figure 3.5: Decomposition of the normalized half-disc D into the central region $z < 1-|s|$ and the two corner regions. In the central region, the candidate b_p is chosen to be linear in the z-direction.

Central region

First consider the central region: $z < 1-|s|$. Equivalently, $z < \min\{1+s, 1-s\}$. In this region both $x = 1+s$ and $y = 1-s$ are larger than z. The candidate b_p is chosen to be linear in the z-direction.

For $p > 0$ the lower boundary value is $b_p(s,0) = 2$. At the top point of the central region we have $(x,y,z) = (1,1,1)$. This point lies on the upper boundary $z = \sqrt{xy}$, and therefore $b_p(0,1) = \varphi_p(1,1,1) = \left(1^{1/p} + 1^{1/p}\right)^p = 2^p$. Thus, for $p > 0$, the central part is obtained by linear interpolation between the value 2 at $z=0$ and the value 2^pat $z=1$. Hence

$$b_p(s,z) = 2 + (2^p - 2)z.$$

Corner regions
Now consider one of the corner regions, for example the region where $s \geq 0$. Then $1 - s \leq 1 + s$, so the smaller coordinate is $t := 1 - s$. The corner region is described by $z \geq t$. We now keep the smaller coordinate t and the overlap z fixed. The upper boundary condition is $z^2 = xy$. If the smaller coordinate is t, then the corresponding upper boundary point has the other coordinate $\frac{z^2}{t}$. However, this point does not necessarily lie on the normalized slice $x + y = 2$, since $t + \frac{z^2}{t}$ is not always equal to 2. To obtain the value on the normalized slice, we correct for this difference in the $x + y$-direction.

Therefore, for $p > 0$ in the corner region, $b_p(s,z) = \left(t^{1/p} + \left(\frac{z^2}{t}\right)^{1/p}\right)^p + \left(2 - t - \frac{z^2}{t}\right)$. Equivalently,

$$b_p(s,z) = 2 + \left[\left(t^{1/p} + \left(\frac{z^2}{t}\right)^{1/p}\right)^p - \left(t + \frac{z^2}{t}\right)\right].$$

Half-disc
Combining the central and corner regions we get the formulas on the normalized slice for $p > 0$:

$$b_p(s,z) = 2 + \begin{cases} \left[\left((1-|s|)^{1/p} + \left(\dfrac{z^2}{1-|s|}\right)^{1/p}\right)^p - \left((1-|s|) + \dfrac{z^2}{1-|s|}\right)\right], & z \geq 1 - |s|, \\ (2^p - 2)z, & z < 1 - |s|. \end{cases}$$

We now rewrite these formulas in compact form on the full cone. First assume that $z > 0$. For $(x,y,z) \in \Omega$, define

$$v := \min\left\{\frac{x}{z}, \frac{y}{z}, 1\right\}.$$

In the central region, $z \leq \min\{x,y\}$, so $v = 1$. In the corner regions, one of x or y is smaller than z, and v records the ratio of this smaller coordinate to z.

For $p > 0$, the second candidate envelope is

$$G_p(x,y,z) = x + y + \left[\left(v^{1/p} + v^{-1/p}\right)^p - \left(v + v^{-1}\right)\right] z, \qquad z > 0. \tag{3.2}$$

Therefore G_p is a one-homogeneous extension of the boundary data for every $p > 0$.

3.6.2. Concavity/convexity check

It remains to determine in which range of p the second candidate envelope G_p is concave or convex.

As before, because G_p is one-homogeneous, Lemma 3.4 shows that it is enough to check this on the normalized slice $x + y = 2$. On this slice we write $x = 1 + s$, $y = 1 - s$, and study $b_p(s,z) = G_p(1+s, 1-s, z)$. The function b_p is defined piecewise. In the central region $z < 1 - |s|$, $b_p(s,z)$ is linear, and therefore both concave and convex.

The only nontrivial part is the corner region $z \geq 1 - |s|$. There we set $t = 1 - |s|$ and write $b_p(s,z) = 2 + \omega(t,z)$, where $\omega(t,z) = \left(t^{1/p} + \left(\frac{z^2}{t}\right)^{1/p}\right)^p - \left(t + \frac{z^2}{t}\right)$. Thus the concavity or convexity of G_p is determined by the curvature of this expression.

The function ω is one-homogeneous in the variables (t,z). Therefore it is enough to study ω on the one-dimensional slice $z = 1$. We define $h(t) := \omega(t,1)$. Then

$$h(t) = \left(t^{1/p} + t^{-1/p}\right)^p - \left(t + t^{-1}\right), \qquad 0 < t \leq 1.$$

The computation of $h''(t)$ in Appendix A gives

$$h''(t) \leq 0 \qquad \text{for} \qquad p \in (1,2),$$

and

$$h''(t) \geq 0 \qquad \text{for} \qquad p \in (0,1] \cup [2,\infty).$$

Hence, for $p > 0$, the corner part is concave for $p \in (1,2)$, and convex for $p \in (0,1] \cup [2,\infty)$. Together with the linearity in the central region, this proves that, for $p > 0$, G_p is concave for $p \in (1,2)$, and convex for $p \in (0,1] \cup [2,\infty)$.

Together with the linearity in the central region, this shows that G_p is concave for

$$p \in (1,2),$$

and convex for

$$p \in (0,1] \cup [2,\infty).$$

It remains to check that the central and corner formulas fit together correctly across their common interface.

Matching across the interfaces

The function b_p is defined by different formulas in different parts of the half-disc D. We also have to check that the formulas fit together correctly on the boundary between the regions, namely on $z = 1-|s|$.

We first check this on the interface. Write $t = 1 - |s|$. Then the interface is given by $z = t$. The corner formulas contain the expression

$$\Phi_p(t,z) = \left(t^{1/p} + \left(\frac{z^2}{t}\right)^{1/p} \right)^p .$$

On the interface $z = t$, we have

$$\Phi_p(t,t) = 2^p t.$$

Moreover, a direct computation gives

$$\partial_t \Phi_p(t,t) = 0, \qquad \partial_z \Phi_p(t,t) = 2^p.$$

For $p > 0$, the corner formula is

$$2 + \omega(t,z),$$

where

$$\omega(t,z) = \Phi_p(t,z) - \left(t + \frac{z^2}{t}\right).$$

On the interface $z = t$, we have

$$t + \frac{z^2}{t} = 2t.$$

Also,

$$\partial_t \left(t + \frac{z^2}{t}\right)\bigg|_{z=t} = 0, \qquad \partial_z \left(t + \frac{z^2}{t}\right)\bigg|_{z=t} = 2.$$

Therefore,

$$\omega(t,t) = (2^p - 2)t, \qquad \partial_t \omega(t,t) = 0, \qquad \partial_z \omega(t,t) = 2^p - 2.$$

So the corner formula $2 + \omega(t,z)$ has the same value and the same first derivatives as the central formula

$$2 + (2^p - 2)z$$

on the interface $z = t$.

Thus the central and corner formulas join smoothly on the interface $z = 1 - |s|$. In other words, b_p is C^1 across the interface. We now explain why this implies concavity or convexity on the whole slice D. Take any line segment contained in D, parametrized by

$$\gamma(\theta) = (s_0, z_0) + \theta(s_1 - s_0, z_1 - z_0), \qquad 0 \leq \theta \leq 1,$$

and define

$$Q(\theta) := b_p(\gamma(\theta)).$$

The interface may divide the interval $[0, 1]$ into several subintervals. On each subinterval on which $\gamma(\theta)$ remains in one of the regions, the function Q has the required curvature. In the concave case,

$$Q''(\theta) \leq 0,$$

so Q' is non-increasing on that subinterval. In the convex case,

$$Q''(\theta) \geq 0,$$

so Q' is non-decreasing on that subinterval.

Thus the central and corner formulas have the same value and the same first derivatives on the interface $z = 1 - |s|$. Hence b_p is C^1 across the interface. Therefore, in the concave case, Q' is non-increasing on the whole interval $[0, 1]$, while in the convex case, Q' is non-decreasing on the whole interval.

It follows that Q is concave, respectively convex, on $[0, 1]$. Since this holds for every line segment contained in D, the function b_p is concave, respectively convex, on D.

3.6.3. Envelope check for G_p

We have constructed the function G_p, checked that it agrees with the boundary data, and determined its concavity and convexity ranges. It remains to explain why G_p is indeed the corresponding envelope. As for F_p, it is enough to verify the envelope property on the normalized slice $D = \Omega \cap \{x + y = 2\}$. By Lemma 3.5, the corresponding one-homogeneous envelope property then follows on the full cone Ω.

On the normalized slice D, write $x = 1 + s$ and $y = 1 - s$. Recall that D is the half-disc $D = \{(1 + s, 1 - s, z) : -1 \leq s \leq 1,\ 0 \leq z \leq \sqrt{1 - s^2}\}$. The construction of G_p divides D into the central region $z \leq 1 - |s|$, and the two corner regions $z \geq 1 - |s|$. We show that in each region G_p is affine along line segments whose endpoints lie on the boundary of D. This is the geometric reason why G_p is an envelope.

First consider the central region for $p > 0$. Let $u = (x, y, z) \in D$, and $z \leq \min\{x, y\}$. Then u lies on the line segment connecting the lower boundary point

$$c_u = \left(\frac{x - z}{1 - z}, \frac{y - z}{1 - z}, 0\right)$$

and the upper boundary point $m = (1, 1, 1)$. Indeed,

$$u = (1 - z)c_u + z\,m.$$

The point c_u belongs to the lower boundary of D, while m belongs to the upper boundary. For $p > 0$, the boundary values are

$$G_p(c_u) = \varphi_p(c_u) = 2, \qquad G_p(m) = \varphi_p(m) = 2^p.$$

Moreover, in the central region for $p > 0$,

$$G_p(u) = 2 + (2^p - 2)z.$$

Hence

$$G_p(u) = (1 - z)G_p(c_u) + zG_p(m).$$

Thus G_p is affine on these central line segments for $p > 0$.

Now consider the right corner region, where $y \leq x$ and $z \geq y$. Set $t := y$. The corresponding upper boundary point before normalization is

$$q = \left(\frac{z^2}{t}, t, z\right),$$

because for this point let $m_q := t + \frac{z^2}{t}$. Since $u \in D$, one has $m_q \leq 2$. Normalizing q to the slice gives the boundary point

$$\tilde{q} := \frac{2}{m_q} q \in D.$$

The point $u = (x, y, z)$ lies on the line segment between the corner point $e_1 = (2, 0, 0)$ and the upper boundary point $\tilde{q}$. More precisely,

$$u = \frac{m_q}{2} \tilde{q} + \left(1 - \frac{m_q}{2}\right) e_1.$$

Since G_p agrees with the boundary data on ∂D, and since it is affine on this segment by construction, we obtain

$$G_p(u) = \frac{m_q}{2} G_p(\tilde{q}) + \left(1 - \frac{m_q}{2}\right) G_p(e_1).$$

Using one-homogeneity of the boundary data,

$$G_p(\tilde{q}) = \varphi_p(\tilde{q}) = \frac{2}{m_q} \left[\left(\frac{z^2}{t} \right)^{1/p} + t^{1/p} \right]^p,$$

and

$$G_p(e_1) = \varphi_p(e_1) = 2.$$

Therefore

$$\begin{aligned} G_p(u) &= \left[\left(\frac{z^2}{t} \right)^{1/p} + t^{1/p} \right]^p + 2 - m_q \\ &= 2 + \left[\left(t^{1/p} + \left(\frac{z^2}{t} \right)^{1/p} \right)^p - \left(t + \frac{z^2}{t} \right) \right]. \end{aligned}$$

This is exactly the corner formula for G_p. Hence G_p is affine on the line segments used in the right corner construction. The left corner region is identical, with the roles of x and y interchanged.

3.7. Two-function inequalities

We now translate the envelopes F_p and G_p back into sharp inequalities for two functions, in the sense of Section2.4. Let $f, g \geq 0$ be measurable functions and define

$$\Gamma_p := \frac{2\|fg\|_{p/2}^{p/2}}{\|f\|_p^p + \|g\|_p^p}.$$

For the bounds coming from G_p, we also define

$$C_p := \frac{\min\{\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}\}}{\|fg\|_{p/2}^{p/2}},$$

with the convention $C_p = 1$ if $\|fg\|_{p/2}^{p/2} = 0$.

Theorem 3.1 (Bounds for $p \in (0, 1] \cup [2, \infty)$). *Let $p \in (0, 1] \cup [2, \infty)$, and let $f, g \geq 0$ be measurable functions in L^p. Then*

$$\|f + g\|_p^p \leq \left[\left(\frac{1 + \sqrt{1 - \Gamma_p^2}}{2} \right)^{1/p} + \left(\frac{1 - \sqrt{1 - \Gamma_p^2}}{2} \right)^{1/p} \right]^p \left(\|f\|_p^p + \|g\|_p^p \right), \tag{3.3}$$

and

$$\|f + g\|_p^p \geq \|f\|_p^p + \|g\|_p^p + \left[\left(C_p^{1/p} + C_p^{-1/p} \right)^p - \left(C_p + C_p^{-1} \right) \right] \|fg\|_{p/2}^{p/2}. \tag{3.4}$$

Proof. Let $x = \|f\|_p^p$, $y = \|g\|_p^p$, and $z = \|fg\|_{p/2}^{p/2}$. For $p \in (0,1] \cup [2,\infty)$, the envelope F_p is concave and G_p is convex. Hence the envelope reduction gives

$$G_p(x,y,z) \le \|f+g\|_p^p \le F_p(x,y,z).$$

Substituting the explicit formula for F_p, with $w = 2z/(x+y) = \Gamma_p$, gives the stated upper bound. Substituting the formula for G_p, with

$$v = \min\left\{\frac{x}{z}, \frac{y}{z}, 1\right\} = C_p,$$

gives the stated lower bound. □

Theorem 3.2 (Bounds for $p \in (1,2)$)**.** *Let $p \in (1,2)$, and let $f, g \ge 0$ be measurable functions in L^p. Then*

$$\|f+g\|_p^p \ge \left[\left(\frac{1+\sqrt{1-\Gamma_p^2}}{2}\right)^{1/p} + \left(\frac{1-\sqrt{1-\Gamma_p^2}}{2}\right)^{1/p}\right]^p \left(\|f\|_p^p + \|g\|_p^p\right), \tag{3.5}$$

and

$$\|f+g\|_p^p \le \|f\|_p^p + \|g\|_p^p + \left[\left(C_p^{1/p} + C_p^{-1/p}\right)^p - \left(C_p + C_p^{-1}\right)\right] \|fg\|_{p/2}^{p/2}. \tag{3.6}$$

Proof. Again set $x = \|f\|_p^p$, $y = \|g\|_p^p$, and $z = \|fg\|_{p/2}^{p/2}$. For $p \in (1,2)$, the roles of the two envelopes are reversed: F_p is convex and G_p is concave. Hence the envelope reduction gives

$$F_p(x,y,z) \le \|f+g\|_p^p \le G_p(x,y,z).$$

Substituting the explicit formulas for F_p and G_p, with $w = \Gamma_p$ and $v = C_p$, gives the stated inequalities. □

3.8. Optimality

Throughout this section we work on the measure space $(X, \mathcal{A}, \mu) = ([0,1], \mathcal{B}, dx)$.

We saw that the envelopes F_p and G_p give valid upper and lower bounds for $\|f+g\|_p^p$. In this section we prove that these bounds are the sharpest bounds we could get, expressed in terms of the three quantities $\|f\|_p^p, \|g\|_p^p$ and $\|fg\|_{p/2}^{p/2}$.

We do this by comparing the envelope functions with the best possible bounds. If the envelope functions coincide with these best possible bound functions, then the obtained bounds are optimal.

Definition 3.4 (Realizable triples and extremal value functions)**.** Let Ω_p denote the set of triples $(x,y,z) \in \Omega$ that can be realized by a pair of admissible functions f, g, that is,

$$\Omega_p := \left\{(x,y,z) \in \Omega : \exists f, g \text{ such that } x = \|f\|_p^p,\ y = \|g\|_p^p,\ z = \|fg\|_{p/2}^{p/2}\right\}.$$

For $(x,y,z) \in \Omega_p$, define

$$\overline{B}_p(x,y,z) := \sup\left\{\|f+g\|_p^p : \|f\|_p^p = x,\ \|g\|_p^p = y,\ \|fg\|_{p/2}^{p/2} = z\right\},$$

and

$$\underline{B}_p(x,y,z) := \inf\left\{\|f+g\|_p^p : \|f\|_p^p = x,\ \|g\|_p^p = y,\ \|fg\|_{p/2}^{p/2} = z\right\}.$$

Before comparing the extremal value functions with the envelope functions, we first check that they scale in the same way. Indeed, if f and g are multiplied by the same factor, then x, y, z and also $\|f+g\|_p^p$, are multiplied by the same factor, this is the one-homogeneity property from Definition 2.12. The best possible upper and lower bounds should therefore have the same property. The next lemma states this for $\overline{B}_p$ and $\underline{B}_p$.

Lemma 3.6 (One-homogeneity)**.** *The functions $\overline{B}_p$ and $\underline{B}_p$ are one-homogeneous on Ω_p.*

Proof. We prove the statement for $\overline{B}_p$. The proof for $\underline{B}_p$ is identical. For $(x, y, z) \in \Omega_p$, define the admissible classes

$$\mathcal{A}(x, y, z) := \left\{(f, g) : \|f\|_p^p = x,\ \|g\|_p^p = y,\ \|fg\|_{p/2}^{p/2} = z\right\}.$$

and

$$\mathcal{A}(tx, ty, tz) := \left\{(F, G) : \|F\|_p^p = tx,\ \|G\|_p^p = ty,\ \|FG\|_{p/2}^{p/2} = tz\right\}.$$

By definition,

$$\overline{B}_p(x, y, z) = \sup_{(f,g)\in\mathcal{A}(x,y,z)} \|f + g\|_p^p.$$

To prove one-homogeneity, we compare this supremum with the corresponding supremum over $\mathcal{A}(tx, ty, tz)$. The idea is to construct a bijection between these two admissible classes. If such a bijection exists and if, under this bijection, the quantity $\|f + g\|_p^p$ is multiplied by t, then the supremum is also multiplied by t. This will show that

$$\overline{B}_p(tx, ty, tz) = t\,\overline{B}_p(x, y, z),$$

which is exactly the one-homogeneity of $\overline{B}_p$.

Fix $t > 0$. For $(f, g) \in \mathcal{A}(x, y, z)$, define

$$T_t(f, g) := (t^{1/p} f, t^{1/p} g).$$

We first check that T_t maps $\mathcal{A}(x, y, z)$ into $\mathcal{A}(tx, ty, tz)$. Indeed, $\|t^{1/p} f\|_p^p = t\|f\|_p^p = tx$, and similarly $\|t^{1/p} g\|_p^p = ty$. Moreover, $(t^{1/p} f)(t^{1/p} g) = t^{2/p} fg$, so $\|(t^{1/p} f)(t^{1/p} g)\|_{p/2}^{p/2} = \|t^{2/p} fg\|_{p/2}^{p/2} = t\|fg\|_{p/2}^{p/2} = tz$. Hence $T_t(f, g) \in \mathcal{A}(tx, ty, tz)$.

The objective value scales in the same way: $\|t^{1/p} f + t^{1/p} g\|_p^p = \|t^{1/p}(f + g)\|_p^p = t\|f + g\|_p^p$. Therefore every admissible pair for (x, y, z) gives an admissible pair for (tx, ty, tz), with objective value multiplied by t. Taking the supremum gives

$$\overline{B}_p(tx, ty, tz) \geq t\,\overline{B}_p(x, y, z).$$

To obtain the reverse inequality, we use that T_t is invertible. Its inverse is

$$T_t^{-1}(F, G) = (t^{-1/p} F, t^{-1/p} G).$$

Indeed, $T_t^{-1}(T_t(f, g)) = T_t^{-1}(t^{1/p} f, t^{1/p} g) = (f, g)$, and $T_t(T_t^{-1}(F, G)) = T_t(t^{-1/p} F, t^{-1/p} G) = (F, G)$. Thus T_t is a bijection from $\mathcal{A}(x, y, z)$ onto $\mathcal{A}(tx, ty, tz)$.

Now let $(F, G) \in \mathcal{A}(tx, ty, tz)$. Since T_t is bijective, there exists $(f, g) \in \mathcal{A}(x, y, z)$ such that $(F, G) = T_t(f, g)$. Hence $\|F + G\|_p^p = t\|f + g\|_p^p \leq t\,\overline{B}_p(x, y, z)$.

Taking the supremum over all $(F, G) \in \mathcal{A}(tx, ty, tz)$, we obtain

$$\overline{B}_p(tx, ty, tz) \leq t\,\overline{B}_p(x, y, z).$$

Combining both inequalities gives

$$\overline{B}_p(tx, ty, tz) = t\,\overline{B}_p(x, y, z).$$

Thus $\overline{B}_p$ is one-homogeneous. The same argument applies to $\underline{B}_p$. □

We have shown that the extremal value functions $\overline{B}_p$ and $\underline{B}_p$ are one-homogeneous on Ω_p. Before using these functions on the full cone Ω, we first prove that every point of Ω is realizable. We also establish the concavity of $\overline{B}_p$ and the convexity of $\underline{B}_p$, which will be needed later.

Lemma 3.7 (Realization of the cone and convexity properties)**.** *For the measure space* $([0,1],\mathcal{B},dx)$*, let* Ω_p *be as defined in Definition 3.4. Then*

$$\Omega_p = \Omega$$

for all $p>0$*. Moreover,* $\overline{B}_p$ *is concave on* Ω*, and* $\underline{B}_p$ *is convex on* Ω*.*

Proof. We first prove that $\Omega_p=\Omega$. Let $(x,y,z)\in\Omega$. Then notice $x\geq 0$, $y\geq 0$ and $0\leq z\leq\sqrt{xy}$.

We need to construct admissible functions f,g on $[0,1]$ such that $\|f\|_p^p = x$ and $\|g\|_p^p=y$ and $\|fg\|_{p/2}^{p/2}=z$.

Assume first that $p>0$. For $s\in[0,\frac{1}{2}]$, define

$$f_s=(2x)^{1/p}\chi_{[s,s+1/2]},\qquad g=(2y)^{1/p}\chi_{[1/2,1]}.$$

Then $\|f_s\|_p^p=\int_s^{s+1/2}2x\,dt=x,$, $\|g\|_p^p=\int_{1/2}^1 2y\,dt=y$ and $\|f_sg\|_{p/2}^{p/2}=2\sqrt{xy}\,\big|[s,s+1/2]\cap[1/2,1]\big|$.

As s varies from 0 to $1/2$, the intersection length varies continuously from 0 to $1/2$. Hence $\|f_sg\|_{p/2}^{p/2}$ ranges over the whole interval $[0,\sqrt{xy}]$. Therefore every $z\in[0,\sqrt{xy}]$ can be realized.

It remains to prove the concavity of $\overline{B}_p$ and the convexity of $\underline{B}_p$. We prove the concavity of $\overline{B}_p$; the proof for $\underline{B}_p$ is analogous.

Let $(x_1,y_1,z_1),(x_2,y_2,z_2)\in\Omega$ arbitrary. Fix $\varepsilon>0$. Choose admissible pairs (f_1,g_1) and (f_2,g_2) such that $\|f_i\|_p^p=x_i$, $\|g_i\|_p^p=y_i$, and $\|f_ig_i\|_{p/2}^{p/2}=z_i$, and

$$\|f_i+g_i\|_p^p\geq\overline{B}_p(x_i,y_i,z_i)-\varepsilon,\qquad i=1,2.$$

We place the first pair on the left half of $[0,1]$ and the second pair on the right half. Define

$$\widetilde{f}_1(t)=2^{1/p}f_1(2t),\qquad \widetilde{g}_1(t)=2^{1/p}g_1(2t),\qquad t\in[0,\tfrac{1}{2}],$$

and set them equal to 0 outside $[0,\frac{1}{2}]$. Similarly, define

$$\widetilde{f}_2(t)=2^{1/p}f_2(2t-1),\qquad \widetilde{g}_2(t)=2^{1/p}g_2(2t-1),\qquad t\in[\tfrac{1}{2},1],$$

and set them equal to 0 outside $[\frac{1}{2},1]$.

Now define

$$f_0:=2^{-1/p}(\widetilde{f}_1+\widetilde{f}_2),\qquad g_0:=2^{-1/p}(\widetilde{g}_1+\widetilde{g}_2).$$

Because the two pairs are supported on disjoint halves of the interval, we get $\|f_0\|_p^p=\frac{x_1+x_2}{2}$, $\|g_0\|_p^p=\frac{y_1+y_2}{2}$, and $\|f_0g_0\|_{p/2}^{p/2}=\frac{z_1+z_2}{2}$. Thus (f_0,g_0) is admissible for the triple

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right).$$

Moreover,

$$\|f_0+g_0\|_p^p=\frac{1}{2}\|\widetilde{f}_1+\widetilde{g}_1\|_p^p+\frac{1}{2}\|\widetilde{f}_2+\widetilde{g}_2\|_p^p.$$

By the scaling in the definitions of $\widetilde{f}_i,\widetilde{g}_i$, we have $\|\widetilde{f}_i+\widetilde{g}_i\|_p^p=\|f_i+g_i\|_p^p,\qquad i=1,2.$

Hence

$$\|f_0+g_0\|_p^p=\frac{1}{2}\|f_1+g_1\|_p^p+\frac{1}{2}\|f_2+g_2\|_p^p.$$

Using the choice of the almost extremizing pairs, we obtain

$$\|f_0+g_0\|_p^p\geq\frac{1}{2}\overline{B}_p(x_1,y_1,z_1)+\frac{1}{2}\overline{B}_p(x_2,y_2,z_2)-\varepsilon.$$

Since (f_0, g_0) is admissible for the averaged triple, it follows that

$$\overline{B}_p\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) \geq \frac{1}{2}\overline{B}_p(x_1,y_1,z_1) + \frac{1}{2}\overline{B}_p(x_2,y_2,z_2) - \varepsilon.$$

Letting $\varepsilon \to 0$, we get midpoint concavity, as defined in Defintion 2.7:

$$\overline{B}_p\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) \geq \frac{1}{2}\overline{B}_p(x_1,y_1,z_1) + \frac{1}{2}\overline{B}_p(x_2,y_2,z_2).$$

Since $\overline{B}_p$ is locally bounded, midpoint concavity implies concavity by Lemma 2.1.

For the lower extremal function, the same construction is used with almost minimizing pairs. It gives

$$\underline{B}_p\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) \leq \frac{1}{2}\underline{B}_p(x_1,y_1,z_1) + \frac{1}{2}\underline{B}_p(x_2,y_2,z_2).$$

Thus $\underline{B}_p$ is midpoint convex.

It remains to justify that the extremal value functions are locally bounded. Because then midpoint convexity implies convexity by Lemma 2.1. Then for every admissible pair,

$$0 \leq \|f+g\|_p^p \leq C_p\left(\|f\|_p^p + \|g\|_p^p\right) = C_p(x+y),$$

where

$$C_p := \max\{1, 2^{p-1}\}.$$

Indeed, for $0 < p \leq 1$ we use $(f+g)^p \leq f^p + g^p$, while for $p \geq 1$ we use $(f+g)^p \leq 2^{p-1}(f^p + g^p)$.

Consequently, both $\overline{B}_p$ and $\underline{B}_p$ are locally bounded on Ω. Since $\underline{B}_p$ is locally bounded, midpoint convexity implies convexity. ☐

By Lemma 3.7, we have $\Omega_p = \Omega$. Hence the extremal value functions $\overline{B}_p$ and $\underline{B}_p$ are defined on the whole cone Ω. Moreover, $\overline{B}_p$ is concave and $\underline{B}_p$ is convex on Ω.

We now verify that these extremal value functions agree with the prescribed boundary data.

Lemma 3.8 (Boundary values). *On the boundary of Ω, the extremal value functions agree with the boundary data:*

$$\overline{B}_p = \underline{B}_p = \varphi_p \qquad \text{on } \partial\Omega.$$

Proof. First consider the lower boundary $z = 0$. This corresponds to the case where f and g have disjoint support. Then $\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p = x + y$.

Hence

$$\overline{B}_p(x,y,0) = \underline{B}_p(x,y,0) = x + y = \varphi_p(x,y,0).$$

This proves the statement on the lower boundary $z = 0$. It remains to check the other part of the boundary, namely the upper boundary $z = \sqrt{xy}$.

Now consider the upper boundary $z = \sqrt{xy}$. This is the equality case in the Cauchy–Schwarz inequality applied to $f^{p/2}$ and $g^{p/2}$. Therefore $f^{p/2}$ and $g^{p/2}$ are proportional almost everywhere, see for more details Preliminaries Section 2.2. Equivalently, $g = \lambda f$ almost everywhere for some $\lambda \geq 0$.

Then

$$y = \|g\|_p^p = \lambda^p \|f\|_p^p = \lambda^p x, \qquad \lambda = \left(\frac{y}{x}\right)^{1/p}.$$

Hence

$$\|f+g\|_p^p = \|(1+\lambda)f\|_p^p = (1+\lambda)^p x = \left(x^{1/p} + y^{1/p}\right)^p.$$

Thus

$$\overline{B}_p(x,y,\sqrt{xy}) = \underline{B}_p(x,y,\sqrt{xy}) = \left(x^{1/p} + y^{1/p}\right)^p = \varphi_p(x,y,\sqrt{xy}).$$

Therefore the extremal value functions agree with the boundary data. ☐

We have now shown that the extremal value functions are defined on all of Ω and agree with the prescribed boundary data on $\partial\Omega$. We next compare them with the concave and convex envelopes constructed from the same boundary data.

Lemma 3.9 (Envelope bounds for the extremal value functions)**.** *Let $\overline{H}_p$ be the concave envelope of the boundary data φ_p on Ω, and let $\underline{H}_p$ be the convex envelope of φ_p on Ω. Then, for every $(x, y, z) \in \Omega$,*

$$\underline{H}_p(x, y, z) \leq \underline{B}_p(x, y, z) \leq \overline{B}_p(x, y, z) \leq \overline{H}_p(x, y, z).$$

Proof. Let $(x, y, z) \in \Omega$, and let f, g be any admissible pair with associated triple (x, y, z). From the envelope reduction Lemma 3.2, applied to the concave envelope $\overline{H}_p$, we have $\|f + g\|_p^p \leq \overline{H}_p(x, y, z)$. Taking the supremum over all admissible pairs gives

$$\overline{B}_p(x, y, z) \leq \overline{H}_p(x, y, z).$$

Similarly, applying the envelope reduction Lemma to the convex envelope $\underline{H}_p$ gives $\underline{H}_p(x, y, z) \leq \|f + g\|_p^p$. Taking the infimum over all admissible pairs gives

$$\underline{H}_p(x, y, z) \leq \underline{B}_p(x, y, z).$$

Finally, by definition of supremum and infimum, $\underline{B}_p(x, y, z) \leq \overline{B}_p(x, y, z)$.

Combining these inequalities gives

$$\underline{H}_p \leq \underline{B}_p \leq \overline{B}_p \leq \overline{H}_p.$$

$\square$

So we see that the extremal value functions are squeezed between the convex and concave envelope bounds. To prove that these bounds are actually sharp, we need to show that this squeeze becomes an equality.

Lemma 3.10 (Identification of the extremal functions with the envelopes)**.** *The extremal value functions coincide with the corresponding envelopes:*

$$\overline{B}_p = \overline{H}_p, \qquad \underline{B}_p = \underline{H}_p \qquad \textit{on } \Omega.$$

Proof. By Lemma 3.7, $\overline{B}_p$ is concave and $\underline{B}_p$ is convex on Ω. By Lemma 3.8, both extremal functions agree with the boundary data on $\partial\Omega$.

Hence $\overline{B}_p$ is a concave majorant of φ_p. Since $\overline{H}_p$ is the smallest concave majorant of the boundary data, we obtain

$$\overline{H}_p \leq \overline{B}_p.$$

On the other hand, Lemma 3.9 gives

$$\overline{B}_p \leq \overline{H}_p.$$

Therefore,

$$\overline{B}_p = \overline{H}_p.$$

Similarly, $\underline{B}_p$ is a convex minorant of the boundary data. Since $\underline{H}_p$ is the largest convex minorant, we obtain

$$\underline{B}_p \leq \underline{H}_p.$$

Together with

$$\underline{H}_p \leq \underline{B}_p$$

from Lemma 3.9, this gives

$$\underline{B}_p = \underline{H}_p.$$

$\square$

Combining Lemmas 3.7, 3.8, 3.9, and 3.10 gives the sharpness of the two-function bounds.

Theorem 3.3 (Optimality of the two-function bounds)**.** *The inequalities obtained from F_p and G_p are sharp.*

For $p \in (0,1] \cup [2,\infty)$,

$$\overline{B}_p = F_p, \qquad \underline{B}_p = G_p,$$

and hence

$$G_p(x,y,z) \le \|f+g\|_p^p \le F_p(x,y,z).$$

For $p \in (1,2)$, the roles are reversed:

$$\overline{B}_p = G_p, \qquad \underline{B}_p = F_p,$$

and hence

$$F_p(x,y,z) \le \|f+g\|_p^p \le G_p(x,y,z).$$

Proof. By Lemma 3.10, we have

$$\overline{B}_p = \overline{H}_p, \qquad \underline{B}_p = \underline{H}_p.$$

From the envelope classification proved in the previous sections,

$$\overline{H}_p = \begin{cases} F_p, & p \in (0,1] \cup [2,\infty), \\ G_p, & p \in (1,2), \end{cases}$$

and

$$\underline{H}_p = \begin{cases} G_p, & p \in (0,1] \cup [2,\infty), \\ F_p, & p \in (1,2). \end{cases}$$

Substituting these identities gives the stated sharp inequalities. Since $\overline{B}_p$ and $\underline{B}_p$ are defined as the supremum and infimum over all admissible pairs with the same associated triple, these bounds cannot be improved using only the quantities $\|f\|_p^p$, $\|g\|_p^p$, and $\|fg\|_{p/2}^{p/2}$.

□

3.9. Equality and extremal cases

With the envelope method, we derived sharp bounds and insight into the cases where equality is attained. In the proof of the envelope reduction lemma, the main inequality comes from Jensen's inequality. Therefore equality can occur when the pointwise triples lie in a region where the relevant envelope function is affine.

More precisely, because the envelope functions are one-homogeneous, it is enough to look at the unnormalised pointwise triples

$$\big(f^p, g^p, (fg)^{p/2}\big).$$

If these triples lie almost everywhere in a region where the corresponding envelope is affine, then the Jensen step is an equality. This gives extremal examples for the sharp bounds.

Remark (Extremal examples for the F_p-bound)*.* The function F_p is affine on the hyperplanes

$$\{z = k(x+y)\} \cap \Omega, \qquad k \in \left[0, \frac{1}{2}\right].$$

Indeed, on such a hyperplane the quantity $w = \frac{2z}{x+y}$ is constant, namely $w = 2k$. Hence, from the formula for F_p,

$$F_p(x,y,z) = \frac{x+y}{2}\left[(1+\sqrt{1-4k^2})^{1/p} + (1-\sqrt{1-4k^2})^{1/p}\right]^p.$$

Therefore, F_p is affine on this hyperplane.

This gives the equality condition

$$(fg)^{p/2} = k(f^p + g^p) \qquad \text{a.e.}$$

for some constant $k \in [0, \frac{1}{2}]$. Indeed, this condition means exactly that the pointwise triples $\left(f^p, g^p, (fg)^{p/2}\right)$ lie in one of the hyperplanes on which F_p is affine.

A concrete class of extremal examples is obtained on $([0,1], \mathcal{B}, dx)$ by taking

$$(f,g) = (a,b)\chi_{[0,c]} + (b,a)\chi_{[c,1]},$$

where $a, b \geq 0$ and $c \in [0,1]$ are chosen appropriately. Explicitly,

$$f(t) = \begin{cases} a, & t \in [0,c], \\ b, & t \in [c,1], \end{cases} \qquad g(t) = \begin{cases} b, & t \in [0,c], \\ a, & t \in [c,1]. \end{cases}$$

For this pair, $\|f\|_p^p = ca^p + (1-c)b^p$ and $\|g\|_p^p = cb^p + (1-c)a^p$.

Moreover, since $fg = ab$ and $f + g = a + b$ on both subintervals, $\|fg\|_{p/2}^{p/2} = (ab)^{p/2}$, $\|f+g\|_p^p = (a+b)^p$. Also, $(fg)^{p/2} = (ab)^{p/2}$ and $f^p + g^p = a^p + b^p$ on both subintervals. Hence

$$(fg)^{p/2} = k(f^p + g^p), \qquad k := \frac{(ab)^{p/2}}{a^p + b^p}.$$

Therefore the pointwise triples lie in an affine region of F_p, and equality holds in the F_p-bound.

Remark (Extremal regions for the G_p-bound). The equality cases for G_p come from the affine regions of G_p. Recall that for $p > 0$,

$$G_p(x,y,z) = x + y + \left[\left(v^{1/p} + v^{-1/p}\right)^p - \left(v + v^{-1}\right)\right] z, \qquad v = \min\left\{\frac{x}{z}, \frac{y}{z}, 1\right\}$$

.

The first affine region is the triangular cone

$$\{(x,y,z) \in \Omega : z \leq \min\{x,y\}\}.$$

On this region, $v = 1$. Therefore, for $p > 0$,

$$G_p(x,y,z) = x + y + (2^p - 2)z,$$

which is affine.

Under the substitution

$$x = f^p, \qquad y = g^p, \qquad z = (fg)^{p/2},$$

the condition $z \leq \min\{x,y\}$ becomes $(fg)^{p/2} \leq \min\{f^p, g^p\}$.

For $p > 0$, this implies $f = g$ on $\{fg > 0\}$.

The other affine regions are the hyperplanes

$$\{z = \gamma x\} \cap \Omega \qquad \text{and} \qquad \{z = \gamma y\} \cap \Omega, \qquad \gamma \geq 1.$$

On these hyperplanes, the parameter v is constant, and hence G_p is affine.

If $(fg)^{p/2} = \gamma f^p$, then, for $p > 0$,

$$g = \lambda f \qquad \text{on } \{f > 0\}$$

for some $\lambda \geq 1$. Similarly, if

$$(fg)^{p/2} = \gamma g^p,$$

then

$$f = \lambda g \qquad \text{on } \{g > 0\}$$

for some $\lambda \geq 1$.

Thus equality in the G_p-bound is obtained when the pointwise triples lie entirely in one of these affine regions. In other words, the main extremal configurations are

$$f = g \quad \text{on the common finite support,}$$

or one of the functions is a constant multiple of the other on the relevant part of the space.

For the purposes of this thesis, these should be read as sufficient extremal configurations. In endpoint cases, especially $p = 1$, equality can occur for more trivial reasons, so the statement should not automatically be interpreted as a complete if-and-only-if classification.

A converse for the equality statement in the G_p-bound

The previous discussion gives sufficient conditions for equality in the G_p-bound. We now explain why, away from degenerate endpoint cases, these conditions are also necessary. The idea is to combine the equality case of Jensen's inequality with the classification of the affine regions of G_p.

Recall that in the proof of the envelope reduction lemma we consider the normalized pointwise triples

$$\xi(t) := \left(\frac{f(t)^p}{(f(t)+g(t))^p}, \frac{g(t)^p}{(f(t)+g(t))^p}, \frac{(f(t)g(t))^{p/2}}{(f(t)+g(t))^p} \right).$$

These triples lie on the boundary of Ω. Since the envelope functions are one-homogeneous, the normalized triples $\xi(t)$ lie on the same rays as the unnormalized triples

$$\big(f(t)^p, g(t)^p, (f(t)g(t))^{p/2}\big).$$

Therefore, for the purpose of studying affine regions, it is equivalent to work with the unnormalized triples.

Assume that equality holds in the G_p-bound. Then equality must occur in the Jensen step of the envelope reduction proof. Hence the essential range of the pointwise triples must be contained in a region where G_p is affine. In other words, equality forces the pointwise triples

$$\big(f^p, g^p, (fg)^{p/2}\big)$$

to lie almost everywhere in one of the affine regions of G_p.

It remains to identify these affine regions. The following affine regions arising from the construction inside Ω are the following:

$$\mathcal{C}_0 := \{(x,y,z) \in \Omega : z \le \min\{x,y\}\},$$

and, for $\gamma \ge 1$,

$$\mathcal{C}_x^\gamma := \{(x,y,z) \in \Omega : z = \gamma x\}, \qquad \mathcal{C}_y^\gamma := \{(x,y,z) \in \Omega : z = \gamma y\}.$$

On the first region, $v = 1$, so G_p reduces to an affine expression. For $p > 0$,

$$G_p(x,y,z) = x + y + (2^p - 2)z.$$

On the other two regions, the parameter v is constant. Indeed, if $z = \gamma x$, then

$$\frac{x}{z} = \frac{1}{\gamma},$$

and the condition $(x,y,z) \in \Omega$ implies $\frac{y}{z} \ge 1$. Hence

$$v = \min\left\{\frac{x}{z}, \frac{y}{z}, 1\right\} = \frac{1}{\gamma}.$$

Thus v is constant on $\mathcal{C}_x^\gamma$, and therefore G_p is affine there. The same argument applies to $\mathcal{C}_y^\gamma$.

We now translate these affine regions back into conditions on f and g.

First suppose that the pointwise triples lie in $\mathcal{C}_0$. Then

$$(fg)^{p/2} \le \min\{f^p, g^p\} \qquad \text{a.e.}$$

For $p > 0$, this gives, on $\{fg > 0\}$, $(fg)^{p/2} \le f^p \implies g \le f$, and $(fg)^{p/2} \le g^p \implies f \le g$.
Hence

$$f = g \qquad \text{on } \{fg > 0\}.$$

Next suppose that the pointwise triples lie in $\mathcal{C}_x^\gamma$. Then

$$(fg)^{p/2} = \gamma f^p \qquad \text{a.e.}$$

For $p > 0$, on $\{f > 0\}$, this gives

$$\left(\frac{g}{f}\right)^{p/2} = \gamma,$$

and hence

$$g = \lambda f, \qquad \lambda := \gamma^{2/p} \ge 1.$$

Similarly, if the pointwise triples lie in $\mathcal{C}_y^\gamma$, then

$$(fg)^{p/2} = \gamma g^p \qquad \text{a.e.}$$

For $p > 0$, this implies

$$f = \lambda g \qquad \text{on } \{g > 0\}$$

for some $\lambda \ge 1$.

Thus the converse follows: if equality holds in the G_p-bound, then the essential range of the normalized pointwise triples must be contained in one of the affine regions of G_p. Consequently, the functions must satisfy one of the corresponding structural conditions:

$$f = g$$

on the common part of their support, or one function must be a constant multiple of the other on the relevant part of the measure space.

This matches the equality cases stated for the G_p-bound: for $p > 0$, equality forces one of the conditions

$$f = g \text{ on } \{fg > 0\}, \qquad g = \lambda f \text{ on } \{f > 0\},\ \lambda \ge 1, \qquad f = \lambda g \text{ on } \{g > 0\},\ \lambda \ge 1.$$

Remark. The converse direction would require a complete classification of all maximal affine regions of the envelope G_p. Such a classification is beyond the scope of this thesis.

4

Extensions to q-powers

In the previous chapter, we obtained a sharp estimate for the p-th power $\|f+g\|_p^p$. In this chapter, we investigate what happens when the final exponent is changed from p to some $q > 2$.

Throughout the chapter, we assume that $p > 2$, $q > 2$, and $f, g \geq 0$. As before, we introduce the quantities

$$x = \|f\|_p^p, \qquad y = \|g\|_p^p, \qquad z = \|fg\|_{p/2}^{p/2}.$$

By Hölder's inequality, $0 \leq z \leq \sqrt{xy}$, so every admissible triple (x, y, z) belongs to the cone

$$\Omega = \{(x, y, z) : x, y \geq 0,\ 0 \leq z \leq \sqrt{xy}\}.$$

The main result of the previous chapter states that

$$\|f+g\|_p^p \leq F_p(x, y, z),$$

where F_p is the concave envelope of the boundary data on Ω. This gives a sharp answer when the final exponent is p. We now want to understand whether this result can also be used to estimate

$$\|f+g\|_p^q.$$

A first estimate follows immediately from the previous chapter:

$$\|f+g\|_p^q = \left(\|f+g\|_p^p\right)^{q/p} \leq F_p(x, y, z)^{q/p}.$$

This estimate is sharp, but it is not especially convenient. The right-hand side still depends on the full three-variable envelope F_p, and this envelope is then raised to the power q/p. As a result, the estimate does not have a simple form in terms of the individual norms of f and g.

For this reason, we consider a different problem. We fix the overlap

$$J = \|fg\|_{p/2}$$

and look for the smallest constant $C_{p,q}(J)$ such that

$$\|f+g\|_p^q \leq C_{p,q}(J)\left(1 + \|f\|_p^q + \|g\|_p^q\right).$$

The quantity on the right-hand side is symmetric in f and g and depends only on their individual L^p-norms. The influence of the overlap is contained entirely in the constant $C_{p,q}(J)$. This turns the problem into a best-constant problem, rather than a new three-variable envelope problem.

4.1. The new constant and the first reduction

As explained above, we fix the overlap $J = \|fg\|_{p/2} > 0$ and compare $\|f+g\|_p^q$ with the symmetric quantity

$$1 + \|f\|_p^q + \|g\|_p^q.$$

We therefore define $C_{p,q}(J)$ to be the smallest constant such that

$$\|f+g\|_p^q \le C_{p,q}(J)\big(1 + \|f\|_p^q + \|g\|_p^q\big) \tag{4.1}$$

holds for all non-negative $f, g \in L^p$ satisfying $\|fg\|_{p/2} = J$.

The constant $C_{p,q}(J)$ then tells us how large the factor must be so that the inequality is true for all pairs f, g with fixed overlap J. Since $\|fg\|_{p/2} = J$, we have $z = \|fg\|_{p/2}^{p/2} = J^{p/2}$. Applying the sharp estimate for the p-th power gives $\|f+g\|_p^p \le F_p(x,y,z)$. Raising both sides to the power q/p, we obtain

$$\|f+g\|_p^q \le F_p(x,y,z)^{q/p}. \tag{4.2}$$

Thus the original functional problem is reduced to comparing the right-hand side of (4.2) with

$$1 + \|f\|_p^q + \|g\|_p^q = 1 + x^{q/p} + y^{q/p}.$$

We now define the constant that describes the worst possible value of this quotient. This is the constant we need in order to make the inequality true for all admissible pairs f, g.

Theorem 4.1 (Exact finite-dimensional reduction)**.** *Let $p > 2$, $q > 2$, $J > 0$, and put $z = J^{p/2}$. Then*

$$C_{p,q}(J) = \sup_{\substack{x,y>0 \\ xy \ge z^2}} \frac{F_p(x,y,z)^{q/p}}{1 + x^{q/p} + y^{q/p}}.$$

Proof. Let $f, g \ge 0$ satisfy $\|fg\|_{p/2} = J$, and define $x = \|f\|_p^p, y = \|g\|_p^p$ and $z = J^{p/2}$. By the sharp envelope inequality and raising both sides to the power q/p, we obtain $\|f+g\|_p^q \le F_p(x,y,z)^{q/p}$. Therefore

$$\frac{\|f+g\|_p^q}{1 + \|f\|_p^q + \|g\|_p^q} \le \frac{F_p(x,y,z)^{q/p}}{1 + x^{q/p} + y^{q/p}}.$$

Taking the supremum over all admissible pairs f, g gives

$$C_{p,q}(J) \le \sup_{\substack{x,y>0 \\ xy \ge z^2}} \frac{F_p(x,y,z)^{q/p}}{1 + x^{q/p} + y^{q/p}}.$$

For the reverse inequality, we use the sharpness of the envelope F_p from the previous chapter. Fix an admissible pair of numbers $x, y > 0$ with $xy \ge z^2$. The sharpness result from the previous chapter implies that this value of the envelope can be approached by admissible pairs of non-negative functions. Thus there exists a sequence of pairs (f_n, g_n) such that

$$\|f_n\|_p^p = x, \qquad \|g_n\|_p^p = y, \qquad \|f_n g_n\|_{p/2}^{p/2} = z,$$

and

$$\|f_n + g_n\|_p^p \to F_p(x,y,z).$$

Hence

$$\frac{\|f_n+g_n\|_p^q}{1 + \|f_n\|_p^q + \|g_n\|_p^q} \longrightarrow \frac{F_p(x,y,z)^{q/p}}{1 + x^{q/p} + y^{q/p}}.$$

Consequently, the corresponding value of the finite-dimensional quotient can be approached by admissible function pairs. Taking the supremum over all admissible x, y gives the reverse inequality.

Therefore

$$C_{p,q}(J) = \sup_{\substack{x,y>0 \\ xy \ge z^2}} \frac{F_p(x,y,z)^{q/p}}{1 + x^{q/p} + y^{q/p}}.$$

□

Remark (Finite-dimensional reduction and sharpness). Theorem 4.1 reduces the original functional problem to a finite-dimensional optimization problem. Instead of optimizing over all admissible functions f and g, we only need to optimize over $x = \|f\|_p^p$ and $y = \|g\|_p^p$, while $z = J^{p/2}$ is fixed.

The reduction is exact. In particular, the right-hand side is not merely an upper bound for $C_{p,q}(J)$, but gives the smallest possible constant in (4.1). The sharpness of F_p ensures that every value of the finite-dimensional quotient can be approached by admissible functions. Therefore, any exact evaluation of the remaining supremum gives the optimal constant in the original functional inequality.

4.2. Case $q = 2$

Before studying the general case with $q > 2$, we first look at the special case $q = 2$. In this case we do not need envelopes. The following proposition determines the optimal constant when $q = 2$.

Proposition 4.1 (The case $q = 2$). *Let $p \geq 2$ and $J = \|fg\|_{p/2}$. Then*

$$\|f+g\|_p^2 \leq C_{p,2}(J)\left(1 + \|f\|_p^2 + \|g\|_p^2\right) \qquad \textit{with} \qquad C_{p,2}(J) = \max\left(1, \frac{4J}{1+2J}\right).$$

Proof. Assume $p \geq 2$, and let $J = \|fg\|_{p/2}$. Since $p/2 \geq 1$, the space $L^{p/2}$ is normed (see Definition 2.2). Hence $\|f+g\|_p^2 = \|(f+g)^2\|_{p/2}$. Using $(f+g)^2 = f^2 + 2fg + g^2$ and the triangle inequality in $L^{p/2}$, we obtain

$$\|f+g\|_p^2 \leq \|f^2\|_{p/2} + 2\|fg\|_{p/2} + \|g^2\|_{p/2}.$$

Since $\|f^2\|_{p/2} = \|f\|_p^2$ and $\|g^2\|_{p/2} = \|g\|_p^2$, this gives

$$\|f+g\|_p^2 \leq \|f\|_p^2 + 2J + \|g\|_p^2. \tag{4.3}$$

Put $S = \|f\|_p^2 + \|g\|_p^2$. By Hölder's inequality and Proposition 2.4,

$$J \leq \|f\|_p\|g\|_p \leq \frac{\|f\|_p^2 + \|g\|_p^2}{2} = \frac{S}{2}.$$

Hence

$$S \geq 2J.$$

Dividing the estimate by $1+S$, we obtain

$$\frac{\|f+g\|_p^2}{1 + \|f\|_p^2 + \|g\|_p^2} \leq \frac{S+2J}{1+S}. \tag{4.4}$$

Since the constant must also work for the worst possible value, we want to find the largest possible right-hand side value of 4.4. Since S must satisfy $S \geq 2J$, the answer depends on the size of J.

If $J \leq \frac{1}{2}$, then $S + 2J \leq S + 1$, so the ratio is at most 1. If $J \geq \frac{1}{2}$, then the function $S \mapsto \frac{S+2J}{1+S}$ is decreasing for $S \geq 2J$. Hence the maximum occurs at $S = 2J$, giving $\frac{2J+2J}{1+2J} = \frac{4J}{1+2J}$.

Thus

$$C_{p,2}(J) = \max\left(1, \frac{4J}{1+2J}\right).$$

It remains to show that this constant is sharp.

First suppose that $J \geq \frac{1}{2}$. Choose a non-negative function h satisfying $\|h\|_p = 1$, and put

$$f = \sqrt{J}\,h, \qquad g = \sqrt{J}\,h.$$

Then $\|fg\|_{p/2} = J$ and

$$\frac{\|f+g\|_p^2}{1 + \|f\|_p^2 + \|g\|_p^2} = \frac{4J}{1+2J}.$$

Thus the second value is attained.

Now suppose that $0 < J \leq \frac{1}{2}$. Again choose $\|h\|_p = 1$, and for $a > 0$ put $f_a = ah$, $g_a = \frac{J}{a}h$. Then

$$\|f_a g_a\|_{p/2} = J,$$

and

$$\frac{\|f_a + g_a\|_p^2}{1 + \|f_a\|_p^2 + \|g_a\|_p^2} = \frac{(a + J/a)^2}{1 + a^2 + (J/a)^2}.$$

As $a \to \infty$, this quotient converges to 1. Hence the value 1 is also sharp, although it is not necessarily attained.

Therefore

$$C_{p,2}(J) = \max\left\{1, \frac{4J}{1+2J}\right\}.$$

$\square$

Remark. Notice that we used the identity $(f+g)^2 = f^2 + 2fg + g^2$ for the proof. However for general $q > 2$, no equally simple expansion is available, which is why the envelope method becomes useful again.

4.3. Reducing the optimization to the denominator

We have now found a constant for the case $q = 2$, and we return to the general case $q > 2$. For this case, a direct expansion is no longer available. We have now reduced the problem to the finite-dimensional quotient in Theorem 4.1. The next step is to understand where the supremum in this quotient can occur. We use the envelopes constructed in Chapter 3.1. Before doing the optimization, it is important to understand where the optimization takes place. The envelope gives control over the numerator of the quotient, visible in 4.1. The explicit formula for the envelope is

$$F_p(x,y,z) = \frac{x+y}{2} A_p\left(\frac{2z}{x+y}\right)^p \qquad \text{with} \qquad A_p(r) = \left(1 + \sqrt{1-r^2}\right)^{1/p} + \left(1 - \sqrt{1-r^2}\right)^{1/p}.$$

We should notice that x and y appear in two different ways. In the envelope formula, they appear together through the sum $x + y$. In the denominator of our quotient (right-hand side 4.3), however, they appear separately as $x^{q/p}$ and $y^{q/p}$. Therefore, we first study how this difference affects the optimization problem.

Lemma 4.1. *Fix $z > 0$. Then $F_p(x,y,z)$ depends on x and y only through the sum $s = x + y$.*

Proof. This follows directly from the formula for F_p. If $s = x + y$, then $F_p(x,y,z) = \frac{s}{2} A_p\left(\frac{2z}{s}\right)^p$. Thus x and y only appear through the sum $s = x + y$. $\square$

This means that once $s = x + y$ and z are fixed, the numerator $F_p(x,y,z)^{q/p}$ is fixed. The only remaining choice is how the sum s is split between x and y. This choice only changes the denominator $1 + x^{q/p} + y^{q/p}$ in 4.1.

Lemma 4.2. *Fix $s = x + y$ and $z > 0$. Then the admissible values of x are*

$$x \in \left[\frac{s - \sqrt{s^2 - 4z^2}}{2}, \frac{s + \sqrt{s^2 - 4z^2}}{2}\right],$$

with $y = s - x$. For this fixed s, maximizing

$$\frac{F_p(x,y,z)^{q/p}}{1 + x^{q/p} + y^{q/p}}$$

is the same as minimizing

$$x^{q/p} + (s-x)^{q/p}.$$

Proof. For fixed s, the numerator is fixed by the previous lemma. Therefore the quotient becomes largest when the denominator becomes smallest. Since $y = s - x$, the condition $xy \ge z^2$ becomes $x(s-x) \ge z^2$ by the defining condition of Ω. Solving this quadratic inequality gives

$$x \in \left[\frac{s - \sqrt{s^2 - 4z^2}}{2}, \frac{s + \sqrt{s^2 - 4z^2}}{2}\right].$$

Therefore, for fixed s, we only need to minimize $x^{q/p} + (s-x)^{q/p}$ on this interval. $\square$

We have now reduced the problem to minimizing $x^{q/p} + (s-x)^{q/p}$ for fixed $s = x + y$ and fixed z. This is the point where the value of q becomes decisive. If $q/p \le 1$, then the function $t \mapsto t^{q/p}$ is concave, so the minimum occurs at the boundary of the admissible interval. If $q/p \ge 1$, then the function $t \mapsto t^{q/p}$ is convex, so the minimum occurs at the balanced point. This is why the optimization naturally splits into the two ranges $2 < q \le p$ and $q \ge p$.

4.4. The range $2 < q \le p$

We first consider the case $2 < q \le p$. In this range the exponent $\alpha = \frac{q}{p}$ satisfies $0 < \alpha \le 1$. Therefore the function $t \mapsto t^\alpha$ is concave.

From the previous section we know that, for fixed $s = x + y$ and fixed z, the numerator of the quotient is fixed. So we only have to minimize the denominator. This means that we have to minimize $x^\alpha + (s-x)^\alpha$ on the admissible interval from Lemma 4.2. If $0 < \alpha < 1$, this function is concave, so its minimum is attained at an endpoint of the interval. If $\alpha = 1$, the function is constant, and an endpoint may still be chosen. These endpoints correspond to the boundary case $xy = z^2$. In our problem $z = J^{p/2}$, so this boundary becomes

$$xy = J^p.$$

On this boundary the envelope agrees with the boundary data. This means that the problem reduces to proportional functions.

Theorem 4.2 (The range $2 < q \le p$)**.** *Assume* $2 < q \le p$. *Then*

$$C_{p,q}(J) = \sup_{u>0} \frac{(u + J/u)^q}{1 + u^q + (J/u)^q}.$$

Proof. Put $\alpha = \frac{q}{p}$. Then $0 < \alpha \le 1$. By Theorem 4.1,

$$C_{p,q}(J) = \sup_{\substack{x,y>0 \\ xy \ge J^p}} \frac{F_p(x, y, J^{p/2})^\alpha}{1 + x^\alpha + y^\alpha}.$$

Fix $s = x + y$. By Lemma 4.2, for fixed s we only need to minimize

$$x^\alpha + (s-x)^\alpha$$

on the admissible interval.

For $0 < \alpha < 1$, the function

$$x \mapsto x^\alpha + (s-x)^\alpha$$

is concave, so its minimum is attained at an endpoint. If $\alpha = 1$, the function equals s and is therefore constant. In either case, an endpoint of the admissible interval may be chosen.

The endpoints are exactly the points for which $x(s-x) = J^p$. Since $y = s - x$, this is equivalent to $xy = J^p$. On this boundary we have $F_p(x, y, J^{p/2}) = (x^{1/p} + y^{1/p})^p$.

Now write

$$x = u^p, \qquad y = \left(\frac{J}{u}\right)^p.$$

Then $xy = J^p$. Moreover,

$$F_p(x, y, J^{p/2})^{q/p} = (u + J/u)^q,$$

and

$$1 + x^{q/p} + y^{q/p} = 1 + u^q + (J/u)^q.$$

Therefore

$$C_{p,q}(J) = \sup_{u>0} \frac{(u+J/u)^q}{1+u^q+(J/u)^q}.$$

□

Theorem 4.3 (Evaluation of the proportional supremum)**.** *Assume that $2 < q \le p$, and define*

$$J_q^* := \left(\frac{1}{2(q-1)}\right)^{2/q}.$$

If $J \ge J_q^*$, then the supremum in Theorem 4.2 is attained at $u = \sqrt{J}$, and

$$C_{p,q}(J) = \frac{2^q J^{q/2}}{1+2J^{q/2}}.$$

If $0 < J < J_q^*$, then there exists a unique $t_J > 0$ satisfying

$$\sinh t_J = 2J^{q/2} \sinh((q-1)t_J).$$

The supremum is attained at the two points

$$u = \sqrt{J}\, e^{t_J} \qquad \text{and} \qquad u = \sqrt{J}\, e^{-t_J},$$

and

$$C_{p,q}(J) = \frac{J^{q/2}(2\cosh t_J)^q}{1+2J^{q/2}\cosh(qt_J)}.$$

Proof. By Theorem 4.2,

$$C_{p,q}(J) = \sup_{u>0} \frac{(u+J/u)^q}{1+u^q+(J/u)^q}.$$

Write $u = \sqrt{J}\, e^t$. Then

$$u + \frac{J}{u} = 2\sqrt{J}\cosh t$$

and

$$u^q + \left(\frac{J}{u}\right)^q = 2J^{q/2}\cosh(qt).$$

Hence

$$C_{p,q}(J) = \sup_{t\in\mathbb{R}} \Phi_{q,J}(t),$$

where

$$\Phi_{q,J}(t) := \frac{J^{q/2}(2\cosh t)^q}{1+2J^{q/2}\cosh(qt)}.$$

Since $J > 0$, the function $\Phi_{q,J}$ is strictly positive, so logarithmic differentiation is valid. Moreover, $\Phi_{q,J}$ is even, so it is enough to consider $t \ge 0$.

Logarithmic differentiation gives

$$\frac{\Phi'_{q,J}(t)}{q\Phi_{q,J}(t)} = \tanh t - \frac{2J^{q/2}\sinh(qt)}{1+2J^{q/2}\cosh(qt)}.$$

Using

$$\sinh t\cosh(qt) - \cosh t \sinh(qt) = -\sinh((q-1)t),$$

we obtain

$$\operatorname{sgn}\Phi'_{q,J}(t) = \operatorname{sgn}\Big(\sinh t - 2J^{q/2}\sinh((q-1)t)\Big).$$

For $t > 0$, define
$$R_q(t) := \frac{\sinh t}{\sinh((q-1)t)}.$$
We first show that R_q is strictly decreasing. Indeed,
$$\frac{d}{dt} \log R_q(t) = \coth t - (q-1)\coth((q-1)t).$$
To determine the sign, consider
$$h(s) := s \coth s, \qquad s > 0.$$
Then
$$h'(s) = \coth s - s \ \mathrm{csch}^2 s = \frac{\sinh s \cosh s - s}{\sinh^2 s}.$$
Since
$$\sinh(2s) > 2s \qquad \text{for } s > 0,$$
we have
$$\sinh s \cosh s - s > 0.$$
Therefore
$$h'(s) > 0,$$
so h is strictly increasing.

Because $q > 2$, we have
$$(q-1)t > t.$$
Hence
$$(q-1)t \coth((q-1)t) > t \coth t.$$
Dividing by $t > 0$, we obtain
$$(q-1)\coth((q-1)t) > \coth t.$$
Consequently,
$$\frac{d}{dt} \log R_q(t) < 0,$$
and therefore R_q is strictly decreasing.

Moreover,
$$\lim_{t \downarrow 0} R_q(t) = \frac{1}{q-1}, \qquad \lim_{t \to \infty} R_q(t) = 0.$$

A positive critical point of $\Phi_{q,J}$ therefore satisfies
$$R_q(t) = 2J^{q/2}.$$

Suppose first that
$$2J^{q/2} \ge \frac{1}{q-1}.$$
Since
$$R_q(t) < \frac{1}{q-1} \qquad \text{for every } t > 0,$$
there is no positive solution of
$$R_q(t) = 2J^{q/2}.$$
Moreover,
$$R_q(t) - 2J^{q/2} < 0 \qquad \text{for every } t > 0,$$
so
$$\Phi'_{q,J}(t) < 0 \qquad \text{for every } t > 0.$$

Thus $\Phi_{q,J}$ is strictly decreasing on $(0,\infty)$, and its maximum is attained at $t = 0$. This corresponds to

$$u = \sqrt{J}.$$

The condition

$$2J^{q/2} \ge \frac{1}{q-1}$$

is equivalent to

$$J \ge \left(\frac{1}{2(q-1)}\right)^{2/q}.$$

Substituting $t = 0$ gives

$$C_{p,q}(J) = \frac{J^{q/2}(2\cosh 0)^q}{1 + 2J^{q/2}\cosh 0} = \frac{2^q J^{q/2}}{1 + 2J^{q/2}}.$$

Now suppose that

$$2J^{q/2} < \frac{1}{q-1}.$$

Since R_q is strictly decreasing from $1/(q-1)$ to 0, there exists a unique $t_J > 0$ such that

$$R_q(t_J) = 2J^{q/2}.$$

Equivalently,

$$\sinh t_J = 2J^{q/2}\sinh((q-1)t_J).$$

For $0 < t < t_J$, we have

$$R_q(t) > 2J^{q/2},$$

and therefore

$$\Phi'_{q,J}(t) > 0.$$

For $t > t_J$, we have

$$R_q(t) < 2J^{q/2},$$

and therefore

$$\Phi'_{q,J}(t) < 0.$$

Hence t_J is the unique positive maximizer. Since $\Phi_{q,J}$ is even, $-t_J$ is also a maximizer.

Returning to the variable u, the two maximizers are

$$u = \sqrt{J}\,e^{t_J} \qquad \text{and} \qquad u = \sqrt{J}\,e^{-t_J}.$$

Substituting $t = t_J$ into $\Phi_{q,J}$ gives

$$C_{p,q}(J) = \frac{J^{q/2}(2\cosh t_J)^q}{1 + 2J^{q/2}\cosh(qt_J)}.$$

□

Remark (When the symmetric proportional case is optimal). The point $u = \sqrt{J}$ corresponds to the symmetric proportional case. By Theorem 4.3, this point is optimal exactly when

$$J \ge \left(\frac{1}{2(q-1)}\right)^{2/q}.$$

For smaller values of J, the symmetric point is not optimal. In that case, the supremum is attained at two asymmetric proportional configurations related by the symmetry $u \longmapsto \frac{J}{u}$.

4.5. The range $q \geq p$

We now consider the case $q \geq p$. In this range the exponent $\alpha = \frac{q}{p}$ satisfies $\alpha \geq 1$. Therefore the function $t \mapsto t^\alpha$ is convex.

From the previous section we know that, for fixed $s = x + y$ and fixed z, the numerator of the quotient is fixed. So we only have to make the denominator as small as possible. This means that we have to minimize $x^\alpha + (s - x)^\alpha$ on the admissible interval from Lemma 4.2. If $\alpha > 1$, the function

$$x \mapsto x^\alpha + (s - x)^\alpha$$

is strictly convex, so its minimum is attained at the balanced point

$$x = y = \frac{s}{2}.$$

If $\alpha = 1$, the function is constant, and the balanced split may still be chosen. Hence the minimum is attained at

$$x = y = \frac{s}{2}.$$

If $q = p$, then $\alpha = 1$, and every split gives the same value. In particular, the balanced split is still allowed.

Theorem 4.4 (The range $q \geq p$). *Assume $q \geq p$. Then*

$$C_{p,q}(J) = \sup_{s \geq 2J^{p/2}} \frac{\left(\frac{s}{2} A_p \left(\frac{2J^{p/2}}{s}\right)^p\right)^{q/p}}{1 + 2\left(\frac{s}{2}\right)^{q/p}}.$$

Equivalently, with

$$r = \frac{2J^{p/2}}{s}, \qquad \textit{we have} \qquad C_{p,q}(J) = \sup_{0 < r \leq 1} \frac{J^{q/2} A_p(r)^q}{r^{q/p} + 2J^{q/2}}.$$

Proof. Put $\alpha = \frac{q}{p}$. Then $\alpha \geq 1$. By Theorem 4.1,

$$C_{p,q}(J) = \sup_{\substack{x,y>0 \\ xy \geq J^p}} \frac{F_p(x, y, J^{p/2})^\alpha}{1 + x^\alpha + y^\alpha}.$$

Fix $s = x + y$. By Lemma 4.2, for fixed s we only need to minimize

$$x^\alpha + (s - x)^\alpha$$

on the admissible interval.

If $\alpha > 1$, the function

$$x \mapsto x^\alpha + (s - x)^\alpha$$

is strictly convex and symmetric around $x = s/2$, so its unique minimum is attained at

$$x = y = \frac{s}{2}.$$

If $\alpha = 1$, the function is constant, so the balanced split may still be chosen.

So the quotient becomes

$$\frac{F_p(s/2, s/2, J^{p/2})^{q/p}}{1 + 2(s/2)^{q/p}}.$$

Since the condition $xy \geq J^p$ becomes $\left(\frac{s}{2}\right)^2 \geq J^p$, we must have $s \geq 2J^{p/2}$. Therefore

$$C_{p,q}(J) = \sup_{s \geq 2J^{p/2}} \frac{F_p(s/2, s/2, J^{p/2})^{q/p}}{1 + 2(s/2)^{q/p}}.$$

Using the explicit formula for the envelope,

$$F_p(x,y,z) = \frac{x+y}{2} A_p \left(\frac{2z}{x+y} \right)^p,$$

we get

$$F_p(s/2, s/2, J^{p/2}) = \frac{s}{2} A_p \left(\frac{2J^{p/2}}{s} \right)^p.$$

This gives the first formula.

For the second formula, put

$$r = \frac{2J^{p/2}}{s}.$$

Then $0 < r \leq 1$, and

$$\frac{s}{2} = \frac{J^{p/2}}{r}.$$

Substituting this into the first formula gives

$$C_{p,q}(J) = \sup_{0<r\leq 1} \frac{J^{q/2} A_p(r)^q}{r^{q/p} + 2J^{q/2}}.$$

$\square$

Remark (Special values). In Theorem 4.4, the point $r = 1$ gives the symmetric proportional value

$$\frac{2^q J^{q/2}}{1 + 2J^{q/2}}.$$

The limit $r \downarrow 0$ gives the disjoint-support value $2^{q/p-1}$. These values are useful to keep in mind, but the theorem shows that the exact constant is given by the full supremum over $0 < r \leq 1$.

Remark (The remaining one-dimensional optimization). Theorem 4.4 gives an exact one-dimensional representation of $C_{p,q}(J)$. Thus the corresponding inequality is sharp, even when the supremum is not available in closed form.

An interior maximizer $r \in (0,1)$ must satisfy

$$q \frac{A_p'(r)}{A_p(r)} = \frac{q}{p} \frac{r^{q/p-1}}{r^{q/p} + 2J^{q/2}}.$$

Equivalently,

$$p \frac{A_p'(r)}{A_p(r)} = \frac{r^{q/p-1}}{r^{q/p} + 2J^{q/2}}.$$

Together with the endpoint values $r = 1$ and $r \downarrow 0$, this equation determines the possible maximizers. In general, however, the equation depends non-trivially on p, q, and J, and does not appear to have a simple closed-form solution.

Remark (The borderline case $q = p$). If $q = p$, then $q/p = 1$. In that case the denominator becomes $1 + x + y = 1 + s$.

So the denominator no longer depends on how s is split between x and y. This explains why the cases $2 < q \leq p$ and $q \geq p$ meet naturally at $q = p$.

5

Conclusion

In this thesis we studied sharpened triangle inequalities for sums of two functions in L^p-spaces. The classical triangle inequality gives a general estimate for $\|f+g\|_p$, but it does not use information about how the two functions overlap. The aim of this thesis was therefore to understand how the overlap between f and g can be used to obtain sharper estimates for the size of their sum.

The main quantities used throughout the thesis were

$$x = \|f\|_p^p, \qquad y = \|g\|_p^p, \qquad z = \|fg\|_{p/2}^{p/2}.$$

The first two quantities measure the sizes of f and g separately, while the third quantity measures their overlap. By the Cauchy–Schwarz inequality, these quantities satisfy

$$0 \le z \le \sqrt{xy}.$$

Thus the possible triples (x, y, z) lie in the admissible cone

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x \ge 0,\ y \ge 0,\ 0 \le z \le \sqrt{xy}\}.$$

This allowed us to translate the analytic problem of estimating $\|f+g\|_p^p$ into a geometric problem on the cone Ω.

The boundary of this cone corresponds to the two extreme cases of overlap. On the lower boundary $z = 0$, the functions have disjoint support when $p > 0$, and the value of $\|f+g\|_p^p$ is $x + y$. On the upper boundary $z = \sqrt{xy}$, the functions are proportional, and the value is

$$\left(x^{1/p} + y^{1/p}\right)^p.$$

These boundary values define the boundary function φ_p. The main task was then to extend this boundary data to the interior of the cone in the sharpest possible way.

The envelope method gives a systematic way to do this. We showed that every concave one-homogeneous extension of the boundary data gives an upper bound for $\|f+g\|_p^p$, while every convex one-homogeneous extension gives a lower bound. Therefore the sharp upper and lower bounds are obtained from the concave and convex envelopes of φ_p.

Following Ivanisvili and Mooney, we studied two explicit envelope functions, denoted by F_p and G_p. The first envelope F_p is obtained by the horizontal chord construction on the normalized slice $x + y = 2$. Its explicit formula is

$$F_p(x, y, z) = \frac{x+y}{2}\left[\left(1+\sqrt{1-w^2}\right)^{1/p} + \left(1-\sqrt{1-w^2}\right)^{1/p}\right]^p, \qquad w = \frac{2z}{x+y}.$$

We showed that F_p is concave for
$$p \in (0,1] \cup [2,\infty),$$
and convex for
$$p \in (1,2).$$
Thus F_p gives the sharp upper bound in the first range and the sharp lower bound in the second range.

The second envelope G_p is obtained from a corner construction. This construction divides the normalized half-disc into a central region and two corner regions. The function G_p is affine on the corresponding line segments used in this construction. For $p > 0$, it is given by

$$G_p(x,y,z) = x + y + \left[\left(v^{1/p} + v^{-1/p}\right)^p - \left(v + v^{-1}\right)\right] z, \qquad v = \min\left\{\frac{x}{z}, \frac{y}{z}, 1\right\}.$$

We showed that G_p has the opposite role from F_p. It is concave for

$$p \in (1,2),$$

and convex for
$$p \in (0,1] \cup [2,\infty).$$
Therefore G_p gives the sharp upper bound exactly in the range where F_p gives the lower bound, and conversely.

Combining these two envelope functions gives the sharp two-function inequalities. For

$$p \in (0,1] \cup [2,\infty),$$

we obtain
$$G_p(x,y,z) \le \|f+g\|_p^p \le F_p(x,y,z).$$
For
$$p \in (1,2),$$
the inequalities are reversed in the sense that

$$F_p(x,y,z) \le \|f+g\|_p^p \le G_p(x,y,z).$$

These inequalities show precisely how the overlap term $\|fg\|_{p/2}^{p/2}$ sharpens the usual triangle inequality.

We also discussed the optimality of these bounds. To do this, we introduced the extremal value functions $\overline{B}_p$ and $\underline{B}_p$, which describe the best possible upper and lower values of $\|f+g\|_p^p$ among all functions with the same associated triple (x,y,z). We showed that these extremal functions have the same one-homogeneity and boundary values as the envelope functions. Using realization of the cone and convexity properties of the extremal functions, we concluded that the envelope bounds are optimal. This means that the inequalities cannot be improved if one only uses the three quantities

$$\|f\|_p^p, \qquad \|g\|_p^p, \qquad \|fg\|_{p/2}^{p/2}.$$

The envelope method also gives insight into equality and extremal cases. Since the main inequality is obtained through Jensen's inequality, equality can occur when the pointwise triples

$$\left(f^p, g^p, (fg)^{p/2}\right)$$

lie almost everywhere in a region where the relevant envelope is affine. For F_p, these affine regions are related to the hyperplanes on which the normalized overlap parameter is constant. For G_p, the affine regions correspond to the central region and the corner regions of the construction. This explains why extremal functions often have a rigid structure, such as being equal on their common support or being proportional on the relevant part of the measure space.

In the final chapter, we studied an extension in which the final power is changed from p to another exponent $q > 2$. In this part, we assumed $p > 2$, so that F_p gives the sharp upper bound for the p-power problem. Fixing

$$J = \|fg\|_{p/2},$$

we studied the smallest constant $C_{p,q}(J)$ such that

$$\|f+g\|_p^q \le C_{p,q}(J)\left(1 + \|f\|_p^q + \|g\|_p^q\right).$$

Using the sharp envelope estimate, we reduced this problem to the exact finite-dimensional optimization problem

$$C_{p,q}(J) = \sup_{\substack{x,y>0 \\ xy \ge J^p}} \frac{F_p(x,y,J^{p/2})^{q/p}}{1 + x^{q/p} + y^{q/p}}.$$

This reduction is sharp: the right-hand side gives the smallest possible constant in the original inequality.

The final optimization is determined by the exponent q/p. The numerator depends on x and y only through their sum, while the denominator depends on them separately.

If $2 < q \le p$, then for fixed $x+y$ the minimum of the denominator may be taken at the boundary $xy = J^p$. Consequently,

$$C_{p,q}(J) = \sup_{u>0} \frac{(u + J/u)^q}{1 + u^q + (J/u)^q}.$$

This supremum can be analysed further. Define

$$J_q^* = \left(\frac{1}{2(q-1)}\right)^{2/q}.$$

If $J \ge J_q^*$, then the symmetric proportional point $u = \sqrt{J}$ is optimal, and

$$C_{p,q}(J) = \frac{2^q J^{q/2}}{1 + 2J^{q/2}}.$$

If $0 < J < J_q^*$, then the maximum is attained at two asymmetric proportional configurations

$$u = \sqrt{J}\, e^{\pm t_J},$$

where $t_J > 0$ is the unique solution of $\sinh t_J = 2J^{q/2} \sinh((q-1)t_J)$.

If $q > p$, then $q/p > 1$, and the denominator is minimized by the balanced split $x = y$.

This gives the exact one-dimensional representation

$$C_{p,q}(J) = \sup_{0<r\le 1} \frac{J^{q/2} A_p(r)^q}{r^{q/p} + 2J^{q/2}},$$

where

$$A_p(r) = \left(1 + \sqrt{1-r^2}\right)^{1/p} + \left(1 - \sqrt{1-r^2}\right)^{1/p}.$$

In the borderline case $q = p$, the denominator depends only on $x + y$, so every split gives the same value and the balanced split may still be chosen. Although the remaining supremum does not appear to have a simple closed form in general, it is an exact sharp one-dimensional characterization of the optimal constant.

Overall, the results of this thesis show that the overlap between two functions can be used to sharpen the classical triangle inequality in a precise and optimal way. The envelope method gives both the sharp inequalities and a geometric explanation of why these inequalities are optimal. It also provides a useful framework for studying related problems, such as the q-power extension considered in the final chapter. The main conclusion is that the three quantities

$$\|f\|_p^p, \qquad \|g\|_p^p, \qquad \|fg\|_{p/2}^{p/2}$$

contain exactly the information needed to obtain the sharp two-function bounds studied in this thesis.

Although the results are obtained in a specific setting, they demonstrate how geometric information, encoded through the overlap parameter, can lead to sharper estimates than those based solely on norms.

References

[1] Paata Ivanisvili and Connor Mooney. “Sharpening the triangle inequality : envelopes between L2 and Lp spaces”. In: *Analysis PDE* 13 (July 2020), pp. 1591–1603. DOI: 10.2140/apde.2020.13.1591.

[2] Eric Carlen et al. “Inequalities for L□-Norms that Sharpen the Triangle Inequality and Complement Hanner’s Inequality”. In: *The Journal of Geometric Analysis* 31 (July 2018). DOI: 10.1007/s12220-020-00425-y.

[3] Asadollah Aghajani and Juha Kinnunen. *New inequalities related to sums of L^p functions in connection with Carbery’s problems*. 2026. arXiv: 2601.21594 [math.FA]. URL: https://arxiv.org/abs/2601.21594.

[4] Paata Ivanisvili. *Bellman function approach to the sharp constants in uniform convexity*. 2015. arXiv: 1402.4690 [math.AP]. URL: https://arxiv.org/abs/1402.4690.

[5] Keith Ball, Eric A. Carlen, and Elliott H. Lieb. “Sharp uniform convexity and smoothness inequalities for trace norms”. In: *Invent. Math.* 115.3 (1994), pp. 463–482. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01231769. URL: https://doi.org/10.1007/BF01231769.

[6] Delft University of Technology. *Lecture Notes AM2090: Real Analysis*. Faculty of Electrical Engineering, Mathematics and Computer Science. Aug. 2024.

[7] Tuomas Hytönen et al. *Analysis in Banach Spaces. Volume I: Martingales and Littlewood-Paley Theory*. Vol. 63. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer, 2016.

[8] Fedor Nazarov, Sergei Treil, and Alexander Volberg. *Bellman functions and two weight inequalities for Haar multipliers*. 1997. arXiv: math/9711209 [math.FA]. URL: https://arxiv.org/abs/math/9711209.

[9] Alexander Volberg. *Bellman function technique in Harmonic Analysis. Lectures of INRIA Summer School in Antibes, June 2011*. 2011. arXiv: 1106.3899 [math.PR]. URL: https://arxiv.org/abs/1106.3899.

[10] Walter Rudin. *Real and Complex Analysis*. 3rd ed. McGraw–Hill, 1987.

[11] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

AI Disclosure

During the preparation of this thesis, I made extensive use of generative AI tools, primarily ChatGPT (OpenAI), as a supporting tool throughout the research and writing process.

The AI assistance included help with structuring and reformulating text, improving clarity and academic language, checking grammar and notation, working with LaTeX, summarizing and explaining mathematical ideas, and suggesting possible approaches to calculations, proofs, and the presentation of results. It was also used to identify unclear passages, possible inconsistencies, and points that required further explanation or verification.

All AI-generated suggestions were reviewed critically before being included. Mathematical arguments, calculations, and statements were checked against the relevant literature and worked through independently. The final selection, interpretation, organization, and presentation of the material, as well as responsibility for the correctness of the thesis, remain my own.

A

Appendix

.0**Derivative computation for F_p** In this appendix we justify the concavity and convexity statement for the function F_p. On the normalized slice $x+y=2$, the function F_p depends only on z. We therefore write

$$u(t) = \left((1+\sqrt{1-t^2})^{1/p} + (1-\sqrt{1-t^2})^{1/p}\right)^p, \qquad 0 \le t \le 1.$$

The goal is to determine the sign of $u''(t)$.

To remove the square root, set $t = \sin(2s)$ with $0 \le s \le \frac{\pi}{4}$. Then $\sqrt{1-t^2} = \cos(2s)$, and therefore

$$\begin{aligned} u(\sin(2s)) &= \left((1+\cos(2s))^{1/p} + (1-\cos(2s))^{1/p}\right)^p \\ &= \left((2\cos^2 s)^{1/p} + (2\sin^2 s)^{1/p}\right)^p \\ &= 2\left(\cos^{2/p} s + \sin^{2/p} s\right)^p. \end{aligned}$$

Hence $\frac{1}{2}u(\sin(2s)) = \left(\cos^{2/p} s + \sin^{2/p} s\right)^p$. Define

$$A(s) := \cos^{2/p} s + \sin^{2/p} s.$$

Differentiating gives

$$u'(\sin(2s))\cos(2s) = pA(s)^{p-1}A'(s),$$

where $A'(s) = \frac{2}{p}\left(\sin^{2/p-1} s\cos s - \cos^{2/p-1} s \sin s\right)$. Equivalently,

$$A'(s) = \frac{2}{p}\cos^{2/p} s\left(\tan^{2/p-1} s - \tan s\right).$$

Differentiating once more and simplifying, one obtains

$$\operatorname{sgn}\big(u''(\sin(2s))\big) = \operatorname{sgn}\big(v(x)\big), \qquad x = \tan^2(s) \in [0,1],$$

where

$$v(x) = x^{2/p-1} + \left(\frac{2}{p}-1\right)x^{1/p-1}(1-x) - 1.$$

The simplification uses the identity

$$\frac{2u''(\sin(2s))\cos^2(2s)}{A(s)^{p-2}\cos^{4/p} s} = \frac{2(1+x)^2}{1-x}\left[x^{2/p-1} + \left(\frac{2}{p}-1\right)x^{1/p-1}(1-x) - 1\right],$$

with $x = \tan^2(s)$. The prefactor is positive for $0 \le x < 1$, so the sign is determined by $v(x)$.

We now determine the sign of v. Clearly, $v(1) = 0$. A direct differentiation gives

$$v'(x) = x^{1/p-2}\left(\frac{2}{p} - 1\right)\left[x^{1/p} - \left(1 + \frac{x-1}{p}\right)\right].$$

The sign of the last bracket follows from the elementary comparison between the graph of $x^{1/p}$ and its tangent line at $x = 1$.

If $p > 2$, then $x^{1/p}$ is concave, and hence $x^{1/p} \le 1 + \frac{x-1}{p}$. Since $\frac{2}{p} - 1 < 0$, this gives $v'(x) \ge 0$. Together with $v(1) = 0$, this implies $v(x) \le 0$for $0 \le x \le 1$. Thus $u'' \le 0$, so u is concave.

If $1 < p < 2$, then $x^{1/p}$ is still concave, but now $\frac{2}{p} - 1 > 0$. Hence $v'(x) \le 0$, and since $v(1) = 0$, we get $v(x) \ge 0$. Thus $u'' \ge 0$, so u is convex.

If $0 < p < 1$, then $x^{1/p}$ is convex, and hence $x^{1/p} \ge 1 + \frac{x-1}{p}$. Since $\frac{2}{p} - 1 > 0$, we get $v'(x) \ge 0$. Therefore $v(x) \le 0$, and u is concave.

Consequently, F_p is concave for $p \in (0, 1] \cup [2, \infty)$, and convex for $p \in (1, 2)$.

.0**Derivative computation for G_p**

In this appendix we justify the concavity and convexity statement for the function G_p. On the normalized slice $x + y = 2$, write $x = 1 + s, y = 1 - s$, and define $b_p(s, z) := G_p(1 + s, 1 - s, z)$. For $p > 0$, the function b_p is given by

$$b_p(s,z) = 2 + \begin{cases} \omega(1-|s|, z), & z \ge 1-|s|, \\ (2^p - 2)z, & z < 1-|s|, \end{cases}$$

where $\omega(t, z) := \left(t^{1/p} + \left(\frac{z^2}{t}\right)^{1/p}\right)^p - \left(t + \frac{z^2}{t}\right)$. The central part is linear, so its curvature is zero. It remains to study the curvature of the corner expression.

The function ω is one-homogeneous in the variables (t, z). Hence, by the same slice argument used in Lemma 3.4, it is enough to study the one-variable function $h(t) := \omega(t, 1) = \left(t^{1/p} + t^{-1/p}\right)^p - \left(t + t^{-1}\right), 0 < t \le 1$. We compute

$$h'(t) = \left(t^{1/p} + t^{-1/p}\right)^{p-1}\left(t^{1/p-1} - t^{-1/p-1}\right) - (1 - t^{-2}).$$

In particular, $h'(1) = 0$. A second differentiation gives

$$\begin{aligned} h''(t) = {} & \frac{p-1}{p}\left(t^{1/p} + t^{-1/p}\right)^{p-2}\left(t^{1/p-1} - t^{-1/p-1}\right)^2 \\ & + \frac{1}{p}\left(t^{1/p} + t^{-1/p}\right)^{p-1}\left((1-p)t^{1/p-2} + (1+p)t^{-1/p-2}\right) - 2t^{-3}. \end{aligned}$$

After simplification, this becomes $h''(t) = 2t^{-3}\left[(1 + t^{2/p})^{p-2}\left(1 + \left(\frac{2}{p} - 1\right)t^{2/p}\right) - 1\right]$. Set $x := t^{2/p}$. For $p > 0$, we have $x \in [0, 1]$. Since $2t^{-3} > 0$, the sign of h'' is the sign of $(1+x)^{p-2}\left(1 + \left(\frac{2}{p} - 1\right)x\right) - 1$. Equivalently, the sign is determined by $g_p(x) := 1 + \left(\frac{2}{p} - 1\right)x - (1 + x)^{2-p}$.

We now determine the sign of g_p on $[0, 1]$. We have $g_p(0) = 0$. Also, $g_p(1) = \frac{2}{p} - 2^{2-p}$. This has the same sign as $2^{p-1} - p$. The function $p \mapsto 2^{p-1}$ is convex and meets the line p at $p = 1$ and $p = 2$. Hence $2^{p-1} \ge p$for $p \in (0, 1] \cup [2, \infty)$, and $2^{p-1} \le p$ for $p \in (1, 2)$. Therefore $g_p(1) \ge 0$ for $p \in (0, 1] \cup [2, \infty)$, and $g_p(1) \le 0$ for $p \in (1, 2)$.

Furthermore, the first term in g_p is linear in x, while the function $x \mapsto (1 + x)^{2-p}$ is convex for $p \in (1, 2)$, and concave for $p \in (0, 1) \cup (2, \infty)$. It follows that $g_p(x) \le 0$ for $p \in (1, 2)$, and $g_p(x) \ge 0$ for $p \in (0, 1] \cup [2, \infty)$. Thus $h''(t) \le 0$ for $p \in (1, 2)$, and $h''(t) \ge 0$ for $p \in (0, 1] \cup [2, \infty)$. Consequently, for $p > 0$, G_p is concave for $p \in (1, 2)$, and convex for $p \in (0, 1] \cup [2, \infty)$.

Consequently, G_p is concave for

$$p \in (1, 2),$$

and convex for

$$p \in (0, 1] \cup [2, \infty).$$