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Incorporating an Interest Rate Smile in an Equity Local Volatility Model

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Abstract

The focus of this paper is on finding a connection between the interest rate and equity asset classes. We propose an equity interest rate hybrid model which preserves market observable smiles: the equity from plain vanilla products via a local volatility framework and the interest rate from caps and swaptions via the Stochastic Volatility Libor Market Model [25]. We define a multi-factor short-rate process implied from the Libor Market Model [21], [7], [24] via an arbitrage-free interpolation [29] and combine it with the local volatility equity model for stochastic interest rates [6]. We show that the interest rate smile has a significant impact on the equity local volatility. The model developed is intuitive and straightforward, enabling consistent pricing of related hybrid products. Moreover, it preserves the non-arbitrage Heath, Jarrow, Morton [19] conditions.

Key words: Local Volatility with Stochastic Interest Rates; Hybrid Model; Stochastic Volatility Libor Market Model; Volatility Smiles;

1 Introduction

Hybrids based on a combination of underlyings from different asset classes are highly popular in the financial markets and accurate modeling is a challenging task. Implied Black & Scholes [9] (BS) volatilities of plain-vanilla equity options and interest rate products, like caps and swaptions, show a typical smile/skew-shaped pattern. In this article we present a flexible hybrid model for pricing products combining both the equity and interest rate assets. Our model is equity and interest-rate smile consistent. Furthermore, it does not violate the Heath, Jarrow, Morton (HJM) [19] arbitrage-free conditions.

Due to the presence of the leverage effect in the equity market the basic BS model is not a good choice for pricing exotic products like barrier options or cliques. On the other hand, well known stochastic volatility equity models are difficult to calibrate to market data in a consistent and stable way. A simpler model than a stochastic volatility model, which preserves the equity market smile is the so-called Local Volatility Model (LVM) developed by Dupire [13] and Derman & Kani [12]. Since the input for the LVM is an implied BS equity volatility surface, the model fits market data in a natural way and it is nonparametric. The basic LVM is very popular for pricing equity based exotic products, but it assumes a deterministic interest rate, and generates therefore prices that are insensitive to IR volatilities. Important steps incorporating interest rates

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volatilities into the LVM were done by Atlan [5] (2006) and Benhamou [6] (2008), defining a Local Volatility Model with Stochastic Interest Rates (LVSIR). Inclusion of the standard short-rate processes like Vašiček [30], Hull & White [20] or Black & Karasinski [8] in the LVM, which is proposed in these references, allows a more accurate pricing of the interest-rate sensitive hybrids. However, the volatility smile effect on the caps and swaptions is not explicitly captured.

For several years the log-normal Libor Market Model (LMM) [7; 21; 24] established itself as the benchmark for interest-rate derivatives. Without enhancements this model is not able to incorporate strike-dependent volatilities of fixed income derivatives, such as caps and swaptions. An important step in the IR modeling came with the Stochastic Volatility LMM (SV-LMM) [2; 3] which allowed the model to be fitted reasonably well, while guaranteeing the model's stability.

In the literature a number of stochastic volatility extensions of LMM were presented, see e.g., Brigo & Mercurio (2007) [10]. The one on which our work is based is developed by Piterbarg [25] where, due to the *"effective skew"* and *"effective volatility"* concepts, the calibration of the model to a whole swaption grid was enabled.

Based on the arbitrage-free interpolation proposed in [29], here we extract a multi-factor short-rate process from the SV-LMM and include it into the LVM framework for the equity component. We extract the short-rate via a continuous tenor structure from the LMM. So, we end up with a local volatility equity model in which we define a stochastic interest rate based on the HJM consistent Libor Market Model. The impact of including the interest rate smile in the local volatility equity model is analyzed in details by means of numerical experiments.

In Section 2 we recall the Local Volatility Model and derive its extension to stochastic interest rates. We show that the results presented in [6] can be obtained without Malliavin calculus. In Section 3 we recall some necessary results from the HJM framework. Also in this section we use the arbitrage-free interpolation from [29] of the zero-coupon bonds (ZCB) under the LMM and derive the corresponding short-rate processes. Then, we perform a number of numerical experiments, that show the impact of the interest rate smile.

2 Local Volatility Model with Stochastic Interest Rate

The LVM developed by Derman & Kani [12] is a successful model which preserves the market observable equity smile. Furthermore, the calibration to implied volatilities is straightforward. Without careful consideration, however, the LVM may give rise to an underpricing of certain exotic options due to flattening forward volatility skews [17]. Improvements for this have been developed in [11].

Interest rate equity hybrid products require a model which also takes the randomness of the interest rates into account. E. Benhamou et al. [6] have shown that, by applying the Malliavin calculus, the LVSIR can be expressed in terms of the basic LVM with a covariance term between the logarithm of the underlying stock price and the spot rates. In this section we derive the LVSIR model without the Malliavin calculus.

Throughout the paper we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ under the usual conditions. By $\mathbb{P}$ we indicate the *real-world* probability measure and $\mathbb{Q}$ is the *risk-free* measure induced by a money-market account $B(t) = \exp(\int_0^t r_s ds)$. Let us now consider a system of stochastic differential equations of the following form:

$$\left\{\begin{array}{rcl} dS_t &=& r_t S_t dt + \sigma(t, S_t) S_t dW_t^S, \\ dr_t &=& \mu(t, r_t)dt + \gamma(t, r_t) dW_t^r, \\ \langle W_t^S, W_t^r \rangle &=& \rho_{S,r} t, \end{array}\right. \tag{2.1}$$

where S_t is an asset process with a local volatility, $\sigma(t, S_t)$; r_t is a stochastic interest rate process with drift $\mu(t, r_t)$ and volatility $\gamma(t, r_t)$. Now, for a given strike K, we define a claim, $C(t, S_t, K)$, by:

$$C(t, S_t, K) = B(0)\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B(t)} \max\left(S_t - K, 0\right) |\mathcal{F}_0\right). \tag{2.2}$$

With Itô's calculus we find for $C(t, S_t, K)$,

$$
\begin{aligned}
dC(t, S_t, K) &= d\left[\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B(t)}\max\left(S_t - K, 0\right)|\mathcal{F}_0\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[d\left(\frac{1}{B(t)}\max\left(S_t - K, 0\right)\right)|\mathcal{F}_0\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left(-r_t\frac{1}{B(t)}\max(S_t - K, 0)dt|\mathcal{F}_0\right) + \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B(t)}d\max(S_t - K, 0)|\mathcal{F}_0\right),
\end{aligned}
\tag{2.3}
$$

assuming the interchange of differentiation and expectation is justified. Now, in order to find the dynamics of $\max(S_t - K, 0)$ we follow the reasoning in [23] and use Itô-Meyer-Tanaka's Theorem:

Theorem 2.1 (Itô-Meyer-Tanaka). *For any continuous semimartingale S_t and any real number K, an increasing continuous process called "local time", $L_t^K(S_t)$, exists, such that:*

$$
\max(S_t - K, 0) = \max(S_0 - K, 0) + \int_0^t \mathbb{1}_{\{S_u - K > 0\}} dS_u + \frac{1}{2}L_t^K(S_t),
$$

for any $t \geq 0$, and where the local time of S_t at K is defined as:

$$
L_t^K(S_t) = \lim_{n\to\infty} n\int_0^t \mathbb{1}_{\{S_u \in [K, K+1/n]\}} d\langle S\rangle_u,
$$

where $\langle\cdot\rangle_t$ denotes the quadratic variation as in [22].

Proof. A proof for this theorem can be found in [14]. □

We see that the local time, $L_t^K(S_t)$, can be expressed as:

$$
L_t^K(S_t) = \int_0^t \delta(S_u - K) d\langle S\rangle_u.
$$

with $\delta(S_u - K)$ the Dirac delta function. So,

$$
d\left(\max(S_t - K, 0)\right) = \mathbb{1}_{\{S_t - K > 0\}} dS_t + \frac{1}{2}\delta(S_t - K)d\langle S\rangle_t. \tag{2.4}
$$

Now, by inserting Equations (2.1) and (2.4) in (2.3) we find:

$$
\begin{aligned}
dC(t, S_t, K) &= \mathbb{E}^{\mathbb{Q}}\left(-r_t\frac{1}{B(t)}\max(S_t - K, 0)dt|\mathcal{F}_0\right) + \mathbb{E}^{\mathbb{Q}}\left(r_t\frac{1}{B(t)}\mathbb{1}_{\{S_t - K > 0\}}S_t dt|\mathcal{F}_0\right) \\
&+ \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B(t)}\sigma^2(t, S_t)S_t^2\delta(S_t - K)dt|\mathcal{F}_0\right).
\end{aligned}
\tag{2.5}
$$

The first term in (2.5) can be expressed as:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(-r_t\frac{1}{B(t)}\max(S_t - K, 0)dt|\mathcal{F}_0\right)dt &= \mathbb{E}^{\mathbb{Q}}\left(r_t\frac{1}{B(t)}\mathbb{1}_{\{S_t - K > 0\}}K dt|\mathcal{F}_0\right) \\
&- \mathbb{E}^{\mathbb{Q}}\left(r_t\frac{1}{B(t)}\mathbb{1}_{\{S_t - K > 0\}}S_t dt|\mathcal{F}_0\right).
\end{aligned}
$$

So, finally, by setting $t = T$ in (2.5) we obtain:

$$
\begin{aligned}
\frac{\partial}{\partial T}C(T, S_T, K) &= \mathbb{E}^{\mathbb{Q}}\left(r_T K\frac{1}{B(T)}\mathbb{1}_{\{S_T - K > 0\}}|\mathcal{F}_0\right) \\
&+ \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B(T)}\delta(S_T - K)\sigma^2(T, S_T)S_T^2|\mathcal{F}_0\right).
\end{aligned}
\tag{2.6}
$$

The next lemma gives a relation between the last term in Equation (2.6) and the market observed European call prices.

Lemma 2.2. *For a given maturity time, $T \geq 0$, and a call option with strike K given by $C(T, S_T, K)$ under the system of SDEs given by Equations (2.1), the following equality holds:*

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B(T)}\delta(S_T - K)\sigma^2(T, S_T)S_T^2|\mathcal{F}_0\right) = K^2\sigma^2(T, K)\frac{\partial^2}{\partial K^2}C(T, S_T, K).$$

Proof. Firstly we recall a property of the time-delayed Dirac delta function $\delta(x)$, i.e.,

$$\int_{-\infty}^{+\infty} f(x)\delta(x - y)dx = f(y). \tag{2.7}$$

We define $X_T := -\int_0^T r_s ds$ and the joint density of the interest rate and asset process at time T by $f_{X_T,S_T}(x, y)$. Now, with Equation (2.7), we find:

$$\begin{aligned}
\mathbb{E}\left(e^{X_T}\delta(S_T - K)\sigma^2(T, S_T)S_T^2|\mathcal{F}_0\right) &= \int_{-\infty}^{+\infty} e^x \left(\int_{-\infty}^{+\infty} \delta(y - K)\sigma^2(T, y)y^2 f_{X_T,S_T}(x, y)dy\right) dx \\
&= K^2\sigma^2(T, K)\int_{-\infty}^{+\infty} e^x f_{X_T,S_T}(x, K)dx \\
&= -K^2\sigma^2(T, K)\int_{-\infty}^{+\infty} e^x \left(\frac{\partial}{\partial K}\int_K^{+\infty} f_{X_T,S_T}(x, y)dy\right) dx \\
&= K^2\sigma^2(T, K)\int_{-\infty}^{+\infty} e^x \left(\frac{\partial^2}{\partial K^2}\int_K^{+\infty} (y - K)f_{X_T,S_T}(x, y)dy\right) dx \\
&= K^2\sigma^2(T, K)\frac{\partial^2}{\partial K^2}C(T, S_T, K).
\end{aligned}$$

This concludes the proof of Lemma 2.2. □

By applying the results from Lemma 2.2 the partial derivative of the call with respect to maturity T in (2.6) can be simplified to:

$$\frac{\partial}{\partial T}C(T, S_T, K) = \mathbb{E}^{\mathbb{Q}}\left(r_T\frac{1}{B(T)}\mathbb{1}_{\{S_T - K > 0\}}K|\mathcal{F}_0\right) + \frac{1}{2}K^2\sigma^2(T, K)\frac{\partial^2}{\partial K^2}C(T, S_T, K).$$

So, by rearranging the terms the local volatility $\sigma^2(T, K)$ is equal to:

$$\sigma^2(T, K) = \frac{\frac{\partial}{\partial T}C(T, S_T, K) - K\mathbb{E}^{\mathbb{Q}}\left(r_T\frac{1}{B(T)}\mathbb{1}_{\{S_T - K > 0\}}|\mathcal{F}_0\right)}{\frac{1}{2}K^2\frac{\partial^2}{\partial K^2}C(T, S_T, K)}. \tag{2.8}$$

As already found in [6], the LVSIR expression (2.8) can be decomposed in terms of the LVM for deterministic rates plus some correction terms. We follow that route and by taking rate $r_s \equiv r$ constant, we evaluate the expectation in Equation (2.8):

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(r\frac{1}{B(T)}\mathbb{1}_{\{S_T - K > 0\}}|\mathcal{F}_0\right) &= re^{rT}\mathbb{E}^{\mathbb{Q}}\left(\mathbb{1}_{\{S_T - K > 0\}}|\mathcal{F}_0\right) \\
&= re^{rT}\mathbb{Q}\left(S_T > K\right) \\
&= -r\frac{\partial}{\partial K}C(T, S_T, K),
\end{aligned} \tag{2.9}$$

where $\mathbb{Q}$ represents the risk-free probability, see the derivations in [26]. So, for constant r Equation (2.8) reduces to the famous Dupire [13] formula, i.e.,

$$\sigma^2_{det}(T, K) = \frac{\frac{\partial}{\partial T}C(T, S_T, K) + Kr\frac{\partial}{\partial K}C(T, S_T, K)}{\frac{1}{2}K^2\frac{\partial^2}{\partial K^2}C(T, S_T, K)},$$

with $\sigma^2_{det}(T,K)$ the "deterministic" local volatility. Relating the local volatility for both the stochastic and deterministic models we find that:

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) - \frac{\mathbb{E}^{\mathbb{Q}}\left(r_T \frac{1}{B(T)} \mathbb{1}_{\{S_T - K > 0\}} | \mathcal{F}_0\right) + r \frac{\partial}{\partial K} C(T, S_T, K)}{\frac{1}{2} K \frac{\partial^2}{\partial K^2} C(T, S_T, K)}. \tag{2.10}$$

We therefore see that, in order to find the volatility in the LVSIR, one can first derive the local volatility for constant interest rate, r, and then determine a *correction term.*

2.1 Relation between Local Volatility Models

Although we have obtained a formula for the local volatility model under a stochastic interest rate in terms of its deterministic version, we see that the main formula (2.10) involves stochastic integration, which has to be done numerically. The *correction term* requires the joint distribution between S_T and r_T. In [27] it was shown that via the arbitrage-free HJM conditions and by application of the Kolmogorov forward equation the density for the joint distribution, $f_{S,r}$, can be obtained. In the derivation below we show how the expectation in (2.10) can be simplified.

Theorem 2.3. *Let us assume that the asset process is given by system (2.1) with the correlation coefficient between the processes given by $\rho_{S,r} \neq 0$. The relation between the local volatility models with and without stochastic interest rate is then given by:*

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) - 2\,cov(r_T, \log S_T | \log S_T = \log K). \tag{2.11}$$

Proof. We start the proof with two helpful lemmas:

Lemma 2.4 (Short-rate process under the forward measure)**.** *Let $f(t,T)$ be an instantaneous forward rate, and r_T be a short-rate at time T, then the expectation of the short-rate process under the T-forward measure is equal to the instantaneous forward rate, i.e.,*

$$\mathbb{E}^T\left(r_T | \mathcal{F}_t\right) = f(t,T).$$

Proof. The proof can be found in [10] pp. 39. □

Lemma 2.5. *Let $f_{S_T}(K)$ be the probability density function of the equity process S_t and $C(T, S_T, K)$ a claim as in (2.2), then the following equalities hold:*

$$\begin{aligned} \frac{\partial^2}{\partial K^2} C(T, S_T, K) &= f_{S_T}(K)\, \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B(T)} | S_T = K\right) \\ &= f_{S_T}(K)\, P(0,T) \mathbb{E}^T\left(1 | S_T = K\right) \\ &= f_{S_T}(K)\, P(0,T), \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \frac{\partial}{\partial K} C(T, S_T, K) &= P(0,T) \frac{\partial}{\partial K} \mathbb{E}^T\left(\max\left(S_T - K, 0\right) | \mathcal{F}_0\right) \\ &= -P(0,T) \mathbb{Q}^T\left(S_T > K\right). \end{aligned} \tag{2.13}$$

Proof. We set again $X_T := -\int_0^T r_s ds$ and define the joint density between X_T and S_T by $f_{X_T,S_T}(x,y)$. It then follows that:

$$\begin{aligned} \frac{\partial^2}{\partial K^2} C(T, S_T, K) &= \frac{\partial^2}{\partial K^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^x \max(s - K, 0) f_{X_T,S_T}(x,s) ds dx \\ &= -\int_{-\infty}^{+\infty} e^x \frac{\partial^2}{\partial K^2} \left(\int_{+\infty}^{K} (s-K) f_{X_T,S_T}(x,s) ds\right) dx \\ &= \int_{-\infty}^{+\infty} e^x f_{S_T}(K) f_{X_T|S_T}(x,K) dx \\ &= f_{S_T}(K) \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} | S_T = K\right), \end{aligned}$$

using $f_{X_T,S_T}(\cdot) = f_{S_T}(\cdot) f_{X_T|S_T}(\cdot)$.

The second part of the lemma, (2.13), comes straightforward from integration rules. $\square$

We now start the proof of Theorem 2.3 with the following change of measure:

$$\eta_T = \frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{P(T,T)}{P(0,T)} \cdot \frac{B(0)}{B(T)},$$

where $B(T)$ is the money-market account and $P(t,T)$ is a zero-coupon bond. With the change of measure the expectation in Equation (2.8) is given by:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left(r_T \frac{1}{B(T)} \mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0\right) &= \mathbb{E}^T\left(r_T \frac{1}{B(T)} \mathbb{1}_{\{S_T-K>0\}} B(T) P(0,T) | \mathcal{F}_0\right) \\ &= P(0,T)\mathbb{E}^T\left(r_T \mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0\right). \end{aligned} \tag{2.14}$$

Therefore, the local volatility in (2.8) reads

$$\sigma^2(T,K) = \frac{\frac{\partial}{\partial T} C(T,S_T,K) - K P(0,T)\mathbb{E}^T\left(r_T \mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0\right)}{\frac{1}{2} K^2 \frac{\partial^2}{\partial K^2} C(T,S_T,K)}. \tag{2.15}$$

By the definition of covariance we have:

$$\mathrm{cov}^T(r_T, \mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0) = \mathbb{E}^T(r_T \mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0) - \mathbb{E}^T(r_T | \mathcal{F}_0) \mathbb{E}^T(\mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0),$$

and with Lemmas 2.4 and 2.5 we find:

$$\begin{aligned} \mathbb{E}^T(r_T \mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0) &= \mathrm{cov}^T(r_T, \mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0) + f(0,T)\mathbb{E}^T(\mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0) \\ &= \mathrm{cov}^T(r_T, \mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0) + f(0,T)\mathbb{Q}^T\left(S_T > K\right) \\ &= \mathrm{cov}^T(r_T, \mathbb{1}_{\{S_T-K>0\}} | \mathcal{F}_0) - \frac{f(0,T)}{P(0,T)} \frac{\partial}{\partial K} C(T,S_T,K). \end{aligned}$$

By including this result in Equation (2.15) we find that the bias between the local volatility models can be expressed as:

$$\sigma^2(T,K) - \sigma^2_{det}(T,K) = -\frac{2P(0,T)\mathrm{cov}^T\left(r_T, \mathbb{1}_{\{S_T>K\}}\right)}{K \frac{\partial^2}{\partial K^2} C(T,S_T,K)}. \tag{2.16}$$

In order to simplify the RHS of (2.16) we use the following lemma:

Lemma 2.6. *For any two random variables $X, Y \in \mathbb{R}$ we have:*

$$cov(X,Y) = \int_{\mathbb{R}} cov(X, \mathbb{1}_{\{Y>K\}}) dK.$$

Proof. We start the proof by assuming that Y is a positive random variable:

$$\begin{aligned} \int_{\mathbb{R}^+} \mathrm{cov}(X, \mathbb{1}_{\{Y>K\}}) dK &= \int_{\mathbb{R}^+} \mathbb{E}(X \mathbb{1}_{\{Y>K\}}) dK - \int_{\mathbb{R}^+} \mathbb{E}X \mathbb{E}(\mathbb{1}_{\{Y>K\}}) dK \\ &= \int_{\mathbb{R}^+} \left(\int_{\{\omega: Y(\omega)>K\}} X d\mathbb{P} \right) dK - \mathbb{E}X \int_{\mathbb{R}^+} P(Y>K) dK \\ &= \int_{\Omega} \int_0^{Y(\omega)} X(\omega) dt d\mathbb{P} - \mathbb{E}X\mathbb{E}Y \\ &= \int_{\Omega} X(\omega) Y(\omega) d\mathbb{P} - \mathbb{E}X\mathbb{E}Y \\ &= \mathbb{E}(XY) - \mathbb{E}X\mathbb{E}Y =: \mathrm{cov}(X,Y), \end{aligned}$$

employing Fubini's Theorem in the third step. By using the properties of covariance we can write for any constant $a \in \mathbb{R}$ and $Y \in \mathbb{R}^+$:

$$\text{cov}(X, Y) = \text{cov}(X, Y + a) = \int_a^{+\infty} \text{cov}(X, \mathbb{1}_{\{Y+a>K\}})dK. \tag{2.17}$$

This concludes the proof of Lemma 2.6. □

Now, by multiplying both sides of Equation (2.16) by $\frac{\partial^2}{\partial K^2} C(T, S_T, K)$, we get:

$$\left(\sigma^2(T, K) - \sigma^2_{det}(T, K)\right) \frac{\partial^2}{\partial K^2} C(T, S_T, K) \quad = \quad -\frac{2}{K} P(0, T)\text{cov}^T \left(r_T, \mathbb{1}_{\{S_T>K\}}\right).$$

Integrating both sides w.r.t. K gives:

$$\int_{\mathbb{R}^+} \left(\left(\sigma^2(T, K) - \sigma^2_{det}(T, K)\right) \frac{\partial^2}{\partial K^2} C(T, S_T, K)\right) dK = -2P(0, T) \int_{\mathbb{R}^+} \text{cov}^T \left(r_T, \mathbb{1}_{\{S_T>K\}}\right) \frac{1}{K} dK. \tag{2.18}$$

By Lemma 2.5, Equation (2.18) can be written as:

$$\int_{\mathbb{R}^+} \sigma^2(T, K) f_{S_T}(K)dK - \int_{\mathbb{R}^+} \sigma^2_{det}(T, K) f_{S_T}(K)dK = -2 \int_{\mathbb{R}} \text{cov}^T \left(r_T, \mathbb{1}_{\{S_T>e^x\}}\right) dx,$$

With $\mathbb{1}_{\{S_T>e^x\}} = \mathbb{1}_{\{\log S_T>x\}}$, we obtain:

$$\int_{\mathbb{R}^+} \sigma^2(T, K) f_{S_T}(K)dK - \int_{\mathbb{R}^+} \sigma^2_{det}(T, K) f_{S_T}(K)dK = -2 \int_{\mathbb{R}} \text{cov}^T \left(r_T, \mathbb{1}_{\{\log S_T>x\}}\right) dx,$$

and by Lemma 2.6 we have:

$$\int_{\mathbb{R}^+} \sigma^2(T, K) f_{S_T}(K)dK - \int_{\mathbb{R}^+} \sigma^2_{det}(T, K) f_{S_T}(K)dK = -2\text{cov}^T(r_T, \log S_T). \tag{2.19}$$

We recognize the LHS integrals in (2.19) as being expectations. So, finally, we obtain:

$$\mathbb{E}\left(\sigma^2(T, S_T) - \sigma^2_{det}(T, S_T) + 2\text{cov}^T(r_T, \log S_T)\right) \quad = \quad 0, \tag{2.20}$$

which, by the property of conditional expectation[1], becomes:

$$\sigma^2(T, K) - \sigma^2_{det}(T, K) \quad = \quad -2\text{cov}^T(r_T, \log S_T | \log S_T = \log K) \text{ a.s.}$$

This concludes the proof of Theorem 2.3. □

2.2 Iterative Algorithm for Local Volatility

We first assume that the correlation $\rho_{S,r}$ between the stock asset S_t and the interest rates r_t can be estimated from historical data and is known a priori. Then, the calculation of the LVSIR can be simplified as follows.

Lemma 2.7. *Suppose the equity asset and short-rate processes are driven by (2.1). The covariance of the two processes* $\log S_t$ *and* r_t *can then be approximated by:*

$$cov(r_t, \log S_t | \log S_t = \log K) \quad = \quad \rho_{S,r} \int_0^t \mathbb{E}\left(\sigma(u, S_u)\gamma(u, r_u)\right) du.$$

[1] $\mathbb{E}(\cdot) = \mathbb{E}\left(\mathbb{E}(\cdot|\mathcal{A})\right)$

Proof. From the Itô derivative for a log-stock price and from the definition of covariance we have:

$$\begin{aligned}
\text{cov}(dr_t, d\log S_t) &= \mathbb{E}(d\log S_t dr_t) - \mathbb{E}(d\log S_t)\mathbb{E}(dr_t) \\
&= \mathbb{E}\Big(\mu(t,r_t)(r_t - \frac{1}{2}\sigma(t,S_t)^2)(dt)^2 + \sigma(t,S_t)\gamma(t,S_t)dW_t^r dW_t^S \\
&\quad +\sigma(t,S_t)\mu(t,r_t)(dt)dW_t^S + \gamma(t,r_t)(r_t - \frac{1}{2}\sigma(t,S_t)^2)(dt)dW_t^r\Big) \\
&= \rho_{S,r}\mathbb{E}\left(\sigma(t,S_t)\gamma(t,S_t)dt\right),
\end{aligned}$$

where three terms equal zero. Now, by integration the expressions under the covariance operator we have:

$$\text{cov}\left(\int_0^t d\log S_u, \int_0^t dr_u\right) = \rho_{S,r}\mathbb{E}\left(\int_0^t \sigma(u,S_u)\gamma(u,S_u)du\right),$$

where the LHS is equal to $\text{cov}\left(\log S_t - \log S_0, r_t - r_0\right) \equiv \text{cov}(\log S_t, r_t)$. □

Although we have presented, in Theorem 2.1, a simplified relationship between the local volatility models, we see that the covariance, $\text{cov}(r_T, \log S_T | \log S_T = \log K)$, involves the distribution of the log-asset, which is itself dependent on the local volatility $\sigma^2(t, S_t)$. This makes the computation more involved. Nevertheless, the issue can be resolved by a basic iteration based on the Fixed-Point Theorem:

$$\widehat{\sigma}_n^2(T,K) = \sigma_{det}^2(T,K) - 2\text{cov}\left(r_T, \log S(T, \hat{\sigma}_{n-1}(T,S_T)) | \log S_T = \log K\right), \tag{2.21}$$

with $n \in \mathbb{N}^+$, and $\hat{\sigma}_0(T,K) = \sigma_{det}(T,K)$. By Lemma 2.7, Equation (2.21) can be simplified to:

$$\widehat{\sigma}_n^2(T,K) = \sigma_{det}^2(T,K) - 2\rho_{S,r}\int_0^T \mathbb{E}\left(\hat{\sigma}_{n-1}(t,S_t)\gamma(t,r_t)\right)dt,\ n \in \mathbb{N}^+,$$

with the same initial condition.

3 Short-rate Processes under HJM

The focus of the current section is to find a well-defined short-rate process which can be included in the local volatility framework defined above. Choosing for the interest rate model one of the well-known stochastic volatility short-rate models, like, for example, the Heston type model (as in [18]) by Fong-Vašiček [15] (1991) is not fully satisfactory. In [1] it was shown that such a model is not truly stochastic, as it generates a volatility-dependent ZCB. Moreover, already the formulation of a short-rate model with a volatility which is not directly observable from the yield curve movement, is quite difficult. In [4] it was shown that one way to incorporate the volatility of the yield curve properly is to use the HJM model [19]. In this section we follow this idea and specify the short-rate model constructed from the Libor Market Model. We describe a procedure of extracting a short-rate process from the discretely tenored Stochastic Volatility Libor Market Model (SV-LMM) while satisfying the Heath, Jarrow and Morton arbitrage-free conditions.

3.1 HJM Model Setup

The HJM interest rate framework provides a general specification of the interest rate term structure [28], which is able to incorporate the main market observable features. The HJM approach to the modeling of the term structure is based on the dynamics of an instantaneous, continuously compounded forward rate, $f(t,T_i)$, for a certain index $i \in \mathbb{N}$, with its dynamics under the real-world measure $\mathbb{P}$, defined by the following SDE:

$$df(t,T_i) = \alpha^{\mathbb{P}}(t,T_i)dt + \Sigma(t,T_i)dW_t^{\mathbb{P}}, \tag{3.1}$$

where $\alpha^{\mathbb{P}}(t, T_i)$ and $\Sigma(t, T_i)$ are well-defined adapted stochastic processes with values in $\mathbb{R}$. By taking a certain maturity $T_i \in \mathbb{R}^+$ and time $0 \leq t \leq T_i$, we define the money-market account, $B(t)$, as

$$B(t) = \exp\left(\int_0^t r_s ds\right) = \exp\left(\int_0^t f(s,s)ds\right), \tag{3.2}$$

where $r_s \equiv f(s,s)$, and a ZCB, which pays 1 unit at maturity T_i:

$$P(t, T_i) = \exp\left(-\int_t^{T_i} f(t,s)ds\right), \forall t \in [0, T_i]. \tag{3.3}$$

From the arbitrage-free principle we know that the ZCB with the money-market account as a numéraire has to be a martingale, i.e. the dynamics of the relative prices have to be driftless. This is one of the important results in the HJM model. The model specifies in particular the relationship between the drift $\alpha^{\mathbb{Q}}(t, T_i)$ in the risk-free neutral measure and the volatility structure $\Sigma(t, T_i)$ for the instantaneous forward rates:

Lemma 3.1 (HJM no-arbitrage instantaneous forward rates drift condition)**.** *Under the $\mathbb{Q}$ measure induced by the money-market account (3.2), the dynamics for the instantaneous forward rates are given by:*

$$df(t, T_i) = \alpha^{\mathbb{Q}}(t, T_i)dt + \Sigma(t, T_i)dW_t^{\mathbb{Q}}, \tag{3.4}$$

where the drift reads:

$$\alpha^{\mathbb{Q}}(t, T_i) = \Sigma(t, T_i)\int_t^{T_i} \Sigma(t,s)ds. \tag{3.5}$$

Proof. The proof can be found in [28] pp. 307. □

Now, we investigate the time evolution of the ZCB. The instantaneous forward rates, $f(t, T_i)$, are stochastically defined functions. As $f(t, T_i)$ is stochastic over time, so is $P(t, T_i)$. The lemma below describes the dynamics of $P(t, T_i)$ for a given $f(t, T_i)$.

Lemma 3.2 (ZCB dynamics under risk-free measure)**.** *For any fixed maturity $T_i \leq T^*$ the dynamics of $P(t, T_i)$ under the spot measure $\mathbb{Q}$ are given by:*

$$\frac{dP(t, T_i)}{P(t, T_i)} = r_t dt - \left(\int_t^{T_i} \Sigma(t, T_i)ds\right) dW_t^{\mathbb{Q}}, \tag{3.6}$$

with $r_t \equiv f(t,t)$.

Proof. The proof can be found in [28] pp. 309. □

By integrating Equation (3.4), and by setting $T_i = t$ we get:

$$r_t \equiv f(t,t) = f(0,t) + \int_0^t \alpha^{\mathbb{Q}}(s,t)ds + \int_0^t \Sigma(s,t)dW_s^{\mathbb{Q}}. \tag{3.7}$$

By applying the Itô-Leibniz integral rule [2] we obtain the following short-rate dynamics:

$$dr_t = \left(\frac{\partial}{\partial t}f(0,t) + \alpha^{\mathbb{Q}}(t,t) + \int_0^t \frac{\partial}{\partial t}\alpha^{\mathbb{Q}}(s,t)ds + \int_0^t \frac{\partial}{\partial t}\Sigma(s,t)dW_s^{\mathbb{Q}}\right) dt + \Sigma(t,t)dW_t^{\mathbb{Q}}. \tag{3.8}$$

Equation (3.8) presents the short-rate dynamics, r_t, as a function of initial yield, $f(0,t)$, and volatility, $\Sigma(s,t)$, for any $s \leq t$.

[2]

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx = f(b,\alpha)\frac{\partial}{\partial\alpha}b(\alpha) - f(a,\alpha)\frac{\partial}{\partial\alpha}a(\alpha) + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}f(x,\alpha)dx$$

3.2 HJM under the Forward Measure

This subsection discusses the HJM drift restrictions under the forward measure. For a given $t \leq T_i \leq T_j$ with $i \leq j$ we have the change of measure, η_{T_j}:

$$\eta_{T_j} = \frac{d\mathbb{Q}^{T_j}}{d\mathbb{Q}} = \frac{P(T_j, T_j)}{P(0, T_j)} \cdot \frac{B(0)}{B(T_j)}, \tag{3.9}$$

with $P(T_j, T_j) = B(0) = 1$. With Equation (3.7), we find

$$\begin{aligned} \eta_{T_j} &= \exp\left(\int_0^{T_j} f(0,s)ds - \int_0^{T_j} r_s ds\right) &(3.10)\\ &= \exp\left(-\int_0^{T_j}\left(\int_0^s \alpha^{\mathbb{Q}}(u,s)du\right)ds - \int_0^{T_j}\left(\int_0^s \Sigma(u,s)dW_u^{\mathbb{Q}}\right)ds\right). &(3.11)\end{aligned}$$

Application of Fubini's Theorem and the HJM drift condition in Equation (3.5) gives:

$$\eta_{T_j} = \exp\left(-\frac{1}{2}\int_0^{T_j}\left(\int_u^{T_j}\Sigma(u,s)ds\right)^2 du - \int_0^{T_j}\left(\int_u^{T_j}\Sigma(u,s)ds\right)dW_u^{\mathbb{Q}}\right). \tag{3.12}$$

By the Girsanov Theorem we have a generating kernel,

$$\phi_t = -\int_t^{T_j}\Sigma(t,s)ds,$$

with which the dynamics change as follows:

$$dW_t^{\mathbb{Q}} = \phi_t dt + dW_t^{T_j}.$$

By the definition of instantaneous forward rates we get:

$$\begin{aligned} df(t,T_i) &= \Sigma(t,T_i)\left(\int_t^{T_i}\Sigma(t,s)ds\right)dt + \Sigma(t,T_i)\left(\phi_t dt + dW_t^{T_j}\right)\\ &= \Sigma(t,T_i)\left(\int_t^{T_i}\Sigma(t,s)ds - \int_t^{T_j}\Sigma(t,s)ds\right)dt + \Sigma(t,T_i)dW_t^{T_j} &(3.13)\\ &= -\Sigma(t,T_i)\left(\int_{T_i}^{T_j}\Sigma(t,s)ds\right)dt + \Sigma(t,T_i)dW_t^{T_j}.\end{aligned}$$

This represents the dynamics of the instantaneous forward rates under the forward measure for $t < T_i \leq T_j$.

The next lemma defines the dynamics of the zero-coupon bonds for the corresponding measure:

Lemma 3.3 (ZCB dynamics under the forward measure)**.** *For a given time $t < T_i \leq T_j$, the ZCB dynamics are given by:*

$$\frac{dP(t,T_i)}{P(t,T_i)} = \Gamma(t,T_i,T_j)dt - \left(\int_t^{T_i}\Sigma(t,s)ds\right)dW_t^{T_j}, \tag{3.14}$$

where:

$$\Gamma(t,T_i,T_j) = f(t,t) - \left(\int_t^{T_i}\alpha^{T_j}(t,s,T_j)ds\right) + \frac{1}{2}\left(\int_t^{T_i}\Sigma(t,s)ds\right)^2.$$

Proof. By the definition of the ZCB in (3.3) and with $Z(t,T_i) = -\int_t^{T_i} f(t,s)ds$ we have the Itô dynamics:

$$\frac{dP(t,T_i)}{P(t,T_i)} = dZ(t,T_i) + \frac{1}{2}\left(dZ(t,T_i)\right)^2. \tag{3.15}$$

Since

$$dZ(t,T_i) = f(t,t)dt - \int_t^{T_i} df(t,s)ds, \tag{3.16}$$

and by using Equation (3.13) the $Z(t,T)$ dynamics read:

$$\begin{aligned} dZ(t,T_i) &= f(t,t)dt - \int_t^{T_i} \left(\alpha^{T_j}(t,s,T_j)dt\right) ds - \left(\int_t^{T_i} \Sigma(t,s)dW_t^{\mathbb{Q}^{T_j}}\right) ds & (3.17)\\ &= \left[f(t,t) - \left(\int_t^{T_i} \alpha^{T_j}(t,s,T_j)ds\right)\right] dt - \left(\int_t^{T_i} \Sigma(t,s)ds\right) dW_t^{T_j}. & (3.18)\end{aligned}$$

Collecting the terms in Equation (3.15) concludes the proof. □

3.3 Short-rate Processes implied by Libor Market Model

Typically, the pricing of hybrid products is based on Monte Carlo sampling from a calibrated model, therefore we require a continuous interest rate process. The LMM is however defined on a discrete tenor structure, so it does not fit immediately to the continuous interest rate model. A way of generalizing the Libor market model from the discrete to a continuous tenor was proposed in [29], where via an arbitrage-free interpolation between the Libors the zero-coupon bond for continuous time was defined. In this section we generalize this idea and use the HJM arbitrage-free condition to extract the short-rate from the interpolated zero-coupon bonds.

We assume a trading time horizon $[0,T^*]$ and a set of times $\{T_k; k \in [1,N], N \in \mathbb{N}\}$ such that $\{0 \leq T_0 < T_k < \cdots < T_N \leq T^*\}$, with a tenor $\tau_k = T_k - T_{k-1}$. We consider the family of simple compounded forward rates, $L(t;T_{k-1},T_k)$, with expiry at time T_{k-1} and maturity T_k.

Definition 3.4 (Simple compounded forward Libor rate)**.** *For a given year fraction τ_k, and $P(t,T_k)$, a risk-free zero-coupon bond maturing at time T_k with a nominal value of 1, the simple Libor forward rate $L(t;T_{k-1},T_k)$ over the period $[T_{k-1},T_k]$ is defined as:*

$$L(t;T_{k-1},T_k) := \tau_k^{-1}\left(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1\right). \tag{3.19}$$

We also define the spot Libor measure:

Definition 3.5 (Spot Libor measure)**.** *For a given time, $T_{m(t)-1} < t \leq T_{m(t)}$, where $m(t) = \min(i : T_i \geq t)$ and a zero-coupon bond, $P(t,T_{m(t)})$, the money-market account is given by*

$$B(t) := P(t,T_{m(t)})\left(\prod_{i=1}^{m(t)} P(T_{i-1},T_i)\right)^{-1}. \tag{3.20}$$

Moreover, for any $t \leq T_{m(t)}$, and $m(t) = \min(i : T_i \geq t)$, under the $\mathbb{Q}^{m(t)+1}$ measure induced by a zero-coupon bond $P(t,T_{m(t)+1})$, we define the drift-less Libor rate dynamics as:

$$dL(t;T_{m(t)},T_{m(t)+1}) = \Lambda(t,\mathbf{L})dW_t^{m(t)+1}, \tag{3.21}$$

where the volatility function $\Lambda(t,\mathbf{L})$ is a certain adapted process. Note that the Libor rate process is fully determined by its volatility structure. So, by taking $\Lambda(t,\mathbf{L}) = \sigma_{m(t)+1}L(t;T_{m(t)},T_{m(t)+1})$ the model collapses to a log-normal one. For $\Lambda(t,\mathbf{L}) = \sigma_{m(t)+1}(t)(L(t;T_{m(t)},T_{m(t)+1}) + \alpha_{m(t)+1})$ it becomes a displaced-diffusion Libor model. We obtain a stochastic volatility model for $\Lambda(t,\mathbf{L})$ when $\sigma_{m(t)+1}(t)$ is modeled by a stochastic process.

3.3.1 Introducing the zero-coupon bond volatility

In [29], it was shown that for a given discrete tenor model, and an arbitrage-free interpolation of the short dated ZCBs, the continuous tenor model is completely specified. We extend this interpolation in which a control of the amount of *external* volatility for the ZCB can be included. We use these results to determine the continuous short-rate process.

Definition 3.6 (ZCB interpolation). *For any $n \geq 0$ and time $t \in (T_{m(t)-1}, T_{m(t)}]$ with $m(t) = \min(i : T_i \geq t)$, the zero-coupon bond $P(t, T_{m(t)})$ is defined by:*

$$P(t, T_{m(t)}) = \frac{1}{1 + (T_{m(t)} - t)\psi(t)}, \tag{3.22}$$

where

$$\psi(t) = \alpha_{t,n} L_{m(t)}(T_{m(t)-1}) + (1 - \alpha_{t,n}) L_{m(t)+1}(t), \tag{3.23}$$

with

$$L_{m(t)+1}(t) := L(t; T_{m(t)}, T_{m(t)+1}), \quad L_{m(t)}(T_{m(t)-1}) := L(T_{m(t)-1}; T_{m(t)-1}, T_{m(t)}),$$

and

$$\alpha_{t,n} = \left(\frac{T_{m(t)} - t}{T_{m(t)} - T_{m(t)-1}} \right)^n. \tag{3.24}$$

The interpolation presented in Definition 3.6 above is consistent with the LMM definition, i.e.:

- $\forall n \in \mathbb{N} \quad \lim_{t \to T^+_{m(t)-1}} \alpha_{t,n} = 1 \quad \text{so} \quad \lim_{t \to T^+_{m(t)-1}} P(t, T_{m(t)}) = \frac{1}{1 + \tau_{m(t)} L(T_{m(t)-1}, T_{m(t)})},$
- $\forall n \in \mathbb{N} \quad \lim_{t \to T^-_{m(t)}} \alpha_{t,n} = 0 \quad \text{so} \quad \lim_{t \to T^-_{m(t)}} P(t, T_{m(t)}) = 1.$

These limits are independent of n. We see from (3.24) that for any $n \geq 0$ the weighting factor $\alpha_{n,t} \in [0, 1]$. In Table 1 the influence of n on the interpolation weights is presented.

Table 1: Weight distribution w.r.t. coefficient n.

n	weight impact	
	$L(T_{m(t)-1}; T_{m(t)-1}, T_{m(t)})$	$L(t; T_{m(t)}, T_{m(t)+1})$
$n = 0$	100%	0%
$n = 1$	50%	50%
$0 < n < 1$	$> 50\%$	$< 50\%$
$n > 1$	$< 50\%$	$> 50\%$

Since we have now defined a continuous tenor structure, with the corresponding ZCBs, we can go one step further and specify an underlying stochastic volatility LMM (SV-LMM). We find the dynamics of the ZCBs and extract the continuously compounded driving short-rate process for it.

Lemma 3.7. *For a given time $t \in (T_{m(t)-1}, T_{m(t)}]$ with $m(t) = \min(i : T_i \geq t)$, the stochastic volatility displaced Libors with dynamics under the $\mathbb{Q}^{m(t)+1}$ measure are given by:*

$$\begin{cases} d\widetilde{L}_t &= \widetilde{\sigma}_t \sqrt{V_t} \varphi(\widetilde{L}_t) dW_t^{m(t)+1}, \\ dV_t &= \zeta(\theta - V_t)\, dt + \epsilon \sqrt{V_t} dW_t^V, \\ \left\langle W_t^{m(t)+1}, W_t^V \right\rangle &= 0, \end{cases} \tag{3.25}$$

where $\widetilde{L}_t := L(t; T_{m(t)}, T_{m(t)+1})$, $\varphi(x) = \beta x(t) + (1-\beta)x(0)$, skew parameter β, volatility parameter $\widetilde{\sigma}_t$, volatility of variance, ϵ, and positive parameters ζ, θ. The dynamics of the interpolated ZCB, as in Definition 3.6, under the $\mathbb{Q}^{m(t)}$ measure are now given by:

$$\frac{dP(t, T_{m(t)})}{P(t, T_{m(t)})} = \xi_{t,n} dt + (1 - \alpha_{t,n}) \kappa_t \widetilde{\sigma}_t \varphi(\widetilde{L}_t) \sqrt{V_t} P(t, T_{m(t)}) dW_t^{m(t)}, \tag{3.26}$$

where:

$$\xi_{t,n} = \theta^1_{t,n} P(t, T_{m(t)}) + \theta^2_{t,n} \frac{\kappa_t V_t \varphi^2(\widetilde{L}_t) \widetilde{\sigma}_t^2}{1 + \widetilde{\tau}_t \widetilde{L}_t} P^2(t, T_{m(t)}), \tag{3.27}$$

$$\theta^1_{t,n} = \widetilde{L}_t + \alpha_{t,n}(1 + n) \left(L_t - \widetilde{L}_t \right), \tag{3.28}$$

$$\theta^2_{t,n} = (\alpha_{t,n} - 1)(\kappa_t \alpha_{t,n}(1 + L_t \widetilde{\tau}_t) - \kappa_t - \widetilde{\tau}_t), \tag{3.29}$$

and $L_t := L(T_{m(t)-1}; T_{m(t)-1}, T_{m(t)})$, $\kappa_t = t - T_{m(t)}$, $\widetilde{\tau}_t = T_{m(t)+1} - T_{m(t)}$, *and* $\alpha_{t,n}$ *as in Definition 3.22.*

Proof. First of all, we note that the stochastic volatility Libor, $\widetilde{L}_t$, under the $\mathbb{Q}^{m(t)}$ measure, is given by:

$$d\widetilde{L}_t = \frac{\widetilde{\tau}_t \varphi^2(\widetilde{L}_t) V_t \widetilde{\sigma}_t^2}{1 + \widetilde{\tau}_t \widetilde{L}_t} dt + \sqrt{V_t} \varphi(\widetilde{L}_t) \widetilde{\sigma}_t dW_t^{m(t)}. \tag{3.30}$$

Application of Itô's Lemma to Equation (3.22) in Definition 3.6 gives:

$$\begin{aligned}
\frac{\partial}{\partial t} P(t, T_{m(t)}) &= \left(\widetilde{L}_t + (L_t - \widetilde{L}_t)\alpha_{t,n} + \kappa_t (L_t - \widetilde{L}_t) \frac{\partial \alpha_{t,n}}{\partial t} \right) P^2(t, T_{m(t)}), \\
\frac{\partial}{\partial \widetilde{L}_t} P(t, T_{m(t)}) &= \kappa_t (1 - \alpha_{t,n}) P^2(t, T_{m(t)}), \\
\frac{\partial^2}{\partial \widetilde{L}_t^2} P(t, T_{m(t)}) &= 2\kappa_t^2 (1 - \alpha_{t,n})^2 P^3(t, T_{m(t)}).
\end{aligned} \tag{3.31}$$

By combining these results we obtain:

$$\begin{aligned}
\frac{dP(t, T_{m(t)})}{P(t, T_{m(t)})} &= \left(\widetilde{L}_t + (L_t - \widetilde{L}_t) \left(\alpha_{t,n} + \kappa_t \frac{\partial \alpha_{t,n}}{\partial t} \right) \right) P(t, T_{m(t)}) dt \\
&+ \kappa_t (1 - \alpha_{t,n}) P(t, T_{m(t)}) d\widetilde{L}_t \\
&+ \kappa_t^2 (1 - \alpha_{t,n})^2 P^2(t, T_{m(t)}) V_t \varphi^2(\widetilde{L}_t) \widetilde{\sigma}_t^2 dt.
\end{aligned} \tag{3.32}$$

Taking $\theta_{t,n}^1$, as defined in Lemma 3.7, we get

$$\begin{aligned}
\frac{dP(t, T_{m(t)})}{P(t, T_{m(t)})} &= \theta_{t,n}^1 P(t, T_{m(t)}) dt + \kappa_t (1 - \alpha_{t,n}) P(t, T_{m(t)}) \frac{\widetilde{\tau}_t \varphi^2(\widetilde{L}_t) V_t \widetilde{\sigma}_t^2}{1 + \widetilde{\tau}_t \widetilde{L}_t} dt \\
&+ \kappa_t^2 (1 - \alpha_{t,n})^2 P^2(t, T_{m(t)}) V_t \varphi^2(\widetilde{L}_t) \widetilde{\sigma}_t^2 dt \\
&+ \kappa_t (1 - \alpha_{t,n}) P(t, T_{m(t)}) \sqrt{V_t} \varphi(\widetilde{L}_t) \widetilde{\sigma}_t dW_t^{m(t)}.
\end{aligned} \tag{3.33}$$

We see that the diffusion terms in Equations (3.33) and (3.26) agree. We find for the drift, $\xi_{t,n}$, of the ZCB dynamics in (3.33):

$$\begin{aligned}
\xi_{t,n} &= \theta_{t,n}^1 P(t, T_{m(t)}) + \kappa_t (1 - \alpha_{t,n}) P(t, T_{m(t)}) \frac{\widetilde{\tau}_t \varphi^2(\widetilde{L}_t) V_t \widetilde{\sigma}_t^2}{1 + \widetilde{\tau}_t \widetilde{L}_t} \\
&+ \kappa_t^2 (1 - \alpha_{t,n})^2 P^2(t, T_{m(t)}) V_t \varphi^2(\widetilde{L}_t) \widetilde{\sigma}_t^2,
\end{aligned}$$

which can be expressed as:

$$\xi_{t,n} = \theta_{t,n}^1 P(t, T_{m(t)}) + P^2(t, T_{m(t)}) \frac{\kappa_t V_t \varphi^2(\widetilde{L}_t) \widetilde{\sigma}_t^2}{1 + \widetilde{\tau}_t \widetilde{L}_t} \left(\frac{(1 - \alpha_{t,n}) \widetilde{\tau}_t}{P(t, T_{m(t)})} + \kappa_t (1 - \alpha_{t,n})^2 (1 + \widetilde{\tau}_t \widetilde{L}_t) \right).$$

By definition of the interpolated $P(t, T_{m(t)})$ and by setting $\theta_{t,n}^2$, as in Lemma 3.7, we finally obtain:

$$\xi_{t,n} = \theta_{t,n}^1 P(t, T_{m(t)}) + \theta_{t,n}^2 P^2(t, T_{m(t)}) \frac{\kappa_t V_t \varphi^2(\widetilde{L}_t) \widetilde{\sigma}_t^2}{1 + \widetilde{\tau}_t \widetilde{L}_t},$$

which concludes the proof of Lemma 3.7. □

Lemma 3.8 (Short-rate r_t under $\mathbb{Q}^{m(t)}$ measure)**.** *For any* $t \in (T_{m(t)-1}, T_{m(t)}]$ *and* $n \geq 0$, *the short-rate process* r_t *under the* $\mathbb{Q}^{m(t)}$ *measure with the interpolated ZCB as in Definition 3.6 is given by:*

$$r_t = \xi_{t,n} - \frac{1}{2}(1 - \alpha_{t,n})^2 \kappa_t^2 \widetilde{\sigma}_t^2 \varphi^2(\widetilde{L}_t) V_t P^2(t, T_{m(t)}), \tag{3.34}$$

with $\xi_{t,n}$, $\alpha_{t,n}$, κ_t, $\widetilde{L}_t$ *as in Lemma 3.7 and* $P(t, T_{m(t)})$ *as in Definition 3.6.*

Proof. In the proof we use the results obtained in Section 3.2, where, by taking $T_i \equiv T_j = T_{m(t)}$ the dynamics for the instantaneous rates under the forward measure read:

$$df(t, T_{m(t)}) = \Sigma(t, T_{m(t)})dW_t^{m(t)}.$$

Moreover, under the $\mathbb{Q}^{m(t)}$ measure, the ZCB dynamics from Equation (3.3) are

$$\frac{dP(t, T_{m(t)})}{P(t, T_{m(t)})} = \Gamma(t, T_{m(t)}, T_{m(t)})dt - \left(\int_t^{T_{m(t)}} \Sigma(t, s)ds\right) dW_t^{m(t)}, \tag{3.35}$$

with

$$\Gamma(t, T_{m(t)}, T_{m(t)}) = r_t + \frac{1}{2}\left(\int_t^{T_{m(t)}} \Sigma(t, s)ds\right)^2. \tag{3.36}$$

Now, the HJM ZCB dynamics should correspond to the dynamics introduced in Lemma 3.7. So, $\Gamma(t, T_{m(t)}, T_{m(t)}) = \xi_{t,n}$. By comparing the diffusion terms we find:

$$-\int_t^{T_{m(t)}} \Sigma(t, s)ds = (1 - \alpha_{t,n})\kappa_t\widetilde{\sigma}_t\varphi(\widetilde{L}_t)\sqrt{V_t}P(t, T_{m(t)}). \tag{3.37}$$

Combining Equations (3.37) and (3.36) concludes the proof. □

3.3.2 Zero external volatility for the interpolated zero-coupon bond

As presented in Table 1 by changing the weighting parameter, n, for the ZCB interpolation it is possible to control the amount of external volatility in the arbitrage-free interpolation.

Since the weighting coefficient n has to be externally chosen, one way of finding the appropriate value for it, is by calibrating the model to already quoted interest rate equity hybrids on the market. In order to show the impact of the interest rate smile on the equity local volatility we consider the model without external volatility, and define the ZCB only in terms of already so called *dead* Libors. By choosing $n = 0$ the full weight is put on the Libor $L(T_{m(t)-1}, T_{m(t)})$. The lemma below discusses the short-rate for this special case.

Lemma 3.9 (Zero volatility short-rate model)**.** *For any time $t \in (T_{m(t)-1}, T_{m(t)}]$, the zero volatility short-rate process, r_t, extracted from the interpolated ZCB (Definition 3.6) is given by:*

$$r_t = \frac{L(T_{m(t)-1}, T_{m(t)})}{1 + (T_{m(t)} - t)L(T_{m(t)-1}, T_{m(t)})}. \tag{3.38}$$

Proof. By taking $n = 0$ in Lemma 3.7 we get

$$r_t = \xi_{t,0} - \frac{1}{2}(1 - \alpha_{t,0})^2\kappa_t^2\widetilde{\sigma}_t^2\widetilde{L}_t^2P^2(t, T_{m(t)}). \tag{3.39}$$

Since $\alpha_{t,0} = 1$, we obtain $\theta^1_{t,0} = \widetilde{L}_t + 1 \cdot (L_t - \widetilde{L}_t) = L_t$, and $\theta^2_{t,0} = 0$, which reads:

$$r_t = \xi_{t,0} = L_tP(t, T_{m(t)}),$$

with $L_t = L(T_{m(t)-1}, T_{m(t)})$ and we have taken $n = 0$ for interpolation. So the zero-coupon bond equals to

$$P(t, T_{m(t)}) = \frac{1}{1 + (T_{m(t)} - t)L(T_{m(t)-1}, T_{m(t)})},$$

which completes the proof. □

We note that the model introduced in Lemma 3.7 collapses for $n = 0$ to the basic short-rate process introduced in [16]. In Figure 1 we show the process trajectories for different n. As we see the generated trajectories are discontinuous at the exercise days T_i. For $n = 0$ the first Libor $L(T_0, T_1)$ is already set, so the short-rate process between times T_0 and T_1 is deterministic i.e., no uncertainty is induced.

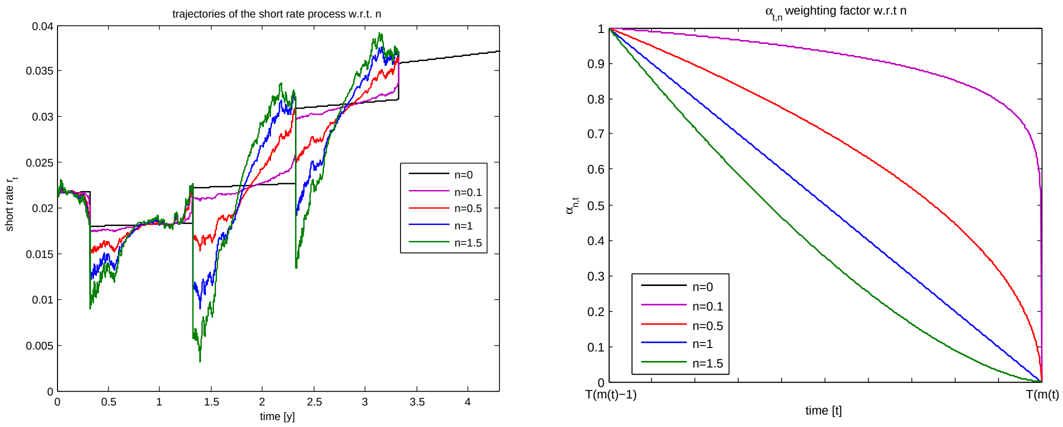

Figure 1: Impact of the weighting factor n on the amount of external volatility included in the interpolation. LEFT: Sample paths w.r.t weighting factor. RIGHT: Weighting function $\alpha_{t,n}$.

4 Numerical experiments

In this section we perform a number of numerical experiments, and examine the impact of different interest rate processes on the local volatility structure in the LVSIR. We compare the results with the Gaussian two-factor short-rate model[3] (G2++). Note that the calibration of the Libor market models, either the log-normal or with the stochastic volatility, is common market practice (see for example [10]). We assume that these Libor market models are input to our framework via Equation (3.34). We give the fully detailed calibration results in Appendix A.

For simplicity, we first assume a zero external volatility (Equation (3.38)) included in the short-rate model as in Section 3.3.2. As a first step in our analysis we check whether the short-rate process given by Equation (3.38) is able to reproduce the smile generated by the SV-LMM. Figure 2 shows an almost perfect match between the smile generated by the r_t process and the

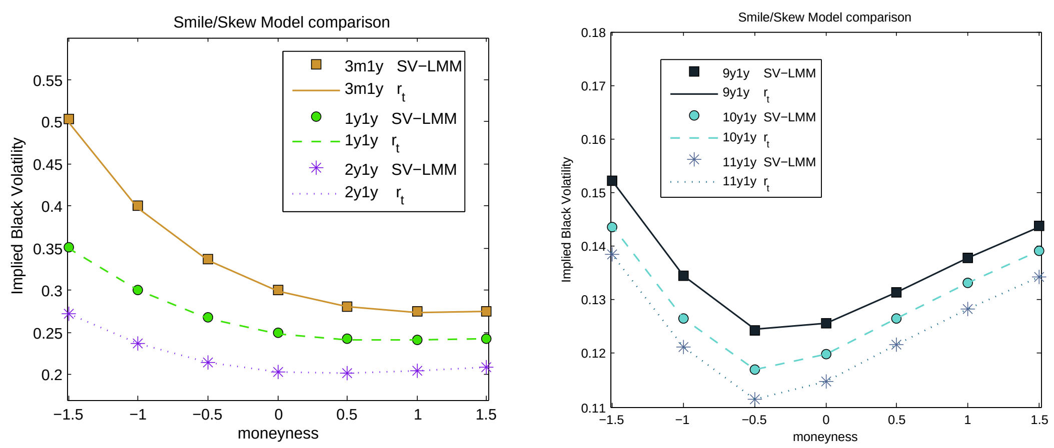

Figure 2: Comparison of Black-Scholes implied volatilities between the smile generating SV-LMM and the implied short-rate process r_t. The short-rate interpolation is done without external volatility, i.e. $n = 0$. The results are presented for a tenor of 1 year with different expiries. LEFT: Short expiry: 3 months-2 years, RIGHT: Long expiry: 9-11 years.

underlying SV-LMM.

We now investigate the impact of the short-rate processes on the equity local volatility model. In Section 2, in Theorem 2.3, we have found a relationship between the deterministic and stochastic

[3] Also known as Hull-White two-factor model.

rates local volatility models, $\sigma^2(T,K) = \sigma^2_{det}(T,K) - 2\text{cov}\left(r_T, \log S_T | \log S_T = \log K\right)$. Since the correlation between the equity and the interest rate Brownian motions, $\rho_{S,r}$, can relatively easily be imposed externally by a historical estimation, we simplify the formula to:

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) - 2\rho_{S,r} \cdot \sigma_{\log S_T}(T)\sigma_{r_T}(T), \tag{4.1}$$

where $\sigma_{\log S_T}(T)$ and $\sigma_{r_T}(T)$ are standard deviations of the log-equity and the short-rate processes, respectively. We note that the local volatility models for deterministic and stochastic rates are equivalent if the corresponding correlation or standard deviations becomes zero. By assuming a non-zero correlation, the difference between the local volatility models increases with an increase of the corresponding sigmas. In order to check this influence we have calibrated the G2++ model, the log-normal LMM and the SV-LMM. Figure 3 shows that for all three calibrated models the first moment of the interest-rate processes, $\mathbb{E}(r_T)$, is close; However, this is not the case for the standard deviation (see the RHS of Figure 3). Whereas the short-rate process of the G2++ model and the process implied by the log-normal LMM have a similar standard deviation, the one implied from the SV-LMM is significantly higher. This observation indicates that the local volatility with an implied short-rate process by the SV-LMM should be lower than for the other models.

Before we evaluate the impact of the stochastic interest rates on the local volatility model in some more depth, we check the convergence of the local volatility. So, we iterate expression (4.1) to determine the volatility $\sigma^2(T,K)$ by means of the fixed point iteration. To determine $\sigma_{det}(T,K)$ first a constant interest rate of $r = 6\%$ is set.

In Table 2 the convergence results are presented by means of the difference of two subsequent approximations, from the iterations $j-1$ and j, for the volatility in the L_2 norm. The table gives a relationship between the correlation, $\rho_{S,r}$, and the accuracy of the approximation: A higher correlation between the equity and the interest rate causes a slow-down of the convergence. However, engineering accuracy is already achieved in $j = 2$ fixed point iterations for all correlation coefficients considered.

Table 2: Convergence of the local volatility for equity models with different stochastic interest rate processes (the constant interest rate to determine $\sigma^2_{det}(T,K)$ is set to $r = 6\%$). The convergence is measured as the Euclidian norm $||\cdot||$ of two consecutive local volatility matrices, j and $j-1$.

$\rho_{S,r}$	2^j	$\|\|\hat{\sigma}_j - \hat{\sigma}_{j-1}\|\|_2$		
%	j	G2++	Log-normal LMM	SV-LMM
10%	1	3.285E-06	4.684E-06	1.148E-05
	2	3.779E-10	1.323E-09	5.569E-09
	3	1.556E-16	2.328E-16	2.447E-15
50%	1	7.772E-04	6.802E-04	1.800E-03
	2	1.539E-06	5.516E-06	2.605E-05
	3	2.3468E-11	7.326E-10	1.108E-08
90%	1	3.300E-02	4.600E-03	1.290E-02
	2	3.812E-05	1.392E-04	7.371E-04
	3	6.976E-09	2.614E-07	4.930E-06

In Figure 4 we present the local volatility obtained for short maturity equity options, on the basis of three different short-rate processes, the G2++ model, a short-rate implied by the log-normal LMM and a short-rate implied from the SV-LMM . The figure shows that for very short maturities of 0.5 years the local volatility curves for deterministic and stochastic short-rate processes are very similar. For an expiry of 3 years the local volatilities with stochastic interest rates are lower than those obtained for the deterministic case, although the difference within the stochastic interest rates models is not significant. By changing the option expiry to 12 years an interesting pattern appears as all the local volatility curves with stochastic interest rates are significantly lower than the one for the deterministic case. Moreover the short-rate implied by the SV-LMM is more pronounced than the other ones.

Tables with all the details are presented in the Appendix B. There it is shown, in Table 7, that the difference between the local volatilities with deterministic and stochastic rates from the SV-

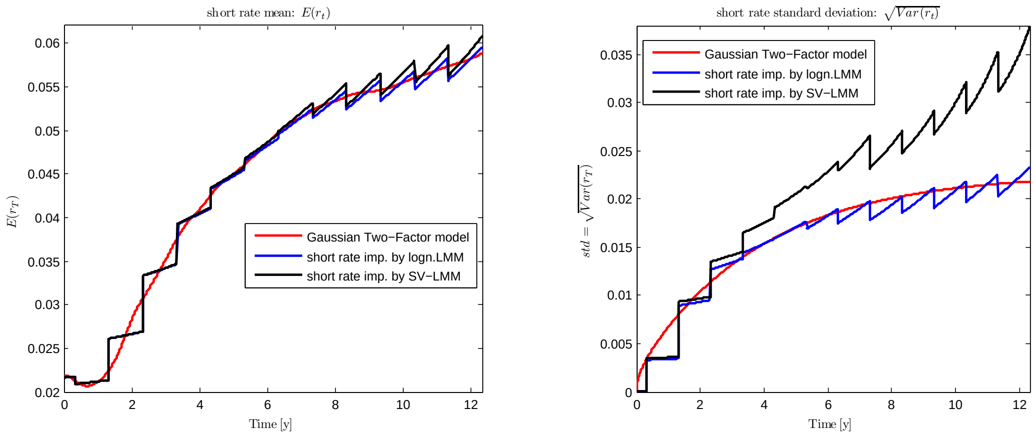

Figure 3: Characteristics of the interest rate processes. LEFT: Expectation of the short-rate processes r_t vs. time. RIGHT: Standard deviation of the short-rate processes vs time.

LMM for long maturity options is about 3.5%; It is about 2.2% for the G2++ and for the model obtained by interpolating the log-normal LMM (see Tables 5 and 6). The correlation between the short-rate and the equity is chosen to be 40% in this experiment. The observed effect can be reduced by a decrease of the correlation parameter or enhanced by an increase.

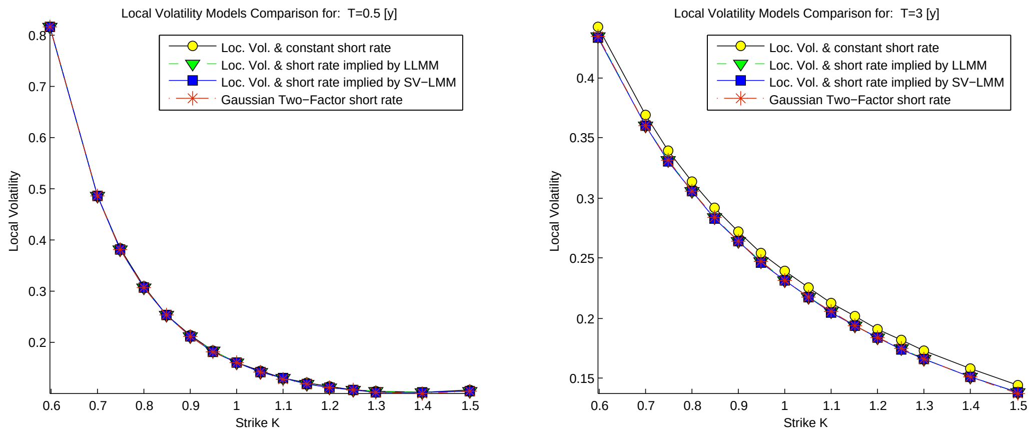

Figure 4: Local volatility equity results for different r_t processes. The simulation was performed with: $S_0 = 1$, $\rho_{S,r} = 40\%$, $r = 6\%$. LEFT: Option expiry of 0.5 years. RIGHT: Option expiry of 3 years.

5 Conclusion

In this paper we have presented a local volatility model with stochastic interest rates. We have shown a way to set up a short-rate process which generates an interest rate smile. Our numerical experiments indicate, especially for contracts with a long time to maturity, a significant impact of the interest rate smile on the local volatilities for the equity.

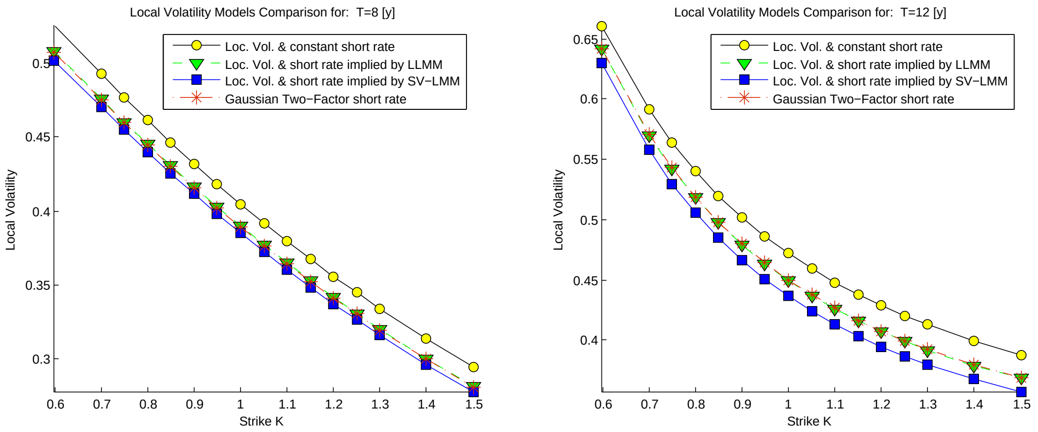

Figure 5: Local volatility equity results for different r_t processes. The simulation was performed with: $S_0 = 1$, $\rho_{S,r} = 40\%$, $r = 6\%$. LEFT: Option expiry of 8 years. RIGHT: Option expiry of 12 years.

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A Calibration Results

In this appendix we give details about the calibration of the models used in Section 4. Figure 6 shows the fit to the market data for at-the-money caplets. Figure 7 presents the implied volatility smile for these caplets and Table 3 illustrates the error of the calibrated SV-LMM. As we can see the models are calibrated with a sufficient accuracy.

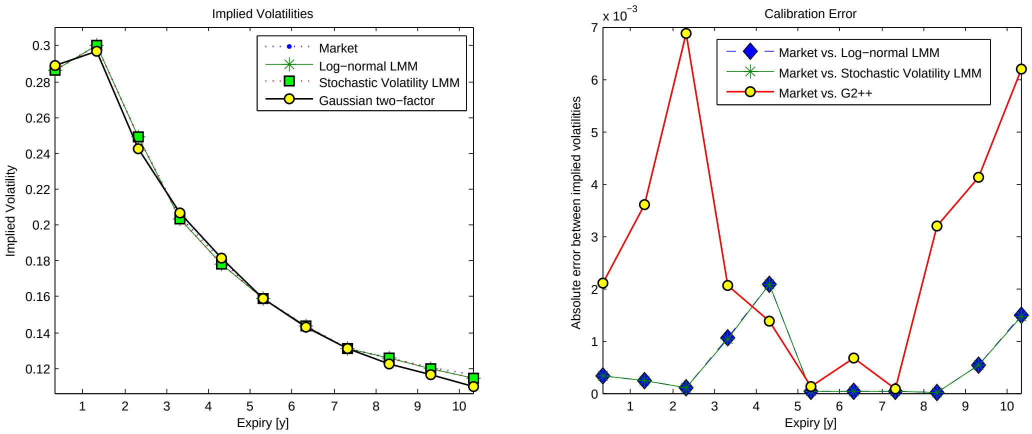

Figure 6: At the money (ATM) Euro Caplet calibration results. LEFT: Euro Caplet implied volatility curves implied by log-normal LMM, stochastic volatility LMM and Gaussian two-factor (G2++) models calibrated to at the money caplet volatility curve (12m) on July 8^{th} 2006. RIGHT: Absolute difference between market and model implied volatilities.

Table 3: Difference between implied volatilities between the market from July 8^{th} 2006 and the calibrated stochastic volatility libor market model (SV-LMM).

Maturity	Moneyness						
date	**-1.5**	**-1**	**-0.5**	**0**	**0.5**	**1**	**1.5**
01-Nov-06	-9.9E-2	-1.4E-2	-9.8E-2	4.0E-4	3.6E-2	2.7E-2	-8.1E-2
01-Nov-07	-1.8E-1	-9.5E-2	-3.1E-2	1.5E-3	1.7E-2	2.2E-2	2.3E-2
03-Nov-08	-8.9E-2	-4.0E-2	-1.2E-2	2.4E-3	9.2E-3	1.1E-2	1.1E-2
02-Nov-09	4.0E-4	2.0E-4	-1.0E-3	-1.2E-7	3.8E-3	8.2E-3	1.1E-2
01-Nov-10	-9.0E-4	-3.4E-3	-4.6E-3	-2.1E-7	7.6E-3	1.5E-2	1.8E-2
01-Nov-11	-1.6E-3	-4.6E-3	-5.1E-3	7.8E-7	7.6E-3	1.4E-2	1.7E-2
01-Nov-12	-3.8E-3	-7.2E-3	-7.0E-3	5.1E-7	8.4E-3	1.5E-2	1.8E-2
01-Nov-13	-3.7E-3	-6.5E-3	-6.4E-3	6.2E-7	8.0E-3	1.4E-2	1.7E-2
03-Nov-14	-3.9E-3	-7.0E-3	-7.1E-3	1.6E-6	8.6E-3	1.5E-2	1.8E-2
02-Nov-15	-4.9E-3	-8.5E-3	-8.4E-3	2.5E-6	9.1E-3	1.6E-2	1.8E-2
01-Nov-16	-5.0E-3	-8.9E-3	-9.1E-3	2.4E-6	9.6E-3	1.7E-2	1.9E-2

B LVSIR: Numerical Results

In this section we present results of the Local Volatility Model with stochastic interest rates. In Table 4 we present the (smoothed) local volatility surface with constant interest rate. In Tables 5, 6 and 7 we give the LVSIR results corresponding to the models presented in Section 4.

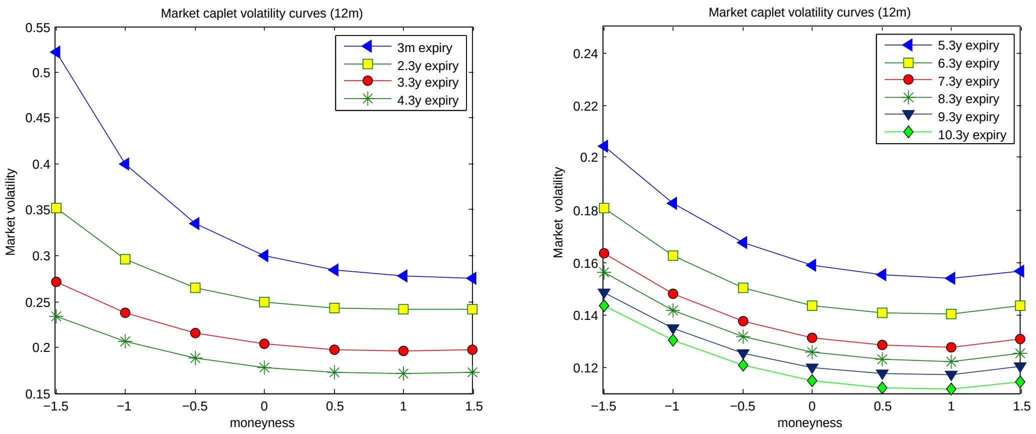

Figure 7: Euro Caplet implied volatility on July 8^{th} 2006.

Table 4: The parametric local volatility for the equity with constant interest rate of $r = 6\%$.

Strike	Maturity												
K	**1m**	**3m**	**6m**	**9m**	**1y**	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**10y**	**12y**
0.6	1.13	0.97	0.82	0.72	0.64	0.47	0.44	0.45	0.46	0.48	0.50	0.53	0.59	0.66
0.7	0.54	0.52	0.49	0.46	0.44	0.38	0.37	0.39	0.41	0.44	0.46	0.49	0.55	0.59
0.75	0.40	0.39	0.38	0.37	0.36	0.34	0.34	0.36	0.39	0.42	0.45	0.48	0.53	0.56
0.8	0.31	0.31	0.31	0.31	0.30	0.30	0.31	0.34	0.37	0.40	0.43	0.46	0.52	0.54
0.85	0.25	0.25	0.25	0.26	0.26	0.27	0.29	0.32	0.35	0.38	0.41	0.45	0.50	0.52
0.9	0.20	0.21	0.21	0.22	0.22	0.25	0.27	0.30	0.33	0.37	0.40	0.43	0.49	0.50
0.95	0.17	0.18	0.18	0.19	0.20	0.22	0.25	0.29	0.32	0.35	0.38	0.42	0.47	0.49
1	0.15	0.15	0.16	0.17	0.18	0.21	0.24	0.27	0.30	0.34	0.37	0.40	0.46	0.47
1.05	0.13	0.14	0.14	0.15	0.16	0.19	0.23	0.26	0.29	0.32	0.36	0.39	0.45	0.46
1.10	0.12	0.12	0.13	0.14	0.15	0.18	0.21	0.24	0.28	0.31	0.34	0.38	0.44	0.45
1.15	0.11	0.11	0.12	0.13	0.14	0.17	0.20	0.23	0.26	0.30	0.33	0.37	0.43	0.44
1.20	0.10	0.11	0.11	0.12	0.13	0.16	0.19	0.22	0.25	0.29	0.32	0.36	0.42	0.43
1.25	0.10	0.10	0.11	0.11	0.12	0.15	0.18	0.21	0.24	0.28	0.31	0.34	0.40	0.42
1.30	0.09	0.10	0.10	0.11	0.12	0.14	0.17	0.20	0.23	0.26	0.30	0.33	0.40	0.41
1.40	0.09	0.10	0.10	0.11	0.11	0.13	0.16	0.18	0.21	0.25	0.28	0.31	0.38	0.40
1.50	0.10	0.10	0.11	0.11	0.11	0.13	0.14	0.17	0.20	0.23	0.26	0.29	0.36	0.39

Table 5: Difference (in %) between local volatility for the equity with constant interest rate $r = 6\%$ and LVM with short-rate driven by *Gaussian two-factor model* (as in [10]) with: $a = 0.543464289$, $\sigma = 0.008656305$, $b = 0.121389268$, $\eta = 0.014003025$, $\rho_{x,y} = -0.98603472$.

Strike	Maturity												
K	**1m**	**3m**	**6m**	**9m**	**1y**	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**10y**	**12y**
0.6	0.02	0.06	0.14	0.23	0.33	0.81	0.87	1.14	1.37	1.56	1.71	1.81	1.91	1.90
0.7	0.02	0.06	0.13	0.21	0.30	0.68	0.83	1.08	1.30	1.48	1.62	1.74	1.91	2.06
0.75	0.02	0.06	0.13	0.20	0.28	0.63	0.81	1.05	1.26	1.44	1.58	1.71	1.90	2.11
0.8	0.02	0.06	0.13	0.20	0.27	0.60	0.79	1.02	1.23	1.40	1.55	1.67	1.89	2.14
0.85	0.02	0.06	0.12	0.19	0.26	0.57	0.77	1.00	1.20	1.37	1.52	1.64	1.87	2.16
0.9	0.02	0.06	0.12	0.19	0.26	0.54	0.76	0.98	1.18	1.34	1.49	1.61	1.85	2.18
0.95	0.02	0.06	0.12	0.19	0.25	0.52	0.75	0.96	1.15	1.32	1.46	1.59	1.83	2.18
1	0.02	0.06	0.12	0.18	0.25	0.51	0.74	0.95	1.13	1.29	1.43	1.56	1.81	2.17
1.05	0.02	0.06	0.12	0.18	0.24	0.50	0.73	0.94	1.12	1.27	1.41	1.54	1.79	2.16
1.10	0.02	0.06	0.12	0.18	0.24	0.49	0.72	0.92	1.10	1.25	1.39	1.51	1.77	2.15
1.15	0.02	0.06	0.12	0.18	0.24	0.48	0.71	0.91	1.08	1.23	1.37	1.49	1.75	2.13
1.20	0.02	0.06	0.12	0.18	0.24	0.48	0.71	0.90	1.07	1.21	1.35	1.47	1.72	2.10
1.25	0.02	0.06	0.12	0.18	0.24	0.48	0.70	0.89	1.06	1.20	1.32	1.44	1.70	2.08
1.30	0.02	0.06	0.12	0.18	0.24	0.48	0.70	0.88	1.04	1.18	1.30	1.42	1.67	2.05
1.40	0.02	0.06	0.12	0.18	0.24	0.49	0.69	0.87	1.02	1.15	1.26	1.37	1.62	1.98
1.50	0.02	0.06	0.12	0.19	0.25	0.52	0.68	0.86	1.00	1.12	1.22	1.33	1.56	1.91

Table 6: Difference (in %) between local volatility for the equity with constant interest rate $r = 6\%$ and LVM with short-rate implied from the *log-normal Libor Market Model*. The simulation was performed without including external volatility i.e. $n = 0$ for ZCB interpolation routine. Remaining parameters are: $S_0 = 1$, $\rho_{S,r} = 40\%$. The output is taken after three covariance iterations.

Strike	Maturity													
K	**1m**	**3m**	**6m**	**9m**	**1y**	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**10y**	**12y**
0.6	0.00	0.00	0.11	0.14	0.17	0.73	0.89	1.15	1.38	1.57	1.69	1.76	1.90	1.96
0.7	0.00	0.00	0.10	0.13	0.15	0.61	0.84	1.08	1.30	1.48	1.60	1.69	1.90	2.12
0.75	0.00	0.00	0.10	0.12	0.14	0.57	0.82	1.05	1.27	1.44	1.56	1.65	1.88	2.17
0.8	0.00	0.00	0.09	0.12	0.14	0.53	0.80	1.02	1.23	1.40	1.53	1.62	1.87	2.20
0.85	0.00	0.00	0.09	0.11	0.13	0.51	0.78	1.00	1.20	1.37	1.49	1.59	1.85	2.22
0.9	0.00	0.00	0.09	0.11	0.13	0.48	0.77	0.98	1.17	1.34	1.46	1.56	1.83	2.23
0.95	0.00	0.00	0.09	0.11	0.13	0.47	0.75	0.96	1.15	1.31	1.43	1.53	1.81	2.23
1	0.00	0.00	0.09	0.11	0.12	0.45	0.74	0.94	1.13	1.28	1.41	1.50	1.79	2.23
1.05	0.00	0.00	0.09	0.11	0.12	0.44	0.73	0.92	1.11	1.26	1.38	1.48	1.77	2.22
1.10	0.00	0.00	0.09	0.11	0.12	0.43	0.72	0.91	1.09	1.24	1.36	1.45	1.75	2.20
1.15	0.00	0.00	0.09	0.11	0.12	0.43	0.71	0.89	1.07	1.22	1.33	1.43	1.72	2.18
1.20	0.00	0.00	0.09	0.11	0.12	0.42	0.70	0.88	1.05	1.19	1.31	1.40	1.70	2.15
1.25	0.00	0.00	0.09	0.11	0.12	0.42	0.69	0.87	1.04	1.17	1.29	1.38	1.67	2.12
1.30	0.00	0.00	0.09	0.11	0.12	0.43	0.68	0.86	1.02	1.16	1.27	1.36	1.64	2.09
1.40	0.00	0.00	0.09	0.11	0.12	0.44	0.67	0.84	0.99	1.12	1.22	1.31	1.59	2.02
1.50	0.00	0.00	0.09	0.11	0.13	0.46	0.66	0.82	0.96	1.08	1.17	1.26	1.53	1.94

Table 7: Difference (in %) between local volatility for the equity with constant interest rate $r = 6\%$ and LVM with short-rate implied from the *Stochastic Volatility Libor Market Model*. The simulation was performed without including external volatility i.e. $n = 0$ for ZCB interpolation routine. Remaining parameters are: $S_0 = 1$, $\rho_{S,r} = 40\%$. The output is taken after three covariance iterations.

Strike	Maturity													
K	**1m**	**3m**	**6m**	**9m**	**1y**	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**10y**	**12y**
0.6	0.00	0.00	0.11	0.15	0.18	0.76	0.95	1.33	1.69	2.00	2.30	2.38	2.78	3.12
0.7	0.00	0.00	0.10	0.13	0.16	0.64	0.91	1.26	1.60	1.90	2.19	2.30	2.81	3.43
0.75	0.00	0.00	0.10	0.13	0.15	0.59	0.88	1.23	1.56	1.85	2.14	2.25	2.80	3.53
0.8	0.00	0.00	0.10	0.12	0.14	0.56	0.87	1.20	1.52	1.81	2.10	2.21	2.79	3.60
0.85	0.00	0.00	0.10	0.12	0.14	0.53	0.85	1.17	1.49	1.77	2.06	2.18	2.77	3.64
0.9	0.00	0.00	0.10	0.12	0.14	0.51	0.83	1.15	1.46	1.73	2.02	2.14	2.75	3.67
0.95	0.00	0.00	0.10	0.12	0.13	0.49	0.82	1.13	1.43	1.70	1.99	2.11	2.72	3.69
1	0.00	0.00	0.09	0.11	0.13	0.47	0.81	1.11	1.41	1.67	1.95	2.08	2.70	3.69
1.05	0.00	0.00	0.09	0.11	0.13	0.46	0.80	1.09	1.39	1.65	1.92	2.05	2.67	3.68
1.10	0.00	0.00	0.09	0.11	0.13	0.46	0.79	1.08	1.37	1.62	1.90	2.02	2.64	3.66
1.15	0.00	0.00	0.09	0.11	0.13	0.45	0.78	1.07	1.35	1.60	1.87	1.99	2.61	3.63
1.20	0.00	0.00	0.09	0.11	0.13	0.45	0.77	1.05	1.33	1.58	1.84	1.96	2.58	3.60
1.25	0.00	0.00	0.09	0.11	0.13	0.45	0.77	1.04	1.32	1.55	1.81	1.93	2.54	3.56
1.30	0.00	0.00	0.09	0.11	0.13	0.45	0.76	1.03	1.30	1.53	1.79	1.90	2.51	3.51
1.40	0.00	0.00	0.09	0.11	0.13	0.46	0.75	1.01	1.27	1.49	1.74	1.85	2.43	3.41
1.50	0.00	0.00	0.09	0.12	0.13	0.48	0.74	0.99	1.24	1.45	1.68	1.79	2.35	3.29