Hantush Well Function Revisited

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Abstract

In this paper we comment on some recent numerical and analytical work to evaluate the Hantush Well Function. We correct an expression found in a Comment by Nadarajah (*J. of Hydrology*, vol. 338, p. 152-153 (2007)) to a paper by Prodanoff et al. (*J. of Hydrology*, vol. 318, p. 173-183 (2006)). We subsequently derived another analytic representation based on a generalized hypergeometric function in two variables and from the hydrological literature we cite an analytic representation by Hunt (*J. of Hydrology*, vol. 33, p. 179-183 (1977)). We have implemented both representations and compared the results. Using a convergence accelerator Hunt's representation of Hantush Well Function is efficient and accurate. While checking our implementations we found that Bear's table of the Hantush Well Function (*"Hydraulics of Groundwater"*, 1979, Table 8-6) contains a number of typographical errors that are not present in the original table published by Hantush (*Transactions, American Geophysical Union*, vol. 36, p. 95-100 (1956)). Finally, we offer a very fast approximation with a maximum relative error of 0.0033 for the parameter range in the table given by Bear.

Keywords: Hantush Well Function, Generalized Incomplete Gamma Function, pumping test, leaky aquifer, closed-form representation, time series analysis.

1. Introduction

Hantush's Well Function (Hantush and Jacob (1955)) may well be the most popular formula in hydrogeological practice, which is remarkable for an inconvenient mathematical expression that classifies as a special case of the Generalized Incomplete Gamma Function. Ever since its first appearance hydrogeologists have searched for methods to compute the well function; several methods will be reviewed in the next paragraph. In this paper, we review existing analytic methods and discuss two analytic representations. At the end of the paper we present a very fast approximation, which may be useful in programs that require many evaluations of the function, such as models for time series analysis (Asmuth et al. (2008); `http://www.menyanthes.nl`). We remark that one may alternatively evaluate the Hantush Well Function by performing a numerical inversion of the Laplace transform (2) or by standard numerical integration, for example using Gaussian quadrature (e.g., the Matlab®-code `quadgk`). Both turn out to be also satisfactory. Other methods to evaluate the Hantush Well Function, sometimes purely numerical, have been published, by e.g., Harris (2008) and Temme (2009).

Besides a proposed numerical integration scheme Prodanoff et al. (2006) presented a review of earlier results. Nadarajah (2007) commented on Prodanoff et al. (2006) to the effect that there was no longer a need for approximate methods, since a closed-form mathematical expression was available, based on an Appell type generalization of the well-known hypergeometric series that allegedly was offered by standard mathematical software. We found some difficulties in evaluating Nadarajah's solution for various reasons which we discuss in this paper (Section 3). Another representation of the Generalized Incomplete Gamma Function uses also an Appell type series generalization of the hypergeometric series (see (13) in Section

4). We rewrite (13) into (24), a representation expressed in the better known Bessel functions I_j and K_0 and discuss its evaluation in Section 6. In Section 5 we consider the closed-form analytic representation by Hunt (1977), which Prodanoff et al. (2006) erroneously called approximate. Using a convergence accelerator, Hunt's representation of Hantush's Well Function is efficient and accurate, which means that one gets high precision for a relatively small number of terms. While checking our implementations we found that Bear's table of the Hantush Well Function (Bear (1979, Table 8-6)) contains a number of typographical errors, which are not present in the original table published by Hantush (1956). We discuss the performance of the two representations mentioned above in Section 7. In Section 8 we end our paper with a very fast but stable approximate expression that is good enough for engineering practice. It is continuous and has continuous first derivatives with respect to its parameters, which is important when it is to be used in an optimization loop. The Appendix contains a Matlab code for this approximation. Our Matlab code for the method by Hunt is available upon request.

2. Hantush Well Function

The Hantush Well Function is defined as

$$W\left(u, \frac{r}{B}\right) = \int_u^\infty \frac{1}{t} \exp\left(-t - \frac{r^2}{4B^2 t}\right) dt. \tag{1}$$

This function was introduced in the field of hydrology by Hantush and Jacob (1955). An application with a number of numerical results in table form was given by Hantush (1956). Bear summarized these results and included a table for this function (Bear (1979, Table 8-6)). The Hantush Well Function was given by Hunt (1977) as a sum over Iterated Exponential Integrals. A recent survey of methods to evaluate the Hantush Well Function was given by Prodanoff et al. (2006). Also, in the mathematical literature attention has been paid to this function, see Harris (2008) and Temme (2009). The Laplace transform of (1) reads

$$\overline{W}\left(s; \frac{r}{B}\right) = \int_0^\infty W\left(u, \frac{r}{B}\right) \exp(-us) du = \tag{2}$$
$$2\left(K_0\left(\frac{r}{B}\right) - K_0\left(\sqrt{1+s}\frac{r}{B}\right)\right)/s,$$

where K_0 is the Modified Bessel Function of the Second Kind, order 0.

3. Result of Nadarajah (2007)

In a recent Comment (Nadarajah (2007)) to the paper by Prodanoff et al. (2006) the author points out that it is possible to express (1) in a closed analytical form as a double sum based on the Appell hypergeometric series of the first kind Φ_1^N. The result of Nadarajah (2007) reads

$$W\left(u, \frac{r}{B}\right) = \begin{cases} K_0(r/B) + I\,, & 0 < u < r/(2B), \\ K_0(r/B) - I\,, & u \geq r/(2B), \end{cases} \tag{3}$$

$$\text{with} \quad I = \sqrt{\frac{2Bu}{r} + \frac{r}{2Bu} - 2}\exp\left(-\frac{r}{B}\right) \times$$
$$\Phi_1^N\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{1}{2} - \frac{Bu}{2r} - \frac{r}{8Bu}, u + \frac{r^2}{4B^2 u} - \frac{r}{B}\right),$$

where K_0 is the Modified Bessel Function of the Second Kind, order 0, and where Φ_1^N is defined according to Nadarajah as

$$\Phi_1^N(a,b,c;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}\,(b)_n\,x^m y^n}{(c)_{m+n}\,m!n!}, \tag{4}$$

$$\text{with } (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1), \quad a \neq 0,-1,-2,\cdots .$$

We introduced the notation Φ_1^N in stead of Φ_1 for reasons below. We discuss this result with a few remarks:

1. Nadarajah (2007) calls the function Φ_1^N the Appell hypergeometric series of the first kind. This is not correct. The Appell hypergeometric series of the first kind is commonly denoted by F_1 and defined with one extra parameter as (see e.g., Horn (1931, p. 383), Gradshteyn and Ryzhik (1965, (9.180)))

$$F_1(a,b,b',c;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}\,(b)_m\,(b')_n\,x^m y^n}{(c)_{m+n}\,m!n!}, \tag{5}$$

whereas the function (4) used by Nadarajah (2007) is one of the other functions in two variables introduced by Horn (1931, p. 383-384) as a generalization of the well-known hypergeometric function.

2. According to Horn (1931, p. 384) and Erdélyi (1954, p. 384) the correct definition for that function Φ_1 reads (see also Srivastava and Karlsson (1985, p. 25, (16)))

$$\Phi_1(a,b,c;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}\,(b)_m\,x^m y^n}{(c)_{m+n}\,m!n!}, \tag{6}$$

with the restriction of $|x| < 1$ and $|y| < \infty$.

Note the different subscript for the term (b) in the numerator in (4) and (6). The definition used by Nadarajah (2007) occurs in the literature by Erdélyi (1953, p. 225, (20)) and Gradshteyn and Ryzhik (1965, p. 1067, (9.261)). This has caused some confusion with respect to the results in which Φ_1 is involved. As can easily be seen, there holds

$$\Phi_1(a,b,c;x,y) = \Phi_1^N(a,b,c;y,x). \tag{7}$$

3. Nadarajah (2007) used a result for some specific integral listed in Prudnikov et al. (1986, (2.3.8.1))

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}(x+z)^{-\rho}\exp(-px)dx = B(\alpha,\beta)z^{-\rho}a^{\alpha+\beta-1}\Phi_1(\alpha,\rho,\alpha+\beta;-a/z,ap), \tag{8}$$

$$\text{with } \operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad |\arg(1-\sigma)| < \pi,$$

$$\text{and} \quad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The function Φ_1 seems not to be defined in that work. This result (8) is related to a similar integral listed in Erdélyi (1954, 4.3, (24))

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-\sigma t)^{-\gamma}\exp(-pt)dt = B(\alpha,\beta)\Phi_1(\alpha,\gamma,\alpha+\beta;\sigma,-p), \tag{9}$$

with $\quad \mathrm{Re}\,\alpha > 0, \quad \mathrm{Re}\,\beta > 0, \quad |\arg(1-\sigma)| < \pi.$

The result (9) is in accordance with (8) (after the scaling $t = x/a$, and the change of parameters $\gamma \to \rho$, $\sigma \to -a/z$, $p \to ap$), except for the minus-sign in the last argument for Φ_1. A careful analytical study reveals that (9) is correct. Gradshteyn and Ryzhik (1965, (3.385)) also gave this result (9), but since Gradshteyn and Ryzhik (1965, p. 1067, (9.261)) used the wrong definition for Φ_1, it seems that their result is in error. There exists a correction for that result (Gradshteyn and Ryzhik (1965, (3.385))) (see `http://www.mathtable.com/gr`) in the sense that the two last arguments for Φ_1 have to be changed. It would have been better to correct the definition of Φ_1 in Gradshteyn and Ryzhik (1965, p. 1067, (9.261)).

4. The final conclusion is that the result given by Nadarajah (2007) for the Hantush Well Function is in error. His formula can be repaired by introducing an extra minus-sign for the last argument ($u+\frac{r^2}{4B^2u}-\frac{r}{B} \to -\left(u+\frac{r^2}{4B^2u}-\frac{r}{B}\right)$) and by requiring that the correct definition of Φ_1 will be used (*i.e.* (6)). Moreover, the function Φ_1 converges only for values $\left|\frac{1}{2}-\frac{Bu}{2r}-\frac{r}{8Bu}\right| < 1$. So, the correct results reads

$$W\left(u,\frac{r}{B}\right) = \begin{cases} K_0(r/B) + I\,, & 0 < u < r/(2B), \\ K_0(r/B) - I\,, & u \geq r/(2B), \end{cases} \tag{10}$$

with $\quad I = \sqrt{\dfrac{2Bu}{r}+\dfrac{r}{2Bu}-2}\exp\left(-\dfrac{r}{B}\right)\times$

$$\Phi_1\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{1}{2}-\frac{Bu}{2r}-\frac{r}{8Bu},-u-\frac{r^2}{4B^2u}+\frac{r}{B}\right),$$

$$\text{for } \left|\frac{1}{2}-\frac{Bu}{2r}-\frac{r}{8Bu}\right| < 1.$$

4. Another Representation of the Hantush Well Function

Chaudhry and Zubair (2002) discussed a result by Vu Kim Tuan for the Generalized Incomplete Gamma Function defined by

$$\Gamma(\alpha,x;b) = \int_x^\infty t^{\alpha-1}\exp\left(-t-\frac{b}{t}\right)dt. \tag{11}$$

So,

$$W\left(u,\frac{r}{B}\right) = \Gamma\left(0,u;\frac{r^2}{4B^2}\right), \qquad \text{or} \tag{12}$$

$$\Gamma(0,x;b) = W\left(x,2\sqrt{b}\right).$$

Again, it turns out that the function $\Gamma(\alpha,x;b)$ can be expressed as a generalized hypergeometric function

in two variables, like (10). The result reads (Chaudhry and Zubair (2002, (2.158)))

$$\Gamma(\alpha,x;b)=\Gamma(\alpha)\sum\nolimits_{n=0}^{\infty}\frac{b^n}{n!(1-\alpha)_n} - \frac{x^\alpha}{\alpha}\Gamma_2\left(-\alpha,\alpha,x,\frac{b}{x}\right), \tag{13}$$

$$\text{with}\ \ \Gamma_2\left(\beta,\beta',x,y\right)= \sum\nolimits_{m=0}^{\infty}\sum\nolimits_{n=0}^{\infty}\frac{(\beta)_{n-m}(\beta')_{m-n}}{m!n!}x^m y^n,$$

where the interpretation for the symbol $(a)_{-k}$ can be given in a natural way as (see Srivastava and Karlsson (1985, p. 16-17))

$$(a)_{-k}=\frac{\Gamma(a-k)}{\Gamma(a)}=\frac{(-1)^k}{(1-a)_k}, \tag{14}$$
$$k=1,2,3,\cdots,\quad a\neq 0,\pm1,\pm2,\pm3,\cdots.$$

The function Γ_2 is another one in the series of generalized hypergeometric functions in two variables, see Horn (1931, p. 384) or Erdélyi (1953, p. 226, (28)).

To express (1) in terms of (13), one has to evaluate the result (13) carefully for $\alpha\to 0$. Therefore, we need the following asymptotic results

$$\lim_{\alpha\to0}(\alpha)_{-k}=\lim_{\alpha\to0}\frac{\Gamma(\alpha-k)}{\Gamma(\alpha)}= \tag{15}$$
$$\lim_{\alpha\to0}\frac{(-1)^k}{(1-\alpha)_k}=\frac{(-1)^k}{k!},$$

$$\frac{x^\alpha}{\alpha}=\frac{\mathrm{e}^{\alpha\ln x}}{\alpha}=\frac{1+\alpha\ln x+O\left(\alpha^2\right)}{\alpha}= \tag{16}$$
$$\frac{1}{\alpha}+\ln x+O(\alpha),\qquad \alpha\to0,$$

$$\Gamma(\alpha)=\frac{1}{\alpha}\mathrm{e}^{-\gamma\alpha}+O(\alpha)=\frac{1}{\alpha}-\gamma+O(\alpha),\quad \alpha\to0, \tag{17}$$

$$\gamma=\lim_{m\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}-\ln m\right)= \tag{18}$$
$$0.5772156649\cdots,$$

$$\frac{1}{(1-\alpha)_n}=\frac{1}{n!(1-\alpha)(1-\alpha/2)\cdots(1-\alpha/n)}= \tag{19}$$
$$\frac{(1+\alpha)(1+\alpha/2)\cdots(1+\alpha/n)}{n!}+O(\alpha^2)=$$
$$\frac{1}{n!}\left(1+\alpha\sum\nolimits_{l=1}^{n}\frac{1}{l}\right)+O(\alpha^2),\qquad \alpha\to0.$$

In (13), the second term gives rise to the expressions

$$\begin{aligned}
&\text{if } n > m\text{: } \lim_{\alpha\to 0} \frac{(-\alpha)_{n-m}(\alpha)_{-(n-m)}}{} = \\
&\qquad \lim_{\alpha\to 0} \frac{\Gamma(-\alpha + \overset{\alpha}{(n-m)})}{\alpha\Gamma(-\alpha)} \frac{\Gamma(\alpha-(n-m))}{\Gamma(\alpha)} \\
&\qquad = -\Gamma(n-m)\frac{(-1)^{n-m}}{(n-m)!} = \frac{(-1)^{m-n}}{m-n}, \\
&\text{if } n = m\text{: } \lim_{\alpha\to 0}(-\alpha)_0(\alpha)_0 = 1, \\
&\text{if } n < m\text{: } \lim_{\alpha\to 0} \frac{(-\alpha)_{-(m-n)}(\alpha)_{m-n}}{} = \\
&\qquad \lim_{\alpha\to 0} \frac{\Gamma(-\alpha - \overset{\alpha}{(m-n)})}{\Gamma(-\alpha)} \frac{\Gamma(\alpha+(m-n))}{\alpha\Gamma(\alpha)} \\
&\qquad = \frac{(-1)^{m-n}}{(m-n)!}\Gamma(m-n) = \frac{(-1)^{m-n}}{m-n}.
\end{aligned}$$

So, we have the result

$$\begin{aligned}
&\Gamma(\alpha, x; b) = \\
&\Gamma(\alpha)\sum_{n=0}^{\infty} \frac{b^n}{n!(1-\alpha)_n} - \frac{x^\alpha}{\alpha}\Gamma_2\left(-\alpha, \alpha, x, \frac{b}{x}\right) = \\
&\Gamma(\alpha)\sum_{n=0}^{\infty} \frac{b^n}{n!(1-\alpha)_n} \\
&-\frac{x^\alpha}{\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-\alpha)_{n-m}(\alpha)_{m-n}}{m!n!} x^m \left(\frac{b}{x}\right)^n.
\end{aligned}$$

This gives for $\alpha \to 0$

$$\begin{aligned}
&\Gamma(\alpha, x; b) = \\
&\left(\frac{1}{\alpha} - \gamma + O(\alpha)\right)\sum_{n=0}^{\infty} \frac{b^n}{n!}\frac{1}{n!}\left(1 + \alpha\sum_{l=1}^{n}\frac{1}{l}\right) \\
&-(1+O(\alpha))\times \\
&\sum\sum_{m,n=0,m\neq n}^{\infty} \frac{(-1)^{m-n}}{(m-n)\,m!n!} x^m \left(\frac{b}{x}\right)^n \\
&-\left(\frac{1}{\alpha} + \ln x + O(\alpha)\right)\sum_{n=0}^{\infty}\frac{1}{n!n!}x^n\left(\frac{b}{x}\right)^n, \\
&\qquad \alpha \to 0.
\end{aligned}$$

So,

$$\begin{aligned}
&\Gamma(0, x; b) = \lim_{\alpha\to 0}\Gamma(\alpha, x; b) = \\
&\sum\sum_{m,n=0,m\neq n}^{\infty} \frac{1}{(n-m)\,m!n!}(-x)^m\left(-\frac{b}{x}\right)^n \\
&+(-\ln x - \gamma)\sum_{n=0}^{\infty}\frac{b^n}{n!n!} + \sum_{n=1}^{\infty}\frac{b^n}{n!n!}\sum_{l=1}^{n}\frac{1}{l}.
\end{aligned} \tag{20}$$

We remark that the following result can be applied (Abramowitz and Stegun (1964, (9.6.13)))

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{b^n}{n!n!}\sum_{l=1}^{n}\frac{1}{l} = \\
&K_0\left(2\sqrt{b}\right) + \left(\ln\left(\sqrt{b}\right) + \gamma\right) I_0\left(2\sqrt{b}\right),
\end{aligned} \tag{21}$$

which gives for $\Gamma(0,x;b)$ the surprisingly simple expression

$$\begin{aligned}
&\Gamma(0,x;b) = \\
&\sum\sum_{m,n=0,m\neq n}^{\infty} \frac{1}{(n-m)\,m!n!}\,(-x)^m \left(-\frac{b}{x}\right)^n \\
&+ K_0\left(2\sqrt{b}\right) + \ln\left(\frac{\sqrt{b}}{x}\right) I_0\left(2\sqrt{b}\right), \qquad (22)
\end{aligned}$$

(remember that $\sum_{n=0}^{\infty} \frac{b^n}{n!n!} = I_0(2\sqrt{b})$). As a bonus one gets from (22) for $b \to 0$ an expression for the Exponential Integral $E_1(x)$, since (in view of (21))

$$\begin{aligned}
&\lim_{b\to 0} \Gamma(0,x;b) = \\
&\lim_{b\to 0} \left(K_0\left(2\sqrt{b}\right) + \ln\left(\frac{\sqrt{b}}{x}\right) I_0\left(2\sqrt{b}\right)\right) \\
&+ \sum_{m=1}^{\infty} \frac{1}{(-m)\,m!}\,(-x)^m \\
&= -\gamma - \ln(x) - \sum_{m=1}^{\infty} \frac{1}{m!m}\,(-x)^m \\
&= E_1(x). \qquad (23)
\end{aligned}$$

The last identity is given in Abramowitz and Stegun (1964, (5.1.11)). Of course, this is in agreement with the definition of $\Gamma(0,x;0)$.

Eq. (22) involves a double sum $\sum\sum_{m,n=0,m\neq n}^{\infty} a_{m,n}$. It is possible to handle this sum by performing summation along diagonals, below ($n > m$) and above ($m > n$) the main diagonal ($m = n$), which itself is excluded from this sum. We make the substitutions $j = n - m$ and $j' = m - n$, respectively, with the following result

$$\begin{aligned}
&\sum\sum_{m,n=0,m\neq n}^{\infty} \frac{1}{(n-m)\,m!n!}\,(-x)^m \left(-\frac{b}{x}\right)^n \\
&= \sum_{j=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{jm!(j+m)!}(-1)^{j+2m}x^{-j}b^{j+m} \\
&\quad - \sum_{j'=1}^{\infty}\sum_{n=0}^{\infty} \frac{1}{j'n!(j'+n)!}(-1)^{j'+2n}x^{j'}b^{n} \\
&= \sum_{j=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{jm!(j+m)!}\times \\
&\qquad \left\{(-x)^{-j}\,b^{j+m} - (-x)^j\,b^m\right\} \\
&= \sum_{j=1}^{\infty} \frac{1}{j}\left(\left(-\frac{b}{x}\right)^j - (-x)^j\right)\times \\
&\qquad \sum_{m=0}^{\infty} \frac{b^m}{m!(j+m)!} \\
&= \sum_{j=1}^{\infty} \frac{1}{j}\left(\left(-\frac{b}{x}\right)^j - (-x)^j\right) b^{-j/2} I_j\left(2\sqrt{b}\right) \\
&= \sum_{j=1}^{\infty} \frac{1}{j}(-1)^j I_j\left(2\sqrt{b}\right)\left(\left(\frac{\sqrt{b}}{x}\right)^j - \left(\frac{x}{\sqrt{b}}\right)^j\right).
\end{aligned}$$

Here, we used the power series expansion for I_j, the Modified Bessel Function of the First Kind, order j, see Abramowitz and Stegun (1964, (9.6.10)). So, the final analytical expression for $\Gamma(0,x;b)$ reads

$$\Gamma(0,x;b) = \sum_{j=1}^{\infty} \frac{1}{j}(-1)^j I_j\left(2\sqrt{b}\right)\left(\left(\frac{\sqrt{b}}{x}\right)^j - \left(\frac{x}{\sqrt{b}}\right)^j\right) + K_0\left(2\sqrt{b}\right) + \ln\left(\frac{\sqrt{b}}{x}\right) I_0\left(2\sqrt{b}\right). \tag{24}$$

Exploring the behaviour of $I_j\left(2\sqrt{b}\right)$ as $b^{j/2}/j!$ for $j \to \infty$, and for fixed argument, it is clear that this sum converges, see among others Temme (1996, (9.37)). It can be remarked that this expression can also be derived using Abramowitz and Stegun (1964, (9.6.33))

$$\exp\left(\frac{1}{2}z\left(t+\frac{1}{t}\right)\right) = \sum_{k=-\infty}^{k=\infty} t^k I_k(z),$$

together with the knowledge that $\Gamma(0,0;b) = 2K_0\left(2\sqrt{b}\right)$. As a by-product we derive the interesting relation

$$\lim_{x\to 0} \sum_{j=1}^{\infty} \frac{1}{j}(-1)^j I_j\left(2\sqrt{b}\right) \times \left(\left(\frac{\sqrt{b}}{x}\right)^j - \left(\frac{x}{\sqrt{b}}\right)^j\right) + \ln\left(\frac{\sqrt{b}}{x}\right) I_0\left(2\sqrt{b}\right) = K_0\left(2\sqrt{b}\right). \tag{25}$$

It has to be remarked that for large values of b together with $\sqrt{b} \gg x$ evaluation of (24) poses numerical problems since loss of precision arises by the fact that the general term becomes very large before it converges to zero for large indices.

N.M. Temme has drawn the authors' attention to the paper by Harris (2008), where (24) occurs in a somewhat different but equivalent form as equation (38) in that paper. In Veling (2009) we have extended this type of expansion for $\Gamma(\alpha,x;b)$ with general α.

5. Yet Another Representation of the Hantush Well Function

Another analytic expression for $\Gamma(0,x;b)$ can be found by expanding the factor $\exp(-b/t)$ in the integrand as a power series

$$\begin{aligned}\Gamma(0,x;b) &= \int_x^{\infty} t^{-1}\exp\left(-t-\frac{b}{t}\right)dt = \\ &\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}\int_x^{\infty} t^{-n-1}\exp(-t)dt = \\ &\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}x^{-n}\int_1^{\infty} t'^{-n-1}\exp(-xt')dt' = \\ &\sum_{n=0}^{\infty}\frac{(-b/x)^n}{n!}E_{n+1}(x), \quad x>0,\end{aligned} \tag{26}$$

and by the substitution $t' = b/t$ one can derive

$$\Gamma(0, x; b) =$$
$$\int_0^{b/x} t'^{-1} \exp\left(-t' - \frac{b}{t'}\right) dt' =$$
$$2K_0\left(2\sqrt{b}\right) - \int_{b/x}^{\infty} t'^{-1} \exp\left(-t' - \frac{b}{t'}\right) dt' =$$
$$2K_0\left(2\sqrt{b}\right) - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} E_{n+1}(b/x), \quad x \geq 0. \tag{27}$$

In the hydrologic literature these expressions have been given already by Hunt (1977). See also Chaudhry et al. (1996, (2.1)). Contrary to what has been suggested in Prodanoff et al. (2006, (10) and (12)), these expressions are no approximations but closed-form analytical expressions. These formulas are rather easy to evaluate, since the Iterated Exponential Integral can be found by a recursion, and it remains to evaluate the function $E_1(x)$ in (26) only once or the functions $K_0(2\sqrt{b})$ and $E_1(b/x)$ in (27) only once. It is appropriate to evaluate (26) for $x \geq \sqrt{b}$, and (27) for $0 < x < \sqrt{b}$. Since the terms in the sum are alternating and decreasing to 0 ($E_n(x) = \frac{\exp(-x)}{(x+n)}(1 + O(n^{-2}))$, $n \to \infty$), the convergence is assured and the error can be estimated by the absolute value of the last term included in the sum. These series can be handled by convergence accelerators like the Euler summation technique suitable for such alternating series.

As a side remark we note that for cases where $\alpha \neq 0$, but an integer $\alpha = m$, the same expansion can be applied with the following results

$$\Gamma(m, x; b) = \tag{28}$$
$$\int_x^{\infty} t^{m-1} \exp\left(-t - \frac{b}{t}\right) dt =$$
$$\sum_{n=0}^{\infty} \frac{(-b/x)^n}{n!} x^m E_{-m+n+1}(x), \quad x > 0,$$

and

$$\Gamma(m, x; b) = 2b^{m/2} K_m\left(2\sqrt{b}\right) \tag{29}$$
$$- \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} x^m E_{m+n+1}(b/x), \quad x \geq 0.$$

In cases where the index of the Expontial Integral E_{-m}, $m \geq 0$, becomes non-positive, that function has to be interpreted as the function α_m, see Abramowitz and Stegun (1964, (5.1.5))

$$E_{-m}(x) = \alpha_m(x) = \int_1^{\infty} t^m \exp(-xt) dt;$$
with $\alpha_0(x) = E_0(x) = \exp(-x)/x$.

The function $E_m(x)$, $m \geq 2$, can be found by a recursion, starting from $E_1(x)$; the function $\alpha_m(x)$, $m \geq 1$, by a recursion starting from $\alpha_0(x)$.

6. Numerical Evaluation Eq. (22)

Here, we describe how formula (22) has been implemented in a Matlab script. An $N \times N-$matrix G will be filled with zeros on the diagonal and with the terms according the double sum as

$$G_{ij} = 0, \qquad i = j, \qquad 1 \leq i, j \leq N, \tag{30}$$
$$G_{ij} = \frac{(-b/x)^{i-1} (-x)^{j-1}}{(i-j)(i-1)!(j-1)!}, \quad i \neq j, \ 1 \leq i, j \leq N.$$

In view of the restrictions posed on by Matlab, the indices start at $i = 1$, $j = 1$. Summation over all elements of this matrix gives an approximation for the double sum in (22). To make a choice for the right value of N, we have to make an error estimation. The sum of the terms not taken into account by the matrix G can be represented by

$$E = B + B' + C + C', \qquad \text{where} \tag{31}$$

$$B = \sum_{m=0}^{N-1} \frac{(-x)^m}{m!} \sum_{n=N}^{\infty} \frac{(-b/x)^n}{(n-m)n!},$$
$$B' = \sum_{m=N}^{\infty} \frac{(-x)^m}{m!} \sum_{n=m+1}^{\infty} \frac{(-b/x)^n}{(n-m)n!},$$
$$C = \sum_{n=0}^{N-1} \frac{(-b/x)^n}{n!} \sum_{m=N}^{\infty} \frac{(-x)^m}{(n-m)m!},$$
$$C' = \sum_{n=N}^{\infty} \frac{(-b/x)^n}{n!} \sum_{m=n+1}^{\infty} \frac{(-x)^m}{(n-m)m!}.$$

The sum B represents the terms for $0 \le m \le N-1$, $n \ge N$; the sum B' the terms $n \ge m+1$, $n \ge N$ and $m \ge N$, and analogously for C and C'. So, E represents all terms not summed. We estimate the various terms in a rather crude, but effective way.

$$|B| \le \sum_{m=0}^{N-1} \left| \frac{(-x)^m}{m!} \sum_{n=N}^{\infty} \frac{(-b/x)^n}{(n-m)n!} \right|$$
$$\le \sum_{m=0}^{N-1} \left| \frac{(-x)^m}{m!} \right| \left| \sum_{n=N}^{\infty} \frac{(-b/x)^n}{(n-m)n!} \right|$$
$$\le \sum_{m=0}^{N-1} \frac{x^m}{m!} \frac{(b/x)^N}{(N-m)N!} \le \sum_{m=0}^{N-1} \frac{x^m}{m!} \frac{(b/x)^N}{N!}.$$

$$|B'| \le \left| \sum_{m=N}^{\infty} \frac{(-x)^m}{m!} \sum_{n=m+1}^{\infty} \frac{(-b/x)^n}{(n-m)n!} \right|$$
$$\le \sum_{m=N}^{\infty} \left| \frac{(-x)^m}{m!} \right| \left| \sum_{n=m+1}^{\infty} \frac{(-b/x)^n}{(n-m)n!} \right|$$
$$\le \sum_{m=N}^{\infty} \frac{x^m}{m!} \frac{(b/x)^{m+1}}{(m+1-m)(m+1)!}$$
$$= \sum_{m=N}^{\infty} \frac{x^m}{m!} \frac{(b/x)^{m+1}}{(m+1)!}.$$

We required the additional condition $b/(x(N+1)) < 1$, (which can be fullfilled for N large enough) to estimate the sums over n by their first term, since the terms in these sums are oscillating and are monotonously decreasing under that condition. This results in

$$|B + B'| \le |B| + |B'| \tag{32}$$
$$\le \sum_{m=0}^{N-1} \frac{x^m}{m!} \frac{(b/x)^N}{N!} + \sum_{m=N}^{\infty} \frac{x^m}{m!} \frac{(b/x)^{m+1}}{(m+1)!}.$$

Since $b/(x(N+1)) < 1$, this equation (32) can be further estimated as

$$\begin{aligned} &|B + B'| \\ &\leq \sum\nolimits_{m=0}^{N-1} \frac{x^m}{m!} \frac{(b/x)^N}{N!} + \sum\nolimits_{m=N}^{\infty} \frac{x^m}{m!} \frac{(b/x)^N}{N!} \\ &= \mathrm{e}^x \frac{(b/x)^N}{N!}. \end{aligned} \tag{33}$$

Analogously, under the additional condition $x/(N+1) < 1$, we find

$$\begin{aligned} &|C + C'| \\ &\leq \sum\nolimits_{n=0}^{N-1} \frac{(b/x)^n}{n!} \frac{x^N}{N!} + \sum\nolimits_{n=N}^{\infty} \frac{(b/x)^n}{n!} \frac{x^N}{N!} \\ &= \mathrm{e}^{b/x} \frac{x^N}{N!}. \end{aligned} \tag{34}$$

So, we find as an absolute errror estimate the rather simple expression

$$\begin{aligned} &|E| \leq \mathrm{e}^x \frac{(b/x)^N}{N!} + \mathrm{e}^{b/x} \frac{x^N}{N!}, \\ &\quad \text{for} \quad b/(x(N+1)) < 1 \quad \text{and} \quad x/(N+1) < 1. \end{aligned} \tag{35}$$

For given x and b the parameter N can be taken as large as necessary to fullfill the estimate

$$\begin{aligned} &|E| \leq \mathrm{e}^x \frac{(b/x)^N}{N!} + \mathrm{e}^{b/x} \frac{x^N}{N!} \leq E_{abs}, \\ &\quad \text{with } E_{abs} \text{ a by the user required absolute error.} \end{aligned} \tag{36}$$

So, for given E_{abs} a value for N can be found, and the matrix G can be filled and summed.

For $E_{abs} = 1.\mathrm{E}{-7}$ we recalculated the values in a table given by Bear (1979, Table 8-6). The values in that table were taken from Hantush (1956), where Hantush made the statement "*It is probable that several of the tabulated values are in error by one unit in the fourth decimal, but very few may be in error by more than two units in the fourth decimal*". We found indeed several corrections. Moreover, some values were taken over not correctly from Hantush (1956); see Table 1 and the legend.

Table 1

r/λ u	0	0.0020	0.0040	0.0050	0.0070	0.0100	0.0200	0.0400	0.0600	0.0800	0.1000
0		12.6611	11.2748	10.8286	10.1557	9.4425	8.0569	6.6731	5.8658	5.2950	4.8541
1.E-6	13.2383	12.4417	11.2711	10.8283	10.1557	9.4425	8.0569	6.6731	5.8658	5.2950	4.8541
2.E-6	12.5451	12.1013	11.2259	10.8174	10.1554	9.4425	8.0569	6.6731	5.8658	5.2950	4.8541
5.E-6	11.6289	11.4384	10.9642	10.6822	10.1291	9.4413	8.0569	6.6731	5.8658	5.2950	4.8541
8.E-6	11.1589	11.0377	10.7151	10.5072	10.0602	9.4314	8.0569	6.6731	5.8658	5.2950	4.8541
1.E-5	10.9357	10.8382	10.5725	10.3963	10.0034	9.4176	8.0569	6.6731	5.8658	5.2950	4.8541
2.E-5	10.2426	10.1932	10.0522	9.9530	9.7126	9.2961	8.0558	6.6731	5.8658	5.2950	4.8541
5.E-5	9.3263	9.3064	9.2479	9.2052	9.0957	8.8827	8.0080	6.6730	5.8658	5.2950	4.8541
7.E-5	8.9899	8.9756	8.9336	8.9026	8.8224	8.6625	7.9456	6.6726	5.8658	5.2950	4.8541
0.0001	8.6332	8.6233	8.5937	8.5717	8.5145	8.3983	7.8375	6.6693	5.8657	5.2950	4.8541
0.0002	7.9402	7.9352	7.9203	7.9092	7.8800	7.8192	7.4972	6.6242	5.8637	5.2949	4.8541
0.0005	7.0242	7.0222	7.0162	7.0118	6.9999	6.9750	6.8346	6.3626	5.8011	5.2849	4.8530
0.0007	6.6879	6.6865	6.6822	6.6791	6.6706	6.6527	6.5508	6.1917	5.7273	5.2619	4.8478
0.0010	6.3315	6.3305	6.3276	6.3253	6.3194	6.3069	6.2347	5.9711	5.6057	5.2087	4.8292
0.0020	5.6394	5.6389	5.6374	5.6363	5.6334	5.6271	5.5907	5.4516	5.2411	4.9848	4.7079
0.0050	4.7261	4.7259	4.7253	4.7249	4.7237	4.7212	4.7068	4.6499	4.5590	4.4389	4.2960
0.0070	4.3916	4.3915	4.3911	4.3908	4.3899	4.3882	4.3779	4.3374	4.2719	4.1839	4.0770
0.0100	4.0379	4.0378	4.0375	4.0373	4.0368	4.0356	4.0285	4.0003	3.9544	3.8920	3.8150
0.0200	3.3547	3.3547	3.3545	3.3544	3.3541	3.3536	3.3501	3.3365	3.3141	3.2832	3.2442
0.0500	2.4679	2.4679	2.4678	2.4678	2.4677	2.4675	2.4662	2.4613	2.4531	2.4416	2.4271
0.0700	2.1508	2.1508	2.1508	2.1508	2.1507	2.1506	2.1497	2.1464	2.1408	2.1331	2.1232
0.1000	1.8229	1.8229	1.8229	1.8229	1.8228	1.8227	1.8222	1.8200	1.8164	1.8114	1.8050
0.2000	1.2227	1.2226	1.2226	1.2226	1.2226	1.2226	1.2224	1.2215	1.2201	1.2181	1.2155
0.5000	0.5598	0.5598	0.5598	0.5598	0.5598	0.5598	0.5597	0.5595	0.5592	0.5587	0.5581
0.7000	0.3738	0.3738	0.3738	0.3738	0.3738	0.3738	0.3737	0.3736	0.3735	0.3732	0.3729
1.0000	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2193	0.2193	0.2191	0.2190
2.0000	0.0489	0.0489	0.0489	0.0489	0.0489	0.0489	0.0489	0.0489	0.0489	0.0489	0.0489
5.0000	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
7.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
8.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 1: *This table shows the corrected table 8-6 from Bear (1979) in the notation of Bear. Note that u corresponds to x and $r/\lambda = \rho$ to $2\sqrt{b}$. In this table differences with the results of Hantush (1956) have been denoted with an underlining of one or (once, $u = 5.\text{E}{-5}$, $r/\lambda = 0.0040$) two digits.*
All other underlinings are numbers taken over not correctly from Hantush (1956) in the book by Bear (1979, Table 8-6).

7. Numerical Evaluation Eqs. (24) and (26) & (27)

We performed some experiments using the two analytic methods (24) and (26) & (27) and we used some kind of automated numerical procedure to find the sum with some specified numerical relative error, which is in fact better than the use of an absolute error criterium. We compared the results using (24) and (26) & (27) for the parameter values of Table 1 with the results presented in Table 1 and found agreement to all digits. It turned out that (24) required more terms than the expressions (26) & (27) for the same relative error estimate. So, for the parameter values in Table 1, the number of terms one needed for a relative precision of 1.E−12 was for the sum (24) some 20 to 40 terms and for (26) & (27) was the maximum number 11 terms. Moreover, during evaluation of the sum (24) some terms can become very large and with opposite sign, so, in those cases loss of precision will occur if one strives for a high precision. If one is satisfied with a relative precision of 1.E−3, then in case (26) & (27) just 5 terms are enough. Moreover, as has been said before, formulas (26) & (27) are easier to evaluate.

8. Fast Approximation

The methods of evaluation treated above may be cumbersome in applications that call upon the Hantush Well Function innumerable times. Such is the case, for instance, in a program for geohydrological time series analysis that we are developing. Therefore, we give here a very efficient approximative formula expressed as a simple expression in terms of the Exponential Integral E_1 and the Modified Bessel Function K_0 for the Hantush Well Function. Firstly, we rewrite,

$$\Gamma(0,x;b) = \int_x^\infty t^{-1}\exp\left(-t-\frac{b}{t}\right)dt = \int_{-\infty}^{\log(\sqrt{b}/x)} \exp\left(-2\sqrt{b}\cosh\tau\right)d\tau, \tag{37}$$

by the substitution $t=\sqrt{b}\exp(-\tau)$. Here, we have put all the information of the integrand in just one single exponential with a complicated argument. We define

$$F(\rho,\tau) = \int_{-\infty}^{\tau} \exp\left(-\rho\cosh\tau'\right)d\tau' = \int_{-\tau}^{\infty} \exp\left(-\rho\cosh\tau'\right)d\tau'. \tag{38}$$

So, the translation between the different forms of the Hantush Well Function are summarized as

$$\begin{aligned}
&\Gamma(0,x;b) = F\left(2\sqrt{b},\log\left(\frac{\sqrt{b}}{x}\right)\right) = W(x,2\sqrt{b}),\\
&F(\rho,\tau) = \Gamma\left(0,\frac{\rho}{2}\exp(-\tau);\frac{\rho^2}{4}\right) = \\
&W\left(\frac{\rho}{2}\exp(-\tau),\rho\right),\\
&W(u,\rho) = \Gamma\left(0,u;\frac{\rho^2}{4}\right) = F\left(\rho,\log\left(\frac{\rho}{2u}\right)\right).
\end{aligned} \tag{39}$$

Thus, $u=x$, $\rho=2\sqrt{b}$ and $\tau=\log(\sqrt{b}/x)$. We know

$$F(\rho,0) = K_0(\rho); \qquad F(\rho,\infty) = 2K_0(\rho).$$

We introduce

$$J\left(\rho,\tau\right)=\int_{\tau}^{\infty}\exp\left(-\rho\cosh\tau'\right)d\tau', \tag{40}$$

and using a symmetry argument we have

$$F\left(\rho,\tau\right)=J\left(\rho,-\tau\right)=2K_0(\rho)-J\left(\rho,\tau\right). \tag{41}$$

We estimate $J(\rho,\tau)$ as follows, for $\tau>0$

$$\begin{aligned}
&J(\rho,\tau)=\int_{\tau}^{\infty}\exp\left(-\rho\cosh\tau'\right)d\tau'\\
&>\int_{\tau}^{\infty}\exp\left(-\rho\cosh\tau'\right)\tanh(\tau')d\tau'\\
&=\int_{\rho\cosh(\tau)}^{\infty}\zeta'^{-1}\exp\left(-\zeta'\right)d\zeta'\\
&=E_1(\rho\cosh(\tau))\equiv J_-(\rho,\tau),
\end{aligned} \tag{42}$$

and also

$$\begin{aligned}
&J(\rho,\tau)=\int_{\tau}^{\infty}\exp\left(-\frac{\rho}{2}\exp(\tau')-\frac{\rho}{2}\exp(-\tau')\right)d\tau'\\
&<\int_{\tau}^{\infty}\exp\left(-\frac{\rho}{2}\exp(\tau')\right)d\tau'\\
&=\int_{\rho/2\exp(\tau)}^{\infty}\zeta'^{-1}\exp\left(-\zeta'\right)d\zeta'\\
&=E_1\left(\frac{\rho}{2}\exp(\tau)\right)\equiv J_+(\rho,\tau).
\end{aligned} \tag{43}$$

Since $J(\rho,\tau)$ is bounded by $J_-(\rho,\tau)$ and $J_+(\rho,\tau)$, it is possible to interpolate between these upper and lower functions as

$$\begin{aligned}
&J_{app}(\rho,\tau)=\\
&wJ_+(\rho,\tau)+(1-w)J_-(\rho,\tau),\qquad 0<w<1.
\end{aligned} \tag{44}$$

So, we know that

$$\begin{aligned}
&E_{app}=\max\left|J_{app}(\rho,\tau)-J(\rho,\tau)\right|\\
&<J_+(\rho,\tau)-J_-(\rho,\tau).
\end{aligned} \tag{45}$$

For an approximation it is appropriate that $J_{app}(\rho,0)=K_0(\rho)$, so, if one makes the choice

$$w(\rho)=\frac{J_-(0,\tau)-K_0(\rho)}{J_-(0,\tau)-J_+(0,\tau)}=\frac{E_1(\rho)-K_0(\rho)}{E_1(\rho)-E_1\left(\frac{\rho}{2}\right)}, \tag{46}$$

we find a very simple, and easy approximation for $\tau>0$

$$\begin{aligned}
&F\left(\rho,\tau\right)=2K_0(\rho)-J\left(\rho,\tau\right)\\
&\cong 2K_0(\rho)-J_{app}\left(\rho,\tau\right),\text{ so}\\
&F\left(\rho,\tau\right)\cong 2K_0(\rho)-w(\rho)E_1\left(\frac{\rho}{2}\exp(\tau)\right)\\
&\qquad\qquad-\left(1-w(\rho)\right)E_1(\rho\cosh(\tau)),
\end{aligned} \tag{47}$$

and for $\tau \leq 0$

$$
\begin{aligned}
&F(\rho,\tau) = J(\rho,-\tau) \cong J_{app}(\rho,-\tau), \text{ so} \\
&F(\rho,\tau) \cong w(\rho) E_1\left(\frac{\rho}{2}\exp(-\tau)\right) \\
&\qquad + (1-w(\rho))\, E_1(\rho\cosh(\tau)),
\end{aligned}
\tag{48}
$$

Translated into the arguments for $\Gamma(0,x;b)$, this results in, for $x < \sqrt{b}$

$$
\begin{aligned}
&\Gamma(0,x;b) \cong 2K_0(2\sqrt{b}) \\
&- \left\{ \left(\frac{E_1(2\sqrt{b}) - K_0(2\sqrt{b})}{E_1(2\sqrt{b}) - E_1(\sqrt{b})} \right) E_1\left(\frac{b}{x}\right) \right. \\
&\quad \left. + \left(\frac{K_0(2\sqrt{b}) - E_1(\sqrt{b})}{E_1(2\sqrt{b}) - E_1(\sqrt{b})} \right) E_1\left(x + \frac{b}{x}\right) \right\},
\end{aligned}
\tag{49}
$$

and for $x \geq \sqrt{b}$

$$
\begin{aligned}
&\Gamma(0,x;b) \cong \\
&\left\{ \left(\frac{E_1(2\sqrt{b}) - K_0(2\sqrt{b})}{E_1(2\sqrt{b}) - E_1(\sqrt{b})} \right) E_1(x) \right. \\
&\quad \left. + \left(\frac{K_0(2\sqrt{b}) - E_1(\sqrt{b})}{E_1(2\sqrt{b}) - E_1(\sqrt{b})} \right) E_1\left(x + \frac{b}{x}\right) \right\}.
\end{aligned}
\tag{50}
$$

It is possible to give an estimate for the maximal error E_{app} (45) under the condition $\frac{\rho}{2}\exp(-\tau) \to 0$ as follows

$$
\begin{aligned}
&E_{app} = J_+(\rho,\tau) - J_-(\rho,\tau) \\
&= E_1\left(\frac{\rho}{2}\exp(\tau)\right) - E_1(\rho\cosh(\tau)) \\
&= E_1\left(\frac{\rho}{2}\exp(\tau)\right) - E_1\left(\frac{\rho}{2}\exp(\tau) + \frac{\rho}{2}\exp(-\tau)\right) \\
&= E_1\left(\frac{\rho}{2}\exp(\tau)\right) - \\
&\left\{ E_1\left(\frac{\rho}{2}\exp(\tau)\right) + \frac{\rho}{2}\exp(-\tau)\frac{dE_1}{dx}\left(\frac{\rho}{2}\exp(\tau)\right) \right. \\
&\quad \left. + O\left(\left(\frac{\rho}{2}\exp(-\tau)\right)^2\right) \right\}, \\
&\text{for } \frac{\rho}{2}\exp(-\tau) \to 0, \quad \tau > 0.
\end{aligned}
\tag{51}
$$

This results into

$$
\begin{aligned}
&E_{app} \\
&= -\frac{\rho}{2}\exp(-\tau)\left(\frac{(-1)}{x}\exp(-x)\right)\Big|_{x=\frac{\rho}{2}\exp(\tau)} \\
&= \exp\left(-2\tau - \frac{\rho}{2}\exp(\tau)\right), \\
&\text{for } \frac{\rho}{2}\exp(-\tau) \to 0, \quad \tau > 0.
\end{aligned}
\tag{52}
$$

The corresponding estimates for the maximal error for $\tau < 0$, can be found by using the equations (51) and (52) with $\tau = -\tau$.

We have evaluated (49) and (50) for the same set of values as in Table 1 (1.E−6 $\leq x \leq$ 8, 0.002 $\leq \rho = r/\lambda \leq$ 0.1) and we found that the maximal absolute error for these parameters was 5.3052E−3 for $W(0.1, 0.1) = 1.8050$ (i.e. $b = (r/\lambda)^2/4 = 0.0025$) and the maximal relative error of 3.3133E−3 for $W(0.2, 0.1) = 1.2155$ (i.e. $b = (r/\lambda)^2/4 = 0.0025$). Moreover, we have extended our analysis for the range of the parameters as given in Kruseman and de Ridder (1994, Annex 4.2) and beyond (1.E−6 $\leq x \leq$ 8, 0.002 $\leq \rho = r/\lambda \leq$ 6) and found that the maximal absolute error for these parameters was 7.6081E−3 for $W(0.07, 0.25) = 1.9867$ (i.e. $b = (r/\lambda)^2/4 = 0.015625$) and the maximal relative error of 1.260E−1 for $W(5, 6) = 2.4562$E−4 (i.e. $b = (r/\lambda)^2/4 = 9$). This last combination of parameters is well at the border of the standard range.

9. Discussion

The Comment (Nadarajah (2007)) to the paper by Prodanoff et al. (2006) triggered this paper and exploring several expressions for the Hantush Well Function we found the equations (26) & (27) the most useful for evaluation. We found that (26) & (27) were about three times faster than evaluation of (24) for the calculation of Table 1. Equations (47) & (48) (or (49) & (50)) may suffice for engineering purposes.

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Appendix: A Matlab code

The following Matlab code evaluates

$$s(r,t) \approx \frac{Q}{4\pi T} W\left(\frac{r^2 S}{4Tt}, \frac{r}{\lambda}\right) = \frac{Q}{4\pi T} \int_{\frac{r^2 S}{4Tt}}^{\infty} \frac{1}{t'} \exp\left(-t' - \frac{r^2}{4\lambda^2 t'}\right) dt',$$

where

s	[L]	approximate transient drawdown during a pumping test of a leaky aquifer,
Q	[$\mathrm{L^3T^{-1}}$]	steady well discharge, starting at $t = 0$,
T	[$\mathrm{L^2T^{-1}}$]	transmissivity of the aquifer,
W	[−]	Hantush's Well Function, see eq. (1),
r	[L]	distance from the well,
S	[−]	storativity of the aquifer,
t	[T]	time,
λ	[L]	leakance, $\lambda = \sqrt{Tc}$.

```
function s = Hantush(T,c,S,r,t,Q)
rho = r/sqrt(T*c);
tau = log(2/rho*t/(c*S));
s = Q/(4*pi*T)*F(rho,tau);

function h = F(rho,tau)
tau = tau(:);
tau(find(tau>100)) = 100;
h_inf = besselk(0,rho);
expintrho = expint(rho);
w = (expintrho-h_inf)/(expintrho-expint(rho/2));
I = h_inf - w*expint(rho/2*exp(abs(tau))) + (w-1)*expint(rho*cosh(tau));
h = h_inf + sign(tau).*I;
```

The first function calls the second function, given by eqs. (47) and (48). The variable t may be a vector in which case the output s will be a vector of the same length; the other parameters are scalars. The relative error of the approximation is at most 3.3 ‰.