A STUDY OF THE BEHAVIOUR OF AND THE FORCES IN A BED PROTECTING MATTRESS

"THE FALLING APRON"

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1. Introduction

In many cases the bottom around a structure that is exposed to current has to be protected by a mattress. The purpose of the mattress is to protect the soil besides the structure from erosion and scour, thus preserving the strength of the foundation of the structure. Examples are the bottom protection around bridge piers, along guide bunds and behind discharge sluices. At the edge of the bottom protection parallel to the current (e.g. along guide bunds) or at the trailing edge (e.g. behind discharge sluice) scourholes will be formed in the unprotected bed. The slopes of these scourholes may be so steep that local soil failure follows. In that case the mattress has to span the local depression caused by the slope failure and maintain its protective function (see Figure 2-1). The mattress should therefore be able to withstand the tension forces that build up in the fabric. In this report a sequence of models is proposed and verified to estimate the maximum force in a mattress spanning a local soil failure. The study was commisioned by CUR in Gouda as a contribution to the design of the guide bunds of the Jamuna Bridge.

2. A first Approximation of the Mattress spanning a Gap

The mattress is spanning the gap created by a slope failure perpendicular to its edge.

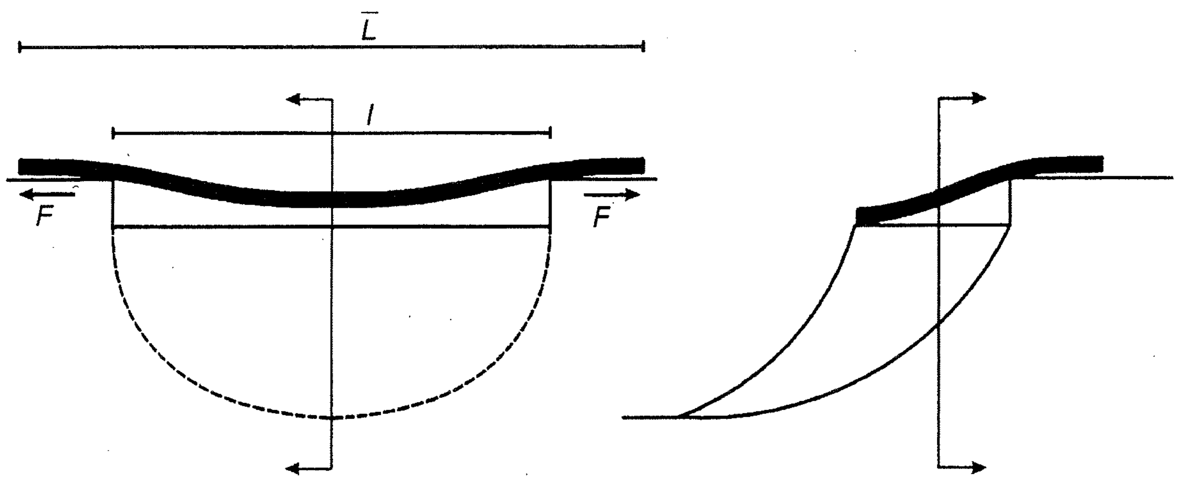


Figure 2-1: A mattress spanning a local soil failure

The maximal friction force *F* exerted by the soil on the mattress is at each side:

$$F = tg\varphi \cdot \frac{\overline{L}}{2} \cdot q \qquad [\text{Nm}^{-1}]$$

Where:	φ	= friction angle sand-mattress	[rad]
	$\overline{L}$	= length of mattress	[m]
	q	= weight of mattress	[Nm^{-2}]

If the sag *t* is small a parabola may approximate the form of the mattress reasonably:

$$t = \frac{q}{2T} x(l - x) \qquad [\text{m}]$$

Where:	T	= tension force in the mattress	[Nm^{-1}]
	l	= free span	[m]

From this equation follows that the tension force equals:

$$T = \frac{q}{2t_{\max}} \cdot \frac{l^2}{4} \qquad [\text{Nm}^{-1}]$$

For:

$$x = \frac{l}{2} \qquad [\text{m}]$$

From the equilibrium of horizontal forces it is clear that the mattress will sag until the tension force *T* just equals the maximal friction force *F*.

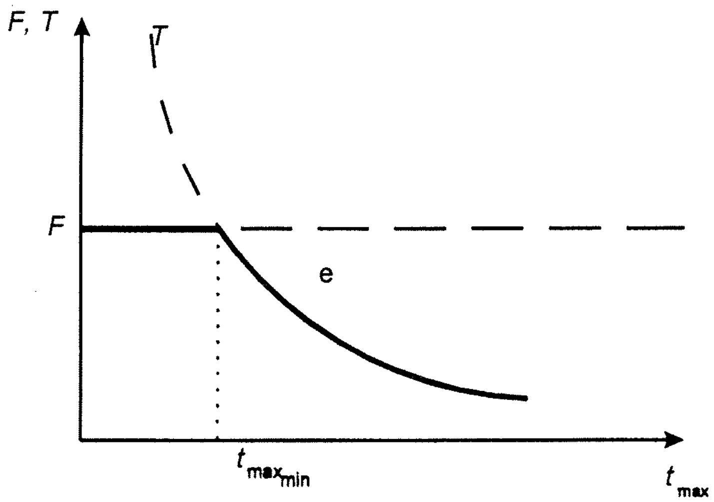


Figure 2-2: The maximum tension force in the mattress as a function of the sag

The equilibrium is described by:

$$F \geq T \qquad [\,Nm^{-1}\,]$$

$$tg\varphi \cdot \frac{\bar{L}}{2} \cdot q = \frac{q}{2t_{max}} \cdot \frac{l^2}{4} \qquad [\,Nm^{-1}\,]$$

The maximum sag in the middle of the span is minimally equal to:

$$\boxed{t_{max_{min}} = \frac{l^2}{4 \cdot tg\varphi \cdot \bar{L}}} \qquad [\,m\,]$$

Consequently the tension force in the mattress cannot exceed:

$$\boxed{F_{max} = tg\varphi \cdot \frac{\bar{L}}{2} \cdot q} \qquad [\,Nm^{-1}\,]$$

For very large sags the parabolic approximation fails. In the limit (*l* = 0), when the mattress hangs vertically in a narrow but deep gap, the tension force equals:

$$T = t \cdot q \qquad [\,Nm^{-1}\,]$$

This means that the sag *t* will always be smaller than a value:

$$t < tg\varphi \cdot \frac{\bar{L}}{2} \qquad [\,m\,]$$

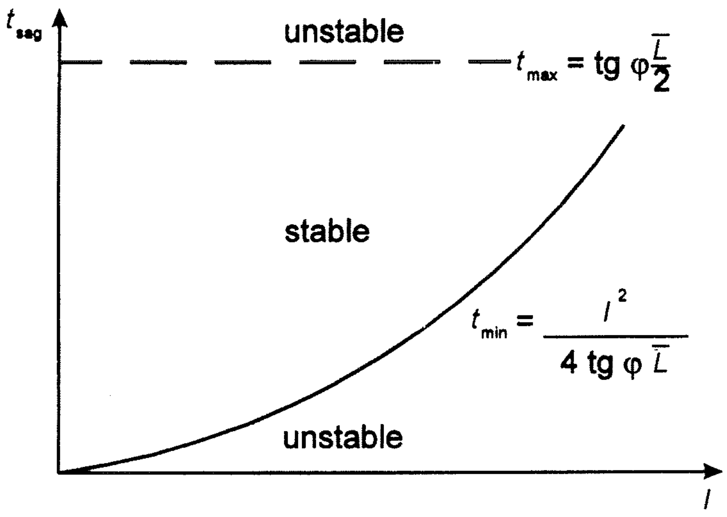


Figure 2-3: The minimum and the maximum sag as a function of the gap width

From these two conditions it appears that equilibrium can be attained at two points t_{min} and t_{max}. Between these points equilibrium is assured.

3. A refined Model of a Mattress spanning a Gap

3.1 The Refinements

The model developed in the previous paragraph will be improved in two steps. First the description of the forces at the supporting corner will be analysed in more detail. This analysis will be confirmed experimentally. Secondly the description op the sag will be improved from a parabola to the more accurate catenary. The improved description of the supporting corner and the sag will be analysed and experimentally verified.

3.2 A refined friction Model for the supporting Corner

The maximum force that may be exerted on a mattress lying on the soil is more complicated than assumed above.

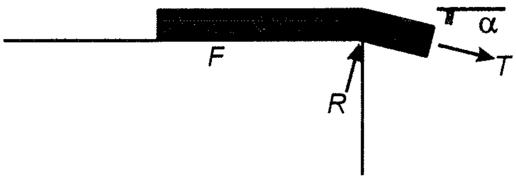


Figure 3-1: The forces at the supporting corner

The friction force *F* and the tension force *T* give rise to a resulting force *R* equal to:

$$\vec{R} = \{(F - T\cos\alpha), T\sin\alpha\} \qquad [\text{Nm}^{-1}]$$

Where: $(F - T\cos\alpha) \geq 0$ [Nm^{-1}]

This force *R* causes an extra friction force equal to:

$$\text{tg}\varphi.|R| = \text{tg}\varphi.\sqrt{(F - T\cos\alpha)^2 + (T\sin\alpha)^2} \qquad [\text{Nm}^{-1}]$$

The total equilibrium is thus expressed as:

$$F + \text{tg}\varphi.\sqrt{(F - T\cos\alpha)^2 + (T\sin\alpha)^2} - T = 0 \qquad [\text{Nm}^{-1}]$$

For the simple case where α = 0° the form reduces to:

$$F - T = 0 \qquad [\text{Nm}^{-1}]$$

If α = 90° the equilibrium is described by:

$$F + \text{tg}\varphi.\sqrt{F^2 + T^2} - T = 0 \qquad [\text{Nm}^{-1}]$$

<u>Experimental verification of the friction model</u>

If a mat of known length $\overline{L}$ hangs from a table the length of the sagging part is bounded by the formula derived above.

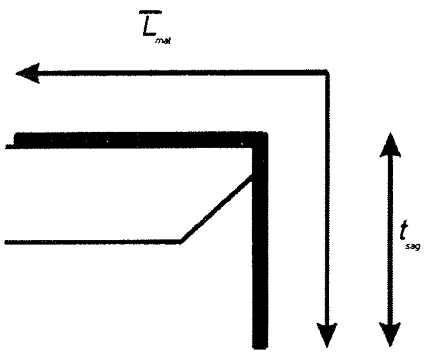


Figure 3-2: The mat hangs from a table

$$F = \text{tg}\varphi.q\left(\bar{L}_{mat} - t_{sag}\right) \qquad [\text{Nm}^{-1}]$$

$$T = q.t_{sag} \qquad [\text{Nm}^{-1}]$$

These values have to be substituted in the equilibrium equation.

$$\text{tg}\varphi.q\left(\bar{L}_{mat} - t_{sag}\right) + \text{tg}\varphi\sqrt{\left(\text{tg}\varphi.q\left(\bar{L}_{mat} - t_{sag}\right)\right)^2 + \left(q.t_{sag}\right)^2} - q.t_{sag} = 0 \qquad [\text{Nm}^{-1}]$$

The only unknown is t_{sag}, which can be numerically solved. From the solution it appears that the ratio t_{sag} / $\bar{L}_{mat}$ is a function of φ.

φ	simple model t_{sag} / $\bar{L}_{mat}$ = tg φ	refined model
20	0.364	0.386
30	0.577	0.602

The length, at which a model mat just kept hanging from the table, was experimentally determined. The friction angle between the mat and the table surface was determined, by sloping the surface, at 20°. Then the maximum sag t was found for various mat lengths. The results are given in Figure 3-3. It is clear that the ratio t_{sag} / $\bar{L}_{mat}$ is well predicted by the model and that the ratio is slightly above tg φ.

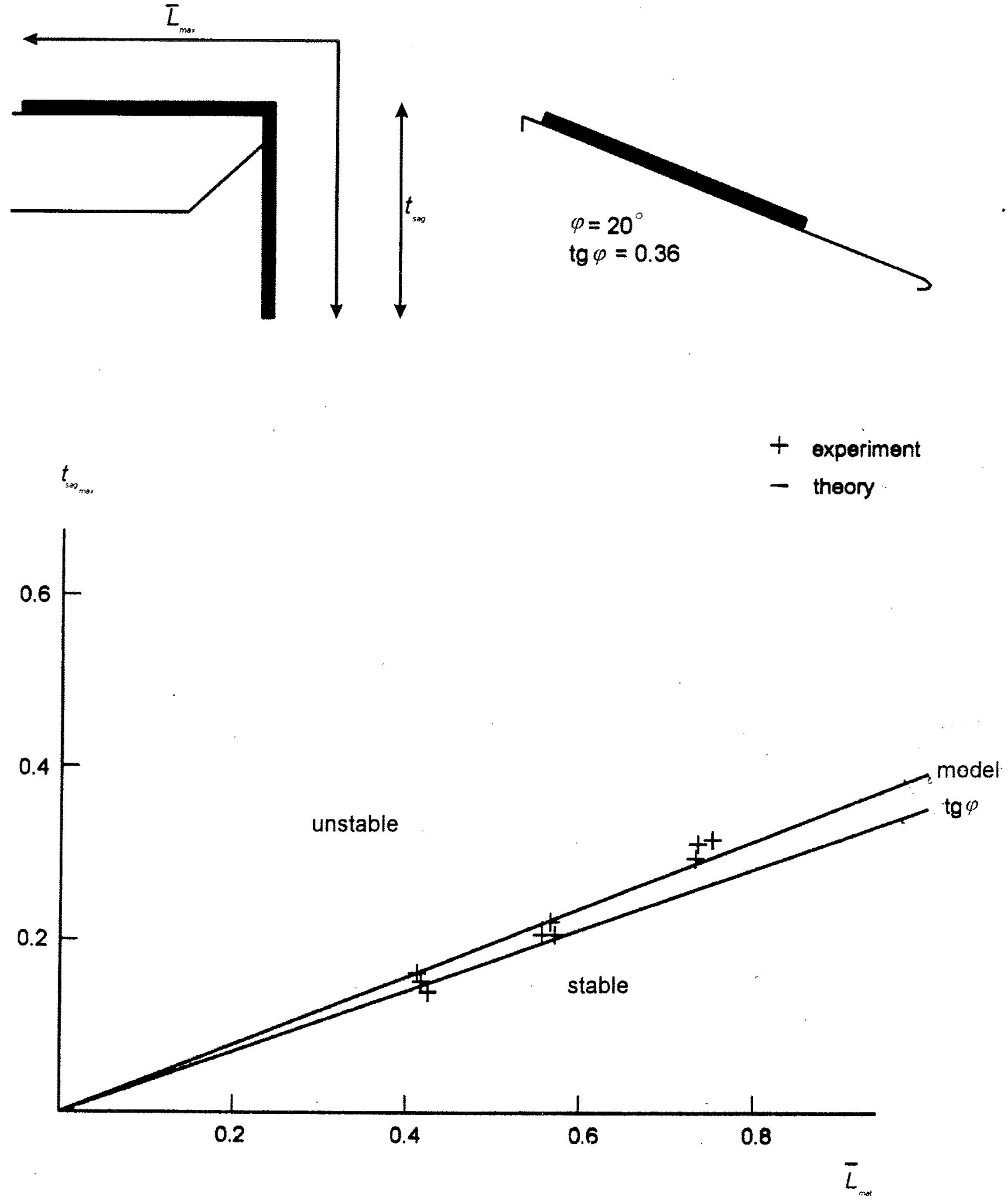


Figure 3-3: The maximal stable sag from a supporting corner as a function of mattress length

3.3 A refined Model for the sagging Mattress spanning a Gap

In the second chapter a parabola approximated the sagging mat. In fact the catenary curve is the exact model for this problem.

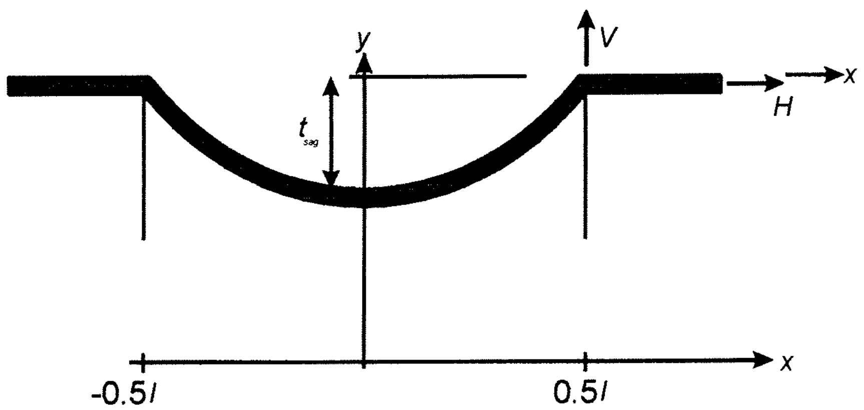


Figure 3-4: The sagging mattress and the system of co-ordinates

It is known that the catenary is described by:

$$y = A\cosh\left(\frac{x}{A}\right) \qquad [\text{m}]$$

Filling in the boundary condition at the support:

$$y = t_{sag} + A = A\cosh\left(\frac{l}{2A}\right) \qquad [\text{m}]$$

Or:

$$A\cosh\left(\frac{l}{2A}\right) - t_{sag} - A = 0 \qquad [\text{m}]$$

From this equation A can be numerically solved.

There from the forces may be calculated by:

$$H = q.A \qquad [\text{Nm}^{-1}]$$

$$V = q.s \qquad [\text{Nm}^{-1}]$$

Where: $s = A\sinh\left(\frac{l}{2A}\right)$ [m]

$$T = \left(t_{sag} + A\right)q \qquad [\text{Nm}^{-1}]$$

If the catenary model for the sagging mat is combined with the model for the maximal friction force at the supports at both sides, the behaviour of the mattress can be studied.

From calculations it appears that there is minimal sag that is sustained by the maximal friction forces. If the mat starts with smaller sag it will slip until the minimal value is reached. When the sag increases further, a point is reached where the weight of the unsupported mat is too large for the friction forces on both sides. The mat will fall in the gap. Between this minimal and maximal value all values of the sag give a stable configuration.

The forces in the mat and the stable area are sketched as a function of t_{sag} in Figure 3-5. The mattress always slips to $t_{sag} \approx 0{,}027$ in this example. Between 0,027 and 0,093 all configurations are stable. If $t_{sag} = 0{,}093$ is exceeded the mattress falls into the gap. In the process the maximal tension force in the mat never rises above the maximal friction force. In the figure the parabolic model is also sketched. The approximation is only acceptable in the area $t_{sag} < t_{sag_{min}}$. The parabolic model underestimates the minimal sag, but due to the limiting aspect of the friction the maximal tension is correctly predicted.

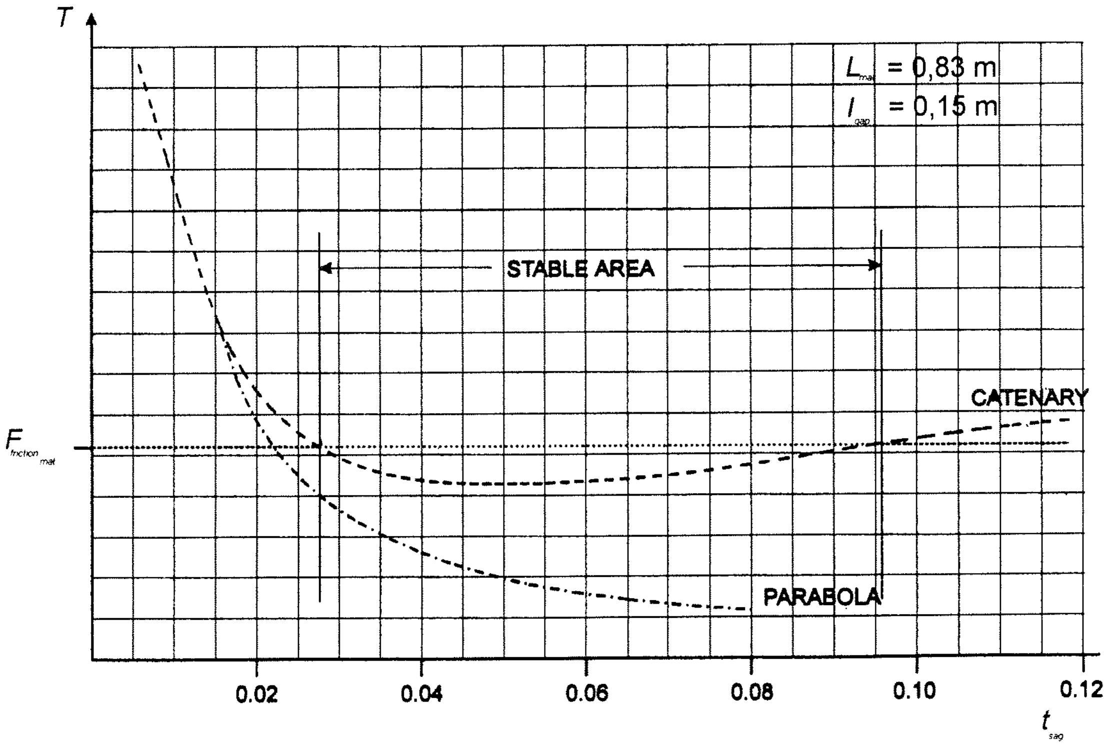


Figure 3-5: The tension force in the mattress as a function of the sag

The minimal and the maximal sag are both a function of the gap width (see Figure 3-6). There is a gap width that cannot be bridged by a mattress of a certain length. In Figure 3-6 this situation is reached at $l = 0{,}21$.

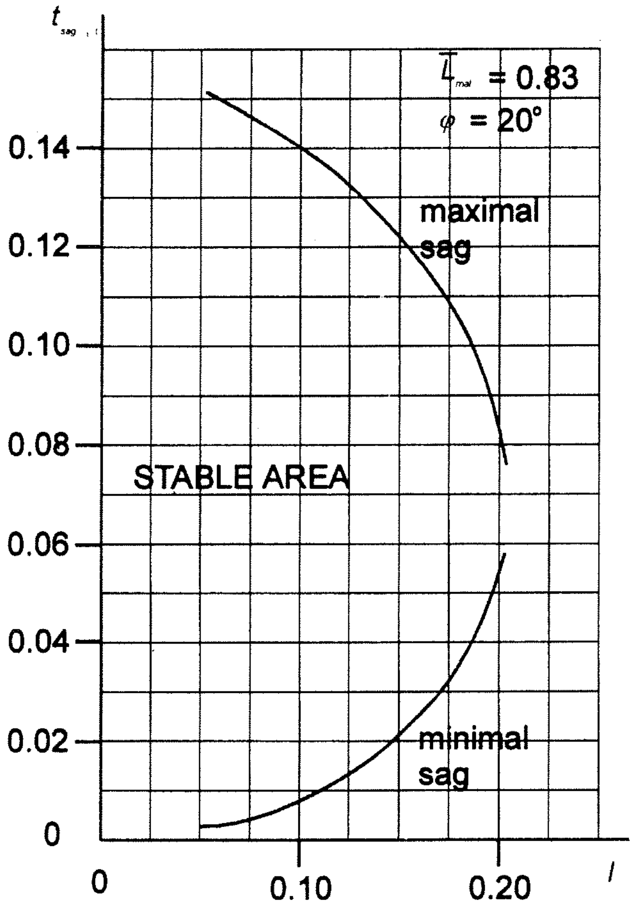


Figure 3-6: The minimal and the maximal stable sag as a function of the gap width

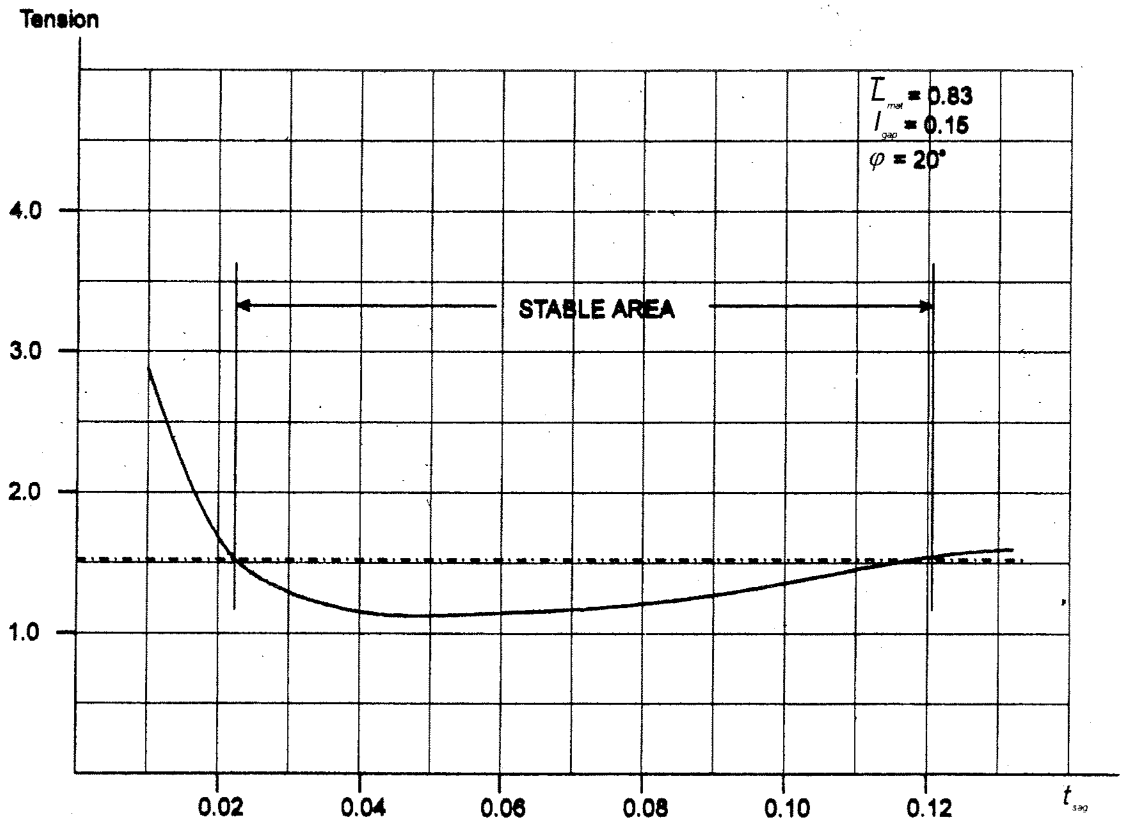


Figure 3-7: The tension force in the mattress as a function of the sag

4. Experimental Verifications of the sagging Mattress Model

The model derived in the previous section is experimentally verified. A model mat was hung between two horizontal surfaces. For various values of the gap width l_{gap} the minimal and the maximal sag were determined.

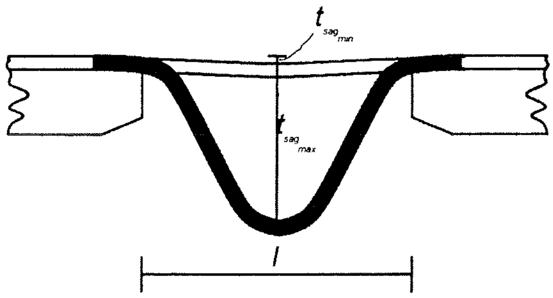


Figure 4-1: A mattress spanning a gap has two positions where equilibrium is just lost

The length of the mattress was 0,83 m. The friction angle between mattress and surface is 20°. The behaviour of the mattress is calculated with the model and sketched in Figure 3-7 for a gap width of l = 0,15 m and in Figure 3-6 as a function of the gap width. In Figure 4-2 the experimental results are plotted as a function of the gap width. The agreement between theory and experiment is acceptable.

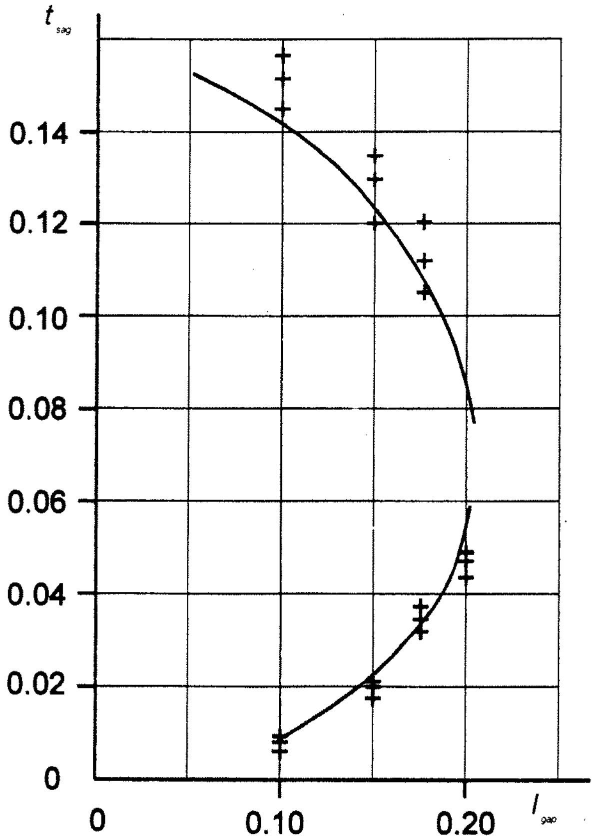


Figure 4-2: The results of experiments compared with the theory

5. Conclusions

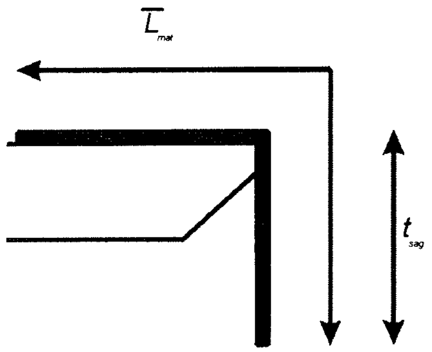
A bed protection mattress spanning a local depression caused by a soil failure will always slip to minimum sag. At sags equal to or larger than this minimum value equilibrium is assured. If however the sag exceeds a certain maximum value, the weight of the sag cannot be supported anymore by the ends lying on the undisturbed bed. The mattress will slide into the gap (see Figure 3-6 and Figure 3-7).

The maximum friction force that can be mobilised (see Figure 3-7) will always limit the tension force in the mattress. In a first approximation the tension force is limited by:

$$T_{max} \leq F_{max} = tg\varphi \cdot \frac{L}{2} \cdot q \qquad [\,Nm^{-1}\,]$$

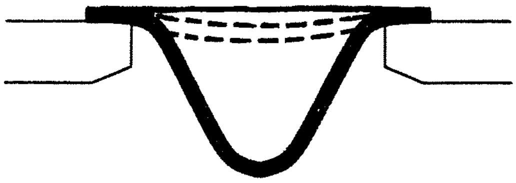
In the report it is shown that in an improved model the maximal tension force may be slightly (+/-5%) greater due to extra friction at the supporting corners.

Appendix I: Experimental Results Friction Model



$\bar{L}_{1_{mat}}$	t_{sag}	t_{sag} / $\bar{L}_{1_{mat}}$
0.83	0.31	0.37
0.83	0.30	0.36
0.83	0.31	0.37
0.56	0.22	0.39
0.56	0.21	0.37
0.56	0.21	0.37
0.42	0.14	0.33
0.42	0.15	0.36
0.42	0.16	0.38

Appendix II: Experimental Results sagging Mat



l_{gap} [m]	t_{min} [cm]	t_{max} [cm]
0.20	4.5 4.0	4.5 4.5
0.175	3.2 3.7 3.4	10.5 12.0 11.2
0.15	1.9 2.0 1.7	13.4 12.9 12.0
0.125	1.2 1.3 1.3	14.0 14.2 14.4
0.10	0.8 0.7 0.6	14.4 15.1 15.6

Appendix III: Program Catenary

```
PROGRAM CATENARY(INPUT,OUTPUT);
CONST
  dr = 0.01745;

VAR
  L_mat,
  l_gap,
  min,max,
  tangent,
  Tension,
  Tension_H,
  Tension_V,
  Tension_max,
  q_mat,
  t_sag,
  x,A          : REAL;

{$I \PASCAL\FUNCT\HYPER.FUN }


FUNCTION G(X : REAL):REAL;
  BEGIN
    G:= X * ( COSH( l_gap/(2*X) )-1) - t_sag;
  END;




PROCEDURE Newton_Raphson( VAR X:REAL);
  CONST
    Precision = 0.000001;
    Eps = 0.001;
    MaxIt = 100;
  VAR
    I                  : INTEGER;
    X_l,X_r,
    Y_l,Y_r,dY,Y       :REAL;

  BEGIN
  I := 0;  X_l := X; Y := G(X);

  WHILE (ABS( Y ) > Precision) AND (I < MaxIt) DO
   BEGIN
    Y_r := G((1+Eps)*X_l);  Y_l := Y;
    dY  := (Y_r - Y_l)/(Eps*X_l);
    X_l := X_l -  Y_l / dY;
    Y   := G(X_l);
{    writeln(X_l:10:3,Y:10:3); }
    I := I+1;
   END;
```

```
    X := X_1;
   END;


  BEGIN
  WRITELN(LST);
  WRITELN(LST,'      CALCULATION OF CATENARY ');
  WRITELN(LST);
  WRITELN(LST);
  WRITE('      L_gap       = ');READLN(L_gap);
  WRITE('      q_cable     = ');READLN(q_mat);
  WRITE('      t_sag_min   = ');READLN(min);
  WRITE('      t_sag_max   = ');READLN(max);



  ClrScr;
  writeln;
  WRITELN(LST,'      L_gap       = ',L_gap:5:3,'  [m]');
  WRITELN(LST,'      q_cable     = ',q_mat:5:3,'  [kN/m]');
  WRITELN(LST,'      t_sag_min   = ',min:5:3,'  [m]');
  WRITELN(LST,'      t_sag_max   = ',max:5:3,'  [m]');

  writeln(lst);

    writeln(LST,'       H          V          T        t_sag     L_cable
  A');
    writeln(LST,'      [kN]       [kN]       [kN]       [m]         [m] ');

  t_sag := min;

  WHILE t_sag <= max DO

   BEGIN
    A:= sqr(l_gap)/(8*t_sag);

    Newton_Raphson(A);

    L_mat     := 2 * A  * SINH( l_gap/(2*A));
    Tension_H := A * q_mat;
    Tension_V := 0.5 * L_mat * q_mat;
    Tension   := ( t_sag + A)* q_mat;


  Writeln(LST,Tension_H:10:3,Tension_V:10:3,Tension:10:3,t_sag:10:3,L_
  mat:10:3,A:10:3);

    t_sag := t_sag + (max - min )/19;
  END;
  WRITELN(LST);
  WRITELN(lst,CHR(12));

  END.
```

Appendix IV: Program Corner Mat

```
PROGRAM CORNER_MAT(INPUT,OUTPUT);
CONST
  dr = 0.01745;

VAR
  L_mat,
  l_gap,
  phi,
  tangent,
  f_friction,
  q_mat,
  t_sag,
  x,A          : REAL;

{$I \PASCAL\FUNCT\HYPER.FUN }


FUNCTION G( t_sag : REAL):REAL;
 VAR
   H,V : REAL;

  BEGIN
    V:= q_mat * t_sag;
    H:= f_friction * q_mat * (L_mat-t_sag);
    G:= V - H - f_friction * sqrt( sqr(H) + sqr(V) ) ;
  END;




PROCEDURE Newton_Raphson( VAR X:REAL);
  CONST
    Eps = 0.001;
    MaxIt = 10;
  VAR
    I                  : INTEGER;
    X_l,X_r,
    Y_l,Y_r,dY,Y       :REAL;

  BEGIN
  I := 0;  X_l := X; Y := G(X);

  WHILE (ABS( Y ) > Eps) AND (I < MaxIt) DO
   BEGIN
    Y_r := G((1+Eps)*X_l);  Y_l := Y;
    dY  := (Y_r - Y_l)/(Eps*X_l);
    X_l := X_l -  Y_l / dY;
    Y   := G(X_l);
    I  := I+1;
   END;
```

```
    X := X_1;
   END;


BEGIN
phi   := 28.0;

q_mat := 10.0;

t_sag :=0.01;

WHILE phi <= 32.0 DO
BEGIN
f_friction := TAN(dr*phi);
ClrScr;
writeln(lst);writeln(lst);writeln(lst);
WRITELN(LST,'      phi    = ',phi  :5:2);
writeln(lst);
writeln(LST,'      L_mat       t_sag    ');

L_mat := 0.1;

WHILE L_mat <= 1.0 DO
 BEGIN
  A:= L_mat /5;

  Newton_Raphson(A);


  t_sag := A  ;

  Writeln(LST,L_mat:10:3,t_sag:10:3);

  L_mat := L_mat + 0.1;
END;
phi := phi + 2.0;
END;

END.
```

Appendix V: Program sagging Mat

```
PROGRAM SAGGING_MAT(INPUT,OUTPUT);
CONST
  dr = 0.01745;

VAR
  I            : INTEGER;
  t_m          : ARRAY[1..2] OF REAL;
  L_mat,
  l_gap,
  phi,
  tangent,
  Tension,
  Tension_H,
  Tension_V,
  Tension_max,
  Friction,
  f,Hulp,
  q_mat,
  t_sag,
  t_old,
  Eq, Eq_old,
  x,A            : REAL;

 {$I \PASCAL\FUNCT\HYPER.FUN }


 FUNCTION G(X : REAL):REAL;
   BEGIN
     G:= X * ( COSH( l_gap/(2*X) )-1) - t_sag;
   END;




 PROCEDURE Newton_Raphson( VAR X:REAL);
   CONST
     Eps = 0.001;
     MaxIt = 10;
   VAR
     I                  : INTEGER;
     X_l,X_r,
     Y_l,Y_r,dY,Y       :REAL;

   BEGIN
   I := 0;  X_l := X; Y := G(X);

   WHILE (ABS( Y ) > Eps) AND (I < MaxIt) DO
    BEGIN
     Y_r := G((1+Eps)*X_l);  Y_l := Y;
     dY  := (Y_r - Y_l)/(Eps*X_l);
```

```
    X_1 := X_1 -  Y_1 / dY;
    Y   := G(X_1);
{    writeln(X_1:10:3,Y:10:3); }
    I := I+1;
   END;
   X := X_1;
  END;


BEGIN
phi   := 20.0;
L_mat := 0.83;
q_mat := 10.0;
l_gap :=0.2;

WRITELN(LST,'L_mat = ',L_mat:5:3);
WRITELN(LST,'q_mat = ',q_mat:5:3);
WRITELN(LST,'phi   = ',phi  :5:3);
WRITELN(LST);
WRITELN(LST,'     l_gap      t_min        t_max');


WHILE l_gap <= 0.21 DO
BEGIN
t_sag :=0.0005;t_m[1]:=0;t_m[2]:=0;
Eq := -3.0; I:= 1;

f := TAN(dr*phi);


WHILE t_sag < 0.25 DO
 BEGIN
  A := sqr(l_gap)/(8*t_sag);
  Newton_Raphson(A);

  Tension_H := A * q_mat;
  Tension_V := A * q_mat * SINH( l_gap/(2*A));
  Tension := ( t_sag + A)* q_mat;
  Friction := f * (q_mat*L_mat/2 - Tension_V);
  Hulp := Friction - Tension_H;
  IF Hulp <0 THEN Hulp :=0.0;
  Tension_max := Friction + f * SQRT( SQR(Hulp) + SQR(Tension_V) );

 Eq:= ( Tension_max - Tension) ;
  IF Eq *Eq_old < 0 THEN
             BEGIN
               t_m[I] :=(t_old*Eq - t_sag*Eq_old)/(Eq - Eq_old);
               I:=2;
             END;
  t_old := t_sag; Eq_old := Eq;
  t_sag := t_sag + 0.0005;
END;

WRITELN(LST,l_gap:10:3,t_m[1]:10:3,t_m[2]:10:3);
l_gap := l_gap + 0.01;
END;
```