Economic Engineering & Statistical Physics

Linking Microeconomics and Macroeconomics using Statistical Physics

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Abstract

Economic engineering models individual agents as inertia elements and can be viewed as a microeconomic theory based on analogs with classical mechanics. In this thesis the economic engineering concept of modeling individual agents is used to model economic systems consisting of many agents, e.g. an entire country. This is done using classical statistical physics. In statistical physics the microscopic movement of individual gas particles as described by classical mechanics and the macroscopic properties of gases as described by thermodynamics are linked. Using this insight, microeconomics and macroeconomics are linked within the economic engineering framework.

The critical contribution of this thesis is finding the analog of Gibbs' interpretation of entropy, calling it the amount of diversification. This is done as follows. A thermodynamic system in equilibrium is seen as the analog of a Pareto optimal economy or macroeconomic equilibrium. The 2^{nd} law of thermodynamics guarantees the existence of thermodynamic equilibrium and it follows that the entropy is maximized in equilibrium. Clausius interpreted the entropy as an "arrow of time" that pushes the system towards equilibrium. In this thesis Adam Smith's invisible hand is then viewed as an economic analog of Clausius' entropy. Gibbs gives a statistical interpretation to entropy. By calling the amount of diversification the analog of the Gibbs entropy, it follows that a Pareto optimal economy is fully diversified. The amount of diversification contains both the distribution of economic rent over agents and the portfolio diversification of agents for different goods.

Based on the diversification analog, the thesis develops several further analogs. The analog of the partition function is called the opportunity function and gives the opportunities for extracting profits from an economic system by trading. From this the economic engineering analog of the free energy follows. The temperature and chemical potential are given economic engineering analogs as well, namely the level of welfare and the disposable income per capita respectively. The thesis is finalized with applications of the theory developed.

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Acknowledgements

Graduating has without a doubt been the most challenging thing I have ever had to do in my (academic) life. For the first time in my life, I had to not only solve a problem, but find a problem worth solving. A problem of which you do not know whether or not you will ever find the solution for, or whether that solution even exists. This process challenges everything you know and everything you think you know. I can without a doubt say that the lessons learned and skills obtained during the graduation process will stay with me forever and will be incredibly useful in my career and life.

I would like to thank my supervisor Dr.ir. Max Mendel for his continued support during this difficult process. I would also like to thank him for allowing me to do research within the field of economic engineering and for allowing me to go beyond the standard curriculum of systems and control engineering. I would also like to thank Ir. Coen Hutters for his support in guiding me through the graduation process and for his critical questions and feedback.
Furthermore I would like to thank my fellow economic engineering students for their feedback during our weekly meetings, the great ideas I got from their research and the (emotional) support they provided when needed. A special thank you to Quinten De Wit, with whom I had weekly meetings in which we presented the latest developments in our research to one another. This helped me tremendously during the most difficult times of my thesis and gave me the motivation to keep moving forward. I am grateful that I was able to go through the graduation experience with such a friend.

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“One often meets his destiny on the road he takes to avoid it.”
— *Master Oogway* (Kung Fu Panda) *&* *Shannon Sharpe* (Undisputed)

Quote slightly adapted from the original quote by Jean de La Fontaine (1621-1695). I choose the quote by fictional character Master Oogway & NFL Hall of Famer Shannon Sharpe because they introduced me to it.
The quote is applicable to me and this thesis, because of my history with statistical physics. After completing my BSc. in applied physics, the statistical physics course was one of the main reasons for not wanting to do a masters in applied physics and choosing for systems & control instead. It is only fitting then, that my thesis subject in economic engineering is about statistical physics and relating it to economic engineering.

Chapter 1

Introduction

1-1 Economic engineering

Economic engineering [6] models individual agents as inertia elements, where each agent is modeled as a separate inertia element. Economic engineering can thus be viewed as a microeconomic theory. Current economic engineering theory models economic systems based on first principles. Analogs between microeconomics and classical mechanics are used to model economic systems in the same way an engineer would model mechanical systems. Once the economic system is modeled, the well-known methods within engineering to predict the behaviour of mechanical systems are applied to the economic system and interpreted economically. Scenario analysis and control can then be performed on the economic system, where both the inputs and disturbances, as well as the outputs of the system are economically interpretable.

For simple economic systems consisting of only a few agents, this method of modeling is very feasible. However, it becomes increasingly difficult to model economic systems consisting of many agents using this approach. Since economic engineering is based on analogs with Newtonian mechanics, Newton's equation of motion needs to be solved for every agent within the system to find the progression of their reservation price(s). For economic systems with many agents this becomes prohibitively difficult.

In this thesis economic systems consisting of many agents and many distinct goods are considered. Instead of modeling a small number of agents, an entire economy is modeled. To do this statistical physics is used.

1-2 Statistical Physics

Statistical physics provides a framework for relating the microscopic properties of individual gas particles to the macroscopic properties of gases. Classical mechanics explains the microscopic movement of gas particles, while thermodynamics explains the macroscopic properties of gases. Initially these branches of physics were not connected and no method existed for deriving thermodynamic relations from Newtonian mechanics. Statistical physics solved this problem and became the bridge between physics on microscopic and physics on macroscopic scale. Statistical physics thus explains thermodynamics as a natural result of statistics, classical mechanics, and quantum mechanics at the microscopic level.In this thesis, quantum mechanics is ignored and only classical statistical physics is used.

In statistical physics thermodynamic state variables like entropy and temperature are given a statistical or microscopic interpretation. By showing how these macroscopic variables are given a microscopic interpretation, physics on the micro and macro scale are connected.

As economic engineering is based on classical mechanics, statistical physics is used in this thesis to connect microeconomics and macroeconomics within the economic engineering framework. This thesis thus develops a macroeconomic theory of economic engineering. Throughout this thesis, analogs are found for the quantities derived in statistical physics and are given economic interpretation. Following these analogies results are interpreted economically as well and conclusions are made.

1-3 Microeconomics & Macroeconomics

Microeconomics is a branch of economics that studies the behavior of individuals and firms in making decisions regarding the allocation of scarce resources and the interactions among these individuals and firms [7]. Microeconomics focuses on the study of individual markets, sectors, or industries. Macroeconomics on the other hand is a branch of economics dealing with performance, structure, behavior, and decision-making of an economy as a whole [8]. While microeconomics focuses on firms and individuals, macroeconomics focuses on the sum total of economic activity, dealing with the issues of growth, inflation, and unemployment and with national policies relating to these issues.

Currently, microeconomics and macroeconomics are two separate branches of economics and no clear connection exists between the two. This disconnect is comparable to the disconnect that existed in physics between classical mechanics and thermodynamics before the invention of statistical physics. Using statistical physics, these branches of economics are connected within the economic engineering framework in this thesis.

1-4 Thesis Outline

This thesis is structured as follows.

Chapter 2 provides the assumptions, definitions and analogs used throughout this thesis.
Two approaches can be taken when reading this thesis. The reader can either read this chapter to become familiar with the terminology, assumptions and analogies used in this thesis or choose to skip this chapter initially and look these up when needed.

Chapter 3 contains literature on economics and economic engineering, as well as thermoeconomics.
In Section 3-1 literature on economic engineering is given. It is also shown how this relates to microeconomic literature. Readers familiar with economic engineering can skip this section.
In Section 3-2 basic literature on macroeconomics is provided. Readers familiar with the basics of macroeconomics can skip this section.
In Section 3-3 previous attempts to link thermodynamics and economics are shown. The content of this section is not needed to understand the rest of the thesis.

Chapter 4 contains literature on the kinetic theory of gases and statistical physics. Readers familiar with these subjects can skip this chapter.

Chapter 5 is the main contribution of this thesis and uses the framework of statistical physics to derive a macroeconomic theory of economic engineering.

Section 6-1 connects the results obtained in Chapter 5 with thermodynamics and macroeconomics. The link between microeconomics and macroeconomics is thus completed in this chapter.
Furthermore, applications of the theory developed in this thesis are given in Sections 6-2 and 6-3.

Finally, in Chapter 7 a summary of the thesis is given, the contributions of this thesis are highlighted in the conclusions section and recommendations for future research are given.

Chapter 2

Assumptions, Definitions & Analogies

The definitions used and main assumptions made in this thesis are given in this chapter, as well as the analogies found between economics and (statistical) physics and the economic engineering interpretation of mathematical functions and operations.

The goal of this chapter is to provide the reader with the necessary background to easily understand the contents of this thesis.

> Two approaches can be taken when reading this thesis. The reader can either read this chapter to become familiar with the terminology, assumptions and analogies used in this thesis or choose to skip this chapter initially and look these up when needed.

The analogies between economics and (statistical) physics found in this thesis, as well as the analogies found in earlier economic engineering research [6] are given in Section 2-1.

Section 2-2 gives the definitions used in this thesis. These are either used in current economic engineering theory [6], current economics literature, current physics literature or newly defined based on analogs from economics and physics.

Section 2-3 contains the main assumptions made to link microeconomics and macroeconomics using analogies with statistical physics.

Finally, the (economic engineering) interpretation of mathematical functions and operations is given in Section 2-4.

2-1 Analogies Between Economics & Physics

In this section the analogies used in this thesis from current economic engineering literature as well the ones found in this thesis between economics and statistical physics are given.

Using these analogies, this thesis should be understandable to readers who are familiar with either (statistical) physics or economic engineering literature.

Analogies for Section 3-1.

Economics	Physics	Symbol
Inventory level	Position	q
Quantity demanded (supplied)	Velocity	$\dot{q}$
Period Costs	Action	$\mathcal{S}$
Running Costs (Disutility)	Lagrangian	L
Variable Costs	Kinetic Co-Energy	T^*
Economic Surplus	Kinetic Energy	T
Potential Surplus	Potential Energy	Φ
Allocated Economic Rent	Hamiltonian	H
Reservation price	Momentum	p
Price elasticity of demand (supply)	Inverse inertia	$\mathcal{E}$
Economic desire (want)	Force	F

Table 2-1: Analogs found between economics and statistical physics relevant for Section 3-1, namely for quantities demanded, inventory, different types of costs and surplus, price, desire and the price elasticity.

Analogies for Subsections 5-2-1 and 5-2-2.

Economics	Physics	Symbol
Number of distinct goods	Number of dimensions	D
Number of agents	Number of particles	N
Reservation price	Momentum	p
Quantity demanded	Velocity	$\dot{q}$
Stock (Inventory) level	Position	q
Price elasticity of demand	Inverse inertia	$\mathcal{E}$
Price-Quantity Space	Phase Space	Γ

Table 2-2: Analogs found between economics and statistical physics relevant for Subsections 5-2-1 and 5-2-2, namely for agents, goods, prices, quantities demanded, inventory, price elasticity and the price-quantity space.

Analogies for Subsection 5-2-3.

Economics	**Physics**	**Symbol**
Economic Surplus	Kinetic Energy	T
Potential Surplus	Potential Energy	Φ
Allocated Economic Rent	Hamiltonian	H
Running Costs (Disutility)	Lagrangian	L
GDP	Internal Energy	U
Economic Rent Dissipation	Energy Dissipation	D
Indifference to goods	Isotropic Gas	
Level of Welfare	Temperature	Θ

Table 2-3: Analogs found between economics and statistical physics relevant for Subsection 5-2-3, namely for surplus, economic rent, utility, GDP and the level of welfare.

Analogies for Section 5-3.

Economics	**Physics**	**Symbol**
Level of Welfare	Temperature	Θ
Level of Poverty	Inverse Temperature	β
Disposable Income per Capita	Chemical Potential	υ,μ
Macroeconomic Equilibrium	Thermodynamic Equilibrium	
Amount of Diversification	Entropy	S
1^{st} fundamental theorem of welfare economics	Existence of thermodynamic equilibrium	
2^{nd} fundamental theorem of welfare economics	2^{nd} law of thermodynamics	
International Trade	Exchange of Heat	
Migration	Exchange of Particles	

Table 2-4: Analogs found between economics and statistical physics relevant for Section 5-3, namely for the level of welfare, poverty, disposable income, equilibrium, information, the fundamental theorems of welfare, international trade and migration.

Analogies for Subsection 5-4-3.

Economics	**Physics**	**Symbol**
Price-Quantity Space Probability Density	Phase Space Probability Density	ϱ
Macroeconomic Equilibrium	Thermodynamic Equilibrium	
1 Cent	Planck's Constant	$¢$, $\hbar$
Price-Quantity Space Value	Phase Space Volume	$\tilde{\omega}$
Possible Distribution of Wealth	Microstate	
Options Available for Diversification	Multiplicity	Ω
Amount of Diversification	Entropy	S
Autonomous Economy	Isolated System	

Table 2-5: Analogs found between economics and statistical physics relevant for Subsection 5-4-3, namely for the probability density function, equilibrium, the quantization of money, value and the amount of diversification.

Analogies for Subsections 5-4-4 and 5-4-5.

Economics	**Physics**	**Symbol**
Constant Economic Rent Ensemble	Microcanonical Ensemble	
Constant Level of Welfare Ensemble	Canonical Ensemble	
Equal Disposable Income Ensemble	Grand Canonical Ensemble	
Opportunity function	Partition Function	Z
Economic Heat Capacity	Heat Capacity	$\mathcal{C}$
Free Economic Rent	Free Energy	F

Table 2-6: Analogs found between economics and statistical physics relevant for Subsections 5-4-4 and 5-4-5, namely for different ensembles, the opportunity function, the economic heat capacity and the free economic rent.

2-2 Definitions Used for Linking Economics & Statistical Physics

This section contains the definitions used in this thesis. These are either used in current economic engineering theory [6], current economics literature, current physics literature or newly defined based on analogs from economics and physics.

The terms defined in this section are used throughout this thesis. The reader of this thesis is advised to become familiar with these terms as some of these might be unknown to readers unfamiliar with classical mechanics, statistical physics or economics literature.

The definitions of the terminology used is given next in alphabetical order. Words written in *cursive* are defined as well.

List of Definitions

1. **Action**: Time integral of the *Lagrangian*. The time evolution of a physical *system* is along the path of stationary action [9].
   The action (3-1) is the economic engineering analog of the total *period costs*, which a rational agent seeks to minimize.

2. **Autonomous Economy**: Economy where no financial aid is given or received.

3. **Bit**: A logical state with one of two possible values, used in information theory. A unit of information [10].

4. **Closed Economy**: *Economic system* where agents are allowed to trade with other economies but migration is not allowed.

5. **Complete Information**: Every agent in a market knows exactly the *reservation price* and *quantity demanded (supplied)* of every other agent [11].

6. **Complete Market**: A market of *perfect competition* where each agent has *complete information* of the entire market [11].

7. **Conjugate Variables**: Pair of variables, one *intensive* and the other *extensive*, that when multiplied give a unit of energy (*economic rent*).

8. **Chemical Potential**: The energy that can be absorbed or released due to a change of the particle number of the given species. Particles move from higher chemical potential to lower chemical potential, which reduces the *free energy*.

9. **Degree of Freedom (economics)**: An independent method of acquiring a surplus available to an agent. Number of degrees of freedom is equal to number of distinct goods for agents who can only acquire a surplus through trade.

10. **Degree of Freedom (physics)**: An independent motion available to a particle. Number of degrees of freedom is equal to number of dimensions for monatomic gas particles.

11. **Diversification, Amount of (economic)**: Amount of possible allocations of the total *allocated economic rent* within an *economic system*. Depends on value of the total *allocated economic rent*, the number of agents and the number of distinct goods within the *economic system*.

12. **Diversification, Product**: Increasing the number of distinct goods within an economy that can be traded and used to acquire a surplus.

13. **Discounting (time)**: Process of determining the present value of a payment or a stream of payments that is to be received in the future [12].
Time discounting was the original application of the Laplace transform in economic engineering [6], [13]. In this thesis discounting over the *economic rent* and population size is introduced.

14. **Disposable Income**: The part of an agents' income that is available for spending after his fixed expenses have been deducted [14].
A *rational agent* operating in an *open economy* will migrate to the *economic system* that allows him to maximize his disposable income.
The disposable income per capita in an *economic system* is the microeconomic interpretation of the analog of the chemical potential in a *thermodynamic system*.

15. **Disutility**: Inverse of the *utility*. Economic engineering analog of the *Lagrangian* (3-2).

16. **Economic Driving Force**: *Want (desire)* on a macroeconomic scale. Something that changes the behaviour of *rational agents* in such a way that *macroeconomic equilibrium* is reached.

17. **Economic Rent**: Those payments to a factor of production that are in excess of the minimum payment necessary to have that factor supplied [15].
The total economic rent is the sum of the *allocated economic rent* and the *dissipated economic rent* (3-21) and is the economic engineering analog of the total energy. The total energy is the sum of the dissipated energy and the Hamiltonian of the system.

18. **Economic Rent (Allocated)**: Part of the total *economic rent* that either makes up the surplus of an agent or can do so in the future.
The allocated economic rent is the economic engineering analog of the Hamiltonian.

19. **Economic Rent (Dissipated)**: Part of the total *economic rent* that can no longer be used to generate a surplus.
The dissipated economic rent is the economic engineering analog of the dissipated energy.

20. **Ensemble**: A collection of *systems* that are in different *microstates* but are macroscopically (macroeconomically) identical.

21. **Ensemble, Stationary**: *Ensemble* that does not change over time and is thus in *macroscopic (macroeconomic) equilibrium*.

22. **Equilibrium, Economic**: Aggregate demand equals aggregate supply [15]. There is no *economic growth*, no net migration and the economy is *autonomous*. Further economic *diversification* is not possible.

23. **Equilibrium, Thermodynamic**: There is no net flow of matter or energy (from one system to another). Entropy is maximized [16], and the (relevant) *thermodynamic potential* is minimized.

24. **Extensive Property**: Property that scales with the amount of particles (agents) within the *thermodynamic (economic) system*.

25. **Free Economic Rent**: Portion of the GDP that can be extracted as profits at a constant *level of welfare*.
The free economic rent is the Legendre transform of the GDP.

26. **Free Energy**: Portion of the internal energy available to perform thermodynamic work at constant temperature [17].
The free energy is a *thermodynamic potential* and is the Legendre transform of the internal energy.

27. **Growth, Economic**: Increase or improvement in the inflation-adjusted market value of the goods and services produced by an economy over a certain period of time. Increase in the (real) GDP.

28. **Heat Capacity (economics)**: The amount of change of the GDP of an economy required to produce an (arbitrarily chosen) unit change in its *level of welfare*.

29. **Heat Capacity (physics)**: The amount of heat to be supplied to an object to produce a unit change in its temperature.

30. **Incentive for Diversification:** Total desire felt by the agents of an *economic system* to be more diversified.
The incentive for diversification in an *economic system* is the macroeconomic interpretation of the analog of the temperature in a *thermodynamic system*.

31. **Incentive for Migration:** Total desire felt by the agents of an *open economic system* to migrate.
The incentive for migration in an *open economic system* is the macroeconomic interpretation of the analog of the chemical potential in an *open thermodynamic system*.

32. **Intensive Property**: Property that does not scale with the amount of particles (agents) within the thermodynamic (economic) system.

33. **Inventory level**: The amount of a certain good an agent has in storage.

34. **Isolated Economy**: *Economic system* where trading with other economies and migration are not allowed.

35. **Lagrangian**: A function that describes the state of a dynamic (mechanical) *system* in terms of position coordinates and their time derivatives and that is equal to the difference between the kinetic co-energy and potential energy (3-2) [9].
Taking the time integral of the Lagrangian produces the *action*, which needs to be stationary throughout the time evolution of the system (3-1).
The Lagrangian is the analog of the *running* costs in economic engineering.

36. **Level of Poverty**: Inverse of the *level of welfare*.

37. **Level of Welfare**: Expectation value of the revenue of a random agent of an *economic system*. Describes the prosperity of an *economic system*.
The level of welfare in an *economic system* is the microeconomic interpretation of the analog of the temperature in a *thermodynamic system*.

38. **Marginal Costs**: The change in the total cost that arises when the quantity produced is incremented; the cost of producing additional quantity.
In economic literature [15] the marginal cost is equal to the *reservation price* of the producer.

39. **Microstate (economics)**: A possible configuration of all *reservation prices* and *inventory levels* of agents within an *economic system*. A microstate is a single point in the *price-quantity space* and is of dimension $2ND$.

40. **Microstate (physics)**: A possible configuration of all momenta and positions of particles in a *thermodynamic system*. A microstate is a single point in the *phase space* [17] and is of dimension $6N$ in 3-Dimensional space.

41. **Multiplicity**: Amount of accessible *microstates* for a given macrostate of a *thermodynamic equilibrium*. Gives the amount of unique ways the internal energy of a *system* can be distributed among the particles, where each particle has a position and a momentum coordinate in each dimension.

42. **Open Economy**: *Economic system* where agents are allowed to trade with other economies and are allowed to migrate to other *economic systems*.

43. **Opportunity Function**: Dimensionless function describing the statistical properties of a *system* in *macroeconomic equilibrium*. Gives the (weighted) expected value of the *options available for diversification*. The economic engineering interpretation of the Laplace transform was used to find its economic meaning, namely the opportunities for extracting profits from an *economic system* by trading.
The opportunity function is the economic engineering analog of the *partition function*.

44. **Options Available for Diversification**: Amount of accessible *economic microstates* for a given macrostate of a *macroeconomic equilibrium*. Gives the amount of unique ways the GDP of an *economic system* can be distributed among the agents, where each agent has an inventory and a *quantity demanded (supplied)* for each distinct good.

45. **Pareto Optimality**: A situation where no individual can be made better off without making at least one individual worse off [11].

46. **Partition Function**: Dimensionless function describing the statistical properties of a system in *thermodynamic equilibrium*.
The partition function is the economic engineering analog of the *opportunity function*.

47. **Perfect Competition**: A market with a large number of agents who are all *rational*, where there are no transaction costs [15] and where the supply for all goods exactly matches the demand.

48. **Period Costs**: Total costs a producer makes in a given period of time. A *rational agent* will seek to minimize his period costs.
The *action* as known in analytical mechanics is the economic engineering analog of the period costs.

49. **Phase Space**: $6N$-Dimensional space of which the (momentum and position) coordinate dimensions represent the variables required to specify the phase or state of a *physical (thermodynamic) system* [17].

50. **Price Elasticity (of demand)**: A measurement of the change in the *quantity demanded* of a good by a consumer in relation to a change in its price [15].
The inverse of the price elasticity of demand is the 2^{nd} derivative of the *running disutility* w.r.t. the *quantity demanded* (3-7).

51. **Price Elasticity (of supply)**: A measurement of the change in the *quantity supplied* of a good by a producer in relation to a change in its price [15].
The inverse of the price elasticity of supply is the 2^{nd} derivative of the *running costs* w.r.t. the *quantity supplied* (3-7).

52. **Price-Quantity Space**: $2ND$-Dimensional space of which the (*reservation price* and *inventory level*) coordinate dimensions represent the variables required to specify the state of an *economic system*.

53. **Probability Density Function**: A probability distribution function giving the distribution of accessible *microstates* in the *phase space (price-quantity space)*.
The probability density function depends on the constraints on the *system*.

54. **Quantity demanded (supplied)**: The total amount of a goods demanded (supplied) by a consumer (producer) in a given period of time.
The quantity demanded is the opposite of the quantity supplied in economic engineering and is analogous to the velocity of a particle.

55. **Rational Agent**: A rational agent is a person that always aims to perform optimal actions based on given premises and information, where the goal in a given period of time is to maximize his total period *utility* (consumer) or minimize his total period costs (producer). A rational agent is a cost minimizing agent.
The optimal decisions made by the rational agent and the optimal path taken by a particle in analytical mechanics given by the principle of stationary action are analogs in economic engineering.
The *Lagrangian* of a particle is the analog of the running costs or *disutility* of the rational agent, which is given in (3-2).

56. **Reservation Price**: The minimum amount that a seller will accept or the highest price a buyer is willing to pay for a good or service.
The reservation price is the change in the *running costs* w.r.t. the *quantity supplied* (3-4), also known as the marginal costs in economic literature [15].

57. **Reversible Process (economics)**: A change in an *economic system* from an initial to a final state carried out in infinitesimal transactions which, when reversed returns both the *economic system* and the external economies to their respective initial states.

58. **Reversible Process (physics)**: A change in a *thermodynamic system* from an initial to a final state carried out in infinitesimal steps which, when reversed returns both the system and the environment to their respective initial states [16].

59. **Running Costs**: Difference between the variable costs and the benefits of ownership of the producer.
The *Lagrangian* of a particle is the analog of the running costs or *disutility* of the *rational agent*, which is given in (3-2).

60. **Social Welfare**: The total well-being of society. In an economy with *rational agents* the social welfare is maximized. This is stated by the 2nd fundamental theorem of welfare economics [11], which is the analog of the 2nd law of thermodynamics.

61. **Surplus, Consumer**: The monetary gain obtained by consumers because they are able to purchase a product for a price that is less than the highest price that they would be willing to pay.

62. **Surplus, Producer**: The amount that producers benefit by selling at a market price that is higher than the least that they would be willing to sell for; this is roughly equal to profit.

63. **System (economics)**: An idealized economic "body" which is further restricted in that all endogenous economic conditions can be discriminated [18]. The system can thus be *isolated* from external economies.

64. **System (physics)**: An idealized body which is further restricted in that it can be isolated from everything else [16].

65. **Thermodynamic Potential (Energy)**: A scalar quantity used to represent the thermodynamic state of a *system*. Constraints on the *system* determine the most useful thermodynamic potential. Expressions of all thermodynamic potentials (e.g. *free energy*) can be derived using a Legendre transform from an expression of the internal energy U.

66. **Utility**: The usefulness or enjoyment a consumer can get from a service or good [15]. The running utility is to the consumer what the *running costs* are to the producer.

67. **Want (Desire)**: Desire felt by an agent to change his quantity demanded (supplied).
A force is the economic engineering analog of a want (desire) as given in (3-8).
The change in the *running costs* with respect to the *inventory level*.

2-3 Assumptions Made for Linking Economics & Statistical Physics

The assumptions made in this thesis to link microeconomics and macroeconomics using the framework of statistical physics are given here.

Unless specifically stated otherwise, the assumptions given next are used throughout this thesis.

List of Assumptions

1. **All agents are rational** [15], [14].
   All choices an agent makes are to minimize his running costs or maximize his utility. This is analogous to a particle in a thermodynamic system whose movement is along the path of stationary action as known in analytical mechanics [9].

2. **Agents are identical**.
   All agents within the economic system have the same price elasticity of demand. As a result agents are indistinguishable. Swapping the inventory levels and reservation prices of two agents does not change the economic microstate.

3. **Agents can only trade**.
   Trading is the only means of acquiring a surplus available to agents. This is analogous to point particles in a thermodynamic system, where they can only have translational kinetic energy. No rotations of a particle around its axis are considered.

4. **Constant Price Elasticity of Demand**.
   The price elasticity of demand of an agent is independent of his quantity demanded. This is consistent with current economic engineering literature where the inertia is the analog of the price elasticity.
   In classical mechanics the inertia of a body is constant and independent of its velocity. In the theory of relativity the inertia is a function of the velocity. The velocity dependency is however only relevant for relativistic velocities, so for magnitudes of the velocity approaching the speed of light.
   In economics literature the price elasticity is not necessarily constant; it can be a function of the quantity demanded [15].
   It is thus assumed that the quantities demanded are sufficiently low, so that all "relativistic" effects are negligible. The price elasticity is constant.
   The assumption of sufficiently low particle velocities is also made in statistical physics literature.

5. **No "Economic Freezing"**.
   The quantity demanded of agents is sufficiently high, so no "economic freezing" occurs. The level of welfare within the economy is high enough to ensure economic activity at all times.
   This is analogous to the temperature of a gas being sufficiently high so quantum mechanical effects are negligible. This assumption is also made in statistical physics literature.

6. **Economy in a Market of Perfect Competition**.
   See the definition of a market with perfect competition: a large number of rational

agents are considered where no transaction costs are taken for trading. There is thus no economic rent dissipation when agents trade or interact. Furthermore, the supply and demand for all goods are perfectly matched, meaning that agents have no desire to hold an inventory.
This is analogous to an ideal gas. Particles interact or collide elastically, meaning that no kinetic energy is dissipated. The inter-molecular forces between particles are also neglected.

7. **Money is Quantized**.
Transactions are always rounded to the nearest smallest quanta of money. For transactions involving physical goods the smallest quanta is usually one cent (1¢). This quantization is used in this thesis.
In physics literature the reduced Planck's $\hbar$ constant is chosen as the smallest quanta of action.

2-4 Interpretation of Mathematical Operations

In this section the (economic engineering) interpretation of mathematical functions and operators used in this thesis is given.
This is done to make mathematical expressions more insightful and to fully understand the meaning of a mathematical operation in economic engineering.

List of Mathematical Operations and their Interpretations

1. **Laplace Transform**
   In physics and engineering the Laplace transform is often presented as a tool to solve differential equations. In economic engineering literature a more applied interpretation exists [6], [13], namely to find the present value of future payments. Future payments are discounted over time to give the present value of the payment. The present value is a function of the (complex) discount rate.
   The Laplace transform is used in this thesis not to discount future payments over time, but to discount options over values of the economic rent. The Laplace transform returns the (weighted) expected value of the options if the discount factor is seen as an exponential probability distribution function.

2. **Legendre Transform**
   Just like the Laplace transform, the Legendre transform is often introduced in physics and engineering classes as a mathematical tool, where the power and elegance of the Legendre transform is omitted. The paper by Zia, et al. [19] does a good job to provide the reader with a better understanding of the Legendre transform.
   The interpretation given to the Legendre transform in [19] is used in this thesis, where the "tool" is used to as an alternative way to provide information, for example how much profits can be made from trading with an economic system.

3. **Logarithm**
   In physics literature the logarithm is rarely given an actual interpretation. Instead, it is viewed as a mathematical function, the inverse of the exponent. In economic engineering the logarithm is given an actual (economic) interpretation, using information theory.
   In information theory the base-2 log gives the amount of choices that need to be made to extract information from a sequence, where in each choice 2 options of equal probability are presented. This information is given in the unit of bits.
   In this thesis the logarithm is taken over the base-e log or natural logarithm. This thus gives the amount of choices that need to be made to extract information from a "sequence", where in each choice $e \approx 2.71828...$ options of equal probability are presented. The logarithm function is thus interpreted as the function providing the amount of choices available or the amount of choices that need to be made.

4. **Z-Transform**
   The Z-transform is to discrete time signals what the Laplace transform is to continuous time signals. The interpretation given to the Laplace transform also holds for the Z-transform.

In this thesis the Z-transform is used to discount options over the population size of an economy, instead of discounting future payments over time as is usually done in economic engineering.

Chapter 3

Economic and Economic-Engineering Literature

This chapter provides an overview of current economic literature, as well as literature of the fields of economic engineering and thermoeconomics.
The economic literature presented will be about microeconomics and macroeconomics.

Economic engineering is a new field of study in which economic systems are modelled using first principles. Current literature of the economic engineering theory of supply and demand, based on microeconomics laws, will be given.
To fully highlight the analogs between the microeconomic laws of supply and demand and the economic engineering theory of supply and demand, literature on these two subjects will be presented simultaneously.

The economic engineering theory of supply and demand will be at the basis of this thesis research, which is to link microeconomics and macroeconomics within economic engineering.

Finally, literature on attempts to connect economics and thermodynamics will be discussed. This field of study was coined thermoeconomics. The inconsistencies in the approaches taken to match economics and thermodynamics will be highlighted here. A past attempt to connect thermodynamics and economics within the field of economic engineering will be discussed here as well.

3-1 Microeconomics & Economic Engineering

Microeconomics is the most well-known branch of economics and focuses on the behaviour of individuals. More elaborately, microeconomics is the study of what is likely to happen when individuals make choices in response to changes in incentives, prices, resources, and/or methods of production. Individual actors are often grouped into microeconomic subgroups, such as buyers and sellers. These groups create the supply and demand for resources from which the well-known laws of supply and demand are derived [7].

Economic engineering is a new field of study in which economic systems are modelled using first principles. The economic systems are modelled based on economic laws in the same way that physical systems are modelled using the laws of physics. Analogs are found between economic systems and physical systems and based on these analogs, the economic systems are modelled identically to how physical systems would be modelled in engineering. These models can then be presented in a domain-neutral environment, e.g. using Bond Graphs and can be controlled to exhibit desired behaviour [6], [13].
The economic engineering theory presented in this chapter is the work of Dr. Ir. M. Mendel and his economic engineering group at the Delft University of Technology (TU Delft) [6]. The group is a sub-part of the Delft Center for Systems and Control (DCSC) in the department of Mechanical, Maritime and Materials Engineering (3mE).

The economic engineering theory of supply and demand is based on the microeconomic theory of supply and demand. Analogs found between the laws of supply and demand and Newtonian physics allow for the modelling of microeconomic systems using the tools well-known in mechanical engineering.

3-1-1 Stock Level, Quantity Demanded & Quantity Supplied

The law of supply and demand explains the interaction between the sellers of a resource and the buyers for that resource. Although these resources are not necessarily physical goods, it is assumed in this thesis that physical goods are being traded. A typical interaction consists of a supplier selling (a part of) his inventory of goods to a demander.

In economic engineering, the stock level or inventory level is given by the symbol q, where $[q] = \#$. The demander can also have an inventory, but is looking to increase his inventory level while the supplier is looking to decrease his inventory.

The quantity demanded (supplied) is given by $\dot{q}$, where $[\dot{q}] = \frac{\#}{\text{day}}, \frac{\#}{\text{week}}, \frac{\#}{\text{year}}$, etc. In economic literature the term quantity is usually reserved for the quantity demanded (supplied), meaning that the quantity is measured in units of the good over a given time interval [20].
Because economists use the term quantity and are often looking at predefined periods of time, many graphs in economic literature will plot the quantity in $\#$ instead of $\frac{\#}{\text{year}}$ and give this quantity the symbol Q. The economic engineering notation is used in this thesis. The symbol q thus indicates the stock or inventory levels. The quantity demanded (supplied) as often given in economic literature with the symbol Q will be the time derivative of q and thus indicated by $\dot{q}$.

In economic engineering physical systems are given economic interpretation. The interaction between a supplier and a demander is replaced in economic engineering by two massive bodies interacting. Just as in economics, both the supplier and demander can have an inventory level. Economic engineering states that the **inventory level** is the analog of **position**. The symbol q is also common in analytical mechanics for a position variable.

Velocity is the time derivative of position and is thus also the analog of the **quantity demanded** $\dot{q}$. When considering translational motion in Newtonian mechanics, the most common unit of q is: $[q] = \mathrm{m}$. From this it follows that $[\dot{q}] = \frac{\mathrm{m}}{\mathrm{s}}$.

A **mass** with a positive (negative) velocity corresponds to a **supplier (demander)**, or agent in general. Just like velocities in Newtonian physics, quantities demanded are relative. A supplier might become a demander when a trades with an even bigger supplier [6].

3-1-2 (Dis)utility and Costs

The goal of the producer (otherwise known as the supplier) is to minimize his costs over a given time period. The consumer (also known as the demander) on the other hand wishes to maximize his utility over a given period of time [15].
These objectives are analogous to the stationary action principle approach taken in analytical mechanics [6],[9].

Consider a system considered in analytical mechanics consisting of a particle with a certain position q_1 at time t_1 and a position q_2 at time t_2. The path γ taken by the particle between times t_1 and t_2 from q_1 to q_2 is the one for which the action $\mathcal{S}(\gamma)$ is stationary:

$$\delta \mathcal{S} = \delta \int_{\gamma} L\left(q, \dot{q}, t\right) dt = \delta \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt = 0, \tag{3-1}$$

where L is the Lagrangian of the mechanical system:

$$L = T^{*} - \Phi, \tag{3-2}$$

with T^{*} being the kinetic co-energy of the system and Φ the potential energy. For energy conserving systems, it holds that $L\left(q, \dot{q}, t\right) = L\left(q, \dot{q}\right)$, i.e. $\frac{\partial L}{\partial t} = 0$.

Economically the **Lagrangian** L is interpreted as the **disutility** from the point of view of the consumer and as the **running costs** of the producer. The natural units of the running costs are $\frac{\$}{\mathrm{day}}, \frac{\$}{\mathrm{week}}, \frac{\$}{\mathrm{year}}$, etc. depending on the period of time considered.

The **action** $\mathcal{S}$ is the analog of the **total period costs** and thus has units \$. An agent going from an inventory level q_1 to q_2 at times t_1 and t_2 respectively will do so along the "path" that minimizes the period costs as shown in (3-1). Such an agent is called rational.

The running costs (disutility) of the producer (consumer) depend on his inventory level q, the quantity supplied $\dot{q}$ and the amount of time t that has passed, i.e. $L = L\left(q, \dot{q}, t\right)$. In an economic system where no goods are being consumed, value is not depreciated and no external surplus is acquired (no free lunch) the running costs do not explicitly depend on time: $L = L\left(q, \dot{q}\right)$.
The running costs L are calculated as shown in (3-2), where T^{*} are the **variable costs** and Φ are the **benefits of ownership**, the economic engineering analogs of the **kinetic co-energy** and the **potential energy** respectively.

3-1-3 Price, Price Elasticity & Economic Desire

Price

Working out the variation in (3-1) leads to the Euler-Lagrange equation [9]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{3-3}$$

Noether's theorem states that for a mechanical system that is invariant under translations in space, the momentum is conserved [9]. If $\frac{\partial L}{\partial q} = 0$ it follows from (3-3) that:

$$p := \frac{\partial L}{\partial \dot{q}}, \tag{3-4}$$

where p is the momentum of the particle associated with translational motion in the direction of q. Momentum is thus defined to be the change if the Lagrangian w.r.t. the generalized velocity.

The expression in (3-4) is given an economic interpretation as well. The change in the running costs w.r.t. the quantity supplied is known in economic literature as the marginal cost of the producer [15]. Economic literature defines the reservation price to be the marginal cost. The **reservation price** is thus the economic engineering analog of **momentum** and is given in units $\frac{\$}{\#}$.

Economists define the reservation price of the demander as the maximum willingness to pay for a good, while the reservation price of the supplier is the lowest price he is willing to accept in a trade for a good [15]. From this it is concluded that the reservation price is agent specific, just how the momentum is specific to each particle or body in a mechanical system.

Noether's theorem applied to the economic system then states that the reservation price remains constant when considering economic systems that are invariant to stock level changes. This means that the reservation prices remain constant when no benefit is obtained from holding an inventory, a conclusion that is in correspondence with economic literature.

Price Elasticity

Taking the 2$^{\text{nd}}$ derivative of the Lagrangian w.r.t. the generalized velocity gives the inertia of the particle, which is a measure of how much the particle opposes a change in its velocity:

$$\mathcal{I} := \frac{\partial^2 L}{\partial \dot{q}^2} = \frac{\partial p}{\partial \dot{q}} > 0, \tag{3-5}$$

where $\mathcal{I}$ is the inertia of the particle and is always positive.

The inertia is given economic meaning as well. Economic literature states that the price elasticity of supply (demand) is the percentage change in quantity supplied (demanded) divided by the percentage change in price [21]. In other words, the elasticity of supply (demand) of a good is determined by how much the supply (demand) of the good changes as the price changes: (**This is not used in Economic Engineering!**)

$$e = \frac{dQ}{dp}\frac{p}{Q}, \tag{3-6}$$

where e is the price elasticity of supply (demand) as used in economic literature, p is the price as defined in (3-4) and Q is the notation used in economic literature for the quantity supplied (demanded) $\dot{q}$. Comparing (3-6) and (3-5), it is concluded that the price elasticity of supply (demand) is the inverse of the inertia, up to a factor $\frac{p}{\dot{q}}$. The definition of the **price elasticity** $\mathcal{E}$ used in economic engineering is thus:

$$\mathcal{E} := \left(\frac{\partial^2 L}{\partial \dot{q}^2}\right)^{-1} = \left(\frac{\partial p}{\partial \dot{q}}\right)^{-1} > 0, \tag{3-7}$$

which is the **inverse of the inertia** in as given in (3-5). The law of diminishing marginal returns (utility) states that indeed the price elasticity should be positive [15].

Economic Desire

Substituting the definition of momentum found in (3-4) into (3-3), we find that:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\partial L}{\partial q}, \tag{3-8}$$

meaning that the change in momentum of the particle w.r.t. time is equal to the change of the Lagrangian w.r.t. translations. For systems of constant inertia the left-hand side of (3-8) is the force $\mathbb{F}$ acting on the particle. For a free particle it holds that $\frac{\partial L}{\partial q} = 0$, meaning that $\mathbb{F} = 0$ and the momentum is a constant, as discussed above.

Economically this means that whenever the running costs are independent of the inventory level prices remain constant. This means that there is no economic desire or want felt by the agent to change his quantity supplied (demanded). Whenever there is a benefit to ownership, the reservation price of the agent will change as shown in (3-8), giving the desire to change the quantity supplied (demanded). The **economic desire** $\mathbb{F}$ is thus the analog of the **force** and given in units $\frac{\$}{\#\cdot\mathrm{day}}$, where any period of time can be used instead of “day”.

3-1-4 Supply Curves & Demand Curves

According to the law of demand the higher the price of a good is, the fewer people will demand that good if all other factors remain equal. In other words, the higher the price, the lower the quantity demanded. The inverse is true for the law of supply, where a higher price corresponds with a higher quantity supplied. As a result, demand curves are typically downward sloping, while supply curves typically slope upwards. Figure 3-1a shows a typical graph of a supply and demand curve. The slopes of the curves shown in Figure 3-1a are constant, which is often assumed in literature but not a requirement. The slope of the supply (demand) curve is the inverse price elasticity of supply (demand).

In physics the relation between momentum p and velocity v is given by $p = \mathcal{I}v$, where $\mathcal{I}$ is the inertia as defined in (3-5). An increase in momentum for a body of constant inertia thus results in an increased velocity of that body. The inertia is thus the slope of a $p - v$ diagram, as shown in Figure 3-1b.

The demand curve has a “negative inertia” since a quantity demanded is interpreted as a negative quantity supplied and thus a velocity in the opposite direction.

Although the inertia of a body is considered constant and positive in Newtonian mechanics, the relation between momentum and velocity in Einstein's relativity theory becomes nonlinear [22], corresponding with nonlinear slopes of demand and supply curves in economics.

3-1-5 Economic Surplus and Economic Rent

A surplus describes the amount of an asset or resource that exceeds the portion that's actively utilized. A surplus can refer to a host of different items, including income, profits, capital, and goods. There are two types of economic surplus, namely consumer surplus and producer surplus.
It is assumed that all individuals (or agents) within an economy are rational. Rational agents will minimize their running costs (maximize their utility) as explained in subsection 3-1-2. As such, all agents will seek to maximize their surplus. The surplus T and the variable costs T^* are related via:

$$T = p\dot{q} - T^*, \tag{3-9}$$

where $p\dot{q}$ is the revenue generated. The surplus and variable costs are defined as:

$$T := \int \dot{q}(p)dp, \tag{3-10}$$

and

$$T^* := \int p(\dot{q})d\dot{q}, \tag{3-11}$$

The **surplus** is the economic engineering analog of the **kinetic energy**. Readers familiar with the harmonic oscillator know that the kinetic energy of the harmonic oscillator is given as:

$$T_{HO} = \frac{1}{2\mathcal{I}}p^2, \tag{3-12}$$

where $\dot{q}(p) = \frac{p}{\mathcal{I}}$ was used in (3-10). For constant inertia the kinetic energy and kinetic co-energy are equal, with the latter being:

$$T^*_{HO} = \frac{1}{2}\mathcal{I}\dot{q}^2. \tag{3-13}$$

In subsection 3-1-2 it was mentioned that $\frac{\partial L}{\partial t} = 0$ for energy conserving systems. Taking the total derivative of L w.r.t. time yields:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}. \tag{3-14}$$

Replacing $\frac{\partial L}{\partial q}$ with $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)$ in accordance with the Euler-Lagrange equation (3-3) yields:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right). \tag{3-15}$$

It thus holds for energy conserving systems that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0. \tag{3-16}$$

Since energy conservation means that $\dot{E} = 0$, the total energy E is:

$$E := \dot{q}\frac{\partial L}{\partial \dot{q}} - L. \tag{3-17}$$

Since the total energy E remains constant during the motion of the system, it is an integral of motion.
The expression given in (3-17) is the Legendre transform of the Lagrangian $L(\dot{q}, q)$ as a function of $\dot{q}$ and produces the Hamiltonian of the system. For energy conserving systems it thus holds that the total energy E is equal to the Hamiltonian H.

Combining (3-2), (3-4) and (3-17) yields the expression for the Hamiltonian of an energy conserving system:

$$H = T(p, q) + \Phi(q). \tag{3-18}$$

The potential energy $\Phi = \Phi(q)$ of the harmonic oscillator is:

$$\Phi_{HO} = \frac{1}{2\mathbb{C}}q^2, \tag{3-19}$$

where $\mathbb{C}$ is the compliance. The Hamiltonian of the harmonic oscillator is:

$$H_{HO} = \frac{1}{2\mathcal{I}}p^2 + \frac{1}{2\mathbb{C}}q^2. \tag{3-20}$$

In economic engineering the harmonic oscillator is the analog of an agent that both trades and has an inventory. The **Hamiltonian** is the economic engineering analog of the **allocated economic rent**.
In the Hamiltonian description (given in (3-18)) the economic engineering analog of the **potential energy** is the **potential surplus**, since goods in inventory can be traded at a later time to increase the **surplus**. This is comparable to a mechanical system where the potential energy can be converted into **kinetic energy** in the future.

For energy conserving systems, the total energy E is equal to the Hamiltonian. For dissipative systems however, the total energy E and Hamiltonian H are related as:

$$E = H + \mathcal{D}, \tag{3-21}$$

where $\mathcal{D}$ is the dissipated energy and H as given in (3-18).

The **total energy** is the economic engineering analog of the **total economic rent**, while the **dissipated energy** is the analog of the **economic rent dissipation** [6].

Hamilton's Equations

The Euler-Lagrange equation (3-3) can now be formulated in terms of the Hamiltonian H given in (3-18) [9]:

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{3-22}$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \tag{3-23}$$

where $\dot{q}$ is the quantity demanded and $\dot{p}$ is the change in the reservation price, which was shown to be the economic desire $\mathbb{F}$.

The Hamilton's equations ((3-22) & (3-23)) are completely equivalent to the Euler Lagrange equation. For systems with D degrees of freedom, the Euler Lagrange equations give D second order differential equations w.r.t. time, while Hamilton's equations yield $2D$ first order differential equations.

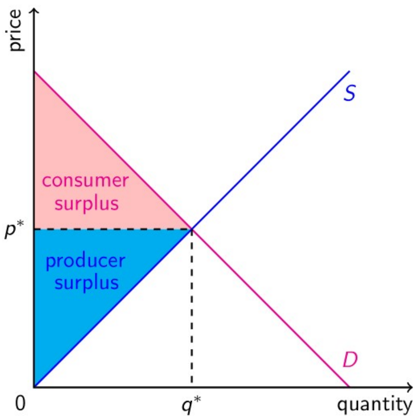
The Hamiltonian formalism will prove to be convenient in this thesis, since the time evolution of prices and inventories can easily be found separately.

Consumer & Producer Surplus

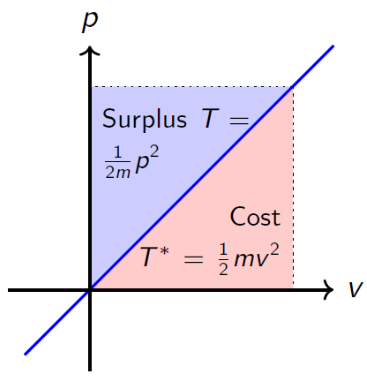
A consumer surplus occurs when the price for a product or service is lower than the highest price a consumer would willingly pay. The area shaded pink in Figure 3-1a is the consumer surplus [23].

A producer surplus on the other hand occurs when goods are sold at a higher price than the lowest price the producer was willing to sell for. The area shaded blue in Figure 3-1a is the producer surplus [23].

In physics, the kinetic energy is the area shaded in purple in Figure 3-1b. This corresponds with the producer surplus.



(a) Graph of a supply and demand curve typically found in economic literature. The quantities supplied and demanded shown on the horizontal axis are measured in units of the good over a given time interval [24] [20].



(b) Graph of the relation between momentum and velocity for a body of Inertia m in Newtonian physics. The area shaded in purple is known as the kinetic energy and is the economic engineering analog of the producer surplus [6]. The area shaded in pink is the kinetic co-energy and is known in economic engineering as the variable costs.

Figure 3-1: Graphs of supply and demand curves in economics and the momentum-velocity relation in physics and economic engineering.

3-2 Macroeconomics

Microeconomics is the study of of the behaviour of the economy as a whole. It examines the forces that affect firms, consumers, and workers in the aggregate. Macroeconomics studies economy-wide phenomena such as inflation, price levels, rate of economic growth, national income, gross domestic product (GDP), and changes in unemployment [8]. It contrasts with microeconomics, which studies individual prices, quantities, and markets.

3-2-1 Macroeconomic Objectives & Policy

Samuelson [14] identifies three main objectives of macroeconomics:

- a high level and rapid growth of outputs,
- a high level of employment with low involuntary unemployment,
- stable prices.

The government of a nation has two major policies that can be used to pursue its macroeconomic goals, namely fiscal policy and monetary policy [14]. Fiscal policy is primarily used to effect long-term economic growth.

Fiscal policy consists of government expenditure and taxation.
Government expenditure can be in two forms, the first of which is in the form of government purchases on goods and services, e.g. construction of roads, salaries for judges, etc. The other form of government expenditure is transfer payments to increase the income of targeted groups.
Taxation reduces the disposable income of individuals and effects private savings and investments.

Monetary policy is conducted by the central bank and determines short-run interest rates. It therefor effects credit conditions, asset prices and exchange rates [14]. Monetary value has an important effect on GDP.

3-2-2 Gross Domestic Product & Disposable Income

Gross Domestic Product

The Gross Domestic Product (GDP) is the most comprehensive measure of a nation's total output of goods and services. The GDP is the sum of consumption (C), gross investments (I), government purchases of goods and services (G) and net imports (X) per year:

$$\text{GDP} = C + I + G + X, \tag{3-24}$$

where $[\text{GDP}] = \frac{\$}{\text{year}}$. GDP is used mainly to measure the overall output of an economy.

Economists measure GDP either by the flow op product approach or the income approach. The GDP using the flow of product approach is calculated as the sum of the flow of final

goods produced times the price, as shown in (3-24).
The income approach to calculate GDP takes the sum of all earnings (due to wages, interest, rent and profit). These two approaches are identical [14].
In this thesis the income approach will be used to calculate GDP since it most naturally fits with the approach taken in economic engineering.

Income can be generated in a variety of ways. The four major factors of production are assumed to be the means through which agents earn an income. The factors of production and their reward are [25]:

- land — rent
- labor — wage
- capital — interest
- entrepreneurship — profit

Disposable Income

Economists define the Disposable Income (DI) as the amount of income households have available to spend. Rational agents will seek to maximize their disposable income, which is calculated as [14]:

$$\text{DI} = \text{GDP} - T - D, \tag{3-25}$$

where T are the taxes payed to the government and D is the depreciation of capital. The National Income (NI) is defined as the difference between GDP and the depreciation.
In Chapter 5 it is argued that the disposable income is the economic engineering analog of the free energy as known in statistical physics and thermodynamics.

3-2-3 Income Distribution

In subsection 3-2-1 it was mentioned that main objectives of a government are maintaining a high level of economic growth and a high level of employment. A major benchmark in determining the success of a government in reaching those goals is a fair distribution of income [14].
Important theoretical and policy concerns include the balance between income inequality and economic growth, and their often inverse relationship [26].

In current macroeconomic literature, the distribution of income is analyzed using the Lorenz Curve, see Figure 3-2 [1]. The Lorenz Curve shows the proportion of overall income or wealth assumed by the bottom $x\%$ of the people of a nation. In an economy with a perfectly equal distribution of income the Lorenz Curve coincides with the "Line of equality".

The income distribution is measured using the Gini coefficient. The Gini coefficient is found by taking the ratio of the area between the line of perfect equality and the observed Lorenz curve to the area between the line of perfect equality and the line of perfect inequality [27]. The line of perfect inequality is the line the Lorenz Curve would lie on if one individual

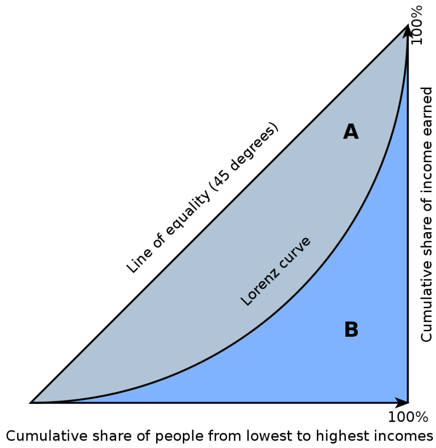


Figure 3-2: Lorenz Curve used in macroeconomic literature to analyze the income distribution within an economy [1].

possessed all the wealth in a nation.
In Figure 3-2 the Gini coefficient G is thus:

$$G = \frac{A}{A+B}. \tag{3-26}$$

A more formal calculation of the Gini coefficient can be made if given a mathematical description of the Lorenz Curve $L(u)$, where $u \in [0, 1]$ and $L(u)$ is the proportion of the total income of the economy that is received by the bottom $100u\%$ of income receivers [27]. Formally the Gini coefficient is:

$$G = 1 - \frac{1}{\mu} \int_0^{y^*} (1 - F(y))^2 \, \mathrm{d}y, \tag{3-27}$$

where y is the income, $F(y)$ denotes the proportion of the population that receives incomes no greater than y, meaning that it is the cumulative probability distribution of income. μ is the mean or average income and y^* is the upper limit of income, which may be infinite.

In Chapter 5 the amount of diversification is derived. This quantity is proposed to be the analog of entropy S and gives the dispersion of economic rent (surplus) within the economy over the population size N and the portfolio distribution of a single agent over the available D distinct goods.
The amount of diversification thus contains all information about the income distribution within an economy and is used in this thesis instead of the Gini coefficient and the Lorenz Curve.

3-2-4 General Equilibrium Theory

General equilibrium theory is a macroeconomic theory that attempts to explain the behaviour of supply, demand and prices in a whole economy, where markets for goods interact with one another [28].

General equilibrium theory differs from the partial equilibrium approach taken in microeconomics where it is assumed that markets do not interact. The microeconomic theory of supply and demand is based on the theory of partial equilibrium. In partial equilibrium, economic equilibrium occurs when the supply and demand match in individual markets. Section 3-1 shows the microeconomic theory of supply and demand in the economic engineering framework.

The general equilibrium theory tries to explain the economy from a "bottom-up" approach, starting with individual markets for goods and agents and adding complexity to the system until the desired number of markets and agents are modelled [28]. As such, the general equilibrium theory is an attempt within current economic literature to link microeconomics and macroeconomics.

A major liability of the general equilibrium approach is that it is often not feasible to be applied to an entire economy. Applying the general equilibrium approach to an entire economy requires exact knowledge of the supply function for all goods as well as the demand functions of all agents for all goods within the economy [28]. This data is then used to compute the equilibrium prices for all goods.

The equilibrium prices are calculated using complex algorithms and solvers that require a lot of computational power [29], [30]. For large numbers of goods and agents, these equilibria cannot always be found.

In Chapter 5 a statistical approach to finding the macroeconomic equilibrium is proposed, that is based on statistical physics.

3-3 Thermoeconomics

Various attempts to link economics and thermodynamics have previously been made. The field of thermoeconomics is a school of economics that attempts to find analogies between thermodynamic and economic systems. A major problem within this field is the lack of consistency across literature for certain analogies. Some of these inconsistencies will be highlighted in this chapter.

This thesis will attempt to link thermodynamics and economics as well. This will be done using the economic engineering framework. The main difference between the thermoeconomics approaches known in literature and the economic engineering approach proposed here is that the economic engineering theory of supply and demand will be used. By using this theory as a basis careful and consistent analogies are made to arrive at the macroeconomic relations that have thermodynamic analogies.

In Subsection 3-3-2 a previous attempt in economic engineering titled "Thermodynamics of Economic Engineering" to find analogies between thermodynamics and economics [18] is analyzed. It will be shown in chapter 5 that an entirely different approach is taken in this thesis.

3-3-1 Thermoeconomic literature

Examples of thermoeconomic research done previously are that of Dragulescu [31], Saslow [32], Yegorov [33] and Rashkovskiy [34].

Dragulescu [31] considers money to be the analog of internal energy U and price to be the analog of temperature T. Economic engineering [6] shows that price cannot be the analog of temperature, since price should have a momentum variable as its analog and temperature is an effort variable.
The requirement for price needing to be a momentum variable is straightforward: economic equilibrium is achieved when prices do not change; likewise Newton showed that in mechanical equilibrium the momentum of a body remains constant.

Saslow [32] considers utility to be the economic engineering analog of the internal energy U and surplus to be the analog of TS, the product of temperature and entropy. He calls temperature the level of economic development and proposes entropy to be a level of economic variety. This definition of temperature does not match well with the statistical physics definition that temperature is the average kinetic energy per particle.
Furthermore, Saslow calls the chemical potential μ the analog of price. This cannot be correct for the same reason that Dragulescu's [31] analogy cannot be correct, namely that the chemical potential is an effort variable and not a momentum variable.

Consistent with the economic engineering definition, Yegorov [33] did consider income to be the analog of internal energy. He however had a more philosophical approach and considered the inverse of pressure p to be the economic freedom f. High economic freedom could be compared to a democracy, while a system with low economic freedom is seen as a dictatorship. These philosophical and qualitative definitions are not used in economic engineering.

Finally, Rashkovskiy [34] also considered money to be the analog of the internal energy. His definition of temperature: the average amount of money per individual is consistent with the statistical physics definition but incorrect as a result of his incorrect definition of money. Furthermore, Rashkovskiy considered pressure to be the analog of price, which cannot be correct since price is a momentum variable and pressure, like chemical potential and temperature, is an effort variable.

Upon review of thermoeconomic literature, I conclude that 2 major mistakes are made:

- Money is seen as an analog of energy,
- Price is given an effort variable as analog.

The realization that economists implicitly consider a quantity demanded as an amount of stock over time instead of just an amount of stock leads to fixing the first mistake. It is then shown that income is the analog of energy and money is the analog of the action.

The mistake to consider price as an effort variable is the result of the desire to link economics and thermodynamics without realizing that no momentum variables exist in the field of thermodynamics. Individual prices thus cannot be considered when comparing economics and thermodynamics. An analogy between microeconomics and thermodynamics thus cannot exist. Only in the field of macroeconomics can analogs with thermodynamics be found. The realization that thermodynamics is actually to Newtonian physics what macroeconomics is to microeconomics solves this conundrum.

3-3-2 Thermodynamics of Economic Engineering

A previous attempt to find analogies between thermodynamics and economic engineering was made by Manders [18]. In his work, Manders' goal was to model economic growth using analogies with thermodynamics.
The goal of this thesis is more general, namely to derive macroeconomic laws using the economic engineering theory of supply and demand. Instead of basing his work on the economic engineering supply and demand model, Manders independently sought analogies between economics and the field of thermodynamics that was previously unexplored from an economic engineering point of view.

In his work [18], Manders also made the analogy between temperature and prices, as frequently done within the field of thermoeconomics. As explained earlier, this is inconsistent with the economic engineering view, where temperature is an effort variable and price is a momentum variable.
Manders defined entropy to be the analog of human capital, explaining it as the effects unmeasured by the factors of production. He explains that in thermodynamics entropy is interpreted as the "one aggregate variable for the unexamined degrees of freedom".
In this thesis, the statistical definition of entropy will be given economic interpretation. Macroeconomic behaviour will be explained by a good understanding of economic behaviour on the micro scale.

Chapter 4

Kinetic Theory of Gases & Statistical Physics

In this chapter relevant literature on the kinetic theory of gases & statistical physics is summarized.
The kinetic theory of gases is a part of statistical physics and perhaps the most intuitive for readers that are familiar with Newtonian mechanics but do not have a pure physics background. Historically, the kinetic theory of gases was the first explicit exercise of the ideas of statistical physics.

The field of statistical physics connects two branches of physics that for a long time had a disconnect [35]. The invention and subsequent use of steam engines, which lead to the industrial revolution, lead to much research being done on the behaviour of gases [36]. From this, the field of thermodynamics was born. Although physicists at the time were quite successful in modelling and predicting the behaviour of gases it was unknown at the time how this new branch of physics was related to the "well-known" branch of Newtonian mechanics. Statistical physics serves as the bridge between these two branches of physics and makes it possible to derive the macroscopic state variables and relations known in thermodynamics from Newtonian models and statistical tools.

The blueprint laid out by statistical physics to connect Newtonian physics and thermodynamics will be summarized in this chapter. This will serve as the foundation to do the same for microeconomics and macroeconomics.

4-1 Motivation for a Statistical Approach

A classical gas is considered, where the gas particles are seen as point particles and their collisions are elastic [37]. Furthermore, it is assumed that the gas is sufficiently dilute and the temperature is high enough to avoid quantum effects but low enough to neglect relativistic effects.

Knowing the exact position and velocity (or momentum) of each particle in a system would provide complete information of the system. Even when ignoring the fact that quantum mechanics tells us that this is impossible, two difficulties arise:

- There is too much information. A typical gas contains in the order of 10^{23} particles.
- The system is sensitive to tiny perturbations of the initial conditions.

Because of this, statistical models are used to describe the state of the system.

4-2 Kinetic Theory of Gases

The derivations of the kinetic theory of gases equations taken in this section follow the lectures of Richard Feynman [38], [39] and graduate level university readers on statistical physics [4], [37], [17].

4-2-1 Position, Velocity, Momentum & Phase Space

A classical gas is considered, as mentioned in section 4-1.
At each point in time every particle in the system has a position coordinate q_α and a momentum p_α for each dimension, where $\alpha \in \{1, 2, \ldots, D\}$. The state of the system is defined by the positions and momenta of all N particles of the system.
In 3-dimensional space every particle thus has 3 position coordinates and 3 momentum coordinates. A system of N particles can thus be described by $6N$ coordinates. All possible states of the system are represented by these $6N$ coordinates. This is called the phase space of the system.
In a D-dimensional world the phase space is thus given by:

$$\Gamma = \{q^{DN}, p^{DN}\}, \tag{4-1}$$

meaning that for N particles in D dimensions the phase space Γ consists of DN position coordinates and DN momentum coordinates [4].
It holds that the momentum, position and velocity of each particle in each dimension are expressed as:

$$\mathbf{p}_i = \begin{bmatrix} p_{i,1} \\ p_{i,2} \\ \vdots \\ p_{i,D} \end{bmatrix}, \qquad \mathbf{q}_i = \begin{bmatrix} q_{i,1} \\ q_{i,2} \\ \vdots \\ q_{i,D} \end{bmatrix}, \qquad \dot{\mathbf{q}}_i = \begin{bmatrix} \dot{q}_{i,1} \\ \dot{q}_{i,2} \\ \vdots \\ \dot{q}_{i,D} \end{bmatrix}, \tag{4-2}$$

where $p_{i,\alpha}$ indicates the momentum of particle i in direction α, $q_{i,\alpha}$ is the position coordinate of particle i in direction α and $\dot{q}_{i,\alpha}$ is the velocity of particle i in direction α.

4-2-2 Energy & Mass

Total Energy

The total energy of the system, also known as the Hamiltonian H, is the sum of the energies of all individual particles [40]. The energy of the particles consists of their kinetic energy, as well as the potential energy due to interactions between particles. The expression for the total energy is [40]:

$$H = \sum_{i=1}^{N} \left[\mathbf{p}_i^{\mathrm{T}} \left(2\underline{\underline{\mathbf{M}_i}} \right)^{-1} \mathbf{p}_i \right] + \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \Phi_{ij} \left(\mathbf{q}_i, \mathbf{q}_j \right), \tag{4-3}$$

where $\mathbf{p}_i$ and $\mathbf{q}_i$ are as given in (4-2) and $\underline{\underline{\mathbf{M}_i}}$ is the mass matrix of particle i.
$\mathbf{p}_i^{\mathrm{T}} \left(2\underline{\underline{\mathbf{M}_i}} \right)^{-1} \mathbf{p}_i$ is the kinetic energy of a single particle.

$\Phi_{ij}\left(\mathbf{q}_i, \mathbf{q}_j\right) = \Phi_{ij}\left(\left|\mathbf{q}_i - \mathbf{q}_j\right|\right)$ is the potential energy due to interaction between particles i and j, which only depends on the distance between the 2 particles. The fraction $\frac{1}{2}$ in front of the double summation in (4-3) is to avoid double counting of potential energy terms.

The expression given in (4-3) is the most general expression of the total energy, where it is not assumed that space is isotropic and that the particles are identical. In (4-3) it is not assumed that the interaction between the particles is negligible, as is done for ideal gases.

Mass Matrix

The mass matrix of particle i is of the form:

$$\underline{\underline{\mathbf{M}_i}} = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1D} \\ m_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ m_{D1} & \dots & \dots & m_{DD} \end{pmatrix}_i , \tag{4-4}$$

which is a square symmetric matrix of dimension D.
Since the particles considered are monatomic, they are seen as point particles. As a result, there are no "cross-directional effects". This means that a change in momentum in one direction (e.g. the $x-$direction) cannot effect a change in velocity in any other direction (e.g. the $y-$direction). As a result the mass matrix is a diagonal matrix.
It is shown in subsection 5-2-3 that the analog of this matrix in economic engineering is not necessarily diagonal. This corresponds with particles consisting of multiple atoms, where rotations can occur.

Each entry of the mass matrix gives the inertia of the particle in the corresponding direction. For clarity, the mass matrix of particle i in 3 dimensions is given:

$$\underline{\underline{\mathbf{M}_i}} = \begin{pmatrix} m_{xx} & 0 & 0 \\ 0 & m_{yy} & 0 \\ 0 & 0 & m_{zz} \end{pmatrix}_i . \tag{4-5}$$

For an isotropic gas it holds that all entries on the diagonal of (4-5) are equal. The mass matrix $\underline{\underline{\mathbf{M}_i}}$ in (4-3) can then be replaced by the scalar mass m_i, which is usually found in literature, where m_i is the entry for all non-zero elements of matrix (4-5).

Internal Energy

The kinetic theory of gases states that if the expectation value of the energy of a single particle is $\langle E_1 \rangle$, then the total energy as given in (4-3) is approximated as:

$$H \approx U = N \langle E_1 \rangle , \tag{4-6}$$

for large N. This approximation is known in literature as the thermodynamic limit [37], [41].

The expectation value of any variable X is calculated as follows:

$$\langle X \rangle = \sum_s X(s) \mathcal{P}(s) , \tag{4-7}$$

i.e. a summation over all possible states where $\langle X \rangle$ indicates the expectation value of the variable of interest, $X(s)$ is the value of X in state s and $\mathcal{P}(s)$ is the probability of state s occurring.

To summarize: the total energy of the system is approximated by multiplying the number of particles with the expectation value of the energy of a single particle.
The approximated total energy U is known in statistical physics and thermodynamics literature as the internal energy of the system.

4-2-3 Temperature

In his kinetic theory of gases lectures, Feynman states that thermal equilibrium between two systems exists when the average kinetic energy of the particles of the two systems is equal [38].
From this the term temperature is introduced and the two systems are said to have the same temperature. A direct relation thus exists between the temperature of a gas and the average kinetic energy.

A straightforward relation between the temperature and expectation value of the kinetic energy would be to define the two to be equal: (**This definition is not used in physics literature!**)

$$\Theta := \langle T_i \rangle, \tag{4-8}$$

where Θ is the temperature of the gas, T_i is the kinetic energy of a single (randomly chosen) particle and $\langle T_i \rangle$ is the expectation value of the particle's kinetic energy. From (4-3) it follows that the kinetic energy T_i of a randomly chosen particle i is:

$$T_i = \mathbf{p}_i^{\mathrm{T}} \left(2\underline{\underline{\mathbf{M}_i}} \right)^{-1} \mathbf{p}_i, \tag{4-9}$$

and where the thermodynamic limit states that $\langle T_i \rangle$ is a good approximation of T_i for all particles.

For historical reasons however, the temperature and kinetic energy are related via the equipartition theorem [38]:

$$\langle T_i \rangle = \frac{f}{2} k\Theta, \tag{4-10}$$

where f is the amount of degrees of freedom available to the particles of the system, k is the Boltzmann constant and Θ is the temperature of the gas. For monatomic gas particles in D-dimensions it holds that $f = D$, since monatomic particles have only translational degrees of freedom.
When particles made up of multiple atoms are considered, $f = D + r + \vartheta$, where r is the amount of rotational degrees of freedom and ϑ is the amount of vibrational degrees of freedom, which are only activated at high temperatures [42].

For an ideal gas, it holds that the potential energy terms of (4-3) are negligible. As such, it holds that $E_i = T_i$, i.e. the energy of a single particle is only its kinetic energy.
For an ideal gas, equations (4-6) and (4-10) can be combined. This yields:

$$U = N\frac{f}{2} k\Theta, \tag{4-11}$$

which is the well-knows expression for the internal energy of an ideal gas in thermodynamics.

The expressions given in (4-10) and (4-11) hold for all systems of which the mass matrices $\underline{\underline{\mathbf{M}_i}}$ of the particles are diagonal. As a result, the temperature and internal energy of non-isotropic systems and systems consisting of monatomic particles with different mass can also be calculated using these expressions.

Since the temperature is independent of the interactions between particles, the relation between the expected value of the kinetic energy and the temperature given in (4-10) holds for any real gas consisting of monatomic gas particles.

4-3 Statistical Physics & Thermodynamics

4-3-1 Maxwell's Demon, Entropy & The 2^{nd} Law of Thermodynamics

Maxwell's Demon

Using the definitions and macroscopic relations defined in section 4-2, Maxwell's Demon argument can now be analyzed.

Carnot [2] showed that it is possible to perform (mechanical) work W when heat flows from a high temperature reservoir Θ_H (or a system of high temperature) to a low temperature reservoir Θ_C (or a system of low temperature), as shown in Figure 4-1.

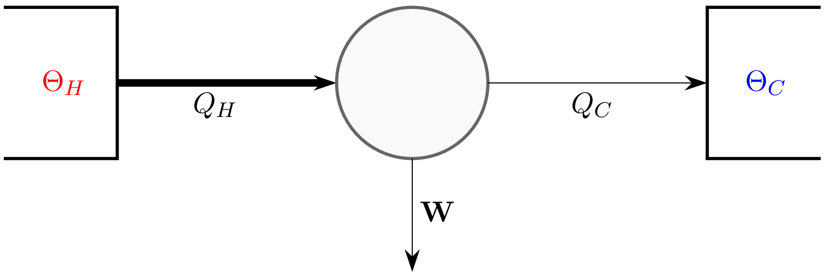


Figure 4-1: Simple view of the principles of a Carnot Engine [2].

In section 4-2 the temperature of a system was shown to be the expectation value of the kinetic energy of a single particle (4-10). Maxwell [43], [38] shows that when two systems of different temperature are connected, thermal equilibrium will eventually be reached.

Maxwell's Demon [3] is a thought experiment in which a small "demon" is able to infinitely fast open and close a partition separating two connected subsystems A and B, see Figure 4-2. No work is performed when operating the partition of negligible mass.
Allowing selective particles to pass from one subsystem to the other, the demon is able to collect particles with a large velocity in subsystem B and those with a small velocity in subsystem A, effectively increasing the temperature of subsystem B. The demon is thus able to create a temperature difference between two subsystems that were initially in thermal equilibrium. The temperature difference can then be used to perform (mechanical) work as explained by Carnot [2].

The Maxwell's Demon thought experiment is a violation of the 2^{nd} law of thermodynamics, since the demon is able to create a flow of heat from a low temperature to a high temperature without performing work.

Entropy

Consider a single molecule in a system consisting of two chambers (e.g. the system considered in Maxwell's Demon Figure 4-2 when the partition is opened). A molecule that can roam through both chambers A and B has twice as many possible positions as a molecule confined to a single chamber. If there were two molecules in the two-chamber setting, each molecule would have twice as many possible positions as it would have in the single chamber case,

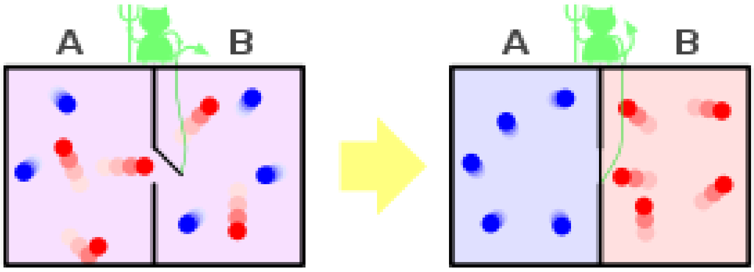


Figure 4-2: View of the Maxwell's Demon thought experiment, where a demon, shown in green, is able to distinguish fast (red) and slow (blue) particles. The demon can operate the partition without performing work and as such is able to create a temperature difference between two subsystems that were initially at thermal equilibrium. Maxwell's Demon is a violation of the 2^{nd} law of thermodynamics [3].

meaning the system as a whole as four times as many possible configurations. The amount of possible configurations (accessible states) available to a system of N molecules is 2^N times as high when two chambers are accessible compared to the single chamber case.

The entropy (4-12) of a system is defined as the logarithm of the number of states accessible to the system [4], [3].
In subsection 4-2-1 it was shown that the states accessible to the system "live" in the phase space (4-1), consisting of the particles positions and momentum. Since a one-to-one relation between momentum and velocity exists, one can also say that the accessible states of the system depends on the range of allowed particle positions and velocities.
The entropy thus increases when the volume increases or when the temperature increases, where the former corresponds with more positions accessible and the latter corresponds with more accessible momenta. Increasing the number of particles also means increasing the number of possible configurations of the system and thus the entropy.
For historical reasons the entropy S at fixed internal energy U, fixed system shape and volume V and fixed number of particles N is expressed as:

$$S = k \ln \left(\Omega \left(U, N, V \right) \right), \tag{4-12}$$

where Ω is the number of states available to the system and is called the multiplicity of the system.

Looking at the definitions of phase space (4-1) and entropy (4-12) it follows that the entropy is the dispersion of energy through a system. The energy dispersal increases for both a higher temperature (more momenta available) or by considering higher dimensional systems, allowing the phase space to expand due to more positional availability's.

Entropy is a macroscopic state variable. Clausius [16] found that for reversible processes it holds that:

$$\mathrm{d}S = \left(\frac{\mathrm{d}Q}{\Theta} \right)_{rev}, \tag{4-13}$$

where $\mathrm{d}S$ is the change in entropy of the system during the process, $\mathrm{d}Q$ is the incoming heat flow during the process and Θ is the temperature. A flow of heat is thus always accompanied by a change of entropy.

A heat flow carries an amount of entropy proportional to the quantity of heat flowing divided by the temperature at which the flow takes place. Hence a flow from a hot body to a cold body raises the entropy of the cold body more than it lowers the entropy of the hot one. A heat flow from a hot to a cold body thus raises the entropy of the universe.

Looking again at the Maxwell's Demon thought experiment, we can conclude that the demon violates the 2^{nd} law of thermodynamics because he is lowering the entropy of room A by a greater amount than it is increasing the entropy of room B. The total entropy of the universe would thus decrease [3].

4-3-2 State Counting & Partition Functions

To determine the entropy using its statistical definition as given in (4-12), the multiplicity Ω needs to be found. Following the definitions, the most straightforward way to determine the number of states available to a system is by taking the quotient of the (hyper)volume of the system in phase space by the volume of a single microstate. Heisenberg [4] showed that the volume of a single microstate in 1-D is bounded fundamentally by:

$$\Delta x \Delta p \geq h, \tag{4-14}$$

where h is Planck's constant.
The multiplicity for a D-dimensional system is then found to be [44]:

$$\Omega = \frac{1}{N!}\frac{1}{h^{DN}}\int_{\text{Vol}}\prod_{i=1}^{N}\mathrm{d}^D x_i \mathrm{d}^D p_i, \tag{4-15}$$

where the $\frac{1}{N!}$ prefactor is necessary if all N particles are indistinguishable.

Explicitly calculating Ω using (4-15) is usually a daunting task and can only be done in practice for extremely simple systems.
For complicated systems, physicists have developed a powerful tool called the "partition function" that is easier to calculate and is related to the multiplicity via the Laplace transform [19]:

$$Z(\beta) = \int \Omega(E)\exp(-\beta E)\,\mathrm{d}E, \tag{4-16}$$

where $Z(\beta)$ is the partition function, $\beta = \frac{1}{k\Theta}$ is the inverse temperature and E is a possible energy level the system can be in.
The multiplicity Ω is found by taking the inverse Laplace transform of the partition function.

Consider a system whose temperature Θ is imposed by an external heat bath, see Figure 4-3. Energy can freely be exchanged between the system and the heat bath, also known as the environment. At all times it holds that the sum of the energy of the system and the energy of the environment is constant.
The probability of finding the system in a certain microstate s of the phase space Γ is [4],[44]:

$$\mathrm{P}_s = \frac{\exp(-\beta E_s)}{Z(\beta)}, \tag{4-17}$$

where it must hold that $\sum_s \mathrm{P}_s = 1$, from which it is concluded that:

$$Z(\beta) = \sum_s \exp(-\beta E_s). \tag{4-18}$$

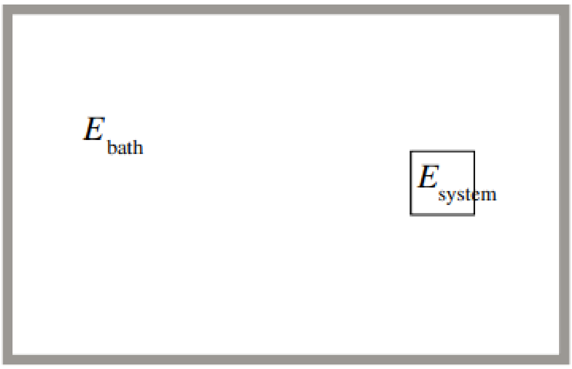


Figure 4-3: A system whose temperature Θ is imposed by an external heat bath [4].

The partition function given in (4-18) is known as the canonical partition function as is used for systems of constant temperature.

The average value of any physical quantity A which assumes the value A_s in state s can now be determined [4]:

$$\langle A \rangle = \frac{\sum_s A_s \exp\left(-\beta E_s\right)}{Z}. \tag{4-19}$$

The average energy, earlier defined to be the internal energy of the system, thus becomes:

$$U = \langle E \rangle = \frac{\sum_s E_s \exp\left(-\beta E_s\right)}{Z} = -\frac{\partial \ln\left(Z\left(\beta\right)\right)}{\partial \beta}, \tag{4-20}$$

4-3-3 Free Energies & Thermodynamic state variables

Now that the partition function has been defined, it is useful to define a quantity closely related function called the "Helmholtz free energy" F:

$$F\left(\beta\right) = -k\Theta \ln\left(Z\left(\beta\right)\right). \tag{4-21}$$

The Helmholtz free energy is a thermodynamic potential, just like the internal energy U.

Whenever a thermodynamic potential is a function of natural variables, all thermodynamic properties of the system can be obtained by partial differentiation [45]. The natural variables of a system are the variables that describe the current state of a system.
The natural variables of the internal energy U are the entropy S, the volume V and the number of particles N. It thus holds that:

$$U = U\left(S, V, N\right). \tag{4-22}$$

For any variation in the internal energy it holds that:

$$\mathrm{d}U = \left(\frac{\partial U}{\partial S}\right)_{V,N} \mathrm{d}S + \left(\frac{\partial U}{\partial V}\right)_{S,N} \mathrm{d}V + \left(\frac{\partial U}{\partial N}\right)_{S,V} \mathrm{d}N, \tag{4-23}$$

where S, V and N are all extensive variables. In general it holds for U that:

$$U = U\left(X_1, X_2, \ldots, X_{\mathcal{N}}\right), \tag{4-24}$$

with X_i the i-th natural variable for a system of $\mathcal{N}$ natural variables and all X_i being extensive. The variation of U then becomes:

$$\mathrm{d}U = \sum_{i=1}^{\mathcal{N}} \left(\frac{\partial U}{\partial X_i}\right)_{X_i \neq X_j} \mathrm{d}X_i. \tag{4-25}$$

For a regular thermodynamic system with only S, V and N as natural variables of U, the differentials shown in (4-23) are:

$$\left(\frac{\partial U}{\partial S}\right)_{V,N} = \Theta \quad ; \quad \left(\frac{\partial U}{\partial V}\right)_{S,N} = -P \quad ; \quad \left(\frac{\partial U}{\partial N}\right)_{V,S} = \mu, \tag{4-26}$$

where Θ is the temperature, P is the pressure and μ is the chemical potential as defined in classical thermodynamics.

Thermodynamic Potentials

As mentioned above in this subsection the Helmholtz free energy F is a thermodynamic potential, just like the internal energy U. These two are related via a Legendre transform with respect to Θ:

$$F = U[\Theta] = U - \Theta S. \tag{4-27}$$

As indicated by the square brackets, Θ is a natural variable of the Helmholtz free energy. This is often done since Θ is an easier variable to keep track of than its conjugate S. It thus holds that $F = F(\Theta, V, N)$.

A Legendre transform is thus a mathematical tool to convert the internal energy U into a thermodynamic potential that is convenient to work with. The free energy is the amount of energy available to the system to perform useful work.

For systems where pressure and temperature are convenient natural variables (e.g. chemical reactions) it is useful to take a Legendre transform w.r.t. Θ and P. We then define a new thermodynamic potential called the Gibbs free energy G, where $G = G(\Theta, P, N)$:

$$G = U[\Theta, P] = U - \Theta S + PV. \tag{4-28}$$

The conjugate variables of the natural variables of F can easily be determined using the total differential method shown in (4-26).

Different system configurations and constraints can lead to many different free energy expressions. The relevant free energy always gives the amount of energy available to the system to perform useful work.

4-3-4 Entropy & Information

In information theory the entropy of a random variable is a measure for the level of uncertainty inherent to the variable's possible outcomes. The entropy definition in information theory is closely related to that of statistical physics. Shannon [10] showed that the entropy of a random variable X with possible outcomes $x_1, x_2, \ldots, x_n$ and corresponding probability $\mathrm{P}(x_1), \mathrm{P}(x_2), \ldots, \mathrm{P}(x_n)$ in information theory is:

$$S(X) = -\sum_{i=1}^{N} \mathrm{P}(x_i) \log (P(x_i)). \tag{4-29}$$

Applying Shannon's definition of entropy to a thermodynamic system, it follows that the entropy is the result of the uncertainty in the momenta and positions of all particles in a system. A system of which the state of each particle is known exactly thus has zero entropy. From this it is concluded that the term TdS in the expressions for thermodynamic potentials such as (4-25) gives the incremental increase of energy "unaccounted for".

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Chapter 5

Deriving Macroeconomics from Microeconomics Using Economic Engineering & Statistical Physics

This chapter serves as the major contribution of this thesis, namely to derive economic variables that have a macroeconomic interpretation using the framework of statistical physics. Literature on statistical physics is given in Chapter 4.
Starting from the economic engineering theory of supply and demand, which is based on microeconomic theories, these macroeconomic variables are derived.

The variables derived in this chapter are given a macroeconomic interpretation in Chapter 6. The link between microeconomics and macroeconomics can then be made.

The macroeconomic variables derived from microeconomics in this chapter are the **GDP**, the **level of welfare**, the **amount of diversification**, the **disposable income per capita** and the **free economic rent**.

The macroscopic economic variables based on economic engineering derived in this chapter are not necessarily limited to macroeconomic applications. Any economic system containing an unmanageable amount of degrees of freedom can be modeled using the macroeconomic laws derived.

5-1 Motivation for Using the Statistical Physics Approach

In this section the motivation for linking microeconomics and macroeconomics using statistical physics is given.

Currently, the fields of microeconomics and macroeconomics are considered separately in economic literature, as explained in Chapter 3. In this chapter the link between microeconomics and macroeconomics in economic-engineering will be made using the tools of statistical physics.

In the introduction of Chapter 4 it was explained that a similar disconnect existed between Newtonian physics and thermodynamics.
Chapter 4 provided an overview as to how physicist were able to derive macroscopic thermodynamic relations by developing the field of statistical physics.

In this chapter it is shown how the economic-engineering model of supply and demand is used to derive macroscopic economic relations by following the steps taken in statistical physics.

> The definitions of- and the analogs between- the physical and economic terminology used in this section to argue the need for a statistical approach to link microeconomics and macroeconomics are given in Chapter 2.

In practice their are two main reasons why a statistical approach to link microeconomics and macroeconomics is useful. These are:

- A lack of computational power & information,
- Money is quantized, so transactions are always rounded.

Elaboration on these points is given next.

Computational Power & Information

For economic systems consisting of a small number of agents and goods, the total allocated economic rent can easily be found since the reservation prices and inventory levels of all agents can be monitored. For systems consisting of many agents and goods however, this becomes prohibitively difficult.
Monitoring the reservation prices and inventory levels of all agents means that every single transaction within an economy must be known. Even if the desire to do this were present, a lack of processing and computational power would render this approach practically impossible.

These difficulties are analogous to the ones encountered in physics. For systems consisting of many particles, the equations of motion for all particles cannot be solved due to a lack of computational power. Due to the lack of computational power, a statistical approach is proposed in physics literature to arrive at macroscopic thermodynamic relations [37]. The same argument is thus made for an economic system.

Quantization of Money

The exact knowledge of an agent's reservation price and quantity demanded cannot be gained from the information his transactions provide.
In practice money is quantized and transactions are rounded. In a regular business transaction one will often round a the amount to be paid to the nearest $\$0.01 = 1¢$, while stocks on the stock market and foreign currencies on the foreign exchange market are often rounded to even smaller fractions of a dollar. In this thesis it is assumed that $\$0.01 = 1¢$ is the smallest quanta of money.
If the actual reservation price of an agent is not an integer times ¢, there is no way to exactly know his reservation price for a good using the information of a transaction for 1 unit of that good.

Similarly, there is no way of knowing what the actual quantity demanded of an agent is, since goods can only bought in quantized units. An agent demanding 1.9 bottles of water per day will most likely register a transaction for 2 bottles of water on a given day, suggesting he demands 2 bottles of water. There is no way of knowing the actual quantity demanded of the agent using the information of the transactions the agent makes.

In physics, Heisenberg's uncertainty principle [46] states that it is not possible to know the exact values of both the momentum and position of a particle. The uncertainty in momentum Δp and uncertainty in position Δq are related as $\Delta p \Delta q \gtrsim \hbar$, where $\hbar$ is the reduced Planck's constant. This is the smallest "volume" in the phase space that can be measured.
In the economic engineering analogy, $\hbar$ has the units of money and can thus be seen as the economic engineering analog of ¢, the smallest possible measurable "volume" in the in the price-quantity space or the smallest unit of money. By taking the product of a price and an inventory level, it follows that a "volume" element in the price-quantity space must have the units of money.

Conclusion of the Motivation

In conclusion, the difficulties in modelling large economic systems are analogous to the difficulties found in physics to model systems of many particles.
The statistical approach taken in physics to derive the macroscopic laws of thermodynamics from microscopic laws of Newtonian mechanics can thus also be used to derive macroeconomic laws in economics from microeconomic laws. This is done using economic engineering [6].

5-2 Kinetic Theory Approach for Deriving Macroeconomics from Microeconomics

The steps taken in this section to derive the macroscopic economic relations are analogous to those of Section 4-2.
Analogies introduced in Subsections 5-2-1 and 5-2-2 are given in Table 2-2.

5-2-1 Agents, Goods & Resources

In the economic engineering theory of supply and demand [6], interactions are between a buyer of goods or resources and a seller. In this proposal, both buyers and sellers are often grouped together in the term economic agents. Any interaction is thus between two economic agents, or agents for brevity.

Examples of goods traded between a buyer and seller are shoes, mobile phones and bread. Examples of resources are labour, capital and land. For each good or resource, each agent has his own reservation price and quantity demanded (supplied).
Similarly, each particle has a component of momentum and velocity in each dimension. Unless otherwise stated, it is assumed that agents are the analogs of monatomic point particles. This means that surplus can only be obtained by trading goods or services.
In this thesis no economic analog of the rotation of a (diatomic) molecule around its axis considered, meaning that there is no economic engineering analog of the rotational kinetic energy.

Since **stock level** and **quantity demanded** are the economic engineering analogs of **position** and **velocity** respectively, it is proposed that a **dimension** in physical space corresponds with a **good** or **resource**.
An economy consists of many different goods and resources. In general we thus consider agents in a $D-$dimensional economy, meaning an economy of D different goods and resources, that corresponds with gas particles in $D-$dimensional space.

In total there are N **agents** within the economy, consistent with the kinetic theory of gases approach where there are N gas **particles** in the system.
These N particles need not necessarily be identical [38].
Since the inverse of **price elasticity of demand (supply)** is the economic engineering analog of the **inertia**, agents being identical means that the price elasticity of all agents is the same. In this thesis this is assumed to be the case.

5-2-2 Stock Level, Quantity Demanded, Price & Price-Quantity Space

At each point in time every agent has a stock level q_α and a reservation price p_α for each good or service α, where $\alpha \in \{1, 2, \ldots, D\}$.
In a D-dimensional economy, each agent thus has D stock level coordinates and D reservation price coordinates. An economy consisting of N agents can thus be described by $2ND$ coordinates. All possible states of the economy are represented by these $2ND$ coordinates and are called economic microstates.

Each possible economic microstate corresponds to a unique point in the **price-quantity space**, which is the economic engineering analog of the **phase space**. The price-quantity space consists of all possible values of stock level and reservation price of a good or resource. The price-quantity space in a D-dimensional economy is given by:

$$\Gamma = \{q^{DN}, p^{DN}\}, \tag{5-1}$$

meaning that for N agents and D goods and resources the phase space Γ consists of DN stock levels and DN reservation prices. This is consistent with the approach taken in statistical physics, as shown in Section 4-2.

The quantity demanded of good or service α, as known in the microeconomic theory of supply and demand, is simply the change in the stock level of good α w.r.t. time.
The reservation price, stock level and quantity demanded for each good of agent i are thus:

$$\mathbf{p}_i = \begin{bmatrix} p_{i,1} \\ p_{i,2} \\ \vdots \\ p_{i,D} \end{bmatrix}, \qquad \mathbf{q}_i = \begin{bmatrix} q_{i,1} \\ q_{i,2} \\ \vdots \\ q_{i,D} \end{bmatrix}, \qquad \dot{\mathbf{q}}_i = \begin{bmatrix} \dot{q}_{i,1} \\ \dot{q}_{i,2} \\ \vdots \\ \dot{q}_{i,D} \end{bmatrix}, \tag{5-2}$$

where $p_{i,\alpha}$ indicates the reservation price of agent i for good α, $q_{i,\alpha}$ is the stock level of agent i for good α and $\dot{q}_{i,\alpha}$ is the quantity demanded of agent i for good α.

The **reservation price** of an agent is the economic engineering analog of the **momentum** of a particle.

Analogies between Economics and Physics

Table 2-2 shows the analogs between economics and physics introduced in Subsections 5-2-1 and 5-2-2.

5-2-3 Economic Rent, GDP & Price Elasticity

The analogs between economics and physics introduced in this subsection are given in Table 2-3.

Economic Rent

Consistent with current economic engineering literature [6], the **allocated economic rent** of an agent is seen as the analog of the **Hamiltonian** of a particle. From this it follows that the total allocated economic rent of all N agents within the economy is the Hamiltonian of the entire system, as shown for a gas in (4-3). For our economic system this becomes:

$$H = \frac{1}{2}\sum_{i=1}^{N}\left[\mathbf{p}_i^{\mathrm{T}}\underline{\underline{\mathcal{E}_i}}\mathbf{p}_i\right] + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\Phi_{ij}\left(\mathbf{q}_i, \mathbf{q}_j\right), \tag{5-3}$$

where $\mathbf{p}_i$ and $\mathbf{q}_i$ are as given in (5-2) and $\underline{\underline{\mathcal{E}_i}}$ is the price elasticity of demand (supply) matrix of agent i.

The term $\frac{1}{2}\mathbf{p}_i^{\mathrm{T}}\underline{\underline{\mathcal{E}_i}}\mathbf{p}_i$ gives the **surplus** of agent i, which is the economic engineering analog of the **kinetic energy**.
$\Phi_{ij}\left(\mathbf{q}_i, \mathbf{q}_j\right) = \Phi_{ij}\left(\left|\mathbf{q}_i - \mathbf{q}_j\right|\right)$ is the benefit of ownership or potential surplus and depends on the difference in stock levels of the interacting agents. The **potential surplus** is the economic engineering analog of the **potential energy**. The fraction $\frac{1}{2}$ in front of the double summation in (5-3) is to avoid double counting of potential surplus terms.
In an economic system where the agents do not care about their inventory levels, the potential surplus is neglected. In a perfect market the quantity supplied for every good equals the demand for that good. Agents operating in a perfect market thus have no desire to hold an inventory, meaning that the potential surplus is negligible.
This is the economic engineering analog of an ideal gas, where the potential energy due to particle interaction is neglected.

In (5-3) it is assumed that no **economic rent dissipation** occurs. The total economic rent within the economy is thus equal to the allocated economic rent and constant over time. This is the analog of a mechanical system where no **damping** occurs. The relation between the total economic rent and the allocated economic rent was given in (3-21).

Price Elasticity Matrix

The price elasticity of demand (supply) matrix of agent i is of the form:

$$\underline{\underline{\mathcal{E}_i}} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \dots & \mathcal{E}_{1D} \\ \mathcal{E}_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathcal{E}_{D1} & \dots & \dots & \mathcal{E}_{DD} \end{pmatrix}_i , \tag{5-4}$$

which is a square symmetric matrix of dimension D.

In economics, complementary and substitution goods exist [15]. As a result, price changes in one good can cause the quantity demanded for another good to change. As a result, the off-diagonal terms of the price elasticity matrix (5-4) are non-zero.

Complementary goods have negative cross-price elasticity while, substitution goods have positive cross-price elasticity. Negative (positive) entries of the price elasticity matrix $\underline{\underline{\mathcal{E}_i}}$ thus correspond with complementary (substitution) goods.

Goods are called independent of they are neither substitutes nor complementary. The cross-price elasticity of independent goods is zero.
The price elasticity matrix in an economy consisting of 3 independent goods thus becomes:

$$\underline{\underline{\mathcal{E}_i}} = \begin{pmatrix} \mathcal{E}_{11} & 0 & 0 \\ 0 & \mathcal{E}_{22} & 0 \\ 0 & 0 & \mathcal{E}_{33} \end{pmatrix}_i . \tag{5-5}$$

If an agent has the same price elasticity for all goods, we say that the agent is **indifferent** to the goods. This is the analog of an **isotropic** gas, where the gas particle has the same inertia in each dimension.
The entries on the diagonal of the elasticity matrix are equal if the agent is indifferent to the goods. In that special case the price elasticity of demand (supply) of the agent can be given by a scalar, similar to how the mass of a particle is a scalar for isotropic gases.

GDP

In Subsection 4-2-2 it is shown that the Hamiltonian of an ideal gas system is approximated by multiplying the number of particles with the expectation value of the energy of a single particle (4-6). This approximation holds because the potential energy term in the expression for the Hamiltonian is neglected for ideal gases. The total Hamiltonian is thus equal to the total kinetic energy.

In macroeconomic literature, the total surplus of an economy is called the Gross Domestic Product (GDP) [14]. The GDP is thus proposed as the economic engineering analog of the total kinetic energy T in this thesis.

In this thesis a perfect market is assumed unless otherwise specified. A perfect market is identified as the analog of an ideal gas. For an economic system with a perfect market the **internal energy** U is thus the analog of the **GDP**.

The GDP is thus found to be:

$$H \approx \text{GDP} = N \langle H_i \rangle, \tag{5-6}$$

where $\langle H_i \rangle$ is the expectation value of the economic rent of a single agent in an economy consisting of N agents.

In general H is a random variable since the reservation prices $\mathbf{p}_i$ are random variables. For isolated economic systems in a perfect market H is constant and exactly equal to the GDP. For non-isolated systems the approximation given in (5-6) holds for large N, which is also assumed in the definition of a perfect market, see sections 2-2 and 2-3.

The approach taken here to determine the GDP matches the one taken in macroeconomic literature using the income approach [14] as shown in Subsection 3-2-2.

When not considering an entire economy but rather a large, arbitrary collection of economic agents the economic engineering analog of the internal energy is not called the GDP, but rather the total allocated economic rent. The approximation of (5-6) however still holds.

5-2-4 Level of Welfare

In Subsection 4-2-3 it was shown that the temperature of a gas is the product of the expectation value of the kinetic energy of a single particle and a constant.

Because the **surplus** is the economic engineering analog of the **kinetic energy**, it is argued that the economic engineering analog of the temperature and the expectation value of the surplus of a single agent should be related as:

$$\langle T_i \rangle = C \cdot \Theta, \tag{5-7}$$

where C is some constant. T_i is the surplus of a single (randomly chosen) agent and $\langle T_i \rangle$ is the expectation value of that agent's surplus.
Θ is found to be the economic engineering analog of the **temperature** and is called the **level of welfare**.

From (5-3) it follows that T_i is given as:

$$T_i = \frac{1}{2} \mathbf{p}_i^{\mathrm{T}} \underline{\underline{\mathcal{E}_i}} \mathbf{p}_i, \tag{5-8}$$

with T_i and T related as:

$$T = \sum_{i=1}^{N} T_i. \tag{5-9}$$

Similarly to how each degree of freedom in a gas adds a factor $\frac{1}{2}k\Theta$ to the expectation value of the kinetic energy, each economic degree of freedom adds factor of $\frac{1}{2}\Theta$ to the expectation value of the surplus. An economic degree of freedom is an independent method of acquiring a surplus available to an agent.

The level of welfare and surplus of a single agent are thus related as:

$$\langle T_i \rangle = \frac{f}{2}\Theta, \tag{5-10}$$

where f is the amount of economic degrees available to the agent. For agents that can only generate a surplus through trading goods, it holds that $f = D$.
Adding more degrees of freedom to an economy means that the (product) diversification of the economy is increased.

In Subsection 5-3-2 it is argued that the difference in the level of welfare between two economies is the driving force behind trade between economies.

Following the analogy with statistical physics, the level of welfare is independent of the potential surplus of the agents. This is also consistent with current economics literature. Although having an inventory does give an agent the potential for increasing his surplus in the future, it is only the actual surplus he enjoys that dictates his level of satisfaction [14].
In Subsection 4-2-3 it was mentioned that the temperature is independent of the potential energy of the particles. The temperature only depends on the kinetic energy of the particles. The analogy between the level of welfare and temperature is thus further highlighted.

Rearranging terms in (5-10) yields:

$$\Theta = \frac{2\langle T_i \rangle}{D} = \frac{\left\langle \mathbf{p}_i^{\mathrm{T}} \mathbf{q}_i \right\rangle}{D}, \tag{5-11}$$

where it is used that $\underline{\underline{\mathcal{E}_i}}\mathbf{p}_i = \mathbf{q}_i$.
The term $\mathbf{p}_i^{\mathrm{T}}\mathbf{q}_i$ is known in economic literature as the revenue.
The numerator of the right-hand side of equation (5-11) is thus the expected value of the revenue of a single agent.

Assuming a perfect market the level of welfare is also given as:

$$\Theta = \frac{\sum_{i=1}^{N} \left\langle \mathbf{p}_i^{\mathrm{T}} \mathbf{q}_i \right\rangle}{ND} = \frac{2\,\mathrm{GDP}}{ND} \tag{5-12}$$

where equations (5-6) and (5-9) were used to rewrite (5-11).
$\sum_{i=1}^{N} \left\langle \mathbf{p}_i^{\mathrm{T}} \mathbf{q}_i \right\rangle$ is the expected value of the total revenue of all agents within the economy, which is two times the GDP.
I propose to interpret the denominator of equation 5-12 as the “size of the economy” ND.

The level of welfare in an economic system increases when either the GDP (or total expected revenue) increases or when the economy decreases in “size”.

The meaning of the “size of the economy” used here is differs from the one commonly used in economic literature [14], where the GDP is meant by the size of the economy. In this thesis, the size of the economy is the product of the population size N and the number of distinct goods D.

The expression given for the level of welfare (5-11) holds for all economic systems without substitution and complementary goods, so whenever the price elasticity matrix given in (5-4) is diagonal.

Equation 5-12 gives the expression of the level of welfare derived from microeconomic variables. In Chapter 6 this is compared to its macroeconomic interpretation.

Analogies between Economics and Physics

Table 2-3 shows the analogs between economics and physics introduced in Subsection 5-2-3.

5-3 Trading and Migration between Economies

In this section the driving forces behind trade and migration between economies are analyzed. Furthermore the fundamental theorems of welfare economics are matched to thermodynamics.

Note: This section uses concepts from thermodynamics, welfare economics and macroeconomics and could have been placed earlier in the thesis or in Chapter 6. It is not placed earlier, because the it was not necessary in the derivations of Section 5-2. The concepts introduced in this section are however fundamental for the derivations of Section 5-4. As a result, I have chosen to include this section in this chapter. In Chapter 6 the findings of this section are connected to the findings of Section 5-4 and the link between microeconomics and macroeconomics is made.

The **level of welfare** Θ is the analog of the **temperature**. In thermodynamics literature a temperature difference is the driving force behind the exchange of heat between two (closed) systems [16].
The difference in the level of welfare of two economies is thus identified as the driving force behind trade between (closed) economies.

The **disposable income per capita** is found to be the analog of the **chemical potential**. A difference in the chemical potential is the driving force behind the exchange of particles between two open systems [16].
The difference in the disposable income per capita is thus identified as the driving force behind migration.

To define and give economic interpretation to the driving forces behind trade and migration, the economic engineering analog of **entropy** S, which is the **amount of diversification**, needs to be well-defined and interpreted economically.

In section 2-2 the definition of the amount of diversification S used in this thesis is given, while the full derivation and economic interpretation is given in Subsection 5-4-3.
Readers unfamiliar with the statistical interpretation of entropy may find it useful to read Subsection 5-4-3 before reading this subsection.

5-3-1 Theorems of Welfare & Laws of Thermodynamics

In this subsection the analogs between thermodynamics and the fundamental theorems of welfare economics are found.

The 1^{st} fundamental theorem of welfare economics states that in economic equilibrium, a set of complete markets, with complete information, and in perfect competition, will be Pareto optimal.
The 1^{st} fundamental theorem thus states the existence of a Pareto optimal macroeconomic equilibrium.

In thermodynamic equilibrium, the entropy is maximized [16] (for a given level of the internal energy of the system) and no macroscopic changes occur.
Figure 5-1 shows the internal energy-entropy curve of a thermodynamic system, indicating the stable equilibria of the system.

The 1st law of thermodynamics states that points on the US-curve are "Pareto optimal" since no particle can gain energy without another particle losing energy.
I propose to view the US-curve as the analog of the Pareto frontier [15] as known in economic literature, where it is used to find the optimal allocation of resources. Each point on the curve of the Pareto frontier is Pareto optimal.

The 1st fundamental theorem of welfare economics states the existence of macroeconomic equilibrium, where the amount of economic diversification is maximized.
The 1st law of thermodynamics ensures that this equilibrium is Pareto optimal.

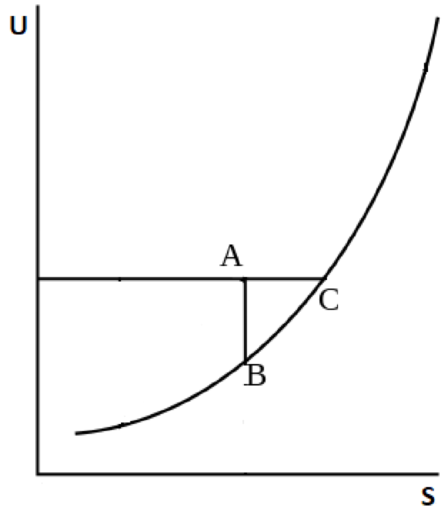

Figure 5-1: Energy, Entropy Curve of a thermodynamic system. For a macroeconomic system this curve shown the possible macroeconomic equilibria. The amount of diversification of an economic system that is initially in point A will spontaneously increase. The system will follow the path AC until macroeconomic equilibrium is reached.

The 2nd law of thermodynamics states that the entropy of a system cannot decrease spontaneously. A system that is not initially in equilibrium (point A in Figure 5-1 will spontaneously move along the "path" AC until thermodynamic equilibrium is reached. The entropy will thus increase until equilibrium is reached.

The 2nd fundamental theorem of welfare economics states that all Pareto optimal outcomes can in principle be reached through market mechanisms [11]. However, since agents are rational, the reached optimum state is the one that maximizes the total social welfare.
Furthermore it is assumed that agents are rational and look to maximize their utility, as argued in Subsection 3-1-2. As a result it is argued that the amount of diversification within an economy does not spontaneously decrease.

The 2nd fundamental theorem of welfare economics is interpreted as to be describing the macroeconomic "invisible hand"[47] that drives the economic system to equilibrium.
The macroeconomic invisible hand is discussed in more detail in the segment "Macroeconomic Invisible Hand" of section 6-1.

Both the 2nd fundamental theorem of welfare economics and the 2nd law of thermodynamics state that an optimal equilibrium will be reached. In this thesis it is argued that an economy that maximizes its social welfare is fully diversified, since it is Pareto optimal.

From the analogies between the 2nd fundamental theorem of welfare economics and the 2nd law of thermodynamics it is concluded that an economic system that is initially not in equilibrium will be "pushed" to a Pareto optimal equilibrium by increasing the amount of diversification.

5-3-2 Driving Force in Trade Between Economies

In this subsection the driving force behind trade between two economic systems is analyzed. The difference in the level of welfare (poverty) is identified as the driving force in trade between economies.

Next, consider economic systems that are closed but not isolated. The definitions of closed and isolated economic systems is given in Section 2-2. Closed economic systems that trade with one another are found to be analogous to two thermodynamic systems that can exchange heat, but do not exchange particles reaching a thermodynamic equilibrium.
Figure 5-2 shows a schematic view of two closed economies trading.

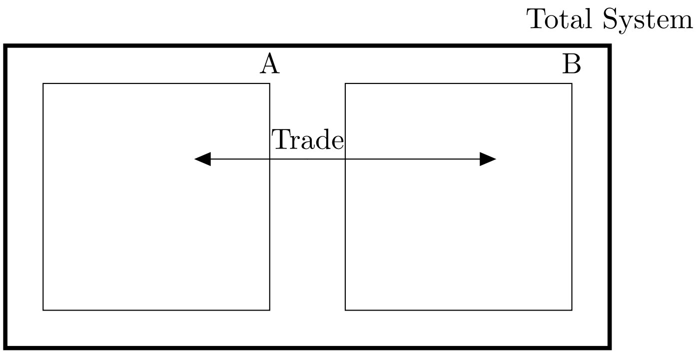

Figure 5-2: Schematic view of two economies A and B that are isolated from the rest of the world, but allowed to trade between themselves. Migration is not allowed. Eventually both economies reach macroeconomic equilibrium with the same level of welfare Θ.

In Subsection 5-3-1 it is argued that the amount of diversification of a economic system cannot spontaneously decrease and is maximized in macroeconomic equilibrium.
It holds for the amount of diversification of the total system (economy A and B) that:

$$S = S_A + S_B, \tag{5-13}$$

where $S = \ln\left(\Omega\left(U, N, D\right)\right) = \ln\left(\Omega\left(\text{GDP}, N, D\right)\right)$ as given in (5-41).

The definition of a closed economy implies that the number of agents is constant. I assume that the number of distinct goods D is also constant.
The amount of diversification in economies A and B thus only depends on the the GDP in each nation. Since the total allocated economic rent $H = H_A + H_B$ is fixed and $\text{GDP} = \langle H \rangle$ it holds for the total GDP that:

$$\text{GDP} = \text{GDP}_A + \text{GDP}_B = \text{constant}, \tag{5-14}$$

and

$$\Delta \text{GDP}_A = -\Delta \text{GDP}_B. \tag{5-15}$$

Combining equation (5-13) and the fact that S is maximized in macroeconomic equilibrium yields:

$$\frac{\partial S}{\partial \text{ GDP}_A} = \frac{\partial S_A}{\partial \text{ GDP}_A} + \frac{\partial S_B}{\partial \text{ GDP}_A} = 0, \tag{5-16}$$

where it was used that D and N are kept constant.
Substitution of (5-15) in (5-16) yields:

$$\frac{\partial S_A}{\partial \text{ GDP}_A} = \frac{\partial S_B}{\partial \text{ GDP}_B}. \tag{5-17}$$

The expression given in (5-17) is the condition for macroeconomic equilibrium between economies A and B. Since the total amount of diversification cannot decrease, economies that trade with one another eventually reach equilibrium.

The term $\frac{\partial S_i}{\partial \text{ GDP}_i}$ I propose to call the level of poverty of economy i.
In an economy with a low level of poverty, the amount of diversification barely increases with an increasing GDP. An economy with a high level of poverty on the other hand will gain a substantial amount of diversification options with an increase in the GDP.

The **level of poverty** of an economy is the analog of the **inverse temperature** β of a thermodynamic system:

$$\beta = \frac{1}{\Theta} = \frac{\partial S}{\partial \text{ GDP}}, \tag{5-18}$$

where Θ is the level of welfare, the analog of temperature.

Economies that have a differing level of welfare (poverty) are not in equilibrium. A net flow of economic rent is thus present. Once equilibrium has been reached the net flow of economic rent is zero and the level of welfare (poverty) is equal in both economies.

I propose to view the difference in the level of welfare (poverty) as the driving force of trade between economies.

How long it takes before macroeconomic equilibrium is reached is an unanswered question in this thesis. To answer this question the analogs between economics and statistical physics must be extended to the domain of non-equilibrium statistical mechanics [48].

5-3-3 Driving Force behind Migration

In this subsection the driving force behind migration is analyzed. The difference in the disposable income per capita is identified as the driving force behind migration.
It is shown that two economies that are allowed to trade with one another and have open borders eventually reach a macroeconomic equilibrium. This is found to be analogous to two thermodynamic systems that can exchange heat and particles reaching a thermodynamic equilibrium.

Consider economic systems that are open, meaning they are allowed to trade with one another and agents are allowed to migrate from one economic system to another.

Figure 5-3 shows a schematic view of two open economies A and B trading and exchanging agents. The two economic systems are open with respect to each other but isolated from the rest of the world, meaning that they do no trade with or have open borders for any other economy.
The definitions of isolated and open economic systems are given in Section 2-2.

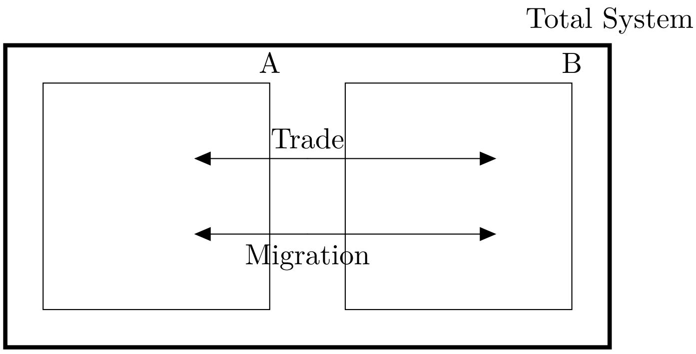

Figure 5-3: Schematic view of two economies A and B that are isolated from the rest of the world, but are allowed to trade between themselves and have open borders. Migration is thus allowed. Eventually both economies reach macroeconomic equilibrium with the same level of welfare Θ and disposable income per capita v.

In Subsection 5-3-1 it is argued that the amount of diversification of a economic system in maximized in macroeconomic equilibrium.
Subsection 5-3-2 argues that two economies that are allowed to trade will eventually reach the same level of welfare (poverty) upon reaching macroeconomic equilibrium.

As is the case for open economic systems the total amount of diversification S is the sum of the amount of diversification in systems A and B respectively (5-13) and with S as defined in (5-41).

In open economies the agents are allowed to migrate from one economic system to the other. Together, economies A and B form an isolated system. From the definition of isolated systems it follows that:

$$N = N_A + N_B = \text{constant}, \tag{5-19}$$

and

$$\Delta N_A = -\Delta N_B. \tag{5-20}$$

Equations (5-14) and (5-15) also hold for the open economies A and B.
It is assumed that the number of distinct goods D is a constant.

The amount of diversification S in each economy depends on the GDP and the number of agents of each economy.
The dependence of S on the GDP is shown in subsection 5-3-2.

Analogous to the derivation of the driving force in trade between two economies, as shown in subsection 5-3-2, maximization of the amount of diversification means that:

$$\frac{\partial S}{\partial N_A} = \frac{\partial S_A}{\partial N_A} + \frac{\partial S_B}{\partial N_A} = 0, \tag{5-21}$$

which results in:

$$\frac{\partial S_A}{\partial N_A} = \frac{\partial S_B}{\partial N_B}. \tag{5-22}$$

This means that the term $\frac{\partial S}{\partial N}$ is equal in economies that are in equilibrium. The change in the amount of economic diversification w.r.t. the change in population size of both economies must thus be equal in macroeconomic equilibrium.

In physics literature the chemical potential is defined as:

$$\frac{\mu}{\Theta} := -\frac{\partial S}{\partial N}, \tag{5-23}$$

where Θ is the temperature and μ is the chemical potential. Since it is known that the temperature Θ is equal in equilibrium, it must hold that the chemical potential in systems A and B is the same in equilibrium.
As is the case with gravitational potential, particles flow from a high to a low chemical potential and reach equilibrium when no potential gradient exists.

Following the analogy between economics and statistical physics, the economic engineering analog of the chemical potential μ of economies A and B must be equal in macroeconomic equilibrium.
Similar to how particles move from a system with a higher to a system with a lower chemical potential when a potential gradient exists, economic agents will move from an economy with a high "economic chemical potential" to a low "economic chemical potential", thus resulting in migration.
Economic literature [49] states that agents migrate for political, environmental, demographic, social and economic reasons. In this only the economic motivation for migration is considered.

In Subsection 3-1-5 it is mentioned that agents are assumed to be rational. It was then argued that the trading decisions agents make are based on the principle of maximizing their utility or minimizing their running costs. Rational agents were found to maximize their surplus T.
In an open economy agents are allowed to both trade and migrate. A rational agent seeking to maximize his utility will not necessarily seek to maximize his surplus. Instead, he will seek to maximize his disposable income.
The disposable income is the part of an agents' income that is available for spending after his fixed expenses have been deducted [14].

A rational agent will thus seek to maximize his **disposable income** and migrate to the economy that allows him to do so. This is analogous to a particle looking to minimize its **chemical potential**.
Once the disposable income per capita of both economies is equal, macroeconomic equilibrium has been reached and the net migration of agents is zero.

> The disposable income per capita is the proposed analog of the negative of the chemical potential of a thermodynamic system. The difference in the disposable income per capita of the economies is identified as the driving force behind migration.

The disposable income per capita is given the symbol υ, where $\upsilon_A = \upsilon_B$ in equilibrium. From (5-22) it follows that:

$$\frac{\upsilon}{\Theta} := \frac{\partial S}{\partial N}, \tag{5-24}$$

where (5-24) is the economic engineering analog of (5-23).

5-4 Statistical Approach for Deriving Macroeconomics from Microeconomics

In this section macroeconomic variables are derived from microeconomics using a statistical approach. This is done by deriving the distribution of economic microstates ϱ of an economic system. Once ϱ is found, the relevant macroeconomic variables of the economic system can be derived.

The constraints on the economy determine the form of the distribution of microstates in the price-quantity space. This distribution is called the probability density function.

In Subsection 5-4-1 the probability density function is introduced.

In Subsection 5-4-2 the conditions the probability density function must adhere to in order to achieve macroeconomic equilibrium are given.

In Subsection 5-4-3 the probability density function is derived for isolated economic systems. The analog of the entropy S is also derived in this subsection. This is the major contribution of this thesis for the field of economic engineering.

In Subsection 5-4-4 the probability density function for closed economic systems is derived. The analogs of free energy F and the heat capacity $\mathcal{C}$ are derived and given an economic interpretation in this subsection.

In Subsection 5-4-5 the probability density function for open economic systems is derived. Furthermore, the expected population size of an economy is derived, as well as how this is related to the disposable income per capita v and the level of welfare Θ.

5-4-1 Probability Density Function for Economic Systems

The probability density function gives the distribution of accessible economic microstates in the price-quantity space. The definitions of the price-quantity space and economic microstates are given in Section 2-2.

> The determination of the probability density function is the fundamental problem of the statistical approach. Depending on the constraints on the economic system, a suitable probability density function is found, from which the macroeconomic state of the economy is found.

To set the stage for deriving a probability density function, some background and mathematical prerequisites are given in the current and following subsection.

In Subsection 3-1-5 the expressions found in literature for the time evolution of the reservation prices and inventories are given. It is shown in Subsection 5-2-2 that $p_{i,\alpha}$ indicates the reservation price of agent i for good α while $q_{i,\alpha}$ is the stock level of agent i for good α.
To find the time evolution of the prices and inventories of all N agents in an economic system consisting of D goods, Hamilton's equations must consist of $2ND$ first order equations:

$$\dot{q}_{i,\alpha} = \frac{\partial H}{\partial p_{i,\alpha}}, \tag{5-25}$$

$$\dot{p}_{i,\alpha} = -\frac{\partial H}{\partial q_{i,\alpha}}, \tag{5-26}$$

where H is the total allocated economic rent as given in (5-3).

Hamilton's equations give the time evolution of the economic system in the price-quantity space for any given initial state, where the initial state is a single point in the price-quantity space from where the trajectory starts [4].

If the reservation prices and stock levels of all agents are known precisely, the exact state of the economic system can be determined. Since agents can only acquire a surplus through trade, the prices and stock levels provide sufficient information to specify the state of the economic system.
Subsection 5-1 argues however that knowing the reservation price and inventory level of all agents becomes prohibitively difficult for large N and D. Furthermore, solving these $2ND$ differential equations requires a prohibitive amount of computational power.

Instead, the probability of finding an economic system in a particular economic microstate $(\mathbf{q}, \mathbf{p})$ is of interest. The goal is thus to find a distribution function that shows how the possible microstates are distributed in the price-quantity space. This distribution function is called the "probability density function" and and represents the "density" of the probability distribution in the price-quantity space.

The probability density function ϱ shows the distribution of the economic microstates in the price-quantity space and the microstates consists of reservation prices and inventory levels of all agents. It can thus be concluded that ϱ must be a function of the inventory levels and the reservation prices. This is also consistent with the statistical physics analogy.
The probability density function must also depend on time, since the time evolution of ϱ provides information on how the economy evolves over time. It must thus hold that:

$$\varrho = \varrho(\mathbf{q}, \mathbf{p}; t), \tag{5-27}$$

where $\mathbf{q}$ and $\mathbf{p}$ are:

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \vdots \\ \mathbf{q}_N \end{bmatrix} \quad , \quad \mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_N \end{bmatrix}, \tag{5-28}$$

with $\mathbf{q}_i$ and $\mathbf{p}_i$ as given in (5-2). The price vector $\mathbf{p}_i$ and inventory level vector $\mathbf{q}_i$ are thus both of length DN.

The probability of finding an economic system at time t where the stock levels are inside a DN-dimensional "box" $\mathrm{d}^D\mathbf{q}_1\mathrm{d}^D\mathbf{q}_2 \ldots \mathrm{d}^D\mathbf{q}_N \equiv \mathrm{d}^{DN}\mathbf{q}$ and the reservation prices in a similar box $\mathrm{d}^D\mathbf{p}_1\mathrm{d}^D\mathbf{p}_2 \ldots \mathrm{d}^D\mathbf{p}_N \equiv \mathrm{d}^{DN}\mathbf{p}$ is:

$$\varrho(\mathbf{q}, \mathbf{p}; t)\, \mathrm{d}^D\mathbf{q}_1\mathrm{d}^D\mathbf{q}_2 \ldots \mathrm{d}^D\mathbf{q}_N \cdot \mathrm{d}^D\mathbf{p}_1\mathrm{d}^D\mathbf{p}_2 \ldots \mathrm{d}^D\mathbf{p}_N = \varrho(\mathbf{q}, \mathbf{p}; t)\, \mathrm{d}^{DN}\mathbf{q}\mathrm{d}^{DN}\mathbf{p}. \tag{5-29}$$

The mathematical form and dependency of the probability density function is analyzed in Subsection 5-4-2.

In this thesis the proposed solution to the difficulties encountered in the general equilibrium approach taken in current economic literature is finding a suitable probability density function based on the constraints on the economic system.

To find the economic equilibrium using the general equilibrium theory requires a lot of computational power. This is discussed in Subsection 3-2-4.
The problem of requiring a high computational power was also encountered in physics. In Subsection 5-1 the motivation for a statistical approach to bypass this problem is given.

Using a statistical approach, only information about the constraints on the economy is required to find the state of the economy in macroeconomic equilibrium.

5-4-2 Probability Density Function Prerequisites for Macroeconomic Equilibrium

In this subsection the conditions the probability density function must adhere to in order to achieve macroeconomic equilibrium are given.
Furthermore it is shown that the probability density function can be used to determine the expected value of macroeconomic state variables.

Macroeconomic Equilibrium

In this thesis an economic system is said to be in a macroeconomic equilibrium when no economic growth occurs and the economy is autonomous.
An autonomous economic system is defined as an economic system where no exogenous economic rent source is present. This means that no financial aid is received.

According to economic literature, economic growth is an autonomous or endogenous increase in the GDP [14], while financial aid is an example of of an exogenous increase in the GDP.

Subsection 5-2-3 argues that the GDP is the economic engineering analog of the internal energy U of a thermodynamic system. The internal energy of a thermodynamic system is a type of thermodynamic potential. The thermodynamic potential of a system in equilibrium remains unchanged. The equilibrium conditions of statistical physics can thus also be applied to an economic system.

Following the analogies found between economics and statistical physics, the equilibrium conditions for thermodynamic equilibrium in physics are applied to economic systems for macroeconomic equilibrium.

For a physical system in equilibrium, the probability density function must remain constant in time, since a probability density function that is increasing in time corresponds with an increase in the phase space volume, meaning an increase in the total energy [4].
The same argument is made for an economic system. The probability density function must remain constant in time for macroeconomic equilibrium, since the total economic rent is constant and time-independent.

Mathematical Description of Macroeconomic Equilibrium

The conditions for macroeconomic equilibrium are found mathematically using Liouville's theorem.

Liouville's theorem in statistical mechanics describes the time evolution of the phase space probability density function $\varrho(\mathbf{q}, \mathbf{p}; t)$. The theorem can thus also be used to describe the time evolution of the price-quantity probability density function of the economic system.
The Liouville equation states that:

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = \frac{\partial \varrho}{\partial t} + \sum_{i=1}^{N} \sum_{\alpha=1}^{D} \left(\frac{\partial \varrho}{\partial q_{i,\alpha}} \dot{q}_{i,\alpha} + \frac{\partial \varrho}{\partial p_{i,\alpha}} \dot{p}_{i,\alpha} \right) = 0, \tag{5-30}$$

meaning that the probability density function is constant along any trajectory in the price-quantity space.
The second term in (5-30) is recognized as the Poisson bracket of ϱ and H using Hamilton's equations ((5-25) & (5-26)):

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = \frac{\partial \varrho}{\partial t} + \{\varrho, H\} = 0. \tag{5-31}$$

A perfect market is assumed, so no economic rent is dissipated within the economy. Also, no external source or sink of economic rent is present. Expression (3-21) shows that the total economic rent E is thus equal to the total allocated economic rent H, meaning that H is a constant. Thus, the total allocated economic rent H does not depend explicitly on time.
Receiving financial aid is a form of an economic rent source, while giving out financial aid is an example of a sink.

An important property of the Poisson bracket is that if $\varrho = \varrho(H)$, the Poisson bracket disappears [46]:

$$\{\varrho(H), H\} = 0. \tag{5-32}$$

Combining (5-31) and (5-32) under the conditions that $\varrho = \varrho(H)$ and H does not explicitly depend on time yields:

$$\frac{\partial \varrho}{\partial t} = 0, \tag{5-33}$$

meaning that the ensemble is stationary.

An ensemble is a term used in statistical physics for a collection of systems which are macroscopically identical. However, many microscopic configurations can lead to the same macroscopic states. A stationary ensemble thus means that from a macroscopic point of view, the system is in equilibrium.

Economically a stationary ensemble means that the economic system is in macroeconomic equilibrium. The condition that the total economic rent does not explicitly depend on time is clear now, since an economy receiving financial aid is not in equilibrium, similar to how a driven mechanical system is not in equilibrium.

To achieve equilibrium, the probability density function ϱ must thus lead to a stationary ensemble and be of the form

$$\varrho(\mathbf{q}, \mathbf{p}; t) = \varrho(\mathbf{q}, \mathbf{p}) = \varrho(H). \tag{5-34}$$

For macroeconomic equilibrium it must hold that the probability density function is independent of time and a function of the total allocated economic rent.

Finding Macroeconomic State Variables in Macroeconomic Equilibrium

The density function is used to find macroeconomic state variables in macroeconomic equilibrium. A macroeconomic quantity $f = f(\mathbf{q}, \mathbf{p})$ that depends on the reservation prices $\mathbf{p}$ and inventory levels $\mathbf{q}$ of the agents within an economy is found by taking its expected value:

$$\langle f \rangle := \iint f(\mathbf{q}, \mathbf{p})\, \varrho(\mathbf{q}, \mathbf{p})\, \mathrm{d}^{DN}\mathbf{q}\, \mathrm{d}^{DN}\mathbf{p}, \tag{5-35}$$

where examples of f are the total allocated economic rent H, the total surplus T and the population size N.

5-4-3 Constant Economic Rent Ensemble & Entropy

In this subsection the probability density function of isolated economic system is derived. Furthermore, the statistical interpretation of entropy for economic systems is given.

The economic engineering analog of the statistical interpretation of **entropy** is the main contribution of this thesis and is called the **amount of diversification**. The amount of diversification for an isolated economic system is derived in this subsection.

Subsection 5-4-2 argues that for economies in equilibrium, the probability density function ϱ must be a function of the total allocated economic rent and no economic rent sources or sinks may be present.

Furthermore, it is argued that an economic ensemble is a collection of all possible economic microstates that lead to a certain macroeconomic state.
Thus, depending on the macroeconomic state and constraints of the economy an appropriate probability density function is chosen. This probability density function shows how the possible economic microstates of the identified ensemble are distributed in the price-quantity space.

Isolated & Perfect Economy

An isolated and stationary economic system is considered.
Stationary means that the economy is in equilibrium as argued in Subsection 5-4-2 and will remain in equilibrium.
Isolated means that the economy does not interact with the rest of the world. No agents enter or leave the economy, there is no growth in the population, no goods are being traded with foreign economies and no financial aid is received. The total allocated economic rent H of the economy is thus constant. Economists can consider this to be an autonomous system or a Robinson Crusoe economy of many agents.

Finally, it is assumed that the economy is in perfect competition [15]. In economic literature a market in perfect competition has a large number of agents who are all rational. There are also no transaction costs.

A perfect market has similarities with an ideal gas. There are a large number of particles and analytical mechanics literature [9] states that the principle of stationary action applies

to each particle. The analogy between rational agents and the principle of stationary action as used in current economic engineering literature was given in 3-1-5.
There is no dissipation of energy due to particle interaction in an ideal gas. In economic literature transaction costs are a form of economic rent dissipation. A perfect market has no transaction costs, consistent with the analogy to an ideal gas.

Distribution of Economic Microstates

In statistical physics literature the *postulate of equal a priori probabilities* states that for a system of fixed energy, each configuration with that energy level is equally likely to be visited by the phase space trajectory in the course of time.
Applied to an isolated and stationary economic system, this postulate states that every economic microstate in the price-quantity space at a particular value of the allocated economic rent H is equally likely to occur.

Although the total allocated economic rent of an isolated economic system is preserved, there is no way of knowing the exact value of H. In Subsection 5-1 it was argued that knowing the exact reservation price and inventory level of all agents is practically impossible.
The analogy to Heisenberg's uncertainty principle was made in the argument of Section 5-1. The uncertainty principle holds not only for uncertainties in momentum and position, but for any pair of conjugate variables [46].
Thus, the same principle also prohibits measuring the exact value of the Hamiltonian of a particle at a specified moment in time. Naturally, the total Hamiltonian of the system cannot be known exactly at a specified moment in time.

Following the analogy of Heisenberg's principle, the economic interpretation is that the exact value of the total allocated economic rent cannot be known. Although H is said to be constant for an isolated economy, its exact value is unknown.
For the total allocated economic rent H it holds that $\mathcal{H} \leq H(\mathbf{q}, \mathbf{p}) \leq \mathcal{H} + \Delta\mathcal{H}$, where $\Delta\mathcal{H}$ is the uncertainty in the total economic rent.

Using the postulate of equal a priori probabilities, the economic microstates accessible to the economy of allocated rent H are uniformly distributed inside a infinitesimal hyper-spherical shell of dimension $2ND$ and width $\Delta\mathcal{H}$. In Figure 5-4 the infinitesimal hyper-spherical shell containing the possible microeconomic states is shown for an economic system consisting of a single agent ($N = 1$) and a single good ($D = 1$), where the hyper-spherical shell reduces to a ring of width $\Delta\mathcal{H}$. Since the economic microstates are uniformly distributed within the hyper-spherical shell, the probability density function must be of the form [17]:

$$\varrho(\mathbf{q}, \mathbf{p}) = \begin{cases} C & \mathcal{H} \leq H(\mathbf{q}, \mathbf{p}) \leq \mathcal{H} + \Delta\mathcal{H} \\ 0 & \text{otherwise} \end{cases}, \tag{5-36}$$

where C is a to be determined constant.
Expression (5-36) shows indeed that the economic microstates are uniformly distributed.

Counting Economic Microstates

Next the possible economic microstates of the economic system of constant allocated rent are counted.

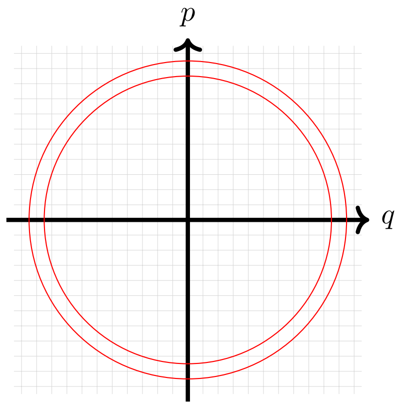

Figure 5-4: Schematic view of the infinitesimal shell of width $\Delta\mathcal{H}$ containing the allowed economic microstates of the system for an economy with one agent and one good. The allowed microstates are uniformly distributed within the shell.

I propose to call the **amount of possible economic microstates** the **amount of options available for economic diversification**.

Since ϱ is a probability distribution function, it must be normalized:

$$1 = \iint \varrho\left(\mathbf{q},\mathbf{p}\right)\, \mathrm{d}^{DN}\mathbf{q}\, \mathrm{d}^{DN}\mathbf{p} = C \iint_{\mathcal{H}\leq H(\mathbf{q},\mathbf{p})\leq\mathcal{H}+\Delta\mathcal{H}} \mathrm{d}^{DN}\mathbf{q}\, \mathrm{d}^{DN}\mathbf{p}\,, \tag{5-37}$$

where the integral is performed over the region $\mathcal{H} \leq H\left(\mathbf{q},\mathbf{p}\right) \leq \mathcal{H} + \Delta\mathcal{H}$. The right-hand side of (5-37) yields:

$$C\left[\tilde{\omega}\left(\mathcal{H}+\Delta\mathcal{H}\right) - \tilde{\omega}\left(\mathcal{H}\right)\right] = 1. \tag{5-38}$$

In physics literature $\tilde{\omega}\left(\mathcal{H}\right)$ is known as the volume of the phase space of region $H\left(\mathbf{q},\mathbf{p}\right) \leq \mathcal{H}$ and $\tilde{\omega}\left(\mathcal{H}+\Delta\mathcal{H}\right)$ is the phase space volume of region $H\left(\mathbf{q},\mathbf{p}\right) \leq \mathcal{H}+\Delta\mathcal{H}$. The phase space volume of the hyper-spherical shell is thus simply the difference between the two, which is the inverse of C.

Using the phase space volume of the hyper-spherical shell, the amount of microstates within the shell is determined. Physics literature [4] argues that microstates cannot be distinguished due to Heisenberg's uncertainty principle if they are located too close to one another in phase space. It is thus proposed to divide phase space in small boxes of hypervolume $\left(\Delta p \Delta q\right)^{DN}$, where it must hold that $\Delta p \Delta q \gtrsim \hbar$.
Furthermore, for gases consisting of identical gas particles, swapping the position of two particles does not change the microstate, since swapping identical particles does not lead to a identifiable change in the configuration of the system. A term of $N!$ is taken to account for the indistinguishability when swapping particles.

The amount of distinguishable microstates contained inside the phase space volume enclosed by the hyper-spherical shell is called the multiplicity $\Omega\left(\mathcal{H}\right)$ and is given as:

$$\Omega\left(\mathcal{H}\right) = \frac{\tilde{\omega}\left(\mathcal{H}+\Delta\mathcal{H}\right) - \tilde{\omega}\left(\mathcal{H}\right)}{N!\,\hbar^{DN}}, \tag{5-39}$$

where the numerator indicates the hypervolume of the hyper-spherical shell in phase space and $\hbar^{DN}$ is the hypervolume of a box containing a single microstate.

The multiplicity is a function of the total energy of the system. If the Hamiltonian of the system increases, the amount of microstates Ω will increase as well.
Although rarely mentioned in physics literature, equation (5-39) shows that the multiplicity depends on N and D as well. This statement is redundant in physics because for an isolated system the number of agents N remains constant and $D = 3$ in physical systems. As a result, both N and D are constant and do not affect the multiplicity.

The derivation of the multiplicity is given an economic interpretation as well.
The "volume" of the hyper-spherical shell in the price-quantity space is the total "hyper-value" or wealth within an economy. For the simplified economy of 1 agent and 1 good Figure 5-4 the "volume" of the hyper-spherical shell reduces to the area of a strip, indicating the total value measured in \$. For an economic system with $2ND$ degrees of freedom, the unit of the "hyper-value" is $\$^{ND}$.

As mentioned in Section 5-1, economic transactions are always quantized. Regular transactions are rounded to the nearest integer of \$0.01 = 1¢, which is measured in \$. Economic microstates for an economy with $N = 1$ and $D = 1$ are thus automatically separated by at least one unit of ¢, the economic engineering analog of $\hbar$. For an economic system with $2ND$ degrees of freedom, the numerator of (5-40) thus has the units of $\$^{ND}$. This is the unit of the denominator as well, since the amount of economic microstates should be a dimensionless quantity.

For an isolated economy of constant total economic rent H, where $\mathcal{H} \leq H(\mathbf{q}, \mathbf{p}) \leq \mathcal{H} + \Delta\mathcal{H}$ the "volume" of the price-quantity space of allowed economic microstates $\tilde{\omega}(\mathcal{H})$ is a hyper-spherical shell of width $\Delta\mathcal{H}$, consistent with the physical analogy.

Next, the multiplicity is found for economic systems and given economic interpretation. The multiplicity is the amount of economic microstates available to the economic system:

$$\Omega(\mathcal{H}) = \frac{\tilde{\omega}(\mathcal{H} + \Delta\mathcal{H}) - \tilde{\omega}(\mathcal{H})}{N!\ ¢^{DN}}. \tag{5-40}$$

In (5-40) it is assumed that agents are identical and indistinguishable, leading to the $N!$ term. This is analogous to physics literature. If every agent is unique, this term can be omitted.

If the agents of the economy can be split into groups where two agents from the same group are indistinguishable but agents from different groups can be differentiated, a suitable constant must replace the $N!$ term in (5-40). Groups of agents can be made for example on the basis of their price elasticity of demand. This is an analog of a mixed gas consisting of particles with different masses.

In economic sense I propose to view the **multiplicity** as the amount of **options available for diversification**. It shows the amount of distinguishable ways the economic rent can be allocated within an economy.
Finding the amount of options available for diversification thus comes down to determining the amount of allowed economic microstates. The different ways of economic diversification are discussed in the segment "Economic Diversification" of Subsection 5-4-3.

For an ideal gas, the numerator of equation (5-39) is calculated by determining the surface area of a hyper-spherical shell in DN dimensions [4]. This result contains the term L^{DN}, where L^D is the D dimensional volume of the system.
To find the numerical value of the numerator of equation (5-40) an economic engineering

analog of the volume is needed. This analog is not found in this thesis.
As a result no quantitative calculations relating to economic engineering analog of the entropy can be made in this thesis.

Economic Diversification

In general there are three ways in which the options available for diversification Ω can increase, namely with an increase in the economic rent, with an increase in the number of agents and with an increase in the number of goods within the economy.

As the total allocated economic rent increases, the amount of configurations of $\mathbf{q}$ and $\mathbf{p}$ that lead to the corresponding value of $\mathcal{H}$ increases. The amount of options in which the economic rent can be distributed thus increases.
For an isolated economy the total allocated economic rent is constant and the amount of options available for diversification does not increase due to an increase of the economic rent.

If N increases, the economic rent must be distributed among more agents. The amount of ways in which the rent can be distributed increases.
For an isolated economy the number of agents is constant and the amount of options available for diversification does not increase due to population growth.

Economies with more product diversification, so a higher number of distinct goods D, have more options available for diversification as well. A higher product diversification means agents can trade more distinct goods and can diversify their inventory more.
In Subsection 3-1-2 it was argued that an agent chooses to follow the "trading path" that maximizes his utility. In an economy with more product diversification the number of "paths" available to the agent increases.

The difference between the statistical approach proposed here and the general equilibrium approach taken in economic literature to find the macroeconomic state of the economy is thus clear.
In the statistical approach the number of ways the economic rent can be distributed to reach a macroeconomic state are counted. Only H, N and D are needed to do this.
Subsection 3-2-4 shows that in the general equilibrium approach all information about the stock level and reservation prices of the agent is needed. Also, the equilibrium state can only be found by brute force calculation.

Example of Economic Diversification Increase

To illustrate the various ways of increasing the economic diversification an example is given. Consider an exchange economy consisting of $N = 2$ agents. Initially there is only 1 distinct good (D=1) available, namely apples. In this example the allocated rent of an agent is the number of apples he owns. The total allocated economic rent $H = \mathcal{H}$ of the economy is thus $\mathcal{H} = H_1 + H_2$ and is equal to the number of apples available within the entire economy.

Initially there are 4 apples, so $H = 4$. If (a_1, a_2) shows the number of apples agents 1 and 2 own respectively, the apples can be distributed as follows: (4,0), (3,1), (2,2), (1,3), (0,4). There are thus 5 options available for diversification.

The first way the amount of options available for diversification Ω can increase is due to an increase in the total allocated economic rent H. If the number of apples within the economy is increased to 5, the apples can be distributed as follows: (5,0), (4,1), (3,2), (2,3), (1,4), (0,5). There are now 6 options available for diversification.

Next, consider population growth. If the economy with 4 apples now has $N = 3$ agents the apples can be distributed as follows: (4,0,0), (3,1,0), (3,0,1), (2,2,0), (2,1,1), (2,0,2), (1,3,0), (1,2,1), (1,1,2), (1,0,3), (0,4,0), (0,0,4), (0,3,1), (0,1,3), (0,2,2). There are now 15 options available for diversification.

The final way the options of available for diversification can increase is due to product diversification. Consider the economy with $N = 2$ agents and $H = 4$ fruits. There are now both apples and pears; 2 of each. For simplicity agents have no preference between apples and pears so the allocated economic rent due to ownership of an apple and pear is equal.
Let (a_1, p_1, a_2, p_2) be the number of apples and pears for agents 1 and 2 respectively. The fruits can now be distributed as follows: (2,2,0,0), (2,1,0,1), (2,0,0,2), (1,2,1,0), (1,1,1,1), (1,0,1,2), (0,2,2,0), (0,1,2,1), (0,0,2,2). There are now 9 options available for diversification. If apples and pears cannot be distinguished, Ω reduces to the 5 options available in the original scenario.

The example given here illustrates that the amount of options available for economic diversification depends on the level of the total allocated economic rent, on the number of agents and the number of distinct goods. From this it follows that $\Omega = \Omega(H, N, D)$.

Amount of Diversification & Entropy

As argued in the segment "Distribution of Economic Microstates" of Subsection 5-4-3, it is assumed that in an isolated economy the economic microstates are uniformly distributed, consistent with the postulate of equal a priori probabilities in statistical physics.
It is argued in the segment "Economic Diversification" of Subsection 5-4-3 that the amount of microstates or options available for diversification increases with increasing H, N and D.

Since each microstate is equally likely to be occupied, the uncertainty in the knowledge of the occupied microstate of the economic system increases when the multiplicity or amount of options available for diversification Ω increases.

> In statistical physics the uncertainty of the occupied microstate of the system is called the entropy. As such, the economic engineering analog of the **entropy** is the **amount of diversification**

The amount of diversification S is given as:

$$S(\mathcal{H}, N, D) = \ln(\Omega(\mathcal{H}, N, D)) = \ln\left(\frac{\tilde{\omega}(\mathcal{H} + \Delta\mathcal{H}) - \tilde{\omega}(\mathcal{H})}{N!\ \cent^{DN}}\right), \tag{5-41}$$

where the natural logarithm of Ω is taken and with Ω as given in (5-40).
The economic engineering interpretation of the logarithm is given in Section 2-4.

Taking the logarithm of the options gives the amount of choices that need to be made to find the economic microstate.

For example, these choices are made by an entrepreneur looking to make a profit by trading with the economy. The entrepreneur must first identify where the economic rent is allocated before he can start trading.
The economic diversification can thus also be seen as the amount of choices that need to be made to find the economic microstate of the economic system. In a more diversified economy, more choices need to be made before the correct state is found.

In economic literature the economic diversification means shifting an economy away from a single source of income toward multiple sources [14].
The term also pops up when considering investment portfolio's, where diversification means that money is invested in different asset classes and securities in order to minimize the overall risk of the portfolio [14].
Finally, a more general definition of economic diversification is variations in the economic status or the use of a broad range of economic activities in a region or country [50].

In this thesis a slightly different definition is used. Here the term diversification means that the possibilities for dispersion of economic rent have increased. This can happen happen in two ways, namely over D and over N.

Dispersion over D tells you how a single agent chooses to diversify his portfolio. If more goods are available within an economic system, an agent will have more diversification options available.

Dispersion over N tells you how the total surplus of an economic system is distributed over the population. This is known in economic literature as the income distribution.

In this thesis, the diversification of an agents portfolio and the distribution of surplus over the population are both contained within the amount of diversification S, which is the analog of entropy.
S is thus the natural variable that contains the information about the diversification of portfolios en distribution of surplus within an economy.

Equation 5-41 gives the expression of the amount of diversification derived from microeconomic variables. In Chapter 6 this is compared to its macroeconomic interpretation.

Amount of Diversification from Information Theory

The amount of diversification can also be found using Shannon's definition of entropy [10] for a continuous distribution, as derived by Jaynes [51]. This is the expression Gibbs used in his derivation of entropy and is given as:

$$S = -\iint \varrho\left(\mathbf{q}, \mathbf{p}\right) \ln\left(\frac{\varrho\left(\mathbf{q}, \mathbf{p}\right)}{\mathcal{K}}\right) \mathrm{d}^{DN}\mathbf{q}\,\mathrm{d}^{DN}\mathbf{p}, \tag{5-42}$$

where

$$\mathcal{K} = \frac{1}{¢^{DN} N!}, \tag{5-43}$$

as proposed by Jaynes and upon substitution of $\hbar$ for $¢$.

Since the probability density function $\varrho\left(\mathbf{q}, \mathbf{p}\right)$ is a constant for an economy with a constant total economic rent H, equation (5-42) becomes:

$$S = \ln\left(\frac{1}{\varrho\left(\mathbf{q}, \mathbf{p}\right)\ ¢^{DN} N!}\right) \iint \varrho\left(\mathbf{q}, \mathbf{p}\right) \mathrm{d}^{DN}\mathbf{q}\,\mathrm{d}^{DN}\mathbf{p}, \tag{5-44}$$

where the integral reduces to unity per definition of the probability density function.
The term in front of the integral is exactly equal to the expression given in (5-41) after substitution of (5-36) and (5-38) in equation (5-44).

The amount of diversification S can thus be determined by following the definitions used in statistical physics or in information theory, as these are equivalent.

5-4-4 Constant Level of Welfare Ensemble & Potential Profits

In this subsection the probability density function of closed economic systems is derived.
Furthermore, the economic engineering analog of the **partition function** is found and is called the **opportunity function**.
Finally, the **maximum profit available** for trading with a closed economy is found, which is found to be the analog of the **change in the (Helmholtz) free energy**.

Closed Economy

Economic systems are considered that are closed but not isolated. A closed economy is defined as an economy that has no population growth and does not allow cross-border migration, but does allow its inhabitants to trade with foreign economies.

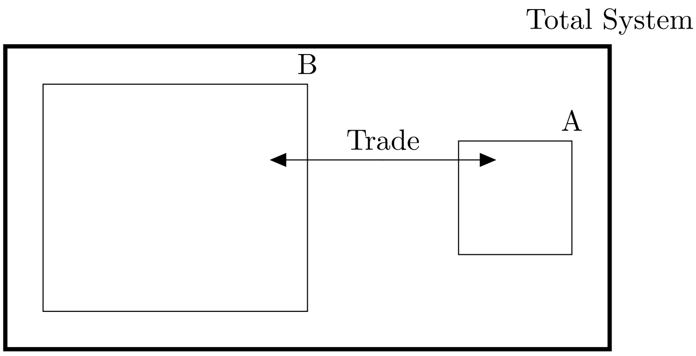

Figure 5-5: Schematic view of two economies A and B that are isolated from the rest of the world, but allowed to trade between themselves. Migration is not allowed. Economy B is considered large enough that trading does not influence its level of welfare.

For simplicity consider two economies A and B as shown in Figure 5-5, where the economy A is of interest and economy B is extremely large. Free trade is allowed between the two economies. Agent migration between the economic systems is not allowed.

Equal Welfare

In Subsections 5-2-4 and 5-3-2 it is argued that if the level of welfare of the two economies is not equal, a net flow of economic rent will be present. The direction of the net flow is from the economy with a higher level of welfare to the economy with a lower level of welfare. Eventually a dynamic equilibrium is reached, where the economies are still allowed to trade, but no net transfer of allocated economic rent exists. This happens when the level of welfare has become equal in both economies.

Since economy B is considered to be extremely large, it is argued that no amount of trading with economy A can significantly affect the level of welfare of economy B. The level of welfare of economy B is thus constant and eventually economy A will reach the same level of welfare. In equilibrium it holds that:

$$\Theta_A = \Theta_B = \Theta. \tag{5-45}$$

This is the analog of a system in thermal contact with a large heat bath of constant temperature. Eventually a thermal equilibrium will exist between the system and the heat bath. Economic literature shows that in international trade, the level of welfare of the economy that has a low level of welfare increases [52].

The two economies A and B are isolated from the rest of the world. The total allocated economic rent $\mathcal{H} = H_A + H_B$ is thus constant. This was argued in Subsection 5-4-3. Following the definition of an isolated system, any change in H_A must mean a change of opposite sign in H_B, as given in (5-15).

Distribution of Economic Microstates

Since the two economies are isolated from the rest of the world, the probability density function derived in Subsection 5-4-3 holds for the combined system AB:

$$\varrho_{AB}\left(\mathbf{q}^A, \mathbf{p}^A; \mathbf{q}^B, \mathbf{p}^B\right) = \begin{cases} C & \mathcal{H} \leq H_A\left(\mathbf{q}^A, \mathbf{p}^A\right) + H_B\left(\mathbf{q}^B, \mathbf{p}^B\right) \leq \mathcal{H} + \Delta\mathcal{H} \\ 0 & \text{otherwise} \end{cases}, \tag{5-46}$$

where $\mathbf{q}^j$ and $\mathbf{p}^j$ indicate the inventory levels and reservation prices (5-28) of all agents and all goods in economy j. Because transactions are rounded to the nearest multiple of ¢ as argued in Subsection 5-1, the total allocated economic rent of the two economic systems cannot be known exactly, but is known up to an uncertainty $\Delta\mathcal{H}$.

Only the statistical distribution of economy A is of interest. To find the probability density function belonging to A, the degrees of freedom corresponding to system B in (5-46) must be integrated out:

$$\varrho_A\left(\mathbf{q}^A, \mathbf{p}^A\right) = \iint \varrho_{AB}\left(\mathbf{q}^A, \mathbf{p}^A; \mathbf{q}^B, \mathbf{p}^B\right) \; \mathrm{d}^{DN}\mathbf{q}^B \, \mathrm{d}^{DN}\mathbf{p}^B, \tag{5-47}$$

which yields [17]:

$$\varrho_A\left(\mathbf{q}^A, \mathbf{p}^A\right) = C \cdot \Omega_B\left(\mathcal{H} - H_A\left(\mathbf{q}^A, \mathbf{p}^A\right)\right), \tag{5-48}$$

where $\Omega_B\left(\mathcal{H} - H_A\left(\mathbf{q}^A, \mathbf{p}^A\right)\right)$ is the multiplicity or options available for diversification of economy B at economic rent $H_B\left(\mathbf{q}^B, \mathbf{p}^B\right) = \mathcal{H} - H_A\left(\mathbf{q}^A, \mathbf{p}^A\right)$ and C is a to be determined constant.

Expression (5-41) of Subsection 5-4-3 shows the relation between the amount of diversification and the options available for diversification. Applying (5-41) to economy B yields:

$$S_B\left(H_B\right) = \ln\left(\Omega_B\left(H_B\right)\right), \tag{5-49}$$

where the dependence on N and D is dropped since agent migration is prohibited and no product diversification is assumed.

Since economy B is much larger than economy A, it is fair to assume that $\mathcal{H} \approx H_B\left(\mathbf{q}^B, \mathbf{p}^B\right) \gg H_A\left(\mathbf{q}^A, \mathbf{p}^A\right)$. Hence (5-49) is approximated by a Taylor expansion around $H_A = 0$:

$$S_B\left(H_B\right) \approx S_B\left(\mathcal{H}\right) - \left(\frac{\partial S_B}{\partial H_B}\right) \cdot H_A\left(\mathbf{q}^A, \mathbf{p}^A\right). \tag{5-50}$$

$\frac{\partial S_B}{\partial H_B}$ is recognized as the inverse of the level of welfare Θ_B of economy B, as shown in (5-18) upon realization that $\langle H \rangle = U = \text{GDP}$.

Combining equations (5-45) and (5-50) yields:

$$S_B\left(H_B\right) = C' - \frac{H_A\left(\mathbf{q}^A, \mathbf{p}^A\right)}{\Theta}, \tag{5-51}$$

where $S_B\left(\mathcal{H}\right) = C'$ is a constant once the economies are in equilibrium. Once the two economies have reached the same level of welfare, the amount of diversification of the total economic system can no longer increase and is thus constant.
Taking the exponential function to both sides of (5-51) yields:

$$\Omega_B\left(H_B\right) = C' \exp\left(-\frac{H_A\left(\mathbf{q}^A, \mathbf{p}^A\right)}{\Theta}\right). \tag{5-52}$$

Combining (5-48) and (5-52) yields the expression for the probability density function of an economy with a constant level of welfare:

$$\varrho_A\left(\mathbf{q}^A, \mathbf{p}^A\right) = \frac{1}{¢^{DN} N!\, Z_\Theta} \exp\left(-\frac{H_A\left(\mathbf{q}^A, \mathbf{p}^A\right)}{\Theta}\right), \tag{5-53}$$

where ¢ and $N!$ have the same role as in the normalization of the constant economic rent ensemble (5-40), namely to distinguish economic microstates.
Z_Θ is the economic engineering analog of the **partition function** called the **opportunity function**, which is a constant that normalizes the probability density function.
The significance of the opportunity function is discussed in the segment "Opportunity Function & Level of Poverty" of Subsection 5-4-4.

The expression for the probability density function (5-53) shows that for a certain level of welfare Θ, high values of economic rent H_A are discounted. This is because for a certain value of Θ there are many more allocations of low economic rent than allocations of high economic rent leading to that level of welfare. The probability of finding agents with a high amount of surplus in an economy with a low level of welfare falls of exponentially.

In a perfect market, the allocated economic rent is equal to the total surplus of all agents. Expression (5-3) shows that the total surplus is a function of $\mathbf{p}^{\mathrm{T}}\mathbf{p}$.
The probability density function of a closed economy with identical agents in a perfect market has the form of a normal distribution with variance $\sigma^2 \propto \Theta$.

In a closed economic system where level of welfare is high, the probability density function is more spread out. For such economic systems, many allocations of economic rent are possible, leading to many possible economic microstates. The amount of diversification thus scales with the variance.

Opportunity Function & Level of Poverty

The probability density function for a closed economy is given in (5-53). The opportunity function Z_Θ is a dimensionless function that normalizes the distribution function and is given as:

$$Z_\Theta = \frac{1}{\mathfrak{c}^{DN} N!} \iint \exp\left(-\frac{H^A\left(\mathbf{q}^A, \mathbf{p}^A\right)}{\Theta}\right) \mathrm{d}^{DN}\mathbf{q}\,\mathrm{d}^{DN}\mathbf{p}. \tag{5-54}$$

In this thesis Z_Θ is called the opportunity function because of its relation to the amount of options available for diversification.
In physics literature it is given that the partition function and the multiplicity are related as [19]:

$$Z_\Theta(\beta) = \int \Omega(\mathcal{H}) \exp(-\beta\mathcal{H})\,\mathrm{d}\mathcal{H}, \tag{5-55}$$

where it is shown that the partition function is the Laplace transform of the multiplicity. β is the inverse of the temperature Θ as given in (5-18).
The **inverse temperature** β is the analog of the **level of poverty**.

The opportunity function contains all information of the distribution of economic microstates and depends on the level of welfare (level of poverty).
An economy with a low level of welfare (high level of poverty) will provide little opportunities for extracting profits trough trading, while the opposite is true for a flourishing economy.

In the economic engineering analogy, the opportunity function is the Laplace transform of the amount of options available for diversification (5-55). The proof of this is given in Section A-1 of the Appendix.

The individual terms in the integral of (5-55) are interpreted using the analogies with statistical physics, while the meaning of the Laplace transform is derived from its use in previous economic engineering research.
In Section 2-4 the meaning of the Laplace transform in economic engineering is given.

The term $\exp(-\beta\mathcal{H})$ in (5-55) is known in physics literature as the Boltzmann distribution [17], which is a probability distribution function. In this thesis it has the role of discounting the options available for diversification $\Omega(\mathcal{H})$.
The integral (5-55) is thus a weighted summation of all options available for diversification, where the weight of $\Omega(\mathcal{H})$ falls off exponentially for increasing values of the total allocated economic rent.
The discounting of $\Omega(\mathcal{H})$ is over the total allocated economic rent. This is a new application of the Laplace transform in economic engineering, where previously cash flows discounted over time were considered [13].

Taking $\exp(-\beta\mathcal{H})$ as a probability distribution function, the opportunity function $Z_\Theta(\beta)$ gives the (weighted) expected value of the options available for diversification $\Omega(\mathcal{H})$ and is interpreted as the opportunities for extracting profits from an economic system by trading.

Free Economic Rent & Free Energy

In statistical physics literature [17] the partition function is related to the free energy of the thermodynamic system. The change in the free energy is the maximum amount of work that can be extracted from the system.

For a system kept at constant temperature Θ, the corresponding free energy is the Helmholtz free energy F, which is the Legendre transform of U [17]:

$$F := U\left[\Theta\right] = U - \Theta S, \tag{5-56}$$

and with

$$\Delta F = W_{\max}, \tag{5-57}$$

where $U = \langle H \rangle$ is the internal energy, which is the expectation value of the Hamiltonian of the total system and S is the entropy of the system.
ΔF is the change in the Helmholtz free energy of the system and $W_{\max}$ is the maximum amount of work that can be extracted from the system.

The economic engineering analog of the **free energy** will be called the **free total economic rent**, or free economic rent for brevity, in this thesis.
In a perfect market the term **free surplus** can also be used as the analog of the free energy.

Using the economic interpretation given to U, Θ and S, the economic engineering analog of the Helmholtz free energy is:

$$F = \text{GDP} - \Theta S. \tag{5-58}$$

The free economic rent of an economy with a constant level of welfare Θ and an amount of diversification S is thus given by (5-58).

Trading results in a change in the GDP of an economy and thus a change in the free economic rent. The **change in the free economic rent** gives the **maximum amount of profits** that can be made from trading with an economy, where the profits made is the analog of the work extracted from a thermodynamic system.

Building upon the analogs between statistical physics and economic engineering introduced in this thesis, the relation between the opportunity function Z_Θ and the free economic rent F is derived next.
To do this, the Shannon entropy [10] for a continuous distribution as derived by Jaynes (5-42) is used [51]. This is the expression Gibbs used in his derivation of entropy. The entropy used by Gibbs applied to the economic system A with a constant level of welfare gives the amount of diversification S_A of economy A:

$$S_A = -\iint \varrho_A\left(\mathbf{q}^A, \mathbf{p}^A\right) \ln\left(\frac{\varrho_A\left(\mathbf{q}^A, \mathbf{p}^A\right)}{\mathcal{K}}\right) \mathrm{d}^{DN}\mathbf{q}\, \mathrm{d}^{DN}\mathbf{p}, \tag{5-59}$$

with $\varrho_A\left(\mathbf{q}^A, \mathbf{p}^A\right)$ as given in (5-53) and where

$$\mathcal{K} = \frac{1}{\cent^{DN} N!}, \tag{5-60}$$

as proposed by Jaynes and upon substitution of $\hbar$ for $\cent$.

Rewriting (5-59) and manipulation of mathematical expressions yields:

$$S_A = \beta \iint H_A \ \varrho_A \left(\mathbf{q}^A, \mathbf{p}^A\right) \ \mathrm{d}^{DN}\mathbf{q}\,\mathrm{d}^{DN}\mathbf{p} + \ln\left(Z_\Theta\right) \iint \varrho_A \left(\mathbf{q}^A, \mathbf{p}^A\right) \mathrm{d}^{DN}\mathbf{q}\,\mathrm{d}^{DN}\mathbf{p} \ . \quad (5\text{-}61)$$

The first integral of (5-61) reduces to $\langle H_A \rangle = \mathrm{GDP}_A$ if $f = H_A$ is chosen in the expression for the expectation value as given in (5-35).
The second integral is equal to unity per definition of the probability density function.

Expression (5-61) becomes:

$$S = \beta \ \mathrm{GDP} + \ln\left(Z_\Theta\right), \quad (5\text{-}62)$$

where the subscript A is dropped for brevity and generality.
Upon realization that $\beta = \frac{1}{\Theta}$, it immediately follows that equations (5-58) and (5-62) are equivalent if and only if

$$F := -\Theta \ln\left(Z_\Theta\right). \quad (5\text{-}63)$$

In equation (5-63) the natural logarithm of the opportunity function Z_Θ is taken. The economic interpretation of taking the natural logarithm is given in Section 2-4.

Taking the natural logarithm of the opportunity function yields the expected amount of choices available to utilize the opportunities of the economy. An economy with a lot of choices to utilize the opportunities presents a lot of profits available for traders.
Equation (5-63) shows that multiplication of $\ln\left(Z_\Theta\right)$ with the level of welfare Θ yields the magnitude of the free economic rent F.

It makes sense that the free economic rent depends on both the level of welfare as well as the expected amount of choices for trading. Having more choices means that more manners for extracting surplus out of an economy are present.
Also, a higher level of welfare means that more surplus can be extracted from the economy.
An economy with a high amount of free economic rent thus presents a high amount of potential profits that can be extracted from trading with the economy.

Free Economic Rent & Disposable Income

The free economic rent can also be interpreted as the total disposable income.
Physics literature [4] shows that the Gibbs free energy G and the number of agents N are related as:

$$G = \mu N, \quad (5\text{-}64)$$

where μ is the chemical potential.

The Gibbs free energy is the free energy of a thermodynamic system where the temperature Θ, number of particles N and pressure P are the controlled parameters.
The Helmholtz free energy on the other hand is the free energy of a thermodynamic system where the temperature Θ, number of particles N and volume V are the controlled parameters.

In this thesis no analogs for the pressure and volume are identified. As a result no distinction between the Gibbs free energy and the Helmholtz free energy can be made in economic engineering. The Gibbs free energy and the Helmholtz free energy are assumed to be the same.

Replacing G with F in (5-64) yields the relation between the free economic rent F and the number of agents N within the economy:

$$F = \upsilon N, \tag{5-65}$$

where υ is the disposable income per capita as argued in Subsection 5-3-3.

From (5-65) it follows that the free economic rent can also be interpreted as the total disposable income.

The change in the total disposable income of an economy is exactly the maximum amount of profits that can be acquired from trading with that economy.
If the disposable income of an economy has become so low that no one within the economy wants to trade, no profits can be extracted from that economy. Such an economy is completely exhausted. This is similar to a thermodynamic system that has reached a minimum amount of free energy, meaning that no more work can be extracted from the system.

Heat Capacity & GDP Fluctuations

Next, the fluctuations of the GDP of a closed economy is analyzed.
Furthermore, the "heat capacity" of a closed economy is derived, which is proposed to be the analog of the heat capacity of a thermodynamic system.
The "economic heat capacity" is a measure for the amount of change in GDP required for an (arbitrarily chosen) unit change in the level of welfare.

Taking the derivative of the opportunity function Z_Θ (5-54) w.r.t. β yields:

$$\frac{\partial Z_\Theta}{\partial \beta} = -\frac{1}{\cent^{DN} N!} \iint H^A \left(\mathbf{q}^A, \mathbf{p}^A\right) \, \exp\left(-\beta \, H^A \left(\mathbf{q}^A, \mathbf{p}^A\right)\right) \, \mathrm{d}^{DN}\mathbf{q} \, \mathrm{d}^{DN}\mathbf{p}. \tag{5-66}$$

Using the definition of the expected value (5-35) and the expression for the probability density function for a closed economy (5-53) it follows that:

$$\mathrm{GDP} = \langle H \rangle = -\frac{1}{Z_\Theta}\frac{\partial Z_\Theta}{\partial \beta} = -\frac{\partial}{\partial \beta}\left(\ln\left(Z_\Theta\right)\right), \tag{5-67}$$

or equivalently

$$\mathrm{GDP} = \Theta^2 \frac{\partial}{\partial \Theta}\left(\ln\left(Z_\Theta\right)\right), \tag{5-68}$$

where the subscript A is again dropped for generality.

The differential in equation 5-68 is the change in the amount of choices to utilize the opportunities w.r.t. the change of the level of welfare. In an economic system with a large GDP, a marginal increase in the level of welfare yields many more choices to utilize the opportunities. The opposite is true for an economic system with a low GDP.

The expected value of the squared value of the total allocated economic rent is:

$$\left\langle H^2 \right\rangle = \frac{1}{Z_\Theta}\frac{\partial^2 Z_\Theta}{\partial \beta^2}. \tag{5-69}$$

The fluctuations in the GDP of the economy are determined by combining equations (5-67) and (5-69):

$$(\Delta \text{ GDP})^2 = \left\langle H^2 \right\rangle - \langle H \rangle^2 = \Theta^2 \left(\frac{\partial \text{ GDP}}{\partial \Theta} \right), \tag{5-70}$$

where Δ GDP is the fluctuation of the GDP, which is the standard deviation of H.

The term $\left(\frac{\partial \text{ GDP}}{\partial\Theta}\right)$ is defined to be:

$$\mathcal{C} := \left(\frac{\partial \text{ GDP}}{\partial\Theta} \right) = \frac{(\Delta \text{ GDP})^2}{\Theta^2}, \tag{5-71}$$

where $\mathcal{C}$ is the "economic heat capacity".
An economy with a large "heat capacity" will experience a small change in the level of welfare when the GDP changes. An economy with a small "heat capacity" on the other hand will experience a large shift in the level of welfare for a similar change in the GDP.

The "economic heat capacity" is an extensive quantity, since Δ GDP increases with an increasing population.
As $\mathcal{C}$ is the economic engineering analog of the heat capacity, it is assumed that the relation between the heat capacity and the specific heat as known in thermodynamics also holds for economic systems:

$$\mathcal{C} = Nc, \tag{5-72}$$

where $\mathcal{C}$ is the heat capacity of the system, N is the number of particles and c is the specific heat of the system, which is an intensive property.
Following the analogy, there thus exists an "economic specific heat", which is an intensive property.

From (5-70), (5-71) and (5-72) it follows that Δ GDP and the number of agents N are related as:

$$\Delta \text{ GDP} \propto \sqrt{N}. \tag{5-73}$$

In Subsection 5-2-3 it is argued that:

$$\text{GDP} \propto N. \tag{5-74}$$

From equations (5-73) and (5-74) it follows that:

$$\frac{\Delta \text{ GDP}}{\text{GDP}} \propto \frac{1}{\sqrt{N}} \quad \longrightarrow \quad 0 \quad \text{as} \quad N \longrightarrow \infty. \tag{5-75}$$

Therefore the relative fluctuations in the GDP are found to be negligible for economies with a large number of agents.

For extremely populated economies it is thus concluded that fluctuations in the GDP are so small that the GDP is approximately constant. The difference between an economy with a constant allocated economic rent as mentioned in Subsection 5-4-3 and an economy with a constant level of welfare as mentioned in Subsection 5-4-4 vanishes.

5-4-5 Equal Disposable Income Ensemble

In this subsection the probability density function of economic systems that allow free trade and have open borders is derived.
The opportunity function of open economies is found and related to the opportunity function of a closed economy as given in (5-54). Furthermore, the expected population size of an economy is derived, as well as how this is related to the disposable income per capita and the level of welfare.

Open Economy

Open economic systems are considered. An open economy is defined as en economy that allows trading with other economies and allows cross-border migration.

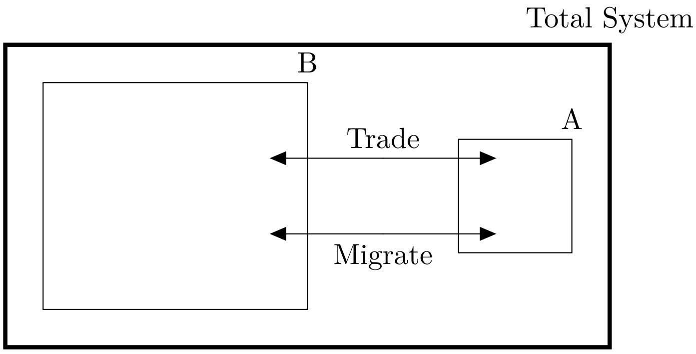

Figure 5-6: Schematic view of two economies A and B that are isolated from the rest of the world, but are allowed to trade between themselves and have open borders. Migration is thus allowed. Economy B is considered large enough that trading does not influence its level of welfare and migration of agents does not influence the disposable income of others.

For simplicity consider two economies A and B as shown in Figure 5-6, where the economy A is of interest and economy B is extremely large. Free trade and migration is allowed between the two economies.
The two economies are isolated from the rest of the world, so all trade and migration happens between the two economies.

Fixed Level of Welfare and Disposable Income

In Subsections 5-3-2 and 5-4-4 it is argued that economies that can trade reach have the same level of welfare once a macroeconomic equilibrium is reached.
From this conclude that open economies in equilibrium must also have the same level of welfare.

Since A and B are open economies, agents are also allowed to migrate between the two economic systems. In Subsection 5-3-3 it is argued that rational agents seek to maximize their disposable income. As a result agents will migrate to the economy that allows them to do so.
The **disposable income per capita** is the proposed analog of the **chemical potential**, a

quantity that particles in a thermodynamic system seek to minimize.
In equilibrium there is no driving force for agent migration, since the disposable income per capita is equal in both economic systems A and B.

Distribution of Economic Microstates

Analogous to the derivation of the probability density function of the closed economy (5-48), the probability density function $\varrho_A\left(\mathbf{q}^A, \mathbf{p}^A, N_A\right)$ of economy A is of the form:

$$\varrho_A\left(\mathbf{q}^A, \mathbf{p}^A, N_A\right) = C \cdot \Omega_B\left(\mathcal{H} - H_A\left(\mathbf{q}^A, \mathbf{p}^A\right), N - N_A\right), \tag{5-76}$$

since the total economic rent and number of agents in the combined system AB must remain constant.
The probability density function ϱ_A now depends not only on the reservation prices and inventory levels of the agents in economy A, but also on the number of agents in A.

$\Omega_B\left(\mathcal{H} - H_A\left(\mathbf{q}^A, \mathbf{p}^A\right), N - N_A\right)$ is the multiplicity or options available for diversification of economy B at economic rent $H_B\left(\mathbf{q}^B, \mathbf{p}^B\right) = \mathcal{H} - H_A\left(\mathbf{q}^A, \mathbf{p}^A\right)$ and population $N_B = N - N_A$ and where C is a to be determined constant.

Again following the derivation of the probability density function of closed system (5-50), the Taylor expansion of the amount of diversification of economy B around $H_A = 0, N_A = 0$ is:

$$S_B\left(H_B, N_B\right) \approx S_B\left(\mathcal{H}, N\right) - \left(\frac{\partial S}{\partial H}\right) \cdot H_A\left(\mathbf{q}^A, \mathbf{p}^A\right) - \frac{\partial S}{\partial N} \cdot N_A. \tag{5-77}$$

Combining (5-6) and (5-18) shows that $\frac{\partial S}{\partial H}$ is the inverse of the level of welfare Θ and (5-24) shows that $\frac{\partial S}{\partial N}$ is the disposable income per capita divided by the level of welfare. Ω_B, the options available for diversification of economy B is thus:

$$\Omega_B\left(H_B, N_B\right) = C' \exp\left(-\frac{H_A\left(\mathbf{q}^A, \mathbf{p}^A\right) + \upsilon N_A}{\Theta}\right). \tag{5-78}$$

Combining equations (5-76) and (5-78) yields the probability density function of an economy with open borders and free trade:

$$\varrho_A\left(\mathbf{q}^A, \mathbf{p}^A, N_A\right) = \frac{1}{\cent^{DN_A} N_A!\, \mathcal{Z}_{\Theta,\upsilon}} \exp\left(-\frac{H_A\left(\mathbf{q}^A, \mathbf{p}^A\right) + \upsilon N_A}{\Theta}\right), \tag{5-79}$$

where $\mathcal{Z}_{\Theta,\upsilon}$ is the opportunity function for an open economy.

The term in the exponent of (5-79) is comparable to the term in the exponent of (5-53), with only a term $\frac{\upsilon N_A}{\Theta}$ added to the exponent.

For a certain level of welfare, the sum of the total economic rent of economy A and the total disposable income is discounted. For a certain value of Θ there are many more allocations of low economic rent plus disposable income than allocations of high economic rent and disposable income leading to that level of welfare.

Opportunity Function for Open and Closed Economies

The probability density function for an open economy is given in (5-79). The corresponding opportunity function $\mathcal{Z}_{\Theta,\upsilon}$ is a dimensionless function that normalizes the probability density function and is given as:

$$\mathcal{Z}_{\Theta,\upsilon} = \sum_{N_A=0}^{\infty} \iint \frac{1}{¢^{DN_A} N_A!} \exp\left(-\frac{H^A\left(\mathbf{q}^A, \mathbf{p}^A\right) + \upsilon N_A}{\Theta}\right) \mathrm{d}^{DN_A}\mathbf{q}\, \mathrm{d}^{DN_A}\mathbf{p}. \tag{5-80}$$

As was the case for the opportunity function of a closed economy Z_Θ (5-54), the opportunity function of an open economy $\mathcal{Z}_{\Theta,\upsilon}$ contains the statistical information required to find the potential profits available.

The opportunity function of a closed economy Z_Θ and the opportunity function of an open economy $\mathcal{Z}_{\Theta,\upsilon}$ are related via the unilateral Z-transform:

$$\mathcal{Z}_{\Theta,\upsilon} = \sum_{N_A=0}^{\infty} Z_\Theta \exp\left(-\beta\upsilon N_A\right), \tag{5-81}$$

where taking $z = \exp\left(\beta\upsilon\right)$ puts equation (5-81) in its usual form of the unilateral Z-transform.

The usual form of the Z-transform is:

$$X\left(z\right) = \mathbb{Z}\left\{x\left[n\right]\right\} = \sum_{n=-\infty}^{\infty} x\left[n\right] z^{-n}, \tag{5-82}$$

where $X\left(z\right)$ is the Z-transform of signal x, $x\left[n\right]$ is the signal value at time step n and z is a complex number. For signals that are only specified for $n \geq 0$ the lower bound of the summation in (5-82) becomes 0 and the unilateral Z-transform is found.

The economic interpretation of the Laplace transform given in Section 2-4 also holds for the Z-transform, because the Z-transform is to discrete time signals what the Laplace transform is to continuous time signals.
Since the number of agents is a discrete variable, a summation over N_A is taken in (5-81) instead of an integral.

In (5-81) the closed economy opportunity function Z_Θ is discounted over the number of agents in economy A. This is in contrast to the usual application of the Laplace transform in economic engineering, where (future) payments are discounted over time [13].
In the segment "Opportunity Function & Level of Poverty" of Subsection 5-4-4 a different type of Laplace transform was introduced. Here, the options available for diversification $\Omega\left(\mathcal{H}\right)$ are discounted over values of the economic rent.

Irrespective of the variable discounted over, the interpretation of the Laplace transform remains consistent.

In the segment "Opportunity Function & Level of Poverty" of Subsection 5-4-4 the closed economy opportunity function Z_Θ is the (weighted) expected value of the options available for diversification $\Omega\left(\mathcal{H}\right)$ and interpreted as the opportunities for extracting profits from an economic system by trading.

The open economy opportunity function can then be given an economic interpretation.
Taking the Laplace transform a second time, this time of the closed economy opportunity function Z_Θ with N_A as the discounting variable, yields the (weighted) expected value of the options available for diversification $\Omega(\mathcal{H})$, where the weighted average is taken over both the economic rent and the agents within the economy.
The economic interpretation of the open economy opportunity function remains consistent, namely the opportunities for extracting profits from the economic system by trading and migration.

Free Economic Rent for Open Systems

The free economic rent for open economic systems Ξ is derived next. The derivation of Ξ is similar to the derivation of the free economic rent for a closed system F as done in the segment "Free Economic Rent & Free Energy" of Subsection 5-4-4.

The free economic rent for open economic systems is the economic engineering analog of the grand potential or grand free energy [9] as known in statistical physics literature.
Open thermodynamic systems are kept at constant temperature and chemical potential. The corresponding thermodynamic potential is the grand potential Ξ, which is the Legendre transform of U with Θ and μ as transformation variables:

$$\Xi := U[\Theta, \mu] = U - \Theta S - \mu N, \tag{5-83}$$

which after substitution of (5-56) in (5-83) yields:

$$\Xi := F - \mu N. \tag{5-84}$$

In the economic engineering analogy Ξ is given as:

$$\Xi := \text{GDP}[\Theta, \upsilon] = \text{GDP} - \Theta S + \upsilon N = F + \upsilon N, \tag{5-85}$$

where S, N and F are now economic symbols as defined earlier, see Section 2-1.

Next, the relationship between the opportunity function $\mathcal{Z}_{\Theta,\upsilon}$ and the free economic rent Ξ for an open economy is derived.

Similar to the derivation of F, the Shannon entropy expression is used [10], [51]. Substitution of the probability density function for an open economy (5-79) in the Shannon entropy yields:

$$S = \beta\,\text{GDP} + \ln(\mathcal{Z}_{\Theta,\upsilon}) + \beta \upsilon N. \tag{5-86}$$

It follows that equations (5-85) and (5-86) are equivalent if and only if

$$\Xi := -\Theta \ln(\mathcal{Z}_{\Theta,\upsilon}). \tag{5-87}$$

Equation (5-87) shows that Ξ has the same form as F (5-63). The economic interpretation is also similar.

In physics literature the grand potential is interpreted as the work that can be extracted from the system by shrinking it down to nothing (putting all the particles and energy back into the reservoir).

Economically the free economic rent of an open economic system Ξ is interpreted as the profits that can be extracted from the economy by trading, taking into account the migration.
$\ln(\mathcal{Z}_{\Theta,\upsilon})$, the expected amount of choices available to utilize the opportunities of the economy, naturally depends on the population size. An increased population size increases the potential trading partners and thus the potential profits.

Population Size Fluctuations

In this segment the average population size and the fluctuations in the population size of an open economy are derived. To do this the steps to determine the fluctuations in the GDP of a closed economy as done in Subsection 5-4-4 are followed.

The derivative of the open economy opportunity function $\mathcal{Z}_{\Theta,\upsilon}$ with respect to the disposable income per capita is:

$$\frac{\partial\, \mathcal{Z}_{\Theta,\upsilon}}{\partial\, \upsilon} = \sum_{N_A=0}^{\infty} \iint \frac{1}{\mathbb{c}^{DN_A} N_A!} \exp\left(-\frac{H^A\left(\mathbf{q}^A, \mathbf{p}^A\right) + \upsilon N_A}{\Theta}\right) \frac{N_A}{\Theta}\; \mathrm{d}^{DN_A}\mathbf{q}\,\mathrm{d}^{DN_A}\mathbf{p}, \tag{5-88}$$

which should be related to the expected population size $\langle N_A \rangle$.

Equation (5-35) shows how the expected value of a function f is calculated when the probability density function depends only on prices and inventory levels. The probability density function derived in (5-79) however depends also on the population size N_A.
The expression for the expected value of f introduced in (5-35) is thus modified:

$$\langle f \rangle := \sum_{N=0}^{\infty} \iint f\left(\mathbf{q}, \mathbf{p}, N\right) \varrho\left(\mathbf{q}, \mathbf{p}, N\right) \mathrm{d}^{DN}\mathbf{q}\; \mathrm{d}^{DN}\mathbf{p}. \tag{5-89}$$

It still holds that $\mathcal{Z}_{\Theta,\upsilon}$ normalizes the probability density function (5-79). This insight, combined with equations (5-88) and (5-89) yields:

$$\frac{\partial\, \mathcal{Z}_{\Theta,\upsilon}}{\partial\, \upsilon} = \langle N \rangle \frac{\mathcal{Z}_{\Theta,\upsilon}}{\Theta}, \tag{5-90}$$

where the subscript A is dropped since it applies to any open economic system.

Rearranging (5-90) yields the expression for the expected population size $\langle N \rangle$ of an open economic system:

$$\langle N \rangle = \Theta \frac{1}{\mathcal{Z}_{\Theta,\upsilon}} \frac{\partial\, \mathcal{Z}_{\Theta,\upsilon}}{\partial\, \upsilon} = \Theta \frac{\partial}{\partial \upsilon} \left(\ln\left(\mathcal{Z}_{\Theta,\upsilon}\right)\right). \tag{5-91}$$

Equation (5-91) shows that the expected population size $\langle N \rangle$ is proportional to the level of welfare. This result is intuitive, since open economies with no restrictions are considered. All agents are free to choose where they wish to settle down. An economy with a higher level of welfare will then naturally have a higher expected population size.

In equation (5-91) $\langle N \rangle$ is also proportional to $\ln(\mathcal{Z}_{\Theta,\upsilon})$. The economic interpretation of taking a logarithm was given in Section 2-4.

$\ln(\mathcal{Z}_{\Theta,\upsilon})$ is the expected amount of choices available to utilize the opportunities of the open economy. This interpretation is identical to the interpretation given to $\ln(Z_\Theta)$ in the segment

"Free Economic Rent & Free Energy" of Subsection 5-4-4, where the only difference in the interpretation of $\ln(Z_\Theta)$ and $\ln(\mathcal{Z}_{\Theta,\upsilon})$ is the dependency on the population size in the latter. Having more choices means that more manners for extracting a surplus out of an economy are present. An increased population size naturally means that more possible trading partners are present.

Analogous to the derivation of $\langle N \rangle$, the expression for $\langle N^2 \rangle$ is derived:

$$\left\langle N^2 \right\rangle = \Theta^2 \frac{1}{\mathcal{Z}_{\Theta,\upsilon}} \frac{\partial^2 \mathcal{Z}_{\Theta,\upsilon}}{\partial \upsilon^2}. \tag{5-92}$$

The fluctuations in the population size ΔN can then be determined using equations (5-91) and (5-92):

$$(\Delta N)^2 = \left\langle N^2 \right\rangle - \langle N \rangle^2 , \tag{5-93}$$

where ΔN is the standard deviation of N.

Finally, by combining equations (5-91), (5-92) and (5-93) it is derived that:

$$\frac{\partial \langle N \rangle}{\partial \upsilon} = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\Theta} = \frac{(\Delta N)^2}{\Theta}, \tag{5-94}$$

where it is concluded that for an open economic system with a high level of welfare, marginal population growth is expected for an increase in the disposable income per capita.
In an economic system with a low level of welfare a significant amount of population growth is expected when the disposable income per capita increases.

Chapter 6

Macroeconomics, Thermodynamics & Applications

In this chapter the results obtained in Chapter 5 are used to complete the link between microeconomics and macroeconomics.
Irreversible processes found in economic literature are linked to the entropy as known in physics.

In Section 6-1 the 1^{st} law of thermodynamics is compared to the expression for the expenditure approach to calculate the GDP given in economic literature.

In Section 6-2 a possible application of the theory derived in Chapter 5 is given, where a trader is trading between two closed economic systems.

In Section 6-3 a possible application of the theory derived in Chapter 5 is given, where the unmodeled costs of a firm are seen as the dissipation of energy in a damper of a mechanical system.
The economic interpretation given to entropy in this thesis is then used to compare the dissipation of heat in the damper, which is caused by an increase in the entropy, to the dissipation of economic rent.

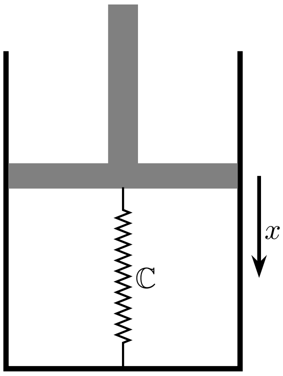

Figure 6-1: Schematic view of a mechanical spring with compliance $\mathbb{C}$ inside an empty cylinder. Once the spring is compressed, the system is isolated. Over time the spring degrades in quality and the compliance increases due to an increase in the entropy. The amount of work that can be extracted from the spring reduces.

6-1 GDP & The 1^{st} Law of Thermodynamics

In this section the well-known expression for the GDP in economic literature is compared to the 1^{st} law of thermodynamics.

Macroeconomic theory [14] gives (3-24) for the calculation of the GDP using the expenditure approach. A simplified version of this expression is used here:

$$\text{GDP} = C + I, \tag{6-1}$$

where the government purchases of goods and services G and net imports X have been neglected.
C is the national consumption and I indicates the gross investments.

In economic literature, consumption is loosely defined as the act of using resources to satisfy current needs and wants [53].
Investments on the other hand are taken to embody all the future satisfactions that will flow from it [53].

The 1^{st} law of thermodynamics is:

$$\Delta U = Q + W, \tag{6-2}$$

where Q is the amount of heat transferred to the system and W is the work done on the system by its surroundings.
The internal energy cannot be measured directly; instead only changes in U are relevant. If absolute zero ($\Theta = 0$) is chosen as the zero point of the internal energy, the Δ in (6-2) is dropped:

$$U = Q + W, \tag{6-3}$$

which is the internal energy of a homogeneous system and has the same form of (6-1).

Figure 6-1 shows a mechanical system consisting of a cylinder closed off by a piston. A spring is attached to the piston on one end and the bottom of the cylinder at the other end. The space inside the cylinder is vacuum.

Initially no heat has been supplied to the system and no work has been done. This is chosen as the zero point of the internal energy: $U = 0$.

The cylinder is then compressed, giving the piston a displacement x. If the compression stroke is adiabatic, meaning no heat is added or produced, the internal energy of the system becomes:

$$U = 0 + \overline{\mathbb{F}}x, \tag{6-4}$$

where $\overline{\mathbb{F}}$ is the average force of compression, $Q = 0$ and $W = W_0 = \overline{\mathbb{F}}x = \frac{1}{\mathbb{C}}x^2$ for a spring with compliance $\mathbb{C}$.

The system is then isolated.
Over time, the spring will degrade in quality and the compliance of the spring increases. On a microscopic level the rearrangement and interaction of molecules inside the spring cause a spontaneous and irreversible increase in the entropy S. The amount of work that can be extracted from the spring reduces.
The internal energy of the system U is still constant, but is now:

$$U = Q(t) + W(t), \tag{6-5}$$

where $W(t) \leq \overline{\mathbb{F}}x = W_0$ for $t > 0$.
The irreversible entropy production thus leads to an in irreversible heat production $Q(t)$. The maximum amount of work that can be extracted from the system $W(t)$ reduces in time.

This exercise is now given a macroeconomic interpretation.
Looking at the definitions of consumption and investments used in macroeconomic literature from a thermodynamics point of view, the consumption is identified as the irreversible spending of the GDP, while the portion of the GDP that is invested is identified as being reversible. The argument behind this interpretation is that capital goods can be disinvested without loss of economic rent, while the same cannot be said for consumed goods.

The work done to displace the piston and compress the spring is seen as an investment I. An economy that initially has a "zero-level" GDP thus receives an investment I_0.
Equation (6-1) then becomes:

$$\text{GDP} = 0 + I_0, \tag{6-6}$$

where initially the national consumption is zero.

Rational agents within the economy seek to maximize their utility and do so through consumption, which is an irreversible process. The amount of diversification S increases due to consumption.
Over time the GDP of the isolated economic system becomes:

$$\text{GDP} = C(t) + I(t), \tag{6-7}$$

where $I(t) \leq I_0$ for $t > 0$.

> Consumption should be viewed as a form of depreciation, where the potential profits that can be extracted from an economy, in the form of a disinvestment, are reduced.

6-1-1 Macroeconomic Entropy & The Invisible Hand

Macroeconomic Entropy

In Chapter 5 the economic engineering analog of the entropy is called the amount of diversification. This terminology followed from the statistical interpretation given to the "economic entropy" derived from microeconomic variables.
In statistical physics the entropy is viewed as a statistical property of a system that gives information about the state of the system.
In thermodynamics entropy is viewed as a state function of the system. Rather than providing statistical information about the system, the entropy predicts that certain processes are irreversible or impossible, despite not violating the conservation of energy.
The thermodynamic interpretation of entropy was first given by Clausius [16]. The statistical interpretation of entropy was given by Gibbs. It is shown in statistical physics literature that the two interpretations of entropy are equal [46].

I thus assume that the amount of diversification S derived in Subsection 5-4-3 is the same as the "macroeconomic entropy" that will be interpreted in this section. It should be noted that the macroeconomic entropy and the amount of diversification are the same thing, similar to how Gibbs and Clausius have different interpretations for the same entropy.

In welfare economics [11] the social welfare is maximized due to the decisions of rational agents. Agents do this by increasing their surplus.
I propose to view the national consumption as a process that converts the total surplus of agents into social welfare. Subsection 5-3-1 argues that an increase in the social welfare corresponds with an increase in the amount of diversification S. From this I conclude that consumption is a process that increases the macroeconomic entropy and is related to the invisible hand of Adam Smith.

Macroeconomic Invisible Hand

Adam Smith [47] called the unobservable market force that helps the demand and supply of goods in a free market to reach equilibrium automatically the invisible hand.

I propose to view the "macroeconomic entropy" as the invisible hand on a macroeconomic scale. Whenever an economy is not in equilibrium, an increase or decrease in the market forces described by the invisible hand will "push" the economy to equilibrium through consumption.

In an economic system where further economic diversification is possible, the amount of diversification S will increase as argued in Chapter 5. An increase in consumption is the cause of this increase in diversification, where eventually macroeconomic equilibrium is reached.

Figure 6-2 shows the energy, entropy curve of a thermodynamic system. All stable thermodynamic equilibrium states lie on the curve, while the region above the curve indicates the unstable states.
A system that is initially in point A will reach a stable equilibrium on the curve between B and C. AB is the free energy of the system and AC is the capacity for entropy increase. The system eventually is "pushed" to a stable equilibrium either by work (path AB), by a spontaneous increase in the entropy (path AC) or any combination of the two.

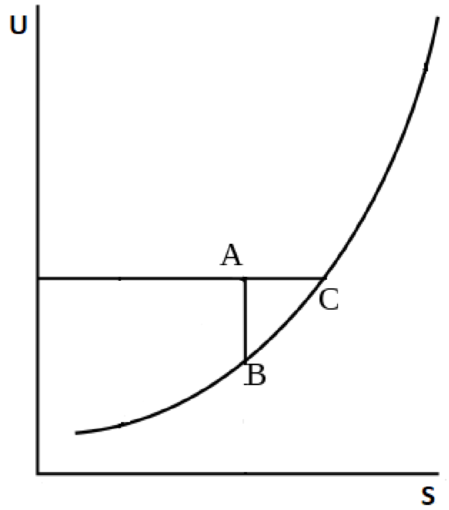

Figure 6-2: Energy, Entropy Curve of a thermodynamic system. For a macroeconomic system this curve shown the possible macroeconomic equilibria. An economic system that is initially in point A will reach a point on the curve between B and C.

Giving Figure 6-2 economic interpretation, an economy that is initially in point A will be pushed to equilibrium, which can happen in two independent ways.
The first (path AB) is due to an external trader that makes a profit by trading with the economy and completely exploiting the opportunities the economy presents for making a profit.

The second method (path AC) for reaching an equilibrium is due to the "invisible hand". The possibilities for further economic diversification are then exhausted due to consumption. Consumption is an irreversible process that causes agents to be more diversified. The amount of diversification is thus identified as the "invisible hand" that pushes a macroeconomic system to equilibrium.

Naturally, the system can reach any point between B and C on the equilibrium curve due to a combination of exploited trading opportunities and increased diversification through consumption.

The invisible hand is identified as the analog of the Clausius entropy or macroeconomic entropy. The analog of Gibbs' interpretation of the entropy is the amount of diversification or microeconomic entropy. Just as in physics, these are the same.

6-1-2 Incentive for Diversification and Migration

The slope of the curve of Figure 6-2 is known in physics literature as the temperature Θ of the system, which follows from its formal definition:

$$\frac{\partial S}{\partial U} = \frac{1}{\Theta} = \beta. \tag{6-8}$$

Writing dU as a total differential yields the 1st law of thermodynamics in differential form:

$$dU = \Theta dS + \mathbb{F}dq + \mu dN, \tag{6-9}$$

where $\mathbb{F}dq$ is a form of work and a term μdN the "chemical work" and is added for open systems.
In economic engineering $\mathbb{F}$ is interpreted as a desire for acquiring goods. Θ and μ should have a similar economic interpretation.

I propose to view Θ macroeconomically as an incentive or desire for diversification. Since everyone has a desire to be more diversified, the incentive for diversification is strictly positive. This result is consistent with classical thermodynamics literature, where only positive values of temperature are well-defined.

> The microeconomic interpretation of the analog of temperature Θ is the level of welfare, while in macroeconomics it is interpreted as the incentive for diversification. The level of welfare and incentive for diversification are two names for the same quantity.

Similarly, I propose to view μ macroeconomically as the incentive or desire for migration. Within an economic system, agents have a certain incentive for migration. An economy with a high incentive for migration will have a net emigration, while an economic system with a low incentive for migration will have a net immigration.

> The incentive for migration is the macroeconomic interpretation given to μ and is the opposite of the disposable income per capita υ, which was identified as the microeconomic interpretation of the chemical potential analog.

In an economic system where the agents have a high disposable income, their incentive to migrate will be low.

> By giving a macroscopic interpretation to the level of welfare, the disposable income per capita and the amount of diversification, a link between microeconomics and macroeconomics has been made.

6-1-3 No Free Lunch

The saying "there ain't no such thing as a free lunch", popularized by economist Milton Friedman, can also be argued using the economic engineering theory derived in this thesis.

The saying is commonly used in economics to emphasize that a person or a society cannot get something for nothing.
This immediately points to a analog with the 1st law of thermodynamics, where it is stated that energy is always conserved and cannot be created or destroyed.
One can thus loosely interpret the 1st law of thermodynamics as saying that it is impossible to get energy for free.
From the analogy with the 1st law, I conclude that the "no free lunch" saying prohibits the existence of "economic perpetual motion" of the 1st kind.

Interpretation of the "no free lunch" saying can however be extended to the 2nd law of thermodynamics as well.
To enjoy or utilize a lunch one must consume it. It is argued in Subsection 6-1-1 that consumption increases the amount of diversification and is an irreversible process.
The consumption is thus the price that one pays to utilize the lunch. In order for the lunch to be "free", it would need to be deconsumed without loss of value. The 2nd law of thermodynamics prevents this from happening.
From the analogy with the 2nd law, I conclude that the "no free lunch" saying also prohibits the existence of "economic perpetual motion" of the 2nd kind.

In this thesis a perfect market is assumed, meaning that there are no transaction costs. In a perfect market an agent could buy and sell a "lunch" without consuming it and effectively return to his initial state.
In reality however a perfect market does not exist and each transaction has costs associated with it. A transaction with no associated costs is a form of a "free lunch" in itself.
The saying "there ain't no such thing as a free lunch"also means that a perfect market with no transaction costs does not exist in reality. From this I conclude that the non-existence of "economic perpetual motion" of the 3rd kind is also contained by the "no free lunch" saying.

> A free lunch in economics would be the analog of perpetual motion in physics, and cannot exist.

6-2 Broker Operating Between Closed Economies

In this section an application of the theory developed in Chapter 5 is given.

Consider two closed economic systems that do not have the same level of welfare or incentive for diversification. Economy A has a higher incentive for diversification than economy B. As a result these economies are willing to make a financial trade for a security (e.g. stocks, bonds or foreign currency).
A broker exploits this incentive difference and acts as the middleman between the two economies, see Figure 6-3.
The broker exploits the fact that the economic systems are not in equilibrium and Pareto optimality has not been reached. As a result the broker can make himself better off (by making a profit) without making anyone else worse off. While doing so he also has unavoidable transaction costs. The non-existence of a broker operating without transaction costs was argued in Subsection 6-1-3.

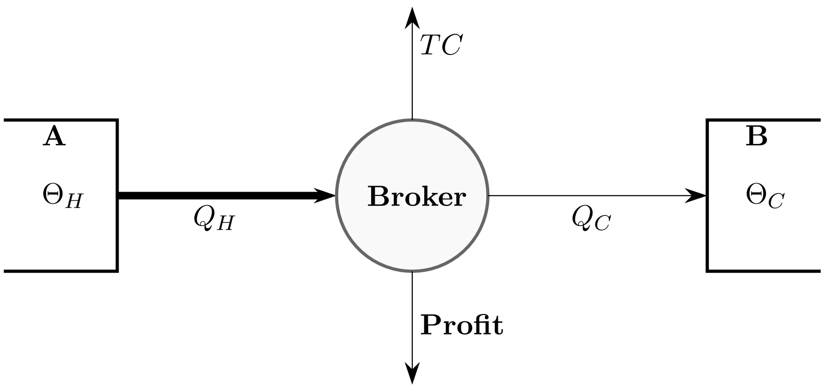

Figure 6-3: Schematic view of a broker operating between two closed economic systems. The broker exploits the difference in incentive for diversification between the economies and acts as the middleman in a financial trade. The profit the broker makes is the difference between his incoming and outgoing cash flow, where transaction costs are taken into account.

In Figure 6-3 a schematic view of the broker operating between economies A and B is given. The profit W of the broker is the difference between his incoming and outgoing cash flows:

$$\text{Profit} = W = Q_H - Q_C - TC, \tag{6-10}$$

where TC are the transaction costs of the broker, Q_H is the cash flow from A to the broker and Q_C is the cash flow from the broker to B. Q_H and Q_C are the revenue and the purchasing costs of the broker respectively. The term TC in (6-10) contains the transaction costs the broker makes for acquiring and selling the goods.

The price the broker pays for securities in economy B is p_B, the average reservation price of economy B for securities. Likewise he sells the securities in economy A for the average reservation price p_A.
The broker does not seek the agents in economy B with the lowest reservation price and the agents with in A with the highest reservation price to trade with.
An external broker that is able to identify individual reservation prices of agents is an economic variant of Maxwell's Demon [3]. Maxwell argued that the demon in his thought experiment

violates the 2^{nd} law of thermodynamics and cannot exist.
From this I conclude that the costs associated with obtaining the information of the prices are higher than the profit the broker can make.

The broker described in Figure 6-3 is analogous to a heat engine in thermodynamics. The "efficiency" of the broker gives the ratio between his profit and his total costs:

$$\eta = \frac{W}{Q_H} = \frac{Q_H - Q_C - TC}{Q_H}. \tag{6-11}$$

The **efficiency** is identified as the analog of the **profit margin** as known in economic literature.

The maximum mathematically possible profit margin would be if the broker were able to trade with zero transaction costs and without increasing the total amount of diversification S. It then holds that:

$$\Delta S = \Delta S_A + \Delta S_B = 0. \tag{6-12}$$

Such an ideal broker would be the economic engineering analog of a Carnot engine [16], but would effectively be receiving a "free lunch".
Analogous to the Carnot engine, the profit margin of the ideal broker is:

$$\eta_{\text{ideal}} = \frac{\Theta_H - \Theta_C}{\Theta_H}. \tag{6-13}$$

For a realistic broker it holds that:

$$\eta < \eta_{\text{ideal}}. \tag{6-14}$$

with η and η_{ideal} as given in (6-13) and (6-14) respectively.
A rational broker will seek to maximize profit margin. The broker will thus aim to minimize his transaction costs and seek to trade between economies with a large difference in their respective levels of welfare.

6-3 Unmodeled Costs & Entropy Production

In this section it is shown how the theory developed in Chapter 5 can be applied to the current economic engineering modeling method based on the microeconomic laws of supply and demand.

Setup and Analogs

Consider a trader that buys and sells a certain good of interest. He also has an inventory in which he can store the good and sell at a later time.

In economic engineering the trader with inventory described here is modeled as a simple mechanical system.

The storage of goods of the trader is modeled as a mechanical spring. The extension q of the spring is the analog of the inventory level of the trader for the good of interest.
The compliance $\mathbb{C}$ of the spring is the economic engineering analog of the storage capacity and the spring force $\mathbb{F}$ is interpreted as the desire felt by the trader to deplete his inventory level.

The quantity demanded (supplied) $\dot{q}$ by the trader is modeled as the velocity of the massive body. The inertia $\mathcal{I}$ is the analog of the inverse price elasticity of demand (supply) $\mathcal{E}$ of the trader.

The trader also makes (unmodeled) costs that are not related to the storage of the good of interest. These costs lead to the dissipation of economic rent $\mathcal{D}$ of the trader. A damper with damping coefficient b is used to model the economic rent dissipation, which is the analog of the dissipated energy.

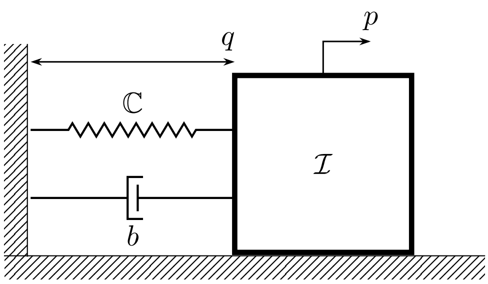

Figure 6-4: Schematic view of a mechanical mass-spring-damper system. The mass with inertia $\mathcal{I}$ is the analog of a trader with price elasticity $\mathcal{E}$, the spring $\mathbb{C}$ is used to model the trader's inventory and the damper b gives the (unmodeled) costs of the trader.

Figure 6-4 shows a mechanical mass-spring-damper system used to model the trader.
The surplus T of the trader is the analog of the kinetic energy of the body, while the potential surplus Φ is the analog of the potential energy.

Modeling Economic Rent Dissipation

The total economic rent E of the trader is the sum of the allocated economic rent H and the dissipated economic rent $\mathcal{D}$, as given in (3-21).

In current economic engineering literature [6] the inertia and spring are seen as storage elements of economic rent, while the damper dissipates economic rent.

The inertia element stores the reservation price p of the trader, while the spring stores his inventory level q.
I propose to view the damper as an element that stores the "economic entropy" S. Since it follows from the 2^{nd} law of thermodynamics that S cannot spontaneously decrease, the "storage of S inside the damper" does not reduce, meaning that the storage of S is irreversible. This is not the case for the two other elements, where the storage of p and q is reversible.

I propose to view the "economic entropy" S as a measure for the amount of dispersion of dissipated economic rent $\mathcal{D}$ among all agents the trader interacted with when acquiring these unmodeled costs.

An agent that trades in a certain good will keep track of his inventory q for that good. However, he will not keep track of all other goods he buys from other agents to remain operational. The total costs he makes in those trade is thus the dissipated economic rent $\mathcal{D}$, which is distributed among the agents he interacted with.

Take for example a car dealer. The car dealer will focus mainly on the amount of cars in stock and the price he wishes to sell them for.
He will be less interested in the amount of paper and pens he uses (e.g. to sign contracts), the amount of coffee he offers potential customers and the amount of soap and water needed to clean the cars. The costs associated with these minor transactions can be combined to give the total unmodeled costs.
The car dealer is not interested in the amount of paper, pens or coffee he has in stock or the price he would be willing to pay for these items. He only cares about the total costs associated with these unmodeled economic activities.

A similar approach is taken in the modeling of mechanical systems. We are interested in the extension of the spring and the momentum of the inertia.
We are not interested in the microscopic movement of oil particles inside a mechanical damper. Instead, only the energy dissipated in the damper is of interest, since this reduces the energy that can be stored in the spring and inertia element.

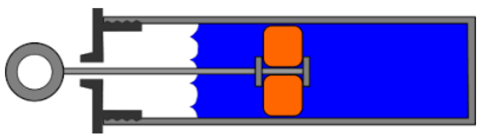

Figure 6-5: Schematic view of a typical mechanical damper. When the damper is compressed, oil particles are randomly dispersed inside the chamber [5].

Figure 6-5 shows a schematic view of dashpot, which is a type of mechanical damper. When the damper is compressed, oil particles are randomly dispersed inside the chamber and the entropy inside the damper increases. This process is comparable to the random movement of particles in a gas. From this I conclude that the macroscopic damper is an application of statistical mechanics. The random demand of unmodeled agents is comparable to the random demand of agents in an economic system as described in Chapter 5.

I propose that to view the analog of the entropy production inside a damper as an increase in the dispersion of dissipated economic rent due to unmodeled economic activity.

Entropy Production Rate

Existing literature on the entropy production rate inside a damper of a mass-spring damper system [54] can be used to model the dissipation of economic rent of the agent.

The economic rent H of the trader with rent dissipation is of the form $H = H(q, p, S)$, meaning that H is now also a function of the dispersion of dissipated economic rent S.
From Hamilton's equations it follows that the change in stock level and reservation price of the good of interest is:

$$\dot{q} = \frac{\partial H(q, p, S)}{\partial p}, \tag{6-15}$$

$$\dot{p} = -\frac{\partial H(q, p, S)}{\partial q} + f(q, p, S), \tag{6-16}$$

where $f(q, p, S)$ is the analog of the friction force due to damping. This is interpreted economically as the increase in the reservation price of the agent due to additional costs made.

The time derivative of $\dot{\mathcal{D}}$ is the dissipated economic rent per day (or per week, month, year, etc.) and is given as:

$$\dot{\mathcal{D}} = \Theta \dot{S} = -\dot{q}\, f \geq 0. \tag{6-17}$$

Since the product of $\dot{q}$ and f must be negative, an inventory depletion corresponds with a desire felt to increase prices, while the agent feels a desire to decrease his prices when the inventory increases. This seems logical from an economic point of view.

Θ is the analog of the temperature and can be interpreted as the average surplus of the unmodeled agents who interacted with the agent modeled by the $\mathcal{I}$ element. This is similar to the level of welfare within the economy as derived in Chapter 5. The dissipated economic rent of the agent modeled by the $\mathcal{I}$ element is thus distributed among the agents he interacted with when making these costs and Θ is the average surplus acquired by those agents.

The entropy production rate can then be used in a bond graph model, which is the preferred method of modeling in economic engineering. The bond graph model of the system given in Figure 6-4 is shown in Figure 6-6.

$\mathbb{F}$ is given by the first term of the right-hand side of (6-16) and indicates the desire of the agent to adjust his reservation price based on his current inventory level. The agent thus changes his reservation price based on his inventory level and the additional costs he makes.
v is given by the right-hand side of (6-15) and indicates the quantity demanded (supplied) by the agent and is equal to the change of his stock level.

In current economic engineering modeling, the red box indicates the R element used to model the costs. No dynamics is taken into account when doing this however. By modeling the damper as a multiport element that dissipates energy and produces entropy, the costs can be modeled more accurately.
By putting entropy into controls, improvements can be made in modeling the costs of agents in economic engineering modeling techniques.

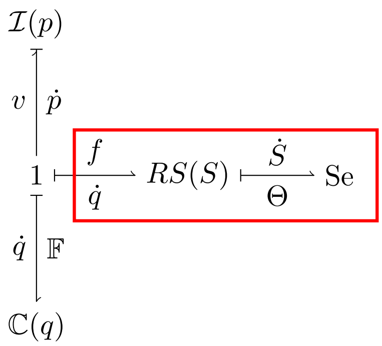

Figure 6-6: Bond graph model of the mass-spring-damper system given in Figure 6-6. The red box indicates the damper R and is modeled as a multiport element.

Chapter 7

Summary, Conclusions & Recommendations

7-1 Summary

The purpose of this thesis is to develop a method for modeling economic systems consisting of many agents within the economic engineering framework. The existing theory of economic engineering is based on analogs with classical mechanics and models individual agents as inertia elements.

In this thesis, statistical physics is used to develop a macroeconomic theory of economic engineering. It is argued that macroeconomic equilibrium is analogous to thermodynamic equilibrium and that an economic analog of entropy exists that pushes the economy to equilibrium. The macroscopic or Clausius interpretation of entropy is found to be analogous to Adam Smith's invisible hand, while the microscopic or statistical interpretation of entropy, otherwise known as the Gibbs entropy, is interpreted as the amount of diversification. From this it is also argued that an economy in equilibrium is Pareto optimal and fully diversified. Furthermore, the amount of diversification is interpreted as giving the dispersion of economic rent over the population size N and the number of distinct goods D, where the dispersion over N is compared to the distribution of income and the dispersion over D is compared to the portfolio diversification of an agent. The amount of diversification is thus linked to two major concepts in economic literature.

Analogs are found for the temperature and chemical potential and given both a microeconomic and macroeconomic interpretation. The macroscopic interpretation of temperature is seen macroeconomically as an incentive for diversification, while the microscopic interpretation of temperature is viewed microeconomically as the average revenue or level of welfare. Similarly, the chemical potential is given a macroeconomic and microeconomic interpretation, with the former being the incentive for migration and the latter being the disposable income per capita. No analogs are found for the canonical variables pressure and volume. A pressure is an effort variable and should be interpreted as an incentive variable, just like the temperature and chemical potential. However, I could not identify a logical analog for volume. In physics the volume of a (rectangular) D-dimensional box is found by simply multiplying the length of the box in each dimension. Multiplying different types of goods makes no sense economically, meaning a clear interpretation of volume could not be found.

It is also shown that depending on the constraints on the economy, the way the options for diversification are distributed in the price-quantity space changes. Non-isolated economic systems are shown to have an opportunity function, which gives the opportunities for extracting surplus from the economic system by trade and/or migration. The opportunity function is identified as the analog of the partition function.
From the opportunity function, the analog of the free energy is determined, which is called the free economic rent. The change in the free energy gives the amount of work that can be extracted from a thermodynamic system. From the analogies made it then follows that the change in the free economic rent gives the maximum amount of profits that be made when trading with an economic system.

Also, a link is made between the fundamental theorems of welfare economics and the laws of thermodynamics. The 1st fundamental theorem states the existence of a macroeconomic equilibrium. Since surplus cannot be created from nothing the 1st law of thermodynamics states that this equilibrium is Pareto optimal. The 2nd fundamental theorem states that an economic system that is not in equilibrium will be "pushed" to a Pareto optimal equilibrium. This matches the 2nd law of thermodynamics, where the existence of equilibrium and a function that is maximized in equilibrium is defined.

Finally, applications of the theory developed are given, showing how a broker operating as the middleman in a financial trade between economies is analogous to a heat engine and how the unmodeled motion of oil particles in a mechanical damper is analogous to the unmodeled reservation prices and inventory levels of goods of an agent.

7-2 Conclusions

The major contribution of this thesis is explaining how economic equilibrium is reached, namely by an increase in the amount of diversification. In his work, Manders [18] argued that thermodynamic equilibrium is analogous to economic equilibrium. However, he did not explain the mechanism for reaching equilibrium. Similarly, Adam Smith [47] argued that invisible market forces push the economy towards equilibrium, which economists today call a Pareto optimal equilibrium. Smith was unable to identify these market forces and explain how the decisions of individual agents lead to equilibrium. This thesis argues that in economic systems consisting of many rational agents who each act in their own self-interest, the economy becomes more diversified. Due to trade between agents, the total economic rent is dispersed through the economy, increasing the amount of diversification. The economy is in equilibrium once the amount of diversification is maximized. This gives a statistical interpretation to the economic entropy, something that is not done by Manders or within the field of thermoeconomics.

The developed idea of diversification is used to interpret (macro)economic phenomena. Manders [18] did not find an analog of the free energy in his thesis. However, the idea of free economic rent could be derived from his work as well, since the free energy is a thermodynamic

quantity. The maximum amount of profits that can be made by trading with an economy then follows. Because of the developed idea of diversification in this thesis, we now understand where the fundamental limit on the available profits comes from. The limit on the available profits is found using the opportunity function and the level of welfare, which both follow from the amount of diversification. Knowing the opportunities provided by an economy, an external trader can decide whether or not trading with that economy is lucrative. Also, more profits can be made when trading with an economy with a high level of welfare, since more surplus is available in such an economy.
If the external trader is the middleman in a trade between two economic systems with a different level of welfare, the profit margin of the trader is also bounded. This is analogous to the efficiency of a heat engine being bounded by Carnot efficiency. A trader that is able to keep the total amount of diversification constant and has no transaction costs operates at Carnot efficiency and maximizes his profit margin. To maximize his profit margin, the trader should aim to minimize his transaction costs and the total increase in diversification. He should also seek to trade between economies with a large difference in their level of welfare.
The concepts of maximum profits available and maximizing the profit margin can now be explained using the amount of diversification interpretation given to economic entropy.

Furthermore, we can now show how policy makers can expect the population size in open economies to change when their policies change the disposable income per capita. It is argued that each individual is rational and seeks to maximize his disposable income. Agents will thus migrate if doing so allows them to maximize their disposable income. This is analogous to a particle moving from high to low chemical potential. The individual behaviour of agents, namely their desire to maximize their disposable income, is thus related to the macroeconomic phenomenon of migration.

Finally, the thesis shows that the random movement of oil particles in a mechanical damper is comparable to the random demand of agents within an economy. The interpretation given to entropy in this thesis can thus be used in economic engineering modeling, where R elements are used to model the dissipated economic rent. The entropy is interpreted as a measure of the dispersion of the dissipated economic rent among unmodeled agents interacted with. The entropy production within a damper is then the increase in the dispersion of dissipated economic rent due to unmodeled economic activity.

7-3 Recommendations

The theory developed in this thesis can be improved upon in several areas.
The effect on prices due to cross-price elasticity was not considered. The fact that goods can be complementary or substitutes was touched upon in the thesis, but not taken into account in the further development of the theory. Finding out if and how this effects the results obtained in this thesis is a possible step in the further development of this theory.

No analog for volume was found in this thesis. As a result, the methods used in statistical physics to determine the volume of a hyper-spherical shell in phase space could not be duplicated, since this requires a well-defined analog of volume. As a result, no quantitative results for the calculation of the amount of diversification could be made. To further develop and fully utilize this theory, a solution for this problem needs to be found.

Classical statistical physics was used to develop this theory. It is interesting to use quantum statistical physics to develop this theory in the future and compare if and how much the obtained results differ.

Analogs with equilibrium statistical mechanics were used in this thesis. The thesis argues the existence of economic equilibrium and that an economy that is initially not in equilibrium will reach equilibrium by become more diversified or by international trade. No statements are made on how long it takes before equilibrium is reached. The field of non-equilibrium statistical mechanics should be used to derive an economic engineering theory of macroeconomics that deals with the time rate of economic processes.

A perfect market was assumed in this thesis, where the potential surplus of agents was neglected. It was thus assumed that agents did not feel a need to keep an inventory for future use. This is the analog of an ideal gas. How relaxing this assumptions affects the theory developed is something that should be researched in the future.

Agents were assumed to be analogs of point particles. Modeling the agents as diatomic particles, meaning they also have a rotational kinetic energy analog, would be an interesting addition to the developed theory.

More research needs to be done on how to setup Lagrangians for damped harmonic oscillators within the economic engineering framework. In this thesis I view the entropy in a mechanical damper as a measure for the amount of dispersion of dissipated economic rent. The random movement of oil particles in a mechanical damper is compared to the random demand of agents in an economy. All unmodeled economic activity can then be modeled as an R-element in economic engineering modeling. More research needs to be done how the entropy production in a damper can be related to the increase in the dispersion of dissipated economic rent due to unmodeled economic activity. The field of economic engineering would gain a lot from this insight and the modeling of costs in economic engineering would improve tremendously.

Appendix A

Appendix

A-1 Partition Function - Multiplicity Laplace Transform Proof

Two economic (thermodynamic) systems A and B. See Figure A-1 for schematic view of the system.

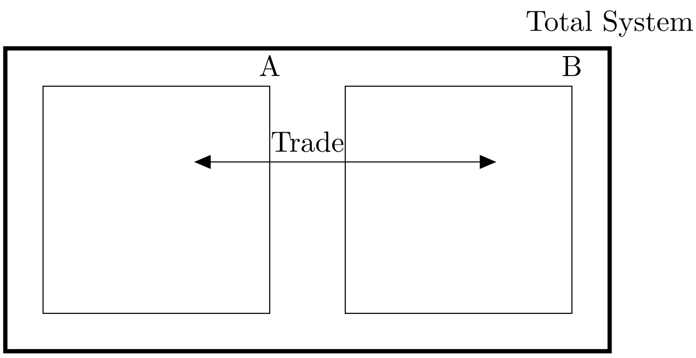

Figure A-1: Schematic view of two economies A and B that are isolated from the rest of the world, but allowed to trade between themselves. Migration is not allowed. Eventually both economies reach macroeconomic equilibrium with the same level of welfare Θ.

The amount of diversification (entropy) S of the entire system is:

$$S = S_A + S_B. \tag{A-1}$$

Together A and B are an isolated system so the total allocated economic rent (Hamiltonian) is constant:

$$H = H_A + H_B = \text{const.} \tag{A-2}$$

The GDP (internal energy) is the expected value of the total allocated economic rent: (Hamiltonian):

$$\text{GDP} = U = \langle H \rangle \,, \tag{A-3}$$

so

$$U = U_A + U_B = \text{constant}, \tag{A-4}$$

and

$$\Delta U_A = -\Delta U_B. \tag{A-5}$$

Closed system, so N_A, N_B are constant, where N is the number of agents (particles).
Amount of diversification S only depends on U.

In equilibrium the amount of diversification (entropy) S of the entire system is maximized:

$$\frac{\partial S}{\partial U_A} = \frac{\partial S_A}{\partial U_A} + \frac{\partial S_B}{\partial U_A} = 0. \tag{A-6}$$

This yields:

$$\frac{\partial S_A}{\partial U_A} = \frac{\partial S_B}{\partial U_B} = \beta, \tag{A-7}$$

where β is the level of poverty (inverse temperature).
In equilibrium the level of poverty (temperature) of the systems is equal.

Amount of diversification S is defined as:

$$S := \ln\left(\Omega\left(U\right)\right), \tag{A-8}$$

where $\Omega\left(U\right)$ is the options available for diversification (multiplicity = number of microstates).
A microstate is a point in the $2ND$ dimensional price-quantity space (phase space), meaning it is a possible allocation of all prices and inventory levels (momenta and positions).

More microstates means more diversification options, meaning it is harder to know what is the occupied microstate of the system.
The amount of diversification (entropy) gives the uncertainty in knowing the occupied microstate: high entropy meaning a lot of possible microstates.

Equations (A-7) and (A-8) yield:

$$\frac{\partial \ln\left(\Omega_A\left(U_A\right)\right)}{\partial U_A} = \beta. \tag{A-9}$$

Since β is constant in equilibrium and does not depend on U_A, (A-9) can be rewritten as:

$$\frac{\partial}{\partial U_A}\left(\ln\left(\Omega_A\left(U_A\right)\right) - \beta U_A\right) = 0, \tag{A-10}$$

meaning that:

$$\ln\left(\Omega_A\left(U_A\right)\right) - \beta U_A = \text{constant}. \tag{A-11}$$

We call the constant in (A-11) the opportunity function $\ln\left(Z_\Theta\left(\beta\right)\right)$.

Equation (A-11) has the form of a Legendre transform:

$$\ln\left(\Omega_A\left(U_A\right)\right) - \beta U_A = \ln\left(Z_\Theta\left(\beta\right)\right), \tag{A-12}$$

compare to the standard form of a Legendre transform [19]:

$$F\left(x\right) - s \cdot x = -G\left(s\right), \tag{A-13}$$

where $F(x) = \ln(\Omega_A(U_A))$, $x = U_A$, $G = -\ln(Z_\Theta(\beta))$ and $s = \beta$.

The paper "Making Sense of the Legendre Transform" [19] shows that for a Legendre transform as given in (A-13) that:

$$x = \frac{\partial G}{\partial s}. \tag{A-14}$$

This yields:

$$U_A = -\frac{\partial \ln(Z_\Theta(\beta))}{\partial \beta}. \tag{A-15}$$

The right-hand side of (A-15) is known in physical literature as the internal energy U of a system kept at constant temperature, which is exactly what we found.

Conclusion: saying that $\ln(\Omega_A(U_A))$ and $-\ln(Z_\Theta(\beta))$ are related via a Legendre transform is justified!

$Z_\Theta(\beta)$ is known in physics literature as the canonical partition function, meaning that is is related to systems kept at constant temperature.

From the Legendre transform (A-12) we find by taking the exponential function on both sides (subscript A dropped):

$$Z_\Theta(\beta) = e^{\ln(\Omega(U))-\beta U} = e^{\ln((\Omega(U)))} \cdot e^{-\beta U}, \tag{A-16}$$

which yields:

$$Z_\Theta(\beta) = \Omega(U)e^{-\beta U}. \tag{A-17}$$

This result is inconsistent with physics literature [19], where it is stated that $Z_\Theta(\beta)$ and $\Omega(U)$ are related via the Laplace transform:

$$Z_\Theta(\beta) = \int \Omega(U)e^{-\beta U} dU. \tag{A-18}$$

The expression found in (A-17) and the one given in literature differ via an integral over dU on the right-hand side.

The exponent of of (A-18) is large, since $U \propto N$ and N is large. Integrals with large exponents can be approximated via the "steepest descent method" [19].
Evaluating (A-18) at equilibrium yields the expression given in (A-17).

It has thus been proven that the options available for diversification Ω and the opportunity function $Z_\Theta(\beta)$ are related via the Laplace transform.

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Glossary

List of Acronyms

3mE	Mechanical, Maritime and Materials Engineering
DCSC	Delft Center for Systems and Control
TU Delft	Delft University of Technology
GDP	Gross Domestic Product
DI	Disposable Income
NI	National Income