Delft University of Technology
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Hydrodynamic Limit for the Symmetric Inclusion Process with Slowly Varying Inhomogeneities

(Nederlandse titel: Hydrodynamisch Limiet voor het Symmetrisch Inclusieproces met Langzaam Variërende Inhomogeniteiten)

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Abstract

In this thesis we study an interacting particle system: the Symmetric Inclusion Process with slowly varying inhomogeneities ($\mathcal{SIP}(\alpha)$). In the $\mathcal{SIP}(\alpha)$ particles display random walk like behaviour subjected to an attractive type of interaction whilst evolving in an inhomogeneous environment. We set out to prove its hydrodynamic limit. The main tool helping us for obtaining the hydrodynamic limit is the self-duality property of the process.

Preface

In front of you is my Master thesis 'Hydrodynamic Limit for the Symmetric Inclusion Process with Slowly Varying Inhomogeneities'. Written as a conclusion of my graduate study Applied Mathematics at the TU Delft.

I would like to thank both my supervisors Prof. dr. F.H.J Redig and his PhD researcher MSc. S. Floreani for their continued support and guidance during my Master thesis. All the valuable insights, feedback and interesting discussions are very much appreciated.

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Chapter 1

Introduction

Interacting particle systems (IPS) are models used to describe phenomena with a high degree of complexity and involving a large number of agents. IPS can be found in the field of statistical mechanics, social sciences, financial markets and many more. In probability theory IPS are used as models in the context of non-equilibrium statistical mechanics. The agents are modeled as particles that evolve in a lattice according to various local rules. The interaction and dynamics of the particles are given on a microscopic level. Often times the rules governing the particle dynamics on the microscopic level are modeled ìn a stochastic manner. Henceforth IPS are stochastic processes. Commonly known models as such are the single particle random walk, symmetric exclusion process (SEP) and the symmetric inclusion process (SIP).

The field of IPS is still relatively young. The works of R.L. Dobrushin and F. Spitzer in 1970 established the first real foundations, after which many others followed suit. The first classical models where general existence and uniqueness had been established are the SEP, stochastic Ising model, Voter model and the contact process. By 1975 elementary properties, and some not so elementary properties, had already been shown. The field really matured in 1985 when Ligget [18] published his renowned book on IPS. After 1985 many more models were introduced and studied. One of which is the SIP.

In this thesis we consider a variation on the SIP: the SIP with slowly varying inhomogeneities ($\mathcal{SIP}(\alpha)$). In the $\mathcal{SIP}(\alpha$) we consider the particles to be inhabiting the lattice sites in $\mathbb{Z}$. Particles jump randomly to neighbouring sites and interact via so called inclusion jumps. These inclusion jumps describe the mutual attraction between particles. This IPS is the direct bosonic analogue to the fermionic SEP. The slowly varying inhomogeneities are modelled by the parameters $(\alpha(x))_{x\in\mathbb{Z}}$. α is given by a smooth and bounded macroscopic profile,

$$\mathbb{R} \ni x \longmapsto \alpha(x) > 0.$$

With $\alpha(x)$ we model the inhomogeneous environment in which the particles evolve. Physically one can think of either composite materials or layered materials. The transition from the microscopic into the macroscopic is facilitated by the slowly varying function $\alpha_N(x) = \alpha\left(\frac{x}{N}\right)$, for each $x \in \mathbb{Z}$ and for any scaling parameter $N \in \mathbb{N}$. When N is large $\alpha_N(x) \approx \alpha_N(x+1)$, hence why we refer to the inhomogeneities as being slowly varying. Complete dynamics of the $\mathcal{SIP}(\alpha)$ is given by the generator,

$$\begin{aligned}\mathcal{L}^N_{\mathcal{SIP}(\alpha)} f(\eta) = &\sum_{x\in\mathbb{Z}} \eta(x)(\alpha_N(x+1) + \eta(x+1))(f(\eta^{x,x+1}) - f(\eta)) \\ &+ \eta(x+1)(\alpha_N(x) + \eta(x))(f(\eta^{x+1,x}) - f(\eta)).\end{aligned}$$

Generator $\mathcal{L}^N_{\mathcal{SIP}(\alpha)}$ will be properly defined in chapter 5.

The main objective is to understand and formally describe the emerging macroscopic dynamics from this microscopic model description. These emergent macroscopic behaviours are best captured by the so called hydrodynamic limit (HDL). The HDL provides us a rigorous manner to derive the emerging partial differential equation (PDE) for the density field. In our case this will be a specific Cauchy problem. In the macroscopic world we seem to observe less complexity as compared with the microscopic world. In the macroscopic world most is described by conservation laws (conservation of energy, conservation of momentum). Microscopically each particle has its own energy and momentum. The only quantity that is conserved is the number of particles in the system. We see the huge discrepancy in complexity between the macroscopic and microscopic world. Nevertheless we expect to be able to prove the HDL of the $\mathcal{SIP}(\alpha)$.

This thesis is structured as follows. First all basic, but necessary, mathematical background is covered. E.g. Markov processes, Markov semigroups, infinitesimal generators, invariant measures, reversible measures, detailed balance and self-adjointness are defined in order to completely understand this thesis. In chapter 3 the $\mathcal{SIP}(\alpha)$ is introduced and its generator is defined. Moreover the invariant measures of the $\mathcal{SIP}(\alpha)$ are computed. Chapter 4 proves the self-duality property of the $\mathcal{SIP}(\alpha)$. Chapter 5 covers the main theorem of this thesis of which concerns itself with the HDL. The proof of the theorem is split into subsections of this chapter. Lastly, we conclude on all of the obtained results and provide suggestions and ideas for future research.

Chapter 2

Preliminaries on Markov Processes

In this thesis we desire to compute the hydrodynamic limit of the rescaled empirical density field. At the core of this matter lies the field of Interacting Particle Systems (IPS). Thus before diving deep into the subject matter it is pivotal to understand the basic concepts underpinning IPS. Most notably the theory of Markov processes, Markov semigroups and generators. Moreover a key theorem, essentially an application of the Hille-Yosida theorem, is introduced.

2.1 Markov Processes

Markov processes are a type of stochastic process that model systems with limited past memory. This property is commonly revered to as the Markov property, or memoryless property. Meaning that the conditional probability of an arbitrary future state, given all its past states, only depends on the present state.

Definition 2.1.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $\mathcal{F}_t = \{X_r \colon r \leq t\}$ *be the* σ*-algebra generated by the random variables* $\{X_r \colon r \leq t\}$*. Stochastic process* $\{X_t \colon t \geq 0\}$*, on probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, satisfies the Markov property if,*

$$\mathbb{E}\left[f(X_t)\middle|\mathcal{F}_s\right] = \mathbb{E}\left[f(X_t)\middle|X_s\right]$$

holds for all bounded, measurable $f \colon \Omega \longrightarrow \mathbb{R}$*. We call* $\{X_t \colon t \geq 0\}$ *a Markov process.*

For better understanding of this definition the following examples are provided.

Example 2.1.1 (Poisson process)**.** *Let* $\{N_t \colon t \geq 0\}$ *be a Poisson process. One can think of* N_t *denoting the number of jumps of a simple random walk on* $\mathbb{Z}$*. [6] So* N_t *is a rate one Poisson process i.e.,*

$$\mathbb{P}\left(N_t = n\right) = \frac{t^n}{n!}\mathrm{e}^{-t}.$$

N_t *has independent Poisson increments, meaning that* $N_{t_i} - N_{t_{i-1}} \perp N_{t_{i-1}} - N_{t_{i-2}}$ *for all* $i = 2, \ldots, n$*. To see that this process is indeed a Markov process notice that,*

$$
\begin{aligned}
&\mathbb{P}\left(N_t = n \middle| N_{t_1} = k_1, \ldots, N_{t_n} = k_n\right) \\
&= \lim_{s \downarrow t_n} \mathbb{P}\left(N_t - N_s = n - k_n \middle| N_{t_1} - N_{t_0} = k_1, N_{t_2} - N_{t_1} = k_2 - k_1, \ldots, N_{t_n} - N_{t_{n-1}} = k_n - k_{n-1}\right) \\
&= \lim_{s \downarrow t_n} \mathbb{P}\left(N_t - N_s = n - k_n\right) = \mathbb{P}\left(N_t - N_{t_n} = n - k_n\right) \overset{\textit{(IND INCR)}}{=} \mathbb{P}\left(N_t - n \middle| N_{t_n} = k_n\right).
\end{aligned}
$$

Where in the second line we use that the probability of an extra jump occurring in $[t_n, s)$ goes to 0 *when $s \downarrow t_n$. The Poisson process $\{N_t : t \geq 0\}$ thus satisfies the Markov property. [2]*

Example 2.1.2 (Brownian Motion)**.** *Let $\{B_t : t \geq 0\}$ be a Brownian motion on $\mathbb{R}$. This means $\{B_t : t \geq 0\}$ satisfies,*

1) $B_0 = 0$,
2) *Independent Gaussian increments, so $W_{t_i} - W_{t_{i-1}} \perp W_{t_{i-1}} - W_{t_{i-2}}$ for all $i = 2, \ldots, n$.*
   And $W_{t_i} - W_{t_{i-1}} \overset{d}{=} \mathcal{N}(0, t_i - t_{i-1})$,
3) *$t \longmapsto W_t$ is continuous in t.*

Verifying that $\{B_t : t \geq 0\}$ is Markov goes similarly as with the Poisson process in 2.1.1.

2.2 Markov Semigroups and Generators

Using the Markov processes as defined in the previous section we are able to introduce various key concepts which play a major role in the study of interacting particle systems. Markov semigroups and generators are such concepts highlighted here. Suppose we are given a measurable function $f : \Omega \longrightarrow \mathbb{R}$, where Ω is the state space of continuous time Markov process $\{X_t : t \geq 0\}$, then we define a family of linear operators $P_t : C(\Omega) \longrightarrow C(\Omega)$, equipped with the supremum norm, by

$$
P_t f(x) := \mathbb{E}\left[f(X_t) \middle| X_0 = x\right] \overset{\text{(set)}}{=} \mathbb{E}_x f(X_t). \tag{2.1}
$$

Notice that we can write $P_t f(x) = \sum_{y \in \Omega} \mathbb{P}\left(X_t = y \middle| X_0 = x\right) f(y)$, whenever Ω is finite. This means that P_t can be interpreted as the matrix $(P_t)_{xy} = P_t(x, y) := \mathbb{P}\left(X_t = y \middle| X_0 = x\right)$. However this interpretation will not be utilized, as it does not generalize to the more general case when Ω is a compact metric space.

Semigroups obey various properties. A few of these are given in proposition 2.2.1. These properties will just be restricted to so called Feller processes. This is a special class of Markov processes for which $(P_t f)(x)$ is continuous as a function of x, such that $f : \Omega \longrightarrow \mathbb{R}$ belongs to $C(\Omega)$ and Ω is a compact metric space. These restrictions are made as allowing for a broader class of functions can be troublesome.

Proposition 2.2.1. *Properties of semigroup P_t [4],*

a) $P_0 = I$, *i.e.* $P_0 f = f, \quad \forall f \in C(\Omega)$
b) *The map $t \longmapsto P_t f$ is right continuous for all $f \in C(\Omega)$ w.r.t. the supremum norm*
c) $P_{t+s} f = P_t(P_s f) = P_s(P_t f) \quad \forall t, s > 0$
d) $f \geq 0 \implies P_t f \geq 0$
e) $P_t 1 = 1$
f) $\sup_x \left|(P_t f)(x)\right| \leq \sup_x \left|f(x)\right|$, *(so $\|P_t\| \leq 1$ for all $t \geq 0$).*

Notice that property *c)* holds true, because of the Markov property.

$$\begin{aligned}(P_{t+s}f)(x) &= \mathbb{E}_x f(X_{t+s}) \\ &= \mathbb{E}\left(\mathbb{E}_x\left[f(X_{t+s})\middle|X_t\right]\right) \\ &= \mathbb{E}_x\mathbb{E}_{X_t} f(X_s) \\ &= \mathbb{E}_x(P_s f)(X_t) = (P_t(P_s f))(x)\end{aligned}$$

For these types of semigroups we are able to find its corresponding operator, which in this instance we refer to as its generator. The generator of a Markov process (Feller process) tells us how the process evolves in an instance of time. Hence it's commonly revered to as the infinitesimal generator of the Markov process, or semigroup.

Definition 2.2.1 (Generator)**.** *Let $\{X_t \colon t \geq 0\}$ be a Feller process. We define the domain $D(A)$ of the infinitesimal generator as [3],*

$$D(A) := \left\{ f \in C(\Omega) \,:\, \lim_{t\downarrow 0} \frac{P_t f - f}{t} \in C(\Omega) \right\}. \tag{2.2}$$

If f belongs to the domain $D(A)$, then

$$Af = \lim_{t\downarrow 0} \frac{P_t f - f}{t}. \tag{2.3}$$

Remark. *To see why we call A the infinitesimal generator, notice from 2.3 that,*

$$P_t f(x) = \mathbb{E}_x f(X_t) = f(x) + tAf(x) + o(t)$$

For arbitrary $f \in D(A)$ and bounded, and where $\frac{o(t)}{t} \longrightarrow 0$. Thus A describes how the Feller process evolves in an infinitesimally small time interval. [3]

A well know result tells us that the generator is the derivative w.r.t. time of the mapping $t \longmapsto P_t f(x)$, formally meaning that $\frac{d}{dt} P_t f(x) = AP_t f(x)$. This result describes a class of PDEs given as follows,

$$\begin{cases} u_t(t,x) = Au(t,x) \\ u(0,x) = f(x) \end{cases} \tag{2.4}$$

where $u(t,x) = P_t f(x)$. The PDEs implied by the semigroup and its generator is a big reason for studying these objects. Another major reason for the study of generators is its link to weak solutions of SDEs, Dirichlet forms and Carré du Champ operators. All of which we will revisit, and properly define in chapter 5 as these are required for proving the hydrodynamic limit of the $\mathcal{SIP}(\boldsymbol{\alpha})$. Here we will just introduce Dynkin's formula.

Definition 2.2.2 (Dynkin's formula)**.** *Let $\{X_t \colon t \geq 0\}$ be a Feller process. If $f \in D(A)$, then*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Af(X_s)ds, \quad t \geq 0 \tag{2.5}$$

is a martingale.

Thus by utilizing Dynkin's formula, and the generator A we are able to associated a class of martingales to $\{X_t \colon t \geq 0\}$. [5]

For an even better grasp on semigroups and generators a few examples are provided. Both examples will turn out to play a vital role for understanding the hydrodynamic limit of the $\mathcal{SIP}(\boldsymbol{\alpha})$.

Example 2.2.1 (Random walk)**.** *Let $\{N_t \colon t \geq 0\}$ be a Poisson process, and let $\{\epsilon_j \colon j \in \mathbb{N}\}$ be a sequence of IID symmetric Bernoulli random variables. Then $X_t := \sum_{j=1}^{N_t} \epsilon_j$ is a simple random walk on $\mathbb{Z}$. Poisson process N_t denotes $\#\{$jumps of X_t up to time $t\}$ [6]. Note that N_t is similar as in Example 2.1.1. Now fix some $x \in \mathbb{Z}$, and let $\mathcal{Z}$ be a random variable such that $\mathcal{Z} \perp \{\epsilon_j \colon j \in \mathbb{N}\}$, and*

$$\mathbb{P}_x(\mathcal{Z} = y) = \begin{cases} \frac{1}{2}, & |x-y| = 1 \\ 0, & \textit{else.} \end{cases}$$

We then compute the generator of $\{X_t \colon t \geq 0\}$ as follows. First note that $X_t \stackrel{d}{=} x\mathbb{1}_{\{N_t=0\}} + \mathcal{Z}\mathbb{1}_{\{N_t=1\}} + X_t\mathbb{1}_{\{N_t \geq 2\}}$, implying that
$P_t f(x) = \mathbb{E}_x f(X_t) \stackrel{(IND)}{=} f(x)\mathbb{P}_x(N_t = 0) + \mathbb{E}_x(f(\mathcal{Z}))\mathbb{P}_x(N_t = 1) + \mathbb{E}_x(f(X_t)\mathbb{1}_{\{N_t \geq 2\}})$, holds for all $f \in C(\Omega)$. Thus,

$$\begin{aligned} Af(x) &:= \lim_{t \downarrow 0} \frac{\mathbb{E}_x f(X_t) - f(x)}{t} \\ &= \lim_{t \downarrow 0} \frac{1}{t} \left[(\mathrm{e}^{-t} - 1) f(x) + \frac{1}{2} t\mathrm{e}^{-t} \sum_{|y-x|=1} f(y) + \mathbb{E}_x(f(X_t)\mathbb{1}_{\{N_t \geq 2\}}) \right]. \end{aligned}$$

Notice that,

$$\begin{aligned} &1) \ \frac{1}{t}(\mathrm{e}^{-t} - 1) f(x) \xrightarrow{t \to 0} -f(x) \\ &2) \ \frac{1}{2}\mathrm{e}^{-t} \sum_{|y-x|=1} f(y) \xrightarrow{t \to 0} \frac{1}{2} \sum_{|y-x|=1} f(y) \\ &3) \ \frac{1}{t}\mathbb{E}_x(f(X_t)\mathbb{1}_{\{N_t \geq 2\}}) \leq \|f\|_\infty \frac{1}{t}\mathbb{P}_x(N_t \geq 2) \xrightarrow{t \to 0} 0. \end{aligned}$$

And thus,

$$Af(x) = \frac{1}{2} \sum_{|y-x|=1} (f(y) - f(x)).$$

Example 2.2.2 (Diffusion Process and Brownian Motion). *A one-dimensional, homogeneous and Markovian diffusion process is described by the following SDE,*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x. \tag{2.6}$$

W_t is the standard Brownian motion on $\mathbb{R}$. Its corresponding generator A is given by,

$$Af(x) = b(x)\frac{d}{dx}f(x) + \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}f(x). \tag{2.7}$$

Where both $b(x)$, and $\sigma(x)$ are C^2 functions. The domain of the generator is $D(A) = \{f \in C_0 \colon f' \in C_0$ and $f'' \in C_0\}$. [2]

In particular if we choose $b(x) \equiv 0$, and $\sigma(x) \equiv 1$ we obtain the generator of the standard Brownian motion on $\mathbb{R}$. To see why this holds true we let $B_t \overset{d}{=} \mathcal{N}(0,t)$ and notice that for f smooth and compactly supported,

$$P_t f(x) = \mathbb{E}_x f(X_t) = \mathbb{E} f(x + B_t) = f(x) + f'(x)\mathbb{E}(B_t) + \frac{1}{2}f''(x)\mathbb{E}(B_t^2) + o(t)$$

with $\frac{o(t)}{t} \xrightarrow{t\to 0} 0$. Implying that for such f,

$$Af(x) = \lim_{t\downarrow 0}\frac{P_t f - f}{t} = \frac{1}{2}f''(x) \text{ uniformly in } x. \tag{2.8}$$

We notice from definitions 2.2.1, 2.1 that in the case of Ω being a finite state space, P_t and its corresponding generator A are matrices. Moreover proposition 2.2.1b) implies that $P_t = \exp(tA) := \sum_{N=0}^{\infty}\frac{t^N}{N!}A^N$. We observe that based on this definition of P_t there should be an one to one correspondence between semigroups and generators. However in general $\exp(tA)$ is not well defined. A manner for resolving this issue is given by theorem 2.2.1, which generalises the relation between semigroups and generators for an infinite state space. This theorem is an application of the so called Hille-Yosida theorem.

Theorem 2.2.1 (Hille-Yosida). *Each Markov semigroup P_t can be associated with its corresponding Markov generator A. [1] The relation is given as follows,*

a) $Af = \lim_{t\to\infty}\frac{P_t f - f}{t}$ *for* $f \in D(A) := \left\{f \colon f \in C(\Omega), \lim_{t\to\infty}\frac{P_t f - f}{t} \text{ exists}\right\}$

b) $P_t = \lim_{N\to\infty}(I - \frac{t}{N}A)^{-N}$, *for* $t \geq 0$.

Moreover,

c) $f \in D(A) \implies P_t f \in D(A)$, *and* $\frac{d}{dt}P_t f = AP_t f = P_t A f$

d) If g is continuous and $\lambda \geq 0$, then $f = \int_0^t e^{-t}P_{\lambda t}g\,dt$ *solves the equation* $f - \lambda Af = g$.

The proof of this theorem will be omitted, but can be found in [4], as it is not important for the subject matter at hand. However the result of this theorem will be utilized. Often times it is most convenient to construct, and describe a process by its generator. Also proving convergence results for the generator is often times easier, as opposed to directly showing this for the corresponding Markov process.

A visual representation of the one-to-one relation is given in figure 2.1. Note that the correct definition for the exponential, for deriving the semigroup, is given in 2.2.1b).

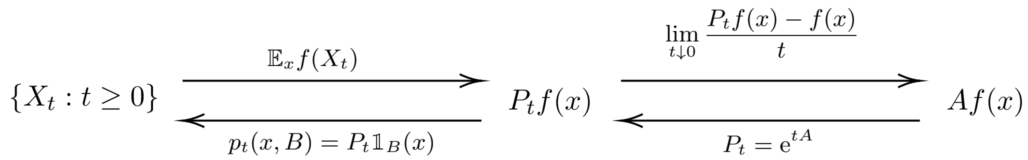

Figure 2.1: Schematic representation of the one-to-one relation between Markov processes, semigroups and generators. [1]

2.3 Invariant Measures and Reversibility

In this section we provide some basic background material needed for studying the $\mathcal{SIP}(\alpha)$. In particular the concepts needed for understanding the hydrodynamic limit. However we still lack some key ingredients for this. Concepts such as invariant measures, reversible measures and detailed balance are exemplary of this. All which will turn out to be very helpful to us for computing the hydrodynamic limit. Here we will introduce these basic concepts, so that we can apply these later on in upcoming chapters.

Again we start of with a Markov process $\{X_t \colon t \geq 0\}$, on a compact metric space Ω. On our space Ω we assume the natural weak topology. I.e.,

$$\mu_N \longrightarrow \mu \iff \int_\Omega f d\mu_N \longrightarrow \int_\Omega f d\mu \tag{2.9}$$

for all $f \in C_b(\Omega)$ [2]. We denote $\mathcal{P}(\Omega)$ as the set of all probability measures on Ω. Moreover assume that our Markov process has some initial distribution μ and corresponding semigroup P_t. We denote μP_t as the distribution of the process at some time $t > 0$, if we start from μ. Formally,

$$\int_\Omega f d(\mu P_t) = \int_\Omega P_t f d\mu \tag{2.10}$$

for all $f \in C_b(\Omega)$ [2]. Measure μ is uniquely determined by this relation. An important subset of $\mathcal{P}(\Omega)$ is the set of measures that are invariant under action of the semigroup P_t. Meaning that they satisfy $\mu P_t = \mu$ a.e. Formally,

Definition 2.3.1 (Invariant measure)**.** *A probability measure $\mu \in \mathcal{P}(\Omega)$ is said to be invariant if,*

$$\int_\Omega P_t f d\mu = \int_\Omega f d\mu \tag{2.11}$$

holds for all $t \geq 0$, and $f \in C_b(\Omega)$. [2]

The r.h.s. of 2.11 determines a unique probability on Ω by the Riesz representation theorem. Therefore if we initialize Markov process $\{X_t : t \geq 0\}$, with corresponding semigroup P_t, from initial measure μ, then for each future time $t > 0$ we have $X_t \overset{d}{=} \mu$.

In general the concept of invariant measures is important, because they play a paramount role in describing the long-term behaviour of Markov chains. Similarly, in our case invariant measures are relevant as they determine the long-term dynamics of our $\mathcal{SIP}(\alpha)$. The following theorems give an alternative characterization of invariant measures. Theorem 2.3.1 states that if some arbitrary measure μ is the weak limit of the distribution of a Markov process, w.r.t. a different measure ν, then this measure is invariant.

Theorem 2.3.1. *Let P_t be some semigroup corresponding to a Feller process on a compact metric space Ω. Assume that the weak limit $\mu = \lim_{t\to\infty} \nu P_t$ exists, for some $\nu \in \mathcal{P}(\Omega)$. Then μ is an invariant measure. [1]*

Proof: We check if measure μ satisfies condition 2.11. By assumption we obtain,

$$\begin{aligned}\int_\Omega P_s f d\mu = \lim_{t\to\infty} \int_\Omega P_s f d(\nu P_t) \overset{(2.10)}{=} \lim_{t\to\infty} \int_\Omega P_t(P_s f) d\nu &= \lim_{t\to\infty} \int_\Omega P_{t+s} f d\nu \\ &\overset{(T=t+s)}{=} \lim_{T\to\infty} \int_\Omega P_T f d\nu \\ &\overset{(2.10)}{=} \lim_{T\to\infty} \int_\Omega f d(\nu P_T) \\ &= \int_\Omega f d\mu, \text{ for any } f \in C_b(\Omega).\end{aligned}$$

■

The following theorem gives yet another characterization of what it means for a measure to be invariant. This characterization is given via the generator A of a Markov semigroup. This can turn out to be handy as it often times shows that proving various results is more straightforward when using generators.

Theorem 2.3.2. *$\mu \in \mathcal{P}(\Omega)$ is an invariant measure iff, [2]*

$$\int_\Omega A f d\mu = 0, \ \forall f \in D(A). \tag{2.12}$$

Proof: Assume 2.12 holds true, then for $f \in D(A)$ we have

$$\int_\Omega (P_t f - f) d\mu = \int_\Omega \int_0^t A P_s f ds \, d\mu = \int_0^t \left(\int_\Omega A P_s f d\mu \right) ds = 0.$$

Implying that, $\int_\Omega P_t f d\mu = \int_\Omega f d\mu$ for all $f \in D(A)$. Conversely, if μ is an invariant measure and f in is the domain of generator A, then

$$\int_\Omega A f d\mu = \lim_{t\to\infty} \frac{1}{t} \int_\Omega (P_t f - f) d\mu = 0.$$

Here we are allowed to exchange limit and integral as $\frac{P_t f - f}{t}$ converges uniformly to Af whenever $f \in D(A)$. ■

Theorems 2.3.1 and 2.3.2 are especially useful for checking if a measure is indeed invariant. However one might be left with the question as to how we find such invariant measures. Most commonly applied are so called reversible measures in order to find invariant measures.

We consider an invariant measure μ on a finite state space Ω, and let $L^2(\mu) := \{f \in C(\Omega)\colon \int_\Omega f^2 d\mu < \infty\}$. Note that both semigroup P_t, and corresponding generator A can be extended to this space of square integrable functions. This extension is valid as by Jensen's inequality we are able to obtain,

$$\int_\Omega (P_t f)^2 \, d\mu \leq \int_\Omega P_t \left(f^2\right) d\mu = \int_\Omega f^2 d(\mu P_t) = \int_\Omega f^2 d\mu < \infty$$

whenever $f \in L^2(\mu)$. The following is a formal definition of what we entail by reversibility of a measure.

Definition 2.3.2 (Reversible measure)**.** *Let $\mu \in \mathcal{P}(\Omega)$ such that Ω is a finite state space, then μ is called a reversible measure if semigroup P_t is self-adjoint$^{(\star)}$ in $L^2(\mu)$, for all $f, g \in L^2(\mu)$. I.e.,*

$$\int_\Omega g \left(P_t f\right) d\mu = \int_\Omega f \left(P_t g\right) d\mu \tag{2.13}$$

$\forall t \geq 0$ and all $f, g \in L^2(\mu)$. [1]

The manner in which reversible measures will be applied, relies on the fact that each reversible measure is also invariant. We then use a relation known as the detailed balance relation for finding invariant measures. Precisely this approach will be taken for finding the invariant measures of our $\mathcal{SIP}(\alpha)$. The concept of detailed balance is explained in 2.3.2. First we show that all reversible measures are indeed invariant.

Proposition 2.3.1. *Let $\mu \in P(\Omega)$ be a reversible measure, then μ is invariant. [1]*

Proof: Let $g \equiv 1$, then by reversibility

$$\int_\Omega P_t f d\mu = \int_\Omega \left(P_t f\right) 1 d\mu \stackrel{(2.13)}{=} \int_\Omega f \left(P_t 1\right) d\mu = \int_\Omega f d\mu$$

holds true for any $f \in L^2(\mu)$. ■

The following proposition gives us the main tool for computing the invariant measures of the $\mathcal{SIP}(\alpha)$.

Proposition 2.3.2 (Detailed balance)**.** *Assume $\{X_t \colon t \geq 0\}$ to be a finite state space continuous-time Markov chain with transition rates $c(x, y)$, then reversibility holds iff the detailed balance relation holds, i.e.,*

$$\mu(x)c(x, y) = \mu(y)c(y, x) \tag{2.14}$$

for all $x, y \in \Omega$. [1]

Thus if the detailed balance relation is shown to hold true, then reversibility holds. Consequently proposition 2.3.1 then implies invariance of the measure. Because we only need the backward direction of proposition 2.3.2 only this is proven here. Proving this requires introducing the concept of a symmetric operator. An operator P_t being symmetric in $L^2(\mu)$ is equivalent to the operator A being self-adjoint on its domain $L^2(\mu)$. If the operator P_t is to be unbounded, then the self-adjointness property is stronger than the symmetry, as self-adjointness implies that the domain is also the domain of the adjoint operator. Meaning that the domain is maximal. For generators of symmetric semigroups of bounded operators on a Hilbert space this is always the case [9]. Suppose that some functionspace is dense in its domain topology, then the self-adjointness$^{(\star)}$ property is equivalent to symmetry [8]. 2.13 gives a characterization of symmetry. Equivalently,

$$\int_\Omega gAf\, d\mu = \int_\Omega fAg\, d\mu, \quad \forall f, g \in D(A). \tag{2.15}$$

The following proposition states the equivalence of reversible measures and the symmetry property.

Proposition 2.3.3. *Let P_t be a Markov semigroup with corresponding generator A, and let $\mu \in \mathcal{P}(\Omega)$. Then,*

$$\mu \text{ is reversible} \iff A \text{ is a symmetric generator (as defined in 2.15)[1]}$$

<u>Proof:</u> Assume μ to be reversible and let $f, g \in D(A)$, then

$$\int_\Omega g\frac{P_tf - f}{t} d\mu = \int_\Omega f\frac{P_tg - g}{t} d\mu.$$

Now we let $t \to 0$ in the Hille-Yoshida theorem 2.2.1a). Which yields us,

$$\int_\Omega gAf\, d\mu = \int_\Omega fAg\, d\mu.$$

I.e. the symmetry property.

Conversely now assume the symmetry property, then

$$\int_\Omega g(f - \lambda Af) d\mu = \int_\Omega f(g - \lambda Ag) d\mu$$

where λ is some positive constant. Note that for $N \in \mathbb{N}$ we can plug in $f = (I - \lambda A)^{-N-1}f$, and $g = (I - \lambda A)^{-N-1}g$. Yielding,

$$\int_\Omega g(I - \lambda A)^{-N} f d\mu = \int_\Omega f(I - \lambda A)^{-N} g d\mu.$$

Letting $\lambda = \frac{t}{N}$ and $N \to \infty$ we obtain reversibility,

$$\int_\Omega g\,(P_tf)\, d\mu = \int_\Omega f\,(P_tg)\, d\mu.$$

■

Using this result we are now ready to give a proof of 2.3.2 (backward direction only).

Proof: [Proposition 2.3.2]
Let $f \equiv \mathbb{1}_x$, and $g \equiv \mathbb{1}_y$ in the definition of symmetry 2.15. The definition now reads $\mu(x)c(x,y) = \mu(y)c(y,x)$, which is precisely the detailed balance relationship. ■

Thus we see that reversibility follows 2.3.2. Hence, in most cases measure μ is uniquely determined by the generator A. Moreover, 2.15 also has a connection with the Carré du Champ operator Γ, which we will come back to later on in this thesis. Γ is given by a symmetric bilinear map defined as, [8]

$$\Gamma(f,g) = \frac{1}{2}\left[A(fg) - gAf - fAg\right] \tag{2.16}$$

Observe that, $\int_\Omega gAf \, d\mu = -\int_\Omega \Gamma(f,g)\, d\mu$.

Remark (Core of the generator)**.** *For the results we have obtained in this section it is often times enough to prove they hold on a core of the generator. A set* $\mathcal{D} \subset D(A)$ *is called a core of the generator* A *if the closure of the graph* $\mathcal{G}(A) = \{(f, Af) \colon f \in \mathcal{D}\}$ *is the complete graph* $G(A) = \{(f, Af) \colon f \in D(A)\}$*. I.e. for all* $(f, Af) \in G(A)$ $\exists (f_N, Af_N) \in \mathcal{G}(A)$ *for* $N \in \mathbb{N}$*, such that* $(f_N, Af_N) \xrightarrow{N\to\infty} (f, Af)$*.*

Chapter 3

Interacting Particle Systems: $\mathcal{SSEP}$ and the $\mathcal{SIP}$

As one can see from the title this chapter will be on two directly opposing particle systems. Opposing as they differ in their interaction properties; $\mathcal{SSEP}$ being repulsive in nature, and $\mathcal{SIP}$ being attractive. However before we delve into the main subject matter, we explore the general concept of an interacting particle system. An interacting particle system is a stochastic process $\{\eta_t \colon t \geq 0\}$, in particular a Markov process, in which the particles inhabit a lattice Σ (some countable infinite graph) in which they evolve according to the dynamics depending on the configuration of particles in a surrounding neighborhood [19]. Our choice for lattice Σ will be $\mathbb{N}^{\mathbb{Z}}$ in case of the symmetric inclusion process. For each $x \in \mathbb{Z}$ the local state we will denote by $\eta(x) \in \mathbb{N}$. I.e. $\eta(x) = \#\{\text{particles at site } x\}$.

The first most obvious question one can ask about stochastic processes concerns their distribution. What is the limiting distribution of the process? Meaning, what are the invariant measures? In this chapter this question will be answered for the $\mathcal{SSEP}$ and $\mathcal{SIP}(\alpha)$.

3.1 Symmetric Simple Exclusion Process

Before properly introducing the $\mathcal{SIP}(\alpha)$ we take a look at a closely related interacting particle system, namely the symmetric simple exclusion process ($\mathcal{SSEP}$). This is a system of indistinguishable particles in $\mathbb{Z}$ which evolves according to a nearest neighbour symmetric continuous-time random walk, in such a manner that no two particles are at the same site. I.e. a jump of a particle, at say position $x \in \mathbb{Z}$, does not take place if its jumpsite, either position $x-1$ or $x+1$, is already occupied (exclusion rule). Because of this particle repulsion we refer to this type of model as fermionic. Hence we are required to know the configuration of the system at each site. This is given by the mapping $\eta \colon \mathbb{Z} \longrightarrow \{0,1\}$, where we have $\eta(x) = 1$ whenever site $x \in \mathbb{Z}$ is occupied by a particle. So $\eta \in \{0,1\}^{\mathbb{Z}}$ is the configuration space of the particles and $\eta(x) = \#\{\text{particles at site } x \in \mathbb{Z}\}$.

The complete dynamics of the $\mathcal{SSEP}$ is captured by its generator $\mathcal{L}_{\mathcal{SSEP}}$. Let $\Omega := \{0,1\}^{\mathbb{Z}}$, which is a compact metric space. Then generator $\mathcal{L}_{\mathcal{SSEP}}$ acts on local functions. A function $f \colon \Omega \longrightarrow \mathbb{R}$ is called local if $f(\eta) = f(\xi)$ whenever $\eta_A = \xi_A$ for some finite set $A \subset \mathbb{Z}$. The minimal set of such sets like A is referred to as the dependence set of f. E.g. $f(\eta) = \eta(0)$ is local with set $A = \{0\}$ as its dependence set. On local functions the generator is given by 3.1.1. More formally, [2]

Definition 3.1.1. *The stochastic process* $\{\eta_t \colon t \geq 0\}$ *is Markovian and evolves according to the generator,*

$$\mathcal{L}_{\mathcal{SSEP}} f(\eta) := \sum_{x \in \mathbb{Z}} \eta(x)(1-\eta(x+1))(f(\eta^{x,x+1}) - f(\eta)) + \eta(x+1)(1-\eta(x))(f(\eta^{x+1,x}) - f(\eta)). \tag{3.1}$$

Where $\mathcal{L}_{\mathcal{SSEP}}$ *acts on bounded local functions* $f \colon \{0,1\}^{\mathbb{Z}} \longrightarrow \mathbb{R}$*, meaning that* f *depends only on a finite number of variables [7].* $\eta^{x,y}$ *denotes the configuration which will be obtained from exchanging* x *and* y*. I.e.,*

$$\eta^{x,y} := \begin{cases} \eta(z), & \text{if } z \notin \{x,y\} \\ \eta(x), & \text{if } z = y \\ \eta(y), & \text{if } z = x \end{cases}$$

They are a plethora of well known results on the symmetric simple exclusion process. Obtaining these types of results will also be our goal for the symmetric inclusion process. Ideas taken from the proofs of these results will also be handy for us. The follow proposition gives us the invariant measures of the $\mathcal{SSEP}$. These invariant measures tell us which measures are unaffected by the action of the semigroup on continuous functions.

Proposition 3.1.1 (Invariant Measures of $\mathcal{SSEP}$)**.** *The Bernoulli measures* $\{\nu_\rho \colon \rho \in [0,1]\}$ *are invariant for the* $\mathcal{SSEP}$*. In particular the symmetric simple exclusion process is self-adjoint w.r.t. each* ν_ρ*. Here* ν_ρ *is the product measure on* $\{0,1\}^{\mathbb{Z}}$*, such that* $\nu_\rho(\{\eta(x) = 1\}) = \rho = 1 - \nu_\rho(\{\eta(x) = 0\})$*. [10]*

The proof of this proposition can be found in [10].

3.2 Symmetric Inclusion Process

The symmetric inclusion process was first introduced in [11], [12], and can be considered as the direct bosonic counterpart to the fermionic symmetric simple exclusion process. The evolution in time of the process is straightforward to describe. We start of initially from some particle distribution on $\mathbb{Z}$, and assume no further restrictions on the number of particles per site $x \in \mathbb{Z}$. Then each particle evolves by equipping it with two exponential clocks. One representing the random walk jump with fixed rate α, and the other describing the inclusion event with corresponding rate 1. Each time the random walk clock activates the particle performs a random walk jump with probability $p(x,y)$, from site $x \longmapsto y \in \mathbb{Z}$. Whenever the inclusion event clock triggers the particle jumps from $y \longmapsto x \in \mathbb{Z}$ with probability $p(y,x)$. Thus giving rise to the following generator L_{SIP} acting on local functions,

$$L_{SIP} f(\eta) = \sum_{x,y \in \mathbb{Z}} p(x,y)\eta(x)(\alpha + \eta(y))(f(\eta^{x,y}) - f(\eta)). \tag{3.2}$$

Where $\eta^{x,y} = \eta - \delta_x + \delta_y$, the configuration obtained by switching a particle at site x to y. Notice that L_{SIP} corresponds with a Feller process whenever $\exists R \in \mathbb{Z}^+$ such that,

$$1)\ c(x,y,\eta) := p(x,y)\eta(x)(\alpha + \eta(y)) = 0,\ \forall x,y \in \mathbb{Z} \text{ and } |x-y| > R \tag{3.3}$$

$$2)\ c(x,y,\eta) = c(0, x-y, \eta). \tag{3.4}$$

Remark. *Note that the SIP can give rise to explosive behaviour. I.e. if the initial configuration has 'too many' particles at infinity, then one can have in finite time an infinite amount of particles arriving at some site $x \in \mathbb{Z}$. This issue is to be resolved by restricting the possible initial particle configurations, and by making certain assumptions on this. Moreover tools such as duality, see 4.1.2, will facilitate to show that the SIP is well defined. We refer the reader to the thesis of M.A.A. Valenzuela [1] for complete details on this.*

We are now ready to properly introduce and define the symmetric inclusion process with an inhomogeneous profile ($\mathcal{SIP}(\alpha)$). This again is an interacting particle system where each particle, say at position x, performs an independent random walk at rate $\alpha(x)$ with transition probability $c(\eta(x), \eta'(x))$. $\alpha\colon \mathbb{Z} \longrightarrow (0, \infty)$ is assumed to be a bounded function of x. This function can be viewed as the macroscopic inhomogeneous profile, inhomogeneous in the sense that it is not a constant function of x. Moreover we assume conditions 3.3 and 3.4 to hold true. We facilitate this by only allowing particle jumps to directly neighbouring sites, and setting $p(x, y) = 0$ whenever $|x - y| > 1$.

The complete dynamics of the $\mathcal{SIP}(\alpha)$ is captured by its generator. The generator of the symmetric inclusion process with a inhomogeneous profile is given by,

$$L_{\text{SIP}} f(\eta) = \sum_{x,y \in S} \eta(x)(\alpha(y) + \eta(y))(f(\eta^{x,y}) - f(\eta)) + \eta(y)(\alpha(x) + \eta(x))(f(\eta^{y,x}) - f(\eta)). \quad (3.5)$$

$S \coloneqq \{x, y \in \mathbb{Z} : |x - y| = 1\} \subseteq \mathbb{Z}$, the set containing all points in $\mathbb{Z}$ at distance 1 from x. For this type of processes the particles are 'invited', at a certain rate, to jump to its nearest neighbour, implying some kind of particle attraction. This type of IPS-models are thus called to be bosonic. More formally,

Definition 3.2.1. *The continuous-time Markov process $\{\eta_t \colon t \geq 0\} \in \mathbb{N}^{\mathbb{Z}} \coloneqq \Sigma$ evolves according to generator $\mathcal{L}_{\mathcal{SIP}(\alpha)}$ given by,*

$$\begin{aligned}\mathcal{L}_{\mathcal{SIP}(\alpha)} f(\eta) = &\sum_{x \in \mathbb{Z}} \eta(x)(\alpha(x+1) + \eta(x+1))(f(\eta^{x,x+1}) - f(\eta)) \\ &+ \eta(x+1)(\alpha(x) + \eta(x))(f(\eta^{x+1,x}) - f(\eta)).\end{aligned} \quad (3.6)$$

Where $f\colon \Sigma \longrightarrow \mathbb{R}$ is a local function. $\eta^{x,x+1} \in \Sigma$ denotes the configuration obtained when a particle jumps from x to $x+1$. Meaning that $\eta^{x,x+1} = \eta - \delta_x + \delta_{x+1}$. This occurs at rate $\eta(x)(\alpha(x+1) + \eta(x+1))$.

The generator captures the complete dynamics of the $\mathcal{SIP}(\alpha)$, but it's very formal. A more intuitive, graphical representation is given in figure 3.1.

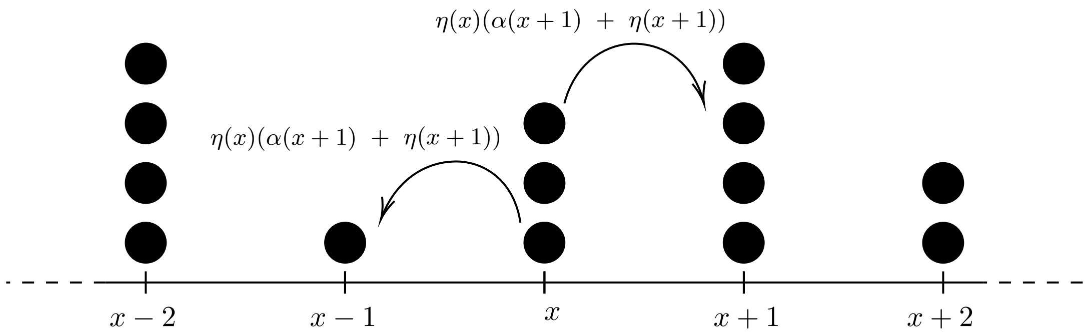

Figure 3.1: One-dimensional schematic description of the Symmetric Inclusion Process with Slowly Varying Inhomogeneities

3.3 Invariant Measures of the $\mathcal{SIP}(\alpha)$

The aim of this section is to compute the invariant measures of the $\mathcal{SIP}(\alpha)$. We can explicitly find the invariant measures by utilising the detailed balance relation. Proposition 2.3.2 shows us the equivalence relation between reversibility and detailed balance. Then by proposition 2.13 we are to conclude invariance. This leaves us to just explicitly compute the detailed balance relation. The following proposition gives the invariant measures of the $\mathcal{SIP}(\alpha)$.

Proposition 3.3.1 (Invariant measures $\mathcal{SIP}(\alpha)$)**.** *The reversible product measures of the $\mathcal{SIP}(\alpha)$ are given by $\mu_\lambda(\eta) = \bigotimes_{x\in\mathbb{Z}} \mu_\lambda^{\alpha(x)}(\eta(x))$, such that*

$$\mu_\lambda^{\alpha(x)}(\eta(x) = N) = \frac{1}{Z(\lambda, \alpha(x))} \frac{\lambda^N}{N!} \frac{\Gamma(\alpha(x) + N)}{\Gamma(\alpha(x))}. \tag{3.7}$$

Where $\lambda > 0$, and $Z(\lambda, \alpha(x)) = (1-\lambda)^{-\alpha(x)}$ is a constant such that each measure $\mu_\lambda^{\alpha(x)}$ indeed defines a probability measure on $\mathbb{N}$.

Proof: Assume that $\mu_\lambda(\eta) = \bigotimes_{x\in\mathbb{Z}} \mu_\lambda^{\alpha(x)}(\eta(x))$ is a reversible product measure. The detailed balance condition reads,

$$\mu(\eta)c(\eta, \eta') = \mu(\eta')c(\eta', \eta).$$

Note that $\eta \longrightarrow \eta' = \eta - \delta_x + \delta_{x+1}$, is occurring at rate $c(\eta, \eta') = \eta(x)(\alpha(x+1) + \eta(x+1))$. Therefore,

$$\mu_\lambda(\eta)(\alpha(x+1) + \eta(x+1))\eta(x) = \mu_\lambda(\eta - \delta_x + \delta_{x+1})(\alpha(x) + \eta(x) - 1)(\eta(x+1) + 1).$$

μ_λ is defined as the product measure $\bigotimes_{x\in\mathbb{Z}} \mu_\lambda^{\alpha(x)}(\eta(x)) = \mu_\lambda^{\alpha(x)}(\eta(x))\mu_\lambda^{\alpha(x+1)}(\eta(x+1))$, implying that,

$$\begin{aligned}&\mu_\lambda^{\alpha(x)}(\eta(x))\mu_\lambda^{\alpha(x+1)}(\eta(x+1))(\alpha(x+1) + \eta(x+1))\eta(x)\\&= \mu_\lambda^{\alpha(x)}(\eta(x) - 1)\mu_\lambda^{\alpha(x+1)}(\eta(x+1) + 1)(\alpha(x) + \eta(x) - 1)(\eta(x+1) + 1).\end{aligned}$$

Collecting all x and $x+1$ terms on each side yields,

$$\frac{\mu_\lambda^{\alpha(x)}(\eta(x))\eta(x)}{\mu_\lambda^{\alpha(x)}(\eta(x)-1)(\alpha(x)+\eta(x)-1)} = \frac{\mu_\lambda^{\alpha(x+1)}(\eta(x+1)+1)(\eta(x+1)+1)}{\mu_\lambda^{\alpha(x+1)}(\eta(x+1))(\alpha(x+1)+\eta(x+1))}.$$

As this holds for each $x \in \mathbb{Z}$ and for all $\eta(x) \in \mathbb{N}$ the fraction has to equal a constant $\lambda > 0$. So that,

$$\frac{\mu_\lambda^{\alpha(x)}(\eta(x))\eta(x)}{\mu_\lambda^{\alpha(x)}(\eta(x)-1)(\alpha(x)+\eta(x)-1)} = \lambda.$$

Rearranging the terms and setting $\mathbb{P}(\eta(x) = N) = \mu_\lambda^{\alpha(x)}(N)$ produces the following,

$$\frac{\mu_\lambda^{\alpha(x)}(N)}{\mu_\lambda^{\alpha(x)}(N-1)} = \lambda \cdot \frac{\alpha(x)+N-1}{N}.$$

Notice that the proposed solution for $\mu_\lambda^{\alpha(x)}$ in 3.7 satisfies the above recursion and thus is indeed the unique solution. Which obtains us the desired result,

$$\mu_\lambda^{\alpha(x)}(N) = \frac{1}{Z(\lambda,\alpha(x))}\frac{\lambda^N}{N!}\frac{\Gamma(\alpha(x)+N)}{\Gamma(\alpha(x))}.$$

$Z(\lambda, \alpha(x))$ is of course just a normalising constant that makes $\mu_\lambda^{\alpha(x)}(N)$ a probability measure. Implying that $Z(\lambda,\alpha(x)) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!}\frac{\Gamma(\alpha(x)+N)}{\Gamma(\alpha(x))}$. Straightforward calculation gives,

$$Z(\lambda,\alpha(x)) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!}\frac{\Gamma(\alpha(x)+N)}{\Gamma(\alpha(x))} = (1-\lambda)^{-\alpha(x)}.$$

Which confirms our claim. ■

Chapter 4

Duality for the $\mathcal{SIP}(\alpha)$

Duality is a very general and broad concept with numerous applications in many areas of mathematics. Sir M.F. Atiyah states, in [13], that: "Duality in mathematics is not a theorem, but a 'principle'. Duality gives two different points of view of looking at the same object."

In the field of interacting particles systems duality is widely applied. Properties of various stochastic processes, e.g. $\mathcal{SSEP}$ and $\mathcal{SIP}$, can be proved by means of duality. Something we also set out to obtain in this chapter. We show that the $\mathcal{SIP}(\alpha)$ is self-dual for the one-particle case, and by means of this duality relation we are able to compute the moments of our symmetric inclusion process.

4.1 Self-Duality for the $\mathcal{SIP}(\alpha)$

An important concept for the study of the $\mathcal{SIP}(\alpha)$ is duality. With duality we can reduce the complexity of infinitely many particles to problems only involving finitely many particles. This makes it one of the more powerful tools we can apply.

This section will focus on the self-duality properties of the $\mathcal{SIP}(\alpha)$. Our goal is to prove the self-duality by explicit computation. First we formally define the concept of duality. This is written in terms of the expectation. This can also be done via the semigroups or more relevantly on the level of generators. An alternate, but more useful definition via the generator of the $\mathcal{SIP}(\alpha)$ is also given.

Definition 4.1.1 (Duality between Markov Processes)**.** *Let η_t and ξ_t be two Markov processes with corresponding state spaces Ω and $\tilde{\Omega}$. Moreover let $D\colon \Omega \times \tilde{\Omega} \longrightarrow \mathbb{R}$ be a bounded measurable function. We call the two Markov processes to be* dual *w.r.t* D *if,*

$$\mathbb{E}_\eta D(\eta_t, \xi) = \mathbb{E}_\xi D(\eta, \xi_t) \tag{4.1}$$

for all $\eta \in \Omega$, $\xi \in \tilde{\Omega}$ and $t \geq 0$. [14]

The definition of the generator of the $\mathcal{SIP}(\alpha)$ is given in 2.2.1. Working with the generator is often times more convenient, thus we will make use of the following definition which defines duality via the generator.

Definition 4.1.2 (Duality between Generators)**.** *Consider two Markov processes with corresponding generators $\mathcal{L}$, and $\tilde{\mathcal{L}}$. We say that generator duality holds with duality function $D\colon \Omega \times \tilde{\Omega} \longrightarrow \mathbb{R}$ if,*

$$\mathcal{L}D(\,\cdot\,,\eta)(\xi) = \tilde{\mathcal{L}}D(\xi,\cdot\,)(\eta) \tag{4.2}$$

for all $\eta \in \Omega$ and $\xi \in \tilde{\Omega}$. Which we will from now on denote as,

$$\mathcal{L}_{left}D(\xi,\eta) = \tilde{\mathcal{L}}_{right}D(\xi,\eta). \tag{4.3}$$

If $\mathcal{L} = \tilde{\mathcal{L}}$, then we refer to this relation as self-duality. [14]

With this definition we should have enough in order to prove self-duality for the $\mathcal{SIP}(\alpha)$. However we do not know what this duality function D is. Thus we start of by identifying the self-duality functions. An effortless manner for obtaining duality functions is with via the so called cheap duality function. First we introduce the following proposition.

Proposition 4.1.1. *Let L be a self dual Markov generator with self-duality function $D\colon \Omega \times \Omega \longrightarrow \mathbb{R}$. Then if S is an operator that commutes with L, the function $\hat{D} = SD$ is also a self-duality function. [1]*

Proof: Let S be an operator commuting with L, then by the self-duality of D we obtain

$$\begin{aligned} L_{\text{left}}\hat{D}(x,\hat{x}) = L_{\text{left}}S_{\text{left}}D(x,\hat{x}) &= S_{\text{left}}L_{\text{left}}D(x,\hat{x}) \\ &= S_{\text{left}}L_{\text{right}}D(x,\hat{x}) \\ &= L_{\text{right}}S_{\text{left}}D(x,\hat{x}) \\ &= L_{\text{right}}\hat{D}(x,\hat{x}). \end{aligned}$$

Thus $L_{\text{left}}\hat{D}(x,\hat{x}) = L_{\text{right}}\hat{D}(x,\hat{x})$ holds for all $x, \hat{x} \in \Omega$. Hence, $\hat{D} = SD$ is also a self-duality function. ■

We know that for a finite state space Ω, L is represented by a square matrix. Consequently, self-duality function D is also a square matrix such that,

$$LD = DL^{\intercal}. \tag{4.4}$$

Consider now μ to be a reversible measure of the underlying Markov semigroup. Then by the detailed balance relation: $\mu(x)L(x,y) = \mu(y)L(y,x)$ for all $x, y \in \Omega$, we notice that diagonal matrix $D(x,y) = \frac{\delta_{x,y}}{\mu(x)}$ satisfies 4.4. In chapter 3 we have shown measures μ_λ to be reversible for the $\mathcal{SIP}(\alpha)$ 3.7. Implying, $D(\xi,\eta) = \prod_i \frac{\delta_{\xi_i,\eta_i}}{\mu_\lambda(\xi_i)}$ to be a cheap self-duality function. Formally,

Proposition 4.1.2. *Measures μ_λ are reversible for the $\mathcal{SIP}(\alpha)$, implying [14],*

$$D(\xi,\eta) = \prod_{x\in\mathbb{Z}} \eta(x)! \frac{\Gamma(\alpha(x))}{\Gamma(\alpha(x)+\eta(x))} \delta_{\xi_x,\eta_x} \tag{4.5}$$

is a cheap self-duality function.

A complete proof can be found in [14]. Here $\Gamma(\cdot)$ denotes the Gamma function. I.e.,

$$\Gamma(t) = \int_0^{\infty} x^{t-1} \mathrm{e}^{-x} dx$$

for $t > 0$. Remember that Γ obeys the recursion $\Gamma(t+1) = t\Gamma(t)$.

However this self-duality function is a bit too simplistic. Remember that in the finite state space case D corresponds with a diagonal matrix. The following proposition gives us the duality functions we are most interested in. They will turn out to be key for computing the hydrodynamic limit of our symmetric inclusion process.

Proposition 4.1.3 (Self-duality $\mathcal{SIP}(\alpha)$)**.** *The symmetric inclusion process with inhomogeneous profile $\alpha(x)$, as defined in 3.2.1, is self-dual with self-duality function*

$$D(\xi, \eta) = \prod_{x \in \mathbb{Z}} d(\xi(x), \eta(x)).$$

Where,

$$d(k, n) := \frac{n!}{(n-k)!} \frac{\Gamma(\alpha(x))}{\Gamma(\alpha(x) + k)} \mathbb{1}_{\{k \leq n\}}, \quad k, n \in \mathbb{N}. \tag{4.6}$$

Remark that when $\xi = \delta_x$, then $D(\xi, \eta) = \frac{\eta(x)}{\alpha(x)}$. [14]

Proof: The complete proof for the general case relies on the abstract form of the generator, which is something outside the scope of this thesis. The complete proof can be found in [14]. The proof presented here will just cover the one-particle case, as this is the main thing we need for the hydrodynamic limit. Meaning that we will show,

$$\mathcal{L}_{\text{left}} D(\delta_x, \eta) = \mathcal{L}_{\text{right}} D(\delta_x, \eta).$$

Where we write $\mathcal{L} = \mathcal{L}_{\mathcal{SIP}(\alpha)}$. Start by noticing that we can rewrite the generator $\mathcal{L}$ as,

$$\begin{aligned} \mathcal{L}f(\eta) = \sum_{x \in \mathbb{Z}} & \eta(x)(\alpha(x+1) + \eta(x+1))(f(\eta^{x,x+1}) - f(\eta)) \\ & + \eta(x)(\alpha(x-1) + \eta(x-1))(f(\eta^{x,x-1}) - f(\eta)). \end{aligned}$$

Before we use this expression for $\mathcal{L}$ note that, for $z \in \mathbb{Z}$ and $\eta = \delta_z$:

$$\begin{aligned} \eta = \delta_z &\longmapsto \delta_{z+1} = \eta' \text{ with rate } \alpha(x+1) \\ \eta = \delta_z &\longmapsto \delta_{z-1} = \eta' \text{ with rate } \alpha(x-1). \end{aligned}$$

Moreover notice that,

$$\begin{aligned} \mathcal{L}\eta(x) &= \eta(x+1)(\alpha(x) + \eta(x)) + \eta(x-1)(\alpha(x) + \eta(x)) \\ &\quad - \eta(x)(\alpha(x+1) + \eta(x+1)) - \eta(x)(\alpha(x-1) + \eta(x-1)) \\ &= \eta(x+1)\alpha(x) + \eta(x-1)\alpha(x) - \eta(x)\alpha(x+1) - \eta(x)\alpha(x-1). \end{aligned}$$

Hence we derive,

$$\begin{aligned}
\mathcal{L}_{\text{right}} D(\delta_x, \eta) &= \mathcal{L}\left(\frac{\eta(x)}{\alpha(x)}\right) \\
&= \eta(x+1) + \eta(x-1) - \frac{\eta(x)}{\alpha(x)}\alpha(x+1) - \frac{\eta(x)}{\alpha(x)}\alpha(x-1) \\
&= \alpha(x+1)\left(\frac{\eta(x+1)}{\alpha(x+1)} - \frac{\eta(x)}{\alpha(x)}\right) + \alpha(x-1)\left(\frac{\eta(x-1)}{\alpha(x-1)} - \frac{\eta(x)}{\alpha(x)}\right) \\
&= \alpha(x+1)\left(D(x+1,\eta) - D(x,\eta)\right) + \alpha(x-1)\left(D(x-1,\eta) - D(x,\eta)\right) \\
&= \mathcal{L}_{\text{left}} D(\delta_x, \eta).
\end{aligned}$$

■

We see that the $\mathcal{SIP}(\alpha)$ is indeed self-dual, at least for the one-particle case. Moreover notice that we can write,

$$\begin{aligned}
\mathcal{L}_{\text{right}} D(\delta_x, \eta) &= \alpha(x+1)\left(\frac{\eta(x+1)}{\alpha(x+1)} - \frac{\eta(x)}{\alpha(x)}\right) + \alpha(x-1)\left(\frac{\eta(x-1)}{\alpha(x-1)} - \frac{\eta(x)}{\alpha(x)}\right) \\
&= \alpha(x+1)(w(x+1) - w(x)) + \alpha(x-1)(w(x-1) - w(x)). \\
&= \hat{\mathcal{L}}_{RW}\left(w(x)\right)
\end{aligned}$$

Where $w(x) = D(x, \eta)$, and $\hat{\mathcal{L}}_{RW}$ is the generator of a simple random walk that jumps from $x \longmapsto x+1 \in \mathbb{Z}$ with rate $\alpha(x+1)$, and from $x \longmapsto x-1 \in \mathbb{Z}$ at rate $\alpha(x-1)$. This fact we will extensively utilise in chapter 5 for the hydrodynamic limit of the $\mathcal{SIP}(\alpha)$.

We end this section by concluding the following theorem from proposition 4.1.3.

Theorem 4.1.1. *Given a finite particle configuration ξ and $\eta \in \Omega$ we have,*

$$\mathbb{E}_{\eta}^{\mathcal{SIP}} D(\xi, \eta_t) = \mathbb{E}_{\xi}^{\mathcal{SIP}} D(\xi_t, \eta). \tag{4.7}$$

It is important to remark that the self-duality function is not unique. The main reason we nevertheless select this specify duality function 4.6 is that it allows particle configuration η to be infinite and ξ finite.

4.2 Application of the $\mathcal{SIP}(\alpha)$ self-duality

As we have seen duality is a handy tool to have in our arsenal. Its powerfulness is not just found in theorems such as 4.7, but also in its applications for the symmetric inclusion process. A first of such is its use in calculating the moments of the $\mathcal{SIP}(\alpha)$. The following proposition gives us the first moment of the $\mathcal{SIP}(\alpha)$ with respect to the invariant measure $\mu_\lambda^{\alpha(x)}$. This result also obtains us the relation between $\mu_\lambda^{\alpha(x)}$ and the self-duality polynomials $d(k,n)$ 4.6.

Proposition 4.2.1. *For all $k \le n$, and $\lambda \in (0,1)$ we have,*

$$\mathbb{E}_{\mu_\lambda^{\alpha(x)}} d(k,\cdot) = \sum_{n=k}^{\infty} d(k,n)\mu_\lambda^{\alpha(x)}(n) = \left(\frac{\lambda}{1-\lambda}\right)^k.$$

Consequently,

$$\int D(\xi,\eta)\mu_\lambda^{\alpha(x)}(d\eta) = \left(\frac{\lambda}{1-\lambda}\right)^{|\xi|}.$$

Proof:

$$\begin{aligned} I(\lambda) := \mathbb{E}_{\mu_\lambda^{\alpha(x)}} d(k,\cdot) &= \sum_{n=k}^{\infty} d(k,n)\mu_\lambda^{\alpha(x)}(n) \\ &= \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\frac{\Gamma(\alpha(x))}{\Gamma(\alpha(x)+k)}\frac{\lambda^n}{n!}\frac{\Gamma(\alpha(x)+n)}{\Gamma(\alpha(x))}(1-\lambda)^{\alpha(x)} \end{aligned}$$

See that we can cancel out certain factors and apply the substitution $m = n-k$. This leaves us with,

$$\begin{aligned} I(\lambda) &= (1-\lambda)^{\alpha(x)} \sum_{m=0}^{\infty} \frac{\lambda^{m+k}}{m!}\frac{\Gamma(\alpha(x)+m+k)}{\Gamma(\alpha(x)+k)} \\ &= (1-\lambda)^{\alpha(x)} \sum_{m=0}^{\infty} \frac{\lambda^{m+k}}{m!}\frac{1}{\Gamma(\alpha(x)+k)} \int_0^\infty t^{\alpha(x)+m+k-1}\mathrm{e}^{-t}dt \\ &= (1-\lambda)^{\alpha(x)}\lambda^k \sum_{m=0}^{\infty} \int_0^\infty \frac{t^{\alpha(x)+k-1}}{\Gamma(\alpha(x)+k)}\frac{(\lambda t)^m}{m!}\mathrm{e}^{-t}dt. \end{aligned}$$

Where the first equality is justified by definition of the Γ-function. Fubini lets us interchange summation and integration, and note that by definition $\sum_{m=0}^{\infty}\frac{(\lambda t)^m}{m!} = \mathrm{e}^{\lambda t}$. Thus,

$$I(\lambda) = (1-\lambda)^{\alpha(x)}\lambda^k \int_0^\infty \frac{t^{\alpha(x)+k-1}}{\Gamma(\alpha(x)+k)}\mathrm{e}^{-(1-\lambda)t}dt.$$

Observe that multiplying the integral with $(1-\lambda)^{\alpha(x)+k}$ we precisely obtain the Γ-distribution. Implying that, after multiplication by $(1-\lambda)^{\alpha(x)+k}$, the integral equates to 1. Hence,

$$
\begin{aligned}
I(\lambda) = \frac{(1-\lambda)^{\alpha(x)+k}}{(1-\lambda)^{\alpha(x)+k}} I(\lambda) &= (1-\lambda)^{\alpha(x)} \lambda^k \frac{1}{(1-\lambda)^{\alpha(x)+k}} \frac{(1-\lambda)^{\alpha(x)+k}}{\Gamma(\alpha(x)+k)} \int_0^\infty t^{\alpha(x)+k-1} \mathrm{e}^{-(1-\lambda)t} dt \\
&= \frac{(1-\lambda)^{\alpha(x)} \lambda^k}{(1-\lambda)^{\alpha(x)+k}} \cdot 1 \\
&= \left(\frac{\lambda}{1-\lambda} \right)^k .
\end{aligned}
$$

This proves the first part of the proposition. To complete the proof note that for $k = 1$ the obtained result reads as, $\frac{\lambda}{1-\lambda} = \int D(\delta_x, \eta) \mu_\lambda^{\alpha(x)}(d\eta)$. Consequently,

$$
\int D(\xi,\eta) \mu_\lambda^{\alpha(x)}(d\eta) = \prod_{x\in\mathbb{Z}} \int D(\xi_x, \eta) \mu_\lambda^{\alpha(x)}(d\eta(x)) = \prod_{x\in\mathbb{Z}} \left(\frac{\lambda}{1-\lambda} \right)^{\xi_x} = \left(\frac{\lambda}{1-\lambda} \right)^{|\xi|} .
$$

■

Chapter 5

Hydrodynamic Limit

In this final chapter we present the main result of this thesis, of which itself concerns with the so called hydrodynamic limit of the $\mathcal{SIP}(\alpha)$. In essence the hydrodynamic limit describes the global behaviour of an interacting particle system over vast space and time scales, where it is assumed that the initial particle distribution varies in space [15]. The hydrodynamic limit tries to make rigours the transition from 'micro' to 'macro'. Before we formalize and substantiate all of these terms we revisit the $\mathcal{SIP}(\alpha)$, in particular its generator $\mathcal{L}_{\mathcal{SIP}}$.

We are interested in describing the evolution of particles in an inhomogeneous environment, for instance in a layered material where each layer has different properties. These inhomogeneities are taken into account by the attractiveness of each site of the lattice, namely the parameters $(\alpha(x))_{x\in\mathbb{Z}}$. More precisely, we assume to have a bounded and smooth macroscopic profile

$$\mathbb{R} \ni x \longmapsto \alpha(x) > 0.$$

For any value of the scaling parameter $N \in \mathbb{N}$ we use this macroscopic profile to facilitate the transition from micro to macro by inducing the inhomogeneities,

$$\alpha_N(x) = \alpha\left(\frac{x}{N}\right) \qquad x \in \mathbb{Z} \tag{5.1}$$

in the microscopic dynamics. The scaling parameter N is there to rescale the mass, time, and space. Notice that when N is sufficiently large, the neighboring particle positions $\frac{x}{N}$ and $\frac{x+1}{N}$ will be very close together at a distance $\frac{1}{N}$, implying that $\alpha_N(x) \approx \alpha_N(x+1)$ due to smoothness of profile α. This is why we refer to the profile being slowly varying, as α varies slowly in space. Formally adaptation of the generator $\mathcal{L}_{\mathcal{SIP}}$ yields the following generator.

Definition 5.0.1. *The continuous-time Markov process $\{\eta_t : t \geq 0\} \in \mathbb{N}^{\mathbb{Z}} := \Sigma$ evolves according to generator $\mathcal{L}^N_{\mathcal{SIP}(\alpha)}$ given by,*

$$\begin{aligned}\mathcal{L}^N_{\mathcal{SIP}(\alpha)} f(\eta) = &\sum_{x\in\mathbb{Z}} \eta(x)(\alpha_N(x+1) + \eta(x+1))(f(\eta^{x,x+1}) - f(\eta)) \\ &+ \eta(x+1)(\alpha_N(x) + \eta(x))(f(\eta^{x+1,x}) - f(\eta)).\end{aligned} \tag{5.2}$$

Where $f : \Sigma \longrightarrow \mathbb{R}$ is a local function. $\eta^{x,x+1} \in \Sigma$ denotes the configuration obtained when a particle jumps from x to $x+1$. Meaning that $\eta^{x,x+1} = \eta - \delta_x + \delta_{x+1}$. This occurs at rate $\eta(x)(\alpha_N(x+1) + \eta(x+1))$.

For convenience we will denote $L_{SIP}^N := \mathcal{L}_{\mathcal{SIP}(\alpha)}^N$.

We begin by initializing our continuous-time Markov process $\{\eta_t\colon t \geq 0\}$ by a probability measure μ on Σ and we call μ the initial distribution. $\{\eta_t\colon t \geq 0\}$ describes the dynamics of our process at microscopic scale in $\mathbb{Z}$. We now desire to go from this microscopic description to a macroscopic one in $\mathbb{R}$, thus rescaling the distance between points by a factor $1/N$. Formally, a macroscopic point $x \in \mathbb{R}$ corresponds with the microscopic point $\lfloor Nx \rfloor \in \mathbb{Z}$. Now in order to formalize going from 'micro' to 'macro' we need to define the rescaled empirical density field.

Definition 5.0.2. *The rescaled empirical density field $\{X_t^N\colon t \geq 0\}$ is a process in $D\left([0,\infty), \mathcal{S}'(\mathbb{R})\right)$ for each $N \in \mathbb{N}$. $D\left([0,\infty), \mathcal{S}'(\mathbb{R})\right)$ is the Skorokhod space of $\mathcal{S}'(\mathbb{R})$-valued cádlág orbits, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz class of rapidly decreasing functions on $\mathbb{R}$. $\mathcal{S}'(\mathbb{R})$ is its topological dual. Given the interacting particle system $\{\eta_t\colon t \geq 0\}$ evolving via 5.0.1, then the rescaled empirical density field is given by [7],*

$$X_t^N = \frac{1}{N} \sum_{x \in \mathbb{Z}} \delta_{\frac{x}{N}} \eta_{tN^2}(x) \quad t \geq 0. \tag{5.3}$$

Moreover, for any test function $\Phi \in \mathcal{S}(\mathbb{R})$ the rescaled empirical density field evaluated at Φ is given by

$$X_t^N(\Phi) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \Phi\left(\frac{x}{N}\right) \eta_{tN^2}(x) \quad t \geq 0. \tag{5.4}$$

Remark. *Notice in 5.3 the time rescaling constant N^2. This rescaling is done as the lattice is shrunk by $\frac{1}{N}$. A particle roughly moves a distance in the order of $t^{\frac{1}{2}}N^{-1}$, in a fixed time t. Implying that if we want to observe macroscale dynamics we have to rescale time t by N^2.*

Note that X_t^N assigns mass $\frac{1}{N}$ at point $\frac{x}{N} \in \mathbb{R}$ whenever at time t site $x \in \mathbb{Z}$ is occupied by a particle. Technically speaking X_t^N is a Radon measure on $\mathbb{R}$. I.e. X_t^N is a nonnegative Borel measure with possibly infinite total mass, but it does assign finite mass to bounded sets. The space of Radon measures is a so called Polish space equipped with the vague topology. Meaning that, in this space of Radon measures, convergence is defined via convergence of integrals of compactly supported functions. The goal now is to derive a law of large numbers type of result for the density field $X_t^N = \int_{\mathbb{R}} \Phi(x) X_t^N(dx)$. Formally speaking we desire to obtain a result such as the following.

$$\text{If } X_0^N(\Phi) \xrightarrow{N\to\infty} \int_{\mathbb{R}} \Phi(x)\rho(x)dx, \text{ then } X_t^N(\Phi) \xrightarrow{N\to\infty} \int_{\mathbb{R}} \Phi(x)\rho(t,x)dx. \tag{5.5}$$

Where $\rho(t,x)$ is the unique weak solution to some Cauchy problem.

The following examples serve to illustrate the reader as to how one goes about obtaining such results.

Example 5.0.1 (IID case)**.** *The most uncomplicated case one can have is if the process η_t is IID with mean ρ, and finite variance bounded by some ρ dependent constant σ_ρ. With convergence we mean that,*

$$\int_{\mathbb{R}} \Phi(x) dX_t^N \xrightarrow{N\to\infty} \int_{\mathbb{R}} \Phi(x)\rho dx \tag{5.6}$$

holds for every test function $\Phi \in C_c^\infty(\mathbb{R})$. We prove consistency of the empirical density field. I.e.,

$$X_t^N(\Phi) \xrightarrow{\mathbb{P}} \int_{\mathbb{R}} \Phi(x)\rho dx. \tag{5.7}$$

We start by noticing that,

$$\mathbb{E}X_t^N(\Phi) \overset{(IID)}{=} \frac{1}{N}\sum_{x\in\mathbb{Z}} \Phi\left(\frac{x}{N}\right) \mathbb{E}\eta_{tN^2}(x) = \frac{1}{N}\sum_{x\in\mathbb{Z}} \Phi\left(\frac{x}{N}\right)\rho \xrightarrow{N\to\infty} \int_{\mathbb{R}} \Phi(x)\rho dx.$$

Moreover by Chebyshevs inequality we obtain,

$$\mathbb{P}\left(\left|X_t^N(\Phi) - \int_{\mathbb{R}} \Phi(x)\rho dx\right| > \epsilon\right) \le \frac{1}{\epsilon^2}\mathbb{V}ar X_t^N(\Phi) \tag{5.8}$$

holds for all $\epsilon > 0$.

Computing the variance yields,

$$\mathbb{V}ar X_t^N(\Phi) \overset{(IID)}{=} \frac{1}{N^2}\sum_{x\in\mathbb{Z}} \Phi^2\left(\frac{x}{N}\right) \mathbb{V}ar(\eta_{tN^2}(x)) \le \frac{\sigma_\rho}{N} \cdot \sum_{x\in\mathbb{Z}} \Phi^2\left(\frac{x}{N}\right) \xrightarrow{N\to\infty} 0 \cdot \int_{\mathbb{R}} \Phi(x)dx = 0.$$

And thus by 5.8 we conclude convergence in probability of our density field. [1]

Example 5.0.2 (IND case)**.** *We have a similar setting as previous example, however we relax the IID assumption to just independence. Moreover assume that $\eta_t \overset{d}{=} \bigotimes \mu_N$ such that,*

$$\mathbb{E}_{\mu_N}\eta(x) = \rho\left(\frac{x}{N}\right)$$
$$\mathbb{V}ar_{\mu_N}(\eta(x)) = \psi\left(\frac{x}{N}\right).$$

We also impose the following requirement on ψ,

$$\int_{\mathbb{R}} \psi(x)\Phi^2(x)dx < \infty \tag{5.9}$$

for all test functions $\Phi \in C_c^\infty(\mathbb{R})$.

Just as previously we can use Chebyshev to bound the probability by the variance. Computation of the variance yields,

$$\mathbb{V}ar X_t^N(\Phi) \overset{(IND)}{=} \frac{1}{N^2}\sum_{x\in\mathbb{Z}} \Phi^2\left(\frac{x}{N}\right) \mathbb{V}ar(\eta(x)) = \frac{1}{N}\cdot\frac{1}{N}\sum_{x\in\mathbb{Z}} \Phi^2\left(\frac{x}{N}\right)\psi\left(\frac{x}{N}\right)$$
$$\xrightarrow{N\to\infty} 0 \cdot \int_{\mathbb{R}} \psi(x)\Phi^2(x)dx \overset{5.9}{=} 0.$$

And thus we again conclude convergence in probability of our density field. [1]

In the remaining part of this chapter we prove a similar like result for our process the $\mathcal{SIP}(\alpha)$. Proving this theorem will be done in two major steps that are outlined in the following sections. First we show the convergence of the expectation with respect to a sequence of initial distributions of $X_t^N(\Phi)$ for any time $t \geq 0$. In order to do so we apply the self-duality property of the $\mathcal{SIP}(\alpha)$. In the second part of the proof we show that a certain Dynkyn martingale has vanishing quadratic variation. For both parts of the proof we need the following assumption.

Assumption 5.0.1. *Let $\rho\colon \mathbb{R} \longrightarrow \mathbb{R}_+$ be a continuous and bounded function. A sequence of probability measure $\{\mu_n\}_{N\in\mathbb{N}}$ on Σ is said to be compatible with ρ if*

$$\mathbb{E}_{\mu_N}(\eta(x)) = \rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) \quad \textit{for all } N \in \mathbb{N}, \tag{5.10}$$

$$\sup_{x\in\mathbb{Z}} \mathbb{E}_{\mu_N}\eta(x)^2 \leq A < \infty. \tag{5.11}$$

And,

$$X_0^N(\Phi) \xrightarrow{\mu_N} \int_{\mathbb{R}} \Phi(x)\rho(x)\alpha(x)dx$$

for all test functions $\Phi \in C_c^\infty(\mathbb{R})$.

We now have all the ingredients needed to state the main theorem of this thesis.

Theorem 5.0.1 (Hydrodynamic Limit $\mathcal{SIP}(\alpha)$)**.** *Let $\rho\colon \mathbb{R} \longrightarrow \mathbb{R}_+$ be a continuous and bounded profile and let $\{\mu_N\}$ be a sequence of probability measure in Σ compatible with ρ (see the above assumption). Then,*

$$X_t^N(\Phi) \xrightarrow{\mathbb{P}_{\mu_N}} \int_{\mathbb{R}} \Phi(x)\rho(t,x)\alpha(x)dx$$

for all test functions $\Phi \in C_c^\infty(\mathbb{R})$, where $[0,\infty) \times \mathbb{R} \ni (t,x) \longrightarrow \rho(t,x)$ solves the following Cauchy problem

$$\begin{cases} \rho_t = 2\alpha'(x)\rho_x + \alpha(x)\rho_{xx} \\ \rho(0,x) = \rho(x). \end{cases} \tag{5.12}$$

Remark. *Note that in our main theorem we consider X_t^N as a measure-valued process obtained as a function of $\eta = \{\eta_t\colon t \geq 0\}$. Here η is our $\mathcal{SIP}(\alpha)$ process.*

Moreover the sequence of probability measures we consider is $\mu = (\mu_N)_{N\in\mathbb{N}}$ on the configuration space $\Sigma = \mathbb{N}^{\mathbb{Z}}$, for all $N \in \mathbb{N}$. The initial measure is denoted by μ_N. $\mathbb{P}_{\mu_N}$ is the probability measure on Skorokhod space $D([0,\infty),\Sigma)$.

5.1 Identification of the Hydrodynamic Equation

This section is dedicated to identifying the hydrodynamic equation by showing the convergence of our density field. Using the self-duality property of the simple inclusion process yields 4.1.3,

$$\begin{aligned}\mathbb{E}_\eta X_t^N(\Phi) &= \mathbb{E}_\eta\left[\frac{1}{N}\sum_{x\in\mathbb{Z}}\Phi\left(\frac{x}{N}\right)\frac{\eta_{tN^2}(x)}{\alpha_N(x)}\alpha_N(x)\right]\\ &\overset{\text{(self-duality)}}{=} \frac{1}{N}\sum_{x\in\mathbb{Z}}\Phi\left(\frac{x}{N}\right)\mathbb{E}_x^{\mathrm{RW}}\left(\frac{\eta_0(X(tN^2))}{\alpha_N(X(tN^2))}\right)\alpha_N(x)\\ &= \frac{1}{N}\sum_{x\in\mathbb{Z}}\Phi\left(\frac{x}{N}\right)\left(S_{tN^2}^{\mathrm{RW}}\psi\right)(x)\alpha_N(x).\end{aligned}$$

Where $S_{tN^2}^{\mathrm{RW}}$ is the semigroup defined as the expectation w.r.t. an one-particle random walk starting from $x\in\mathbb{Z}$, and $\psi(x) := \frac{\eta(x)}{\alpha_N(x)}$. I.e. $\psi(x)$ is the self-duality function.

In order to make progress further we need to examine this semigroup. In particular we want to examine adjointness properties. Lemma 5.3.2 gives us that $S_{tN^2}^{\mathrm{RW}}$ is self-adjoint, as self-adjointness of generators implies self-adjointness of the associated semigroups (theorem 19.25 [17]). In our case it means that $S_{tN^2}^{\mathrm{RW}}$ is self-adjoint on $l^2(\mathbb{Z},\alpha_N)$ with inner product $\langle f,g\rangle_{l^2(\mathbb{Z},\alpha_N)} = \sum_{x\in\mathbb{Z}} f(x)g(x)\alpha_N(x)$. Hence we derive,

$$\begin{aligned}\mathbb{E}_\eta X_t^N(\Phi) = \frac{1}{N}\langle\Phi_N, S_{tN^2}^{\mathrm{RW}}\psi\rangle_{\alpha_N} \overset{\text{(self-adjoint)}}{=} \frac{1}{N}\langle S_{tN^2}^{\mathrm{RW}}\Phi_N,\psi\rangle_{\alpha_N} &= \frac{1}{N}\sum_{x\in\mathbb{Z}}(S_{tN^2}^{\mathrm{RW}}\Phi_N)(x)\psi(x)\alpha_N(x)\\ &= \frac{1}{N}\sum_{x\in\mathbb{Z}}(S_{tN^2}^{\mathrm{RW}}\Phi_N)(x)\eta(x).\end{aligned}$$

Where $\Phi_N = \Phi\left(\frac{x}{N}\right)$. Notice that if we integrate out, w.r.t. the initial measure μ_N, and use assumption 5.0.1 we obtain the following.

$$\begin{aligned}\mathbb{E}_{\mu_N}(X_t^N(\Phi)) = \int_{\mathbb{R}}\mathbb{E}_\eta X_t^N(\Phi)\mu_N(d\eta(x)) &= \frac{1}{N}\sum_{x\in\mathbb{Z}}(S_{tN^2}^{\mathrm{RW}}\Phi_N)(x)\mathbb{E}_{\mu_N}(\eta(x))\\ &= \frac{1}{N}\sum_{x\in\mathbb{Z}}(S_{tN^2}^{\mathrm{RW}}\Phi_N)(x)\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right).\end{aligned}$$

We see that the only manner for acquiring convergence of this summation is by convergence of the semigroup or equivalently the convergence of the corresponding generator [17]. Precisely Lemma 5.3.1 gives us that the random walk generator converges to some limiting generator which corresponds to a diffusion process on $\mathbb{R}$. Thus we have that semigroup $S_{tN^2}^{\mathrm{RW}}$ converges uniformly to the limiting semigroup $\mathcal{S}_t$. Here $\mathcal{S}_t$ is the semigroup that corresponds to the diffusion process $\bar{x}(t)$, such that $\bar{x}(t)$ is the solution to the stochastic differential equation: $dX_t = 2\alpha'(X_t)dt + \sqrt{2\alpha(X_t)}dW_t$. Moreover from lemma 5.3.1 and [17], we also conclude that

$$\frac{X(tN^2)}{N} \xrightarrow{d} \bar{x}(t) \qquad \text{as} \quad N\to\infty.$$

We are now left to show that indeed,
$\frac{1}{N}\sum_{x\in\mathbb{Z}}(S_{tN^2}^{\mathrm{RW}}\Phi)\left(\frac{x}{N}\right)\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) \xrightarrow{N\to\infty} \int_{\mathbb{R}}(\mathcal{S}_t\Phi)(x)\rho(x)\alpha(x)dx.$

To see why this result holds true first note that the limiting integral, by definition of the Riemann integral, has a representation as the limit of its Riemann sum. I.e.,

$$I(N) := \frac{1}{N}\sum_{x\in\mathbb{Z}}(\mathcal{S}_t\Phi)\left(\frac{x}{N}\right)\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) \xrightarrow{N\to\infty} \int_{\mathbb{R}}(\mathcal{S}_t\Phi)(x)\rho(x)\alpha(x)dx.$$

Using this representation we can derive the following,

$$\begin{aligned}
&\Big|\frac{1}{N}\sum_{x\in\mathbb{Z}}(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) - \int_{\mathbb{R}}(\mathcal{S}_t\Phi)(x)\rho(x)\alpha(x)dx\Big| \\
&\overset{\Delta-\text{inequality}}{\leq} \Big|\frac{1}{N}\sum_{x\in\mathbb{Z}}(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) - I(N)\Big| + \Big|I(N) - \int_{\mathbb{R}}(\mathcal{S}_t\Phi)(x)\rho(x)\alpha(x)dx\Big|.
\end{aligned}$$

Implying we only need to show that the first term becomes arbitrarily small, as the other term, by definition of the Riemann integral, converges. We are left to show that,

$$\begin{aligned}
&\Big|\frac{1}{N}\sum_{x\in\mathbb{Z}}(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) - \frac{1}{N}\sum_{x\in\mathbb{Z}}(\mathcal{S}_t\Phi)\left(\frac{x}{N}\right)\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right)\Big| \\
&\leq \frac{1}{N}\sum_{x\in\mathbb{Z}}\Big|(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right) - (\mathcal{S}_t\Phi)\left(\frac{x}{N}\right)\Big|\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) \xrightarrow{N\to\infty} 0.
\end{aligned}$$

Using that $\rho\left(\frac{x}{N}\right)$ is bounded by some constant $M < \infty$, we have that

$$\begin{aligned}
&\frac{1}{N}\sum_{x\in\mathbb{Z}}\Big|(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right) - (\mathcal{S}_t\Phi)(x)\Big|\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) \\
&\leq \frac{M}{N}\sum_{x\in\mathbb{Z}}\Big|(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right) - (\mathcal{S}_t\Phi)(x)\Big|\alpha\left(\frac{x}{N}\right).
\end{aligned}$$

We are now left to prove that indeed,

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}\Big|(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right) - (\mathcal{S}_t\Phi)\left(\frac{x}{N}\right)\Big|\alpha\left(\frac{x}{N}\right) \xrightarrow{N\to\infty} 0.$$

Proving this will require some analytical machinery. Nagy, in [16], applies a variation of Scheffé's lemma.

Lemma 5.1.1 (Scheffé). *Suppose that $f \in L^1(\mathbb{R})$ and let $f_N \in L^1(\mathbb{R})$, $N \in \mathbb{N}^+$ be a sequence of functions satisfying,*

$$
\begin{aligned}
&(a) f_N \geq 0,\\
&(b) f_N(x) \xrightarrow{N\to\infty} f(x),\ \forall x \in \mathbb{R} \text{ and}\\
&(c) \int_{\mathbb{R}} f_N(x)dx \xrightarrow{N\to\infty} \int_{\mathbb{R}} f(x)dx.
\end{aligned}
$$

Then, $\int_{\mathbb{R}} \left|f_N(x) - f(x)\right|dx \xrightarrow{N\to\infty} 0.$

Before we apply this lemma 5.1.1 we need to clearly define the sequence f_N and potential limiting function f, in such manner that both f_N and f are $L^1(\mathbb{R})$ functions. Consider,

$$
\begin{aligned}
f_N(y) &= \sum_{x\in\mathbb{Z}} (S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right)\mathbb{1}_{[\frac{x}{N},\frac{x+1}{N})}(y)\\
f(y) &= (\mathcal{S}_t\Phi)(y)\alpha(y).
\end{aligned}
$$

Note that $f_N, f \in L^1(\mathbb{R})$. Now we are ready to apply Scheffé's lemma. We check all conditions of the lemma step by step.

Condition (a) is trivially true, as each term $(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) \geq 0$.

We also claim condition (b) to hold true. I.e., $f_N(y) \xrightarrow{N\to\infty} f(y)$ pointwise for each $y \in \mathbb{R}$.

Proof: Lemma 5.3.1 gives us the generator uniform convergence $L^{\mathrm{RW}}_N \xrightarrow{N\to\infty} \mathcal{L}$. This uniform convergence is there also on the level of the semigroups [17], i.e.
$\left(S^{\mathrm{RW}}_{tN^2}\Phi\right)(x) \xrightarrow{N\to\infty} (\mathcal{S}_t\Phi)(x)$ uniform in x. Implying that $\left(S^{\mathrm{RW}}_{tN^2}\Phi\right)(y_N) \xrightarrow{N\to\infty} (\mathcal{S}_t\Phi)(y)$ in case $y_N \to y$ uniformly as $N \to \infty$. For our case we want to show
$\left(S^{\mathrm{RW}}_{tN^2}\Phi\right)(y_N)\alpha(y_N) \xrightarrow{N\to\infty} (\mathcal{S}_t\Phi)(y)\alpha(y)$ pointwise in y_N. Intuitively this should hold as we already have uniform convergence of the semigroup, and moreover by assumption α is a smooth and bounded function.

Let $\epsilon > 0$ arbitrarily and choose $y_N = \frac{\lfloor Ny\rfloor}{N}$, so that clearly $y_N \xrightarrow{N\to\infty} y$ uniformly in y. Then from some sufficiently large n onward we have,

$$
\begin{aligned}
|f_n(y) - f(y)| &= \left|\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n)\alpha(y_n) - \alpha(y)\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n) + \alpha(y)\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n) - (\mathcal{S}_t\Phi)(y)\alpha(y)\right|\\
&\leq \left|\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n)\right|\left|\alpha(y_n) - \alpha(y)\right| + \left|\alpha(y)\right|\left|\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n) - (\mathcal{S}_t\Phi)(y)\right|, \qquad \forall n \geq N.
\end{aligned}
$$

Due to the uniform convergence of the semigroup we know that if n is sufficiently large, then $\forall n \geq N_1 : \left|\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n) - (\mathcal{S}_t\Phi)(y)\right| \leq \frac{\epsilon}{2M}$, where M is a constant that bounds $\left|\alpha(y)\right|$. This leaves the first term. Bounding the first term does not seem obvious as we do not know if $\left|\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n)\right|$ is bounded. To see why it is indeed bounded notice that from a sufficiently large n onward we have $\left|\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n) - (\mathcal{S}_t\Phi)(y)\right| \leq 1\ \ \forall n \geq N_2$, due to the uniform convergence of the semigroup. Thus, $\left|\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n)\right| \leq 1 + \left|(\mathcal{S}_t\Phi)(y)\right| \leq 1 + K$. $\left|(\mathcal{S}_t\Phi)(y)\right|$ can be bounded by some finite constant K as it is the finite limit of the sequence $\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n)$, and thus $\left|\left(S^{\mathrm{RW}}_{tn^2}\Phi\right)(y_n)\right|$

is bounded. $\left|\alpha(y_n) - \alpha(y)\right|$ can be made smaller than $\frac{\epsilon}{2(1+K)}$ $\forall n \geq N_3$, due to the continuity of α. Implying that the first term can also be made arbitrarily small. And thus, for all $\epsilon > 0$ and $\forall n \geq N = \sup\{N_1, N_2, N_3\}$,

$$\begin{aligned} |f_n(y) - f(y)| &\leq \left| \left(S^{\text{RW}}_{tn^2}\Phi\right)(y_n)\right| \left|\alpha(y_n) - \alpha(y)\right| + \left|\alpha(y)\right| \left| \left(S^{\text{RW}}_{tn^2}\Phi\right)(y_n) - \left(\mathcal{S}_t\Phi\right)(y)\right| \\ &\leq (1+K)\cdot\frac{\epsilon}{2(1+K)} + M\cdot\frac{\epsilon}{2M} = \epsilon. \end{aligned}$$

Hence, we have now shown that $f_N(y) \xrightarrow{N\to\infty} f(y)$ uniformly in y, which is an even stronger result than originally claimed. ■

Now it remains to prove that condition (c) also holds true. I.e. we need to verify that indeed,

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}(S^{\text{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) \xrightarrow{N\to\infty} \int_{\mathbb{R}}(\mathcal{S}_t\Phi)(x)\alpha(x)dx.$$

Proof: To see why this result holds true we write out the definition of the semigroup and use the reversibility of the random walk with rate $\alpha\left(\frac{x}{N}\right)$. Deriving,

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}(S^{\text{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) = \frac{1}{N}\sum_{x\in\mathbb{Z}}\alpha\left(\frac{x}{N}\right)\sum_{y\in\mathbb{Z}}P^{(N)}_{tN^2}\left(\frac{x}{N},\frac{y}{N}\right)\Phi\left(\frac{y}{N}\right).$$

Where, $P^{(N)}_{tN^2}\left(\frac{x}{N},\frac{y}{N}\right) = \mathbb{P}^{\text{RW}(\alpha_N)}\left(X(tN^2) = x \middle| X(0) = y\right)$. For the measure $\mathbb{P}^{\text{RW}(\alpha_N)}$ we have shown reversibility w.r.t α_N. I.e,

$$\alpha\left(\frac{x}{N}\right)P^{(N)}_{tN^2}\left(\frac{x}{N},\frac{y}{N}\right) = \alpha\left(\frac{y}{N}\right)P^{(N)}_{tN^2}\left(\frac{y}{N},\frac{x}{N}\right).$$

Applying this detailed balance relation, and interchanging the order of summation yields,

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}\alpha\left(\frac{x}{N}\right)\sum_{y\in\mathbb{Z}}P^{(N)}_{tN^2}\left(\frac{x}{N},\frac{y}{N}\right)\Phi\left(\frac{y}{N}\right) = \frac{1}{N}\sum_{y\in\mathbb{Z}}\alpha\left(\frac{y}{N}\right)\Phi\left(\frac{y}{N}\right)\sum_{x\in\mathbb{Z}}P^{(N)}_{tN^2}\left(\frac{y}{N},\frac{x}{N}\right).$$

Note that $\sum_{x\in\mathbb{Z}}P^{(N)}_{tN^2}\left(\frac{y}{N},\frac{x}{N}\right) = 1$, as $P^{(N)}_{tN^2}$ is a probability measure. Thus,

$$\begin{aligned} \frac{1}{N}\sum_{x\in\mathbb{Z}}(S^{\text{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) = \frac{1}{N}\sum_{y\in\mathbb{Z}}\alpha\left(\frac{y}{N}\right)\Phi\left(\frac{y}{N}\right) &= \frac{1}{N}\sum_{x\in\mathbb{Z}}\alpha\left(\frac{x}{N}\right)\Phi\left(\frac{x}{N}\right) \\ &\xrightarrow{N\to\infty} \int_{\mathbb{R}}\Phi(x)\alpha(x)dx. \end{aligned}$$

To conclude the proof we notice that the limiting generator $\mathcal{S}_t$ is stationary w.r.t the measure $\mu^\alpha(dx) = \alpha(x)dx$. Hence obtaining the desired result,

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}(S^{\text{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right) \xrightarrow{N\to\infty} \int_{\mathbb{R}}(\mathcal{S}_t\Phi)(x)\alpha(x)dx.$$

■

Now that we have shown all conditions of lemma 5.1.1 to hold true, we can conclude that $\frac{1}{N}\sum_{x\in\mathbb{Z}}\left|(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)-(\mathcal{S}_t\Phi)\left(\frac{x}{N}\right)\right|\alpha\left(\frac{x}{N}\right)\xrightarrow{N\to\infty}0$. Earlier we have reasoned that this suffices in order to deduce that, $\frac{1}{N}\sum_{x\in\mathbb{Z}}\left|(S^{\mathrm{RW}}_{tN^2}\Phi)\left(\frac{x}{N}\right)-(\mathcal{S}_t\Phi)\left(\frac{x}{N}\right)\right|\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right)\xrightarrow{N\to\infty}0$. Hence proving that,

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}(\mathcal{S}_t\Phi)\left(\frac{x}{N}\right)\rho\left(\frac{x}{N}\right)\alpha\left(\frac{x}{N}\right)\xrightarrow{N\to\infty}\int_{\mathbb{R}}(\mathcal{S}_t\Phi)(x)\rho(x)\alpha(x)dx.$$

A quick recap now before we move on with the next part of the proof. Up till now we shown that,

$$\mathbb{E}_{\mu_N}X_t^N(\Phi)\xrightarrow{N\to\infty}\int_{\mathbb{R}}(\mathcal{S}_t\Phi)(x)\rho(x)\alpha(x)dx\overset{5.3.1}{=}\int_{\mathbb{R}}\Phi(x)\rho(t,x)\alpha(x)dx.$$

Where $\rho(t,x):=\mathcal{S}_t\rho(x)$ satisfies the PDE,

$$\rho_t=\mathcal{L}\rho=2\alpha'(x)\rho_x+\alpha(x)\rho_{xx}. \tag{5.13}$$

Notice that in particular we have the weak form of the PDE,

$$\int_{\mathbb{R}}\rho(t,x)\Phi(x)\alpha(x)dx-\int_{\mathbb{R}}\rho(0,x)\Phi(x)\alpha(x)dx-\int_0^t\int_{\mathbb{R}}\rho(s,x)(\mathcal{L}\Phi)(x)\alpha(x)dxds=0. \tag{5.14}$$

In the present setting there exists a unique strong solution of PDE 5.13, which satisfies 5.14.

5.2 Proof of the HDL: Vanishing Quadratic Variation

In this section we prove the hydrodynamic limit of our density field. We start off by calculating the so called Dynkin martingale 2.2.2. In our case Dynkin's martingale is given by

$$M^f_{tN^2}=f(\eta_{tN^2})-f(\eta_0)-\int_0^{tN^2}\mathcal{L}f(\eta_s)ds,$$

where $f(\eta_s)=\frac{1}{N}\sum_{x\in\mathbb{Z}}\Phi\left(\frac{x}{N}\right)\eta_s(x)$. Notice that,

$$\begin{aligned}\mathcal{L}f(\eta_s)&=\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta_s(x)(\alpha_N(x+1)+\eta_s(x+1))\left(\Phi\left(\frac{x+1}{N}\right)-\Phi\left(\frac{x}{N}\right)\right)\\&\quad+\eta_s(x+1)(\alpha_N(x)+\eta_s(x))\left(\Phi\left(\frac{x}{N}\right)-\Phi\left(\frac{x+1}{N}\right)\right)\\&=\frac{1}{N}\sum_{x\in\mathbb{Z}}(\eta_s(x)\alpha_N(x+1)-\eta_s(x+1)\alpha_N(x))\left(\Phi\left(\frac{x+1}{N}\right)-\Phi\left(\frac{x}{N}\right)\right)\\&=\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta_s(x)\alpha_N(x+1)\left(\Phi\left(\frac{x+1}{N}\right)-\Phi\left(\frac{x}{N}\right)\right)+\eta_s(x)\alpha_N(x-1)\left(\Phi\left(\frac{x-1}{N}\right)-\Phi\left(\frac{x}{N}\right)\right)\\&=\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta_s(x)L^{\mathrm{RW}}_N\Phi\left(\frac{x}{N}\right)\overset{5.3.1}{=}\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta_s(x)\mathcal{L}\Phi\left(\frac{x}{N}\right)+\mathcal{O}\left(\frac{1}{N^2}\right).\end{aligned}$$

We thus have that,

$$M^f_{tN^2} = X^N_t(\Phi) - X^N_0(\Phi) - \int_0^{tN^2} \frac{1}{N} \sum_{x\in\mathbb{Z}} \frac{\eta_s(x)}{\alpha_N(x)} (\mathcal{L}\Phi)\left(\frac{x}{N}\right) \alpha_N(x) ds.$$

Notice that the r.h.s. corresponds to the discretized version of the weak form of the hydrodynamic equation as mentioned in 5.14 and thus, informally, $\frac{\eta_{tN^2}(\lfloor xN \rfloor)}{\alpha_N(\lfloor xN \rfloor)} \longrightarrow \rho(t,x)$. Thus to prove the hydrodynamic limit we need first to show that martingale $M^f_{tN^2}$ vanishes in probability as $N\to\infty$. More precisely we need to show that for all $\delta > 0$,

$$\mathbb{P}_{\mu_N}\left(\sup_{t\in[0,T]} \left|M^f_{tN^2}\right| > \delta\right) \xrightarrow{N\to\infty} 0. \tag{5.15}$$

Applying Markov's inequality yields,

$$\mathbb{P}_{\mu_N}\left(\sup_{t\in[0,T]} \left|M^f_{tN^2}\right| > \delta\right) \le \frac{1}{\delta^2}\mathbb{E}_{\mu_N}\left(\sup_{t\in[0,T]} {M^f_{tN^2}}^2\right) \le \frac{4}{\delta^2}\mathbb{E}_{\mu_N}\left({M^f_{TN^2}}^2\right) = \frac{4}{\delta^2}\left[M^f_{TN^2}, M^f_{TN^2}\right].$$

Where in the second inequality we use the maximal Doob inequality. In the last equality we use that ${M^f_{TN^2}}^2 - \left[M^f_{TN^2}, M^f_{TN^2}\right]$ is also a martingale, and thus has constant expectation and equal to zero. We proceed now by calculating the quadratic variation of M^f_T. The quadratic variation of our martingale M^f_T is given by Carré du Champ 2.16.

$$\left[M^f_{TN^2}, M^f_{TN^2}\right] = \int_0^{TN^2} \Gamma(f)(\eta_s) ds = \int_0^{TN^2} \left(L^N_{SIP}(f^2) - 2fL^N_{SIP}(f)\right)(\eta_s) ds.$$

Calculation yields,

$$\begin{aligned}
\Gamma(f)(\eta_s) &= \sum_{x\in\mathbb{Z}} \eta_s(x)(\alpha_N(x+1)+\eta_s(x+1))(f^2(\eta_s^{x,x+1}) - f^2(\eta_s)) \\
&\quad + \eta_s(x+1)(\alpha_N(x)+\eta_s(x))(f^2(\eta_s^{x+1,x}) - f^2(\eta_s)) \\
&\quad - 2f(\eta_s)\left(\eta_s(x)(\alpha_N(x+1)+\eta_s(x+1))(f(\eta_s^{x,x+1}) - f(\eta_s))\right. \\
&\quad + \eta_s(x+1)(\alpha_N(x)+\eta_s(x))(f(\eta_s^{x+1,x}) - f(\eta_s)) && 16 \\
&= \sum_{x\in\mathbb{Z}} \eta_s(x)(\alpha_N(x+1)+\eta_s(x+1))(f^2(\eta_s^{x,x+1}) - f^2(\eta_s)) \\
&\quad + \eta_s(x)(\alpha_N(x-1)+\eta_s(x-1))(f^2(\eta_s^{x,x-1}) - f^2(\eta_s)) \\
&\quad - 2f(\eta_s)\left(\eta_s(x)(\alpha_N(x+1)+\eta_s(x+1))(f(\eta_s^{x,x+1}) - f(\eta_s))\right. \\
&\quad + \eta_s(x)(\alpha_N(x-1)+\eta_s(x-1))(f(\eta_s^{x,x-1}) - f(\eta_s)) && 16 \\
&= \sum_{x\in\mathbb{Z}} \eta_s(x)(\alpha_N(x+1)+\eta_s(x+1))(f^2(\eta_s^{x,x+1}) - f^2(\eta_s) - 2f(\eta_s)f(\eta_s^{x,x+1}) + 2f^2(\eta_s)) \\
&\quad + \eta_s(x)(\alpha_N(x-1)+\eta_s(x-1))(f^2(\eta_s^{x,x-1}) - f^2(\eta_s) - 2f(\eta_s)f(\eta_s^{x,x-1}) + 2f^2(\eta_s)) \\
&= \sum_{x\in\mathbb{Z}} \eta_s(x)(\alpha_N(x+1)+\eta_s(x+1))(f(\eta_s^{x,x+1}) - f(\eta_s))^2 \\
&\quad + \eta_s(x)(\alpha_N(x-1)+\eta_s(x-1))(f(\eta_s^{x,x-1}) - f(\eta_s))^2.
\end{aligned}$$

More simplified,

$$\Gamma(f)(\eta_s) = \sum_{\substack{x\in\mathbb{Z}\\ y\in\{-1,1\}}} \eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))(f(\eta_s^{x,x+y})-f(\eta_s))^2.$$

Remember we have $f(\eta_s) = \frac{1}{N}\sum_{x\in\mathbb{Z}}\Phi\left(\frac{x}{N}\right)\eta_s(x)$. So that,

$$\Gamma(f)(\eta_s) = \frac{1}{N^2}\sum_{\substack{x\in\mathbb{Z}\\ y\in\{-1,1\}}} \eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\left(\Phi\left(\frac{x+y}{N}\right)-\Phi\left(\frac{x}{N}\right)\right)^2.$$

Applying Taylor's theorem yields,

$$\begin{aligned}\Gamma(f)(\eta_s) &= \frac{1}{N^2}\sum_{\substack{x\in\mathbb{Z}\\ y\in\{-1,1\}}} \eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\left(\frac{y}{N}\Phi'\left(\frac{x}{N}\right)+\frac{y^2}{2N^2}\Phi''(\zeta)\right)^2\\ &\leq \frac{2}{N^2}\sum_{\substack{x\in\mathbb{Z}\\ y\in\{-1,1\}}} \eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\left(\frac{y^2}{N^2}\Phi'\left(\frac{x}{N}\right)^2+\frac{y^4}{4N^4}\Phi''(\zeta)^2\right).\end{aligned}$$

Where ζ is some point between $\frac{x}{N}$ and $\frac{x+y}{N}$. $y^2=1$ for both $y=\pm1$. Thus,

$$\begin{aligned}\left[M^f_{TN^2}, M^f_{TN^2}\right] &\leq \int_0^{TN^2}\frac{2}{N^4}\sum_{\substack{x\in\mathbb{Z}\\ y\in\{-1,1\}}} \eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\Phi'\left(\frac{x}{N}\right)^2 ds\\ &\quad+\int_0^{TN^2}\frac{1}{2N^6}\sum_{\substack{x\in\mathbb{Z}\\ y\in\{-1,1\}}} \eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\Phi''(\zeta)^2 ds.\end{aligned}$$

Taking expectations w.r.t. the initial measure μ_N, and bounding the integral lets us obtain,

$$\begin{aligned}\mathbb{E}_{\mu_N}\left(\left[M^f_{TN^2}, M^f_{TN^2}\right]\right) &\leq \sup_{s\in[0,TN^2]}\frac{2T}{N}\cdot\frac{1}{N}\sum_{\substack{x\in\mathbb{Z}\\ y\in\{-1,1\}}}\left|\mathbb{E}_{\mu_N}\left[\eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\right]\right|\Phi'\left(\frac{x}{N}\right)^2\\ &\quad+\sup_{s\in[0,TN^2]}\frac{T}{2N^3}\cdot\frac{1}{N}\sum_{\substack{x\in\mathbb{Z}\\ y\in\{-1,1\}}}\left|\mathbb{E}_{\mu_N}\left[\eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\right]\right|\sup_{\xi_y(x)}\Phi''(\xi_y(x))^2.\end{aligned}$$

Where we used that we can bound $\left|\Phi''(\zeta)\right| \leq \sup_{\xi_y(x)\in\left(\frac{x}{N},\frac{x+y}{N}\right)}\left|\Phi''(\xi_y(x))\right| \leq A < \infty$, as Φ is a test function. Suppose that now we are able to show that $\left|\mathbb{E}_{\mu_N}\left[\eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\right]\right| \leq C < \infty$, then we would able to obtain,

$$\begin{aligned}\mathbb{E}_{\mu_N}\left(\left[M^f_{TN^2}, M^f_{TN^2}\right]\right) &\leq \frac{4TC}{N}\cdot\frac{1}{N}\sum_{x\in\mathbb{Z}}\Phi'\left(\frac{x}{N}\right)^2+\frac{TC}{N^3}\cdot\frac{1}{N}\sum_{x\in\mathbb{Z}}\sup_{\xi_y(x)\in\left(\frac{x}{N},\frac{x+y}{N}\right)}\Phi''(\xi_y(x))^2\\ &\xrightarrow{N\to\infty} 0\cdot\int_{\mathbb{R}}\Phi'(x)^2dx+0\cdot\int_{\mathbb{R}}\sup_{\xi_y(x)\in\left(\frac{x}{N},\frac{x+y}{N}\right)}\Phi''(\xi_y(x))^2dx = 0.\end{aligned}$$

We see that the desired result can be obtained if only it holds that, $\left|\mathbb{E}_{\mu_N}\left[\eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\right]\right| \le C < \infty$. Thus only this remains to be proven.

$$\begin{aligned}\mathbb{E}_{\mu_N}\left[\eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\right] &= \mathbb{E}_{\mu_N}\left[\alpha_N(x+y)\eta_s(x)\right] + \mathbb{E}_{\mu_N}\left[\eta_s(x)\eta_s(x+y)\right] \\ &\le M\cdot\mathbb{E}_{\mu_N}\left[\eta_s(x)\right] + \int_\Sigma \mathbb{E}_\eta\left[\eta_s(x)\eta_s(x+y)\right] d\mu_N(\eta)\end{aligned}$$

Computing the r.h.s. yields,

$$\begin{aligned}
&= M\int_\Sigma \mathbb{E}_\eta\left[\frac{\eta_s(x)}{\alpha_N(x)}\right]\alpha_N(x)d\mu_N(\eta) + \int_\Sigma \mathbb{E}_\eta\left[\frac{\eta_s(x)}{\alpha_N(x)}\cdot\frac{\eta_s(x+y)}{\alpha_N(x+y)}\right]\alpha_N(x)\alpha_N(x+y)d\mu_N(\eta) \\
&\le M^2\int_\Sigma \mathbb{E}_\eta D(\delta_x,\eta_s)d\mu_N(\eta) + M^2\int_\Sigma \mathbb{E}_\eta D(\delta_x+\delta_{x+y},\eta_s)d\mu_N(\eta) \\
&\overset{\text{(Duality)}}{=} M^2\int_\Sigma \mathbb{E}^{\text{SIP}}_x D(\delta_{X(s)},\eta_0)d\mu_N(\eta) + M^2\int_\Sigma \mathbb{E}^{\text{SIP}}_{x,x+y} D(\delta_{X^1(s)}+\delta_{X^2(s)},\eta_0)d\mu_N(\eta) \\
&= M^2\cdot\mathbb{E}^{\text{SIP}}_x\left[\int_\Sigma D(\delta_{X(s)},\eta_0)d\mu_N(\eta)\right] + M^2\cdot \\
&\int_\Sigma \mathbb{E}^{\text{SIP}}_{x,x+y}\left[\frac{\eta(X^1(s))}{\alpha_N(X^1(s))}\frac{\eta(X^2(s))}{\alpha_N(X^2(s))}\mathbb{1}_{\{X^1(s)\neq X^2(s)\}} + \frac{\eta(X^1(s))}{(\alpha_N(X^1(s))+1)}\frac{(\eta(X^1(s))-1)}{\alpha_N(X^1(s))}\mathbb{1}_{\{X^1(s)=X^2(s)\}}\right]d\mu_N(\eta) \\
&\le M^2\cdot\mathbb{E}^{\text{SIP}}_x\left[\int_\Sigma \frac{\eta(X(s))}{\alpha_N(X(s))}d\mu_N(\eta)\right] + M^2\cdot \\
&\int_\Sigma \mathbb{E}^{\text{SIP}}_{x,x+y}\left[\frac{\eta(X^1(s))}{\alpha_N(X^1(s))}\frac{\eta(X^2(s))}{\alpha_N(X^2(s))}\mathbb{1}_{\{X^1(s)\neq X^2(s)\}} + \frac{\eta(X(s))^2}{\alpha_N(X(s))^2}\mathbb{1}_{\{X^1(s)=X^2(s)\}}\right]d\mu_N(\eta) \\
&\le M^2\cdot\mathbb{E}^{\text{SIP}}_x\left[\mathbb{E}_{\mu_N}\left[\frac{\eta(X(s))}{\alpha(X(s))}\right]\right] + M^2\int_\Sigma \mathbb{E}^{\text{SIP}}_{x,x+y}\left[\frac{\eta(X^1(s))}{\alpha_N(X^1(s))}\frac{\eta(X^2(s))}{\alpha_N(X^2(s))}\right]d\mu_N(\eta) \\
&\le M^2\cdot\sup_{x\in\mathbb{Z}}\mathbb{E}_{\mu_N}\left[\frac{\eta(X(s))}{\alpha_N(X(s))}\right] + M^2\cdot\mathbb{E}^{\text{SIP}}_{x,x+y}\left[\int_\Sigma \frac{\eta(X^1(s))}{\alpha_N(X^1(s))}\frac{\eta(X^2(s))}{\alpha_N(X^2(s))}d\mu_N(\eta)\right] \\
&\overset{\text{(C.S)}}{\le} M^2\cdot\sup_{x\in\mathbb{Z}}\mathbb{E}_{\mu_N}\left[\frac{\eta(X(s))}{\alpha_N(X(s))}\right] + M^2\cdot\sup_{x,y\in\mathbb{Z}}\sqrt{\mathbb{E}_{\mu_N}\left[\left(\frac{\eta(X^1(s))}{\alpha_N(X^1(s))}\right)^2\right]\mathbb{E}_{\mu_N}\left[\left(\frac{\eta(X^2(s))}{\alpha_N(X^2(s))}\right)^2\right]}
\end{aligned}$$

Where we have used that $\alpha_N(x)$ is uniformly bounded, i.e. $0 < m \le |\alpha_N(x)| \le M < \infty$. Note that the only manner that we can bound the expression above further only when $\sup_{x\in\mathbb{Z}}\mathbb{E}_{\mu_N}\left[\left(\frac{\eta(x)}{\alpha_N(x)}\right)^\beta\right] < \infty$, for $\beta\in\{1,2\}$. By invoking our main assumption 5.0.1 this can be realized. Thus obtaining,

$$\left|\mathbb{E}_{\mu_N}\left[\eta_s(x)(\alpha_N(x+y)+\eta_s(x+y))\right]\right| \le M^2(A+A^2) := C < \infty.$$

Which gives the desired result.

To conclude our proof of the hydrodynamic limit we follow standard techniques. We need to show that,

1. Tightness holds for the sequence of distributions of the processes $\{X^N_t : t\ge 0\}_N$
2. All limit points coincide and are supported by the unique path $X(t,dx) = \rho(x,t)\alpha(x)dx$, with ρ the unique weak (and in particular strong) bounded and continuous solution of 5.12.

We skip the details as the proof of 1. is a simple adaptation of the proof presented in the thesis of M.A. Ayala Valenzuela [1], and 2. follows by the uniqueness of the solution of 5.14.

5.3 Intermediate Results for the Hydrodynamic Limit

Lemma 5.3.1. *Let L_N^{RW} be the generator of the random walk $X^N(t) = \frac{X(tN^2)}{N}$ as defined in 5.16 and let $\mathcal{L}$ be the generator of the diffusion $x(t)$ given in 5.13. Then,*

$$\sup_{x\in\mathbb{R}} \left| L_N^{RW} f(x) - \mathcal{L} f(x) \right| \xrightarrow{N\to\infty} 0.$$

Proof: The generator corresponding to the random walk $X^N(t) = \frac{X(tN^2)}{N}$ is given by,

$$L_N^{\text{RW}} f(x) = N^2 \alpha\left(x + \frac{1}{N}\right)\left(f\left(x+\frac{1}{N}\right) - f(x)\right) + N^2\alpha\left(x - \frac{1}{N}\right)\left(f\left(x - \frac{1}{N}\right) - f(x)\right). \tag{5.16}$$

This generator converges to some limiting generator $\mathcal{L}$ of the diffusion process $dX_t = 2\alpha'(X_t)dt + \sqrt{2\alpha(X_t)}dW_t$. The first step for deriving this result is by applying Taylor's theorem, around the point x, to the rates $\alpha\left(x \pm \frac{1}{N}\right)$, and functions $f\left(x \pm \frac{1}{N}\right)$. Obtaining,

- $f\left(x \pm \frac{1}{N}\right) = f(x) \pm \frac{1}{N} f'(x) + \frac{1}{2N^2} f''(x) \pm \frac{1}{6N^3} f'''(\zeta^{\pm})$
- $\alpha\left(x \pm \frac{1}{N}\right) = \alpha(x) \pm \frac{1}{N}\alpha'(x) + \frac{1}{2N^2}\alpha''(\xi^{\pm}).$

For some $\zeta^{\pm}, \xi^{\pm}$ in between x and $x \pm \frac{1}{N}$ respectively. Using these expressions we evaluate 5.16. Deriving,

$$\begin{aligned} L_N^{\text{RW}} f(x) = N^2 &\left(\alpha(x) + \frac{1}{N}\alpha'(x) + \frac{1}{2N^2}\alpha''(\xi^+)\right)\left(f(x) + \frac{1}{N}f'(x) + \frac{1}{2N^2}f''(x) + \frac{1}{6N^3}f'''(\zeta^+) - f(x)\right) \\ + N^2 &\left(\alpha(x) - \frac{1}{N}\alpha'(x) + \frac{1}{2N^2}\alpha''(\xi^-))\right)\left(f(x) - \frac{1}{N}f'(x) + \frac{1}{2N^2}f''(x) - \frac{1}{6N^3}f'''(\zeta^-) - f(x)\right). \end{aligned}$$

Note that the $f(x)$ terms in both expansions cancel out. Carefully working out the brackets yields,

$$\begin{aligned} L_N^{\text{RW}} f(x) = N^2 \Big[&\alpha(x)\left(\frac{1}{N^2}f''(x) + \frac{1}{6N^3}\left(f'''(\zeta^+) - f'''(\zeta^-)\right)\right) \\ &+ \frac{1}{N}\alpha'(x)\left(\frac{2}{N}f'(x) + \frac{1}{6N^3}\left(f'''(\zeta^+) + f'''(\zeta^-)\right)\right) \\ &+ \frac{1}{2N^2}\left(\frac{1}{N}f'(x)\left(\alpha''(\xi^+) - \alpha''(\xi^-)\right) + \frac{1}{2N^2}f''(x)\left(\alpha''(\xi^+) + \alpha''(\xi^-)\right)\right) \\ &+ \frac{1}{6N^3}\left(f'''(\zeta^+)\alpha''(\xi^+) - f'''(\zeta^-)\alpha''(\xi^-)\right)\Big]. \end{aligned} \tag{1}$$

$$\begin{aligned} \implies L_N^{\text{RW}} f(x) = {} &\alpha(x) f''(x) + \frac{1}{6N}\left(f'''(\zeta^+) - f'''(\zeta^-)\right) + 2\alpha'(x) f'(x) + \frac{1}{6N^2}\left(f'''(\zeta^+) + f'''(\zeta^-)\right) \\ &+ \frac{1}{2N}f'(x)\left(\alpha''(\xi^+) - \alpha''(\xi^-)\right) + \frac{1}{4N^2}f''(x)\left(\alpha''(\xi^+) + \alpha''(\xi^-)\right) \\ &+ \frac{1}{12N^3}\left(f'''(\zeta^+)\alpha''(\xi^+) - f'''(\zeta^-)\alpha''(\xi^-)\right). \end{aligned} \tag{1}$$

We can take f to be $C_b^3(\mathbb{R})$, and by assumption α is bounded and $C^\infty(\mathbb{R})$. Implying $|f^{(k)}(\zeta^\pm)| \leq C$ and $|\alpha^{(k)}(\xi^\pm)| \leq M$, for $k = 1, 2, 3$. Yielding,

$$\left|L_N^{\text{RW}} f(x)\right| \leq \alpha(x) f''(x) + \frac{C}{3N} + 2\alpha'(x) f'(x) + \frac{C}{3N^2} + \frac{MC}{N} + \frac{MC}{2N^2} + \frac{MC}{6N^3}.$$

And thus,

$$\begin{aligned}
\sup_{x\in\mathbb{R}}\left|L_N^{\text{RW}} f(x) - \mathcal{L}f(x)\right| &\leq \sup_{x\in\mathbb{R}}\left|\alpha(x) f''(x) + \frac{C}{3N} + 2\alpha'(x) f'(x) + \frac{C}{3N^2} + \frac{MC}{N} + \frac{MC}{2N^2} + \frac{MC}{6N^3} - \mathcal{L}f\right| \\
&= \sup_{x\in\mathbb{R}}\left|\frac{C}{3N} + \frac{C}{3N^2} + \frac{MC}{N} + \frac{MC}{2N^2} + \frac{MC}{6N^3}\right| \\
&\xrightarrow{N\to\infty} 0.
\end{aligned}$$

And thus we conclude that $L_N^{\text{RW}} f(x) \xrightarrow{N\to\infty} \mathcal{L}f(x) = \left(2\alpha'(x)\frac{d}{dx} + \alpha(x)\frac{d^2}{dx^2}\right) f(x)$ uniformly in x. ■

Remark. *The uniform convergence as shown in lemma* 5.3.1 *holds for* $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ *such that* $f \in C^\infty(\mathbb{R})$. *Implying that in particular it also holds* $f_N \in l^\infty\left(\frac{\mathbb{Z}}{N}\right)$ *for each* $N \in \mathbb{N}$, *where* $f\colon \frac{\mathbb{Z}}{N} \longrightarrow \mathbb{R}$.

Lemma 5.3.2. *Generator L_N^{RW} 5.16, the generator of the random walk $X^N(t) = \frac{X(tN^2)}{N}$, is self-adjoint in $l^2(\mathbb{Z}, \alpha)$.*

Proof: Consider the random walk operator $A^{\alpha(x)}f(x) = \alpha(x+1)(f(x+1) - f(x)) + \alpha(x-1)(f(x-1) - f(x))$. This suffices to conclude reversibility of the measure $\nu(\{x\}) = \alpha(x + \frac{1}{N})$ for the semigroup $S_{tN^2}^{\mathrm{RW}}$. And thus the self-adjointness of L_N^{RW}.

Note, that the detailed balance relation reads as,

$$\begin{aligned} \nu(\{x\})c(x,x') &= \nu(\{x'\})c(x',x) \\ \alpha(x)\alpha(x+1) &= \alpha(x+1)\alpha(x). \end{aligned}$$

Thus clearly the measure $\nu(\{x\}) = \alpha(x)$ is reversible for the $\mathrm{RW}(\alpha(x))$. We conclude now the desired result from proposition 2.3.3. ■

The reversibility property is also shared by the limiting generator $\mathcal{L}$. We will show the reversibility of the measure $\mu(dx) = \alpha(x)dx$ for the diffusion process $\mathcal{X}_t$ with generator $\mathcal{L} = \left(2\alpha'(x)\frac{d}{dx} + \alpha(x)\frac{d^2}{dx^2}\right)$, by showing the self-adjointness of $\mathcal{L}$.

Proposition 5.3.1. *Generator $\mathcal{L}$ is a self-adjoint operator in $L^2(\mu)$, with $\mu(dx) = \alpha(x)dx$.*

Proof: Let $f(x), g(x) \in D(\mathcal{L})$, and consider the inner products $\langle \mathcal{L}f, g\rangle_{L^2(\alpha)}$ and $\langle f, \mathcal{L}g\rangle_{L^2(\alpha)}$. We will show that $\mathcal{L}$ is symmetric w.r.t. this inner product. By straightforward calculation we show that both equate to the same integral expression. So,

$$\begin{aligned} \langle \mathcal{L}f, g\rangle_{L^2(\alpha)} &= \int_{\mathbb{R}} \left[2\alpha'(x)f'(x) + \alpha(x)f''(x)\right] g(x)\alpha(x)dx \\ &= \int_{\mathbb{R}} 2\alpha(x)\alpha'(x)g(x)f'(x)dx + \int_{\mathbb{R}} \alpha^2(x)g(x)df'(x). \end{aligned}$$

The first term we let be for what it is, and we'll focus on the second integral. Note that by integrating by parts we derive,

$$\begin{aligned} \int_{\mathbb{R}} \alpha^2(x)g(x)df'(x) &= \alpha^2(x)g(x)f'(x)\Big|_{x=-\infty}^{x=+\infty} - \int_{\mathbb{R}} f'(x)d(\alpha^2(x)g(x)) \\ &= 0 - \int_{\mathbb{R}} f'(x)\left[2\alpha(x)\alpha'(x)g(x) + \alpha^2(x)g'(x)\right] dx. \end{aligned}$$

Where the first term vanishes due to $f, g \in D(\mathcal{L})$. Putting all together gives,

$$\langle \tilde{L}f, g\rangle_{L^2(\alpha)} = -\int_{\mathbb{R}} \alpha^2(x)f'(x)g'(x)dx.$$

Observe that $\langle \mathcal{L}f, g\rangle_{L^2(\alpha)} = \int_{\mathbb{R}} [2\alpha'(x)f'(x) + \alpha(x)f''(x)]\, g(x)\alpha(x)dx$, and $\langle f, \mathcal{L}g\rangle_{L^2(\alpha)} = \int_{\mathbb{R}} f(x)\,[2\alpha'(x)g'(x) + \alpha(x)g''(x)]\,\alpha(x)dx$ are expressions which are symmetrical for f and g. Thus we can similarly derive,

$$\langle f, \mathcal{L}g\rangle_{L^2(\alpha)} = -\int_{\mathbb{R}} \alpha^2(x)f'(x)g'(x)dx.$$

Implying that $\langle \mathcal{L}f, g\rangle_{L^2(\alpha)} = \langle f, \mathcal{L}g\rangle_{L^2(\alpha)}$, and thus indeed generator $\mathcal{L}$ is symmetric. In order to conclude that $\mathcal{L}$ is also self-adjoint we notice that $\mathcal{L}$ is the generator of a diffusion process, i.e. a Markov process, thus implying that $\mathcal{L} = \mathcal{L}^\star$ in $L^2(\alpha)$. ■

Chapter 6

Conclusion

In this thesis we have introduced an inhomogeneous version of the symmetric inclusion process, where the inhomogeneities are given by a slowly varying profile (given by bounded and smooth function α). Using stochastic self-duality we have been able to show the hydrodynamic limit of the $\mathcal{SIP}(\alpha)$. It is given by

$$\rho_t(t,x) = 2\alpha'(x)\rho_x(t,x) + \alpha(x)\rho_{xx}(t,x),$$

where $\rho(t,x)$ is the macroscopic density at time $t > 0$ and position $x \in \mathbb{R}$. The self-duality property of $\mathcal{SIP}(\alpha)$ allowed us to identify the above PDE by studying the scaling of the single particle random walk in an inhomogeneous slowly varying environment. By following standard techniques the hydrodynamic limit was then proved.

Moreover we have shown that the invariant measures of the $\mathcal{SIP}(\alpha)$ are given by the product measure $\mu_\lambda(\eta) = \bigotimes_{x\in\mathbb{Z}} \mu_\lambda^{\alpha(x)}$ such that,

$$\mu_\lambda^{\alpha(x)}(\eta(x) = N) = \frac{1}{Z(\lambda, \alpha(x))} \frac{\lambda^N}{N!} \frac{\Gamma(\alpha(x) + N)}{\Gamma(\alpha(x))}.$$

And where $\lambda > 0$ and $Z(\lambda, \alpha(x)) = (1-\lambda)^{-\alpha(x)}$ is a normalizing constant.

Several questions remain open and subject of future research. E.g. What can we say on the fluctuations around the hydrodynamic limit? What is the behaviour of the boundary driven $\mathcal{SIP}(\alpha)$, i.e. the process on a finite chain $\{1, \ldots, N\}$ coupled with a left and a right reservoir injecting and absorbing particles connected, respectively, with the left and the right point of the chain. These are but a few to mention.

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