M, N, and H

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DOI
10.1016/j.topol.2025.109539

Publication date
2025

Document Version
Final published version

Published in
Topology and its Applications

Citation (APA)
Brian, W., Dow, A., & Hart, K. P. (2025). M, N, and H. *Topology and its Applications*, *373*, Article 109539. https://doi.org/10.1016/j.topol.2025.109539

Topology and its Applications 373 (2025) 109539

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

$\mathbb{M}^*$, $\mathbb{N}^*$, and $\mathbb{H}^*$

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ARTICLE INFO

Article history:
Received 7 May 2025
Received in revised form 1 August 2025
Accepted 1 August 2025
Available online 8 August 2025

To the memory of Peter Nyikos

MSC:
primary 54D40, 03E35
secondary 06D50, 3C20

Keywords:
Čech-Stone remainders
Autohomeomorphisms
Continuum Hypothesis
Applications of elementarity

ABSTRACT

Let $\mathbb{M} = \mathbb{N} \times [0,1]$. The natural projection $\pi : \mathbb{M} \to \mathbb{N}$, which sends (n,x) to n, induces a projection mapping $\pi^* : \mathbb{M}^* \to \mathbb{N}^*$, where $\mathbb{M}^*$ and $\mathbb{N}^*$ denote the Čech-Stone remainders of $\mathbb{M}$ and $\mathbb{N}$, respectively.

We show that CH implies every autohomeomorphism of $\mathbb{N}^*$ lifts through the natural projection to an autohomeomorphism of $\mathbb{M}^*$. That is, for every homeomorphism $h : \mathbb{N}^* \to \mathbb{N}^*$ there is a homeomorphism $H : \mathbb{M}^* \to \mathbb{M}^*$ such that $\pi^* \circ H = h \circ \pi^*$. This complements a recent result of the second author, who showed that this lifting property is not a consequence of ZFC.

Combining this lifting theorem with a recent result of the first author, we also prove that CH implies there is an order-reversing autohomeomorphism of $\mathbb{H}^*$, the Čech-Stone remainder of the half line $\mathbb{H} = [0,\infty)$.

1. Introduction

Let $\mathbb{I}$ denote the unit interval $[0,1]$, and let $\mathbb{M} = \mathbb{N} \times \mathbb{I}$. Let $\pi : \mathbb{M} \to \mathbb{N}$ denote the natural projection $(n,x) \mapsto n$. Moving to the Čech-Stone compactifications, π extends to a continuous surjection $\beta\pi : \beta\mathbb{M} \to \beta\mathbb{N}$. Because π is surjective and $\pi^{\leftarrow}(n)$ is compact for every n, the map $\beta\pi$ restricts to a surjection from $\mathbb{M}^* = \beta\mathbb{M} \setminus \mathbb{M}$ onto $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. Let $\pi^* = \beta\pi \restriction \mathbb{M}^*$. In other words, π^* is the continuous surjection $\mathbb{M}^* \to \mathbb{N}^*$ induced by π.

This paper is organized around two main theorems. The first states that, assuming the Continuum Hypothesis (henceforth CH), every autohomeomorphism of $\mathbb{N}^*$ can be "lifted" through π^* to an autohomeomorphism of $\mathbb{M}^*$:

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[1] The first author is supported in part by NSF grant DMS-2154229.

https://doi.org/10.1016/j.topol.2025.109539

Theorem 1. *Assuming* CH*, if h is an autohomeomorphism of $\mathbb{N}^*$, then there is an autohomeomorphism H of $\mathbb{M}^*$ such that $\pi^* \circ H = h \circ \pi^*$.*

This theorem complements a recent result of the second author in [3], where he shows that the conclusion of Theorem 1 fails consistently. Specifically, this lifting property fails in the model of Veličković from [15] in which $\mathsf{MA}_{\aleph_1}$ holds and $\mathbb{N}^*$ has a nontrivial autohomeomorphism. Theorem 1 and the main theorem of [3], taken together, answer Question 2.4 in [4].

It is not difficult to see that all trivial autohomeomorphisms of $\mathbb{N}^*$ lift through π^*. (We give the easy argument in Section 2.) Thus the conclusion of Theorem 1 holds if all autohomeomorphisms of $\mathbb{N}^*$ are trivial, which is implied by forcing axioms like PFA, or even just OCA_T. Thus this lifting property provides a rare example of a statement about Čech-Stone remainders that follows from forcing axioms and from CH, but not from ZFC.

Our second theorem concerns yet another Čech-Stone remainder. Let $\mathbb{H} = [0, \infty)$ denote the space of nonnegative real numbers, and let $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$. The order on $\mathbb{H}$ induces a quasiorder on certain subsets of $\mathbb{H}^*$. This is explained further in Section 4 below (see also the survey of $\mathbb{H}^*$ by the third author, [7]). An old folklore question about $\mathbb{H}^*$ is whether there is an autohomeomorphism of $\mathbb{H}^*$ that reverses this order. Our second theorem shows that, consistently, there is.

Theorem 2. CH *implies there is an order-reversing autohomeomorphism of $\mathbb{H}^*$.*

On the other hand, a recent result of Vignati (see [16, Theorem C]) states that $\mathsf{OCA}_T + \mathsf{MA}$ implies all autohomeomorphisms of $\mathbb{H}^*$ are trivial. An easy argument (which is given in Section 4 below) shows that all trivial autohomeomorphisms of $\mathbb{H}^*$ are order-preserving. Thus the existence of an order-reversing autohomeomorphism of $\mathbb{H}^*$ is independent of ZFC.

Theorem 2 could really be called a corollary. It follows relatively easily from two other theorems: Theorem 1 stated above, and a recent result of the first author in [1], which states that, assuming CH, the shift map and its inverse are conjugate in the autohomeomorphism group of $\mathbb{N}^*$.

The next two sections are devoted to $\mathbb{M}^*$ and $\mathbb{N}^*$, and the proof of Theorem 1. The fourth and final section of the paper contains some background material on $\mathbb{H}^*$ and a proof of Theorem 2.

2. More on $\mathbb{M}^*$ and $\mathbb{N}^*$

The aim of this section is to introduce some ideas and notation concerning $\mathbb{M}^*$ and $\mathbb{N}^*$, and to prove two relatively easy positive results similar to Theorem 1. These two results form part of the motivation for proving Theorem 1.

An *almost permutation* of $\mathbb{N}$ is a bijection from one co-finite subset of $\mathbb{N}$ to another. A *trivial autohomeomorphism* of $\mathbb{N}^*$ is a homeomorphism h induced by an almost permutation f of $\mathbb{N}$, in the sense that $h = \beta f \restriction \mathbb{N}^*$, or equivalently, the action of h on the clopen subsets of $\mathbb{N}^*$ is simply $h[A^*] = (f[A])^*$, for all $A \subseteq \mathbb{N}$. Similarly, if f is a homeomorphism between two co-compact subsets of $\mathbb{M}$ then $H_f = \beta f \restriction \mathbb{M}^*$ is an autohomeomorphism of $\mathbb{M}^*$, and any such autohomeomorphism of $\mathbb{M}^*$ is called *trivial*.

Proposition 3. *If h is a trivial autohomeomorphism of $\mathbb{N}^*$, then there is a trivial autohomeomorphism H of $\mathbb{M}^*$ such that $\pi^* \circ H = h \circ \pi^*$.*

Proof. Let h be a trivial autohomeomorphism of $\mathbb{N}^*$. Fix an almost permutation f of $\mathbb{N}$ such that $h = \beta f \restriction \mathbb{N}^*$. Define $g : \mathrm{dom}(f) \times \mathbb{I} \to \mathbb{M}$ by setting $g(n, x) = (f(n), x)$. Observe that g is a homeomorphism from one co-compact subset of $\mathbb{M}$ to another, and therefore $H_g = \beta g \restriction \mathbb{M}^*$ is a trivial autohomeomorphism of $\mathbb{M}^*$. Because $\pi \circ g = f \circ \pi$, we have $\beta\pi \circ \beta g = \beta f \circ \beta\pi$, and then restricting to the Čech-Stone remainders, $\pi^* \circ H_g = h \circ \pi^*$. $\square$

The existence of a nontrivial autohomeomorphism of $\mathbb{N}^*$ is independent of ZFC. On the one hand, Walter Rudin proved in [11] that CH implies there are $2^{\mathfrak{c}}$ autohomeomorphisms in total, though of course only $\mathfrak{c}$ of them can be trivial. On the other hand, Shelah proved in [12, Chapter 4], via an oracle-c.c. iteration, that it is consistent to have all autohomeomorphisms trivial. Building on Shelah's work, Shelah and Steprāns showed in [13] that PFA implies all autohomeomorphisms are trivial. Veličković showed in [15] that $\mathsf{OCA}_T + \mathsf{MA}$ suffices, though it is consistent with $\mathsf{MA}_{\aleph_1}$ to have nontrivial autohomeomorphisms. (Here OCA_T denotes Todorčević's Open Coloring Axiom, defined in [14], now sometimes called OGA.) Building on work of Moore in [10], DeBondt, Farah, and Vignati showed in [2] that OCA_T alone implies all autohomeomorphisms of $\mathbb{N}^*$ are trivial. Combined with Proposition 3, this shows Theorem 1 remains true when CH is replaced with OCA_T.

Corollary 4. *Assuming OCA_T, if h is an autohomeomorphism of $\mathbb{N}^*$, then there is an autohomeomorphism H of $\mathbb{M}^*$ such that $\pi^* \circ H = h \circ \pi^*$.* □

In light of this, the main point of Theorem 1 is not simply that the conclusion is consistent, but specifically that it follows from CH. This is good to know for two reasons: because it contributes to the longstanding program of understanding the behavior of Čech-Stone remainders under CH, and because it enables us to prove Theorem 2. Note that the conclusion of Theorem 2 is not implied by OCA_T (see Proposition 12 below), and indeed, we do not currently know how to obtain an order-reversing autohomeomorphism of $\mathbb{H}^*$ except via CH.

For each $n \in \mathbb{N}$, let $\mathbb{I}_n = \pi^{\leftarrow}(n) = \{n\} \times \mathbb{I}$. These are the connected components of $\mathbb{M}$. Analogously, for each $u \in \mathbb{N}^*$ let $\mathbb{I}_u = (\pi^*)^{\leftarrow}(u)$. Equivalently,

$$\mathbb{I}_u = \bigcap_{A \in u} (\pi^*)^{\leftarrow}\left[\mathrm{cl}_{\beta\mathbb{N}} A\right] = \bigcap_{A \in u} \mathrm{cl}_{\beta\mathbb{M}}(\pi^{\leftarrow}[A]).$$

These are the connected components of $\mathbb{M}^*$ (see [7, Corollary 2.2]).

In particular, if H is an autohomeomorphism of $\mathbb{M}^*$, then H permutes the set $\{\mathbb{I}_u : u \in \mathbb{N}^*\}$ of connected components of $\mathbb{M}^*$. Let ρ_H denote the corresponding permutation of $\mathbb{N}^*$, so that $H(\mathbb{I}_u) = \mathbb{I}_{\rho_H(u)}$ for all $u \in \mathbb{N}^*$.

If B^* is clopen in $\mathbb{N}^*$, then $(\pi^*)^{\leftarrow}[B^*]$ is clopen in $\mathbb{M}^*$, by the continuity of π^*. But the converse is also true: if C is clopen in $\mathbb{M}^*$, then $C = (\pi^*)^{\leftarrow}[B^*]$ for some clopen $B^* \subseteq \mathbb{N}^*$. To see this, note that if C is clopen then C and $\mathbb{M}^* \setminus C$ are both compact, so $\pi^*[C]$ and $\pi^*[\mathbb{M}^* \setminus C]$ are both compact as well. But these sets are disjoint, because $(\pi^*)^{\leftarrow}(u) = \mathbb{I}_u$ is connected for each $u \in \mathbb{N}^*$, which means either $\mathbb{I}_u \subseteq C$ or $\mathbb{I}_u \subseteq \mathbb{M}^* \setminus C$. Hence $\pi^*[C]$ and $\pi^*[\mathbb{M}^* \setminus C]$ are complementary closed sets, hence clopen, and $C = (\pi^*)^{\leftarrow}[\pi^*[C]]$ and $\mathbb{M}^* \setminus C = (\pi^*)^{\leftarrow}[\pi^*[\mathbb{M}^* \setminus C]]$.

Proposition 5. *If H is an autohomeomorphism of $\mathbb{M}^*$, then ρ_H is an autohomeomorphism of $\mathbb{N}^*$ such that $\pi^* \circ H = \rho_H \circ \pi^*$.*

Proof. That $\pi^* \circ H = \rho_H \circ \pi^*$ follows from the definition of ρ_H, so we need only show ρ_H is an autohomeomorphism of $\mathbb{N}^*$. Because ρ_H is bijective (again, by definition) and $\mathbb{N}^*$ is compact, it suffices to show ρ_H is continuous. Let $A \subseteq \mathbb{N}$, so that A^* is a basic clopen subset of $\mathbb{N}^*$. Then $(\pi^*)^{\leftarrow}[A^*]$ is a clopen subset of $\mathbb{M}^*$. Because H is a homeomorphism, $H^{\leftarrow}\left[(\pi^*)^{\leftarrow}[A^*]\right]$ is clopen as well. By the paragraph preceding this proposition, this means there is some clopen $B^* \subseteq \mathbb{N}^*$ such that $H^{\leftarrow}\left[(\pi^*)^{\leftarrow}[A^*]\right] = (\pi^*)^{\leftarrow}[B^*]$. But

$$H^{\leftarrow}\left[(\pi^*)^{\leftarrow}[A^*]\right] = (\pi^* \circ H)^{\leftarrow}[A^*] = (\rho_H \circ \pi^*)^{\leftarrow}[A^*] = (\pi^*)^{\leftarrow}\left[\rho_H^{\leftarrow}[A^*]\right],$$

so $(\pi^*)^{\leftarrow}\left[\rho_H^{\leftarrow}[A^*]\right] = (\pi^*)^{\leftarrow}[B^*]$, which implies $\rho_H^{\leftarrow}[A^*] = B^*$. □

In other words, this proposition states that autohomeomorphisms of $\mathbb{M}^*$ project downward through π^* to autohomeomorphisms of $\mathbb{N}^*$. That is, Theorem 1 remains true, without even needing to assume CH, if we switch the roles of $\mathbb{M}^*$ and $\mathbb{N}^*$. The opposite direction, lifting upward through π^* rather than projecting downward, is more subtle, and this more difficult direction is the content of Theorem 1.

We would like to thank our anonymous referee for a thoughtful report that helped us to improve the paper.

3. A proof of Theorem 1

Proof of Theorem 1: Let h be an autohomeomorphism of $\mathbb{N}^*$. Using CH, we aim to construct an autohomeomorphism H of $\mathbb{M}^*$ such that $\pi^* \circ H = h \circ \pi^*$. Our construction of H needs a few ingredients.

The first is a map $h^+ : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ with the property that for all subsets A of $\mathbb{N}$ we have $h[A^*] = h^+(A)^*$.

The second is a suitable base for the closed sets of $\mathbb{M}^*$ that is a distributive lattice with respect to $\cup$ and $\cap$. We shall describe H dually by specifying an automorphism of that base.

Let $\mathcal{B}$ be a countable distributive lattice base for the closed sets of $\mathbb{I}$, say the lattice generated by the family of closed intervals with rational end points. We identify members of $\mathcal{B}^{\mathbb{N}}$ with closed subsets of $\mathbb{M}$ in the obvious way: if $B \in \mathcal{B}^{\mathbb{N}}$ then

$$F_B = \bigcup \big\{ \{k\} \times B(k) : k \in \mathbb{N} \big\}$$

In this way the power $\mathcal{B}^{\mathbb{N}}$ determines a base for the closed sets of $\mathbb{M}$, and hence also for the closed sets of $\beta\mathbb{M}$, as the following lemma implies.

Lemma 6. *If F and G are closed and disjoint subsets of $\mathbb{M}$ then there are members B and C of $\mathcal{B}^{\mathbb{N}}$ such that $F \subseteq F_B$, $G \subseteq F_C$ and $F_B \cap F_C = \emptyset$.* □

Because for closed subsets F and G of $\mathbb{M}$ we have $\mathrm{cl}_{\beta\mathbb{M}} F \cap \mathbb{M}^* = \mathrm{cl}_{\beta\mathbb{M}} G \cap \mathbb{M}^*$ if and only if $\{n : F \cap \mathbb{I}_n \neq G \cap \mathbb{I}_n\}$ is finite, we see that the reduced power $\mathcal{B}^{\mathbb{N}}/\mathit{fin}$ determines a base for the closed sets of $\mathbb{M}^*$: the family $\{F_B^* : B \in \mathcal{B}^{\mathbb{N}}\}$. We have $F_B^* \subseteq F_C^*$ if and only if $\{k : B(k) \subseteq C(k)\}$ is co-finite. The latter condition also defines a partial order on $\mathcal{B}^{\mathbb{N}}/\mathit{fin}$, which we denote $\leq$.

It follows that if we let B^* denote the equivalence class of $B \in \mathcal{B}^{\mathbb{N}}$ in $\mathcal{B}^{\mathbb{N}}/\mathit{fin}$ we get the equivalence

$$B^* \leq C^* \quad \text{ if and only if } \quad F_B^* \subseteq F_C^*$$

for $B, C \in \mathcal{B}^{\mathbb{N}}$.

The algebraic structure of the lattice $\mathcal{B}^{\mathbb{N}}/\mathit{fin}$ is determined completely by its partial order, so it will suffice to define an automorphism of the partially ordered set $(\mathcal{B}^{\mathbb{N}}/\mathit{fin}, \leq)$.

We define this automorphism by defining a partial map $\varphi : \mathcal{B}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$ with the following properties. (The construction is detailed below.)

(1) If $B \in \mathcal{B}^{\mathbb{N}}$ then there are unique $C \in \operatorname{dom}\varphi$ and $D \in \operatorname{ran}\varphi$ such that $B =^* C$ and $B =^* D$; this uniqueness ensures that φ determines a well-defined surjection from $\mathcal{B}^{\mathbb{N}}/\mathit{fin}$ to itself.
(2) If B and C are in $\operatorname{dom}\varphi$, and if $A_1 = \{k : B(k) \subseteq C(k)\}$ and $A_2 = \{k : C(k) \subseteq B(k)\}$, then $\{k : \varphi(B)(k) \subseteq \varphi(C)(k)\} =^* h^+(A_1)$ and $\{k : \varphi(C)(k) \subseteq \varphi(B)(k)\} =^* h^+(A_2)$.

In condition (2) the set $A_3 = \mathbb{N} \setminus (A_1 \cup A_2)$ is the set of k where $B(k)$ and $C(k)$ are incomparable. It follows that $h^+(A_3)$ is mod finite equal to the set of k where $\varphi(B)(k)$ and $\varphi(C)(k)$ are incomparable.

Lemma 7. *The conditions above ensure that φ induces an automorphism of the partially ordered set $(\mathcal{B}^{\mathbb{N}}/\mathit{fin}, \leq)$.*

Proof. Since $\operatorname{dom}\varphi$ and $\operatorname{ran}\varphi$ intersect every equivalence class in exactly one point we automatically obtain a surjective map φ^* from $\mathcal{B}^{\mathbb{N}}/\mathit{fin}$ to itself.

In order to see that φ^* is injective assume $\varphi^*(B^*) = \varphi^*(C^*)$. This means that $\{k : \varphi(B)(k) = \varphi(C)(k)\}$ is co-finite, hence so are $\{k : \varphi(B)(k) \subseteq \varphi(C)(k)\}$ and $\{k : \varphi(B)(k) \subseteq \varphi(C)(k)\}$. But this means, with the notation as in condition (2) above, that $h^+(A_1)$ and $h^+(A_2)$ are co-finite too. Because h^+ represents h the sets A_1 and A_2 must then be co-finite as well, and we conclude that $\{k : B(k) = C(k)\}$ is co-finite and hence that $B =^* C$.

Similarly, using that A_1 is co-finite if and only if $h^+(A_1)$ is co-finite, we obtain that $B^* \leq C^*$ iff $\varphi(B^*) \leq \varphi^*(C^*)$, and so φ^* is an automorphism. □

Given $A \subseteq \mathbb{N}$, define $\mathbb{M}^*_A = \left(\bigcup_{n\in A} \mathbb{I}_n\right)^* = \operatorname{cl}_{\beta\mathbb{M}}\left(\bigcup_{n\in A} \mathbb{I}_n\right) \setminus \mathbb{M}$.

Lemma 8. *The autohomeomorphism H of $\mathbb{M}^*$ determined by φ^* satisfies $h \circ \pi^* = \pi^* \circ H$.*

Proof. To show that $\pi^* \circ H = h \circ \pi^*$ we let $A \subseteq \mathbb{N}$ and show that

$$\pi^*\big[H[\mathbb{M}^*_A]\big] = h\big[\pi^*[\mathbb{M}^*_A]\big].$$

Define B and I in $\mathcal{B}^{\mathbb{N}}$ by

$$B(k) = \begin{cases} \mathbb{I} & \text{if } k \in A \\ \emptyset & \text{if } k \notin A \end{cases}$$

and $I(k) = \mathbb{I}$ for all k; in our construction we shall have $I \in \operatorname{dom}\varphi$ and $\varphi(I) = I$.

Then $A = \{k : I(k) \subseteq B(k)\}$ and so

$$\{k : \mathbb{I} \subseteq \varphi(B)(k)\} = \{k : \varphi(I)(k) \subseteq \varphi(B)(k)\} =^* h^+(A).$$

Likewise $\mathbb{N} \setminus A = \{k : B(k) \subseteq \emptyset\}$ and we obtain

$$\{k : \varphi(B)(k) = \emptyset\} =^* h^+(\mathbb{N} \setminus A) =^* \mathbb{N} \setminus h^+(A).$$

We deduce that $H[\mathbb{M}^*_A] = \mathbb{M}^*_{h^+(A)}$ and so

$$\pi^*\big[H[\mathbb{M}^*_A]\big] = \pi^*[\mathbb{M}^*_{h^+(A)}] = h^+(A)^* = h[A^*] = h\big[\pi^*[\mathbb{M}^*_A]\big]. \quad \square$$

3.1. The construction

Using CH, fix an enumeration $\langle B_\alpha : \alpha \in \omega_1\rangle$ of $\mathcal{B}^{\mathbb{N}}$. In a recursion of length ω_1 we construct two sequences $\langle C_\alpha : \alpha \in \omega_1\rangle$ and $\langle D_\alpha : \alpha \in \omega_1\rangle$ of members of $\mathcal{B}^{\mathbb{N}}$. These will be such that

$$\varphi = \big\{\langle C_\alpha, D_\alpha\rangle : \alpha \in \omega_1\big\}$$

is the map that we seek.

To begin the construction we let

- $C_0 = D_0 = \langle \emptyset : n \in \mathbb{N} \rangle$, and
- $C_1 = D_1 = \langle \mathbb{I} : n \in \mathbb{N} \rangle$.
- $C_2 = D_2 = \langle [0, \frac{1}{2}] : n \in \mathbb{N} \rangle$.

The first two conditions take care of the maximum and minimum of $\mathcal{B}^{\mathbb{N}}/\mathit{fin}$ and facilitate the proof of Lemma 8. The third condition ensures that our autohomeomorphism H will preserve the order on each of the $\mathbb{I}_u$. We have not yet said what this order is, and that is because one does not need to worry about it yet. This third condition does not affect the rest of the proof in this section, but we include it because it will be useful in Section 4 when finding an order-reversing autohomeomorphism of $\mathbb{H}^*$.

Next let $\alpha \geq 3$ and assume we have $\varphi_\alpha = \big\{\langle C_\beta, D_\beta\rangle : \beta \in \alpha\big\}$ such that φ_α satisfies conditions (1) and (2) above up to α, that is

(1) If $\gamma < \beta < \alpha$ then $\{k : C_\gamma(k) \neq C_\beta(k)\}$ and $\{k : D_\gamma(k) \neq D_\beta(k)\}$ are infinite; this ensures the uniqueness clause in (1).
(2) If $\gamma < \beta < \alpha$ and $A_1 = \{k : C_\gamma(k) \subseteq C_\beta(k)\}$ and $A_2 = \{k : C_\gamma(k) \supseteq C_\beta(k)\}$ then $\{k : D_\gamma(k) \subseteq D_\beta(k)\} =^* h^+(A_1)$ and $\{k : D_\gamma(k) \supseteq D_\beta(k)\} =^* h^+(A_2)$.

We extend φ_α to $\varphi_{\alpha+1}$, as follows.

- If α is even let C_α be the first term of the sequence $\langle B_\xi : \xi \in \omega_1\rangle$ that satisfies $B_\xi \neq^* C_\beta$ for all $\beta < \alpha$. We show how to determine D_α so as to satisfy the conditions above up to and including α.
- If α is odd let D_α be the first term of the sequence $\langle B_\xi : \xi \in \omega_1\rangle$ that satisfies $B_\xi \neq^* D_\beta$ for all $\beta < \alpha$. We show how to determine C_α so as to satisfy the conditions above up to and including α.

We shall only deal with the even case; the argument in the odd case is the mirror image of that in the even case.

We stop before we start, however. It turns out that the second assumption on the recursion is too weak.

To illustrate the problem, assume that there are γ and β below α such that the set A of k with $C_\gamma(k) \subseteq C_\alpha(k) \subseteq C_\beta(k)$ is infinite. Then we shall need that $D_\gamma(k) \subseteq D_\alpha(k) \subseteq D_\beta(k)$ for all but finitely many $k \in h^+(A)$. We shall certainly have $D_\gamma(k) \subseteq D_\beta(k)$ for all but finitely many $k \in h^+(A)$, so there seems to be no problem specifying $D_\alpha(k)$ for these values of k.

But it is very well possible that for all $k \in A$ (or at least infinitely many) the set $C_\gamma(k)$ is a subset of the *interior* of $C_\beta(k)$. In that case there is a δ such that for all these k we have $B_\delta(k) \cup C_\beta(k) = \mathbb{I}$ and $B_\delta(k) \cap C_\gamma(k) = \emptyset$. (For example, $B_\delta(k)$ could be the complement of the interior of $C_\beta(k)$ for each k.)

If the first such δ is larger than α, then in our choice of D_α it is certainly possible (under the conditions on the recursion as currently stated) that we ensured $D_\alpha(k) \subseteq D_\beta(k)$ for enough $k \in h^+(A)$, but did not ensure that $D_\alpha(k)$ is a subset of the *interior* of $D_\beta(k)$.

However "$C_\alpha(k)$ is in the interior of $C_\beta(k)$" is expressible in terms of the lattice operations (and is witnessed by the later B_δ). Hence we should try to ensure "$D_\alpha(k)$ is in the interior of $D_\beta(k)$" often enough as well. But the latter is impossible in case we did not ensure enough times that $D_\gamma(k)$ is in the interior of $D_\beta(k)$.

To fix this problem we need to find a way to "look ahead" to later stages of the construction, but without explicitly using ordinals larger than α. That way is via quantifiers and elementarity.

To stay with our example we note that "$B(k)$ is in the interior of $C(k)$" is expressible as $(\exists x)\psi\big(B(k), C(k), x\big)$, where $\psi(y, z, x)$ is "$y \cap x = \emptyset \wedge z \cup x = \mathbb{I}$".

So, given $\gamma, \beta < \alpha$ we should also look at $A = \{k : (\exists x)\psi(C_\gamma(k), C_\beta(k), x(k))\}$ and ensure that also $h^+(A) =^* \{k : (\exists x)\psi(D_\gamma(k), D_\beta(k), x(k))\}$. This then will help us build D_α so that $D_\alpha(k)$ is in the interior of $D_\beta(k)$ often enough.

We strengthen condition (2) so that it covers all formulas of lattice theory and all finite sets of ordinals.

$(*)_\alpha$ For every formula χ in the language of lattice theory with free variables $x_1, \ldots, x_n$, and for every tuple $(\beta_1, \ldots, \beta_n)$ of ordinals below α, the set $A = \{k : \chi(C_{\beta_1}(k), \ldots, C_{\beta_n}(k))\}$ satisfies $h^+(A) =^* \{k : \chi(D_{\beta_1}(k), \ldots, D_{\beta_n}(k))\}$.

In the example above we would have $\chi(x_1, x_2)$ equal to $(\exists x)\psi(x_1, x_2, x)$.

For the three base cases of our recursion given above, observe that for $\beta = 0, 1, 2$, we have $C_\beta(k) = D_\beta(k)$ for all $k \in \omega$. Consequently, for any formula χ as above, we have $\{k : \chi(C_{\beta_1}(k), \ldots, C_{\beta_n}(k))\}$ equal to either $\emptyset$ or ω (since there is no actual dependence on k), and $\{k : \chi(D_{\beta_1}(k), \ldots, D_{\beta_n}(k))\}$ equal to the same set. In particular, this implies $(*)_3$ holds, and we have a solid basis for our recursion.

3.2. An application of elementary equivalence and saturation

Before we start to build D_α we need an intermediate result. This will involve some model theory, especially elementary equivalence and saturation. The results that we need can be found in [8, Chapter 10] or [9, Chapter 8].

We fix $u \in \mathbb{N}^*$ for the moment, put $v = h(u)$, and consider the ultrapowers $\mathbb{B}_u = \mathcal{B}^{\mathbb{N}}/u$ and $\mathbb{B}_v = \mathcal{B}^{\mathbb{N}}/v$.

The structures $(\mathbb{B}_u, \bar{C})$ and $(\mathbb{B}_v, \bar{D})$ are elementarily equivalent. Here $\bar{C}$ is the sequence of elements of $\mathbb{B}_u$ determined by the sequence $\langle C_\beta : \beta \in \alpha\rangle$, and likewise $\bar{D}$ is determined in $\mathbb{B}_v$ by $\langle D_\beta : \beta \in \alpha\rangle$.

The reason is that when χ is a formula with free variables $x_1, \ldots, x_n$ and if $\beta_1, \ldots, \beta_n$ are members of α such that $\mathbb{B}_u \models \chi(C_{\beta_1}, \ldots, C_{\beta_n})$ then the set $A_1 = \{k : \mathcal{B} \models \chi(C_{\beta_1}(k), \ldots, C_{\beta_n}(k))\}$ belongs to u. Then $h^+(A_1)$ belongs to v, and hence so does $A_2 = \{k : \mathcal{B} \models \chi(D_{\beta_1}(k), \ldots, D_{\beta_n}(k))\}$. But this then implies that $\mathbb{B}_v \models \chi(D_{\beta_1}, \ldots, D_{\beta_n})$.

The ultrapower $\mathbb{B}_v$ is saturated and so [8, Lemma 10.1.3] or [9, Lemma 8.1.3] applies, which guarantees the existence of an element D of $\mathbb{B}_v$ such that $(B_u, \bar{C}, C_\alpha)$ and $(\mathbb{B}_v, \bar{D}, D)$ are elementarily equivalent.

This means that a local version of $(*)_{\alpha+1}$ holds at the points u and v, with D in place of D_α:

> if χ is a formula from the language of lattices with free variables $x_1, \ldots x_n$, x_{n+1}, and if $\beta_1, \ldots, \beta_n$ are members of α then the two sets $A_1 = \{k : \mathcal{B} \models \chi(C_{\beta_1}(k), \ldots, C_{\beta_n}(k), C_\alpha(k))\}$ and $A_2 = \{k : \mathcal{B} \models \chi(D_{\beta_1}(k), \ldots, D_{\beta_n}(k), D(k))\}$ satisfy $A_1 \in u$ if and only if $A_2 \in v$.

For every $u \in \mathbb{N}^*$ we choose D_u such that $(B_u, \bar{C}, C_\alpha)$ and $(\mathbb{B}_v, \bar{D}, D_u)$ are elementarily equivalent.

Now let χ be a formula, with free variables $x_1, \ldots, x_n$, and let $\beta_1, \ldots, \beta_n$ be elements of $\alpha + 1$. By the rules of interpretation we know that for every $u \in \mathbb{N}^*$ we have *either* $\mathbb{B}_u \models \chi(C_{\beta_1}, \ldots, C_{\beta_n})$, *or* $\mathbb{B}_u \models \neg\chi(C_{\beta_1}, \ldots, C_{\beta_n})$.

Then the former implies that $\mathbb{B}_v \models \chi(D_{\beta_1}, \ldots, D_{\beta_n})$, and the latter implies that $\mathbb{B}_v \models \neg\chi(D_{\beta_1}, \ldots, D_{\beta_n})$, where, in both cases, we insert D_u at the positions where $\beta_i = \alpha$.

Translated to subsets of $\mathbb{N}$ this becomes if $A = \{k : \mathcal{B} \models \chi(C_{\beta_1}(k), \ldots, C_{\beta_n}(k))\}$ belongs to u then $\{k : \mathcal{B} \models \chi(D_{\beta_1}(k), \ldots, D_{\beta_n}(k))\}$ and $h^+(A)$ both belong to v.

3.3. Making D_α

We build D_α out of bits and pieces of the elements D_u chosen above.

The idea will be to make D_α in a recursion of length ω each time adding finitely many coordinates of finitely many D_u in such a way that the higher the coordinates the more formulas these decide. In the end D_α should then decide every formula almost everywhere.

We also have to take care of the ordinals below α so we start by fixing an enumeration of the set of pairs $\langle \chi, \bar{\beta} \rangle$ of formulas and finite sequences of ordinals in $\alpha + 1$ as $\langle \langle \chi_m, \bar{\beta}_m \rangle : m \in \mathbb{N} \rangle$. We assume that the number of free variables in χ_m is always equal to the length of the sequence $\bar{\beta}_m$, call this number p_m.

Each pair $\langle \chi_m, \bar{\beta}_m \rangle$ determines a partition of $\mathbb{N}$ into two sets

$$A_{m,0} = \{k : \mathcal{B} \models \neg \chi_m(C_{\beta_{m,1}}(k), \dots, C_{\beta_{m,p_m}}(k))\}$$

and

$$A_{m,1} = \{k : \mathcal{B} \models \chi_m(C_{\beta_{m,1}}(k), \dots, C_{\beta_{m,p_m}}(k))\}.$$

Using the enumeration we make a sequence of partitions $\langle \mathcal{P}_m : m \in \mathbb{N} \rangle$ of $\mathbb{N}$, as follows.

For each sequence $s \in 2^m$ let $A_s = \bigcap_{l<m} A_{l,s(l)}$, and put $\mathcal{P}_m = \{A_s : s \in 2^m\}$. By default we take $\bigcap \emptyset = \mathbb{N}$, so that $\mathcal{P}_0 = \{\mathbb{N}\}$.

The map h^+ transforms these partitions in almost-partitions, that is, the union $\bigcup \mathcal{P}_m$ is co-finite, and if $s, t \in 2^m$ and $s \neq t$ then $h^+(A_s) \cap h^+(A_t)$ is finite. This implies that there is a natural number N_m such that $\{h^+(A) \setminus N_m : A \in \mathcal{P}_m\}$ is a partition of $\mathbb{N} \setminus N_m$.

In fact, by raising N_m if necessary we can ensure that the map $A \mapsto h^+(A) \setminus N_m$ is an isomorphism between the Boolean algebras generated by $\{A_{l,i} : l < m, \text{ and } i \in 2\}$ and $\{h^+(A_{l,i}) \setminus N_m : l < m, \text{ and } i \in 2\}$ respectively.

3.3.1. Other partitions

Let us fix m for the time being.

The definition of h^+ implies that for every $u \in \mathbb{N}^*$ and $s \in 2^m$ we have $A_s \in u$ iff $h^+(A_s) \in h(u)$.

By the local version of $(*)_{\alpha+1}$ above we know that for every u and s we have $A_s \in u$ iff the set $B_{u,s}$ belongs to $h(u)$, where $B_{u,s}$ is the set of those k that satisfy for all $l < m$:

- $\mathcal{B} \models \chi_l(D_{\beta_{l,1}}(k), \dots, D_{\beta_{l,p_l}}(k))$ when $s(l) = 1$, and
- $\mathcal{B} \models \neg\chi_l(D_{\beta_{l,1}}(k), \dots, D_{\beta_{l,p_l}}(k))$ when $s(l) = 0$.

In both cases we substitute D_u for $D_{\beta_{l,i}}$ whenever $\beta_{l,i} = \alpha$.

Note that this implies that $h^+(A_s) \cap B_{u,s} \in h(u)$ iff $A_s \in u$. It follows that if $s \in 2^m$ then the family $\{B^*_{u,s} : u \in A^*_s\}$ covers $h[A^*_s]$, hence there is a finite subset F_s of A^*_s such that $\{B^*_{u,s} : u \in F_s\}$ covers $h[A^*_s]$. This means that $h^+(A_s) \setminus \bigcup_{u \in F_s} B_{u,s}$ is finite.

Shrinking the sets $B_{u,s}$ if necessary, we may (and do) assume

- for each s, if $u, u' \in F_s$ and $u \neq u'$ then $B_{u,s} \cap B_{u',s} = \emptyset$,
- $B_{u,s} \subseteq h^+(A_s)$ for all $u \in F_s$, and
- the sets $h^+(A_s) \setminus \bigcup_{u \in F_s} B_{u,s}$ remain finite.

Recall that $\{h^+(A) \setminus N_m : A \in \mathcal{P}_m\}$ is a partition of $\mathbb{N}$. Together with the three just-listed assumptions about the $B_{u,s}$, this implies that the whole family $\mathcal{Q}_m = \bigcup_{s \in 2^m} \{B_{u,s} \setminus N_m : u \in F_s\}$ is pairwise disjoint, and its union is co-finite; we increase N_m if necessary so that the latter union contains $\mathbb{N} \setminus N_m$. In short, the family $\mathcal{Q}_m$ is a partition of $\mathbb{N} \setminus N_m$ that is a refinement of $\{h^+(A) : A \in \mathcal{P}_m\}$.

3.3.2. Building D_α

Using the construction above we obtain a sequence $\langle \mathcal{Q}_m : m \in \mathbb{N} \rangle$ of almost-partitions and a sequence $\langle N_m : m \in \mathbb{N} \rangle$ of natural numbers such that $\{Q \setminus N_m : Q \in \mathcal{Q}_m\}$ is a partition of $\mathbb{N} \setminus N_m$ and, without loss of generality, $N_m < N_{m+1}$ for all m.

As indicated above we build D_α piece by piece: more precisely, for every m we define D_α on the interval $[N_m, N_{m+1})$ using the D_u with $u \in \bigcup\{F_s : s \in 2^m\}$.

Fix such an m. If $k \in [N_m, N_{m+1})$ then there is one pair (s, u) with $s \in 2^m$ and $u \in F_s$ such that $k \in B_{u,s}$. Define $D_\alpha(k) = D_u(k)$. Then, by the very choice of $B_{u,s}$ we get, for all $l < m$:

- $\mathcal{B} \models \chi_l(D_{\beta_{l,1}}(k), \dots, D_{\beta_{l,p_l}}(k))$ when $s(l) = 1$, and
- $\mathcal{B} \models \neg\chi_l(D_{\beta_{l,1}}(k), \dots, D_{\beta_{l,p_l}}(k))$ when $s(l) = 0$.

By the choice of N_m above we find that for $\langle l, i\rangle \in m \times 2$ we have $h^+(A_{l,i}) \setminus N_m = \bigcup\{B_{u,s} \setminus N_m : s \in 2^m, u \in F_s, s(l) = i\}$. It follows that

$$h^+(A_{l,1}) \cap [N_m, N_{m+1}) = \big\{k : \mathcal{B} \models \chi_l(D_{\beta_{l,1}}(k), \dots, D_{\beta_{l,p_l}}(k))\big\} \cap [N_m, N_{m+1})$$

and

$$h^+(A_{l,0}) \cap [N_m, N_{m+1}) = \big\{k : \mathcal{B} \models \neg\chi_l(D_{\beta_{l,1}}(k), \dots, D_{\beta_{l,p_l}}(k))\big\} \cap [N_m, N_{m+1})$$

3.3.3. Verification of $(*)_{\alpha+1}$

Let $l \in \mathbb{N}$; we show that $(*)_{\alpha+1}$ holds for the pair $\langle \chi_l, \bar{\beta}_l\rangle$. We have

$$A_{l,1} = \{k : \mathcal{B} \models \chi_l(C_{\beta_{l,1}}(k), \dots, C_{\beta_{l,p_l}}(k))\},$$
$$B_{l,1} = \{k : \mathcal{B} \models \chi_l(D_{\beta_{l,1}}(k), \dots, D_{\beta_{l,p_l}}(k))\}.$$

We must show that $B_{l,1} =^* h^+(A_{l,1})$.

But our construction above ensures $B_{l,1} \cap [N_m, N_{m+1}) = h^+(A_{l,1}) \cap [N_m, N_{m+1})$ whenever $m > l$. This clearly suffices, and this completes the proof. □

Remark 9. The proof above is based on Wallman's representation theorem for distributive lattices, see [17]. That paper established a many-valued duality between certain distributive lattices and compact spaces: to every distributive lattice there corresponds a compact space, and to every compact space there may correspond various lattices, e.g., the full family of closed sets, or any base for its closed sets that forms a lattice.

A true duality for compact spaces is due to Gelfand and Kolomogorov in [6]: here the ring of continuous functions is the algebraic counterpart of the compact space. One can give a version of our proof based on this duality, where the ring $C(\mathbb{M}^*)$ of continuous functions on $\mathbb{M}^*$ is represented as the quotient of the ring $C^*(\mathbb{M})$ by the ideal of functions that have limit zero at infinity. That is, one can construct an automorphism of $C(\mathbb{M}^*)$ whose dual is the desired autohomeomorphism of $\mathbb{M}^*$.

In this ring-theoretic approach, one could take the subring R of the ring $C(\mathbb{I})$ generated by the constant functions and the functions d_B for $B \in \mathcal{B}$, where $d_B(x) = d(x, B)$. The power $R^{\mathbb{N}}$ represents a subring of $C(\mathbb{M})$, and the bounded elements of $R^{\mathbb{N}}$ are the analogue of $\mathcal{B}^{\mathbb{N}}$ in the preceding proof.

4. An order-reversing autohomeomorphism of $\mathbb{H}^*$

We begin this section with a description of the standard quasiorder on the connected components $\mathbb{I}_u$ of $\mathbb{M}^*$.

Given a sequence $\bar{x} = \langle x_n : n \in \mathbb{N}\rangle \in \mathbb{I}^{\mathbb{N}}$ and $u \in \mathbb{N}^*$, define

$$\bar{x}_u = u\text{-}\lim_n(n, x_n),$$

where the limit is taken in $\beta\mathbb{M}$. Equivalently, this is the unique element of the set $\bigcap_{A\in u} \mathrm{cl}_{\beta\mathbb{M}}\{(n, x_n) : n \in A\}$. Let $C_u \subseteq \mathbb{I}_u$ denote the set of all points of this form:

$$C_u = \{\bar{x}_u : \bar{x} \in \mathbb{I}^{\mathbb{N}}\}.$$

This is a proper subset of $\mathbb{I}_u$ (see [7]). Observe that C_u has a natural linear order:

$$\bar{x}_u \leq_u \bar{y}_u \quad \Leftrightarrow \quad \{n : x_n \leq y_n\} \in u.$$

The fact that this is a total order of C_u can be seen as an instance of Łoś' Theorem, because C_u is really just the ultrapower $\mathbb{I}^{\mathbb{N}}/u$. Observe that C_u has a least element $\bar{0}_u$ and a greatest element $\bar{1}_u$, where $\bar{0}$ denotes the constant sequence $\langle 0, 0, 0, \dots\rangle$ and $\bar{1}$ the constant sequence $\langle 1, 1, 1, \dots\rangle$. Recall that a continuum C is *irreducible* between $x, y \in C$ if there is no proper subcontinuum of C containing both x and y.

Proposition 10. *The set $C_u \setminus \{\bar{0}_u, \bar{1}_u\}$ is dense in $\mathbb{I}_u$, and its subspace topology is the same as the order topology induced by $\leq_u$. Furthermore,*

(1) *$\mathbb{I}_u$ is irreducible between $\bar{0}_u$ and $\bar{1}_u$, and if $p \in C_u \setminus \{\bar{0}_u, \bar{1}_u\}$ then $\mathbb{I}_u \setminus \{p\}$ has two connected components, one containing $\bar{0}_u$ and the other $\bar{1}_u$.*
(2) *If $x, y \in C_u$ and $x <_u y$, then every subcontinuum of $\mathbb{I}_u$ that contains y and $\bar{0}_u$ also contains x, and inversely, every subcontinuum of $\mathbb{I}_u$ that contains x and $\bar{1}_u$ also contains y.*

See [7, Section 2] for a proof. This proposition implies that $\{\bar{0}_u, \bar{1}_u\}$ is a topologically definable subset of $\mathbb{I}_u$. In particular, any autohomeomorphism $H : \mathbb{M}^* \to \mathbb{M}^*$ must map $\{\bar{0}_u, \bar{1}_u\}$ to $\{\bar{0}_{\rho_H(u)}, \bar{1}_{\rho_H(u)}\}$. (Recall from page 3 that ρ_H is defined so that $H(\mathbb{I}_u) = \mathbb{I}_{\rho_H(u)}$ for all $u \in \mathbb{N}^*$, and if H is an autohomeomorphism of $\mathbb{M}^*$ then ρ_H is an autohomeomorphism of $\mathbb{N}^*$.)

Part (2) of this proposition enables us to extend the total order $\leq_u$ on C_u to a quasiorder on $\mathbb{I}_u$, also denoted $\leq_u$: for any $x, y \in \mathbb{I}_u$, define $x \leq_u y$ if and only if every subcontinuum of $\mathbb{I}_u$ that contains y and $\bar{0}_u$ also contains x, if and only if every subcontinuum of $\mathbb{I}_u$ that contains x and $\bar{1}_u$ also contains y.

Let us note that $\leq_u$ is not a partial order. Let us write $x \equiv_u y$ to mean that $x \leq_u y$ and $y \leq_u x$. For each $x \in \mathbb{I}_u$, the set $L_x = \{y \in \mathbb{I}_u : y \equiv_u x\}$ is called the *layer* of x in $\mathbb{I}_u$. If $x \in C_u$, then it turns out $L_x = \{x\}$. For points $x \in \mathbb{I}_u \setminus C_u$ there are two possibilities: $L_x = \{x\}$ or $|L_x| = 2^{\mathfrak{c}}$. There are always x for which the second possibility obtains; under CH there are points outside C_u that have a one-point layer, but in the Laver model all points outside C_u have non-trivial layers, see [5]. The layers of $\mathbb{I}_u$ can be quite large, and topologically complex: all are indecomposable continua for example. The quotient of $\mathbb{I}_u$ by $\equiv_u$ is the Dedekind completion of $(C_u, \leq_u)$, with its usual order topology. So one may think of $\mathbb{I}_u$ as something like the Dedekind completion of C_u, but where some of the gaps have been filled not with single points, but with complicated compacta.

An autohomeomorphism $H : \mathbb{M}^* \to \mathbb{M}^*$ is *order-preserving* if

$$\text{if } x, y \in \mathbb{I}_u \text{ and } x \leq_u y, \text{ then } H(x) \leq_{\rho_H(u)} H(y),$$

and H is *order-reversing* if

$$\text{if } x, y \in \mathbb{I}_u \text{ and } x \leq_u y, \text{ then } H(y) \leq_{\rho_H(u)} H(x).$$

Equivalently, H is order-preserving if $H(\bar{0}_u) = \bar{0}_{\rho_H(u)}$ and $H(\bar{1}_u) = \bar{1}_{\rho_H(u)}$ for all $u \in \mathbb{N}^*$, and it is order-reversing if $H(\bar{0}_u) = \bar{1}_{\rho_H(u)}$ and $H(\bar{1}_u) = \bar{0}_{\rho_H(u)}$ for all u.

Observe that an autohomeomorphism H of $\mathbb{M}^*$ need be neither order-preserving nor order-reversing: it may have $H(\bar{0}_u) = \bar{0}_{\rho_H(u)}$ for some u and $H(\bar{0}_u) = \bar{1}_{\rho_H(u)}$ for other u. For example, given $A \subseteq \mathbb{N}$, consider the homeomorphism $\mathsf{flip}_A : \mathbb{M} \to \mathbb{M}$ that flips the intervals in $A \times \mathbb{I}$:

$$\mathsf{flip}_A(n,x) = \begin{cases} (n, 1-x) & \text{if } n \in A, \\ (n,x) & \text{if } n \notin A. \end{cases}$$

The trivial autohomeomorphism of $\mathbb{M}^*$ induced by flip_A is order-preserving if A is finite, it is order-reversing if A is co-finite, and it is neither order-preserving nor order-reversing if A and $\mathbb{N} \setminus A$ are both infinite.

Lemma 11. *Assuming* CH*, if h is an autohomeomorphism of $\mathbb{N}^*$, then there is an order-preserving autohomeomorphism H of $\mathbb{M}^*$ such that $\pi^* \circ H = h \circ \pi^*$.* □

This lemma merely re-states Theorem 1, but with the added requirement that H be order-preserving. To prove the lemma, simply note that our proof of Theorem 1 in the previous section already produces an order-preserving map. (The construction ensures that H maps $K = \left(\mathbb{N} \times [0, \frac{1}{2}]\right)^*$ to itself, which means that for every u, $\bar{0}_u \in K$ cannot map to $\bar{1}_{\rho_H(u)} \notin K$, and therefore must map to $\bar{0}_{\rho_H(u)}$.)

Observe that $\mathbb{H}$ is obtained naturally as a topological quotient of $\mathbb{M}$: just glue the rightmost point of each connected component $\mathbb{I}_n$ to the leftmost point of the next component $\mathbb{I}_{n+1}$. More precisely, define an equivalence relation $\sim$ on $\mathbb{M}$ by setting $(n,1) \sim (n+1,0)$ for all $n \in \mathbb{N}$ (and of course $(n,x) \sim (n,x)$ for all $(n,x) \in \mathbb{M}$). The quotient space $\mathbb{M}/\sim$ is $\mathbb{H}$.

One can obtain $\mathbb{H}^*$ as a quotient of $\mathbb{M}^*$ in a similar fashion. First, let σ denote the autohomeomorphism of $\mathbb{N}^*$ induced by the successor map $n \mapsto n+1$ (which is an almost permutation of $\mathbb{N}$). Explicitly, if $u \in \mathbb{N}^*$ then $\sigma(u)$ is the ultrafilter generated by $\{A+1 : A \in u\}$. Next, like with $\mathbb{M}$ and $\mathbb{H}$, our quotient mapping $\mathbb{M}^* \to \mathbb{H}^*$ is defined by gluing the rightmost point of each connected component $\mathbb{I}_u$ to the leftmost point of the "next" component, $\mathbb{I}_{\sigma(u)}$. More precisely, define an equivalence relation $\sim$ on $\mathbb{M}^*$ by setting $\bar{1}_u \sim \bar{0}_{\sigma(u)}$ for all $u \in \mathbb{N}^*$ (and of course $x \sim x$ for all $x \in \mathbb{M}^*$). Then the quotient space $\mathbb{M}^*/\sim$ is $\mathbb{H}^*$ (see [7, Theorem 2.4]). Let $Q : \mathbb{M}^* \to \mathbb{H}^*$ denote this quotient mapping.

For each $u \in \mathbb{N}^*$, our quotient mapping $Q : \mathbb{M}^* \to \mathbb{H}^*$ restricts to an injection on $\mathbb{I}_u$. Because $\mathbb{I}_u$ is compact and Q continuous, this means $Q \restriction \mathbb{I}_u$ is an embedding $\mathbb{I}_u \to \mathbb{H}^*$. In other words, we may (and do) think of the $\mathbb{I}_u$ as subspaces of $\mathbb{H}^*$. For each $u \in \mathbb{N}^*$, let $\mathbb{I}_u^Q = Q[\mathbb{I}_u]$ denote this copy of $\mathbb{I}_u$ in $\mathbb{H}^*$. Alternatively,

$$\mathbb{I}_u^Q = \bigcap_{A \in u} \mathrm{cl}_{\beta\mathbb{H}} \bigcup \{[n, n+1] : n \in A\}.$$

Via this identification of $\mathbb{I}_u$ with $\mathbb{I}_u^Q$, each $\mathbb{I}_u^Q \subseteq \mathbb{H}^*$ has a natural quasi-order, the push-forward of the quasi-order $\leq_u$ on $\mathbb{I}_u$, which we still denote by $\leq_u$ in $\mathbb{I}_u^Q$.

Let us note that the $\mathbb{I}_u^Q$ are examples of what are called *standard subcontinua* of $\mathbb{H}^*$. These are special connected subsets of $\mathbb{H}^*$ whose structure and interrelationships determine much about the topology of $\mathbb{H}^*$ (see [7, Section 5]). It is not difficult to see that the ordering $\leq_u$ described above for $\mathbb{I}_u^Q$ matches the usual quasi-order defined on a standard subcontinuum of $\mathbb{H}^*$.

Let us say that a homeomorphism $H : \mathbb{H}^* \to \mathbb{H}^*$ is *order-reversing* if there is a permutation ρ of $\mathbb{N}^*$ such that

- $H[\mathbb{I}_u^Q] = \mathbb{I}_{\rho(u)}^Q$ for all $u \in \mathbb{N}^*$, and
- if $x \leq_u y$ in $\mathbb{I}_u^Q$, then $H(y) \leq_{\rho(u)} H(x)$ in $\mathbb{I}_{\rho(u)}^Q$.

In other words, an order-reversing autohomeomorphism of H is one that permutes the $\mathbb{I}^Q_u$ while reversing their order.

Proposition 12. *No trivial autohomeomorphism of $\mathbb{H}^*$ is order-reversing.*

Proof. Let H be a trivial autohomeomorphism of $\mathbb{H}^*$. We aim to show H is not order-reversing. Because H is trivial, there is a homeomorphism $f : C \to D$, where C and D are co-compact subsets of $\mathbb{H}$, such that $H = \beta f \restriction \mathbb{H}^*$. Observe that f must be order-preserving on a tail of $\mathbb{H}$: i.e., if $a < b$ and a, b are sufficiently large, then $f(a) < f(b)$.

For each $n \in \mathbb{N}$, fix a_n, b_n such that $n \leq a_n < b_n \leq n+1$. Let $\bar{a} = \langle a_n : n \in \mathbb{N}\rangle$ and $\bar{b} = \langle b_n : n \in \mathbb{N}\rangle$, and let $f(\bar{a}) = \langle f(a_n) : n \in \mathbb{N}\rangle$ and $f(\bar{b}) = \langle f(b_n) : n \in \mathbb{N}\rangle$.

Fix $u \in \mathbb{N}^*$ and suppose $H[\mathbb{I}^Q_u] = \mathbb{I}^Q_v$ for some $v \in \mathbb{N}^*$. (Otherwise H is not order-reversing.) Let $x = Q(\bar{a}_u)$ and $y = Q(\bar{b}_u)$. Because $a_n < b_n$ for all n, we have $\bar{a}_u <_u \bar{b}_u$ in $\mathbb{I}_u$, which means $x = Q(\bar{a}_u) <_u Q(\bar{b}_u) = y$ in $\mathbb{I}^Q_u$.

Now $H(x) = \beta f(Q(\bar{a}_u)) = Q(f(\bar{a})_v)$, and similarly $H(y) = Q(f(\bar{b})_v)$. But because f is order-preserving on a tail, we have $f(a_n) < f(b_n)$ for all sufficiently large n, and therefore $f(\bar{a})_v \leq_v f(\bar{b})_v$. Hence $H(x) = Q(f(\bar{a})_v) \leq_v Q(f(\bar{b})_v) = H(y)$ in $\mathbb{I}^Q_v$, and this means that H is not order-reversing. □

Corollary 13. *It is consistent that no autohomeomorphism of $\mathbb{H}^*$ is order-reversing.*

Proof. As we mentioned already in Section 2, a recent result of Vignati in [16] states that $\mathsf{OCA}_T + \mathsf{MA}$ implies all autohomeomorphisms of $\mathbb{H}^*$ are trivial. The corollary follows from this and the previous proposition. □

Proof of Theorem 2. As before, let σ denote the trivial autohomeomorphism of $\mathbb{N}^*$ induced by the successor function $n \mapsto n+1$ on $\mathbb{N}$. By a recent theorem of the first author (the main theorem of [1]), CH implies σ and σ^{-1} are conjugate in the autohomeomorphism group of $\mathbb{N}^*$. In other words, CH implies there is an autohomeomorphism f of $\mathbb{N}^*$ such that $f \circ \sigma = \sigma^{-1} \circ f$. Fix some such f and, using CH again and applying Lemma 11, fix an order-preserving autohomeomorphism F of $\mathbb{M}^*$ such that $\pi^* \circ F = f \circ \pi^*$.

Recall the homeomorphism $\mathsf{flip}_{\mathbb{N}} : \mathbb{M} \to \mathbb{M}$ is defined by

$$\mathsf{flip}_{\mathbb{N}}(n, x) = (n, 1 - x)$$

for all $(n, x) \in \mathbb{M}$. Let $\mathsf{flip}^*_{\mathbb{N}} = \beta\,\mathsf{flip}_{\mathbb{N}} \restriction \mathbb{M}^*$ denote the trivial autohomeomorphism of $\mathbb{M}^*$ induced by $\mathsf{flip}_{\mathbb{N}}$. Clearly $\mathsf{flip}^*_{\mathbb{N}}$ is order-reversing on each $\mathbb{I}_u$, and in particular,

$$\mathsf{flip}^*_{\mathbb{N}}(\bar{1}_u) = \bar{0}_u \qquad \text{and} \qquad \mathsf{flip}^*_{\mathbb{N}}(\bar{0}_u) = \bar{1}_u$$

for every $u \in \mathbb{N}^*$.

Because $\mathsf{flip}^*_{\mathbb{N}}$ is an order-reversing autohomeomorphism of $\mathbb{M}^*$ and F is an order-preserving autohomeomorphism of $\mathbb{M}^*$, their composition

$$H = F \circ \mathsf{flip}^*_{\mathbb{N}}$$

is an order-reversing autohomeomorphism of $\mathbb{M}^*$. In particular, observe that

$$H(\bar{1}_u) = F \circ \mathsf{flip}^*_{\mathbb{N}}(\bar{1}_u) = F(\bar{0}_u) = \bar{0}_{f(u)}$$

and

$$H(\bar{0}_u) = F \circ \mathsf{flip}^*_{\mathbb{N}}(\bar{0}_u) = F(\bar{1}_u) = \bar{1}_{f(u)}$$

for every $u \in \mathbb{N}^*$.

Recall the equivalence relation $\sim$ on $\mathbb{M}^*$ described earlier in this section, whose non-singleton equivalence classes are the sets of the form $\{\bar{1}_u, \bar{0}_{\sigma(u)}\}$ for $u \in \mathbb{N}^*$. By our choice of f, for any given $u \in \mathbb{N}^*$ we have

$$H(\bar{1}_u) = \bar{0}_{f(u)} \qquad \text{and} \qquad H(\bar{0}_{\sigma(u)}) = \bar{1}_{f(\sigma(u))} = \bar{1}_{\sigma^{-1}(f(u))}.$$

In particular, H maps the $\sim$-equivalence class $\{\bar{1}_u, \bar{0}_{\sigma(u)}\}$ to the $\sim$-equivalence class $\{\bar{1}_{\sigma^{-1}(f(u))}, \bar{0}_{f(u)}\}$ (i.e., the class $\{\bar{1}_v, \bar{0}_{\sigma(v)}\}$ where $v = \sigma^{-1}(f(u))$). In other words, H respects the equivalence classes of the relation $\sim$. Consequently,

$$[x]_\sim \mapsto [H(x)]_\sim$$

is a well-defined mapping $\mathbb{M}^*/\sim \to \mathbb{M}^*/\sim$. Let h denote this mapping. Recalling that Q denotes the quotient mapping from $\mathbb{M}^*$ to $\mathbb{M}^*/\sim$, we have $Q \circ H = h \circ Q$.

We claim h is an order-reversing autohomeomorphism of $\mathbb{H}^* \cong \mathbb{M}^*/\sim$. The fact that h is an autohomeomorphism follows from the fact that H is, and that $h \circ Q = Q \circ H$. Next, recall that for every $u \in \mathbb{N}^*$, the quotient mapping Q restricts to a homeomorphism $\mathbb{I}_u \to \mathbb{I}_u^Q$. Because the sets of the form $\mathbb{I}_u$ are the connected components of $\mathbb{M}^*$, the homeomorphism H maps each $\mathbb{I}_u$ homeomorphically to $\mathbb{I}_{\rho_H(u)} = \mathbb{I}_{f(u)}$. Thus, for each $u \in \mathbb{N}^*$, the map $h = Q \circ H \circ Q^{-1}$ is a composition of homeomorphisms $\mathbb{I}_u^Q \to \mathbb{I}_u \to \mathbb{I}_{f(u)} \to \mathbb{I}_{f(u)}^Q$. Thus, setting $\rho = f$, h satisfies the first clause in the definition of an order-reversing autohomeomorphism of $\mathbb{H}^*$.

Finally, we wish to show that if $x, y \in \mathbb{I}_u^Q$ and $x \leq_u y$ then $h(y) \leq_{f(u)} h(x)$. Fix $u \in \mathbb{N}^*$ and $x, y \in \mathbb{I}_u^Q$ with $x \leq_u y$. As in the previous paragraph, $h = Q \circ H \circ Q^{-1}$ is a composition of homeomorphisms $\mathbb{I}_u^Q \to \mathbb{I}_u \to \mathbb{I}_{f(u)} \to \mathbb{I}_{f(u)}^Q$. Both Q and Q^{-1} preserve the order of x and y, while H reverses it, so $h(y) \leq_{f(u)} h(x)$. □

Remark 14. Peter had a knack of coming up with colorful descriptions of various constructions. We thought it apt to give a Peter-esque description of the order-reversing homeomorphism in this fashion.

One can think of $\mathbb{M}$ as a sequence of domino tiles. One obtains the map $Q : \mathbb{M} \to \mathbb{H}$, described above by tipping all tiles over to the right so that for every n the top of $\mathbb{I}_n$ touches the bottom of $\mathbb{I}_{n+1}$; after some welding the map Q is done.

If one tips all tiles over to the left and again does some welding to join the bottom of $\mathbb{I}_n$ and the top of $\mathbb{I}_{n+1}$ one obtains another map from $\mathbb{M}$ onto $\mathbb{H}$: the composition $Q \circ \mathsf{flip}$.

Both maps yields maps from $\mathbb{M}^*$ onto $\mathbb{H}^*$ that may be interpreted as tipping over the domino tiles $\mathbb{I}_u$, either all to the right, or all to the left, and doing the analogous welding.

Our results show that under CH there is an autohomeomorphism of $\mathbb{H}^*$ that rearranges the components $\mathbb{I}_u$ and flips them all over.

Clearly there is no autohomeomorphism of $\mathbb{H}$ itself that does this for the tiles $\mathbb{I}_n$, hence this autohomeomorphism is non-trivial.

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