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ON A DECOUPLED ALGORITHM FOR POROELASTICITY AND ITS RESOLUTION BY MULTIGRID

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Abstract. *In this paper we present a reformulation of the poroelasticity system and a decoupled algorithm for the numerical solution. This reformulation enables us to construct a highly efficient multigrid method, confirmed by a realistic experiment.*

1 INTRODUCTION

The quasi–static Biot model for soil consolidation [1] can be formulated as a system of partial differential equations for the displacements of the medium and the pressure of the fluid. One assumes the material's solid structure to be linearly elastic, initially homogeneous and isotropic, the strains imposed within the material are small. We denote by $\overline{\mathbf{u}} = (u, v, p)^T$ the solution vector, consisting of the displacement vector $\mathbf{u} = (u, v)^T$ and pore pressure of the fluid p. The governing equations read

$$-\mu\tilde{\Delta}\mathbf{u} - (\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \alpha\operatorname{grad} p = \mathbf{0}, \qquad \mathbf{x}\in\Omega, \qquad (1)$$

$$\alpha\frac{\partial}{\partial t}(\operatorname{div}\mathbf{u}) - \frac{\kappa}{\eta}\Delta p = f(\mathbf{x},t), \qquad 0 < t \le T\,, \qquad (2)$$

where λ and μ are the Lamé coefficients; κ is the permeability of the porous medium, η the viscosity of the fluid, α is the Biot-Willis constant (which we will take equal one) and $\tilde{\Delta}$ represents the vectorial Laplace operator. The quantity $\operatorname{div}\mathbf{u}\,(\mathbf{x},t)$ is the dilatation, i.e. the volume increase rate of the system, a measure of the change in porosity of the soil. The source term $f(\mathbf{x},t)$ represents a forced fluid extraction or injection process, respectively, see [1].

For simplicity, we assume here that $\partial\Omega$ is rigid (zero displacements) and permeable (free drainage), so that we have homogeneous Dirichlet boundary conditions,

$$\mathbf{u}(\mathbf{x},t)=0, \quad p(\mathbf{x},t)=0, \quad \mathbf{x}\in\partial\Omega. \tag{3}$$

Before fluid starts to flow and due to the incompressibility of the solid and fluid phases, the initial state satisfies

$$\operatorname{div}\mathbf{u}(\mathbf{x},0)=0, \quad \mathbf{x}\in\Omega. \tag{4}$$

Physically, when a load is applied in a poroelastic medium, the pressure suddenly increases and a boundary layer appears in the early stages of the time-dependent process. In the case of an unstable discretization, unphysical oscillations can occur in the first time steps of the solution. After this phase, the solution shows a much smoother behaviour. In this paper we present a reformulated version of the system of poroelasticity equations. We show that problem (1)-(4) can be brought in a form which is favourable for (almost) decoupled iterative solution. Besides, working with the reformulated system, stable numerical solutions are obtained on a standard collocated grid [2]. A numerical 2D experiment confirms the stability, accuracy and the efficient multigrid treatment of the resulting transformed system.

The paper is organised as follows, the transformation of the poroelastic system plus a corresponding algorithm are presented in Section 2. In Section 3 a numerical poroelastic experiment, using a multigrid method, is presented indicating the efficiency of the solution algorithm.

2 TRANSFORMED PROBLEM

Let us rewrite problem (1)-(4) as

$$A\frac{\partial\mathbf{u}}{\partial t}+\operatorname{grad}\frac{\partial p}{\partial t}=\mathbf{0}, \quad \text{in}\,\Omega, \tag{5}$$

$$\operatorname{div}\frac{\partial\mathbf{u}}{\partial t}-\frac{\kappa}{\eta}\Delta p=f, \quad \text{in}\,\Omega, \tag{6}$$

$$\mathbf{u}=\mathbf{0},\ p=0, \quad \text{on}\,\partial\Omega, \tag{7}$$

$$\operatorname{div}\mathbf{u}(\mathbf{x},0)=0, \quad \mathbf{x}\in\Omega, \tag{8}$$

where $A=-\mu\tilde{\Delta}-(\lambda+\mu)\operatorname{grad}\operatorname{div}$.

We transform problem (5-8) to an equivalent problem. Firstly, applying the divergence operator to (5) and the operator $(\lambda+2\mu)\Delta$ to (6), adding the resulting equations and taking into account the equality

$$-(\lambda+2\mu)\,\Delta\operatorname{div}=\operatorname{div}A,$$

we obtain

$$-\Delta \frac{\partial p}{\partial t} + (\lambda + 2\mu)\frac{\kappa}{\eta}\Delta^2 p = -(\lambda + 2\mu)\Delta f. \tag{9}$$

Secondly, by applying operator $(\lambda+\mu)\,\mathrm{grad}$ to (6) and by adding the resulting equation to (5) we get

$$-\mu\tilde{\Delta}\frac{\partial \mathbf{u}}{\partial t} + \mathrm{grad}\,\frac{\partial p}{\partial t} - (\lambda + \mu)\,\frac{\kappa}{\eta}\,\mathrm{grad}\,\Delta p = (\lambda + \mu)\,\mathrm{grad}\,f. \tag{10}$$

With the new variables $q = -\Delta p$ and $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$, we deal with the transformed system

$$-\mu\tilde{\Delta}\mathbf{v} + \mathrm{grad}\frac{\partial p}{\partial t} + (\lambda + \mu)\,\frac{\kappa}{\eta}\,\mathrm{grad}\,q = (\lambda + \mu)\,\mathrm{grad}\,f, \tag{11}$$

$$q + \Delta p = 0, \tag{12}$$

$$\frac{\partial q}{\partial t} - (\lambda + 2\mu)\frac{\kappa}{\eta}\Delta q = -(\lambda + 2\mu)\Delta f, \tag{13}$$

$$\mathbf{v} = \mathbf{0},\ p = 0,\ \mathrm{div}\,\mathbf{v} + \frac{\kappa}{\eta}q = f, \quad \text{on}\,\partial\Omega. \tag{14}$$

plus initial conditions. Therefore, if $(\mathbf{u}, p)$ is a solution of problem (5-8) then $(\mathbf{v}, p, q)$ is solution of problem (11-14).

We prove that both problems really are equivalent. If $(\mathbf{v}, p, q)$ is solution of problem (11-14) then $(\mathbf{u}, p)$ is solution of problem (5-8).

By applying the divergence operator to (11) and the use of equality $\mathrm{div}\tilde{\Delta} = \Delta\mathrm{div}$, we find

$$-\mu\Delta\mathrm{div}\mathbf{v} - \frac{\partial q}{\partial t} + (\lambda + \mu)\,\frac{\kappa}{\eta}\,\Delta\,q = (\lambda + \mu)\,\Delta\,f. \tag{15}$$

Adding (15) and (13), we obtain

$$\mu\Delta\left(\mathrm{div}\mathbf{v} + \frac{\kappa}{\eta}q - f\right) = 0, \quad \text{in}\,\Omega. \tag{16}$$

By using boundary conditions (14) we deduce equation (6). Equation (5) is obtained by applying the operator $(\lambda + \mu)\mathrm{grad}$ to (6) and using (11).

Note that problem (11-14) is coupled over the boundary of the domain. Let us consider a semi-discretization in time with step time $\tau = T/M$ with M a positive integer. For $1 \le m \le M - 1$ and assuming that $\mathbf{v}^m$, p^m and q^m are known, the following iterative scheme is proposed.

1. Solve:

$$\begin{cases} \left(\frac{q^{m+1}-q^m}{\tau}\right) - (\lambda + 2\mu)\frac{\kappa}{\eta}\Delta q^{m+1} = -(\lambda + 2\mu)\Delta f^{m+1}, & \text{in}\,\Omega, \\ \mathrm{div}\,\mathbf{v}^m + \frac{\kappa}{\eta}q^{m+1} = f^{m+1} & \text{on}\,\partial\Omega. \end{cases}$$

2. Solve

$$\begin{cases} -\Delta p^{m+1} = q^{m+1} & \text{in}\, \Omega, \\ p^{m+1} = 0, & \text{on}\, \partial\Omega. \end{cases}$$

3. Solve

$$\begin{cases} -\mu \tilde{\Delta} \mathbf{v}^{m+1} = \text{grad}\left((\lambda+\mu)\, f^{m+1} - \frac{p^{m+1}-p^m}{\tau} - (\lambda+\mu)\, \frac{\kappa}{\eta}\, q^{m+1} \right), & \text{in}\, \Omega, \\ \mathbf{v}^{m+1} = \mathbf{0}, \quad \text{on}\, \partial\Omega. \end{cases}$$

Notice that the boundary condition in step (1) is lagging behind one time step. Without additional iteration the scheme presented here will therefore be of $\mathcal{O}(\tau)$.

First of all, when working with the reformulated system stable numerical solutions are obtained on a standard *collocated* grid. Secondly, the operators to be inverted in the algorithm above are only scalar Laplace type operators, for which standard multigrid for scalar equations works extremely well. In particular, multigrid method with highest efficiency, based on a red-black point-wise Gauss-Seidel smoother, GS-RB, and well-known choices for the remaining multigrid components [4] can be used for all choices of λ, μ, and κ. These include the direct coarse grid discretization of the PDE, full weighting and bilinear interpolation, as the restriction and prolongation operators, respectively.

3 Numerical Experiment

The example considered here is a true 2d footing problem (see also [3]). The simulation domain is a 100 by 100 meters block of porous soil, as in Figure 1.

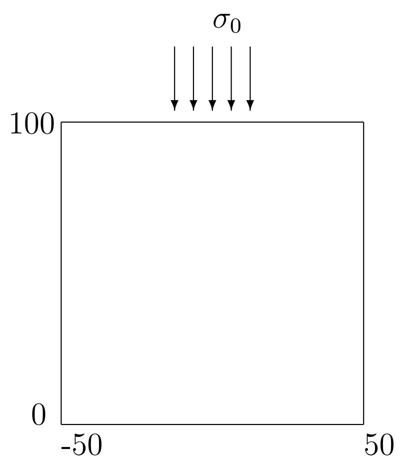

Figure 1: Computational domain for the footing problem

At the base of this domain the soil is assumed to be fixed while at some upper part of the domain a uniform load of intensity σ_0 is applied in a strip of length $40m$. The whole domain is assumed free to drain. Therefore, the boundary data is given as follows:

$$\begin{aligned} p = 0, &\quad \text{on} \quad \partial\Omega, \\ \sigma_{xy} = 0, \sigma_{yy} = -\sigma_0, &\quad \text{on} \quad \Gamma_1 = \{(x,y) \in \partial\Omega, /\ |x| \leq 20, y = 100\}, \\ \sigma_{xy} = 0,\ \sigma_{yy} = 0, &\quad \text{on} \quad \Gamma_2 = \{(x,y) \in \partial\Omega, /\ |x| > 20, y = 100\}, \\ \mathbf{u} = \mathbf{0}, &\quad \text{on} \quad \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2). \end{aligned} \tag{17}$$

The material properties of the porous medium are given in Table 1 where λ and μ are related to the Young's modulus E and the Poisson's ratio ν by the following expressions

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

The uniform load is taken as $\sigma_0 = 10^4 N/m^2$.

Property	Value	Unit
Young's modulus	3×10^4	N/m^2
Poisson's ratio	0.2	-
Permeability	10^{-7}	m^2
Fluid viscosity	10^{-3}	Pas

Table 1: Material parameters for the poroelastic problem.

In Figure 2 the solution of the pressure is presented. The unphysical oscillations for small t that were present in the numerical results in [3], do not occur here with this new formulation.

Finally, the multigrid convergence factor for the decoupled system is found to be 0.06 for the equations for p and q, while for the other two equations, with the stress boundary condition, it is found to be 0.12. The corresponding CPU times on a Pentium IV 2.6 Ghz are $1''$ on a 128^2-grid and $4''$ per time step on a 256^2-grid.

4 Conclusion

In this paper we provide a fast and accurate numerical solver for the incompressible variant of the poroelasticity equations. The system is transformed so that a stable discretization can be obtained on a collocated grid. A multigrid iteration has been defined based on the decoupled version of the poroelasticity system after the transformation. It is sufficient to choose a highly efficient multigrid method for a scalar Poisson type equation for the overall solution of the problem. With standard geometric transfer operators, a direct coarse grid discretization and a point-wise red-black Gauss-Seidel smoother, an efficient multigrid method is developed for all relevant choices of the problem parameters.

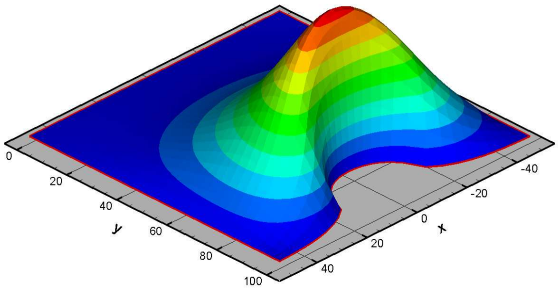

Figure 2: Numerical solution for displacements and pressure for 2D poroelasticity reference problem, 32^2-grid.

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