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RESEARCH

The ECD-BIE Estimation Theorem in Spectral Form

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Abstract
The theory of best integer equivariant (BIE) estimation provides the minimum mean squared error estimator within the class of integer equivariant (IE) estimators and has found increasing application in mixed-integer models such as those arising in carrier-phase GNSS positioning. Recent developments extended BIE estimation to the broader class of elliptically contoured distributions (ECDs), thereby allowing the use of heavier-tailed models. In this contribution, we further develop this theory by deriving a spectral form of the ECD-BIE theorem. Building on the periodic representation of IE-estimators, whereby each estimator can be expressed as a correction to the best linear unbiased estimator (BLUE) that is integer-periodic in the ambiguity domain, we show how this spatial representation can be transformed into a Fourier-based spectral form. This establishes twin representations of ECD-BIE estimators in both spatial and spectral domains. The relative merits of these formulations are discussed and illustrated for the normal, contaminated normal, and multivariate Student distributions.

Keywords Integer-Equivariant (IE) Estimation · Global Navigation Satellite Systems (GNSS) · Mixed-integer model · Best Linear Unbiased Estimation (BLUE) · Fourier ambiguity resolution · Best Integer Equivariant (BIE) Estimator · Spectral-BIE

1 Introduction

The theory of best integer equivariant (BIE) estimation was introduced in Teunissen (2003). The BIE-estimator is within the class of integer equivariant (IE) estimators, the estimator with smallest mean squared error. As the IE-class includes both the class of admissible integer (I) estimators and the class of linear unbiased (LU) estimators,

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the precision of the unbiased BIE estimator is never poorer than that of the best linear unbiased estimator (BLUE) and also never poorer than that of the integer least-squares estimator (ILS). This property is likely the reason for the increasing popularity of the BIE-estimator, in particular when dealing with mixed-integer models, such as is the case with carrier-phase GNSS Strang et al. 1997; Hofmann-Wellenhof et al. 2008; Leick et al. 2015; Teunissen 2017; Morton et al. 2020). Such GNSS applications and performance comparisons can be found in Verhagen and Teunissen (2005); Wen et al. (2012); Odolinski and Teunissen (2020b, a); Brack et al. (2023b); Yang et al. (2023); Brack et al. (2023a). Sequential and vectorial bootstrapping versions of the BIE-estimator have also been developed (Brack et al. 2014; Teunissen et al. 2021; Brack 2025).

A further extension of the IE-theory was realized in Teunissen (2020) with the introduction of BIE-estimators for the general class of elliptically contoured distributions (ECDs). These BIEs are particularly of interest if one wants to work with distributions having tails that are heavier than that of the normal distribution (Cabane et al. 1981). Applications and GNSS performance comparisons for some of these ECD-BIEs can be found in Odolinski and Teunissen (2022); Wu and Shaofeng (2026). In this contribution we will extend this ECD-BIE theory by providing a spectral form of the ECD-BIE theorem. In doing so, we provide in Section 2 a brief review of the principle of integer equivariant estimation and in particular show that every IE-estimator can be written as a correction on the BLUE, whereby the correction is $\mathbb{Z}^n$-periodic in its ambiguity slot. It is this representation that forms the basis of Fourier ambiguity resolution as introduced in Teunissen (2026).

To exploit the periodic representation of IE-estimators, we first present the general spatial form of the ECD-BIEs in Section 3, followed by necessary multivariate Fourier theory in Section 4. This theory is applied in Section 4 so as to Fourier-transform the spatial ECD-BIE representation into the new spectral form. As a result we obtain new representations for each of the ECD-BIE estimators. This therefore establishes the existence of two different expressions for the same BIE-estimator, one in the spatial domain and one in the spectral domain. The relative merits of the two BIE expressions are discussed and further exemplified with three analytically worked out spectral BIE estimators, namely for the normal distribution, for the contaminated normal distribution and for the multivariate Student distribution. Finally, a summary and conclusions are given in Section 5.

The following notation is used throughout. $\mathsf{E}(.)$ and $\mathsf{D}(.)$ stand for the mathematical expectation and dispersion operators, respectively. $\mathcal{N}_m(\mu, Q)$ denotes an m-dimensional, normally distributed random vector, with mean (expectation) μ and variance matrix (dispersion) Q, and $T_m(\mu, \Sigma, d)$ denotes an m-dimensional, multivariate Student or t distribution, with mean μ and variance matrix $\frac{d}{d-2}\Sigma$, having $d > 2$ degrees of freedom. The Best Linear Unbiased Estimator (BLUE) of a parameter vector x is denoted as $\hat{x}$ and its BIE-estimator as $\hat{x}_{\mathrm{BIE}}$. The range space of a matrix M is denoted as $\mathcal{R}(M)$, and $\mathbb{R}^p$ and $\mathbb{Z}^p$ denote the p-dimensional spaces of real- and integer numbers, respectively. The Σ-weighted squared norm is denoted as $||.||^2_\Sigma = (.)^T\Sigma^{-1}(.)$. The Σ-orthogonal projector onto $\mathcal{R}(M)$ is denoted as $P_M = MM^+$, with BLUE-inverse $M^+ = (M^T\Sigma^{-1}M)^{-1}M^T\Sigma^{-1}$. Thus, $P^\perp_M = I - P_M$ is the orthogonal projector that projects orthogonally on the orthogonal complement of $\mathcal{R}(M)$.

2 Integer Equivariant (IE) Estimation

The class of integer-equivariant (IE) estimators was introduced in Teunissen (2003). Let the mean of the m-vector of observables be mixed-integer parametrized as

$$\mathsf{E}(y) = Aa + Bb \ , \ a \in \mathbb{Z}^n, b \in \mathbb{R}^p \tag{1}$$

with $[A, B] \in \mathbb{R}^{m\times(n+p)}$ given and of full rank, and let our interest lie in estimating the linear function

$$\theta = L_a a + L_b b \ , \ \theta \in \mathbb{R}^q \tag{2}$$

It then seems natural to require that the estimator obey the integer remove–restore principle. For example, in the estimation of GNSS integer ambiguities, one would expect that adding an arbitrary number of cycles to the carrier-phase observations results in the estimated ambiguities shifting by the same integer amount. For the estimator of θ, this implies that adding Az to y, for arbitrary $z \in \mathbb{Z}^n$, should produce a corresponding shift of $L_a z$ in the estimate. Likewise, it is reasonable to require that adding $B\beta$ to y, for arbitrary $\beta \in \mathbb{R}^p$, results in a shift of $L_b\beta$. After all, such additions to the observations should not influence the integer component of the estimator. Imposing these two requirements leads to the following definition of integer equivariant estimation.

Definition (*IE-estimation*) The estimator $\hat{\theta}_{\mathrm{IE}} = F_\theta(y)$, $F_\theta : \mathbb{R}^m \mapsto \mathbb{R}^q$, is an integer equivariant estimator of $\theta = L_a a + L_b b \in \mathbb{R}^q$ if $F_\theta(y + Az + B\beta) = F_\theta(y) + L_a z + L_b\beta, \forall y \in \mathbb{R}^m, z \in \mathbb{Z}^n, \beta \in \mathbb{R}^p$.

As the mixed-integer model Eq. 1 is typical for carrier-phase GNSS, we will refer in the following to $a \in \mathbb{Z}^n$ as the ambiguity-vector and to $b \in \mathbb{R}^p$ as the baseline-vector. An important result from IE-theory is that every IE-estimator can be written as a corrected BLUE, whereby the correction is a function of the misclosure vector and periodic in its ambiguity slot (Teunissen 2003).

Theorem 1 *(IE representation) Let $\hat{a}$ and $\hat{b}$ be the BLUEs of a and b, respectively, and let $t = D^T y$ be the misclosure vector of Eq. 1, for some $m \times (m - n - p)$ basis matrix D satisfying $D^T(A, B) = 0$. Then every IE-estimator of $\theta = L_a a + L_b b$ can be represented as*

$$\check{\theta} = L_a\hat{a} + L_b\hat{b} + \Pi_\theta(\hat{a}, t) \tag{3}$$

for some periodic function $\Pi_\theta(x + z, t) = \Pi_\theta(x, t), \forall z \in \mathbb{Z}^n$.

Note that $\hat{\theta} = L_a\hat{a} + L_b\hat{b}$ is the BLUE solution of θ. It is the solution one would get when the integerness of the ambiguities is not imposed. The above result shows therefore that the impact of ambiguity resolution is only felt through a function that is periodic in its ambiguity slot. The IE-estimators of ambiguities and baseline are obtained, with $L_a = I_n$, $L_b = 0$ and $L_a = 0$, $L_b = I_p$, as

$$\begin{aligned} \check{a} &= \hat{a} + \Pi_a(\hat{a}, t) \\ \check{b} &= \hat{b} + \Pi_b(\hat{a}, t) \end{aligned} \tag{4}$$

Thus every ambiguity-resolved baseline $\breve{b}$ can be obtained by means of a periodic correction $\Pi_b(\hat{a}, t)$ on the float baseline $\hat{b}$. That is, the difference between the float solution and the ambiguity-resolved solution is always periodic.

The following examples demonstrate that the class of IE-estimators is a broad class of estimators.

Example 1 If we choose $\Pi_\theta(\hat{a}, t) = 0$, then it directly follows from the above theorem that the BLUE is an IE-estimator.

Example 2 We now show that every linear unbiased estimator (LUE) is an IE-estimator. Recall that any LUE of θ can be written as $\bar{\theta} = L_a\bar{a} + L_b\bar{b}$, with $[\bar{a}^T, \bar{b}^T]^T = [L^T(A, B)]^{-1}L^T y$ for some L. What remains to be shown is therefore that the differences between a LUE and a BLUE are linear functions of the misclosure vector t, i.e. $[(\bar{a} - \hat{a})^T, (\bar{b} - \hat{b})^T]^T = Mt$ for some M. This follows directly from Lemma 2 of Teunissen and Khodabandeh (2013).

Example 3 If we choose $\Pi_\theta(\hat{a}, t) = -[L_a + L_b Q_{\hat{b}\hat{a}} Q_{\hat{a}\hat{a}}^{-1}](\hat{a} - \breve{a})$ for some arbitrary admissible ambiguity estimator $\breve{a}$, then $\breve{\theta} = L_a\breve{a} + L_b\breve{b}$. This shows that every admissible integer (I) estimator is also an IE-estimator (Teunissen 2003).

The above shows that both the class of admissible integer estimators (I-class) and the class of linear unbiased estimators (LU-class) are subsets of the class of integer equivariant estimators,

$$\mathrm{I} \subset \mathrm{IE} \quad \text{and} \quad \mathrm{LU} \subset \mathrm{IE} \tag{5}$$

This has the important consequence that optimality in IE automatically carries over to I and LU. Probably the most popular optimal estimator in the LU-class is the best linear unbiased estimator (BLUE), where 'best' is taken in the minimum mean squared error (MMSE) sense (Rao 1973; Koch 1999; Teunissen 2024). Using the same criterium, but now in the larger class of integer equivariant estimators, leads to the best integer equivariant (BIE) estimator. The general BIE formula is due to Teunissen (2003). Here however, we will specialize to the still broad, but practically important subclass of elliptically contoured distributions. Examples of such distributions are the multivariate normal distribution, the contaminated normal distribution and the multivariate Student-distribution.

3 BIE for Elliptically Contoured Distributions

Elliptically contoured distributions are defined as follows Cabane et al. (1981):

Definition (*Elliptically contoured distribution (ECD)*) A random vector $y \in \mathbb{R}^m$ is said to have an ECD, denoted as $y \sim \mathrm{ECD}_m(\bar{y}, \Sigma_{yy}, g)$, if its PDF is of the form

$$f_y(y) = |\Sigma_{yy}|^{-1/2} g(||y - \bar{y}||^2_{\Sigma_{yy}}) \tag{6}$$

where $\bar{y} \in \mathbb{R}^m$, $\Sigma_{yy} \in \mathbb{R}^{m\times m}$ is positive definite, and $g : \mathbb{R} \mapsto [0, \infty)$ is a decreasing function that satisfies $\int_{\mathbb{R}^m} g(y^T y)dy = 1$.

Note that the contours of constant density of an ECD are ellipsoids, $(y - \bar{y})^T \Sigma_{yy}^{-1}(y - \bar{y}) = \text{constant}$. Also note, since Eq. 6 is symmetric with respect to $\bar{y}$, that $\bar{y}$ in Eq. 6 is the mean of y, $\mathsf{E}(y) = \bar{y}$. It can be shown that the variance matrix of y is a scaled version of Σ_{yy}, $\mathsf{D}(y) = \sigma^2 \Sigma_{yy}$. The function $g(.)$ of Eq. 6 will be referred to as the generating function of the ECD. Several of the properties of ECDs are similar to those of the multivariate normal distribution. For instance, linear functions of an ECD are again an ECD. Also marginal and conditional PDFs of ECDs are an ECD.

The best integer equivariant estimator for elliptically contoured distributions was introduced in Teunissen (2020) and is given by the following theorem.

Theorem 2 *(The ECD-BIE) Let* $y \sim \text{ECD}_m(Aa + Bb, \Sigma_{yy}, g)$ *have the PDF* $f_y(y) = |\Sigma_{yy}|^{-1/2} g(||y - Aa - Bb||^2_{\Sigma_{yy}})$, $a \in \mathbb{Z}^n$, $b \in \mathbb{R}^p$. *Then the BIE-estimator of* $\theta = L_a a + L_b b$ *is given as* $\hat{\theta}_{\text{BIE}} = L_a \hat{a}_{\text{BIE}} + L_b \hat{b}_{\text{BIE}}$, *with*

$$\begin{aligned} \hat{a}_{\text{BIE}} &= \textstyle\sum_{z \in \mathbb{Z}^n} \frac{z f(\hat{a} - z)}{\sum_{u \in \mathbb{Z}^n} f(\hat{a} - u)} \text{ and} \\ \hat{b}_{\text{BIE}} &= \hat{b} - \Sigma_{\hat{b}\hat{a}} \Sigma_{\hat{a}\hat{a}}^{-1} (\hat{a} - \hat{a}_{\text{BIE}}) \end{aligned} \tag{7}$$

where $f(x) = \int_0^\infty g(||\hat{e}||^2_{\Sigma_{yy}} + ||x||^2_{Q_{\hat{a}\hat{a}}} + r^2) r^{p-1} dr$ *and* $\hat{e} = y - A\hat{a} - B\hat{b}$, *with* $\hat{a}$ *and* $\hat{b}$ *being the BLUEs of a and b, respectively.*

To appreciate the 'best' property of the above BIE-estimator, recall that the class of linear unbiased estimators is a subset of the class of integer equivariant estimators, LU $\subset$ IE. This implies that the mean squared error (MSE) of the best linear unbiased estimator (BLUE) is never better than that of its corresponding BIE estimator. Also, since both estimators are unbiased, it implies that the BLUE-precision is never better than the BIE-precision,

$$Q_{\hat{\theta}_{\text{BIE}}\hat{\theta}_{\text{BIE}}} \leq Q_{\hat{\theta}_{\text{BLUE}}\hat{\theta}_{\text{BLUE}}} \tag{8}$$

In the context of GNSS it would therefore make sense to *always* compute the BIE baseline estimator, as its precision will never be poorer than that of the 'float' baseline solution $\hat{b}$. Likewise, since the class of integer-estimators is also a subset of the class of integer equivariant estimators, I $\subset$ IE, it follows that the BIE estimator is also MSE-superior to any admissible integer-estimator, thus also to such popular estimators as integer rounding (IR), integer bootstrapping (IB), and integer least-squares (ILS).

4 ECD-BIE Theorem in Spectral Form

In this section we will exploit the periodic respresentation of IE-estimators and show how multivariate Fourier-theory can be applied to transform the ECD-BIE theorem of the previous section into spectral form. As a result we obtain new representations for each of the ECD-BIE estimators.

4.1 Multivariate Fourier Theory

Although Fourier theory is usually presented for scalar functions, the generalization to the multidimensional case, using vector functions, goes quite naturally. Some first results of Fourier theory are summarized here. Let the two multivariate functions $f, F : \mathbb{R}^n \to \mathbb{C}$ be each others Fourier transform pair, denoted as $f(x) \rightleftharpoons F(s)$. Then

$$\begin{aligned} F(s) &= \int_{\mathbb{R}^n} f(x) \exp(-j2\pi s^T x) dx \\ f(x) &= \int_{\mathbb{R}^n} F(s) \exp(+j2\pi s^T x) ds \end{aligned} \tag{9}$$

with j denoting the imaginary unit ($j^2 = -1$). If needed, Euler's formula can be used to decompose the complex exponential in terms of cosines and sines, $\exp(+j2\pi\alpha) = e^{+j2\pi\alpha} = \cos(2\pi\alpha) + j\sin(2\pi\alpha)$. Note, if $f(x)$ is real and even, $f(x) = f(-x)$, then $F(s)$ is real. Next to the above Fourier-pair, we have for functions $g : \mathbb{R}^n \to \mathbb{C}$ that are $\mathbb{Z}^n$-periodic, $g(x + z) = g(x)$, the following Fourier-pair,

$$\begin{aligned} G_z &= \int_{\mathcal{P}_0} g(x) \exp(-j2\pi z^T x) dx \\ g(x) &= \sum_{z\in\mathbb{Z}^n} G_z \exp(+j2\pi z^T x) \end{aligned} \tag{10}$$

This result is valid for any pull-in region $\mathcal{P}_0$. For proofs of the above pairs, we refer to textbooks such as e.g., Tolstov (1976); Champeney (1987); Osborne (2010). We now list some specific properties of Fourier-transform pairs. They will be needed in our proof of the spectral ECD-BIE theorem of the next section. We have:

$$\begin{array}{lll} (i) & f(x) & \rightleftharpoons F(s) \\ (ii) & f(Mx) & \rightleftharpoons |M|^{-1} F(M^{-T} s) \\ (iii) & f(x) = \phi(||x||^2) & \rightleftharpoons F(s) = \Phi(||s||^2) \\ (iv) & xf(x) & \rightleftharpoons \frac{1}{2\pi} j\partial_s F(s) \\ (v) & \sum_{z\in\mathbb{Z}^n} f(x - z) & = \sum_{z\in\mathbb{Z}^n} F(z) \exp(+2\pi j z^T x) \end{array} \tag{11}$$

Property (ii) shows how the Fourier-transform changes under stretching in the spatial domain. It shows that stretching in the spatial domain (x) causes compression in the frequency domain (s), and vice versa, while maintaining energy conservation. Thus if you make a signal tighter (faster) in space/time, its spectrum must spread out (become broader) in frequency. Property (iii) states if $f(x)$ is a radial function, i.e. a function in $||x||$, then so is its Fourier transform, i.e. a function in $||s||$. Combined with property (ii) this can be understood as follows. As a radial function depends only on $||x||$, it is invariant under rotations, $f(Rx) = f(x)$. Because the Fourier transform integral is symmetric with respect to rotation, rotating the input x to Rx simply rotates the integrand, leaving the result unchanged, confirming it is radial. Property (iv) shows that multiplication in space/time domain corresponds with taking derivatives in frequency domain. The last property (v), bridges discrete sampling and continuous Fourier analysis. It is Poisson's summation formula, which equates a sum of periodic translations of a function (periodization) to a Fourier series whose coefficients are samples of the Fourier transform.

4.2 The Spectral ECD-BIE Theorem

We will now derive the spectral form of the ECD-BIE Theorem 2 using the properties of Eq. 11. We rewrite Eq. 7 as

$$\hat{a}_{\mathrm{BIE}} = \hat{a} - \sum_{z\in\mathbb{Z}^n} \frac{(\hat{a}-z)\,f(\hat{a}-z)}{\sum\limits_{u\in\mathbb{Z}^n} f(\hat{a}-u)} \tag{12}$$

and first work on the denominator, using Poisson's summation equation. According to (v) we may write the denominator as $\sum_{z\in\mathbb{Z}^n} f(\hat{a}-z) = \sum_{z\in\mathbb{Z}^n} F(z)\exp(+2\pi\, j z^T\hat{a})$. Since the function $f(x)$ is real, even and radial, also $F(s)$ is real, even and radial. As $F(s)$ is real and even, we have $\sum_{z\in\mathbb{Z}^n} f(\hat{a}-z) = \sum_{z\in\mathbb{Z}^n} F(z)\cos(2\pi z^T\hat{a})$, which we can rewrite as

$$\sum_{z\in\mathbb{Z}^n} f(\hat{a}-z) = F(0) + 2\sum_{z\in\Omega} F(z)\cos(2\pi z^T\hat{a}) \tag{13}$$

in which $\Omega\subset\mathbb{Z}^n$ is the half-grid, taking only one of each pair $\{-z,+z\}$.

We now consider the numerator in the ratio of Eq. 12. Using properties $(i\,v)$ and (v) of Eq. 11, we may write $\sum_{z\in\mathbb{Z}^n}(\hat{a}-z)f(\hat{a}-z) = \sum_{z\in\mathbb{Z}^n}\frac{1}{2\pi}\,j\,\partial_s F(s)_{|z}\exp(+2\pi\, j z^T\hat{a})$. As $F(s)$ is radial, since $f(x)$ is radial, it follows that $\partial_s F(s)$ is an odd function. With $\partial_s F(s)$ being real and odd, it follows that $\sum_{z\in\mathbb{Z}^n}(\hat{a}-z)f(\hat{a}-z) = -\sum_{z\in\mathbb{Z}^n}\frac{1}{2\pi}\partial_s F(s)_{|z}\sin(2\pi z^T\hat{a})$, which we may rewrite as

$$\sum_{z\in\mathbb{Z}^n}(\hat{a}-z)\,f(\hat{a}-z) = -\tfrac{1}{\pi}\sum_{z\in\Omega}\partial_s F(s)_{|z}\sin(2\pi z^T\hat{a}) \tag{14}$$

If we substitute Eqs. 13 and 14 into Eq. 12, we may finally formulate the spectral form of the ECD-BIE theorem as follows:

Spectral ECD-BIE Theorem: Let $y\sim \mathrm{ECD}_m(Aa+Bb,\Sigma_{yy},g)$ have the PDF $f_y(y) = |\Sigma_{yy}|^{-1/2}g(||y-Aa-Bb||^2_{\Sigma_{yy}})$, $a\in\mathbb{Z}^n$, $b\in\mathbb{R}^p$. Then the spectral form of the BIE-estimator of $\theta = L_a a + L_b b$ is given as $\hat{\theta}_{\mathrm{BIE}} = L_a\hat{a}_{\mathrm{BIE}} + L_b\hat{b}_{\mathrm{BIE}}$, with

$$\hat{a}_{\mathrm{BIE}} = \hat{a} + \frac{1}{\pi}\frac{\sum_{z\in\Omega}\partial_s F(s)_{|z}\sin(2\pi z^T\hat{a})}{F(0)+2\sum_{z\in\Omega}F(z)\cos(2\pi z^T\hat{a})} \tag{15}$$

in which $\Omega\subset Z^n$ is the half-grid, taking only one of each integer pair $\{-z,+z\}$, and $F(s)\rightleftharpoons f(x) = \int_0^\infty g(||\hat{e}||^2_{\Sigma_{yy}} + ||x||^2_{\Sigma_{\hat{a}\hat{a}}} + r^2)r^{p-1}dr$.

This is a general spectral BIE-theorem that holds true for any elliptically contoured distribution that the observable vector y of the mixed-integer model Eq. 1 might have. With this new result we now have two different expressions available for the same BIE-estimator, the classical spatial respresentation of theorem 2 (cf. Eq. 7) and the spectral respresentation Eq. 15. The importance of their relative merit lies in the behaviour of the coefficients used in the two different expansions. As these coefficients are driven

by the two functions $f(x)$ and $F(s)$, it are the characteristics of these two functions that drive the convergence behaviour of the two expansions. We will exemplify this further in the next section where we will specialize to some specific and often used distributions.

4.3 Spectral-BIE For Some Specific ECDs

To determine the spectral-BIE for some specific ECDs, we need to specify their generating functions $g : \mathbb{R} \mapsto [0, \infty)$, followed by the determination of the Fourier-transform of $f(x) = \int_0^\infty g(||\hat{e}||^2_{\Sigma_{yy}} + ||x||^2_{\Sigma_{\hat{a}\hat{a}}} + r^2) r^{p-1} dr$. We will determine the Fourier-transform $F(s)$ of $f(x)$, together with its partial derivative, for the multivariate normal distribution, the contaminated normal distribution, and the multivariate Student distribution.

4.3.1 Spectral BIE for the Multivariate Normal Distribution

The elliptically contoured PDF $f_y(y) = |\Sigma_{yy}|^{-1/2} g(||y - Aa - Bb||^2_{\Sigma_{yy}}$ becomes that of a multivariate normal distribution $\mathcal{N}_m(Aa + Bb, \Sigma_{yy})$, when the generating function $g(\alpha)$ is chosen as

$$g(\alpha) = (2\pi)^{-m/2} e^{-\frac{1}{2}\alpha} \tag{16}$$

Thus in this case, Σ_{yy} is the variance matrix of y. The following result provides the spectral BIE-coefficients in case y is normally distributed.

Corollary 1 *(*Spectral BIE for normal distribution*) Let $y \sim \mathcal{N}_m(Aa + Bb, \Sigma_{yy})$, $a \in \mathbb{Z}^n$, $b \in \mathbb{R}^p$. Then the spectral BIE-coefficients of Eq. 15 are given as*

$$\begin{aligned} F(z) &= c_n \exp(-\tfrac{1}{2} 4\pi^2 ||z||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}}) \\ \partial_s F(s)_{|z} &= -4\pi^2 c_n \exp(-\tfrac{1}{2} 4\pi^2 ||z||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}}) \Sigma_{\hat{a}\hat{a}} z \end{aligned} \tag{17}$$

with c_n a common constant that cancels in their ratio, $\Sigma_{\hat{a}\hat{a}} = (\bar{A}^T \Sigma^{-1}_{yy} \bar{A})^{-1}$, and $\bar{A} = P_B^\perp A$.

Proof By substituting the generating function Eq. 16 into the expression $f(x) = \int_0^\infty g(||\hat{e}||^2_{\Sigma_{yy}} + ||x||^2_{\Sigma_{\hat{a}\hat{a}}} + r^2) r^{p-1} dr$, one obtains for the case $B \neq 0$: $f(x) = (2\pi)^{-m/2} \exp\{-\frac{1}{2}||\hat{e}||^2_{\Sigma_{yy}}\} \exp\{-\frac{1}{2}||x||^2_{\Sigma_{\hat{a}\hat{a}}}\} \int_0^\infty \exp\{-\frac{1}{2} r^2\} r^{p-1} dr$, of which only the x-dependent part is relevant to consider. Using the Gaussian Fourier-transform, also known from its characteristic function, $\int \exp\{-\frac{1}{2}||x||^2_{\Sigma_{\hat{a}\hat{a}}}\} \exp(-j2\pi x^T s) dx = (2\pi)^{n/2} |\Sigma_{\hat{a}\hat{a}}|^{1/2} \exp(-\frac{1}{2} 4\pi^2 ||s||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}})$, the result follows. □

With this result we now have two different expressions available for the same normal-distribution based BIE-estimator. Both involve infinite sums, which is, of course, computationally impractical. However, it is here where the relative merit of the two BIE expressions shows itself. Although both involve infinite sums over the whole

space of integers, we note that in the spatial representation the sum is governed by the function $f(\hat{a}-z) \propto \exp\{-\frac{1}{2}||\hat{a}-z||^2_{\Sigma_{\hat{a}\hat{a}}}\}$, while in the spectral representation, it is governed by $F(z) \propto \exp(-\frac{1}{2}4\pi^2||z||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}})$. In both cases, sufficient accuracy can be maintained if the infinite sums are replaced by finite sums over integers that are collected from sufficiently large search ellipsoids (Teunissen 2003). In case of the spatial representation, one would then collect integers that satisfy $||\hat{a}-z||^2_{\Sigma_{\hat{a}\hat{a}}} \leq r^2$, while for the spectral representation the inequality $||z||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}} \leq \rho^2$ is used for the integer collection. This shows that the spatial representation has its advantage when the ambiguities are precise ($\Sigma_{\hat{a}\hat{a}}$ small) and the frequency representation when the ambiguities have a poor precision ($\Sigma^{-1}_{\hat{a}\hat{a}}$ small).

4.3.2 Spectral BIE for the Contaminated Normal Distribution

The contaminated normal distribution is a *mixture distribution* composed of two normal distributions that share the same mean while having proportional variance matrices. In general, a mixture distribution arises when a random vector y is generated from one of two underlying random vectors, x_1 and x_2. The selection of the generating vector occurs randomly according to prescribed probabilities: x_1 is chosen with probability ϵ and x_2 with probability $1-\epsilon$, where $0 \leq \epsilon \leq 1$. Once the selection is made, the observation of y is obtained as a realization of the chosen random vector.

The PDF of the so created random vector y can be expressed as a convex combination of the density functions $f_{x_1}(y)$ and $f_{x_2}(y)$ of the two random vectors: $f_y(y) = (1-\epsilon)f_{x_1}(y) + \epsilon f_{x_2}(y)$. Thus if $f_{x_1}(y) = |\Sigma_{yy}|^{-1/2} g(||y-\bar{y}||^2_{\Sigma_{yy}})$ and $f_{x_2}(y) = |\delta\Sigma_{yy}|^{-1/2} g(||y-\bar{y}||^2_{\delta\Sigma_{yy}})$ ($\delta > 1$) are two ECDs with same mean but proportional variance matrices, then $f_y(y)$ is a contaminated ECD.

The contaminated ECD is again an ECD, with generating function $(1-\epsilon)g(\alpha) + \epsilon\delta^{-m/2} g(\frac{\alpha}{\delta})$. Thus with $g(\alpha) = (2\pi)^{-m/2}\exp\{-\frac{1}{2}\alpha\}$ being the generating function for the multivariate normal distribution, the generating function for the contaminated normal distribution is given as

$$g(\alpha) = (2\pi)^{-\frac{m}{2}}\left[(1-\epsilon)e^{-\frac{1}{2}\alpha} + \epsilon\delta^{-\frac{m}{2}}e^{-\frac{1}{2\delta}\alpha}\right] \tag{18}$$

Usually δ is chosen large and ϵ small. The idea is that the main distribution $N_m(\bar{y}, \Sigma_{yy})$ is slightly 'contaminated' by the wider distribution $N_m(\bar{y}, \delta\Sigma_{yy})$. This results in a distribution with heavier tails than the main one.

The following result provides the spectral BIE-coefficients in case y has a contaminated normal distribution.

Corollary 2 *(*Spectral BIE for contaminated normal*) Let* $y \sim (1-\epsilon)\mathcal{N}_m(Aa + Bb, \Sigma_{yy}) + \epsilon\mathcal{N}_m(Aa + Bb, \delta\Sigma_{yy})$, $a \in \mathbb{Z}^n$, $b \in \mathbb{R}^p$, $0 \leq \epsilon \leq 1$, $\delta > 1$. *Then,*

for $B \neq 0$, the spectral BIE-coefficients of Eq. 15 are given as

$$\begin{aligned} F(z) &= c_{cn}[c_1 \exp(-\tfrac{1}{2}4\pi^2||z||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}}) + c_2 \exp(-\tfrac{1}{2}4\pi^2\delta||z||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}})] \\ \partial_s F(s)_{|z} &= -4\pi^2 c_{cn}[c_1 \exp(-\tfrac{1}{2}4\pi^2||z||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}}) + c_2\delta \exp(-\tfrac{1}{2}4\pi^2\delta||z||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}})]\Sigma_{\hat{a}\hat{a}} z \end{aligned} \tag{19}$$

with c_{cn} a common constant that cancels in their ratio, $c_1 = (1-\epsilon)\exp\{-\frac{1}{2}||\hat{e}||^2_{\Sigma_{yy}}\}$, and $c_2 = \epsilon\delta^{(n+p-m)/2}\exp\{-\frac{1}{2\delta}||\hat{e}||^2_{\Sigma_{yy}}\}$.

Proof see Appendix. □

Note that Eq. 19 reduces to the normal case Eq. 17 when ϵ is set to zero.

4.3.3 Spectral BIE for the Multivariate Student Distribution

The random vector $y \in \mathbb{R}^m$ is said to have a multivariate t distribution with $d > 2$ degrees of freedom, denoted as $y \sim T_m(\bar{y}, \Sigma_{yy}, d)$, if its PDF is given as Kibria and Joarder (2006); Roth (2013)

$$f_y(y) = \frac{\Gamma(\frac{m+d}{2})}{(d\pi)^{\frac{m}{2}}\Gamma(\frac{d}{2})|\Sigma_{yy}|^{1/2}}\left[1 + \frac{||y-\bar{y}||^2_{\Sigma_{yy}}}{d}\right]^{-\frac{m+d}{2}} \tag{20}$$

in which $\Gamma(.)$ denotes the gamma-function. The mean of y is $\mathsf{E}(y) = \bar{y}$ and its variance matrix is $\mathsf{D}(y) = \frac{d}{d-2}\Sigma_{yy}$. Also the multivariate t distribution is an elliptically contoured distribution. Its generating function is given as

$$g(\alpha) = c_t\left[1 + \frac{\alpha}{d}\right]^{-(m+d)/2} \text{ with } c_t = \frac{\Gamma(\frac{m+d}{2})}{(d\pi)^{\frac{m}{2}}\Gamma(\frac{d}{2})} \tag{21}$$

The following result provides the spectral BIE-coefficients in case y has a multivariate Student distribution.

Corollary 3 *(*Spectral BIE for multivariate *t) Let $y \sim T_m(Aa + Bb, \Sigma_{yy}, d)$, $a \in \mathbb{Z}^n$, $b \in \mathbb{R}^p$. Then the spectral BIE-coefficients of (15) are given as*

$$\begin{aligned} F(z) &= \bar{\bar{c}}_t||z||^{\nu}_{\Sigma^{-1}_{\hat{a}\hat{a}}} K_\nu(c||z||_{\Sigma^{-1}_{\hat{a}\hat{a}}}) \\ \partial_s F(s)_{|z} &= -\bar{\bar{c}}_t c||z||^{\nu-1}_{\Sigma^{-1}_{\hat{a}\hat{a}}} K_{\nu-1}(c||z||_{\Sigma^{-1}_{\hat{a}\hat{a}}})\Sigma_{\hat{a}\hat{a}} z \end{aligned} \tag{22}$$

with $\bar{\bar{c}}_t$ a common constant that cancels in their ratio, $\nu = (m-n-p+d)/2$, $c = 2\pi\sqrt{d + ||\hat{e}||^2_{\Sigma_{yy}}}$, $\Sigma_{\hat{a}\hat{a}} = (\bar{A}^T\Sigma^{-1}_{yy}\bar{A})^{-1}$, $\bar{A} = P^{\perp}_B A$, and $K_\nu(.)$ the modified Bessel function of the second kind of order ν.

Proof see Appendix. □

As in the normal and contaminated-normal case, the spectral BIE coefficients for the Student-distribution are functions of $||z||_{\Sigma_{\hat{a}\hat{a}}^{-1}}$. Hence, the integer candidate vectors can again be collected from searching inside an ellipsoid $||z||^2_{\Sigma_{\hat{a}\hat{a}}^{-1}} \leq \rho^2$. The 'radius' ρ can then be chosen so that the boundary coefficient is smaller than a tolerance τ: $\rho^{\nu} K_{\nu}(c\rho) \leq \tau$. When compared to the normally-distributed case, the spectral BIE-coefficients of the Student-distribution have an exponential decay in $||z||_{\Sigma_{\hat{a}\hat{a}}^{-1}}$. For large α we have namely that $K_{\nu}(\alpha)$ behaves as $\sqrt{\frac{\pi}{2\alpha}} \exp(-\alpha)$, from which follows that the BIE-coefficient $||z||^{\nu}_{\Sigma_{\hat{a}\hat{a}}^{-1}} K_{\nu}(c||z||_{\Sigma_{\hat{a}\hat{a}}^{-1}})$ behaves as $||z||^{\nu-\frac{1}{2}}_{\Sigma_{\hat{a}\hat{a}}^{-1}} \exp(-c||z||_{\Sigma_{\hat{a}\hat{a}}^{-1}})$. This decay is however slower than in the normally-distributed case, which is $\exp(-\frac{1}{2}4\pi^2||z||^2_{\Sigma_{\hat{a}\hat{a}}^{-1}})$. Hence, although still strongly summable, the spectral Student BIE-coefficients are less localized than their normal counterparts. They will become more localized when d increases, whereby in the limit $d \to \infty$ they approach the spectral BIE-coefficients of the normal distribution.

5 Summary and Conclusions

The theory of best integer equivariant (BIE) estimation provides the minimum mean squared error estimator within the class of integer equivariant (IE) estimators and therefore forms an important framework for mixed-integer estimation problems. Since the IE-class contains both the class of admissible integer estimators and the class of linear unbiased estimators, the unbiased BIE-estimator is guaranteed to be at least as precise as the best linear unbiased estimator (BLUE) and the integer least-squares estimator (ILS). This favorable precision property has contributed to the increasing use of BIE-estimation in carrier-phase GNSS applications.

Recent developments extended the BIE theory beyond the normal distribution to the broader class of elliptically contoured distributions (ECDs), thereby allowing for models with heavier tails. Building on this extension, the present contribution has introduced a spectral formulation of the ECD-BIE theorem. The key step in this development was the recognition of Teunissen (2003) that every IE-estimator can be written as a correction to the BLUE, where the correction term is $\mathbb{Z}^n$-periodic in the ambiguity argument. This periodic representation provides the basis for a Fourier treatment of IE-estimators as introduced in Teunissen (2026).

Using this structure, the spatial representation of the ECD-BIE estimator was first reviewed and subsequently transformed into a spectral representation through multivariate Fourier theory. This establishes the existence of two equivalent expressions for the same BIE-estimator: one in the spatial domain and one in the spectral domain. These twin formulations provide complementary insight into the behavior of BIE-estimators and offer alternative computational perspectives.

The relative merits of the spatial and spectral representations were discussed and illustrated for three analytically tractable cases, namely the normal distribution, the contaminated normal distribution, and the multivariate Student distribution. These

examples demonstrate how the spectral form can provide useful analytical structure and may offer advantages depending on the characteristics of the underlying distribution.

In summary, the spectral ECD-BIE formulation presented in this contribution further strengthens the theoretical foundations of integer equivariant estimation. By linking the spatial and frequency-domain perspectives, the results provide additional analytical tools for studying BIE-estimators and open new possibilities for their application in mixed-integer estimation problems, including high-precision GNSS carrier-phase positioning.

Appendix

Proof of Corollary 2 (Spectral BIE for contaminated normal) As we are now dealing with a linear combination of two normal distributions, the proof goes along the same lines as that of Corollary 1, be it that we need to use the integral $\int_0^\infty \exp\{-\frac{1}{2\delta}r^2\}r^{p-1}dr = 2^{p/2-1}\delta^{p/2}\Gamma(p/2)$ twice, and the Fourier-transform $\int \exp\{-\frac{1}{2\delta}||x||^2_{\Sigma_{\hat{a}\hat{a}}}\}\exp(-2j\pi x^T s)dx = (2\pi)^{n/2}|\Sigma_{\hat{a}\hat{a}}|^{1/2}\delta^{n/2}\exp(-\frac{1}{2}4\pi^2\delta||s||^2_{\Sigma^{-1}_{\hat{a}\hat{a}}})$ twice, namely for $\delta = 1$ and $\delta \neq 1$. □

Proof of Corollary 3 (Spectral BIE of multivariate t) The task is to determine the Fourier-transform of the function

$$f(x) = \int_0^\infty g(||\hat{e}||^2_{\Sigma_{yy}} + ||x||^2_{\Sigma_{\hat{a}\hat{a}}} + r^2)r^{p-1}dr \tag{23}$$

for the generating function of the Student-distribution. First we solve for the integral in Eq. 23 and rewrite the function $f(x)$ in a form that is suitable for Fourier transformation. Substitution of Eq. 21 into Eq. 23 gives

$$f(x) = c_t d^b \int_0^\infty (a+||x||^2_{\Sigma_{\hat{a}\hat{a}}}+r^2)^{-b}r^{p-1}dr \text{ with } a = d+||\hat{e}||^2_{\Sigma_{yy}}, b = \tfrac{m+d}{2} \tag{24}$$

Using the integral representation of the Beta-function, we have $\int_0^\infty(\beta+r^2)^{-b}r^{p-1}dr = \frac{1}{2}\beta^{-(b-\frac{p}{2})}B(\frac{p}{2}, b-\frac{p}{2})$, see Gradshteyn and Ryzhik (2007) p.908 (note typo in (22) of Teunissen (2020): $p \to p/2$), which allows us to solve the integral in Eq. 24 as

$$f(x) = \bar{c}_t(a + ||x||^2_{\Sigma_{\hat{a}\hat{a}}})^{-b_1} \text{ with } \bar{c}_t = \tfrac{1}{2}c_t d^b B(\tfrac{p}{2}, b_1), b_1 = b - \tfrac{p}{2} \tag{25}$$

Its Fourier-transform is given by the integral

$$F(s) = \bar{c}_t \int_{\mathbb{R}^n} (a + ||x||^2_{\Sigma_{\hat{a}\hat{a}}})^{-b_1}\exp(-j2\pi s^T x)dx \tag{26}$$

which by the change of variables, $x = \Sigma_{\hat{a}\hat{a}}^{1/2} v$ and $\bar{s} = \Sigma_{\hat{a}\hat{a}}^{1/2} s$, can be written in terms of the Fourier-transform of a radial function as

$$F(\Sigma_{\hat{a}\hat{a}}^{-1/2}\bar{s}) = \bar{c}_t |\Sigma_{\hat{a}\hat{a}}|^{1/2} \int_{\mathbb{R}^n} (a + ||v||_{I_n}^2)^{-b_1} \exp(-j2\pi \bar{s}^T v) dv \tag{27}$$

When computing the Fourier transform of a radial function, the n-dimensional transform reduces to a Hankel-transform, which in our case is in the form of a Basset-type integral (Gradshteyn and Ryzhik 2007), thus producing the solution of Eq. 27 as

$$F(\Sigma_{\hat{a}\hat{a}}^{-1/2}\bar{s}) = \bar{\bar{c}}_t ||\bar{s}||_{I_n}^{\nu} K_\nu(2\pi\sqrt{a}||\bar{s}||_{I_n}) \text{ with } \bar{\bar{c}}_t = \bar{c}_t \frac{(2\pi)^{b_1} |\Sigma_{\hat{a}\hat{a}}|^{1/2}}{2^{b_1-1}\Gamma(b_1)} a^{\frac{n}{4}-\frac{b_1}{2}}, \nu = b_1 - \frac{n}{2} \tag{28}$$

in which $K_\nu(.)$ is the modified Bessel function of the second kind. It is a monotonically decreasing function that decays exponentially as its argument goes to infinity. From Eq. 28, we finally obtain the sought-for Fourier-transform and its derivative as

$$\begin{aligned} F(s) &= \bar{\bar{c}}_t ||s||_{\Sigma_{\hat{a}\hat{a}}^{-1}}^{\nu} K_\nu(c||s||_{\Sigma_{\hat{a}\hat{a}}^{-1}}), \text{ with } c = 2\pi\sqrt{a} \\ \partial_s F(s) &= -\bar{\bar{c}}_t c ||s||_{\Sigma_{\hat{a}\hat{a}}^{-1}}^{\nu-1} K_{\nu-1}(c||s||_{\Sigma_{\hat{a}\hat{a}}^{-1}}) \Sigma_{\hat{a}\hat{a}} s \end{aligned} \tag{29}$$

where use was made of the identity $\frac{d}{dr} r^\nu K_\nu(cr) = -cr^\nu K_{\nu-1}(cr)$. Note that $F(s)$ and $\partial_s F(s)$ have the constant $\bar{\bar{c}}_t$ in common. Hence, it will cancel in the ratio of the BIE. □

Author Contributions PT developed the theory and wrote the paper.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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