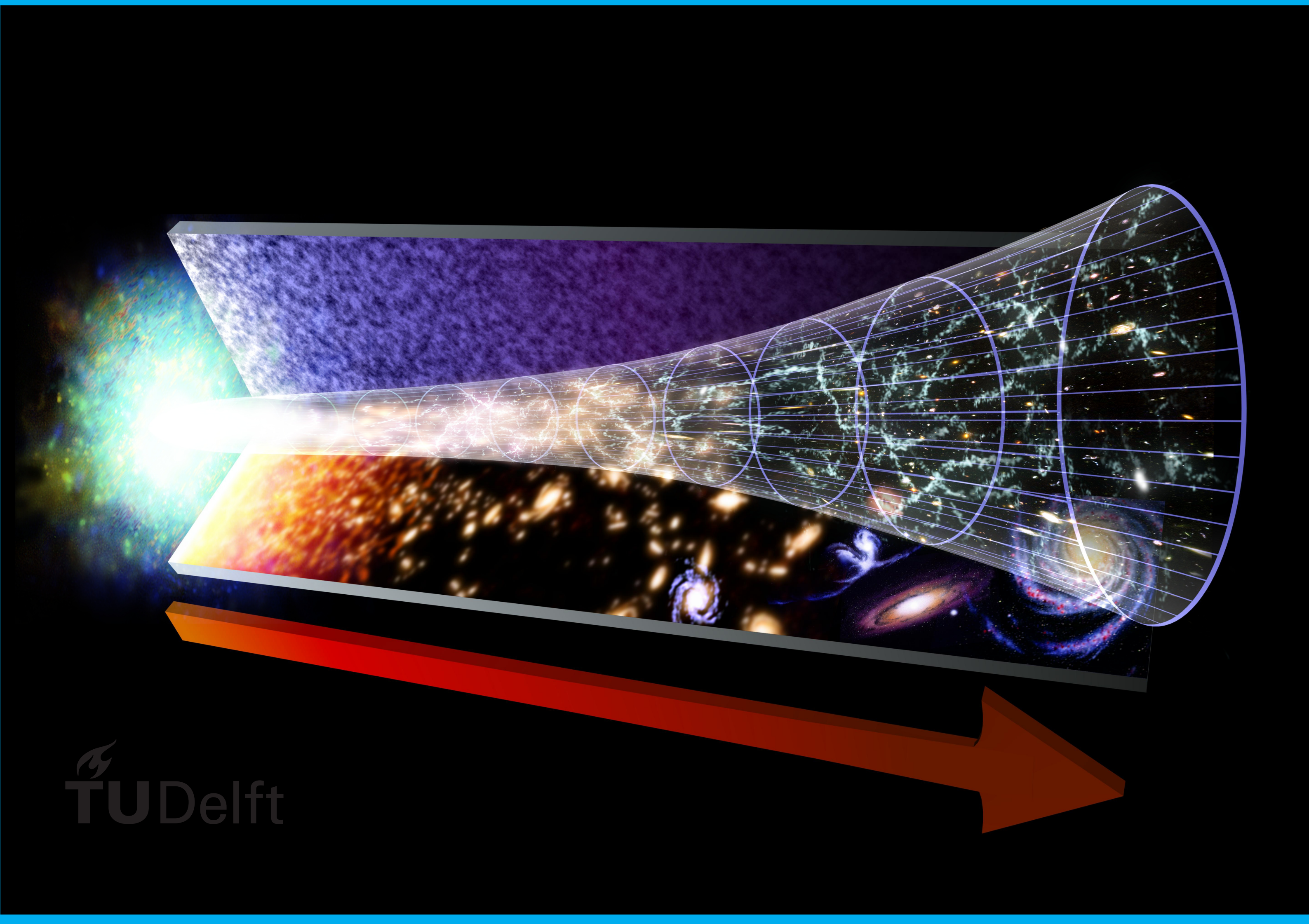
Photon Statistics in Millimeter-Submillimeter Wave Astronomy

T. S. Lopes

Bachelor's thesis

This image represents the evolution of the Universe, starting with the Big Bang. The red arrow marks the flow of time.
Image taken from [1]



Photon Statistics in Millimeter-Submillimeter Wave Astronomy

by

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Abstract

When looking up to the sky and trying to detect the photons of far away stars that are still in formation, we can get access to knowledge we never had before. In order to do this, we must make sure we truly understand the nature of how photons behave in accordance to each other and to our detectors.In quantum optics we can see that these photons have a very dubious character. They are both particle and wave and neither. This makes studying how they are distributed over the spectrum of the sub-millimeter and millimeter wave astronomy a very big challenge.
We know that this very particle-wave duality is what makes up the noise that is fundamental to photons. This duality also makes them obey the Bose-Einstein statistics. Hence due to their particle-like behavior, we have a noise component that resembles that of a Poisson distribution. Very much like raindrops falling down from the sky. On the other hand we have the wave nature of the photons, this causes them to arrive in bunches rather than these random raindrops. It is observed that photons arriving at a detector are corrrelated.
In this thesis we will be investigating how to incorporate the noise of the bunching of the photons into the model of TiEMPO. This model simulates the signal processing of a measurement done by the wide-band spectrometer named DESHIMA. Due to the fact that DESHIMA operates in a wide-band frequency range, we do theoretical research that explains the fundamental theory behind calculating the photon noise over this wide-band range.
We show that taking the wide-band integral of the photon noise is mathematically equivalent of summing the narrow-band approximation for infinitely many sub-bands and adding them up top each other. This approach is the previous method of calculating the photon noise over the wide-band. Due to this method being valid, the question of how these variances over these smaller sub-bands can be additive? Since we are dealing with the detection of photons which as previously stated is a correlated signal. By modeling a simplified version of wide-band photon detection, we have come to the conclusion that due to the small coherence time these photons are independent in the wide-band signal. The photons in these smaller sub-bands of the wide-band signal can also be viewed as statistically independent. If we decrease the frequency bandwidth, we increase the coherence time. Thus measuring the signal over this sub-band equates to having a larger uncertainty in time. Hence when a photon is detected in this sub-band, due to the large coherence time we have that the knowledge of when this photon arrived is mostly lost. Making the time correlations irrelevant to a measurement of an integration time this long.

List of abbreviations and symbols

Abbreviation or symbol	Meaning	Unit (if applicable)
t_c	Coherence time	in seconds (s)
DESHIMA	Deep Spectroscopic HighRedshift Mapper	-
SMGs	Sub-Millimeter Bright Galaxies	-
MKID	Microwave Kinectic Inductance Detector	-
σ^2	Variance of a variable	for the equation 1.1 it is in Watts squared per Hertz ($\frac{W^2}{Hz}$)
τ	Integration time	in seconds (s)
n_{rad}	Photon occupation of signal going into detector	-
η	Power transmission of filter	-
ν	Frequency	in Hertz (HZ)
P_ν	Power density function	in Watt per Hertz (W/Hz)
$\Delta\nu$	frequency bandwidth	in Hertz (Hz)
TiEMPO	Time-dependent End-to-end Model for Post-process Optimization	-
ASTE	Atacama Submillimeter Telescope Experiment	-
h	Planck Constant	$6.62607004 * 10^{-34} \frac{m^2 kg}{s}$
k_B	Boltzmann constant	$1.38064852 * 10^{-23} \frac{m^2 kg}{Ks^2}$
T	Temperature	in Kelvin (K)
NEP_{ph}	Photon noise limited Noise Equivalent Power(NEP)	in $\frac{W}{\sqrt{Hz}}$
c	speed of light in vacuum	299 792 458 meter per second (m/s)

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1
Introduction

The galaxy is a very interesting place. Ever since the beginning of human civilisation we have all been looking at the sky. No matter where we are from or what our culture is, we all have thought about how those stars came to be. Science as we know it has progressed immensely over the last 500 hundred years, so has our understanding of the space around us. In this relatively short time we have developed observational tools, like telescopes, that have aided us in answering these fundamental questions. The light from far away stars can tell us a lot. Where the light is from, how long it has been travelling and possibly a lot more. The nature of light exists in duality between waves and particles. Hence when making pictures of distant galaxies these two contradictory natures also have to be taken into account. The further our understanding of how the nature of light affects a measurement, the more accurately our readings of the sky will be.This is why the quantum mechanical behavior of photons needs to be taken into account.

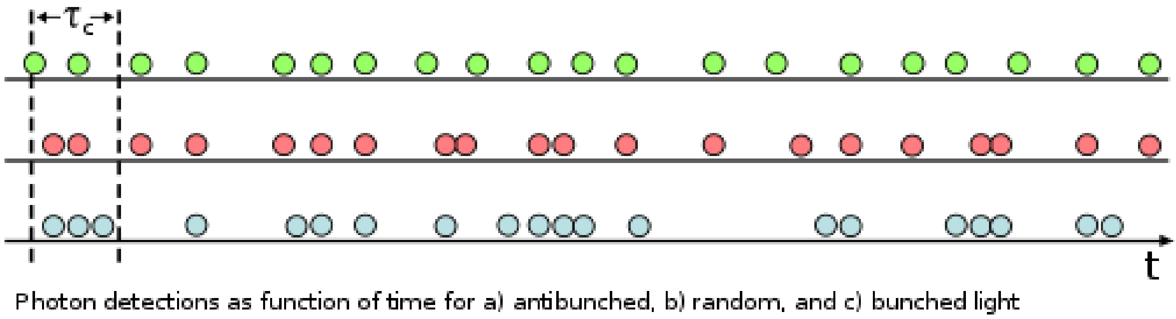


Figure 1.1: A representation of photons bunching. τ_c denotes the coherence time. We have three scenarios of how the probability of detecting another photon within τ_c. We have a) Where the photons are bunched. The probability of detecting another photon within τ_c is higher than detecting another photon after tau_c. Then there is also b) in which the photons are arriving as randomly fluctuating particle. For the last scenario c) we have a phenomenon called anti-bunching. This is the opposite of photon bunching. The probability of detecting another photon within τ_c is smaller than detecting another photon after τ_c. Taken from [2].

1.1. Photon bunching

The arrival times of photons at a detector in the non-classical interpretation is not completely random. When a photon is detected at a certain location in space, there is a significantly higher probability of detecting another photon with a time delay the coherence time t_c,rather than detecting a photon with a time delay bigger than t_c. This phenomena is known as photon bunching. When taking this photon bunching into account, we can conclude that photons arriving at a certain place are correlated to one another. It is this very correlation that makes up a part the noise of photon detection.The noise is not equal to that of independently arriving photons over a certain time, but now also has to incorporate that the photons arrive in bunches or pairs. This bunching is the effect of the wave nature of light, it is namely due to interference. Within the classical picture, we assume light to only be a particle . Detecting light is in this case reminiscent of counting raindrops on a certain surface for a certain amount of time. Then we can view photons as arriving completely at random and it would follow the Poisson distribution, just like those raindrops [3].

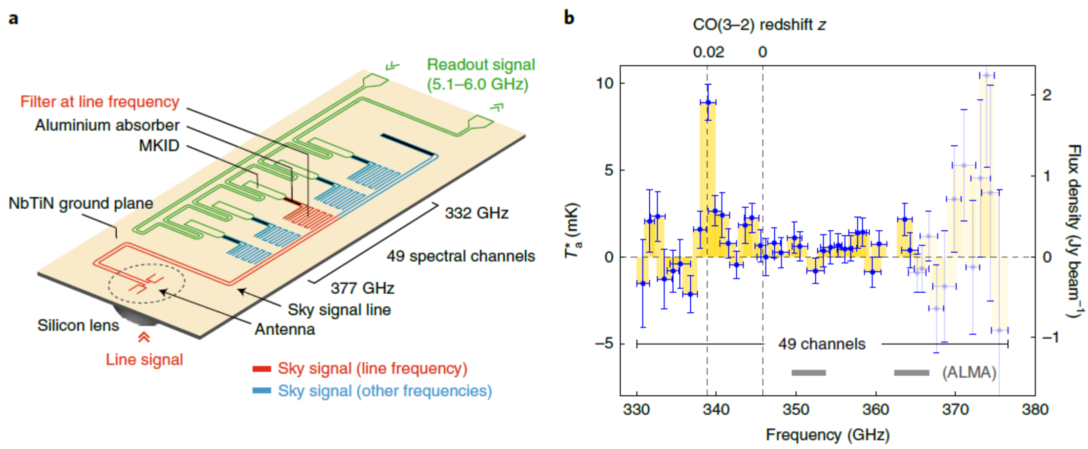


Figure 1.2: A representation of the deshima 1 chip. Taken from [4].

1.2. DESHIMA

DESHIMA which is short for the Deep Spectroscopic High-Redshift Mapper is designed to detect the light which comes from the formation process of a star. This light comes from the thick clouds of dust in a star's formation process in which the dust particles absorb the optical and UV light. Then these atoms radiate out this light in the millimeter to sub-millimeter wavelength range of the infrared spectrum. The galaxies for which this applies are called the Submm-bright galaxies (SMGs) [4]. These signals are measured in a wide-band frequency range of 220-440 GHz. This wide-band frequency range is divided into 347 separate channels by an array of THz band-pass filters. It is to be noted that DESHIMA is the first "octave bandwidth sub-mm wave superconducting on-chip filter-bank spectrometer" [5].
Unfortunately photons do not arrive at the DESHIMA spectrometer like raindrops. The MKIDs connected to the channels are designed to detect the sub-millimeter wave signals from far away galaxies. There are two limits for the photon noise in astronomical observations: namely photons behaving like particles or photons behaving like waves. When photons behave like particles, we speak of Poisson noise. This is because the photons which now act as particles arrive at the detector completely at random much like the aforementioned rain droplets. The other limit in which the photons arriving can be considered as waves, has a noise term dominated by photon bunching noise. This is due to the photons displaying interference-like behavior which causes them to arrive in bunches rather than at random. Our spectrometer which operates in the millimeter-submillimeter wavelength range, which is exactly between these two limits. Therefore the noise term of our observations have both a bunching noise term and a Poisson noise term.

1.3. Mathematics

A lot of improvements can be made in this area between the Poisson noise and the bunching noise of an astronomical signal . We have therefore that a lot of the research within the field is for either of those limits. Zmuidzinas came up with the expression for the wide-band photon noise in the sub-millimeter and millimeter wavelength range of the spectrum [6].

$$\sigma^2 = \frac{1}{\tau}\int \frac{1}{\tau}\eta(\nu)n_{rad}(\nu)[1+\eta(\nu)n_{rad}(\nu)]d\nu \tag{1.1}$$

The integral is done over the bandwidth of the detector expressed in ν for the frequency. The photon occupation number is n_{rad} and η the power transmission of the filter can also be found. Here it would also suffice to use quantum efficiency for η. The current calculation of the noise term in the model for DESHIMA is by using the narrow-band approximation of equation 1.1 for 1500 bins. Each bin needs a random number generation of a Gaussian distribution to simulate the power over the bin including the photon noise. Hence this is why we have that the integral would make the simulation of the noise and the calculation of the noise run much faster. For, the entire integral instead of 1500 randomly generated numbers, only one is necessary to compute the noise over the entire bandwidth of the wide-band signal. The reason being that the former way to calculate the noise has been proven to work very accurately as can be seen in the former model [7], we have to prove in this thesis that this approach is at least equivalent if not more accurate than the former one.

1.4. Goals of thesis

In this thesis we will be investigating the quantum optical background behind the integral in equation 1.1. This will be done by assessing the origin and the physics behind the integral. We also will provide a proof of sorts why implementing the integral as the noise calculation for the simulation of the DESHIMA detector is more time efficient and with proper numerical mathematics can be made a lot more accurate over time. We can show that these two approaches are mathematically equivalent and that the integral is theoretically justified. What is also discussed in this thesis is the validity of taking the integral to calculate the variance of the photon signal over a wide-band. The photon noise must be independent for each sub-band for this to be allowed. However the correlation of the photon noise due to photon bunching cannot be ignored. We take measures to show that these two contradictory natures do not negate one another.

The main research question of this thesis is: How can we calculate the photon-noise limited sensitivity for a wide band measurement, expressed in Noise Equivalent Power (NEP), of a microwave kinetic inductance detector (MKID) for terahertz astronomy?

2

The Detector

In the introduction we have already been introduced to the DESHIMA detector. The Deep Spectroscopic High-redshift Mapper (DESHIMA) is a wide-band spectrometer designed for the Atacama Submillimeter Telescope Experiment (ASTE).

2.1. The Detector

DESHIMA makes use of a microwave kinetic inductance detectors (MKIsD). This type of detector works as follows: First it absorbs photons with high energy in a superconducting film. Hence there are excitations in the super conducting film, namely quasiparticles. This film is situated in a high frequency planar resonant circuit. Both the kinetic inductance and the surface resistance of the film are increased due the absorption of a photon. The resonance frequency of the circuit decreases and the amplitude is altered. With the help of a constant microwave resonance signal going through the detector, we can retrieve the energy of the photons by measuring the change of amplitude and phase of the resonance microwave signal [8].

2.2. What we are trying to measure

Galaxies are created by the coming together of gas and by extracting gas from its environment. This causes new stars to be born and this also causes matter to collapse into a black hole.

In this process stars will form in these massive gas clouds. The visible light and ultraviolet light waves are absorbed by the cloud and they are re-radiated as (mm-submm) infrared light waves. These submm-mm infrared waves make it such that by the red shift due to the Doppler effect, we can tell how far away this stars sending out this signal are and their age. With this information we might be able to map the universe in three dimensions instead of two, with the added dimension being time. Now also the age of galaxies can be determined using this technique [4].

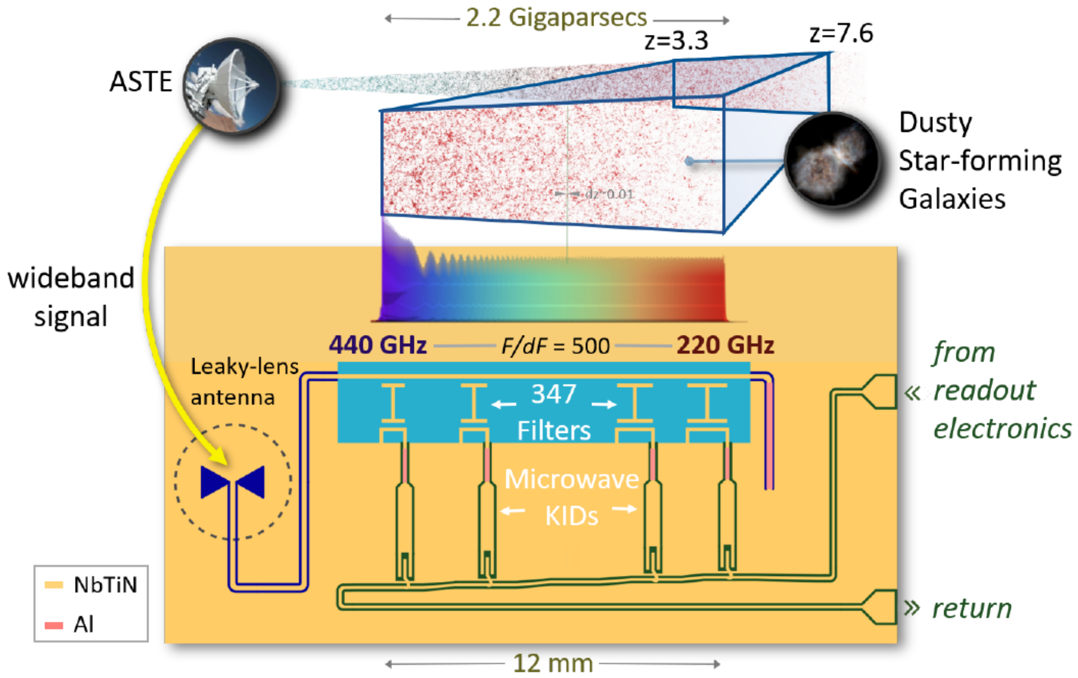


Figure 2.1: A schematic representation of the DESHIMA 2 chip. This image is taken from [7].

In figure 2.1 We see a schematic drawing of the DESHIMA 2 chip. It has a wide-band range of 220 to 440 GHz. This is divided into 347 filter channels. Each of these 347 channels is connected to an MKID. First the light from far away dust galaxies is detected by the ASTE telescope. Then it passes through a leaky lens which picks up the signal and then it goes through one of these 347 channels depending on the frequency of the signal. After it passes through the bandpass filter in the right channel, the signal is collected and processed by the MKID. This gives out an electronic read out signal.

2.3. TIEMPO

For the scope of this project, we will highlight the method used to determine the noise level in the python based simulation of the DESHIMA project called TiEMPO, the Time-dependent End-to-end Model for Post-process Optimization [7] [9]. In this model, the photon noise level is calculated as follows [7][9]:

The incoming astronomical signal is modeled as a Lorentzian distribution function which represents the filter power response. This Lorentzian distribution is in the frequency domain and has a bandwidth of 210 - 450 GHz. This signal is then divided into 1500 equivalent bins. The power in these 1500 sub-bandwidths is calculated by using the power density function of a black-body for a single mode[9]:

$$P_\nu = \frac{h\nu}{\exp\frac{h\nu}{k_B T} - 1} \tag{2.1}$$

In equation 2.1 we have that the power density is denoted by P_ν, h represents the Planck constant, ν stands for the frequency of the photon, k_B is the Boltzmann constant and T stands for the temperature.

So for each bin the power density is calculated with the central frequency in the bin. So $P_{bin,i} = P_{\nu,bini}\Delta\nu$. This is the average power in the frequency domain of each bin. Now we need to calculate the noise to retrieve the power of each bin for each time step. This noise is simulated by a Gaussian distribution with $P_{bin,i}$ as a mean and the standard deviation is equal to the NEP_{ph} for that bin frequency. This translates to generating a random number for each Gaussian distribution of each bin and adding this random number to each average power of that very bin [9][7].

So the total power is [9]:

$$P_{MKID} = \sum_{i=1}^{1500} P_{bin,iwithnoise} \tag{2.2}$$

2.4. Integral

The integral introduced by Zmuidzinas as equation 1.1 in the introduction chapter shows us that the photon noise of an astronomical signal can be modeled faster and easier. By taking the integral, so many computation steps can be skipped. This helps in making the TiEMPO model more time-efficient and more memory-efficient. Also the integral will give a more accurate result It requires also less approximations.

3

Optics

Because we are dealing with an astronomical signal, we can have a deeper look at how light behaves.

3.1. Coherence

Before establishing the quantum optical phenomenon of photon bunching, let us first look at cohernce in general. Coherence in general is defined as a process which is characterised by a well-defined deterministic phase relation. In other words the phase does not fluctuate randomly.[10]. Light has two types of coherence, temporal coherence and spatial coherence.

3.2. temporal coherence

Firstly consider a quasi-monochromatic light source denoted by s. Thus meaning the light has a very narrow bandwidth $\Delta\nu$ with respect to the mean frequency of the light denoted by $\langle\nu\ \nu$. Now the beam is split into two in a Michelson interferometer as seen in figure 3.1 . In this apparatus we have that each beam travels to a mirror. These two mirrors placed in such a way that there is a path difference between the two beams when they reunite at the beam splitter. The mirrors are placed at different distances from the beam splitter. This path difference is denoted by $\Delta l = c\Delta t$. The speed of light is denoted by c and the Δt the delay time of the second beam. From there on, both beams are reflected back onto a screen. If the path difference Δl is small enough, we can see interference fringes forming on the screen. This is an indication that both beams have a manifestation of temporal coherence. We have that the delay time of the second beam δt is smaller than the coherence time t_c. Conversely, if the interference pattern on the screen, as a result of the two beams returning from the Michelson interferometer, has no fringes, we can conclude that no temporal coherence takes place [11].

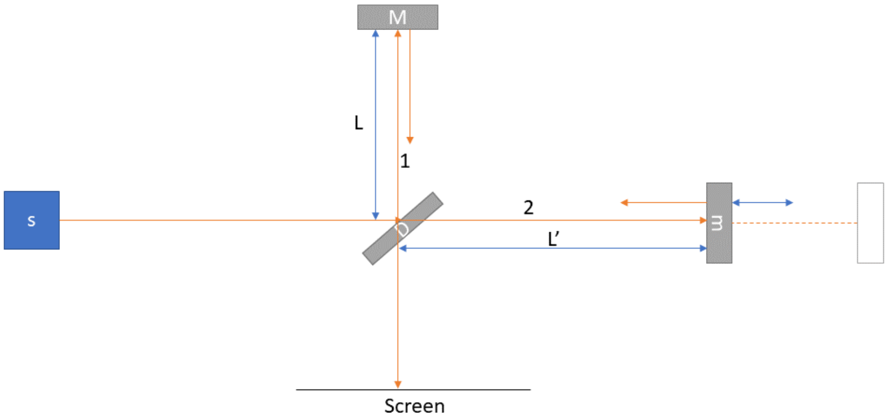


Figure 3.1: A schematic representation of Michelson. The the monochromatic beam coming from light source s passes through a beam splitter at D. From the beam splitter one beam (2) travels to a fixed mirror M which is positioned at a distance L from D and beam 1 travels to a moveable mirror m which is positioned at a distance of L' from D. Both Beam 1 and beam 2 are reflected back to the beam splitter by the mirrors and come together again at point D. From there they are unified again into one beam and travel towards the screen on which a interference pattern arises. Image inspired by figure 4.1 in [11]

Due to experimentation we know that fringes will only appear when [11]:

$$\Delta t \Delta \nu \leq 1 \tag{3.1}$$

Therefore the coherence time t_c must be defined by the following equation:

$$t_c \frac{1}{\Delta \nu} \tag{3.2}$$

3.3. the relation between the coherence time and the bandwidth

When conducting an experiment to detect temporal coherence for varying optical bandwidth $\Delta\nu$, we have that a direct relation can be shown for the bandwidth and the coherence time. From equation 3.2 we have that the coherence time t_c is inversely proportional to $\Delta\nu$. In an experiment conducted by Philips, Kleiman and Davis in 1967 [12], it has been shown that as the bandwidth of the light was narrowed by using an interferometer,as a result the photon bunching takes place over a longer time interval [11]. We can see this clearly when we look at figure 3.2. In this figure we can see, the narrower the bandwidth the longer the photon bunching persists. In addition we can conclude that the photon bunching occurences are a lot less concentrated and spread more over the time interval. This shows that the photons are in less bunches over a tighly packed bunches when we make the bandwidth more narrow. This relation between bunching and detection bandwidths will be discussed more in a later stage of this thesis.

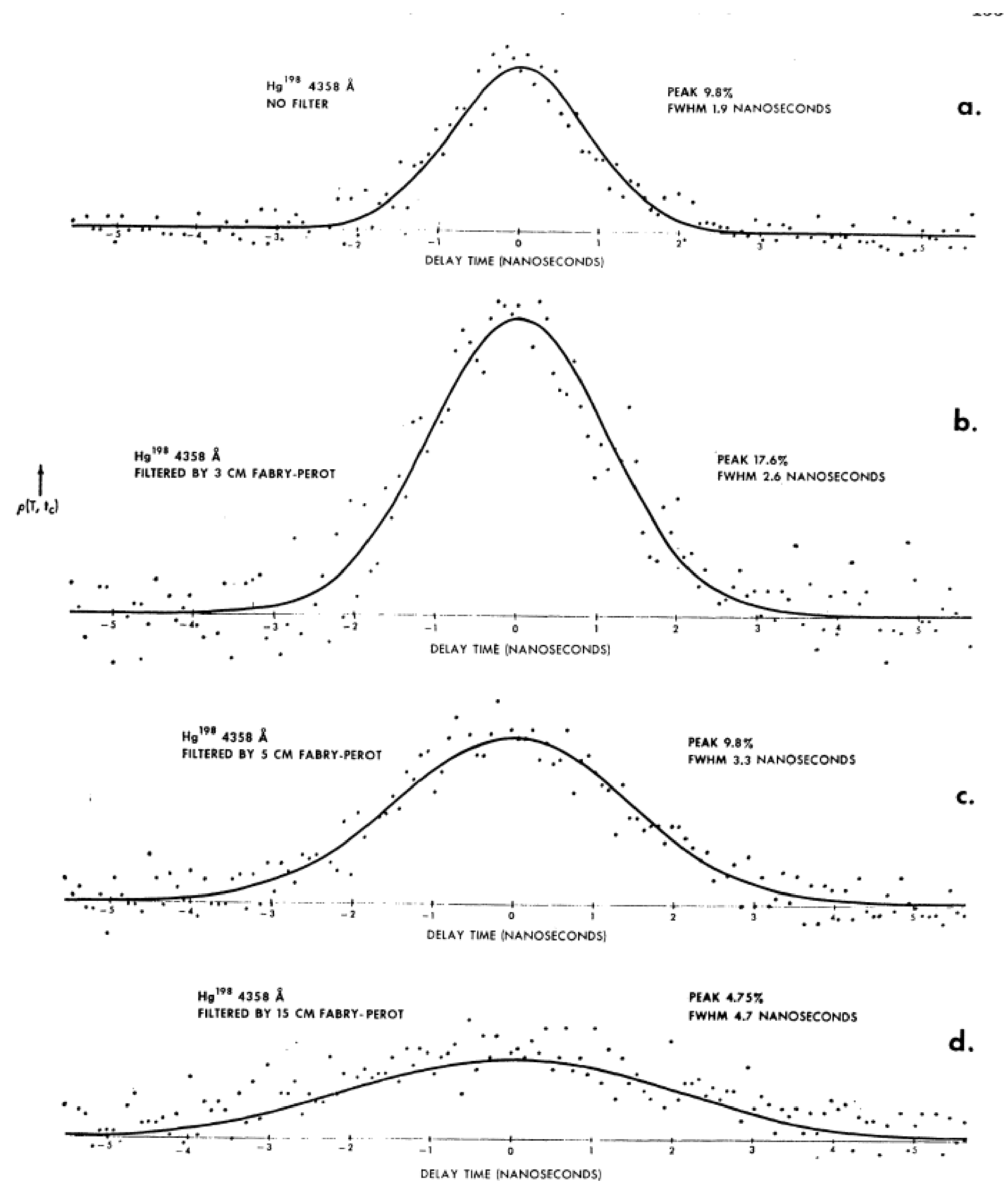


Figure 3.2: Here the probability distribution of photon bunching or coincidences is shown vs the time delay. a) We see the unfiltered signal. b) The signal has a lot higher occurrence of photons bunching and it happens over a longer time interval. c) The signal is filtered to have an even narrower bandwidth than b), but the bunching now happens more spread out over the time interval, but also persists at higher delay times than before. d) We have that the signal is now filtered to the narrowest bandwidth. The photon bunching is going for even longer delay times than c) and also the photon bunching distribution has flattened a lot more. Image taken from [12].

4

Nature of photon bunching

Now we will look at a non-classical approach to the fluctuations of the arrival times of photons in a measurement.

4.1. Variance of quantum light

When we talk about photon bunching, we must explain in further detail what exactly happens on a subatomic level. By looking at the moments of the total number of photons, we can derive using moment generating function that the variance of the number or photons is as follow:

$$\langle \Delta n^2 \rangle = \langle n \rangle + \langle (\Delta U^2) \rangle \tag{4.1}$$

Where n is the total number of photons and U is a random variable. This expression is similar to what Einstein found in 1909. His interpretation showed that the variance of the total number of photons consisted of two terms respectively, one derived from the fluctuations of classical particles and one derived from fluctuating classical waves. It is noteworthy that in equation 4.1 the first term on the right-hand side is the variance of randomly fluctuating particles and the second is the variance of fluctuating waves [11].

Here we clearly see the evidence of the particle-wave duality of photons. This quantum-mechanical property is the fundamental basis of photon bunching.

4.2. Light fluctuations

We can see that the joint probability of two detections at different locations and times are not independent for incoherent light. Each detection holds information of the EM field that influences the next detection. It can be said that intensity fluctuations cause photon detections at two different time or space-time points to be correlated. [11].

The nature of the particle-wave duality is the basis behind photon bunching. Hence photons are not classically independent particles, but also show wave-like behavior like interference. We know that the photons are correlated. Otherwise they would be distinguishable and follow the Boltzmann statistics. This however is not the case. Photons are not independent and they obey the Bose-Einstein statistics. They also are indistinguishable from one another. The extra noise term in equation 4.1 can be considered a consequence of waves interfering [13].

4.3. Heisenberg Uncertainty Principle

Hanbury Brown and Twiss calculated the correlation between the emission times of photo-electrons at different points on the wavefront of a plane wave. Their experiment shows that this is a quantum mechanical phenomenon. They noted that the interference effects, which cause the wave noise in photon detection, are rooted in the Heisenberg uncertainty principle [13]. This uncertainty principle is defined as follows [3]:

$$\Delta E \Delta t = h \tag{4.2}$$

$$\Delta\nu\Delta t = \frac{1}{2\pi} \tag{4.3}$$

The ΔE means the uncertainty in energy and Δt means the uncertainty in time. Equation 4.2 means that when you are certain or have gained more knowledge of the time or arrival time of a photon, you lose more information about the energy of that same photon. In other words the larger the uncertainty in time , the lower the uncertainty in energy and vice versa. Similarly in equation 4.3 the difference in frequency $\Delta\nu$ and the the difference in time or arrival time Δt are correlated in the same way. We have due to the Fourier Transform that the frequency and the time domain have a inversely proportional relationship. This means that relation in equation 4.3 can also be explained classically.

In chapter 2 the relation between bunching and the bandwidth of a signal was already explored. Its conclusion was simple, the narrower the bandwidth, the bigger the coherence time and the more spread out and longer these bunched pairs of photons will still appear.

4.4. Incoherent light

It is important to note that DESHIMA spectrometer is detecting incoherent thermal light. We can approximate this by black-body radiation as we are dealing with dust and gas which are in the process of creating a star.

Let us take a black-body radiation field. Then the general expression for the joint probability of the $p(n)$ where n is the set occupation numbers of is as follows [11]:

$$p\left(\{n\}\right) = \frac{1}{\left[1 + \frac{\langle n\rangle}{\mu}\right]^{\mu}\left[1 + \frac{\mu}{\langle n\rangle}\right]^{\langle n\rangle}} \tag{4.4}$$

Equation 4.4 has been derived from the assumption that the average occupation numbers $< n >$ of all the μ occupied modes are equal. When we use the fact that photons are indistinguishable and calculate the variance for the of the occupation number n, we have the following expression [11]:

$$\left\langle(\Delta n)^2\right\rangle = \langle n\rangle\left(1 + \frac{\langle n\rangle}{\mu}\right) \tag{4.5}$$

In this paper we will only be looking at the single mode case that $\mu = 1$.

The Harry Brown and Twiss (HBT) experiment could not yet experimentally show the photon bunching effect in incoherent light. The former statements are all made about light coming from a partially coherent light source in order to broaden our understanding of the behaviour of the photons This was done much later by Boitier and his peers in 2009. In this experimental setup correlations between by observing the degree of second order coherence $g^{(2)}(\tau)$ expressed as follows [10]:

$$g^{(2)}(\tau) = \frac{\left\langle \hat{E}^{(-)}(t+\tau)\hat{E}^{(-)}(t)\hat{E}^{(+)}(t)\hat{E}^{(+)}(t+\tau)\right\rangle}{\left\langle \hat{E}^{(-)}(t)\hat{E}^{(+)}(t)\right\rangle^2} \tag{4.6}$$

In equation 4.6 we have that this $g^{(2)}(\tau)$ term is actually the probability of detecting a second photon at a time τ after having observed a first photon [14]. We have that $\hat{E}^{(-)}(t)$ and $\hat{E}^{(+)}(t)$ are the complex electric field operators and that when $g^{(2)}(0) = 2$ photon bunching occurs. It is sufficient to show that $g^{(2)}(0) > 1$ [10], to indicate that the photons are correlated. This is exactly what is shown by the experiment performed by Boitier and his peers [15]. Instead of a thermal light source, two devices mimicking the black-body radiators were used.

5

Narrow-band vs Wide-band

5.1. Narrow-Band approximation

5.1.1. Narrow-band expression for the detector output variance.

The key to calculating the Noise Equivalent Power (NEP) of a signal is the variance in the average arrival rate of the photons arriving at the detector over an integration time τ. This is given by the following formula [6]:

$$\sigma^2 = \frac{1}{\tau}\int \eta(\nu)n_{rad}(\nu)[1+\eta(\nu)n_{rad}(\nu)]d\nu. \tag{5.1}$$

In this formula, n_{rad} represents the photon occupation number for the radiation that goes into the detector and η represents the power transmission of the detector or the quantum efficiency of the detector. These quantities are dependent on the frequency ν. The integral is taken over the entire bandwidth of the signal. As the bandwidth of the entire signal can be very large, i.e. of order of a couple hunderd GHz, the wide-band can be split into multiple narrower band-widths. The integral can be computed more easily on these narrow bands than on the collective bigger bandwidth due to the following approximation.

For this narrow-band approximation of equation 5.1 with Δnu being the bandwidth, the following assumptions are made :

1. $\Delta\nu << \nu$
2. n_{rad} and η are constant.

With these approximations the variance in equation 5.1 takes the form of the following equation [6]:

$$\sigma^2 = \frac{1}{\tau}\eta n_{rad}(1+\eta n_{rad})\Delta\nu \tag{5.2}$$

5.1.2. From the narrow-band variance to the narrow-band NEP formula.

We take the expression used to calculate the NEP of the detector in previous papers about the DESHIMA detector [4]:

$$NEP_{photon} = \sqrt{\frac{2P_{RAD}h\nu(1+\eta n_{rad})}{\eta}} \tag{5.3}$$

We want to show that this can be obtained from the equation 5.2.

First we take equation 5.2 to express the uncertainty of the power output of the signal over an integration time τ. The input power of the signal over a narrow-bandwidth with a constant n_{rad} and η is given by [6]

$$P_{RAD} = h\nu n_{rad}\Delta\nu = h\nu\frac{<d>}{\eta} \tag{5.4}$$

with $< d >= \eta n_{rad} \Delta\nu$ being the mean arrival rate of the photons at the detector with the assumptions of the narrow-band approximation.
It is noticeable that the input power of the signal at the detector is proportional to the mean arrival rate. Hence we can use the variance of the mean arrival rate of the photons at the detector in equation 5.2 to express the uncertainty of the input power after an integration time τ as follows [6]

$$\sigma_P = \frac{h\nu}{\sqrt{\Delta\nu\tau}} \sqrt{\frac{n_{rad}(1+\eta n_{rad})}{\eta}\Delta\nu} \tag{5.5}$$

Now we will show that you can obtain the original formula for the NEP used in previous papers for the DESHIMA detector as expressed in equation 5.3 from the equation 5.5 we have just derived above. For the purpose of calculating the NEP, we take $\tau = 0.5s$.

$$\sigma_P = \frac{h\nu}{\sqrt{\Delta\nu \cdot \frac{1}{2}}} \sqrt{\frac{n_{rad}(1+\eta n_{rad})}{\eta}\Delta\nu} \tag{5.6}$$

$$= \sqrt{2h^2\nu^2\Delta\nu \frac{n_{rad}(1+\eta n_{rad})}{\eta}} \tag{5.7}$$

$$= \sqrt{2(h\nu n_{rad}\Delta\nu)h\nu \cdot (\frac{1}{\eta} + n_{rad})} \tag{5.8}$$

$$= \sqrt{2(h\nu n_{rad}\Delta\nu) \cdot (\frac{h\nu}{\eta}) + n_{rad}h\nu} \tag{5.9}$$

Now we use equation 5.4

$$= \sqrt{2P_{RAD}(\frac{h\nu}{\eta} + \frac{P_{RAD}}{\Delta\nu})} \tag{5.10}$$

We now see that equation 5.10 is equivalent to equation 5.3 and that equations 5.1 and 5.3 are consistent. The only problem might arise from a difference in units. The NEP in equation 5.5 has a unit of [W] and the NEP coming from equation 5.3 has a unit of [W/$\sqrt{(Hz)}$]. This difference in units will be explained in the derivation of the wide-band NEP.

5.2. Wide band NEP

The next step is to derive the expression for the photon limited noise equivalent power (NEP) in the general case, so not only for the narrow-band case. By ignoring the approximations made at first we can ultimately follow the same approach as the aforementioned narrow band case for the problem. We start from taking the formula from the variance of the arrival rate of the photons that are absorbed by detector in a time interval τ in the Zmuidzinas paper equation 41, namely equation 5.1 in this paper [6].

First we recall the definition of the NEP. The NEP is the uncertainty in input power after an integration time τ = 0.5 s [6] or after a post detection bandwidth of 1 Hz . The NEP is the signal power incident on the detector required to to produce $S/N = 1$, with S/N we refer to the signal-to-noise ratio.[6].

5.2.1. Derivation of wide band NEP

First we define the specific power absorbed in the detector as

$$P_\nu d\nu = hn_{abs}(\nu)\nu d\nu \tag{5.11}$$

Here ν is defined as the frequency, h stands for the Planck constant, $n_{abs}(\nu)$ now stands for occupation number of the absorbed signal for a single mode.We know that an MKID is a real and non-ideal detector. Therefore the number of absorbed photons cannot be equal to the number of photons reaching the detector. Hence we express the photon occupation number of photons absorbed by the detector n_abs as :

$$n_{abs}(\nu) = \eta n_{rad}(\nu) \tag{5.12}$$

The average variance in the number of photons per mode is equal to

$$< \Delta n_{abs}^2 >= n_{abs}(n_{abs} + 1) = \eta n_{rad}(1 + \eta n_{rad})$$

Because we know the expression for P_ν, we can say that the mean energy fluctuation per bandwidth is given by $h^2\nu^2 < \Delta n_{abs}^2 >$.[16]

We use the assumption that the fluctuations in energy in different infrared bandwidths are uncorrelated. So their mean square fluctuations in energy can be added to each other to obtain the noise equivalent power for the absorbed power signal of the detector per unit post-detection bandwidth B [16].

$$NEP^2 = 2\int \frac{h^2\nu^2\eta(\nu)n_{abs}(\nu)[1 + \eta(\nu)n_{abs}(\nu)]}{\eta^2}d\nu \tag{5.13}$$

The NEP is the optical NEP. This means that the the power at the input of the detector that would produce the same signal level as the noise source [17].

5.2.2. From Zmuidzinas to Richards and Lamarre

We have that the uncertainty in the average power absorbed in a detector during an integration time is equal to what is derived from the uncertainty described in equation 5.1. This done by multiplying both sides of the equation 5.1 with $(h\nu)^2$ [6]:

$$\sigma_p^2 = \frac{1}{\tau}\int (h\nu)^2 n_{rad}(\nu)\eta(\nu)(1 + n_{rad}(\nu)\eta(\nu))d\nu \tag{5.14}$$

This uncertainty gives you the statistical NEP which is σ_p in this case. This means that the statistical NEP is given by $\sigma_p = \sqrt{\sigma_p^2}$. This expression for the NEP gives you the power dissipated into the detector due to the noise. However unlike the optical NEP in equtaion 5.5, we have that there is no dependence of integration time or post-detection bandwidth [17]. If we define the NEP in equation 5.14 to be for an integration time 0.5 s, we have that σ_p is equivalent to the NEP in equation 5.13. This will be equal to the optical NEP when it is defined for a specific integration time. That is when the dependence of the integration time τ in equation 5.14 disappears. We get the following:

$$\sigma_p^2 = 2\int (h\nu)^2 n_{rad}(\nu)\eta(\nu)(1 + n_{rad}(\nu)\eta(\nu))d\nu \tag{5.15}$$

Now the only difference between equation 5.15 and 5.13 is that the NEPs are electrical and optical respectively. Hence we only need to divide equation 5.15 by η^2 and we can see that both NEPs are equivalent [17].

6

The Variance over a wide-band signal

The DESHIMA spectrometer has 347 logarithmically spaced bandpass filters that have central frequencies ranging from 220-440 GHz. This makes it so that the total detection bandwidth of one spectral channel have a range of up to a GHz. This makes it so that if one would wish to calculate the noise of a single spectral channel in a single measurement at one go, the narrow-band formula for the NEP will be ineffective. Here we need to use the wide-band form of the NEP in order to calculate the noise in one integral. The property of an integral is the summing over infinitesimal sub-intervals in order to achieve the value of the integral over a big interval. Hence for the computation to be done in one go, we need to used the wide-band variant of the formula for the NEP.

The goal is to prove that taking the infinite sum over the narrow-band approximated NEPs of infinitesimally small sub-intervals is equivalent to taking the wide-band NEP over the whole range of the wideband detection from 220 GHz to 440 Ghz.

6.1. Formulation of the mathematical problem

We have that the measurement over the entire bandwidth of a single bandpass filter is a random variable X. This random variable X has a sample space Ω that represents each possible outcome of a photon being detected by the detector. This is a random process and has a certain probability distribution. We are investigation whether splitting this random variable into two sub-bands can be statistically justified.

This project is about an astronomical detector observing light. In astronomy the main task is to measure the flux or arrival rate of photons. These measurements always come with a fundamental uncertainty as the arrival time of these photons follows the Bose-Einstein statistics. The photons experience photon bunching which makes them correlated to one another.Measuring the arrival rate of the incoming photons gives the following variance [6]:

$$\sigma^2 = \int f(\nu) d\nu \tag{6.1}$$

Here we have that $f(\nu) = \frac{1}{\tau}\eta(\nu)n_{rad}(\nu)[1 + \eta(\nu)n_{rad}(\nu)]$.

The integral is done over the bandwidth of the detector expressed in ν for the frequency. The photon occupation number is n_{rad} and η the power transmission of the filter can also be found.

6.1.1. Narrow-band variance

In the field of submillimeter-wave astronomy, the narrow-band approximation of this formula is often used for convenience.

$$\sigma^2 = \frac{1}{\tau}\eta n_0(1 + \eta n_0)\Delta\nu \tag{6.2}$$

The bandwidth of the signal is expressed as $\Delta\nu$. For example, the TiEMPO model developed by the THz Sensing Group uses the narrow-band form to generate an artificial time-domain noisy signal [7]. There are two approaches for calculating the wide-band noise:

1. Divide the band into sufficiently narrow frequency bins, calculate the power of the bin with the noise already incorporated into it, and take the sum over all the bins. We repeat this process for3 each time step and make a time series pof the noise over

2. Calculate the integral to directly produce a single time-series that represents the total frequency band such as in equation 6.1.

Till now, approach (1) has been adopted. We want to investigate if (1) and (2) are indeed mathematically equivalent. Here the signal has been in 1500 so called "bins". From those bins the variances of each individual bin is used to calculate the variance of the total signal. We will show that when the signal in these separate bins are statistically independent from one another this approach is mathematically correct.

We have that our signal is being detected in the time domain, but the read out signal is in the frequency domain. This is because due to the Fourier transform that takes a time dependent signal and transforms it into a frequency dependent signal. In figure 6.1 we see a random time dependent signal being split into 2 in the frequency domain.

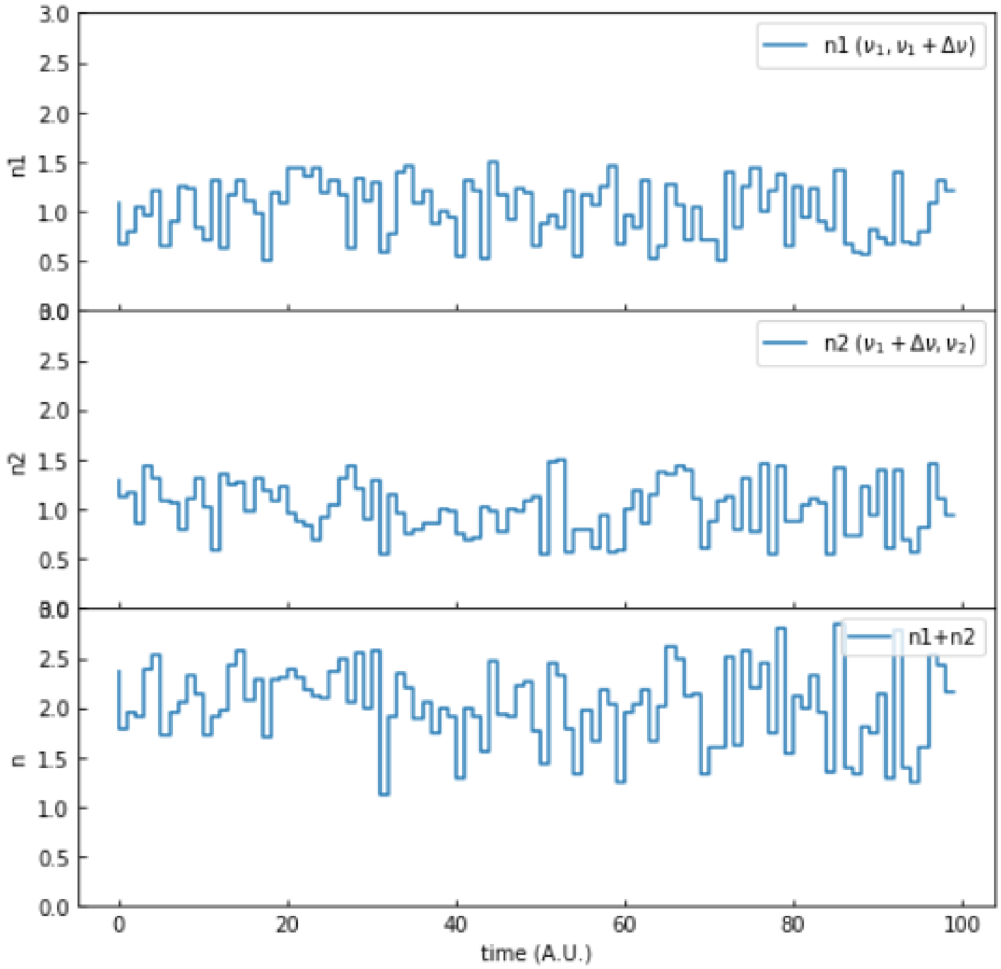


Figure 6.1: A representation of a random signal in time signal n(t) being split into n1(t) and n2(t). Image taken from private communications with Akira Endo.

6.2. Are variances of random variables additive?

The theorem in mathematics is as follows: Let X and Y be two independent random variables. We denote the random variable $Z = X + Y$. We have that

$$Var(Z) = Var(X+Y) = Var(X) + Var(Y) \tag{6.3}$$

6.2.1. proof of theorem

$$Var(X_1 + X_2) = \left\langle ((X_1 + X_2) - \langle X_1 + X_2 \rangle)^2 \right\rangle \quad (6.4)$$

$$= \langle X_1 + X_2 \rangle^2 - \left\langle (X_1 + X_2)^2 \right\rangle \quad (6.5)$$

$$= \langle X_1 \rangle^2 + 2 \langle X_1 X_2 \rangle + \langle X_2 \rangle^2 - \langle X_1^2 \rangle - \langle X_2^2 \rangle - 2 \langle X_1 X_2 \rangle \quad (6.6)$$

$$= \langle X_1 \rangle^2 - \langle X_1^2 \rangle + \langle X_2 \rangle^2 - \langle X_2^2 \rangle \quad (6.7)$$

$$= Var(X_1) + Var(X_2) \quad (6.8)$$

6.3. Proof

Let us define the random variable of detecting a photon as X. We define the even X as the experiment for the detection of a photon in this wide-band bandwidth B. In this case our sample space Ω consists of all the possible outcomes of detecting the photon on this wide-band frequency bandwidth B. In this case we have that $X \subset \Omega$. We define X_1 as the experiment detection of a photon is sub-band B_1 and we define X_2 as the experiment of the detection of a photon is sub-band B_2. We have that $B_1 + B_2 = B$. We assume X_1 and X_2 to be independent from each other and we have that sample space Ω_1 and Ω_2 are disjoint from one another. It follows that the random variable X is equivalent to the random variable $X_1 + X_2$ and that the sample space $\Omega = \Omega_1 \cup \Omega_2$. We have that

$$Var(X) = Var(X_1 + X_2) = Var(X_1) + Var(X_2) \quad (6.9)$$

Since we have that X_1 and X_2 are independent, we have that by the proof

6.4. Do both approaches give the same result?

When we look at the appendix, we see the code for the way of the 1500 bins and the way of the integral.The code of the integral may also be more efficient. Because the noise is still calculated by using a mathematical algorithm, we have that the computational way of taking 1500 bins and averaging the noise over them should give a similar result as using the numerical python method of integration.Only this is not the case yet. I have attempted two code both ways, but no real and accurate conclusions can me made from this code yet.

7

Solving the paradox

7.1. Paradox

When talking about the noise calculation for the TiEMPO simulation of the DESHIMA MKID detector, we have been talking about wide-band calculation in the occupation number of photons. There is one seemingly contradictory issue with the way our wideband NEP is calculated. As we know the variance of the photon occupation number is the core behind the NEP calculation. When we use that this is equal to

$$\sigma^2 = \frac{1}{\tau}\int \eta(\nu)n_{rad}(\nu)[1+\eta(\nu)n_{rad}(\nu)]d\nu \tag{7.1}$$

This is a consequence of assuming that the photon occupation number is independent in one from the occupation number in another frequency sub-bandwidth. The sub-bandwidths in question are infinitesimally small. Therefore this appears to clash with the entire concept of photon bunching. Photons that bunch are in no way independent from each other. When they arrive within the coherence time, they are bunching and therefore dependent of one another. Is there a way for these photons in these infinitesimally small sub-bandwidths to be considered independent of each other?

7.2. The absence of bunching

In the absence of bunching the entire equation for the variance of the occupation number becomes more simple. We have now that the mean square fluctuation in photon number is equal to: $<\Delta n^2>= n.$ By assuming bunching plays no role here, the photons arriving in different sub-bands are independent of one another. We now can for certain say that it is allowed to add the variances in photon number of different sub-bands together. Hence we get the noise term for the power looks as follows

$$\sigma^2 = \frac{1}{\tau}\int \eta(\nu)^2 n_{rad}(\nu)^2 d\nu \tag{7.2}$$

This corresponds to a part of the integral we see in equation 7.1.

7.3. Adding bunching to the mix

When taking photon bunching into an account when we talk about the calculation of the noise term, we can first make two assumptions:

1. The photon noise is white.
2. The spectrum that is being observed is flat

When making those assumptions we can graphically represent our wide-band signal as indicated by figure 7.1. Here the green box represents the measuring bandwidth and the blue dots represent photons. We have time t on the vertical axis and the frequency ν on the horizontal axis. All the photons that are bunched are in the same green box. For the wide-band situation in particular, we have that the coherence time $t_{c,1} = \frac{1}{\Delta f_1}$.

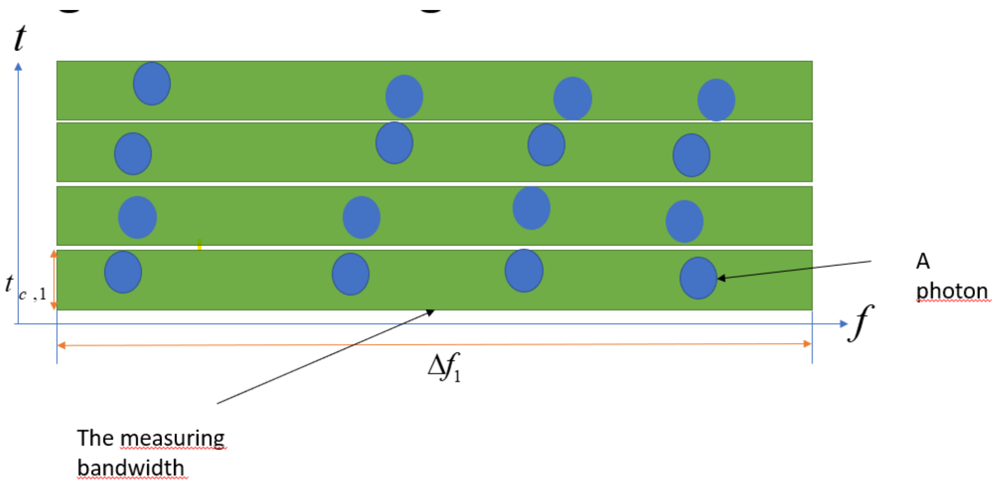


Figure 7.1: A representation of a the wide-band signal that is to be measured

When we divide this wide-band into two sub-bands we get the following situation:

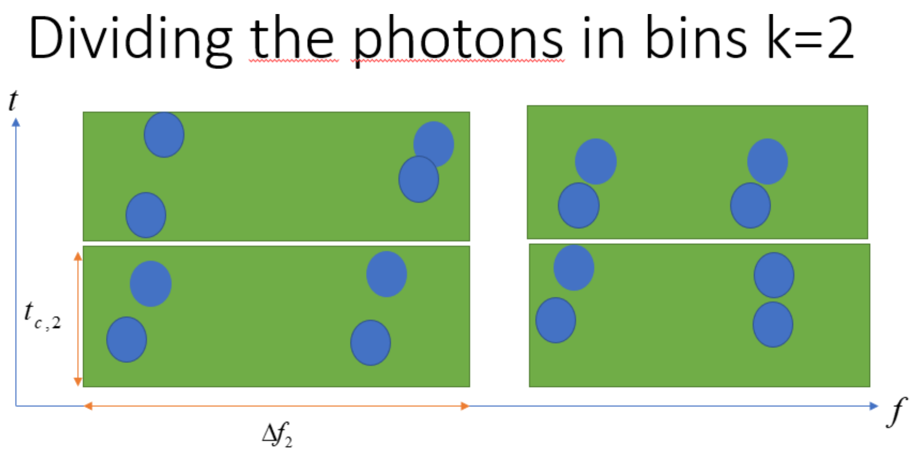


Figure 7.2: A representation of a the wide-band signal that is split into two bins

In figure 7.2 we can see that now the photons that were bunched, i.e. in the same green box in the figure, has changed. We can also see that the number of photons that have been bunched together, i.e. 4 in the figure, has been conserved. Now we have that the coherence time $t_{c,2} = 2t_{c,1} = \frac{1}{\Delta f_2} = \frac{2}{\Delta f_1}$. So our new coherence time $t_{c,2}$ is half the original coherence time of the wide band signal $t_{c,1}$ and our new bandwidth has been halved.

By continuing this narrative by induction we can get to the point where the binds are infinitesimally small.The green boxes are very long and very thin.

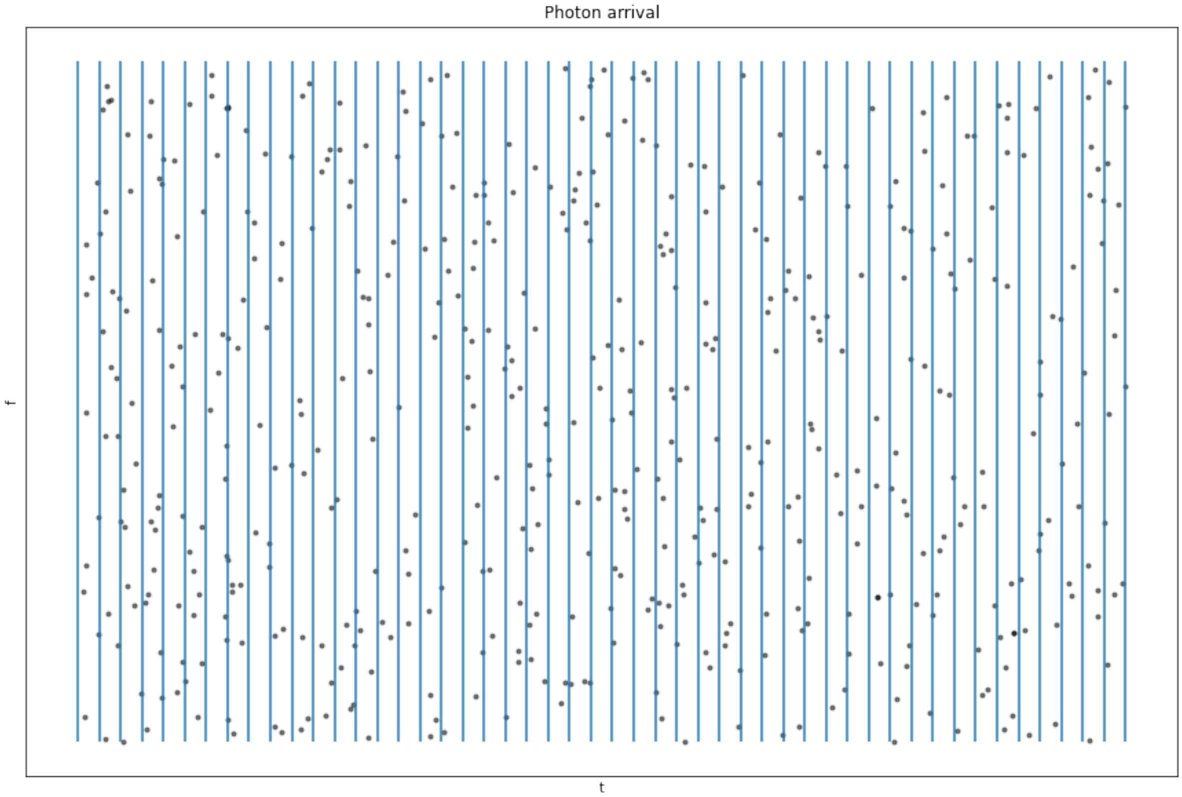


Figure 7.3: A representation of a the wide-band signal that is split into infinitesimally small bins. Photon number still conserved in theory.

We have that for $k >> 1$, we have the coherence time is $t_{c,k} = 2^{k-1} * t_{c,1}$. This means our coherence time is close to infinity. And the corresponding bandwidth Δf_k must be close to 0. This is the equivalent case to taking our Riemann integral. We now assume that the mean photon number within each bin is preserved. This means that the same number of photons on average are bunched whether we take the wide-band frequency bandwidth or the narrow-band bins.As photons are indistinguishable, we have that the number of the bunched photons is conserved. However we due to the uncertainty principle which can be explained both classically and quantum mechanically, we have no way of knowing which photons are exactly bunched at the moment of detection. Hence we can say that the photons that are dependent are indistinguishable from each other. Thus this makes them impossible to tell which photons are bunched with which photons. Therefore, we can conclude that when we use the infinitesimally narrow frequency sub-bands, that this has the same mathematical properties as taking the wide-band integral. Because these photons in different sub-bands are independent from each other, we can treat them as independent in the wide-band case. This proves that they can mathematically be seen as independent.

7.4. General Model

In the previous section, we made the assumption that the spectrum of the signal we were trying to measure was flat. This does not need to be the case. Let us make the same case as the previous time. We still have that the noise is white as the sum of the variances is not anymore dependent on the frequency. This time we will show the same proof as for the flat spectrum. We now have a condition that the non-flat spectrum is constant in time.

We take $k = 1$ again.

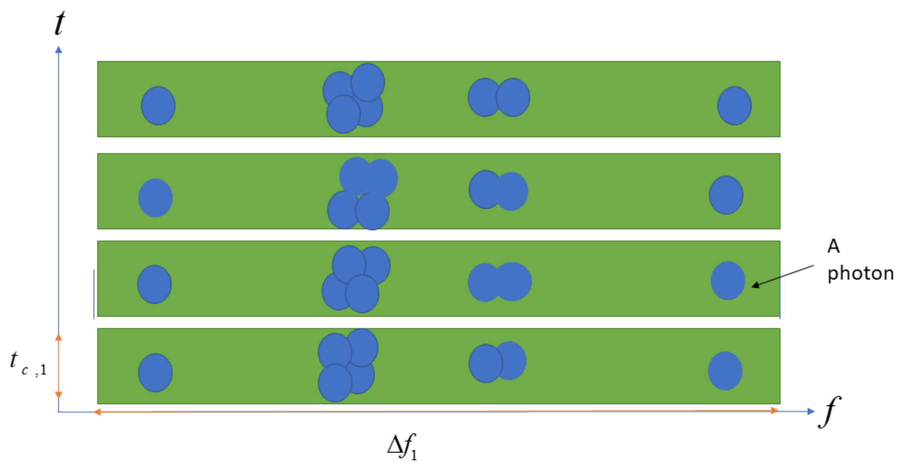


Figure 7.4: A representation of wide-band signal of non flat spectrum

In figure 7.4 we see again that the green boxes represent the bunched photons, we have that the photon density in each box is still preserved due to the time constant signal for all the bunched photons. The non-flat spectrum is reflected in the fact that the photons are not distributed evenly over the frequency bandwidths. If we split the measuring bandwidth Δf_1 in two, we again have that the coherence time will be twice as small again. However, because the spectrum and thus also the photon density of the signal is not flat, we have that the number of bunched photons will also not be the same for every box. Let the number of bins be $k = 2$.

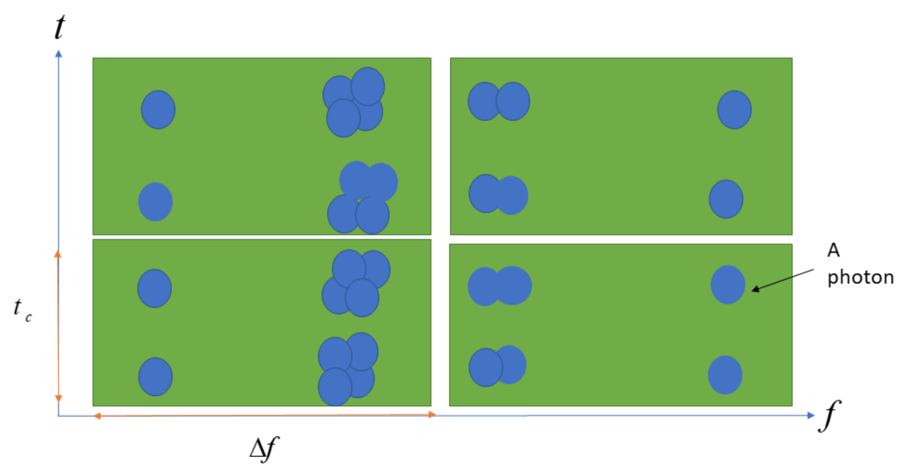


Figure 7.5: A representation of wide-band signal of non flat spectrum divided into 2 sub-bands

We can clearly see the photon number is not preserved in every box, however the number of boxes is preserved. So how does this work without the preservation of occupation number per green box.

Let us divide all the sub-bands intro two again in figure 7.6:

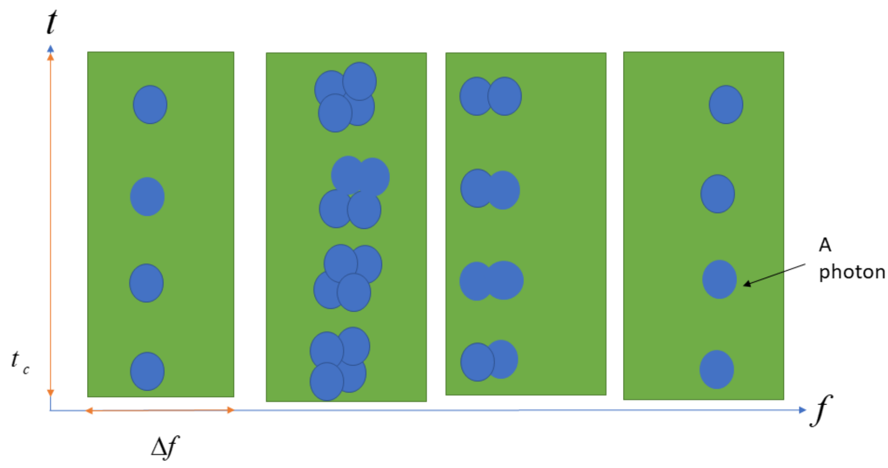


Figure 7.6: A representation of wide-band signal of non flat spectrum divided into 4 sub-bands

The number of photons that are bunched per box changes every time. If we were to repeat this process a k»1 amount of times, we would have infinitely small sub-bands and infinitely large coherence times. As long as we have a time constant non-flat spectrum, it doesn't matter which photons are in coherence box. Due to the Heisenberg uncertainty principle or the uncertainty prinvciple based on the Fourier transform, we have when $k >> 1$, we have an infinitely small frequency band-width on which two photons are correlated. The coherence time of these small subbands is infinite. However since we integrate over the frequency domain, this is irrelevant to our calculation.In the frequency domain we still preserve the independence of the wide-band signal. Thus we know that we can treat photons as statistically independent for frequency bandwidths.

Why bunching needs to be taken into account for the noise calculation and not for the integration? We know that bunching is a fundamental property of photons. It exists due to the particle-wave duality of light that was first introduced by quantum mechanics.

7.5. The mathematical explanation

The principle what we are working with, is based upon the Riemann Integral in mathematics. The mathematical definition of a Riemann integral is as follows:

To define a Riemann integral, we first have to define a partition: "A partition of an interval [a, b] is a finite sequence of numbers of the form

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Each $[x_i, x_{i+1}]$ is called a sub-interval of the partition.

A tagged partition $P(x,t)$ of an interval $[a,b]$ is a partition together with a finite sequence of numbers $t_0, ..., t_{n-1}$ for which $t_i \in [x_i, x_i + 1]$ with $i \in [0, n-1]$. So that means that it is a partition that contains an element in each sub-interval. We can say that the sub-bands are sub-intervals of the wide-band signal.

Now we get to the definition of the Riemann integral: Let f be a real-valued function defined on the integral $[a,b]$. The Riemann sum of f with respect to tagged partition $x_0, ..., x_n$ together with $t_0, ..., t_n - 1$ is

$$\sum_{i=0}^{n-1} f(t_i)\,(x_{i+1} - x_i)\,.$$

We have that every term of the sum is equal to the product of the value of the function f at a certain point and the length of the sub-interval. Graphically this is equivalent to the area of the triangle of the function value at t_i, namely $f(t_i)$, and the width of the sub-interval $[x_i, x_i + 1]$, which is $x_{i+1} - x_i$.

Now when we define the actual Riemann integral, the general idea is as follows. ”The Riemann integral is the limit of the Riemann sums as the partitions get finer”. If this limit exists, we have tat the function is integrable. The finer the partition, the more close the Riemann sum is to the actual integral.

The Riemann integral has to satisfy the following condition: For all $\epsilon > 0$, there exists a $\delta > 0$ such that for any tagged partition $x_0, \ldots, x_n$ and $t_0, .., t_{n-1}$ with a norm smaller than δ, we have

$$\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) - s < \varepsilon$$

, where s is the value of the Riemann integral.

With these definitions we can see that when we use the sub-bands as sub-intervals and the rectangles are just the approximation of the function value over this very small sub-band that this is essentially equivalent to taking the Riemann integral over the wide-band signal.[18]

8

Discussion

8.1. The integral

In chapter 5 we discuss the wide-band integral as opposed to the a summation narrow-band approximated noise terms. We have seen that the two are actually mathematically justified. The power of taking the wide-band integral as we have seen in equation 1.1, is that it reduces the number of Gaussian random numbers that would need to be generated when taking 1500 bins. As seen in the appendix, it suffices to generate only 1 random number. The code in the appendix is only includes as an illustration of how a code with the wide-band integral used numerically could look like. The results coming from the code are incorrect and therefore in no way, shape or form an accurate representation of the result of the simulation. The code is only provided to give an example of how long the coding would be and how different the coding would be fro the original numerical approximation using 1500 bins and also 1500 Gaussian randomly generated numbers. Due to the complexity of the TiEMPO model and the code, this will need to be one in the future. Also the way the integral is implemented is not a representative of a correct numerical method to integration, which has been chosen with careful research and experimentation in the TiEMPO model. What it does show that indeed a shorter way of calculating the noise by the integral can be implemented. This advantage in memory and computation speed can be used to make the simulation less lengthy and more memory efficient. We have also concluded that taking the integral is valid due to the the variances of uncorrelated random variables being additive. Hence from this moment forth this approach is mathematically justified whenever we assume that photons arriving in different frequency bandwidths are independent, no matter the size of the bandwidth. This is at least true when the entire detection is done over a relatively large bandwidth. With respect to the millimeter and sub-millimeter photon statistics, we know that using the Bose-Einstein statistics to be the most accurate.

8.2. Paradox

We have seen that the independence of the photons in different bandwidths is validated by the fact that we are using a wide-band frequency range over which our detection is done. Due to this wide-band we know that the coherence time of the arriving photons is very small. Hence the photons are arriving independently from one another, as our measuring time is a lot bigger that the coherence time [3].

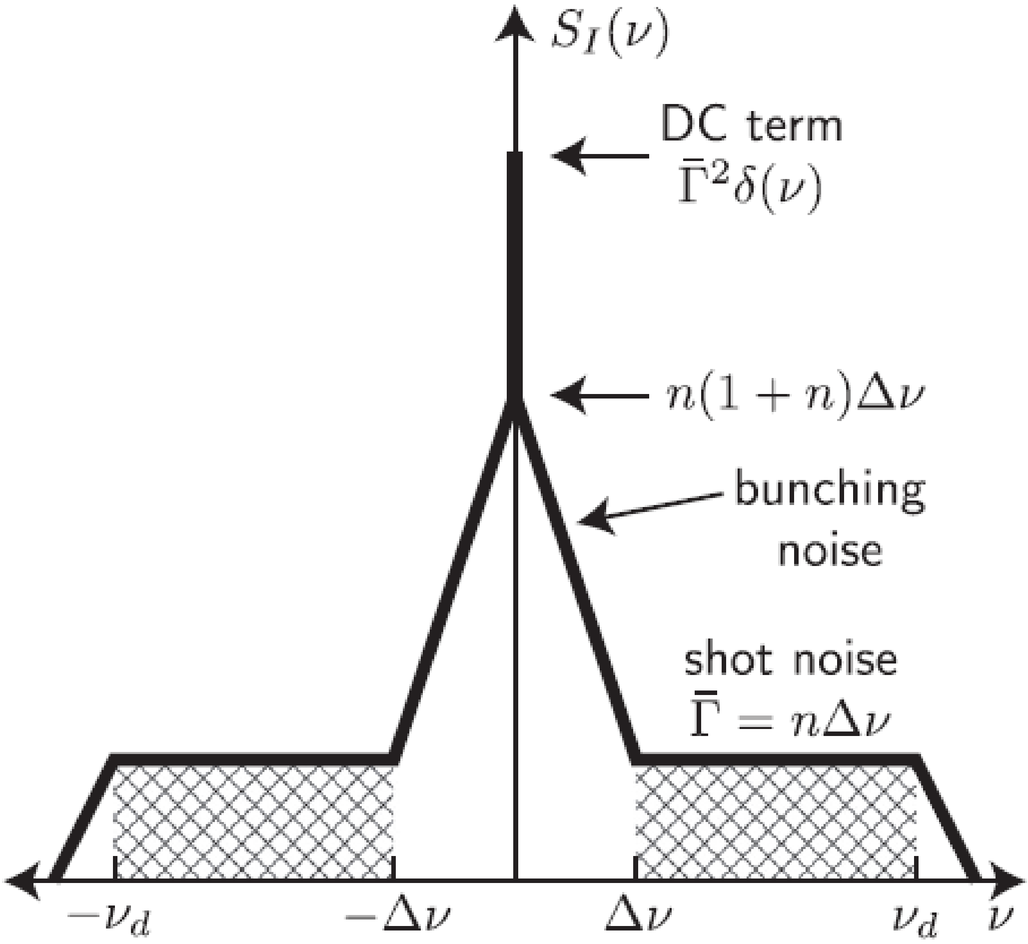


Figure 8.1: The Fourier transform of the noise of an ideal photon detector around a carrier frequency in the center of the plot. We have that the Poisson noise components are in the shaded bands of the plot. You can clearly see that the spectrum of the Poisson noise is white and flat. The photon bunching component is given by the triangular shaped spectrum in the middle of the spectrum. The third component at the very center is the Dc photon current. This is directly proportional to the mean arrival rate of the photons. This picture is based on a constant occupation number n over an optical bandwidth $\Delta\nu$. The mean photon arrival rate is equal to $\bar{\Gamma} = n\Delta\nu$. The frequency bandwidth of the detector is denoted by ν_dTaken from [19]

Because we know that the noise is white and therefore independent of frequency, two questions arise. How come the photon bunching noise is frequency dependent and the total noise is white. when we take the summation over the entire bandwidths and thus also these small sub-bands, we get that the sum over all the intervals is not dependent of the frequency anymore. This principle is also discussed by Zmuidzinas in a later paper[19]. Due to the high spectral resolution of DESHIMA which is $\frac{\nu}{\Delta\nu}\ 1000$. This means that the wide-band noise signal is dominated by the Poisson noise of the photons [20]. The photon noise which is subjected to random fluctuations in time due to bunching. this is because the arrival rate of the photons is randomly fluctuating as a result of photon bunching. The flat white noise spectrum is undergoing something akin to Amplitude Modification (AM). This means that modulation bands are added below and above carrier frequency. As seen in figure **??**. These so called side bands appear on all the Fourier components of the Poisson noise. The net result of summing all the Fourier components of the photon noise will therefore still be a white spectrum. That is how bunching noise is allowed to hide under the white noise [19].

When we add these Fourier noise terms together, we have that the total sum is still a white flat spectrum.[19]. Here we see that the Fourier transform explains why the noise is independent.

8.3. Quantum interpretation

Following the explanation in chapter 7, we have that due to the uncertainty principle, the dependence of the photons in different sub-bands is lost in the translation from the time domain to the frequency domain.This is also in accordance to the Heisenberg uncertainty principle. This problem mentioned above also has a quantum interpretation. Namely that due to the Heisenberg uncertainty principle, the photons that bunch over the time domain, do not have the same correlation in the frequency domain due to the wide-band range of measurement and that in gaining knowledge on the power or energy of the photon, this in turn cost the information of the arrival time and time correlations of the photons. We also have that the sampling rate of 160Hz is too large to pick up the time correlation of the wide-band signal [20].

9
Conclusion

In this thesis we have looked at the photon statistics of millimeter and sub-millimeter wave astronomy. Our main focus has been on enriching our theoretical knowledge about the effect of photon bunching on the sensitivity of DESHIMA the spectrometer. In order to achieve this, the narrow-band approximation has been shown to be equivalent to the wide-band integral of the photon noise of a detector. This means that up till now we have been calculating the noise of the astronomical signals in the millimeter and sub-millimeter range of the infrared radiation spectrum correctly. We have concluded that the best representation of the integral is seen in the form of equation 5.15. This is because this shows a noise term of an astronomical signal independent of the integration time or measuring bandwidth. This expression also gives us the NEP when we take the square root.

An acceptable solution to the paradox of whether the assumption that was made in a lot of literature about photon detection and bunching noise was valid and why that is. This assumption is that photons detected over a wide-band frequency range are statistically independent from other from another photon in a different sub-bandwidth. This sub-bandwidth is allowed to be infinitely small, in order to take the integral. It has been shown that the assumption is indeed correct, because based on this assumption the integral in equation **??** is valid and the approach that we have calculated the noise term for DESHIMA in TiEMPO was also correct. It has been demonstrated that taking these infinitely small sub-bands does not alter the statistical independence of the photons in the wide-band frequency. This statistical independence in the wide-band range is due to the length of the integration time of the measurement.This has to be substantially higher than the coherence time. This coherence time is small due to the uncertainty principle that can be seen as an effect of Fourier transformation or the Heisenberg principle. The uncertainty in the Fourier transform has shown that the correlation of the photons in time is lost when we measure the signal over a wide-band. This wide-band makes it so that the coherence time is very small and thus negligible and therefore can the photons be seen as independent from the photons in the other frequency bandwidths. We also have that the sampling rate of 160Hz is too large to pick up the time correlation of the wide-band signal. This does not change when we take the the narrow-band. We then have that the coherence time is so long that the photons arriving in bunches cannot be distinguished from one another and they are equivalent to the photons that are bunched in the wide-band signal. This also points to statistical independence among the photons in different sub-bands.

It seems like the temporal correlation of the photons in a wide-band signal simply gets lost due to the small uncertainty in energy and frequency. Now this can be explained by both quantum uncertainty principle or the Fourier uncertainty principle.

9.1. In the future

For the future of this research it would be advisable to do a thorough numerical mathematical analysis on how this integral is best applied to the TiEMPO simulation. A lot can be learned about the stability of the integral and the coherence of the integral in this model. The accuracy of the numerical approximation of the integral can also be researched and this could provide more accuracy for the model of TiEMPO as a whole.

The mathematical proof of why the integral of equation 5.15 converges for the wide-band range or any finite frequency bandwidth would also give this approach more mathematical foundation. It would also show the validity of the integral in the context of astronomy or physics in general.

The implementation of the wide-band integral into TiEMPO is also something that still needs to be done. This would allow more accuracy for the model and give better understanding on how the theory translates into numerical approximations. Then the model or simulation for this paradox could also be investigated in a side-by-side comparison of the performance of the model in with the narrow-band approximation and the wide-band integral.

Investigating whether the quantum optical interpretation of photon bunching and the paradox associated with the wide-band calculation of the photon noise or the semi-classical picture is more accurate solving this integral would also give more insight on the nature of photon bunching.

A

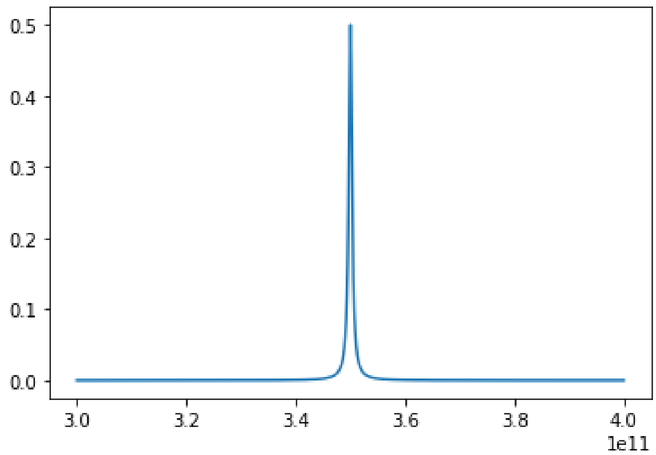
The code of the wide-band and narrow-band noise calculation

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.constants as sc
from astropy.modeling.models import Lorentz1D
from scipy.integrate import quad
import statistics
from scipy.stats import poisson
```

Coding for the problem

We begin by taking the Lorentzian as the η function, namely the quantum efficiency of the filter. We take 300 GHz and 400 GHz as the boundary frequencies to this wide-band. The full width at half maximum is 700 kHz.

```
a=300*10**9 #lower frequency limit
b=400*10**9#upper frequency limit
amp=0.5 #amplitude
x_0=350*10**9 #location of peak
fwhm=0.7*10**9 # full width at half maximum
numbin=1500 # number of bins
s1 = Lorentz1D(amp,x_0,fwhm)
r=np.linspace(a,b,numbin*10) # creat a grid of 15000 values to take the
#integral of the lorentz distribution function over
#weight,err2=quad(s1,a,b)
def eta(f):
    return s1(f)
#filter response
plt.figure()
plt.plot(r,eta(r))
plt.show() #plotting lorentzian
```




```
h = 6.62607004e-34 # Planck constant
k = 1.38064852e-23 # Boltzmann constant
T = 40 #temperature in Kelvin

def n_rad(f): #photon occupation number of atmospheric incoming signal
    return (np.exp((h*f)/(k*T))-1)**-1
def psd(f): #power density spectrum
    return h*f*n_rad(f)

def Powerdens(f):  # power density from the power of each bin
    #including the filter response
    return psd(f)*eta(f) #Also from esmee though I might have
    #not used it right

df=(b-a)/numbin
def NEP(f): # noise equivalent power
    return np.sqrt((2*P_bin(f)*df*h*f*(1+(eta(f)*n_rad(f))))/eta(f))


bins=np.array_split(r,numbin) #splitting array of 15000 values into
#subarrays with 10 subsequential numbers each
#print(bins)
```

```
#def derivLor(f):
#return (16*(f-x_0)*fwhm)/(np.pi*((4*(f-x_0)**2+fwhm**2)**2)) #unnecessary


#plt.figure()
#plt.plot(r,NEP(r))
#plt.show()
#print(np.sqrt(sum((NEP(r)**2))))
```

```
bins_meanlist=[]
delta_flist=[]
P_avg_list=[]
for elt in bins:
    m=np.mean(elt)
    d=max(elt)-min(elt)
    bins_meanlist.append(m)
    delta_flist.append(d)
#this loop takes the mean value of each bin subarray and
#makes a list of the means of each bin
# also delta_flist is a list made of the bandwidths of each bin
bins_mean=np.array(bins_meanlist) #array of bin means
delta_f=np.array(delta_flist) # array of bandiwths of each bin
Pbin=Powerdens(bins_mean)*delta_f #all bin means are used to generate
#the average power density at that particular bin mean.
#Then it is multiplied by the bandwidth to convert it into the average power
Noise_bins=np.zeros(numbin)
for i in range(0,numbin):
    Noise_bins[i]=np.random.normal(Pbin[i],NEP(bins_mean[i]))
# for each bin mean a random number is generated
#from the normal distribution using the average power and the nep of that particular bin
sigma1=np.sqrt(sum((NEP(bins_mean)**2))) # first noise simulation
print("The first noise simulation leads to a total standard deviation of "+  str(sigma1))
```

```
The first noise simulation leads to a total standard deviation of 1.4421974449554 57e-16
```

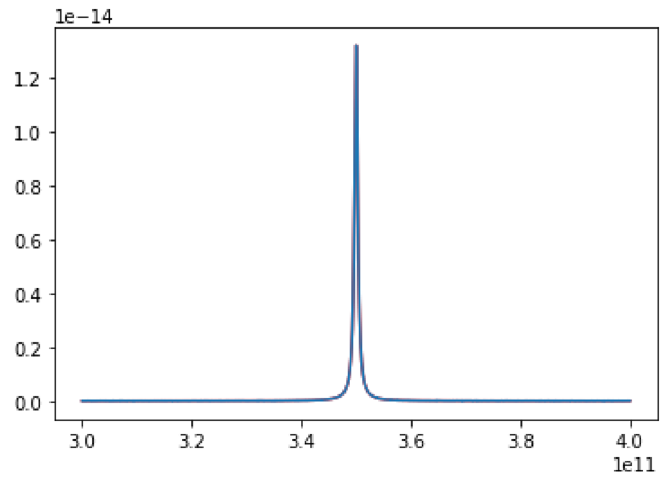
```
def integrand(x):
    return (2*h**2*x**2*n_rad(x)*eta(x)*(n_rad(x)*eta(x)+1))/(eta(x)**2)
#making zmuidzinas integral
ans, err = quad(integrand, a, b) # integrating over the wideband
sigma2=np.sqrt(ans) #result of the integral
def int2(x):
    return h*eta(x)*n_rad(x)*x
P_average,err1=quad(int2, a,b)
noise2=np.random.normal(P_average,sigma2) # the generated noise
```

```
print(sigma2)
print(noise2)
```

```
1.6727474131925326e-14
2.2025192095836662e-13
```

```
plt.plot(bins_mean,Pbin,color='r') #signal without noise
plt.plot(bins_mean,Noise_bins) # signal with noise
```

```
[<matplotlib.lines.Line2D at 0x20b4f287c50>]
```




```
#def derivLor(f):
#return (16*(f-x_0)*fwhm)/(np.pi*((4*(f-x_0)**2+fwhm**2)**2)) #unnecessary
```

```
#plt.figure()
#plt.plot(r,NEP(r))
#plt.show()
#print(np.sqrt(sum((NEP(r)**2)))
```

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