

PRODUCTS OF COMMUTING BOOLEAN ALGEBRAS OF PROJECTIONS AND BANACH SPACE GEOMETRY

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1. Introduction

The study of bounded Boolean algebras (briefly, B.a.) of projections in Banach spaces (intimately connected to the theory of spectral operators, [11]) was initiated in the penetrating work of W. G. Bade in the 1950s, [3, 4]. In the Hilbert space setting, a fundamental result of J. Wermer states that the smallest B.a. of projections which contains a given pair of bounded, commuting Boolean algebras of projections is itself bounded, [31]. The search for analogues of this result in Banach spaces has had some far reaching consequences. Counterexamples, separately due to S. Kakutani and C. A. McCarthy, came quickly; see [11, pp. 2098–2099], for example. Accordingly, much subsequent research concentrated on identifying various classes of Banach spaces in which the conclusion does hold. C. A. McCarthy established that all L_p -spaces, for $1 \leq p < \infty$, have this property and (together with W. Littman and N. Rivière) also their complemented subspaces; see [11, pp. 2099–2100], for instance. This is also the case for all Grothendieck spaces with the Dunford–Pettis property, [25], and the class of all hereditary indecomposable Banach spaces, [26]. The most significant recent results which identify large (and new) classes of Banach spaces with the property that the B.a. generated by every pair of commuting, bounded Boolean algebras of projections in the space is again bounded, are due to T. A. Gillespie, [13]. He showed that this is always the case for arbitrary Banach lattices, for closed subspaces of any p -concave Banach lattice (with p finite), for complemented subspaces of any \mathcal{L}_∞ -space, and for all Banach spaces with local unconditional structure (briefly, l.u.st.).

The aim of this paper is to make a further contribution to the above discussed problem. Our viewpoint is that the geometry of the underlying Banach space is not the only relevant ingredient; an important property of the individual Boolean algebras concerned (when available) can also play a fundamental role. This is the notion of *R-boundedness*, introduced by E. Berkson and T. A. Gillespie in [5] (where it is called the *R*-property), but already implicit in earlier work of J. Bourgain, [6]. Since its conception in the mid-1990s, *R*-boundedness has played an increasingly important role in various branches of functional analysis, operator theory, harmonic analysis and partial differential equations; see, for example, [2, 8, 9, 18], and the references therein.

Let us describe a sample of our results. The basic fact is simple: the B.a. generated by any pair of bounded, commuting Boolean algebras of projections, at least one of which is *R*-bounded, is again bounded. In practice, the effectiveness of

this observation lies in the ability of being able to decide about the R -boundedness of particular Boolean algebras of projections. In this regard it turns out, for the class of Banach spaces with property (α) introduced by G. Pisier in [24], that every bounded B.a. of projections is automatically R -bounded. This class includes all Banach spaces with l.u.st. and having finite cotype. For Banach lattices (which always possess l.u.st.), having finite cotype is equivalent to p -concavity for some finite p which, in turn, is equivalent to property (α) . A larger class of Banach spaces, for which it is possible to decide precisely when property (α) is present, are the GL-spaces (due to Y. Gordon and D. R. Lewis, [14]); these spaces also have property (α) if and only if they have finite cotype. All Banach spaces with l.u.st. are GL-spaces, but not conversely. It is shown that in any GL-space the product of every pair of commuting, bounded Boolean algebras of projections (no R -boundedness needed!) is again bounded.

For a Dedekind σ -complete Banach lattice E something remarkable occurs. The R -boundedness of the *particular* B.a. $\mathcal{B}(E)$ of all *band projections* is equivalent to the space having finite cotype. As noted above, this is also equivalent to having property (α) . Accordingly, every bounded B.a. of projections in E is R -bounded precisely when *just* $\mathcal{B}(E)$ is R -bounded. Since c_0 and ℓ_∞ fail to have finite cotype, we can conclude that the Boolean algebras $\mathcal{B}(c_0)$ and $\mathcal{B}(\ell_\infty)$ are not R -bounded. The techniques used to establish these facts have further consequences. Given any Banach space X and any bounded B.a. of projections \mathfrak{M} in X , it is always possible to equip the cyclic space $\mathfrak{M}[x]$ with a Banach lattice structure, for each $x \in X$. It is shown that *Bade-completeness* of the B.a. \mathfrak{M} (in the sense of [3]) is equivalent to each Banach lattice $\mathfrak{M}[x]$, for $x \in X$, having order-continuous norm. For a bounded B.a. of projections \mathfrak{M} in X , by applying the previous criterion in each Banach lattice $\mathfrak{M}[x]$, for $x \in X$, it is established that the strong operator closure $\overline{\mathfrak{M}}_s$ of \mathfrak{M} is Bade-complete whenever \mathfrak{M} is R -bounded. A consideration of $\mathcal{B}(c_0)$ shows, even in the presence of a cyclic vector, that the converse is false in general.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a (complex) Banach space. The Banach space of all bounded linear operators in X is denoted by $\mathcal{L}(X)$. A Boolean algebra (briefly, B.a.) of projections in X is a commuting family $\mathfrak{M} \subseteq \mathcal{L}(X)$ of projections such that $PQ \in \mathfrak{M}$ and $I - P \in \mathfrak{M}$ whenever $P, Q \in \mathfrak{M}$. Here, I denotes the identity operator in X . Note that \mathfrak{M} is indeed a B.a. with respect to the lattice operations $P \wedge Q = PQ$, $P \vee Q = P + Q - PQ$ and complementation $P^c = I - P$. The B.a. \mathfrak{M} is called *bounded* if

$$\|\mathfrak{M}\| = \sup\{\|P\| : P \in \mathfrak{M}\} < \infty.$$

For any B.a. \mathfrak{M} of projections the strong operator closure $\overline{\mathfrak{M}}_s$ is also a B.a. of projections, which is bounded whenever \mathfrak{M} is bounded. Recall that a B.a. \mathfrak{M} is *Bade-complete* if \mathfrak{M} is complete as an abstract B.a. and $P_\tau \uparrow P$ with respect to the order of \mathfrak{M} implies that $P_\tau x \rightarrow Px$ for all $x \in X$. Every Bade-complete B.a. is bounded and strongly closed, but the converse is in general not true (consider the B.a. \mathfrak{M} in $L_\infty([0, 1])$ of all operators of multiplication by χ_F with $F \subseteq [0, 1]$ measurable; then \mathfrak{M} is strongly closed and bounded but not Bade-complete). For a detailed account of Boolean algebras of projections we refer the reader to [11, 27]. Given two commuting Boolean algebras \mathfrak{M} and \mathfrak{N} of projections in X , there exists

a smallest B.a. in $\mathcal{L}(X)$ containing both Boolean algebras; it is called the B.a. generated by \mathfrak{M} and \mathfrak{N} or the *product* of \mathfrak{M} and \mathfrak{N} . This product B.a. is denoted by $\mathfrak{M} \vee \mathfrak{N}$. Moreover, elements of $\mathfrak{M} \vee \mathfrak{N}$ all have the form $\sum_{i,j=1}^{m,n} \eta_{ij} P_i Q_j$ where $\sum_{i=1}^m P_i = \sum_{j=1}^n Q_j = I$ with $P_i \in \mathfrak{M}$, $Q_j \in \mathfrak{N}$ and $\eta_{ij} \in \{0, 1\}$ for $1 \leq i \leq m$, $1 \leq j \leq n$ and $m, n \in \mathbb{N} \setminus \{0\}$.

A sequence $\Delta = \{\Delta_n\}_{n=1}^\infty$ of bounded linear projections in a Banach space X is called a *Schauder decomposition* of X if:

- (a) $\Delta_n \Delta_m = 0$ whenever $n \neq m$;
- (b) $x = \sum_{n=1}^\infty \Delta_n x$ for all $x \in X$.

A Schauder decomposition $\{\Delta_n\}_{n=1}^\infty$ is *unconditional* if the series in (b) is unconditionally convergent for all $x \in X$. Given a Schauder basis $\{e_n\}_{n=1}^\infty$ of X we denote the corresponding coordinate functionals by $\{e_n^*\}_{n=1}^\infty \subseteq X^*$, where X^* is the dual space of X . Then the one-dimensional coordinate projections $\{P_n\}_{n=1}^\infty$, defined by $P_n x = \langle x, e_n^* \rangle e_n$ for all $x \in X$, form a Schauder decomposition of X , which is unconditional if and only if $\{e_n\}_{n=1}^\infty$ is an unconditional basis of X . The following characterizations of unconditional decompositions is important. Detailed proofs of the next three results can be found in §2.1 of [33].

PROPOSITION 2.1. *A Schauder decomposition $\{\Delta_n\}_{n=1}^\infty$ of the Banach space X is unconditional if and only if there exists a constant $C > 0$ such that*

$$\left\| \sum_{k=1}^n \varepsilon_k \Delta_k x \right\| \leq C \left\| \sum_{k=1}^n \Delta_k x \right\| \quad (1)$$

for all choices of $\varepsilon_k \in \{-1, 1\}$, $x \in X$ and $n \in \mathbb{N}$.

In the setting of Proposition 2.1, the smallest $C > 0$ satisfying (1) is the unconditional constant of the decomposition $\{\Delta_n\}_{n=1}^\infty$ and is denoted by C_Δ .

We denote by $\{r_j\}_{j=1}^\infty$ the sequence of Rademacher functions on the interval $[0, 1]$ (so, $\{r_j\}_{j=1}^\infty$ is a sequence of independent identically distributed symmetric $\{-1, 1\}$ -valued random variables).

LEMMA 2.2. *A Schauder decomposition $\{\Delta_n\}_{n=1}^\infty$ of a Banach space X is unconditional if and only if for every (some) $1 \leq p < \infty$ there exists a constant $C > 0$ such that*

$$C^{-1} \left\| \sum_{k=1}^n \Delta_k x \right\| \leq \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) \Delta_k x \right\|^p dt \right)^{1/p} \leq C \left\| \sum_{k=1}^n \Delta_k x \right\| \quad (2)$$

for all $x \in X$ and all $n \in \mathbb{N}$. In this case (2) is satisfied with $C = C_\Delta$.

PROPOSITION 2.3. *Let $\{\Delta_n\}_{n=1}^\infty$ be an unconditional decomposition of a Banach space X . For every bounded sequence $\lambda = \{\lambda_n\}_{n=1}^\infty$ in \mathbb{C} and every $x \in X$, the series*

$$T_\lambda x = \sum_{n=1}^\infty \lambda_n \Delta_n x$$

is (unconditionally) convergent in X and $\|T_\lambda x\| \leq 2C_\Delta \|\lambda\|_\infty \|x\|$ for all $x \in X$. If λ is a real sequence, then the factor 2 can be omitted in the last estimate.

There is a close connection between (unconditional) Schauder decompositions and Boolean algebras of projections. Indeed, suppose that $\{\Delta_n\}_{n=1}^\infty$ is a Schauder decomposition in X . Let Σ_0 be the algebra of all finite and cofinite subsets of \mathbb{N} . For $F \in \Sigma_0$ define $\Delta_F = \sum_{n \in F} \Delta_n$ whenever F is finite and $\Delta_F = I - \Delta_{\mathbb{N} \setminus F}$ otherwise. Then

$$\mathfrak{M}_\Delta^f = \{\Delta_F : F \in \Sigma_0\} \quad (3)$$

is a B.a. of projections in X . The next lemma follows from Propositions 2.1 and 2.3.

LEMMA 2.4. *The B.a. \mathfrak{M}_Δ^f is uniformly bounded for the operator norm (briefly, bounded) if and only if $\{\Delta_n\}_{n=1}^\infty$ is an unconditional decomposition. In this case, for every subset $A \subseteq \mathbb{N}$ the series*

$$\Delta_A x = \sum_{n \in A} \Delta_n x$$

is unconditionally convergent for all $x \in X$ and $\Delta_A : X \rightarrow X$ is a projection satisfying $\|\Delta_A\| \leq C_\Delta$. Moreover,

$$\mathfrak{M}_\Delta = \{\Delta_A : A \subseteq \mathbb{N}\} \quad (4)$$

is a Bade-complete B.a. of projections in X .

On the other hand, suppose that \mathfrak{M} is a bounded B.a. of projections in X . If $P_1, \dots, P_n \in \mathfrak{M}$ is any finite collection of projections satisfying $\sum_{k=1}^n P_k = I$, then $\{P_k\}_{k=1}^n$ is an unconditional decomposition of X with unconditional constant at most $2\|\mathfrak{M}\|$. So, by Lemma 2.2 we have

$$(2\|\mathfrak{M}\|)^{-1} \|x\| \leq \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) P_k x \right\|^p dt \right)^{1/p} \leq 2\|\mathfrak{M}\| \|x\| \quad (5)$$

for all $x \in X$ and all $1 \leq p < \infty$. The following version of (5) will also be useful: for all choices of $x_1, \dots, x_n \in X$ we have

$$(2\|\mathfrak{M}\|)^{-1} \left\| \sum_{k=1}^n P_k x_k \right\| \leq \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) P_k x_k \right\|^p dt \right)^{1/p} \leq 2\|\mathfrak{M}\| \left\| \sum_{k=1}^n P_k x_k \right\|. \quad (6)$$

Indeed, since the $\{P_k\}_{k=1}^n$ are necessarily pairwise disjoint, this inequality follows immediately from (5) applied to $x = \sum_{k=1}^n P_k x_k$, using the fact that $P_k x = P_k x_k$ for all k .

Suppose now that $\Delta = \{\Delta_n\}_{n=1}^\infty$ and $\Delta' = \{\Delta'_n\}_{n=1}^\infty$ are two commuting Schauder decompositions of a Banach space X , that is, $\Delta_n \Delta'_m = \Delta'_m \Delta_n$ for all n and m . It is clear (for all m and n) that the operator $\Delta_n \Delta'_m$ is a bounded projection in X and that $\bigcup_{m,n} \text{Ran}(\Delta_n \Delta'_m)$ is dense in X . Let $\{D_k\}_{k=1}^\infty$ be the collection $\{\Delta_n \Delta'_m : n, m = 1, 2, \dots\}$ ‘enumerated via squares’, that is,

$$\{D_k\}_{k=1}^\infty = \{\Delta_1 \Delta'_1, \Delta_2 \Delta'_1, \Delta_2 \Delta'_2, \Delta_1 \Delta'_2, \Delta_3 \Delta'_1, \dots\}. \quad (7)$$

Then the partial sum projections $\{\sum_{k=1}^N D_k : N = 1, 2, \dots\}$ are uniformly bounded, from which it follows that $D = \{D_k\}_{k=1}^\infty$ is a Schauder decomposition of X . Unconditionality of both the decompositions Δ and Δ' need not imply the unconditionality of D . In this connection the following observation is of some interest.

PROPOSITION 2.5. Let $\Delta = \{\Delta_n\}_{n=1}^\infty$ and $\Delta' = \{\Delta'_n\}_{n=1}^\infty$ be two commuting unconditional decompositions of a Banach space X . Let the Schauder decomposition $D = \{D_k\}_{k=1}^\infty$ be given by (7). Then the following statements are equivalent:

- (i) the product B.a. $\mathfrak{M}_\Delta \vee \mathfrak{M}_{\Delta'}$ is bounded;
- (ii) the product B.a. $\mathfrak{M}_\Delta^f \vee \mathfrak{M}_{\Delta'}^f$ is bounded;
- (iii) the decomposition $D = \{D_k\}_{k=1}^\infty$ is unconditional.

Proof. Since $\mathfrak{M}_\Delta^f \vee \mathfrak{M}_{\Delta'}^f \subseteq \mathfrak{M}_\Delta \vee \mathfrak{M}_{\Delta'}$, it is clear that (i) implies (ii). To show that (ii) implies (iii), first note that every projection in \mathfrak{M}_D^f can be written as $\sum_{(n,m) \in F} \Delta_n \Delta'_m$, with F a finite subset of $\mathbb{N} \times \mathbb{N}$, or is the complement of such a projection. Consequently, if $\mathfrak{M}_\Delta^f \vee \mathfrak{M}_{\Delta'}^f$ is bounded, then \mathfrak{M}_D^f is bounded as well. By Lemma 2.4 we conclude that D is unconditional. Finally, if we assume that D is an unconditional decomposition, then \mathfrak{M}_D is a bounded (even Bade-complete) B.a. Since $\mathfrak{M}_\Delta \vee \mathfrak{M}_{\Delta'} \subseteq \mathfrak{M}_D$, it follows that $\mathfrak{M}_\Delta \vee \mathfrak{M}_{\Delta'}$ is bounded. So, (iii) implies (i). \square

To illustrate the above situation, let X and Y be Banach spaces and α be a uniform cross norm on the tensor product $X \otimes Y$ (that is, $\alpha(x \otimes y) = \|x\| \|y\|$ for all $x \in X$ and $y \in Y$, and if $u : X \rightarrow X$ and $v : Y \rightarrow Y$ are bounded linear operators, then $\|u \otimes v\| \leq \|u\| \|v\|$). Denote by $X \widehat{\otimes}_\alpha Y$ the norm completion of $(X \otimes Y, \alpha)$. If $u \in \mathcal{L}(X)$ and $v \in \mathcal{L}(Y)$, then $u \otimes v$ is a bounded linear operator on $(X \otimes Y, \alpha)$ and hence, extends uniquely to a bounded linear operator on $X \widehat{\otimes}_\alpha Y$; this unique extension is also denoted by $u \otimes v$. Suppose that $\{P_n\}_{n=1}^\infty$ is a Schauder decomposition of X . Define the projections $\{\Delta_n\}_{n=1}^\infty$ in $X \widehat{\otimes}_\alpha Y$ by $\Delta_n = P_n \otimes I$. Since $\bigcup_{n=1}^\infty \text{Ran}(\Delta_n)$ is dense in $X \otimes Y$ (hence, also dense in $X \widehat{\otimes}_\alpha Y$) and

$$\left\| \sum_{n=1}^N \Delta_n \right\| = \left\| \left(\sum_{n=1}^N P_n \right) \otimes I \right\| \leq \left\| \sum_{n=1}^N P_n \right\|$$

for all $N \in \mathbb{N}$, it follows that $\{\Delta_n\}_{n=1}^\infty$ is a Schauder decomposition of $X \widehat{\otimes}_\alpha Y$. Similarly, if $\{Q_n\}_{n=1}^\infty$ is a Schauder decomposition of Y and we define $\Delta'_n = I \otimes Q_n$ for all $n \in \mathbb{N}$, then $\{\Delta'_n\}_{n=1}^\infty$ is a Schauder decomposition of $X \widehat{\otimes}_\alpha Y$ as well. Clearly $\Delta_n \Delta'_m = \Delta'_m \Delta_n$ for all n and m . As observed above, the sequence $\{D_k\}_{k=1}^\infty$ defined by (7) is a Schauder decomposition of $X \widehat{\otimes}_\alpha Y$. In case the decompositions $\{P_n\}_{n=1}^\infty$ and $\{Q_n\}_{n=1}^\infty$ are both unconditional, Proposition 2.5 gives necessary and sufficient conditions for $\{D_k\}_{k=1}^\infty$ to be an unconditional decomposition of $X \widehat{\otimes}_\alpha Y$. In particular, if $\{e_n\}_{n=1}^\infty$ and $\{f_m\}_{m=1}^\infty$ are Schauder bases of X and Y respectively, then $\{e_n \otimes f_m\}_{n,m=1}^\infty$, ‘enumerated via squares’ as in (7), is a Schauder basis of $X \widehat{\otimes}_\alpha Y$. In this setting Proposition 2.5 can be used to obtain criteria guaranteeing that $\{e_n \otimes f_m\}_{n,m=1}^\infty$ is an unconditional basis of $X \widehat{\otimes}_\alpha Y$.

We end this section by recalling the notion of R -boundedness.

DEFINITION 2.6. Let X be a Banach space. A non-empty collection $\mathcal{T} \subseteq \mathcal{L}(X)$ is called R -bounded if there exists a constant $M \geq 0$ such that

$$\left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) T_j x_j \right\|^2 dt \right)^{1/2} \leq M \left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^2 dt \right)^{1/2} \quad (8)$$

for all $T_1, \dots, T_n \in \mathcal{T}$, all $x_1, \dots, x_n \in X$ and all $n \in \mathbb{N} \setminus \{0\}$.

If $\mathcal{T} \subseteq \mathcal{L}(X)$ is R -bounded, then the smallest constant $M \geq 0$ for which (8) holds will be denoted by $M_{\mathcal{T}}$ and is called the R -bound of \mathcal{T} . Clearly, every R -bounded collection is uniformly bounded in $\mathcal{L}(X)$. For more information concerning R -boundedness we refer to [8, 33]. In particular, the strongly closed absolute convex hull of any R -bounded collection \mathcal{T} is also R -bounded (with the same R -bound in real spaces and with R -bound at most $2M_{\mathcal{T}}$ in complex spaces).

3. Boolean algebras and R -boundedness

In this section we consider two results, related to R -boundedness, which concern the boundedness of products of commuting Boolean algebras of projections.

THEOREM 3.1. *Let \mathfrak{M} be an R -bounded B.a. of projections in a Banach space X . Then $\mathfrak{M} \vee \mathfrak{N}$ is bounded whenever \mathfrak{N} is a bounded B.a. of projections in X commuting with \mathfrak{M} . Moreover,*

$$\|\mathfrak{M} \vee \mathfrak{N}\| \leq 4\|\mathfrak{N}\|^2 M_{\mathfrak{M}}.$$

Proof. Every element of $\mathfrak{M} \vee \mathfrak{N}$ can be written as $\sum_{k=1}^n P_k Q_k$, for some $n \in \mathbb{N}$, projections $Q_1, \dots, Q_n \in \mathfrak{N}$ satisfying $\sum_{k=1}^n Q_k = I$, and $P_1, \dots, P_n \in \mathfrak{M}$. For $x \in X$ it follows from (6), applied in the B.a. \mathfrak{N} , that

$$\begin{aligned} \left\| \sum_{k=1}^n P_k Q_k x \right\| &\leq 2\|\mathfrak{N}\| \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) P_k Q_k x \right\|^2 dt \right)^{1/2} \\ &\leq 2\|\mathfrak{N}\| M_{\mathfrak{M}} \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) Q_k x \right\|^2 dt \right)^{1/2} \\ &\leq (2\|\mathfrak{N}\|)^2 M_{\mathfrak{M}} \|x\|. \end{aligned}$$

This shows that $\mathfrak{M} \vee \mathfrak{N}$ is bounded and $\|\mathfrak{M} \vee \mathfrak{N}\| \leq 4\|\mathfrak{N}\|^2 M_{\mathfrak{M}}$. \square

We now consider a class of Banach spaces, introduced by G. Pisier (see [24, Definition 2.1]), in which every bounded B.a. of projections is automatically R -bounded.

DEFINITION 3.2. A Banach space X has *property* (α) if there exists a constant $\alpha \geq 0$ such that

$$\int_0^1 \int_0^1 \left\| \sum_{j=1}^m \sum_{k=1}^n \varepsilon_{jk} r_j(s) r_k(t) x_{jk} \right\|^2 ds dt \leq \alpha^2 \int_0^1 \int_0^1 \left\| \sum_{j=1}^m \sum_{k=1}^n r_j(s) r_k(t) x_{jk} \right\|^2 ds dt$$

for every choice of $x_{jk} \in X$, $\varepsilon_{jk} \in \{-1, 1\}$ and for all $m, n \in \mathbb{N}$. In this case, the smallest possible constant α in the previous inequality is denoted by α_X .

It is shown in [24, Proposition 2.1], that every Banach space with *l.u.st.* and having finite *cotype* necessarily has property (α) . In particular, every Banach lattice, which automatically has *l.u.st.* (see for example [10, Theorem 17.1]), with finite *cotype* has property (α) . As observed in [24, Remark 2.2], a Banach space X with property (α) cannot contain the ℓ_{∞}^n uniformly. For a Banach space, the property of not containing the ℓ_{∞}^n uniformly is equivalent to having finite *cotype*,

a deep result due to B. Maurey and G. Pisier, [22] (see also [10, §14]). Consequently, in Banach spaces with l.u.st., having property (α) is equivalent to having finite cotype.

THEOREM 3.3. *Let X be a Banach space with property (α) . Then every bounded B.a. \mathfrak{M} of projections in X is R -bounded with R -bound $M_{\mathfrak{M}} \leq 4\|\mathfrak{M}\|^2\alpha_X$.*

Proof. Let $P_1, \dots, P_n \in \mathfrak{M}$ and $x_1, \dots, x_n \in X$ be given. There exist mutually disjoint projections $Q_1, \dots, Q_N \in \mathfrak{M}$ with $\sum_{k=1}^N Q_k = I$ and $\alpha_{jk} \in \{0, 1\}$ such that $P_j = \sum_{k=1}^N \alpha_{jk} Q_k$ for all $j = 1, \dots, n$. Hence,

$$\int_0^1 \left\| \sum_{j=1}^n r_j(s) P_j x_j \right\|^2 ds = \int_0^1 \left\| \sum_{j=1}^n \sum_{k=1}^N \alpha_{jk} r_j(s) Q_k x_j \right\|^2 ds. \quad (9)$$

For each $s \in [0, 1]$ it follows from (6) that

$$\begin{aligned} \left\| \sum_{j=1}^n \sum_{k=1}^N \alpha_{jk} r_j(s) Q_k x_j \right\|^2 &= \left\| \sum_{k=1}^N Q_k \left(\sum_{j=1}^n \alpha_{jk} r_j(s) x_j \right) \right\|^2 \\ &\leq (2\|\mathfrak{M}\|)^2 \int_0^1 \left\| \sum_{k=1}^N r_k(t) Q_k \left(\sum_{j=1}^n \alpha_{jk} r_j(s) x_j \right) \right\|^2 dt \\ &= (2\|\mathfrak{M}\|)^2 \int_0^1 \left\| \sum_{j=1}^n \sum_{k=1}^N \alpha_{jk} r_k(t) r_j(s) Q_k x_j \right\|^2 dt. \end{aligned}$$

Combining this with (9) we have

$$\int_0^1 \left\| \sum_{j=1}^n r_j(s) P_j x_j \right\|^2 ds \leq (2\|\mathfrak{M}\|)^2 \int_0^1 \int_0^1 \left\| \sum_{j=1}^n \sum_{k=1}^N \alpha_{jk} r_k(t) r_j(s) Q_k x_j \right\|^2 ds dt.$$

Since X has property (α) , we conclude that

$$\int_0^1 \left\| \sum_{j=1}^n r_j(s) P_j x_j \right\|^2 ds \leq (2\|\mathfrak{M}\|)^2 \alpha_X^2 \int_0^1 \int_0^1 \left\| \sum_{j=1}^n \sum_{k=1}^N r_k(t) r_j(s) Q_k x_j \right\|^2 ds dt.$$

Given $s \in [0, 1]$, it follows from (6) that

$$\begin{aligned} \int_0^1 \left\| \sum_{j=1}^n \sum_{k=1}^N r_k(t) r_j(s) Q_k x_j \right\|^2 dt &= \int_0^1 \left\| \sum_{k=1}^N r_k(t) Q_k \left(\sum_{j=1}^n r_j(s) x_j \right) \right\|^2 dt \\ &\leq (2\|\mathfrak{M}\|)^2 \left\| \sum_{k=1}^N Q_k \left(\sum_{j=1}^n r_j(s) x_j \right) \right\|^2 \\ &= (2\|\mathfrak{M}\|)^2 \left\| \sum_{j=1}^n \sum_{k=1}^N r_j(s) Q_k x_j \right\|^2 \\ &= (2\|\mathfrak{M}\|)^2 \left\| \sum_{j=1}^n r_j(s) x_j \right\|^2 \end{aligned}$$

and hence, that

$$\int_0^1 \left\| \sum_{j=1}^n r_j(s) P_j x_j \right\|^2 ds \leq (2\|\mathfrak{M}\|)^4 \alpha_X^2 \int_0^1 \left\| \sum_{j=1}^n r_j(s) x_j \right\|^2 ds.$$

This shows that \mathfrak{M} is R -bounded with R -bound $M_{\mathfrak{M}} \leq (2\|\mathfrak{M}\|)^2 \alpha_X$. \square

Combined with Theorem 3.1 the previous theorem immediately yields the following result.

COROLLARY 3.4. *Let X be a Banach space with property (α) and \mathfrak{M} and \mathfrak{N} be two bounded commuting Boolean algebras of projections in X . Then $\mathfrak{M} \vee \mathfrak{N}$ is bounded and $\|\mathfrak{M} \vee \mathfrak{N}\| \leq 16\|\mathfrak{M}\|^2\|\mathfrak{N}\|^2\alpha_X$.*

Let Y be a Banach space with l.u.st. and having finite cotype. Since Y then has property (α) , so does any closed subspace X of Y . So, Corollary 3.4 applies in X . Consequently, Theorem 2.6 in [13] is a special case of Corollary 3.4 (after noting that for a Banach lattice Y , being p -concave for some $1 \leq p < \infty$ is equivalent to having finite cotype, [10, Theorem 16.17]).

Concerning some relevant examples, note that c_0 and ℓ_∞ (for instance) have l.u.st. but fail to have finite cotype (and hence, also fail to have property (α)). The von Neumann–Schatten ideals \mathfrak{S}_p , for $1 < p < \infty$, are Banach spaces with finite cotype but, for $p \neq 2$, fail to have property (α) (hence, also fail l.u.st.); see Corollary 3.4 above and [13, Remark 2.10]. For every $p > 2$, it is known that there exist closed subspaces of L_p (hence they have property (α)) which fail to have l.u.st.; see page 19 of [24].

The following particular example is also relevant. Let X be a Banach space and (Ω, Σ, μ) be a finite measure space. For each $1 \leq p < \infty$ consider the Banach space $L_p(\mu; X)$ of all X -valued Bochner p -integrable functions on Ω . For $x \in X$ and $g \in L_p(\mu)$, define the function $g \otimes x \in L_p(\mu; X)$ by $(g \otimes x)(\omega) = g(\omega)x$ for $\omega \in \Omega$. Given $A \in \Sigma$, define a projection P_A in $L_p(\mu; X)$ by $f \mapsto \chi_A f$. Then $\mathfrak{M} = \{P_A : A \in \Sigma\}$ is a Bade-complete B.a. of projections in $L_p(\mu; X)$ with $\|\mathfrak{M}\| = 1$. An application of Kahane's Inequality (see for example [10, 11.1]), Fubini's Theorem and the Contraction Principle (see for example [10, 12.2]) shows that \mathfrak{M} is R -bounded. However, since $x \mapsto \mathbf{1} \otimes x$ is an isometric embedding of X into $L_p(\mu; X)$, we see that $L_p(\mu; X)$ cannot have property (α) whenever X fails to have this property.

4. Boolean algebras in GL -spaces

Given Banach spaces X and Y and $1 \leq p < \infty$, we denote by $\Pi_p(X, Y)$ the ideal of all p -absolutely summing operators from X into Y , which is a Banach space with respect to the p -summing norm π_p (see [10, § 2], for the relevant definitions). Recall that a bounded linear operator $T : X \rightarrow Y$ is p -factorable ($1 \leq p \leq \infty$) if there exist a measure space (Ω, Σ, μ) and bounded linear operators $B : X \rightarrow L_p(\mu)$ and $A : L_p(\mu) \rightarrow Y^{**}$ such that $Tx = ABx$ for all $x \in X$. For such an operator T define $\gamma_p(T) = \inf\{\|A\| \|B\|\}$, where the infimum is taken over all possible operators A and B satisfying this condition. The ideal of all p -factorable operators

from X into Y is denoted by $\Gamma_p(X, Y)$; it is a Banach space with respect to the norm γ_p (see [10, §§ 7 and 9] for details).

DEFINITION 4.1. A Banach space X is a *Gordon–Lewis space* (briefly, GL-space) if $\Pi_1(X, \ell_2) \subseteq \Gamma_1(X, \ell_2)$. If X is a GL-space, then there exists a constant $c \geq 0$ such that $\gamma_1(T) \leq c\pi_1(T)$ for all $T \in \Pi_1(X, \ell_2)$; the smallest constant $c \geq 0$ with this property is denoted by $\text{gl}(X)$.

A discussion of GL-spaces can be found in [10, § 17]. In particular, it is shown that every Banach space with l.u.st. is a GL-space (a result due to Y. Gordon and D. R. Lewis, [14]). The converse is false. The Banach spaces Z_p (for $1 < p < \infty$), constructed by N. Kalton and N. T. Peck, [17], admit an unconditional Schauder decomposition (into two-dimensional subspaces), have finite cotype and are GL-spaces but fail to have l.u.st., [16]. For further (non-isomorphic) examples, see also [19].

The proof of the following result uses ideas from the proof of the Theorem on page 365 of [10]. This result (that is Theorem 4.2 below) includes [13, Theorem 2.5] as a special case since Banach lattices have l.u.st. and hence, are GL-spaces.

THEOREM 4.2. *Let \mathfrak{M} and \mathfrak{N} be two bounded commuting Boolean algebras of projections in a GL-space X . Then $\mathfrak{M} \vee \mathfrak{N}$ is also bounded and there exists a (universal) constant $K \geq 0$ such that*

$$\|\mathfrak{M} \vee \mathfrak{N}\| \leq K \text{gl}(X) \|\mathfrak{M}\|^2 \|\mathfrak{N}\|^2.$$

Proof. Every element in $\mathfrak{M} \vee \mathfrak{N}$ is of the form $\sum_{j=1}^m \sum_{k=1}^n \alpha_{jk} P_j Q_k$, for finitely many projections $P_1, \dots, P_m \in \mathfrak{M}$ and $Q_1, \dots, Q_n \in \mathfrak{N}$ with $\sum_{j=1}^m P_j = \sum_{k=1}^n Q_k = I$, scalars $\alpha_{jk} \in \{0, 1\}$ and $n, m \in \mathbb{N}$. We have to show that

$$\left\| \sum_{j=1}^m \sum_{k=1}^n \alpha_{jk} P_j Q_k \right\| \leq K \text{gl}(X) \|\mathfrak{M}\|^2 \|\mathfrak{N}\|^2 \quad (10)$$

for all such choices of P_j , Q_k and α_{jk} , for some universal constant $K \geq 0$. So, fix such a choice. Let $x_0 \in X$ and $x_0^* \in X^*$. Defining $\Lambda = \{1, \dots, m\} \times \{1, \dots, n\}$, we denote elements of $\ell_2(\Lambda)$ by $a = (a_{jk})_{j,k} = (a_{jk})_{j=1,k=1}^{m,n}$ with $a_{jk} \in \mathbb{C}$. Define the linear operator $A : X \rightarrow \ell_2(\Lambda)$ by

$$Ax = (\langle P_j Q_k x, x_0^* \rangle)_{j,k}, \quad \text{for } x \in X.$$

Furthermore, define the linear operator $B : X^* \rightarrow \ell_2(\Lambda)$ by

$$Bx^* = (\langle P_j Q_k x_0, x^* \rangle)_{j,k}, \quad \text{for } x^* \in X^*.$$

We claim that

$$\pi_1(A) \leq C \|\mathfrak{M}\| \|\mathfrak{N}\| \|x_0^*\| \quad (11)$$

for some (universal) constant $C \geq 0$. Indeed, define the linear space S of functions on $[0, 1]^2$ by

$$S = \text{span}\{r_j(s)r_k(t) : 1 \leq j \leq m, 1 \leq k \leq n\}.$$

Furthermore, define $S_\infty = (S, \|\cdot\|_\infty)$ and $S_1 = (S, \|\cdot\|_1)$. So, we may consider $S_\infty \subseteq L_\infty([0, 1]^2)$ and $S_1 \subseteq L_1([0, 1]^2)$. Let $J : S_\infty \rightarrow S_1$ be defined by $Jf = f$ for

all $f \in S_\infty$. It follows from [10, Examples 2.9] that $\pi_1(J) = 1$. Define the operator $W : X \rightarrow S_\infty$ by

$$(Wx)(s, t) = \sum_{j=1}^m \sum_{k=1}^n \langle P_j Q_k x, x_0^* \rangle r_j(s) r_k(t)$$

for all $x \in X$ and $(s, t) \in [0, 1]^2$. Then, for all $(s, t) \in [0, 1]^2$, we have

$$\begin{aligned} |(Wx)(s, t)| &= \left| \sum_{j=1}^m \sum_{k=1}^n \langle P_j Q_k x, x_0^* \rangle r_j(s) r_k(t) \right| \\ &= \left| \left\langle \left(\sum_{j=1}^m r_j(s) P_j \right) \left(\sum_{k=1}^n r_k(t) Q_k \right) x, x_0^* \right\rangle \right| \\ &\leq \left\| \sum_{j=1}^m r_j(s) P_j \right\| \left\| \sum_{k=1}^n r_k(t) Q_k \right\| \|x\| \|x_0^*\| \\ &\leq (2\|\mathfrak{M}\|)(2\|\mathfrak{N}\|) \|x\| \|x_0^*\|, \end{aligned}$$

where the last inequality follows by writing $\sum_{j=1}^m r_j(s) P_j = \sum_j^+ P_j - \sum_j^- P_j$. Here $\sum_j^+ P_j$ and $\sum_j^- P_j$ are the sums taken over all j for which $r_j(s)$ equals 1 or -1 respectively, and similarly for $\sum_{k=1}^n r_k(t) Q_k$. Accordingly,

$$\|Wx\|_\infty \leq 4\|\mathfrak{M}\| \|\mathfrak{N}\| \|x\| \|x_0^*\|.$$

This shows that

$$\|W\| \leq 4\|\mathfrak{M}\| \|\mathfrak{N}\| \|x_0^*\|.$$

Next, define the operator $V : S_1 \rightarrow \ell_2(\Lambda)$ by

$$V \left(\sum_{j=1}^m \sum_{k=1}^n a_{jk} r_j(s) r_k(t) \right) = (a_{jk})_{j,k}.$$

It is a consequence of the Khinchin inequality (see for example [10, 1.10]) that

$$\left(\sum_{j=1}^m \sum_{k=1}^n |a_{jk}|^2 \right)^{1/2} \leq A_1^{-2} \int_0^1 \int_0^1 \left| \sum_{j=1}^m \sum_{k=1}^n a_{jk} r_j(s) r_k(t) \right| ds dt$$

for all scalars $(a_{jk})_{j=1,k=1}^{m,n}$ (where A_1 is a universal constant). Consequently, $\|V\| \leq A_1^{-2}$. Since $A = VJW$, this implies that $\pi_1(A) \leq 4A_1^{-2} \|\mathfrak{M}\| \|\mathfrak{N}\| \|x_0^*\|$, which proves (11) with $C = 4A_1^{-2}$. Via a similar argument we find that

$$\pi_1(B) \leq C \|\mathfrak{M}\| \|\mathfrak{N}\| \|x_0\|. \quad (12)$$

Define the linear operator $M_\alpha : \ell_2(\Lambda) \rightarrow \ell_2(\Lambda)$ by $M_\alpha((a_{jk})_{j,k}) = (\alpha_{jk} a_{jk})_{j,k}$. Clearly, $\|M_\alpha\| \leq 1$. Consider the composition

$$\ell_2(\Lambda) \xrightarrow{A^*} X^* \xrightarrow{B} \ell_2(\Lambda) \xrightarrow{M_\alpha} \ell_2(\Lambda),$$

where A^* is the Banach space dual operator of A . A simple computation shows that

$$A^* a = \sum_{j=1}^m \sum_{k=1}^n a_{jk} P_j^* Q_k^* x_0^*$$

for all $a = (a_{jk})_{j,k} \in \ell_2(\Lambda)$. From the definition of B it follows that

$$BA^*a = \left(\left\langle P_j Q_k x_0, \sum_{r=1}^m \sum_{s=1}^n a_{rs} P_r^* Q_s^* x_0^* \right\rangle \right)_{j,k} = (a_{jk} \langle P_j Q_k x_0, x_0^* \rangle)_{j,k}$$

and hence, that

$$M_\alpha BA^*a = (\alpha_{jk} a_{jk} \langle P_j Q_k x_0, x_0^* \rangle)_{j,k}$$

for all $a = (a_{jk})_{j,k} \in \ell_2(\Lambda)$. This shows that $M_\alpha BA^*$ is a multiplication operator on $\ell_2(\Lambda)$ with *trace* given by

$$\text{tr}(M_\alpha BA^*) = \sum_{j=1}^m \sum_{k=1}^n \alpha_{jk} \langle P_j Q_k x_0, x_0^* \rangle. \quad (13)$$

It follows from [10, Lemma 6.14] that

$$|\text{tr}(M_\alpha BA^*)| \leq \iota_1(M_\alpha BA^*) \leq \|M_\alpha\| \iota_1(BA^*) \leq \iota_1(BA^*),$$

where ι_1 denotes the 1-*integral norm*. From [10, Theorem 5.16(a)], it follows that

$$\iota_1(BA^*) \leq \pi_1(B) \iota_\infty(A^*) = \pi_1(B) \gamma_1(A), \quad (14)$$

where in the last equality we have used the fact that $\iota_\infty = \gamma_\infty$ (trivial) and $\gamma_\infty(A^*) = \gamma_1(A)$ (by [10, Proposition 7.2]). Since, by hypothesis, X is a GL-space we also have $\gamma_1(A) \leq \text{gl}(X) \pi_1(A)$ and so

$$\iota_1(BA^*) \leq \text{gl}(X) \pi_1(A) \pi_1(B).$$

Using (11) and (12) we conclude that

$$\iota_1(BA^*) \leq C^2 \text{gl}(X) \|\mathfrak{M}\|^2 \|\mathfrak{N}\|^2 \|x_0\| \|x_0^*\|.$$

In combination with (13) and (14) this shows that

$$\left| \left\langle \sum_{j=1}^m \sum_{k=1}^n \alpha_{jk} P_j Q_k x_0, x_0^* \right\rangle \right| \leq C^2 \text{gl}(X) \|\mathfrak{M}\|^2 \|\mathfrak{N}\|^2 \|x_0\| \|x_0^*\|.$$

This inequality holds for all $x_0 \in X$ and $x_0^* \in X^*$ and so

$$\left\| \sum_{j=1}^m \sum_{k=1}^n \alpha_{jk} P_j Q_k \right\| \leq C^2 \text{gl}(X) \|\mathfrak{M}\|^2 \|\mathfrak{N}\|^2,$$

for all choices of $\alpha_{jk} \in \{0, 1\}$. Accordingly, (10) holds with $K = C^2$. \square

Combining Theorem 4.2 with the observations made in Proposition 2.5 we obtain the following result.

COROLLARY 4.3. *Let $\{\Delta_m\}_{m=1}^\infty$ and $\{\Delta'_n\}_{n=1}^\infty$ be two commuting unconditional decompositions of a GL-space X . Then the product decomposition $\{\Delta_m \Delta'_n\}_{m,n=1}^\infty$ as given by (7) is also an unconditional decomposition of X .*

From the remarks made at the end of §2 it is now clear that the Theorem on page 365 of [10] is a special case of Corollary 4.3.

As observed in the remarks following Definition 3.2, a Banach space X with l.u.st. has property (α) if and only if X has finite cotype. A close inspection of the

proof of [24, Proposition 2.1] shows that actually the following result holds. In view of this result and Corollary 3.4, the result of Theorem 4.2 is only of interest in GL-spaces with trivial cotype.

THEOREM 4.4. *A GL-space X has property (α) if and only if X has finite cotype.*

By the discussion after Definition 4.1 and Theorem 4.2, it follows that the GL-spaces Z_p ($1 < p < \infty$), due to Kalton and Peck, all have property (α) . There also exist Banach spaces with property (α) which fail to be GL-spaces. Indeed, for the circle group \mathbb{T} it is known that for every $p > 2$ there exists a $\Lambda(2)$ -set $F \subseteq \mathbb{Z}$ which fails to be a $\Lambda(p)$ -set, [7, Theorem 2]. Accordingly, if

$$L_{p,F}(\mathbb{T}) = \{f \in L_p(\mathbb{T}) : \widehat{f}(n) = 0 \text{ for all } n \notin F\},$$

then the trigonometric system $\{e^{int}\}_{n \in F}$ is not an unconditional basis for $L_{p,F}(\mathbb{T})$, [24, p. 14]. Hence, $L_{p,F}(\mathbb{T})$ fails to be a GL-space, [24, Theorem 3.1]. However, since $L_p(\mathbb{T})$ has property (α) , so does its closed subspace $L_{p,F}(\mathbb{T})$.

5. The Boolean algebra of band projections

In this section we investigate conditions under which the B.a. $\mathcal{B}(E)$ of all *band projections* in a Dedekind σ -complete Banach lattice is R -bounded. Actually, we consider a slightly more general situation. For the general theory of Banach lattices we refer to [23, 28, 34]. We consider only real Banach lattices, but all results extend easily to complex Banach lattices.

Given a Banach lattice E , denote by $Z(E)$ the *centre* of E , that is,

$$Z(E) = \{\pi \in \mathcal{L}(E) : \exists \lambda \in [0, \infty) \text{ such that } |\pi x| \leq \lambda |x| \ \forall x \in E\},$$

which is a commutative subalgebra of $\mathcal{L}(E)$. The space $Z(E)$ is itself a vector lattice with respect to the lattice operations given by

$$(\pi_1 \vee \pi_2)u = (\pi_1 u) \vee (\pi_2 u) \quad \text{and} \quad (\pi_1 \wedge \pi_2)u = (\pi_1 u) \wedge (\pi_2 u)$$

for all $u \in E^+$ (the *positive cone* of E) and all $\pi_1, \pi_2 \in Z(E)$. Moreover, $|\pi x| = |\pi|(|x|)$ for all $\pi \in Z(E)$ and $x \in E$. In particular, for $\pi \in Z(E)$ and $\lambda \in [0, \infty)$ the inequality $|\pi| \leq \lambda I$ is equivalent to the requirement that $|\pi x| \leq \lambda |x|$ for all $x \in E$. For any $\pi \in Z(E)$, its operator norm is given by

$$\|\pi\| = \inf\{\lambda \in [0, \infty) : |\pi| \leq \lambda I\}.$$

Consequently, the unit ball of $Z(E)$ is equal to the order interval $[-I, I] = \{\pi \in Z(E) : |\pi| \leq I\}$. The B.a. of band projections in E consists precisely of all idempotent elements in $Z(E)$. If E is an L_p -space ($1 \leq p \leq \infty$), then $Z(L_p) \cong L_\infty$, acting on L_p via multiplication; the band projections correspond to multiplication by characteristic functions. If E is Dedekind σ -complete, then it is a consequence of the Freudenthal spectral theorem that, if $|x| \leq |y|$ in E , then there exists $\pi \in Z(E)$ satisfying $\pi y = x$ and $|\pi| \leq I$. Consequently, $Z(E)$ is non-trivial. However, there exist (infinite-dimensional) Banach lattices such that $Z(E) = \{\lambda I : \lambda \in \mathbb{R}\}$ (see [15] and [32]). To avoid this latter pathology it is convenient to introduce the following class of Banach lattices.

DEFINITION 5.1. The centre $Z(E)$ of a Banach lattice E is called *rich* if, whenever $|x| \leq |y|$ in E there exists a sequence $\{\pi_n\}_{n=1}^\infty$ in $Z(E)$ such that $\|\pi_n y - x\| \rightarrow 0$ as $n \rightarrow \infty$.

Replacing π_n (in the above definition) by $(\pi_n \wedge I) \vee (-I)$ we may assume, without loss of generality, that $|\pi_n| \leq I$ for all n . The general form of the following result will be needed in the next section.

LEMMA 5.2. Let E be a Banach lattice. Suppose there exists a subset $\mathcal{T} \subseteq [-I, I]$ such that:

- (i) \mathcal{T} is R -bounded with R -bound $M_{\mathcal{T}}$;
- (ii) whenever $x, y \in E$ satisfy $|x| \leq |y|$, there exists a sequence $\{\pi_j\}_{j=1}^\infty$ in \mathcal{T} with $\pi_j y \rightarrow x$ as $j \rightarrow \infty$.

Then, for any pairwise disjoint system $\{v_1, v_2, \dots, v_{2^n}\}$ in E , we have

$$\left\| \sum_{k=1}^{2^n} v_k \right\| \geq M_{\mathcal{T}}^{-1} \sqrt{n} \min_{k=1, \dots, 2^n} \|v_k\|.$$

Proof. Define $w = \sum_{k=1}^{2^n} v_k$. Let $\{(\varepsilon_{1j}, \dots, \varepsilon_{nj}) : j = 1, \dots, 2^n\}$ be an enumeration of all possible n -tuples $(\varepsilon_1, \dots, \varepsilon_n)$ with each $\varepsilon_k = \pm 1$. For $k = 1, \dots, n$ we define $x_k = \sum_{j=1}^{2^n} \varepsilon_{kj} v_j$. By disjointness,

$$|x_k| = \sum_{j=1}^{2^n} |\varepsilon_{kj} v_j| = \sum_{j=1}^{2^n} |v_j| = |w| \quad \text{for all } k.$$

Moreover, if $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, then

$$\begin{aligned} \left| \sum_{k=1}^n \alpha_k x_k \right| &= \left| \sum_{k=1}^n \sum_{j=1}^{2^n} \alpha_k \varepsilon_{kj} v_j \right| = \left| \sum_{j=1}^{2^n} \left(\sum_{k=1}^n \alpha_k \varepsilon_{kj} \right) v_j \right| \\ &= \sum_{j=1}^{2^n} \left| \sum_{k=1}^n \alpha_k \varepsilon_{kj} \right| |v_j|. \end{aligned}$$

There exists $j_0 \in \{1, \dots, 2^n\}$ such that $|\sum_{k=1}^n \alpha_k \varepsilon_{kj_0}| = \sum_{k=1}^n |\alpha_k|$ and hence,

$$\left\| \sum_{k=1}^n \alpha_k x_k \right\| \geq \left(\sum_{k=1}^n |\alpha_k| \right) \|v_{j_0}\| \geq \left(\sum_{k=1}^n |\alpha_k| \right) \min_{j=1, \dots, 2^n} \|v_j\|.$$

This implies, in particular, that $\|\sum_{k=1}^n r_k(t) x_k\| \geq n \min_{j=1, \dots, 2^n} \|v_j\|$ for all $t \in [0, 1]$ and hence, that

$$\left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2 dt \right)^{1/2} \geq n \min_{j=1, \dots, 2^n} \|v_j\|. \quad (15)$$

By (ii) above, for each $k = 1, \dots, n$ there is a sequence $\{\pi_{k,m}\}_{m=1}^\infty$ in \mathcal{T} such that

$\pi_{k,m}w \rightarrow x_k$ as $m \rightarrow \infty$. The R -boundedness of \mathcal{T} implies that

$$\begin{aligned} \int_0^1 \left\| \sum_{k=1}^n r_k(t) \pi_{k,m} w \right\|^2 dt &\leq M_{\mathcal{T}}^2 \int_0^1 \left\| \sum_{k=1}^n r_k(t) w \right\|^2 dt \\ &= M_{\mathcal{T}}^2 \left(\int_0^1 \left| \sum_{k=1}^n r_k(t) \right|^2 dt \right) \|w\|^2 \\ &= M_{\mathcal{T}}^2 n \|w\|^2 \end{aligned}$$

for all $m = 1, 2, \dots$. Since

$$\left\| \sum_{k=1}^n r_k(t) \pi_{k,m} w \right\|^2 \rightarrow \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2$$

as $m \rightarrow \infty$ (for all $t \in [0, 1]$), it follows from Fatou's Lemma that

$$\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2 dt \leq M_{\mathcal{T}}^2 n \|w\|^2.$$

Combined with (15) this shows that

$$M_{\mathcal{T}} \sqrt{n} \|w\| \geq n \min_{j=1, \dots, 2^n} \|v_j\|$$

and so $\|w\| \geq M_{\mathcal{T}}^{-1} \sqrt{n} \min_{j=1, \dots, 2^n} \|v_j\|$. □

REMARK 5.3. For each $n \in \mathbb{N}$, denote by \mathcal{T}_n the order interval $[-I, I]$ in $Z(\ell_{\infty}^n)$. In Lemma 5.2, choosing $\{v_1, \dots, v_{2^n}\}$ to be the standard basis vectors of $\ell_{\infty}^{2^n}$ we find that $M_{\mathcal{T}_n} \geq \sqrt{n}$. Consequently, $M_{\mathcal{T}_n} \rightarrow \infty$ as $n \rightarrow \infty$. Since $[-I, I]$ ($\subseteq Z(\ell_{\infty}^n)$) is the absolute convex hull of the B.a. $\mathcal{B}(\ell_{\infty}^n)$ of all band projections in ℓ_{∞}^n , it follows that $M_{\mathcal{B}(\ell_{\infty}^n)} = M_{\mathcal{T}_n}$. This implies, in particular, that the Boolean algebras $\mathcal{B}(\ell_{\infty})$ and $\mathcal{B}(c_0)$ are not R -bounded. In the case of c_0 , we note that this B.a. is even Bade-complete and has a cyclic vector.

COROLLARY 5.4. Let E be a Banach lattice, $n \in \mathbb{N}$ and $u : \ell_{\infty}^{2^n} \rightarrow E$ be a lattice isomorphism (into) with inverse $u^{-1} : u(\ell_{\infty}^{2^n}) \rightarrow \ell_{\infty}^{2^n}$. Under the same assumptions as in Lemma 5.2 we have $\|u\| \|u^{-1}\| \geq M_{\mathcal{T}}^{-1} \sqrt{n}$.

Proof. Denote by $\{e_1, \dots, e_{2^n}\}$ the standard basis in $\ell_{\infty}^{2^n}$ and put $v_j = u(e_j)$ for $j = 1, \dots, 2^n$. Since

$$1 = \|e_j\|_{\infty} = \|u^{-1}(v_j)\|_{\infty} \leq \|u^{-1}\| \|v_j\|$$

for all j , it follows that $1 \leq \|u^{-1}\| \min_{j=1, \dots, 2^n} \|v_j\|$. Hence,

$$\|u^{-1}\| \geq \left(\min_{j=1, \dots, 2^n} \|v_j\| \right)^{-1}. \quad (16)$$

On the other hand, it follows from Lemma 5.2 that

$$\begin{aligned} M_T^{-1} \sqrt{n} \min_{k=1, \dots, 2^n} \|v_k\| &\leq \left\| \sum_{j=1}^{2^n} v_j \right\| = \left\| \sum_{j=1}^{2^n} u(e_j) \right\| = \left\| u \left(\sum_{j=1}^{2^n} e_j \right) \right\| \\ &\leq \|u\| \left\| \sum_{j=1}^{2^n} e_j \right\|_{\infty} = \|u\|. \end{aligned} \quad (17)$$

A combination of (16) and (17) yields $\|u\| \|u^{-1}\| \geq M_T^{-1} \sqrt{n}$. \square

Before formulating the next result recall that a Banach lattice E is a *KB-space* if every monotone, norm bounded sequence in E is convergent (see for example [23, Definition 2.4.11]). Every KB-space has *order-continuous norm* (as follows from [23, Theorem 2.4.2]) and so, in particular, is Dedekind complete. A Banach lattice E is a KB-space if and only if it does not contain a lattice copy of c_0 (see [23, Theorem 2.4.12] or [34, Theorem 117.4]).

COROLLARY 5.5. *Let E be a Banach lattice. Under the same assumptions as Lemma 5.2 it follows that E cannot contain the ℓ_{∞}^n uniformly (equivalently, E has finite cotype). Moreover, E is a KB-space.*

Proof. Recall that a Banach lattice E is said to contain the ℓ_{∞}^n uniformly as a sublattice if there exists a constant $C \geq 1$ such that for every $n \in \mathbb{N} \setminus \{0\}$ there exists a linear lattice isomorphism $u_n : \ell_{\infty}^n \rightarrow E$ with inverse $u_n^{-1} : u_n(\ell_{\infty}^n) \rightarrow \ell_{\infty}^n$ and $\|u_n\| \|u_n^{-1}\| \leq C$. From Corollary 5.4 it is clear, under the present assumptions, that E cannot contain the ℓ_{∞}^n uniformly as a sublattice. However, this also implies that E cannot contain the ℓ_{∞}^n uniformly (this result is implicit in [10, Proposition 16.16 and Scholium 16.17]; it is explicitly stated in [28, Theorem 8.13]). Hence, E has finite cotype (see [10, Theorem 14.1]). Consequently, E does not contain a (lattice) copy of c_0 and hence, is a KB-space. \square

Next we consider the converse of the above corollary.

LEMMA 5.6. *Let E be a Banach lattice having finite cotype. Then the order interval $[-I, I]$ in $Z(E)$ is R -bounded.*

Proof. Since E has finite cotype, there is a constant $K \geq 1$ such that

$$K^{-1} \left\| \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} \right\| \leq \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \leq K \left\| \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} \right\| \quad (18)$$

for all $x_1, \dots, x_n \in E$ and all $n \in \mathbb{N} \setminus \{0\}$ (see for example [10, Theorem 16.18]). Here the element $(\sum_{k=1}^n |x_k|^2)^{1/2}$ is defined via the *Krivine functional calculus* for Banach lattices (see for example [10, Chapter 16] or [20, §1.d]).

Let $x_1, \dots, x_n \in E$ and $\pi_1, \dots, \pi_n \in Z(E)$ be given with $|\pi_k| \leq I$ for all k . Since $|\pi_k x_k| \leq |x_k|$ for all k , it follows from properties of the Krivine calculus that

$$\left(\sum_{k=1}^n |\pi_k x_k|^2 \right)^{1/2} \leq \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2}.$$

Using (18) we see that

$$\begin{aligned} \int_0^1 \left\| \sum_{k=1}^n r_k(t) \pi_k x_k \right\| dt &\leq K \left\| \left(\sum_{k=1}^n |\pi_k x_k|^2 \right)^{1/2} \right\| \leq K \left\| \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} \right\| \\ &\leq K^2 \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt. \end{aligned}$$

By Kahane's inequality, this shows that $[-I, I]$ is R -bounded. \square

REMARK 5.7. An alternative proof of Lemma 5.6 is as follows. If E is a Banach lattice having finite cotype, then E has property (α) (see the discussion following Definition 3.2). Hence, by Theorem 3.3, the B.a. $\mathcal{B}(E)$ of all band projections in E is R -bounded. As noted in the last line of the proof of Corollary 5.5, having finite cotype implies that E is a KB-space (and hence, E is Dedekind complete). By the same argument as given in the first part of the proof of Theorem 5.8 below it follows that $[-I, I]$ is R -bounded.

The above facts yield the following result.

THEOREM 5.8. *For a Banach lattice E the following statements are equivalent:*

- (i) *E is Dedekind σ -complete and the B.a. $\mathcal{B}(E)$ of all band projections is R -bounded;*
- (ii) *E has rich centre and the order interval $[-I, I]$ in $Z(E)$ is R -bounded;*
- (iii) *E has finite cotype;*
- (iv) *E has property (α) .*

Proof. If E is Dedekind σ -complete, then $Z(E)$ is also Dedekind σ -complete and $\mathcal{B}(E)$ is the B.a. of all components of I in $Z(E)$. By the Freudenthal spectral theorem, the order interval $[-I, I]$ is the norm closure of the absolute convex hull of $\mathcal{B}(E)$, and so the R -boundedness of $\mathcal{B}(E)$ implies that $[-I, I]$ is R -bounded. Hence, (i) implies (ii). Corollary 5.5 shows that (ii) implies (iii). Finally, if E has finite cotype then, by Lemma 5.6, $[-I, I]$ is R -bounded. In particular, $\mathcal{B}(E)$ is then R -bounded. Moreover, as observed in Lemma 5.6, E is a KB-space which implies that E is Dedekind complete. \square

REMARK 5.9. (a) As shown in [33, Theorem 2.2.14], if X is a Banach space with non-trivial type and $\mathcal{T} \subseteq \mathcal{L}(X)$ is R -bounded, then $\mathcal{T}^* = \{T^* : T \in \mathcal{T}\}$ is R -bounded in $\mathcal{L}(X^*)$. At that time, it was unclear if the condition of X having non-trivial type was necessary for this conclusion. However, for $X = \ell_1$ it follows from Theorem 5.8 that $\mathcal{T} = \mathcal{B}(\ell_1)$ is R -bounded but $\mathcal{T}^* = \mathcal{B}(\ell_\infty)$ is not R -bounded (see Remark 5.3). Accordingly, the condition of X having non-trivial type cannot be omitted (in general) in the theorem cited from [33].

(b) For a Dedekind σ -complete Banach lattice E , combining Theorem 3.3 and Theorem 5.8, we conclude that every B.a. of projections in E is R -bounded precisely when the particular B.a. $\mathcal{B}(E)$ is R -bounded!

Next we discuss some further characterizations (related to R -boundedness) of Banach lattices having finite cotype. For this purpose we recall some definitions and introduce some notation. Let X be a Banach space and $\{x_n\}_{n=1}^\infty$ be a sequence in X . The series $\sum_{n=1}^\infty x_n$ is called *almost unconditionally convergent* (or the sequence $\{x_n\}_{n=1}^\infty$ *almost unconditionally summable*) if the series $\sum_{n=1}^\infty r_n(t)x_n$ is norm convergent in X for almost all $t \in [0, 1]$. For a sequence $\{x_n\}_{n=1}^\infty$ in X the following two statements are equivalent (see for example [10, Theorem 12.3]):

- (i) the sequence $\{x_n\}_{n=1}^\infty$ is almost unconditionally summable;
- (ii) the series $\sum_{n=1}^\infty r_n(\cdot)x_n$ is norm convergent in the Bochner space $L_p([0, 1], X)$ for some (all) $1 \leq p < \infty$.

Given a Banach space X , denote by $\text{rad}(X)$ the collection of all almost unconditionally summable sequences $\{x_n\}_{n=1}^\infty$ (with $x_n \in X$ for all n), which is a vector space with respect to the coordinatewise operations. Furthermore, define

$$\text{Rad}(X) = \left\{ \sum_{n=1}^\infty r_n(\cdot)x_n : \{x_n\}_{n=1}^\infty \in \text{rad}(X) \right\},$$

which is a linear space of (equivalence classes of) *strongly measurable* X -valued functions on $[0, 1]$. Actually, $\text{Rad}(X)$ is a closed linear subspace of $L_p([0, 1], X)$, for each $1 \leq p < \infty$. By Kahane's inequality (see for example [10, 11.1]), the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent on $\text{Rad}(X)$ for all $1 \leq p, q < \infty$. We will consider $\text{Rad}(X)$ equipped with the norm $\|\cdot\|_2$. The mapping $\{x_n\}_{n=1}^\infty \mapsto \sum_{n=1}^\infty r_n(\cdot)x_n$ is a bijection from $\text{rad}(X)$ onto $\text{Rad}(X)$. For $\{x_n\}_{n=1}^\infty \in \text{rad}(X)$, define

$$\|\{x_n\}\|_{\text{rad}(X)} = \left\| \sum_{n=1}^\infty r_n(\cdot)x_n \right\|_2.$$

Equipped with the norm $\|\cdot\|_{\text{rad}(X)}$, the space $\text{rad}(X)$ is a Banach space.

Denote by $\ell_\infty(X)$ the Banach space of all bounded sequences $\{x_n\}_{n=1}^\infty$ in X equipped with the norm $\|\{x_n\}\|_\infty = \sup_n \|x_n\|$. Let $c_{00}(X)$ be the subspace of $\ell_\infty(X)$ consisting of all sequences which are eventually zero. Then $c_{00}(X) \subseteq \text{rad}(X) \subseteq \ell_\infty(X)$ and $c_{00}(X)$ is a dense subspace of $\text{rad}(X)$ relative to $\|\cdot\|_{\text{rad}(X)}$.

Let E be a Banach lattice. With respect to the order defined coordinatewise, $\ell_\infty(E)$ is a Banach lattice in which $c_{00}(E)$ is an order ideal. We consider conditions on E which imply that $\text{rad}(E)$ is an order ideal in $\ell_\infty(E)$.

LEMMA 5.10. *Let E be a Dedekind σ -complete Banach lattice in which the B.a. $\mathcal{B}(E)$ of all band projections is R -bounded. Then $\text{rad}(E)$ is an order ideal in $\ell_\infty(E)$ and there exists a lattice norm on $\text{rad}(E)$ which is equivalent to $\|\cdot\|_{\text{rad}(E)}$.*

Proof. By Freudenthal's spectral theorem, the order interval $[-I, I]$ is the closure of the absolute convex hull of $\mathcal{B}(E)$ in $Z(E)$. Hence, $[-I, I]$ is R -bounded. Take $x = \{x_n\}_{n=1}^\infty$ in $\text{rad}(E)$ and $y = \{y_n\}_{n=1}^\infty$ in $\ell_\infty(E)$ such that $|y| \leq |x|$. For each n there exists $\pi_n \in [-I, I]$ such that $y_n = \pi_n x_n$. Consequently,

$$\begin{aligned} \left\| \sum_{j=m}^n r_j(\cdot)y_j \right\|_{L_2([0,1],E)} &= \left\| \sum_{j=m}^n r_j(\cdot)\pi_j x_j \right\|_{L_2([0,1],E)} \\ &\leq M_{[-I,I]} \left\| \sum_{j=m}^n r_j(\cdot)x_j \right\|_{L_2([0,1],E)} \end{aligned}$$

for all $m < n$. This implies that the series $\sum_{j=1}^{\infty} r_j(\cdot)y_j$ is norm convergent in $L_2([0,1], E)$ and

$$\left\| \sum_{j=1}^{\infty} r_j(\cdot)y_j \right\|_{L_2([0,1], E)} \leq M_{[-I, I]} \left\| \sum_{j=1}^{\infty} r_j(\cdot)x_j \right\|_{L_2([0,1], E)}.$$

Hence, $y \in \text{rad}(E)$ and

$$\|y\|_{\text{rad}(E)} \leq M_{[-I, I]} \|x\|_{\text{rad}(E)}. \quad (19)$$

This shows, in particular, that $\text{rad}(E)$ is an order ideal in $\ell_{\infty}(E)$.

For $x \in \text{rad}(E)$ define

$$\|x\| = \sup \{ \|y\|_{\text{rad}(E)} : y \in \text{rad}(E), |y| \leq |x| \},$$

in which case $\|x\| \geq \|x\|_{\text{rad}(E)}$. It follows from (19) that $\|x\| \leq M_{[-I, I]} \|x\|_{\text{rad}(E)}$. It is easily verified that $\|\cdot\|$ is a lattice norm on $\text{rad}(E)$. \square

LEMMA 5.11. *Let E be a Banach lattice and suppose there exists a lattice norm $\|\cdot\|$ on $c_{00}(E)$ equivalent to $\|\cdot\|_{\text{rad}(E)}$. Then E is a KB-space (in particular, E is Dedekind complete) and the B.a. $\mathcal{B}(E)$ is R -bounded.*

Proof. Let $P_1, \dots, P_n \in \mathcal{B}(E)$ and $x_1, \dots, x_n \in E$. Put $y_j = P_j x_j$ for $j = 1, \dots, n$. Define the elements $x, y \in c_{00}(E)$ by $x = (x_1, \dots, x_n, 0, 0, \dots)$ and $y = (y_1, \dots, y_n, 0, 0, \dots)$. Since $|y_j| \leq |x_j|$ for all j , we have $|y| \leq |x|$ and so, $\|y\| \leq \|x\|$. By hypothesis, there exists a constant $C \geq 1$ such that

$$C^{-1} \|z\|_{\text{rad}(E)} \leq \|z\| \leq C \|z\|_{\text{rad}(E)}$$

for all $z \in c_{00}(E)$. Hence,

$$\|y\|_{\text{rad}(E)} \leq C \|y\| \leq C \|x\| \leq C^2 \|x\|_{\text{rad}(E)}.$$

By the definition of $\|\cdot\|_{\text{rad}(E)}$, this implies that

$$\left\| \sum_{j=1}^n r_j(\cdot) P_j x_j \right\|_{L_2([0,1], E)} \leq C^2 \left\| \sum_{j=1}^n r_j(\cdot) x_j \right\|_{L_2([0,1], E)}.$$

So, we conclude that $\mathcal{B}(E)$ is R -bounded.

To show that E is a KB-space suppose, on the contrary, that E contains a vector sublattice F which is norm and lattice isomorphic with c_0 . Then $c_{00}(F)$ is a vector sublattice of $c_{00}(E)$ and the restriction of $\|\cdot\|$ to $c_{00}(F)$ is equivalent to $\|\cdot\|_{\text{rad}(F)}$. Consequently, from the first part of the proof applied in the Banach lattice F , it follows that the B.a. $\mathcal{B}(F)$ is R -bounded. Since F is norm and lattice isomorphic with c_0 , the B.a. $\mathcal{B}(c_0)$ is also R -bounded. By Remark 5.3 this is a contradiction. Therefore, E does not contain a lattice copy of c_0 and hence E is a KB-space. \square

LEMMA 5.12. *Let E be a Banach lattice such that $\text{rad}(E)$ is an order ideal in $\ell_{\infty}(E)$. Then there exists a lattice norm on $c_{00}(E)$ equivalent to $\|\cdot\|_{\text{rad}(E)}$.*

Proof. We require some notation. For each $k \in \mathbb{N}$ define the linear operator $T_k : \ell_\infty(E) \rightarrow \ell_\infty(E)$ of *right translation* by k -steps via

$$(T_k x)_l = \begin{cases} 0 & \text{for } 1 \leq l \leq k, \\ x_{l-k} & \text{for } l > k, \end{cases}$$

for all $x = (x_1, x_2, \dots)$ in $\ell_\infty(E)$. Moreover, for $F \subseteq \mathbb{N} \setminus \{0\}$ and $x \in \ell_\infty(E)$ define $\chi_F x \in \ell_\infty(E)$ by $(\chi_F x)_l = x_l$ if $l \in F$ and $(\chi_F x)_l = 0$ otherwise. From the definition of $\|\cdot\|_{\text{rad}(E)}$ and the Contraction Principle (see for example [10, 12.2]) it follows that $T_k x \in \text{rad}(E)$ and $\chi_F x \in \text{rad}(E)$ whenever $x \in \text{rad}(E)$ and that $\|T_k x\|_{\text{rad}(E)} = \|x\|_{\text{rad}(E)}$ and $\|\chi_F x\|_{\text{rad}(E)} \leq \|x\|_{\text{rad}(E)}$.

We claim that there exists a constant $C > 0$ satisfying

$$\sup \{ \|y\|_{\text{rad}(E)} : y \in c_{00}(E), |y| \leq |x| \} \leq C \|x\|_{\text{rad}(E)} \quad (20)$$

for all $x \in c_{00}(E)$. Indeed, suppose that (20) fails to hold for every $C > 0$. Then, for every $n \in \mathbb{N} \setminus \{0\}$ there exist $x^{(n)} \in c_{00}(E)$ and $y^{(n)} \in c_{00}(E)$ with $|y^{(n)}| \leq |x^{(n)}|$ such that $\|y^{(n)}\|_{\text{rad}(E)} > n^3 \|x^{(n)}\|_{\text{rad}(E)}$. Without loss of generality we may assume that $\|x^{(n)}\|_{\text{rad}(E)} = 1$ for all n . Write $x^{(n)} = (x_1^{(n)}, \dots, x_{k_n}^{(n)}, 0, 0, \dots)$ and let $K_n = k_1 + \dots + k_n$ for $n \geq 1$ with $K_0 = 0$. Since $\|T_{K_n} x^{(n)}\|_{\text{rad}(E)} = \|x^{(n)}\|_{\text{rad}(E)} = 1$ for all $n \geq 1$, the series $w = \sum_{n=1}^{\infty} n^{-2} T_{K_n} x^{(n)}$ is absolutely convergent in $\text{rad}(E)$. The sequence $T_{K_n} x^{(n)}$ is supported in $(K_{n-1}, K_n]$ and so, $\chi_{(K_{n-1}, K_n]} w = n^{-2} T_{K_n} x^{(n)}$ for all $n \geq 1$. Define $z \in \ell_\infty(E)$ by $z = \sum_{n=1}^{\infty} n^{-2} T_{K_n} y^{(n)}$, which is a pointwise convergent series on $\mathbb{N} \setminus \{0\}$ with disjointly supported terms. Since $|y^{(n)}| \leq |x^{(n)}|$ for all $n \geq 1$, it follows that $|z| \leq |w|$. By hypothesis, this implies that $z \in \text{rad}(E)$ and hence,

$$n \leq \frac{1}{n^2} \|y^{(n)}\|_{\text{rad}(E)} = \left\| \frac{1}{n^2} T_{K_n} y^{(n)} \right\|_{\text{rad}(E)} = \|\chi_{(K_{n-1}, K_n]} z\|_{\text{rad}(E)} \leq \|z\|_{\text{rad}(E)}$$

for all $n \geq 1$. This is a contradiction and, accordingly, there exists a constant $C > 0$ such that (20) holds for all $x \in c_{00}(E)$.

Defining

$$\|x\| = \sup \{ \|y\|_{\text{rad}(E)} : y \in c_{00}(E), |y| \leq |x| \}$$

for all $x \in c_{00}(E)$, we see from (20) that $\|\cdot\|$ is a lattice norm on $\text{rad}(E)$ equivalent to $\|\cdot\|_{\text{rad}(E)}$. \square

Collecting together the above facts we obtain the following result, which complements Theorem 5.8.

THEOREM 5.13. *For a Banach lattice E the following statements are equivalent:*

- (i) E is Dedekind σ -complete and the B.a. $\mathcal{B}(E)$ of all band projections is R -bounded;
- (ii) $\text{rad}(E)$ is an order ideal in $\ell_\infty(E)$ and there exists a lattice norm on $\text{rad}(E)$ equivalent to $\|\cdot\|_{\text{rad}(E)}$;
- (iii) $\text{rad}(E)$ is an order ideal in $\ell_\infty(E)$ (equivalently, whenever a sequence $\{x_n\}_{n=1}^\infty$ is almost unconditionally summable in E and $y_n \in E$ satisfy $|y_n| \leq |x_n|$ for all $n \geq 1$, then $\{y_n\}_{n=1}^\infty$ is almost unconditionally summable);
- (iv) there exists a lattice norm on $c_{00}(E)$ equivalent to $\|\cdot\|_{\text{rad}(E)}$.

Note that all of the above statements are equivalent to E having finite cotype (see Theorem 5.8) and imply that E is a KB-space (see Lemma 5.11).

REMARK 5.14. The condition of the Banach lattice E being a KB-space is by itself not sufficient to imply that the B.a. $\mathcal{B}(E)$ is R -bounded. Indeed, take $1 < p < \infty$ and let E be the ℓ_p -direct sum of the Banach lattices $E_n = \ell_\infty^n$ ($n = 1, 2, \dots$). By [1, Theorem 12.6], the dual space E^* is canonically isometric with the ℓ_q -direct sum (where $p^{-1} + q^{-1} = 1$) of the Banach lattices ℓ_1^n ($n = 1, 2, \dots$). Applying this theorem again we find that E is reflexive. Hence, E is a KB-space, [23, Theorem 2.4.15]. It is clear that E contains the ℓ_∞^n uniformly, so E does not have finite cotype and hence, $\mathcal{B}(E)$ is not R -bounded (see Theorem 5.8). However, since E has order-continuous norm, it is evident that $\mathcal{B}(E)$ is Bade-complete.

6. R -boundedness and bade-completeness

The main result of the present section is: in any Banach space the strong operator closure of any R -bounded B.a. of projections is always Bade-complete. For simplicity we consider real Banach spaces, but the main results carry over immediately to complex spaces. For the convenience of the reader, we start this section by recalling a fundamental construction which equips the cyclic subspaces of a bounded B.a. of projections with a canonical Banach lattice structure. This will enable us to apply the results of the previous section.

Let \mathfrak{M} be a bounded B.a. of projections in a Banach space X . Denote by $\Omega = \Omega_{\mathfrak{M}}$ the (compact, Hausdorff and totally disconnected) Stone space of \mathfrak{M} and let the spectral measure $P : \mathfrak{F}_\Omega \rightarrow \mathcal{L}(X)$ be given by the Boolean isomorphism of the algebra \mathfrak{F}_Ω of all closed-open subsets of Ω onto \mathfrak{M} . Denoting by $\text{sim}(\mathfrak{F}_\Omega)$ the space of all *simple functions* based on the algebra \mathfrak{F}_Ω , it turns out that the corresponding integration map $J : \text{sim}(\mathfrak{F}_\Omega) \rightarrow \mathcal{L}(X)$, given by $J(s) = \int_\Omega s dP$ for all $s \in \text{sim}(\mathfrak{F}_\Omega)$, is an algebra homomorphism which satisfies

$$\|s\|_\infty \leq \|J(s)\| \leq 2\|\mathfrak{M}\| \|s\|_\infty. \quad (21)$$

Since $\text{sim}(\mathfrak{F}_\Omega)$ is dense in $C(\Omega)$, the map J extends to an algebra homomorphism $J : C(\Omega) \rightarrow \mathcal{L}(X)$, still satisfying (21). Consequently, J is a Banach algebra isomorphism from $C(\Omega)$ onto $\overline{\langle \mathfrak{M} \rangle}_u$, the uniformly closed subalgebra of $\mathcal{L}(X)$ generated by \mathfrak{M} . Given $x \in X$ the evaluation map $J_x : C(\Omega) \rightarrow X$ is defined by $J_x(f) = J(f)x$ for all $f \in C(\Omega)$. Note that

$$J_x(fg) = J(f)J_x(g), \quad \text{for } f, g \in C(\Omega). \quad (22)$$

Denoting by $\mathfrak{M}[x]$ the *cyclic subspace* corresponding to x , that is, the norm closure in X of the subspace $\{J(s)x : s \in \text{sim}(\mathfrak{F}_\Omega)\}$, it is clear that J_x takes its values in $\mathfrak{M}[x]$. The following (well-known) observation plays a key role. For convenience we indicate the proof.

LEMMA 6.1. *Let $|g| \leq |f|$ in $C(\Omega)$. Then $\|J_x g\| \leq 2\|\mathfrak{M}\| \|J_x f\|$ for all $x \in X$.*

Proof. Fix $x \in X$. Choose sequences $\{t_n\}_{n=1}^\infty$ and $\{s_n\}_{n=1}^\infty$ in $\text{sim}(\mathfrak{F}_\Omega)$ such that $t_n \rightarrow g$ and $s_n \rightarrow f$ in $C(\Omega)$ as $n \rightarrow \infty$. Replacing t_n with $(t_n \wedge |s_n|) \vee (-|s_n|)$, we may assume that $|t_n| \leq |s_n|$ for all n . There then exists $r_n \in \text{sim}(\mathfrak{F}_\Omega)$ such that

$t_n = r_n s_n$ and $|r_n| \leq 1$ for each n . Hence, $J_x(t_n) = J(r_n)J_x(s_n)$ and so, by (21), $\|J_x(t_n)\| \leq 2\|\mathfrak{M}\| \|J_x(s_n)\|$. By letting $n \rightarrow \infty$, the result follows. \square

Let $x \in X$. For $f \in C(\Omega)$ define

$$p_x(f) = \sup\{\|J_x g\| : g \in C(\Omega), |g| \leq |f|\}.$$

Then $p_x : C(\Omega) \rightarrow [0, \infty)$ is a lattice semi-norm and, by Lemma 6.1, we have

$$\|J_x f\| \leq p_x(f) \leq 2\|\mathfrak{M}\| \|J_x f\|, \quad \text{for } f \in C(\Omega). \quad (23)$$

Clearly, (21) implies that $p_x(f) \leq 2\|\mathfrak{M}\| \|x\| \|f\|_\infty$ for all $f \in C(\Omega)$. Defining the subspace $N_x = \{f \in C(\Omega) : J_x f = 0\}$, it follows from (23) that

$$N_x = \{f \in C(\Omega) : p_x(f) = 0\}.$$

This fact and (22) imply that N_x is a closed order ideal in $C(\Omega)$. Consider now the quotient space $C(\Omega)/N_x$, equipped with the quotient vector lattice structure. The equivalence class in $C(\Omega)/N_x$ corresponding to a function $f \in C(\Omega)$ is denoted by \bar{f} . For $\bar{f} \in C(\Omega)/N_x$ we define $\|\bar{f}\|_1 = p_x(f)$. Then $\|\cdot\|_1$ is a lattice norm on $C(\Omega)/N_x$. From (23) it follows that there exists a linear mapping $\bar{J}_x : C(\Omega)/N_x \rightarrow X$ satisfying $\bar{J}_x \bar{f} = J_x f$ and

$$\|\bar{J}_x \bar{f}\| \leq \|\bar{f}\|_1 \leq 2\|\mathfrak{M}\| \|\bar{J}_x \bar{f}\|, \quad \text{for } \bar{f} \in C(\Omega)/N_x. \quad (24)$$

Let $L_1(Px)$ denote the completion of the vector lattice $C(\Omega)/N_x$ with respect to $\|\cdot\|_1$, which is then a Banach lattice. Then (24) implies that \bar{J}_x extends uniquely to a norm isomorphism (into) $\bar{J}_x : L_1(Px) \rightarrow X$ satisfying also (24). Consequently, \bar{J}_x is a norm isomorphism from $L_1(Px)$ onto the cyclic subspace $\mathfrak{M}[x]$. We now use \bar{J}_x to transfer the Banach lattice structure from $L_1(Px)$ to $\mathfrak{M}[x]$. The corresponding norm on $\mathfrak{M}[x]$ will be denoted by $\|\cdot\|_{\mathfrak{M}[x]}$, so that $\|\bar{J}_x \bar{f}\|_{\mathfrak{M}[x]} = \|\bar{f}\|_1$ for all $\bar{f} \in L_1(Px)$. Note that (24) is now equivalent to the estimate

$$\|y\| \leq \|y\|_{\mathfrak{M}[x]} \leq 2\|\mathfrak{M}\| \|y\|, \quad \text{for } y \in \mathfrak{M}[x].$$

Hence, the Banach lattice norm $\|\cdot\|_{\mathfrak{M}[x]}$ is equivalent on $\mathfrak{M}[x]$ with the original given norm on X . Moreover, the element $x \geq 0$ is a weak order unit in the Banach lattice $\mathfrak{M}[x]$ and the map $J_x : C(\Omega) \rightarrow \mathfrak{M}[x]$ is a Riesz homomorphism. Denote by $\mathfrak{M}\langle x \rangle$ the subspace $J_x(\text{sim}(\mathfrak{F}_\Omega))$ of $\mathfrak{M}[x]$, that is,

$$\mathfrak{M}\langle x \rangle = \left\{ \sum_{j=1}^n \alpha_j P_j x : \alpha_j \in \mathbb{R}, P_j \in \mathfrak{M}, j = 1, \dots, n; n \in \mathbb{N} \right\}.$$

From the construction it is clear that $\mathfrak{M}\langle x \rangle$ is a dense vector sublattice of $\mathfrak{M}[x]$.

Let $\overline{\mathfrak{M}}_s$ be the strong operator closure of \mathfrak{M} in $\mathcal{L}(X)$, which is also a bounded B.a. of projections. Then $\overline{\mathfrak{M}}_s[x] = \mathfrak{M}[x]$ for all $x \in X$. A moment's reflection shows that the lattice structures and the norms $\|\cdot\|_{\overline{\mathfrak{M}}_s[x]}$ and $\|\cdot\|_{\mathfrak{M}[x]}$ are identical (one only needs to verify this on $\mathfrak{M}\langle x \rangle$). In general, $\overline{\mathfrak{M}}_s$ need not be Bade-complete (take for \mathfrak{M} the B.a. of band projections in $X = \ell_\infty$ which is strongly closed but is not Bade-complete).

LEMMA 6.2. *Let \mathfrak{M} be a bounded B.a. of projections in a Banach space X and $x \in X$. Suppose that $\{y_n\}_{n=1}^\infty$ is a disjoint sequence in $\mathfrak{M}\langle x \rangle$ satisfying*

$0 \leq y_n \leq x$ for all n . Then there exists a disjoint sequence $\{P_n\}_{n=1}^\infty$ in \mathfrak{M} such that $0 \leq y_n \leq P_n x$ for all n .

Proof. We start with the following observation. Given $y \in \mathfrak{M}\langle x \rangle$ satisfying $0 < y \leq x$, there exist $Q \in \mathfrak{M}$ and $\beta \in [0, \infty)$ such that $0 < y \leq Qx \leq \beta y$. Indeed, there exists $s \in \text{sim}(\mathfrak{F}_\Omega)$ such that $J_x(s) = y$. Since J_x is a lattice homomorphism, we may assume that $s = \sum_{j=1}^n \alpha_j \chi_{F_j}$, where $0 < \alpha_j \leq 1$ for all $j = 1, \dots, n$ and F_1, \dots, F_n are mutually disjoint in \mathfrak{F}_Ω . Defining $F = \bigcup_{j=1}^n F_j$, $Q = P(F)$ and $\beta = (\min_j \alpha_j)^{-1}$, it is clear that $0 \leq s \leq \chi_F \leq \beta s$ in $\text{sim}(\mathfrak{F}_\Omega)$. Hence, $0 \leq J_x(s) \leq J_x(\chi_F) \leq \beta J_x(s)$, that is, $0 < y \leq Qx \leq \beta y$.

Let $\{y_n\}_{n=1}^\infty$ be as in the statement of the lemma. For each n there exist $G_n \in \mathfrak{F}_\Omega$ and $\beta_n \in [0, \infty)$ such that $Q_n = P(G_n)$ satisfies $0 < y_n \leq Q_n x \leq \beta_n y_n$. This implies that $\{Q_n x\}_{n=1}^\infty$ is a disjoint sequence in $\mathfrak{M}\langle x \rangle$. Since J_x is a lattice homomorphism, it follows that

$$Q_m Q_n x = J_x(\chi_{G_m} \wedge \chi_{G_n}) = J_x(\chi_{G_m}) \wedge J_x(\chi_{G_n}) = (Q_m x) \wedge (Q_n x) = 0$$

for all $m \neq n$. The sequence $\{P_n\}_{n=1}^\infty$, defined inductively by $P_1 = Q_1$ and $P_n = Q_n(I - \sum_{k=1}^{n-1} P_k)$ for $n \geq 2$, has the desired properties. \square

PROPOSITION 6.3. *Let \mathfrak{M} be a bounded B.a. of projections in a Banach space X . The following statements are equivalent:*

- (i) every disjoint sequence $\{P_n\}_{n=1}^\infty \subseteq \mathfrak{M}$ converges strongly to zero in $\mathcal{L}(X)$;
- (ii) for every $x \in X$, the Banach lattice $\mathfrak{M}[x]$ has order-continuous norm;
- (iii) $\overline{\mathfrak{M}}_s$ is Bade-complete.

Proof. First assume that $P_n \rightarrow 0$ strongly in $\mathcal{L}(X)$ as $n \rightarrow \infty$ for every disjoint sequence $\{P_n\}_{n=1}^\infty$ in \mathfrak{M} . Since x is a strong order unit in $\mathfrak{M}\langle x \rangle$, it follows from Lemma 6.2 that every order-bounded, disjoint sequence in $\mathfrak{M}\langle x \rangle$ converges to zero. Hence, by a theorem of P. Meyer-Nieberg, every order-bounded increasing sequence in $\mathfrak{M}\langle x \rangle$ is Cauchy (see [23, Corollary 2.3.6; 34, Theorem 104.2]; in [34], this latter property is referred to as the ρ -Cauchy condition of the norm, [34, Lemma 103.1]; in [21] the ρ -Cauchy condition is called *property (A, iii)*). In [21, Theorem 64.1], it is shown that property (A, iii) carries over from a normed vector lattice to its completion. Hence, in the present situation, we conclude that $\mathfrak{M}[x]$ satisfies the ρ -Cauchy condition. However, in a Banach lattice the ρ -Cauchy condition is equivalent to order continuity of the norm, [34, Corollary 103.8]. Consequently, (i) implies (ii).

Now assume that $\mathfrak{M}[x]$ has order-continuous norm for every $x \in X$. By [11, Lemma XVII.3.4], to prove that $\overline{\mathfrak{M}}_s$ is Bade-complete it suffices to show that every monotone increasing net of elements from $\overline{\mathfrak{M}}_s$ converges strongly to an element of $\overline{\mathfrak{M}}_s$. So, let $\{P_\tau\}$ be an upwards directed system in $\overline{\mathfrak{M}}_s$ and fix $x \in X$. Then $0 \leq P_\tau x \uparrow \leq x$ in $\overline{\mathfrak{M}}_s[x] = \mathfrak{M}[x]$ and, since the norm in $\mathfrak{M}[x]$ is order continuous, it follows that there exists $y \in \mathfrak{M}[x]$ such that $P_\tau x \uparrow y$ and $\|P_\tau x - y\| \rightarrow 0$ [23, Theorem 2.4.2]. This implies that $P_\tau \rightarrow Q$ strongly for some $Q \in \mathcal{L}(X)$. Clearly, $Q \in \overline{\mathfrak{M}}_s$. So, $\overline{\mathfrak{M}}_s$ is Bade-complete. Accordingly, (ii) implies (iii).

Finally, if $\overline{\mathfrak{M}}_s$ is Bade-complete, then it follows from [11, Lemma XVII.3.4], applied to the increasing sequence $\{\sum_{k=1}^n P_k\}_{n=1}^\infty$, that \mathfrak{M} satisfies (i). \square

There are several sufficient conditions which guarantee that the norm in $\mathfrak{M}[x]$ is order continuous for all $x \in X$. We mention in particular the following.

(1) None of the cyclic subspaces $\mathfrak{M}[x]$ contains a copy of c_0 (which is, in particular, the case if X itself does not contain a copy of c_0). Indeed, in this case each of the cyclic subspaces is a KB-space and hence, all have order-continuous norm ([23, Theorem 2.4.12]; note that a Banach lattice does not contain a Banach space copy of c_0 if and only if it does not contain a lattice copy of c_0 , [1, Theorem 14.12]). This observation yields, in particular, the result of [12, Theorem 1].

(2) For each $x \in X$ the set

$$\left\{ \sum_{j=1}^n \alpha_j P_j x : |\alpha_j| \leq 1, \sum_{j=1}^n P_j = I, P_j \in \mathfrak{M}, j = 1, \dots, n; n \in \mathbb{N} \right\}$$

is *relatively weakly compact*. Indeed, this implies that order intervals in $\mathfrak{M}[x]$ are relatively weakly compact which in turn is equivalent to order continuity of the norm in $\mathfrak{M}[x]$, [23, Theorem 2.4.2].

We now discuss another sufficient condition on \mathfrak{M} for $\overline{\mathfrak{M}}_s$ to be Bade-complete. In fact, we show that $\overline{\mathfrak{M}}_s$ is Bade-complete for any R -bounded B.a. \mathfrak{M} of projections in X . We start with the following preliminary observations. Given any bounded B.a. \mathfrak{M} , it is clear that every operator $T \in \overline{\langle \mathfrak{M} \rangle}_u$ leaves each cyclic subspace $\mathfrak{M}[x]$, for $x \in X$, invariant. For $x \in X$ we denote the restriction of $T \in \overline{\langle \mathfrak{M} \rangle}_u$ to $\mathfrak{M}[x]$ by T_x ; so $T_x \in \mathcal{L}(\mathfrak{M}[x])$.

LEMMA 6.4. *Let \mathfrak{M} be any bounded B.a. of projections in a Banach space X . Then*

$$\{T_x : T \in \overline{\langle \mathfrak{M} \rangle}_u\} \subseteq Z(\mathfrak{M}[x]), \quad \text{for } x \in X.$$

Proof. Given $x \in X$ and $T \in \overline{\langle \mathfrak{M} \rangle}_u$, we have to show that there exists $\lambda \in [0, \infty)$ such that $|Ty| \leq \lambda|y|$ for all $y \in \mathfrak{M}[x]$. By density considerations, it is sufficient to obtain this estimate for all elements y of the form $y = J_x s$ for some $s \in \text{sim}(\mathfrak{F}_\Omega)$. Let $f \in C(\Omega)$ satisfy $T = Jf$. Since J_x is a lattice homomorphism from $C(\Omega)$ into $\mathfrak{M}[x]$, we have

$$\begin{aligned} |Ty| &= |(Jf)(J_x s)| = |J_x(fs)| = J_x|fs| \leq \|f\|_\infty J_x|s| \\ &= \|f\|_\infty |J_x s| = \|f\|_\infty |y|. \end{aligned}$$

So, $\lambda = \|f\|_\infty$ has the required property. \square

LEMMA 6.5. *With \mathfrak{M} as in Lemma 6.4, suppose that $x \in X$ and $y, z \in \mathfrak{M}[x]$ satisfy $|y| \leq |z|$. Then there exists a sequence $\{f_n\}_{n=1}^\infty$ in $C(\Omega)$ with $\|f_n\|_\infty \leq 1$ for all n , such that $(Jf_n)_x z \rightarrow y$ in $\mathfrak{M}[x]$ as $n \rightarrow \infty$.*

Proof. Choose sequences $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ in $\text{sim}(\mathfrak{F}_\Omega)$ with $|s_n| \leq |t_n|$ for all n , such that $J_x s_n \rightarrow y$ and $J_x t_n \rightarrow z$ in $\mathfrak{M}[x]$ as $n \rightarrow \infty$ (cf. the proof of Lemma 6.1). We can then write $s_n = f_n t_n$ for appropriate $f_n \in \text{sim}(\mathfrak{F}_\Omega)$ satisfying $|f_n| \leq 1$

for all n . Using $(Jf_n)(J_xt_n) = J_xs_n$ we have

$$\begin{aligned} \|(Jf_n)_x z - y\| &\leq \|(Jf_n)z - (Jf_n)(J_xt_n)\| + \|(Jf_n)(J_xt_n) - y\| \\ &\leq \|Jf_n\| \|z - J_xt_n\| + \|J_xs_n - y\| \\ &\leq 2\|\mathfrak{M}\| \|z - J_xt_n\| + \|J_xs_n - y\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad \square$$

We now come to the main result of this section.

THEOREM 6.6. *Let \mathfrak{M} be an R -bounded B.a. of projections in a Banach space X . Then its strong closure $\overline{\mathfrak{M}}_s$ is Bade-complete.*

Proof. Define $\mathcal{T} \subseteq \overline{(\mathfrak{M})}_u$ by $\mathcal{T} = \{Jf : f \in C(\Omega), \|f\|_\infty \leq 1\}$. Since \mathcal{T} is the closed absolutely convex hull of \mathfrak{M} and \mathfrak{M} is R -bounded, it follows that \mathcal{T} is also R -bounded. For any $x \in X$ let $\mathcal{T}_x \subseteq Z(\mathfrak{M}[x])$ be defined by $\mathcal{T}_x = \{T_x : T \in \mathcal{T}\}$. It is clear that \mathcal{T}_x is R -bounded. From a combination of Corollary 5.5 and Lemma 6.5 it follows that the Banach lattice $\mathfrak{M}[x]$ has finite cotype. In particular, $\mathfrak{M}[x]$ is a KB-space and so, has order-continuous norm. By Proposition 6.3 we conclude that $\overline{\mathfrak{M}}_s$ is Bade-complete. \square

REMARK 6.7. (a) As shown in the proof of the above theorem, if \mathfrak{M} is an R -bounded B.a. of projections in a Banach space X , then all cyclic subspaces have finite cotype. This does not necessarily imply that X itself has finite cotype. Indeed, let $X = L_2([0, 1], c_0)$ and \mathfrak{M} be the B.a. of all multiplication operators by characteristic functions χ_F with $F \subseteq [0, 1]$ measurable. Then \mathfrak{M} is strongly closed and R -bounded (see the remarks at the end of §3). Hence, all cyclic subspaces of X have finite cotype. However, X itself does not have finite cotype since it contains a copy of c_0 . Note that this does not contradict the results of Theorem 5.8 or Corollary 5.5, since \mathfrak{M} is not equal to the Boolean algebra of all band projections in the Banach lattice X .

The following example is also relevant. Consider the B.a. of all *row projections* in the Schatten p -class $X = \mathfrak{S}_p$, for any $1 < p < \infty$. Then \mathfrak{M} is Bade-complete (hence, bounded) but fails to be R -bounded (if $p \neq 2$); see for example [33, §5.3]. Nevertheless, as shown in [30], the space X has finite cotype and so each cyclic subspace $\mathfrak{M}[x]$, for $x \in X$, has finite cotype (and is even independent of x).

(b) According to Theorem 6.6 above and [12, Theorem 2], in every Banach space containing a copy of c_0 , there exists a strongly closed, bounded B.a. of projections which fails to be R -bounded. If X is any separable Banach space containing a copy of c_0 , then there even exists a Bade-complete B.a. of projections in X which fails to be R -bounded. Indeed, c_0 is then complemented in X , say $X = Y \oplus c_0$, [29]. Then

$$\mathfrak{M} = \{P \oplus Q : Q \in \mathcal{B}(c_0), P \in \{0_Y, I_Y\}\}$$

is a Bade-complete B.a. of projections in X . Since $\mathcal{B}(c_0)$ is not R -bounded in $\mathcal{L}(c_0)$ (cf. Remark 5.3), \mathfrak{M} also fails to be R -bounded in $\mathcal{L}(X)$.

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