

Tractable Stochastic Model Predictive Control using Conditional Value at Risk Optimization

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Tractable Stochastic Model Predictive Control using Conditional Value at Risk Optimization

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The undersigned hereby certify that they have read and recommend to the Faculty of
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TRACTABLE STOCHASTIC MODEL PREDICTIVE CONTROL USING CONDITIONAL
VALUE AT RISK OPTIMIZATION

by

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Abstract

A numerically tractable Stochastic Model Predictive Control (SMPC) strategy using Conditional Value at Risk ($CVaR$) optimization for discrete-time linear time-invariant systems, with state and input constraints, subject to additive uncertainty, is presented. SMPC strategies make use of the probabilistic description of uncertainty to define *chance constraints* which allow a certain admissible level of constraint violation. SMPC strategies require the initial state of a system to be within a particular set, referred to as *feasibility set*, probabilistically, such that the derived control input, when applied to the system, gives rise to states that are also within the feasibility set satisfying all chance constraints on the system. This leads to *recursive feasibility* of the SMPC strategy. Such strategies are restrictive in nature when the uncertainty in the system is unbounded, as in the case of White Gaussian noise. In such a case, the feasibility set is very small and leads to a strategy that is very conservative. To reduce this conservatism, some constraint violations are permitted. However, such violations affect the closed-loop behaviour of the system leading to performance degradation. This performance degradation can be quantified as a penalty on the system for violating constraints, and intuitively, it can be thought of as a *risk* taken by the system in that undesirable state. An approach following the *exact penalty method* is proposed using the $CVaR$ function to determine the penalty cost. The same optimal solution as the original constrained problem is obtained from a single unconstrained minimization. Since accurate computation of the expected value of risk using the $CVaR$ function is not possible, a scenario-based approximation of the $CVaR$ is used to obtain an overall tractable and computationally efficient SMPC strategy. An extensive simulation study of the double integrator system is provided to present the functionality of the proposed method.

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Janani Venkatasubramanian

कर्मण्येवाधिकारस्ते मा फलेषु कदाचन ।
मा कर्मफलहेतुर्भूर्मा ते सङ्गोऽस्त्वकर्मणि ॥

*"karmany-evādhikāras te mā phaleṣhu kadāchana
mā karma-phala-hetur bhūr mā te saṅgo 'stv akarmani"*

*In action alone is your rightful-interest, never in its fruit. Let not your motive
be the fruit of action; nor let your attachment be to inaction.*

–Bhagavad Gita, Chapter 2, Verse 47; Translated by Georg Feuerstein

Chapter 1

Introduction

In industry, it is imperative for a control system to efficiently deal with constraints on the system in the presence of disturbances while providing a strategy that achieves optimal performance. The optimal control solution is a simple linear state feedback for the case of linear systems and quadratic performance indices and is computed easily by solving the steady state Riccati equation [1]. However, this optimal analytic solution is difficult to achieve in the case of constrained systems. In the pursuit of optimality, approximate solutions are considered, and this is the most important reason for the success of Model Predictive Control (MPC). MPC provides an appropriate trade-off between optimality and computational effort through the implementation of the receding horizon principle while systematically handling constraints on the system. The philosophy of MPC is simple; predict the behaviour of a system given its model, measurements of the current state of the system and a hypothetical control input trajectory or feedback control policy. The control inputs are parametrized by a finite number of variables which denote a finite number of degrees of freedom. The predicted cost of the problem is optimized over these variables. The control input is applied to the system in a receding horizon fashion wherein only the first element of the predicted control input sequence is applied to the system at the current time instant. The horizon is shifted at the next time instant and the optimization problem is carried out again to obtain a new sequence of control inputs. The receding horizon strategy is instrumental in reducing the gap between the predicted response and the actual response of the system.

Classical MPC faces a more challenging problem in the presence of uncertainty as the predicted behaviour and actual behaviour of the system can differ significantly. This motivates the formulation of Robust MPC (RMPC) and Stochastic MPC (SMPC). Robust MPC addresses the presence of bounded uncertainty in control problems. RMPC strategies take into account every possible future realization of uncertainty while optimizing over the control policy although the future realizations of uncertainty are unknown to the controller at the current time. Since RMPC does not discriminate between realizations of uncertainty based on their likelihood of occurrence, the intensity of computation grows with problem size and length of prediction horizon. In all practicality, it is most often the case that some of these realizations of uncertainty are more likely to occur compared to the other realizations from

the pre-defined uncertainty set. As an example, for uncertainty that is represented as model parameters, some realizations may be close to the nominal value of these parameters and hence, maybe more likely to occur than other realizations that lie more towards the boundary of the uncertainty set. Such model uncertainty can be represented as a stochastic variable with a known probability distribution. The probability distribution of uncertainty is most commonly addressed in the cost function. The MPC cost can be defined as the expected value of the quadratic cost over the uncertainty distribution. This leads to the formulation of SMPC strategies.

The objective of this thesis is to formulate a tractable SMPC strategy using Conditional Value at Risk (*CVaR*) optimization that is computationally efficient while dealing with the constraints and the uncertainty in the system. The following sections elaborate on the motivation for this formulation of SMPC and ideas that stem from related works. This is followed by an overview of the contributions in this thesis work and a compendium of related works. Finally, the structure of the thesis is provided.

1-1 Motivation

The MPC algorithm requires the initial state to belong to a *feasibility set* which is a set of all initially feasible states such that there exists a sequence of control inputs for which all future predicted states are also feasible, thereby making the controller *recursively feasible*. This feasibility set is known as the Controlled Positively Invariant (CPI) set. The elements of this set are all initial states of the system for which a sequence of control inputs exist such that the predicted states satisfy all the state constraints on the system. This hard requirement often results in an empty initially feasible set for white Gaussian noise disturbance. Since white Gaussian noise is unbounded, it is always possible that there exists a realization of noise that violates hard constraints. In these circumstances, chance constraints are imposed on the system which is an advisable trade-off between constraint satisfaction and optimal performance [2, 3, 4]. For example, in engineering, chance constraints have been widely used in power systems management to deal with the uncertainties that come with energy availability. The constraints are modelled to ensure that a power plant can meet the energy demand to atleast a certain confidence level [5]. Outside of engineering, chance constraints have been used in finance for risk management to deal with the uncertainties that arise while investing due to volatile market conditions [4, 6]. Recently, chance constraints have been applied to mid term supply chain planning at multiple sites. The production level at each site is determined by a chance constraint to avoid inventory depletion and and shortage due to demand uncertainty [7].

Chance constraints can be defined as probabilistic constraints where the constraints on the state are to be satisfied with atleast a priori specified probability level. Chance constraints can also be specified as expectation constraints where the constraints on the state have to be satisfied in expectation. However, there are multiple numerical issues with the evaluation of chance constraints. For example, it is not possible to accurately estimate the probability that a particular state violates the chance constraint. Monte-Carlo simulation is the only way to estimate the probability of violation of the chance constraint at a particular state. The computational demand of this estimate grows with increasing requirement of accuracy leading to intractability of the problem. Moreover, the set of feasible states defined by a

chance constraint is usually non-convex which makes the optimization subject to the chance constraint problematic [8]. A general method to obtain a tractable convex approximation of the chance constraint is the scenario approach based on Monte Carlo sampling techniques [9, 10, 11, 12]. The dynamics of the stochastic system are characterized by a finite set of uncertainty realizations giving rise to a set of affine constraints. However, when applied to MPC, the new state measurement at every time instant may not be feasible with respect to the set of approximate affine constraints, thereby leading to the loss of recursive feasibility of the MPC strategy and an unsolvable optimization problem. Hence, there is a need to formulate a strategy that deals with constraints on the system subject to unbounded uncertainty that controls the plant optimally with respect to the performance index and keeps the state within the feasible set as much as possible. Consequently, performance degradation is unavoidable. Nonetheless, it must be quantified and retained within a certain admissible level.

The performance index determines the performance of the system when the state is within the feasible set. A simple method to quantify the performance degradation due to constraint violation at an infeasible state is by adding an extra cost on the system during constraint violation [13, 14]. This extra cost or penalty is determined by a penalty function which penalizes the state that causes constraint violation. The penalty on the infeasible state cannot be made arbitrarily high as a choice of infinite penalty will make the set of feasible states empty almost always. Penalty methods are pervasive in optimization literature to approximate a constrained optimization problem as an unconstrained optimization problem [15, 16]. To ensure that the performance degradation remains within an admissible level, it is important to choose a penalty function that takes into account the stochastic nature of the system. Furthermore, the unconstrained problem with the augmented performance index including the penalty function must be a close approximation of the original problem with chance constraints. Since the optimization problem is solved in closed-loop, it is necessary that this strategy is computationally efficient and numerically tractable. Taking into consideration the above requirements, there is a need to formulate a suitable SMPC strategy.

1-2 Research Question

A tractable SMPC strategy must satisfy the chance constraints on the state of the system in the presence of uncertainty and control the plant optimally with respect to the performance index. The state of the system must be kept within the feasible set as much as possible. To account for the performance degradation due to constraint violation, a penalty function is used. The key considerations that contribute to the research question with respect to these requirements are listed as follows.

- How should we choose a penalty function to account for performance degradation?
- Since the uncertainty in the system is unbounded, it is nearly impossible to evaluate the probability of violation of constraints of a particular state. Hence, it may be necessary to use sampled-approximation techniques. Thus, is the sampled-approximation of the strategy sample efficient?
- The proposed tractable SMPC strategy is implemented using the receding horizon principle. Hence, what are the consequences on the closed-loop performance of the proposed strategy?

1-3 Contributions

The contributions in this thesis include a numerically tractable SMPC strategy using Conditional Value at Risk (*CVaR*) optimization. The chance constrained problem is approximated as a penalized unconstrained problem by using an exact penalty function [16]. This ensures that the optimal solution of the chance constrained problem is achieved through a single minimization of the penalized problem. The *CVaR* function is a coherent risk measure that measures the risk that a system would face at an undesirable state. The *CVaR* is prevalent in literature due to its features of convexity, monotonicity and numerical tractability, and is widely known as the tightest convex approximation of the chance constraint. Hence, a weighted *CVaR* function can be used as the penalty function to determine performance degradation and take appropriate measures to allow only a certain admissible level of constraint violation. An efficient sampled-approximation of the *CVaR* is determined by interpreting the probabilistic constraints as average-in-time rather than point-wise in time and maintaining the violations averaged over time below a specified level. This thesis elaborates on the problem formulation and closed-loop behaviour of the formulated strategy of SMPC. Finally, a numerical example is provided to illustrate the presented technique.

1-4 Related Works

A comprehensive exposition of MPC is provided in [17, 18]. Initial MPC strategies did not guarantee nominal closed-loop stability in the absence of uncertainty due to accounting system behaviour only over a finite horizon. This limitation was overcome by adding terminal conditions on the state of the system to ensure that the system reached a desired steady state value or a subset of feasible states. Closed-loop properties of stability and convergence of MPC is elaborated in [19, 20, 21, 22, 23]. A preferable terminal condition is that the system state at the end of the finite horizon belongs to a subset of the state space with a property that once the subset of state space is entered, the state of the constrained system will never leave the set. In this regard, stabilizing feedback laws were proposed which defined the control input at time instants beyond the initial finite horizon. This introduces the concept of set invariance [24, 25] and the dual-mode prediction paradigm [26, 27, 28, 29]. The dual-mode prediction paradigm typically divides the prediction horizon into two intervals, where the first interval optimizes over control inputs for the fixed finite horizon, and the second interval defines a fixed stabilising feedback law over the subsequent infinite horizon. These conditions and the strategy causes the controlled system to be stable in closed-loop operation. However, it is important to ensure an acceptable degree of robustness in the control strategy in the presence of uncertainty. This motivates the formulation of SMPC which utilizes the probability distribution of uncertainty. Several approaches have been made to obtain a tractable formulation of the stochastic optimal control problem [30]. Most of these approaches are classified based on the dynamics of the system - linear dynamics or non-linear dynamics. Stochastic tube approaches for linear systems are presented in [31, 32, 33, 34]. Stochastic tubes guarantee recursive feasibility and hence, ensure closed-loop stability. Approaches based on optimization over arbitrary functions is not tractable. This problem is addressed in [35, 36, 37] where the control policy is parametrized as an affine function of the disturbance inputs that have been observed till the current time instance,

from which a convex set of decision variables is derived. This helps in establishing input-to-state stability for the closed-loop system. A series of articles [38, 39, 40, 41, 42] discuss SMPC for linear systems with affine-disturbance feedback control policies. These approaches discuss a tractable and recursively feasible receding horizon control (RHC) policy that ensure mean squared boundedness of the closed-loop system. A significant development in the area of approaches based on stochastic programming is the scenario approach [9, 11, 43, 10, 12]. Furthermore, constraint tightening approaches are presented in [44, 45]. SMPC makes use of the probabilistic description of uncertainty to describe chance constraints. Chance constraints may also be probabilistic constraints [2, 3, 46, 4] or expectations constraints [47, 48]. Since the feasible set of chance constraints is usually non-convex, convex approximations of chance constraints are presented in [8]. The *CVaR* function, widely known as the tightest convex approximation of the chance constraint, is a coherent risk measure. A detailed description of risk measures and the approximation of chance constraints using *CVaR* are given in [6, 49, 50, 51]. Penalty methods and exact penalty functions, which are used to obtain the optimal solution using a single unconstrained minimization, are explained in detail in [15, 16]. The work in this thesis also derives ideas from [13, 14, 52, 53, 54].

1-5 Structure of the Thesis

The structure of the thesis is as follows. In chapter 2, relevant mathematical preliminaries that forms an appropriate background of work related to the problem formulation in the thesis is presented. The topics covered in this chapter include a general formulation of SMPC for discrete-time linear time-invariant systems, convex approximations of chance constraints with a focus on the scenario approach and *CVaR*, and penalty methods in optimization. Chapter 3 gives a detailed exposition of the problem formulation for a tractable SMPC strategy using *CVaR* optimization with an evaluation of theory by means of a numerical example. Suitable areas of application of the tractable SMPC strategy are also discussed. Finally, chapter 4 provides a discussion of the results and concluding remarks.

Technical Preliminaries

This chapter presents the technical preliminaries that elaborates on selected topics preceding the problem formulation of the thesis work. A generic formulation of SMPC is provided with a focus on chance constraints and performance index. Recursive feasibility of a SMPC algorithm and concepts of stability and convergence based on the forms of the performance index are discussed. A discussion on convex approximations of chance constraints, with an emphasis on the scenario approach and *CVaR* is provided. Finally, penalty methods, with a focus on the exact penalty method, is provided.

2-1 General Formulation of Stochastic MPC for Discrete-Time Linear Time-Invariant Systems

A linear time-invariant system that is subject to stochastic disturbances is considered. The system is described by the following state space model [18].

$$x_{k+1} = F(x_k, u_k, w_k) = A_k x_k + B_k u_k + E_k w_k \quad (2-1)$$

where $x_k \in \mathcal{X} \subset \mathbb{R}^{n_x}$ is the state, $u_k \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the control input, and $w_k \in \mathbb{R}^{n_w}$ is the additive disturbance at time k . The successor state is x_{k+1} at time $k + 1$. \mathcal{U} is a non-empty measurable compact convex control set and \mathcal{X} is a closed convex set that contains the origin in its interior that are defined by the constraints on the state and the control input. The disturbance input $w_k \in \mathbb{R}^{n_w}$ is an exogenous disturbance with unknown current and future values but known probability distribution. It is a random vector in a probability space (Ω, \mathcal{F}, P) with support $\mathcal{W} \subset \mathbb{R}^{n_w}$. Moreover, it is assumed for $k \neq j$, w_k and w_j are statistically independent. In the stochastic MPC problem formulation, the matrices A_k and B_k denote multiplicative model parameters. The additive disturbance input w_k , along with A_k and B_k , can be expressed as a linear expansion over a known basis set

$$(A_k, B_k, w_k) = (A^{(0)}, B^{(0)}, 0) + \sum_{j=1}^{\rho} (A^{(j)}, B^{(j)}, w^{(j)}) \delta_k^{(j)} \quad (2-2)$$

where, $\mathbb{E}(A_k) = A^{(0)}$ and $\mathbb{E}(B_k) = B^{(0)}$. The expected value of the disturbance input $\mathbb{E}(w_k)$ is taken as zero, that is $w^{(0)} = 0$, and $\delta_k^{(j)}$ is a scalar random variable. The probability distribution of $\delta_k^{(j)}$ is known. The vector $\delta = (\delta^{(1)}, \dots, \delta_k^{(\rho)})$ has different realizations at each time instant but is assumed to be identically distributed for each k . Without loss of generality, the following properties of δ are assumed.

$$\mathbb{E}(\delta_k) = 0, \quad \mathbb{E}(\delta_k \delta_k^T) = I \quad (2-3)$$

Here, δ_k has a mean value of zero and a covariance matrix equal to the identity matrix. It is also assumed that δ_k and δ_i are statistically independent for $k \neq i$. This assumption simplifies the computation of predicted costs based on the expected value of sum of stage costs, which in turn simplifies stability analysis of the system based on the cost function. A non zero expected value $\mathbb{E}(\delta_k)$ and a non zero expected value $\mathbb{E}(w_k)$ is taken into account by using a suitable state translation.

Let N , a positive integer, be the prediction horizon. The system is controlled by a feedback controller, i.e., at each k , the input u_k is the function of the state x_k . A feedback policy $\boldsymbol{\pi} = \{\pi_0(\cdot), \dots, \pi_{N-1}(\cdot)\}$, a sequence of measurable control laws is employed for each $\pi_i : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$, $i = 0, \dots, N-1$. The control laws π belong to a class of controllers Ψ which is a set of continuous maps that map the origin of the state space to into the zero input, $\pi(0) = 0$. The control input u_i is selected as $\pi_i(x_i)$ at the i th stage. When the system is subject to unbounded disturbances and constraints on the input, i.e., \mathcal{U} is bounded, the system is globally asymptotically stable when the matrix pair (A_k, B_k) is stabilizable and the eigenvalues of the system matrix A_k lie on or inside the unit circle for all k .

Assumption 2.1. *The matrix pair (A, B) is stabilizable and all eigenvalues of the system matrix A lie on or inside the unit circle.*

2-1-1 Performance Index

The optimal control problem (OCP) is defined in terms of a performance index $\hat{J}_N(x_k, \mathbf{u}, \mathbf{w})$, also known as the predicted cost, that is evaluated over a horizon of N steps and is solved at each time step. The predicted cost is given as,

$$\hat{J}_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}, \mathbf{w}_{k \rightarrow k+N}) = p(x_{N|k}) + \sum_{i=0}^{N-1} q(x_{i|k}, u_{i|k}) \quad (2-4)$$

where $\mathbf{u}_{k \rightarrow k+N|k} = \{u_{0|k}, \dots, u_{N-1|k}\}$, $\mathbf{w}_{k \rightarrow k+N|k} = \{w_{0|k}, \dots, w_{N-1|k}\}$, the function $q : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}_+$ gives the cost per stage and the function $p : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_+$ is the terminal cost. On account of the stochasticity of w_k , the predicted cost $\hat{J}_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}, \mathbf{w}_{k \rightarrow k+N|k})$ is stochastic. Since the probability distribution of w_k is known, it is appropriate to use a predicted cost that takes into consideration the stochastic nature of the model uncertainty through the expectation of the predicted cost. The expected cost is given as,

$$J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) = \mathbb{E} \left[\left(p(x_{N|k}) + \sum_{i=0}^{N-1} q(x_{i|k}, u_{i|k}) \right) \middle| x_{0|k} = x_k \right] \quad (2-5)$$

The terminal cost ensures that the system is closed-loop stable and the controller found is stabilizing. Very commonly in literature, the stage costs and the terminal cost are given in

terms of weighted l_2 norms of the state and the control input to give a cumulative quadratic cost.

$$J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) = \mathbb{E} \left[\left(x_{N|k}^T P x_{N|k} + \sum_{i=0}^{N-1} (x_{i|k}^T Q x_{i|k} + u_{i|k}^T R u_{i|k}) \right) \middle| x_{0|k} = x_k \right] \quad (2-6)$$

where Q and R are positive definite matrices which are the state and control input weighting matrices respectively, and P is a positive definite matrix that weights the terminal state at the end of the horizon. For unconstrained systems, the optimal control solution is a simple linear state feedback for the case of linear systems and quadratic performance indices and is computed easily by solving the steady state Riccati equation [1]. The predicted control sequence ensures that the predicted state converges to zero as $k \rightarrow \infty$. However, for an unconstrained problem with additive uncertainty, a feedback law cannot lead to convergence of the state to zero identically. In this case, the expected value of the stage cost tends to a finite limit.

2-1-2 Chance Constraints

State and input constraints can be defined in numerous ways. SMPC exploits the probabilistic uncertainty descriptions to define chance constraints on the state [5, 55]. These constraints require the state constraints to be satisfied with at least a priori specified probability level [2, 3, 46, 4], or to be satisfied in expectation [47, 48]. Chance constraints may be defined in terms of expected values as,

$$\mathbb{E}(G(x_{i|k}, w_{i|k})) \leq 0, \quad i = 0, 1, \dots, N-1 \quad (2-7)$$

Probabilistic constraints that are point-wise in time can be defined as,

$$P(G(x_{i|k}, w_{i|k}) \leq 0) \geq 1 - \alpha_i, \quad i = 0, \dots, N-1 \quad (2-8)$$

where, $P(A)$ denotes the probability of some event A .

Alternatively, probabilistic constraints can be imposed over N time steps,

$$P(G(x_{i|k}, w_{i|k}) \leq 0) \geq 1 - \alpha, \quad i = 0, \dots, N-1 \quad (2-9)$$

where $G(x, w)$ may be vector-valued and, α and N are some given probability and horizon, respectively. For the probabilistic constraints defined point-wise in time, it is required that the probability that each element of $G(x_{i|k}, w_{i|k})$ exceeding the value zero be less than α_i for $i = 0, 1, \dots, N-1$. For probabilistic constraints imposed over N time steps, the expected value of each element of $G(x, w)$ exceeding zero over N time steps should be less than αN . Constraints may be constructed by combining constraints of these forms.

Given the performance index and constraints, the SMPC problem is formulated as a finite horizon optimal control problem (FHOCP) by introducing chance constraints on the state.

$$\begin{aligned} J_k^*(x_k) &= \min_{\mathbf{u}_{k \rightarrow k+N|k}} J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) \\ \text{s.t. } & x_{i+1|k} = A x_{i|k} + B u_{i|k} + E w_{i|k}, \quad i = 0, \dots, N-1 \\ & u_{i|k} \in \mathcal{U}, \quad i = 0, \dots, N-1 \\ & P(G(x_{i|k}, w_{i|k}) \leq 0) \geq 1 - \alpha, \quad i = 0, \dots, N-1 \\ & x_{0|k} = x_k \end{aligned} \quad (2-10)$$

where $J_k^*(x_k)$ is the optimal value function under the optimal control policy $\mathbf{u}_{k \rightarrow k+N|k}^*$. The receding horizon policy is implemented by applying the first element of the optimal control input sequence u_k^* to the system at time k .

2-2 Feasibility and Stability of Stochastic MPC of Discrete-Time Linear Time-Invariant Systems

This section provides an overview of closed-loop concepts of recursive feasibility, convergence and stability. The property of recursive feasibility is dependent on constraints on the system, which are probabilistic in nature in SMPC strategies. The requirement of future feasibility of probabilistic constraints induces constraints on the state that must be satisfied for a predefined subset or all realizations of uncertainty. Once feasibility of the problem at all times is ensured, convergence and stability is discussed. A general way to discuss stability is based on the cost function of the problem. Stability with respect to different cost functions - expectation cost and mean-variance cost are discussed. Supermartingale convergence, which is another approach for analysing stability, is provided.

2-2-1 Recursive Feasibility

This subsection analyses the property of recursive feasibility of a SMPC strategy wherein the predicted performance of the strategy is guaranteed to remain feasible at all future sampling instants if it is initially feasible. Recursive feasibility is dependent on the satisfaction of constraints by the state. In stochastic MPC strategies, probabilistic or expectation constraints are used and are usually regarded as 'soft' constraints since they are not expected to hold for all possible realization of uncertainty. Nevertheless, for the problem to remain feasible, it is required that all conditions and constraints of the system are met with surety, irrespective of whether these constraints are to hold for all possible realizations of uncertainty or for a predefined subset of uncertainty. Probabilistic or expectation constraints are feasible when the state belongs to a predefined subset of the state space. This imposes further conditions or constraints on the system for feasibility of the problem. These conditions can be explicitly incorporated in the optimization problem to ensure robust feasibility. Another approach to handling recursive feasibility is to allow the optimization problem to become infeasible whenever necessary. This approach includes a penalty on constraint violation that is added to the MPC cost [56, 57], or directly minimize a measure of distance of the state from the feasible set whenever the problem is infeasible [58].

Consider the system as in (2-1) and the constraints of the following form

$$P(G(x_{i|k}, u_{i|k}, w_{i|k}) \leq 0) \geq 1 - \alpha, \quad i = 0, \dots, N - 1 \quad (2-11)$$

A Stochastic MPC controller is recursively feasible if and only if for all initially feasible states $x_{0|k} = x_k$ and $k \geq 0$, and for all optimal control input sequences, the optimization problem remains feasible for all time [18], [59]. For recursive feasibility, the set of all initially feasible states must be a controlled positively invariant (CPI) set, which is defined as [24, 25],

Definition 2.1. *Controlled Positively Invariant Set* A set $\mathcal{X}_N \subseteq \mathbb{R}_{n_x}$ is a controlled positively invariant set for the system dynamics from (2-1) and constraints in (2-9), such that there exists a $u_k \in \mathcal{U} \subset \mathbb{R}^{n_u}$ such that $P(G(x_k, u_k, w_k) \leq 0) \geq 1 - \alpha$ and $F(x_k, u_k, w_k) \in \mathcal{X}_N$.

The term 'positively' refers to the fact that only forward evolutions of the system in (2-1) will be considered. This implies that for any $x_k \in \mathcal{X}_N$, there must also exist a feasible $\mathbf{u}_{k \rightarrow k+N|k} = \{u_{0|k}, \dots, u_{N-1|k}\}$, $u_{0|k} = u_k$, such that $P(G(x_k, u_k, w_k) \leq 0) \geq 1 - \alpha$ and $x_{k+1} = F(x_k, u_k, w_k) \in \mathcal{X}_N$. The set \mathcal{X}_N is the maximal controlled positively invariant (MCPI) set if it contains all other CPI sets. This CPI set of initially feasible states is defined as,

$$\mathcal{X}_N = \{x_k : \exists \mathbf{u}_{k \rightarrow k+N|k} \text{ s.t. } P(G(x_{i|k}, u_{i|k}, w_{i|k}) \leq 0) \geq 1 - \alpha, i = 0, \dots, N-1, k \geq 0 \text{ and } u_{i|k} \in \mathcal{U}\} \quad (2-12)$$

Feasibility is ensured if the predicted state sequence at time k satisfies $x_{i|k} \in \mathcal{X}$ for all possible realizations of uncertainty sequence $\{\delta_{0|k}, \dots, \delta_{i-1|k}\}$, $i > 0$. The set of all possible realization δ_k , i.e., the support of δ is $\Delta \subseteq \mathbb{R}^p$ is given as,

$$Pr(\delta_k \in \Delta) = 1 \text{ and } Pr(\delta_k \notin \Delta) = 0 \quad (2-13)$$

The constraints on the system can now be written as,

$$P(G(x_{0|k}, u_{0|k}, w_{0|k}) \leq 0) \geq 1 - \alpha \quad (2-14a)$$

$$P\left(\max_{\delta_k, \dots, \delta_{k+i-1} \in \Delta} G(x_{i|k}, u_{i|k}, w_{i|k}) \leq 0\right) \geq 1 - \alpha, i = 1, 2, \dots \quad (2-14b)$$

which are made robust with respect to $\{\delta_k, \dots, \delta_{i-1|k}\}$ but remain stochastic with respect to $w_{i|k}$. To verify recursive feasibility, suppose the control input sequence $\{u_{0|k}, u_{1|k}, \dots\}$ at time k satisfies the constraints in (2-14), and the elements of the sequence of control inputs at time $k+1$ be $u_{i|k+1} = u_{i+1|k}$, $i = 0, 1, \dots$. For $\delta_k \in \Delta$ and $i = 1$, (2-14a) can be written as,

$$P(G(x_{0|k+1}, u_{0|k+1}, w_{0|k+1}) \leq 0) \geq p \quad (2-15)$$

and implies $x_{k+1} \in \mathcal{X}$. When $i = j+1$, (2-14b) can be written as,

$$P\left(\max_{\delta_{k+1}, \dots, \delta_{k+j} \in \Delta} G(x_{j|k+1}, u_{j|k+1}, w_{j|k+1}) \leq 0\right) \geq p, j = 1, 2, \dots \quad (2-16)$$

and implies $x_{j|k+1} \in \mathcal{X}$ for all $\{\delta_k, \dots, \delta_{k+j}\} \in \Delta \times \dots \times \Delta$. This shows that the conditions in (2-14) provide a recursively feasible set of constraints that ensure the satisfaction of constraints in (2-9). These conditions are necessary and sufficient for recursive feasibility. Since a maximization over a subset of uncertainty parameters δ is involved in (2-14b), recursive feasibility can only be guaranteed for model uncertainty that has finite support. In practical applications, restricting to uncertainty with finite support is not a limiting factor since control systems are usually not subjected to unbounded uncertainty. The disturbance input w , however, is not required to have a finite support.

On imposing the constraints in (2-9) on the predicted state and control input trajectories over a finite horizon, an appropriate terminal constraint is used. It is required that $x_{N|k} \in \mathcal{X}_f$ for all realizations of the uncertain sequence $\{\delta_k, \dots, \delta_{k+N-1}\}$ over the N step horizon. Using a linear terminal control law $u_k = Kx_k$, it is required that for all $x_k \in \mathcal{X}_f$,

$$(A(\delta) + B(\delta)K)x_k + Ew(\delta) \in \mathcal{X}_f, \forall \delta \in \Delta \quad (2-17)$$

and

$$P(F(x_k, Kx_k, w_k) \leq 0) \geq 1 - \alpha \quad (2-18)$$

Here, \mathcal{X}_f is a robustly invariant subset of the feasible set \mathcal{X} .

2-2-2 Convergence and Stability

Closed loop behaviour of the system is analysed based on the optimal value of the predicted cost. The optimal value of the performance index is obtained by minimising it with respect to probabilistic constraints that are constructed to ensure that the system is recursively feasible. The performance index considered in the section takes the form of an expectation of a quadratic cost as defined in (2-6). Using the dual-mode prediction paradigm [26, 27, 28, 29, 18], the predicted control sequence is parametrized as,

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \quad i = 0, 1, \dots \quad (2-19)$$

where $\mathbf{c}_{k \rightarrow k+N|k}$ is the vector of optimization variables. Point-wise probabilistic constraints of the form given in (2-11) are imposed on the system. For state $x \in \mathcal{X}_f$, the constraint can be written as,

$$Pr(G((\Phi(q)x + Dw(q)), K(\Phi(q)x + Dw(q)), 0) \leq 0) \geq 1 - \alpha \quad (2-20)$$

for $x \in \mathcal{X}_f$, $\Phi(q) = A(q) + B(q)K$. The MPC algorithm with a cost $J(x_k, \mathbf{c}_{k \rightarrow k+N|k})$ and appropriate probabilistic constraints that satisfy recursive feasibility can now be re-written as,

$$\begin{aligned} J_k^*(x_k) &= \min_{\mathbf{c}_{k \rightarrow k+N|k}} J(x_k, \mathbf{c}_{k \rightarrow k+N|k}) \\ \text{s.t. } &P(G(x_{1|k}, Kx_{1|k} + c_{1|k}, w_{1|k}) \leq 0) \geq 1 - \alpha \\ &P\left(\max_{\delta_k, \dots, \delta_{k+i-1} \in \Delta} G(x_{i+1|k}, Kx_{i+1|k} + c_{i+1|k}, w_{i+1|k}) \leq 0\right) \geq 1 - \alpha, \quad i = 1, \dots, N-2 \\ &P\left(\max_{\delta_k, \dots, \delta_{k+N-2} \in \Delta} G(x_{N|k}, 0, 0) \leq 0\right) \geq 1 - \alpha \end{aligned} \quad (2-21)$$

where $x_{N|k} \in \mathcal{X}_f$ for all $\{\delta_k, \dots, \delta_{k+N-1}\} \in \Delta \times \dots \times \Delta$.

Expectation Cost

In the absence of constraints and under $u_k = Kx_k$, the state in (2-1) satisfies the asymptotic condition that $\lim_{k \rightarrow \infty} \mathbb{E}_0(x_k) = 0$ and $\lim_{k \rightarrow \infty} \mathbb{E}_0(x_k x_k^T) = \Theta$ where Θ is the solution of

$$\Theta - \mathbb{E}((A_k + B_k K)\Theta(A_k + B_k K)^T) = E\mathbb{E}(w_k w_k^T)E^T \quad (2-22)$$

if

$$P - \mathbb{E}((A_k + B_k K)^T P (A_k + B_k K)) \succ 0 \quad (2-23)$$

where, P which is positive definite. The infinite horizon quadratic cost is given by,

$$\sum_{i=0}^{\infty} \mathbb{E}_0(\|x_k\|_Q^2 + \|u_k\|_R^2) \quad (2-24)$$

The expected value of the stage cost converges to a steady state value when the control input is $u_{i|k} = Kx_{i|k}$. This is given by,

$$l_{ss} = \lim_{i \rightarrow \infty} \mathbb{E}_k(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) = \text{tr}(\Theta(Q + K^T R K)) \quad (2-25)$$

The predicted cost is given by,

$$J(x_k, \mathbf{c}_{k \rightarrow k+N|k}) = \sum_{i=0}^{\infty} \mathbb{E}_k(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - l_{ss}) \quad (2-26)$$

From the definition of the predicted cost,

$$\mathbb{E}_k(J(x_{k+1}, \mathbf{c}_{k+1 \rightarrow k+1+N|k})) \leq J^*(x_k) - (\|x_k\|_Q^2 + \|u_k\|_R^2 - l_{ss}) \quad (2-27)$$

Here, since optimality at time $k+1$ implies $J^*(x_{k+1}) \leq J(x_{k+1}, \mathbf{c}_{k+1 \rightarrow k+1+N|k})$ for any realization of $\delta_k \in \Delta$, it follows that,

$$\mathbb{E}_k(J^*(x_{k+1})) \leq J^*(x_k) - (\|x_k\|_Q^2 + \|u_k\|_R^2 - l_{ss}) \quad (2-28)$$

On taking the expectation of (2-28) conditioned over x_0 , the following inequality can be derived.

$$\frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E}_0(\|x_k\|_Q^2 + \|u_k\|_R^2) \leq l_{ss} + \frac{1}{r} (J^*(x_0) - \mathbb{E}_0(J^*(x_r))) \quad (2-29)$$

where, $\mathbb{E}_0(\mathbb{E}_k(J^*(x_{k+1}))) = \mathbb{E}_0(J^*(x_{k+1}))$ and $r > 0$. As $r \rightarrow \infty$, the second term on the RHS of the above inequality vanishes. The closed loop system then satisfies the quadratic stability condition,

$$\lim_{r \rightarrow \infty} \frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E}_0(\|x_k\|_Q^2 + \|u_k\|_R^2) \leq l_{ss}. \quad (2-30)$$

and $P(G(x_{1|k}, u_{1|k}, w_{1|k}) \leq 0) \geq 1 - \alpha$ for all $k > 0$. This bound implies that the control law for the algorithm in (2-21) converges asymptotically to $u_k = Kx_k$ and the state x_k converges with probability 1 to the minimal robustly positively invariant set under this feedback law as $k \rightarrow \infty$.

Mean-Variance Cost

The cost used in the MPC algorithm in (2-21) is now give by a mean-variance predicted cost [58],

$$\begin{aligned} J(x_k, \mathbf{c}_{k \rightarrow k+N|k}) &= \sum_{i=0}^{\infty} (\|x_{i|k^{(0)}}\|_Q^2 + \|u_{i|k^{(0)}}\|_R^2) \\ &+ \kappa^2 \sum_{i=0}^{\infty} \mathbb{E}_k(\|x_{i|k} - x_{i|k^{(0)}}\|_Q^2 + \|u_{i|k} - u_{i|k^{(0)}}\|_R^2 - l_{ss}) \end{aligned} \quad (2-31)$$

where l_{ss} is as given in (2-25) and κ is a constant. When $\kappa^2 \neq 1$, the mean-variance cost is given by [60],

$$\begin{aligned} J(x_k, \mathbf{c}_{k \rightarrow k+N|k}) &= (1 - \kappa^2) \sum_{i=0}^{\infty} (\|x_{i|k^{(0)}}\|_Q^2 + \|u_{i|k^{(0)}}\|_R^2) \\ &+ \kappa^2 \sum_{i=0}^{\infty} \mathbb{E}_k (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - l_{ss}) \end{aligned} \quad (2-32)$$

The cost can be written as,

$$J(x_k, \mathbf{c}_{k \rightarrow k+N|k}) = \begin{bmatrix} z_k \\ 1 \end{bmatrix} \begin{bmatrix} W_z & w_{z1} \\ w_{z1}^T & w_1 \end{bmatrix} \begin{bmatrix} z_k \\ 1 \end{bmatrix} \quad (2-33)$$

where $z_k = (x_k, c_k)$ and W_z , w_{z1} and w_1 are given by,

$$W_z = (1 - \kappa^2) \bar{W}_z + \kappa^2 \hat{W}_z \quad \begin{cases} \bar{W}_z - \Psi^{(0)T} \bar{W}_z \Psi^{(0)} = \hat{Q} \\ \hat{W}_z - \mathbb{E}(\Psi_k^T \hat{W}_z \Psi_k) = \hat{Q} \end{cases} \quad (2-34)$$

and

$$\begin{aligned} w_{z1}^T (I - \Psi^{(0)}) &= \mathbb{E}(w_k^T [E^T \ 0] \hat{W}_z \Psi_k) \\ w_1 &= -tr(\Theta \hat{W}_x) \end{aligned} \quad (2-35)$$

with \hat{W}_x and $\Psi^{(0)}$ given as,

$$\hat{W}_x = \begin{bmatrix} I_{n_x} & 0 \end{bmatrix} \hat{W}_z \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix}, \quad \Psi^{(0)} = \begin{bmatrix} \Phi^{(0)} & B^{(0)} D \\ 0 & M \end{bmatrix} \quad (2-36)$$

where,

$$D = \begin{bmatrix} I_{n_u} & 0 & \dots & 0 \end{bmatrix} \text{ and } M = \begin{bmatrix} 0 & I_{n_u} & 0 & \dots & 0 \\ 0 & 0 & I_{n_u} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_{n_u} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

The cost in (2-31),

$$J(x_k, \mathbf{c}_{k \rightarrow k+N|k}) = x_k^T W_x x_k + \mathbf{c}_k^T W_c \mathbf{c}_k + 2w_{x1}^T x_k + 2w_{c1}^T \mathbf{c}_k + w_1 \quad (2-37)$$

The structure of Ψ_k and $\Psi^{(0)}$ in (2-35) and (2-34) implies that W_c and w_{c1} have a block structure and is given by $W_c = \text{diag}\{S, \dots, S\}$ and $w_{c1}^T = [v^T \dots v^T]$ where $S \in \mathbb{R}^{n_u \times n_u}$ and $v \in \mathbb{R}^{n_u}$ with $S \succ 0$. Equivalently, the cost is given as,

$$J(x_k, \mathbf{c}_{k \rightarrow k+N|k}) = \|\mathbf{c}_{k \rightarrow k+N|k}\|_{W_c}^2 + 2w_{c1}^T \mathbf{c}_{k \rightarrow k+N|k} \quad (2-38)$$

Scalars β , γ and a matrix $P \succ 0$ can be found such that a bound

$$\mathbb{E}_k (\|x_{k+1}\|_P^2) \leq \|x_k\|_P^2 - \|x_k\|^2 + \beta^2 (\|c_{0|k}^*\|_S^2 + 2v^T c_{0|k}^*) + \gamma^2 \mathbb{E}(\|w_k\|^2) \quad (2-39)$$

holds [58]. Satisfaction of constraints as in (2-21) and recursive feasibility follows from the feasibility of $\mathbf{c}_{k+1 \rightarrow k+1+N} = \{c_{1|k}^*, \dots, c_{N-1|k}^*, 0\}$ at time $k+1$. By the optimization of the performance index subject to constraints implies, $J^*(x_{k+1}) \leq \|\mathbf{c}_{k+1 \rightarrow k+1+N|k}\|_{W_c}^2 + 2w_{c1}^T \mathbf{c}_{k+1 \rightarrow k+1+N|k}$ for all $q_k \in \mathcal{Q}$. W_c and w_{c1} have a block structure and therefore,

$$\mathbb{E}_k(J^*(x_{k+1})) \leq \|\mathbf{c}_{k+1 \rightarrow k+1+N|k}\|_{W_c}^2 + 2w_{c1}^T \mathbf{c}_{k+1 \rightarrow k+1+N|k} = J^*(x_k) - (\|c_{0|k}^*\|_S^2 + 2v^T c_{0|k}^*) \quad (2-40)$$

and hence from (2-39),

$$\mathbb{E}_k(\|x_{k+1}\|_P^2) \leq \|x_k\|_P^2 - \|x_k\|^2 + \beta^2(J^*(x_k) - \mathbb{E}_k(J^*(x_{k+1}))) + \gamma^2 \mathbb{E}(\|w_k\|^2) \quad (2-41)$$

Taking expectations and summing over $k = 0, \dots, r-1$,

$$\begin{aligned} \frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E}_0(\|x_k\|^2) &\leq \gamma^2 \mathbb{E}(\|w_k\|^2) + \frac{1}{r} (\|x_0\|_P - \mathbb{E}_0(\|x_r\|_P^2)) \\ &\quad + \frac{\beta^2}{r} (J^*(x_0) - \mathbb{E}_k(V^*(x_r))) \end{aligned} \quad (2-42)$$

As $r \rightarrow \infty$,

$$\lim_{r \rightarrow \infty} \frac{1}{r} \sum_{k=0}^r \mathbb{E}_0(\|x_k\|^2) \leq \gamma^2 \mathbb{E}(\|w_k\|^2) \quad (2-43)$$

and $Pr_k(G(x_{1|k}, u_{1|k}, w_{1|k}) \leq 0) \geq 1 - \alpha$ for all $k > 0$, which is the quadratic stability condition for the closed loop system for some finite scalar γ . This shows that a finite upper bound on the gain between the mean-square value of the additive disturbance and that of the closed loop system state exists. This result, however, is weak as it does not demonstrate how the gain depends on the distribution of multiplicative model uncertainty. Using the conditions in (2-34) and (2-35) and the cost in (2-32), the result in (2-30) can be generalized as,

$$\mathbb{E}_k(J^*(x_{k+1})) \leq J^*(x_k) - (\|x_k\|_Q^2 + \|u_k\|_R^2 - \kappa^2 l_{ss}) \quad (2-44)$$

The closed loop quadratic stability condition can now be given as,

$$\lim_{r \rightarrow \infty} \frac{1}{r} \sum_{k=0}^r \mathbb{E}_0(\|x_k\|_Q^2 + \|u_k\|_R^2) \leq \kappa^2 l_{ss} \quad (2-45)$$

when $\kappa > 1$.

Supermartingale Convergence Analysis

The quadratic stability bounds exposed in (2-30) and (2-45) are valid in the presence of additive uncertainty and guarantee asymptotic convergence of the state to a neighbourhood around the origin. In the case that an additive disturbance input is not present, state convergence is guaranteed on the basis of bounds on the evolution of the optimal value of the cost. The analysis is based on a sequence of optimal cost values that forms a supermartingale [61, 62].

An ellipsoidal set is defined as,

$$\omega_\kappa = \{x : x^T Q x \leq \kappa^2 l_{ss}\} \quad (2-46)$$

and given a sequence of states $\{x_0, x_1, \dots\}$, a sequence is defined as $\{\hat{x}_0, \hat{x}_1, \dots\}$ where $\hat{x}_0 = x_0$ and

$$\hat{x}_k = \begin{cases} x_k & \text{if } x_i \notin \omega_k \text{ for all } i < k, k > 0 \\ \hat{x}_{k-1} & \text{if } x_i \in \omega_k \text{ for some } i < k, k > 0 \end{cases} \quad (2-47)$$

If x_k satisfies (2-28) for $\kappa = 1$ or (2-44) for $\kappa > 1$ then,

$$\mathbb{E}_k(J^*(\hat{x}_{k+1})) \leq J^*(\hat{x}_k) - (\|\hat{k}_k\|_Q^2 - \kappa^2 l_{ss}) \leq J^*(\hat{x}_k) \quad (2-48)$$

if $x_i \notin \omega_k$ for all $i \leq k$ whereas $J^*(\hat{x}_{k+1}) = J^*(\hat{x}_k)$ if $x_i \in \omega_k$ for any $i \leq k$. The sequence $\{J^*(\hat{x}_0), J^*(\hat{x}_1), \dots\}$ is a supermartingale, i.e., a sequence of random variables with the property that $\mathbb{E}_k(J^*(\hat{x}_{k+1})) \leq J^*(\hat{x}_k)$ for all $k \geq 0$. A result from [62] in the above context states that for the optimization problem in (2-21) with either the cost given in (2-26) for $\kappa = 1$ or the cost in (2-31) for $\kappa > 1$, the state of the closed loop system satisfies $x_k \in \omega_k$ for some k with probability 1. To prove this, a function $l(x)$ is defined as

$$l(x) = \begin{cases} \|x\|_Q^2 - \kappa^2 l_{ss} & \text{if } x \notin \omega_k \\ 0 & \text{if } x \in \omega_k \end{cases} \quad (2-49)$$

Here, $l(x) > 0$ if and only if $x \notin \omega_k$. From 2-28, 2-44 and 2-47, for all $k > 0$,

$$\mathbb{E}_k(J^*(\hat{x}_{k+1})) - J^*(\hat{x}_k) \leq -l(\hat{x}_k) \quad (2-50)$$

and summing over all $k < r$ yields, for any $r > 0$,

$$\sum_{k=0}^{r-1} \mathbb{E}_0(l(\hat{x}_k)) \leq J^*(x_0) - \mathbb{E}_0(J^*(\hat{x}_r)) \quad (2-51)$$

The RHS of the above inequality has a finite upper bound as $J^*(x)$ is bounded from below for all x . By the Borel-Cantelli lemma [63], $l(\hat{x}_k) \rightarrow 0$ with probability 1 and hence $\hat{x}_k \rightarrow \omega_k$ with probability 1. This implies that the state trajectory of the closed loop system converges to the set ω_k . Although the subsequent state may not stay in ω_k , successive states must continually return to ω_k . The convergence of \hat{x} to ω_k with probability 1 is equivalent to convergence in probability [62] since $Pr(l(\hat{x}_k) \geq \epsilon) \rightarrow 0$ as $k \rightarrow \infty$ for all $\epsilon > 0$.

For problems that have soft constraints that may be violated with a set frequency, analogous stability and convergence results can be obtained. In applications that involve uncertainty with unbounded support, satisfaction of probabilistic constraints cannot be guaranteed. For these types of problems, supermartingale-like conditions as in (2-28) or (2-26) can be imposed which ensure quadratic stability conditions.

2-3 Convex Approximations of Chance Constraints

Chance constraints can be defined as probabilistic constraints or expectation constraints with probabilistic constraints appearing more commonly in literature. Consider a Chance Constrained Problem (CCP) of the form,

$$\begin{aligned} & \min_{x \in \mathcal{X}} f(x) \\ & \text{s.t. } P(G(x, w) \leq 0) \geq 1 - \alpha \end{aligned} \quad (2-52)$$

where, $f(x)$ is the performance index, w is a random vector in a probability space (Ω, \mathcal{F}, P) with support $\mathcal{W} \subset \mathbb{R}^{n_w}$, α is a number between 0 and 1, $G(x, w) = (g_1(x, w), \dots, g_m(x, w)) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}^m$. Such a constraint is a compromise with the requirement of enforcing a hard constraint $G(x, w) \leq 0$ for all $w \in \mathcal{W}$. Enforcement of such a hard constraint may be very costly or even impossible. However, there are a few significant numerical complications of the optimization problem subject to chance constraints. The disturbance w is usually multi-dimensional. In such a case, it is difficult to check whether the chance constraint is satisfied at the given point x as there are no efficient ways to compute the corresponding probabilities accurately. Firstly, Monte-Carlo simulation is the only way to estimate the probability of a chance constraint to be violated at a given point and the cost of this computation increases as α decreases. The chance constrained version of the randomly perturbed constraint $G(x, w)$, even for the case when $g(x, w)$ is a simple bilinear constraint, is extremely computationally intractable. Secondly, the feasible set for a chance constraint is usually non-convex which makes optimization subject to this constraint problematic [8].

A general method to build a computationally tractable approximation of the CCP is given by the scenario approach based on Monte Carlo sampling techniques [9, 10, 11, 12]. Several other approximations such as the Bernstein approximation, quadratic approximation and the *CVaR* approximation have been proposed [8]. In particular, the scenario approach and the *CVaR* approximation have found wide usage in literature. The main advantage of the scenario approach is that no restrictions are imposed on the distribution of w . However, since the approximation itself is random, the solution may not satisfy the chance constraint. Therefore, it is important to determine the sample size N_s such that the solution satisfies the original CCP with some probability $1 - \beta$. The *CVaR* approximation on the other hand, finds wide usage in literature due to its features of numerical tractability, convexity and monotonicity [49]. Furthermore, it has been widely accepted as the tightest convex approximation of the chance constraint.

2-3-1 The Scenario Approach

The scenario approach is a sample-based approach for obtaining tractable solutions for stochastic programming problems [9, 10, 11, 12]. The basic idea in a sample-based approach is to characterise the dynamics of the stochastic system by a finite set of realizations of uncertainties. Typically, sample-based approaches do not require convexity assumptions with respect to uncertainty. A sample-based approach allows for approximating a CCP as a deterministic optimal control problem with the property that as the number of samples approaches infinity, the approximation becomes exact. This leads to a high computational cost and is prohibitive for practical applications. A significant development in the direction of compensating this drawback is the scenario approach, which provides an insight into the sample complexity, i.e., the number of samples that are required for an adequate approximation of the chance constraints.

In the scenario approach, N_s sampled instances $\Omega = \{w^1, \dots, w^{N_s}\}$ of the uncertainty vector are used to approximate the CCP in (2-52) to get,

$$\begin{aligned} \min_{x \in \mathcal{X}} f(x) \\ \text{s.t. } G(x, w^j) \leq 0, j = 1, \dots, N_s \end{aligned} \quad (2-53)$$

These sampled instances of the uncertainty are independent and are known as '*scenarios*'. The solution derived in this approach satisfies all unforeseen constraints except for a user chosen fraction that rapidly tends to zero as N_s increases. With a selected violation parameter $\alpha \in (0, 1)$ and a confidence parameter $\beta \in (0, 1)$, if N_s is chosen so as to satisfy the condition [9],

$$\sum_{i=0}^{\rho-1} \binom{N_s}{i} \alpha^i (1-\alpha)^{N_s-i} \leq \beta \quad (2-54)$$

then, with probability no smaller than $1-\beta$, the solution satisfies all constraints in the original CCP but at most an α -fraction, i.e., $P(w : F(x^*, w) \not\leq 0) \leq \alpha$. Here, ρ is the support rank of the decision space [64]. An explicit expression for N_s as a function of β and α is given as [11],

$$N_s \geq \frac{2}{\alpha} \left(\ln \frac{1}{\beta} + \rho \right) \quad (2-55)$$

If β is neglected, the solution x^* is robust against uncertainty from set \mathcal{W} upto the selected level α . By choosing a large N_s , α can be made small. It is to be noted that x^* is a random quantity as it depends on the constraints extracted corresponding to Ω . It is possible that the extracted constraints do not represent the uncertainty set very well. In this case, the portion of unseen constraints violated by x^* will be larger than α . Parameter β controls the probability that this happens and the final result that x^* violates at most an α -fraction of constraints holds with probability $1-\beta$. Theoretically, β plays an important role as selecting $\beta = 0$ yields $N_s = \infty$. In practice, by (2-55), β can be selected to be a very small number such as 10^{-10} or 10^{-20} and N_s still does not grow significantly. The main feature of these results in [9] and [11] is that the probability of constraint violation rapidly decreases to 0 as the number of scenarios grows.

The scenario approach to Stochastic MPC

A major challenge in SMPC is the solution of the FHOCP must satisfy the chance constraints at every time step. Various recent scenario-based approaches for SMPC are proposed in [64, 65, 57, 66, 67]. The advantage of Scenario-based SMPC (SCMPC) is that it renders the uncertain system into multiple deterministic affine systems by substituting the individual scenarios. This finite horizon scenario problem (FHSCP) is much simpler compared to the FHOCP. The closed-loop behaviour of the system may be erratic at times due to highly unlikely outliers in the sampled scenarios. This is obviated by a-posteriori scenario removal. State constraints corresponding to $R > 0$ scenarios are removed after outcomes of all samples have been observed. However, the sample complexity N_s must be appropriately increased. The pair (N_s, R) is called sample-removal pair. The FHSCP is solved for possible combinations of choosing R out of N_s scenarios. The combination that yields the lowest cost of all solutions is selected. Since it is required to choose R out of N_s instances of the FHSCP, this leads to prohibitive sample complexities for large values of R . An upper bound on N_s that depends on the support rank of the chance constraints is derived for a fixed R .

Definition 2.2. *Support Rank* (a) The unconstrained subspace \mathcal{L}_i of a constraint $i \in \{0, \dots, N-1\}$ in (2-10) is the largest linear subspace (in the set inclusion sense) of the search space \mathbb{R}^{Nn_u} that remains unconstrained by all sampled instances of i , almost surely. (b) The support rank

of a constraint $i \in \{0, \dots, N-1\}$ in (2-10) is given as

$$\rho_i := Nn_u - \dim \mathcal{L}_i \quad (2-56)$$

where $\dim \mathcal{L}_i$ represents the dimension of the unconstrained subspace \mathcal{L}_i .

It is to be noted that the support rank of a chance constraint is an inherent property of the chance constraint and is not affected by the simultaneous presence of other chance constraints.

Let $V_k|x_k$ denote the first step violation probability, i.e., the probability with which the first predicted state, from (2-1), falls out of the set \mathcal{X} that are defined by the constraints.

$$V_k|x_k := P(Ax_k + Bu_k + Ew_k \notin \mathcal{X}|x_k) \quad (2-57)$$

The control input u_k and $V_k|x_k$ depend on the scenarios $\Omega_k = \{w_k^1, \dots, w_k^{N_s}\}$ that are submitted at time k . The notation $u_k(\Omega_k)$ and $V_k|x_k(\Omega_k)$ shall be used to emphasize this fact. The violation probability $V_k|x_k$ can be considered as a random variable on the probability space $(\mathcal{W}^{N_s N}, P^{N_s N})$ with support in $[0, 1]$. Here, $\mathcal{W}^{N_s N}$ and $P^{N_s N}$ denote the $N_s N$ th product of the set \mathcal{W} and the measure P . The distribution of $V_k|x_k(\Omega_k)$ is unknown but it is possible to derive an upper bound on this distribution. To derive this upper bound, a few assumptions are made. It is assumed that the set of feasible inputs \mathcal{U} is bounded and convex and the state constrained set \mathcal{X} is convex. It is also assumed that the stage cost function $q(\cdot)$ and terminal cost function $p(\cdot)$ in (2-5) are convex functions. Furthermore, with the assumption that the disturbance input w is White Gaussian noise and $\alpha \in [0, 1]$ being any violation level [64, Theorem 6.7],

$$P^{N_s N}(V_k|x_k(\omega_k) > \alpha) \leq U_{N_s, \rho}(\alpha) \quad (2-58)$$

$$U_{N_s, \rho}(\alpha) := \min \left\{ 1, \left(\frac{R + \rho_1 - 1}{R} \right) B(\alpha; N_s, R + \rho_1 - 1) \right\} \quad (2-59)$$

where $B(\cdot; \cdot, \cdot)$ represents the Beta distribution.

$$B(\alpha; N_s, R + \rho_1 - 1) := \sum_{j=0}^{R + \rho_1 - 1} \binom{N_s}{j} \alpha^j (1 - \alpha)^{N_s - j} \quad (2-60)$$

For a fixed value of R , N_s is selected such that $U_{N_s, \rho}(\alpha) \leq \beta$, where β is a desired confidence probability level and $P^{N_s N}(V_k|x_k(\omega_k) > \alpha) \leq \beta$ holds.

However, this approach to SMPC is conservative when applied in a receding horizon fashion. The focus is either on obtaining a robust solution [65, 57], or the chance constraints are over-satisfied by the closed loop system [64, 66, 67]. This conservatism of SCMPC is addressed by interpreting the chance constraints as a time average, rather than pointwise-in-time with a high confidence, which is much less restrictive. This also reduces the sample complexity by exploiting the nature of the structural properties of the FHOCP. The result in (2-58) is used to obtain the bound on the expectation [12],

$$\mathbb{E}^{N_s N}[V_k|x_k] := \int_{\mathcal{W}^{N_s N}} V_k|x_k(\omega_k) dP^{N_s N} \quad (2-61)$$

where, the operator $\mathbb{E}^{N_s N}$ is the expectation operator on $(\mathcal{W}^{N_s N}, P^{N_s N})$. A reformulation via the indicator function $\mathbf{1} : \mathcal{W}^{N_s N} \rightarrow \{0, 1\}$ yields,

$$\begin{aligned} \mathbb{E}^{N_s N}[V_k|x_k] &= \int_{[0,1]} \int_{\mathcal{W}^{N_s N}} \mathbf{1}(V_k|x_k(\omega_k) > \alpha) dP^{N_s N} d\alpha \\ &= \int_{[0,1]} P^{N_s N}[V_k|x_k(\omega_k) > \alpha] d\alpha \\ &\leq \int_{[0,1]} U_{N_s, \rho}(\alpha) d\alpha \end{aligned} \quad (2-62)$$

A sample-removal pair (N_s, R) is admissible if its substitution in (2-62) yields,

$$\mathbb{E}^{N_s N}[V_k|x_k] \leq \epsilon \quad (2-63)$$

where ϵ is a desired violation level. The admissibility of the sample-removal pair can be tested by performing the numerical integration in (2-62). The integral value monotonically decreases with N_s and monotonically increases with R . If R is fixed, then N_s can be determined easily, example, by a bisection method. If $R = 0$, the integration can be replaced by the simple analytic formula,

$$\mathbb{E}^{N_s N}[V_k|x_k] \leq \frac{\rho}{N_s + 1} = \epsilon \quad (2-64)$$

where ϵ is a specified level.

Let $M_k := \mathbf{1}_{\mathcal{X}^c}(x_{k+1})$ denote a random variable indicating that $x_{k+1} \notin \mathcal{X}$, i.e., $\mathbf{1}_{\mathcal{X}^c} : \mathbb{R}^{n_x} \rightarrow 0, 1$ is the indicator function on the complement \mathcal{X}^c of \mathcal{X} . At each time step k , there are a total of $D = (N_s N + 1)$ random variables, i.e., the scenarios and the disturbance $\{\Omega_k, w_k\} \in \mathcal{W}^{(N_s N + 1)} = \mathcal{W}^D$. Define $W_k = \{w_0, \Omega_0, \dots, w_k, \Omega_k\} \in \mathcal{W}^{(k+1)D}$ for any $t \in \{0, \dots, T-1\}$. These variables allow for the expression of the variables $x_k(W_{t-1})$, $V_k(W_{t-1}, \Omega_k)$ and $M_k(W_t)$ to be expressed in terms of the elementary uncertainties. Observe that $M_k \in \{0, 1\}$ is a Bernoulli random variable with random parameter V_k because,

$$\begin{aligned} \mathbb{E}[M_k|W_{k-1}, \Omega_k] &= \int_{\mathcal{W}} M_k(W_k) dP(\Omega_k) \\ &= V_k(W_{k-1}, \Omega_k) \end{aligned} \quad (2-65)$$

Then, with the admissible sample complexity N_s , the expected time-average of closed loop constraint violations remains below the specified level ϵ ,

$$\mathbb{E}^{TD} \left[\frac{1}{T} \sum_{k=0}^{T-1} M_k \right] \leq \epsilon \quad (2-66)$$

for any $T \in \mathbb{N}$ and the operator \mathbb{E}^{TD} is the expectation operator on $(\mathcal{W}^{TD}, P^{TD})$.

2-3-2 Risk Functions and Conditional Value at Risk (CVaR)

Let (Ω, \mathcal{F}) be a sample space equipped with sigma algebra \mathcal{F} on which uncertain outcomes $(Z = Z(w))$ are defined. A *risk function* $\rho(z)$ maps Z onto the extended real line $\mathbb{R} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ [6]. The space \mathcal{Z} of allowable random functions $Z(w)$ for which $\rho(Z)$ is

defined is chosen as $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$ ¹. It is assumed that \mathcal{Z} is a linear space of \mathcal{F} -measurable functions, the risk functions $\rho : \mathcal{Z} \rightarrow \mathbb{R}$ are proper, i.e., $\rho(Z) > -\infty$ for all $Z \in \mathcal{Z}$ and the domain $\text{dom}(\rho) := \{Z \in \mathcal{Z} : \rho(Z) < +\infty\}$.

The following axioms are associated with risk function. For $Z_1, Z_2 \in \mathcal{Z}$, the point-wise partial order is denoted as $Z_2 \succeq Z_1$ which means $Z_2(w) \geq Z_1(w)$ for all $w \in \Omega$.

(A1) *Convexity*: $\rho(\alpha Z_1 + (1 - \alpha)Z_2) \leq \alpha\rho(Z_1) + (1 - \alpha)\rho(Z_2)$ for all $Z_1, Z_2 \in \mathcal{Z}$ and all $\alpha \in [0, 1]$.

(A2) *Monotonicity*: If $Z_1, Z_2 \in \mathcal{Z}$ and $Z_2 \succeq Z_1$, then $\rho(Z_2) \geq \rho(Z_1)$.

(A3) *Translation Equivalence*: If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$.

(A4) *Positive Homogeneity*: If $a > 0$ and $Z \in \mathcal{Z}$, then $\rho(aZ) = a\rho(Z)$

The risk functions satisfying axioms (A1)-(A4) are called *coherent risk measures*. The popular notion Value at Risk (*VaR*), widely used in the fields of statistics and finance, is defined as,

$$\text{VaR}_{1-\alpha}(Z) := \min_{\eta \in \mathbb{R}} \{ \eta : P(Z \leq \eta) \geq 1 - \alpha \} \quad (2-67)$$

However, *VaR* has undesirable characteristics of lack of convexity. The more popular coherent risk measure, *CVaR*, attempts to address the shortcomings of *VaR*. While *VaR* represents the worst-case loss with a probability, *CVaR* represents the expected loss if the worst case threshold is crossed, i.e., it represents the expected loss that will occur beyond the *VaR* threshold. *CVaR* is defined as,

$$\text{CVaR}_{1-\alpha}(Z) = \mathbb{E}[Z \geq \text{VaR}_{1-\alpha}(Z)] \quad (2-68)$$

which is,

$$\text{CVaR}_{1-\alpha}(Z) = \frac{1}{\alpha} \int_{Z \geq \text{VaR}_{1-\alpha}(Z)} ZP(dw) \quad (2-69)$$

$$\text{CVaR}_{1-\alpha}(Z) := \inf_{\eta \in \mathbb{R}} \left(\eta + \frac{1}{\alpha} \mathbb{E}[(Z - \eta)_+] \right) \quad (2-70)$$

Considering the chance constraint in the CCP in (2-52), the *VaR* of the random function $F(x, w)$ is given as,

$$\text{VaR}_{1-\alpha}(G(x, w)) := \min_{\eta \in \mathbb{R}} \{ \eta : P(G(x, w) \leq \eta) \geq 1 - \alpha \} \quad (2-71)$$

The corresponding *CVaR* is given as,

$$\text{CVaR}_{1-\alpha}(G(x, w)) = \min_{\eta \in \mathbb{R}} \left(\eta + \frac{1}{\alpha} \mathbb{E}[(G(x, w) - \eta)_+] \right) \quad (2-72)$$

¹ $\mathcal{L}_p(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ denotes the linear space of all \mathcal{F} -measurable functions $\phi : \Omega \rightarrow \mathbb{R}^n$ such that $\int_{\Omega} \|\phi(w)\|^p dP(w) \leq +\infty$. An element of $\mathcal{L}_p(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ is a class of functions $\phi(w)$ which may differ from each other on sets of P-measure zero. For $n = 1$, the space is denoted as $\mathcal{L}_p(\Omega, \mathcal{F}, P)$. It is assumed that $p \in [1, +\infty)$, P is a probability measure on (Ω, \mathcal{F}) and expectations are taken with respect to P.

It can be verified that $CVaR_{1-\alpha}(G(x, w)) \rightarrow VaR_{1-\alpha}(G(x, w))$ as $\alpha \downarrow 0$. If $G(x, w)$ is convex for almost every w , then $CVaR_{1-\alpha}(G(x, w))$ is also convex. As $CVaR$ is considered as an approximation of VaR , and as $CVaR$ is also convex, the chance constraint in (2-52) can be replaced by the $CVaR$ constraint as [50],

$$CVaR_{1-\alpha}(G(x, w)) \leq 0 \quad (2-73)$$

Furthermore, since it is difficult to evaluate a conditional expectation when the probability distribution is continuous over an infinite support of the random vector, a Monte-Carlo or a sampled approach is taken. A number of independent and identically distributed samples of w, w^1, \dots, w^{N_s} , called 'scenarios', are extracted. The sample-average approximation of $CVaR$ is now given as,

$$CVaR_{1-\alpha}(G(x, w)) = \min_{\eta \in \mathbb{R}} \left(\eta + \frac{1}{\alpha N_s} \sum_{j=1}^{N_s} ((G(x, w^j) - \eta)_+) \right) \quad (2-74)$$

2-4 Exact Penalty Method for Unconstrained Optimization

The basic idea of penalty methods is to eliminate some or all constraints and add a penalty term to the cost cost function. This penalty term penalizes infeasible states by prescribing a high cost to these states. Penalty methods have been proposed in literature that solve a constrained minimization problem by means of a single unconstrained problem [68, 69, 16]. A penalty method may yield a Lagrange multiplier of the problem in a single minimization and it may require a second minimization to yield an optimal solution. This holds true for a wide class of differentiable penalty functions. Non-differentiability is an essential characteristic of a penalty function if the penalty method is to yield an optimal solution in a single minimization [16]. Consider the following problem,

$$\begin{aligned} \min f(x) \\ \text{s.t. } x \in \mathcal{X} \subset \mathbb{R}^{n_x}, f_i(x) \leq 0, i = 1, \dots, m \end{aligned} \quad (2-75)$$

The function $f_i : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ is assumed to be real valued and convex and the set \mathcal{X} is assumed to be closed and convex. The following assumptions are made.

Assumption 2.2. *Problem (2-75) has a non-empty and compact set of optimal solutions.*

Assumption 2.3. *Problem (2-75) has atleast one Lagrange multiplier (or Kuhn-Tucker vector as defined in [70]).*

With assumption 2.2, the ordinary perturbation function, which is a closed proper convex function, is given as,

$$q(u) = \inf \{ f(x) : x \in \mathcal{X}, f_i(x) \leq u_i, i = 1, \dots, m \} \quad (2-76)$$

The ordinary dual function, which is a closed proper concave function, is given as,

$$g(\lambda) = \inf_u \{ q(u) + \sum_{i=1}^m \lambda_i u_i \} \quad (2-77)$$

Furthermore, $q(0) = \min_{u \leq 0} q(u) = \sup_{\lambda} g(\lambda)$. The dual function g is maximized at points which are Lagrange multipliers of the problem and assumption 2.3 guarantees existence of atleast one such point.

Consider penalty functions $p : \mathbb{R} \rightarrow \mathbb{R}$ which satisfy the following conditions:

(C1) p is convex.

(C2) $p(t) = 0$ for all $t \leq 0$ and $p(t) > 0$ for all $t > 0$.

Now, consider the penalized problem

$$\inf_{x \in \mathcal{X}} \left\{ f(x) + \sum_{i=1}^m p_i(f_i(x)) \right\} \quad (2-78)$$

where the the penalty functions $p_i : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the conditions (C1) and (C2). Let $\tilde{x} \in \mathcal{X}$ be a solution of the problem (2-78). Denote by $p_i^* : \mathbb{R} \rightarrow (-\infty, +\infty]$ the convex conjugate function of p_i

$$p_i^*(t^*) = \sup_t \{tt^* - p_i(t)\} \quad (2-79)$$

The following relationship may be verified.

$$\begin{aligned} \inf_{x \in \mathcal{X}} \left\{ f(x) + \sum_{i=1}^m p_i(f_i(x)) \right\} &= \inf_u \left\{ q(u) + \sum_{i=1}^m p_i(u_i) \right\} \\ &= \max_{\lambda} \left\{ g(\lambda) - \sum_{i=1}^m p_i^*(\lambda_i) \right\} \end{aligned} \quad (2-80)$$

where the last equality follows by the direct application of Fenchel-duality theorem [70]. Assuming that \tilde{x} exists, the following proposition is made.

Proposition 2.1. (a) For \tilde{x} to also be a an optimal solution of the original problem (2-75), it is required that,

$$\lim_{t \rightarrow 0^+} \frac{p(t)}{t} \geq \bar{\lambda} \quad (2-81)$$

for some Lagrange multiplier $\bar{\lambda}$ of problem (2-75).

(b) For problem (2-75) and (2-78) to have the same solutions, it is sufficient that

$$\lim_{t \rightarrow 0^+} \frac{p(t)}{t} > \bar{\lambda} \quad (2-82)$$

Proof. (a) If \tilde{x} is an optimal solution of problem (2-75), from (2-80),

$$\bar{f} = f(\tilde{x}) + \sum_{i=1}^m p_i(f_i(\tilde{x})) = \max_{\lambda} \left\{ g(\lambda) - \sum_{i=1}^m p_i^*(\lambda_i) \right\} \quad (2-83)$$

where \bar{f} denotes the optimal value of problem (2-75). Let $\bar{\lambda}$ be any vector attaining the maximum above. Then,

$$\bar{f} + \sum_{i=1}^m p_i^*(\bar{\lambda}_i) = g(\bar{\lambda}) \leq \bar{f} \quad (2-84)$$

Since $p_i^*(t^*) \geq 0$ for all t^* , we get,

$$p_i^*(\bar{\lambda}_i) = 0, \quad i = 1, \dots, m, \quad g(\bar{\lambda}) = \bar{f} \quad (2-85)$$

showing that $\bar{\lambda}$ must be a Lagrange multiplier of problem (2-75). Now the relation $p_i^*(\bar{\lambda}_i) = 0, i = 1, \dots, m$ by the definition of p_i, p_i^* implies (2-81).

- (b) If \bar{x} is an optimal solution of problem (2-75), then by (2-82) and by the definition of a Lagrange multiplier,

$$\begin{aligned} f(\bar{x}) + \sum_{i=1}^m p_i(f_i(\bar{x})) &= f(\tilde{x}) + \sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}) \\ &\leq f(x) + \sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}) \leq f(x) + \sum_{i=1}^m p_i(f_i(\bar{x})), \quad \forall x \in \mathcal{X} \end{aligned} \quad (2-86)$$

Hence \bar{x} is also a solution of problem (2-78). Conversely, if \tilde{x} is a solution of problem (2-78), then \tilde{x} is either a feasible point in which case it is also a solution of problem (2-75), or it is infeasible in which case $f_i(\tilde{x}) > 0$ for some i . Since $p_i(t) > 0$ for all $t > 0$, then (2-82) for any solution \bar{x} of problem (2-75),

$$\begin{aligned} f(\tilde{x}) + \sum_{i=1}^m p_i(f_i(\tilde{x})) &> f(\tilde{x}) + \sum_{i=1}^m \bar{\lambda}_i f_i(\tilde{x}) \\ &\geq \bar{f} = f(\bar{x}) + \sum_{i=1}^m p_i(f_i(\bar{x})) \end{aligned} \quad (2-87)$$

which is a contradiction. Hence, problems (2-75) and (2-78) have exactly the same optimal solutions. □

Proposition 2.1(b) generalizes a known result for the penalty function $p(t) = c \max\{0, t\}$ with c being the penalty parameter which has been discussed in [71, 72, 73]. It is sufficient that c be greater than some Lagrange multiplier of the problem for the original problem (2-75) and the penalized problem (2-78) to have the same optimal solutions. A useful upper bound to the maximum magnitude of the Lagrange multipliers can be obtained if an interior point to the constraints and a lower bound to the optimal value are known. It can be inferred from proposition 2.1(a) shows that unless some Lagrange multiplier is zero and the problem is essentially unconstrained, an optimal solution to problem (2-75) cannot be obtained by solving problem (2-78) by using a differentiable penalty function.

Tractable SMPC using Conditional Value at Risk Optimization

A detailed formulation of a tractable Stochastic MPC strategy using *CVaR* optimization for discrete-time linear time-invariant systems with constraints on the state and the input is presented in this chapter. An approach to incorporate soft constraints on the state through an exact penalty method is introduced. The penalty functions make use of *CVaR* to penalize infeasible states. This penalty is added as an extra cost on the system to account for closed-loop performance degradation due to violation of constraints by infeasible states. The closed-loop behaviour of this formulation is discussed and the formulation is evaluated by means of a numerical example. Finally, a brief overview of suitable applications of this formulation is provided.

3-1 Problem Formulation

A linear time-invariant system that is subject to stochastic disturbances is considered. The system is described by the following state space model.

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k + Ew_k \\z_k &= Cx_k + Du_k\end{aligned}\tag{3-1}$$

Here, $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input and $z_k \in \mathbb{R}^{n_z}$ is the output at time k . The successor state is x_{k+1} at time $k+1$. The disturbance input $w_k \in \mathbb{R}^{n_w}$ is an exogenous disturbance with unknown current and future values but known probability distribution. It is a random vector in a probability space (Ω, \mathcal{F}, P) with support $\mathcal{W} \subset \mathbb{R}^{n_w}$.

Assumption 3.1. *The disturbance input $(w_k)_{k \in \mathbb{N}_{\geq 0}}$ is assumed to be a realization of a stochastic process with $w_k \in \mathcal{G}(0, Q_w)$ where \mathcal{G} denotes a family of Gaussian distributed random variables with zero mean and covariance matrix Q_w . Moreover, for $k \neq j$, w_k and w_j are statistically independent. Thus, the additive disturbance is Gaussian white noise.*

Assumption 3.1 simplifies the computation of predicted costs based on the expected value of sum of stage costs, which in turn simplifies stability analysis of the system based on the cost function. The matrices A , B , E , C and D are of suitable dimensions with real elements.

The system is subject to constraints on the state and control input. Consider constraints of the form,

$$\mathcal{X} := \{x \in \mathcal{R}^{n_x} : g(x, w) \leq 0\}, \quad \mathcal{U} := \{u \in \mathcal{R}^{n_u} : h(u) \leq 0\} \quad (3-2)$$

where $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}^m$ and $h : \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{m_2}$. The constraints on the state x define a closed convex set $\mathcal{X} \subset \mathcal{R}^{n_x}$ that contains the origin in its interior. The constraints on the control input u define a non-empty measurable compact convex control set $\mathcal{U} \subset \mathcal{R}^{n_u}$.

Let N , a positive integer, be the prediction horizon. The system is controlled by a feedback controller, i.e., at each k , the input u_k is the function of the state x_k . A feedback policy $\boldsymbol{\pi} = \{\pi_0(\cdot), \dots, \pi_{N-1}(\cdot)\}$, a sequence of measurable control laws is employed for each $\pi_i : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$, $i = 0, \dots, N - 1$. The control laws π belong to a class of controllers Ψ which is a set of continuous maps that map the origin of the state space to into the zero input, $\pi(0) = 0$. The control input u_i is selected as $\pi_i(x_i)$ at the i th stage. Starting at time $k = 0$, and assuming that $w_k = 0, \forall k$, the system has an initial state $x_0 \in \mathbb{R}^{n_x}$. Suppose the state is generated by (2-1), if there exists a controller $\pi \in \Psi$ such that

$$x_k \rightarrow 0 \text{ as } k \rightarrow \infty$$

then, the state x_0 is a null controllable point in the state space. The set of all null controllable points define a set in the state space known as the recoverable set. The recoverable set is the whole state space only when the system is globally asymptotically stable. When the system is subject to unbounded disturbances and constraints on the input, i.e., \mathcal{U} is bounded, the system is globally asymptotically stable when the matrix pair (A, B) is stabilizable and the eigenvalues of the system matrix A lie on or inside the unit circle.

Assumption 3.2. *The matrix pair (A, B) is stabilizable, the matrix pair (A, C) is observable, and all eigenvalues of the system matrix A lie on or inside the unit circle.*

Retaining the state within the feasible set \mathcal{X} for the entire prediction horizon requires the initial state to belong to a set of feasible initial states given as,

$$\mathcal{X}_N = \{x_k : \exists \mathbf{u}_{k \rightarrow k+N|k} \text{ s.t. } x_{i|k} \in \mathcal{X} \text{ and } u_{i|k} \in \mathcal{U} \forall k \geq 0, i = 0, 1, \dots, N - 1\} \quad (3-3)$$

This requirement is often too conservative and results in poor performance. In most industrial applications, the best performance is usually achieved near the boundary of the feasible set \mathcal{X} , and thus, violation of the hard constraints is unavoidable due to the uncertain nature of the system. In such cases, chance constraints can be viewed as a compromise with the requirement to enforce hard constraints in an uncertain system which may be very expensive or even impossible. Chance constraints on the state trajectory is proposed as,

$$P(x \in \mathcal{X}) \geq 1 - \alpha \quad (3-4)$$

and considering state constraints of the form given in (3-2),

$$P(g(x, w) \leq 0) \geq 1 - \alpha \quad (3-5)$$

where, the chance constraint requires the probability of any predicted state x not belonging to the set \mathcal{X} to be less than α . The optimal control problem is defined in terms of a performance index $J_N(x_k, \mathbf{u}_{k \rightarrow k+N|k}, \mathbf{w}_{k \rightarrow k+N|k})$ that is evaluated over the horizon of N steps and is solved at each time step, where $\mathbf{u}_{k \rightarrow k+N|k} = \{u_{0|k}, \dots, u_{N-1|k}\}$ and $\mathbf{w}_{k \rightarrow k+N|k} = \{w_{0|k}, \dots, w_{N-1|k}\}$. The predicted cost is given as,

$$\hat{J}_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}, \mathbf{w}_{k \rightarrow k+N|k}) = p(x_{N|k}) + \sum_{i=0}^{N-1} q(x_{i|k}, u_{i|k}) \quad (3-6)$$

where, the function $q : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}_+$ gives the cost per stage and the function $p : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_+$ is the terminal cost. On account of the stochasticity of w_k , the predicted cost $\hat{J}_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}, \mathbf{w}_{k \rightarrow k+N|k})$ is stochastic. Since the probability distribution of w_k is known, it is appropriate to use a predicted cost that takes into consideration the stochastic nature of the model uncertainty through the expectation of the predicted cost. The expected cost is given as,

$$\bar{J}_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) = \mathbb{E} \left[\left(p(x_{N|k}) + \sum_{i=0}^{N-1} q(x_{i|k}, u_{i|k}) \right) \middle| x_{0|k} = x_k \right] \quad (3-7)$$

The terminal cost ensures that the system is closed-loop stable and the controller found is stabilizing.

Assumption 3.3. *The functions $p(\cdot)$ and $q(\cdot)$ are assumed to be convex functions.*

Given the performance index and constraints, the chance constrained problem can be formulated as,

$$\begin{aligned} J_k^*(x_k) &= \min_{\mathbf{u}_{k \rightarrow k+N|k}} \bar{J}_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) \\ \text{s.t. } & x_{i+1|k} = Ax_{i|k} + Bu_{i|k} + Ew_{i|k}, \quad i = 0, \dots, N-1 \\ & P(x_{i|k} \in \mathcal{X}) \geq 1 - \alpha \\ & u_{i|k} \in \mathcal{U}, \quad i = 0, \dots, N-1 \\ & x_{0|k} = x_k \end{aligned} \quad (3-8)$$

where, $J_k^*(x_k)$ is the optimal value function under the optimal control policy $\mathbf{u}_{k \rightarrow k+N|k}^*$. The receding horizon policy is implemented by applying the first element of the optimal control input sequence u_k^* to the system at time k . However, there are a few significant numerical complications of this optimization problem with respect to chance constraints [8]. Firstly, Monte-Carlo simulation is the only way to estimate the probability of a chance constraint to be violated at a given point and the cost of this computation increases as α decreases. Secondly, the feasible set for a chance constraint is usually non-convex which makes optimization subject to this constraint problematic. The chance constraint can be approximated by the scenario approach to render the constraint as a combination of multiple affine constraints. However, as the initial state at each sampling time is a new state measurement, the assumption that $x_0 \in \mathcal{X}$ is strong. The SCMPC may not have an initial feasible state which leads to an empty feasible set. This renders the SCMPC as unsolvable.

A simple method to obviate this problem is to penalize infeasible states by incorporating a penalty function in the performance index giving rise to a soft constraint approach [15]. A

similar approach is elaborated in [13, 14], where an additional cost penalizes the state when there is a 'high probability' of constraint violation. However, the probability of constraint violation is not quantified in this approach. A contribution in this thesis is to incorporate an exact penalty method while keeping the state constraint violation in the system within certain admissible levels. A computationally efficient approximation is used to maintain admissible violation levels to achieve a tractable SMPC strategy in closed-loop.

The expected cost including the penalty function for the added cost in the case of constraint violation is of the form,

$$J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}, \mathbf{w}_{k \rightarrow k+N}) = \bar{J}_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}, \mathbf{w}_{k \rightarrow k+N}) + \sum_{i=0}^{N-1} c.p_{\mathcal{X}}(x_{i|k}) \quad (3-9)$$

where, $p_{\mathcal{X}} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ is the penalty function and c is the penalty parameter. The penalty on a feasible state which does not violate constraints is set to zero, i.e., $p_{\mathcal{X}}(x) = 0$, $x \in \mathcal{X}$ and for an infeasible state that violates constraints, $p_{\mathcal{X}}(x) \geq 0$, $x \notin \mathcal{X}$. However, the penalty on the infeasible state cannot be arbitrarily high as a choice of an infinite penalty even for a very large violation will make the set of feasible states almost always empty. Hence, the following assumption is made.

Assumption 3.4. *The penalty function for constraint violation is a finite valued convex function, $p_{\mathcal{X}} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_+$ with $p_{\mathcal{X}}(x) = 0$, $x \in \mathcal{X}$.*

If the penalty function $p_{\mathcal{X}}(x)$ is an *exact penalty function* and the value of c is greater than some Lagrange multiplier of the problem, then the penalty method yields the optimal solution through a single unconstrained optimization [16]. The simplest form of an exact penalty function would be $c \max\{0, g(x, w)\}$ where the value of c is greater than some Lagrange multiplier of the problem. An upper bound on the Lagrange multiplier can be found by making use of Slater's condition [74]. However, since the constraints on the state of the system take the form of chance constraints, it is required to choose a penalty function that is motivated by the presence of stochasticity in the system. The requirement for the penalty function to be a convex function is necessary for the penalty method to be exact. The *CVaR* function is widely known in literature as the tightest convex approximation of the chance constraint [8]. Furthermore, intuitively, the *CVaR* accounts for the risk that the system will face at an undesirable state and it can be used to determine performance degradation due to violation of constraints at the infeasible state. Hence, the *CVaR* can be suitably incorporated as a penalty function.

Given the chance constraint in (3-5), the Value at Risk (VaR) of the random function $g(x, w)$ is given as,

$$VaR_{1-\alpha}(g(x, w)) := \min_{\eta \in \mathbb{R}} \{ \eta : P(g(x, w) \leq \eta) \geq 1 - \alpha \} \quad (3-10)$$

While *VaR* represents the worst-case loss with a probability, *CVaR* represents the expected loss if the worst case threshold is crossed, i.e., it represents the expected loss that will occur beyond the *VaR* threshold. The *CVaR* is defined as the conditional expectation of $g(x, w)$ exceeding *VaR*,

$$CVaR_{1-\alpha}(g(x, w)) := \mathbb{E}[g(x, w) | g(x, w) \geq VaR_{1-\alpha}(g(x, w))] \quad (3-11)$$

which can also be formulated as,

$$\min_{\eta \in \mathbb{R}} \left(\eta + \frac{1}{\alpha} \mathbb{E}[(g(x, w) - \eta)_+] \right) \quad (3-12)$$

The chance constraint in (3-5) can be replaced by the *CVaR* constraint as,

$$CVaR_{1-\alpha}(g(x, w)) \leq 0 \quad (3-13)$$

As required in *CVaR*, an exact evaluation of the expected value of a random function $g(x, w)$ is either impossible or prohibitively expensive [51] and hence a simple sampled-average approximation is used. The sample-average approximation of *CVaR* is now given as,

$$CVaR_{1-\alpha}(g(x, w)) = \min_{\eta \in \mathbb{R}} \left(\eta + \frac{1}{\alpha N_s} \sum_{j=1}^{N_s} ((g(x, w^j) - \eta)_+) \right) \quad (3-14)$$

for which a number of independent and identically distributed samples of w, w^1, \dots, w^{N_s} , called 'scenarios', are extracted. The *CVaR* constraint is now reduced to a combination of multiple affine constraints.

The sample complexity or the number of scenarios N_s depends on the support rank of the problem, which is usually the number of decision variables of the problem Nn_u . The optimal control sequence $\mathbf{u}_{k \rightarrow k+N|k}^*$, when applied to the system following the receding horizon principle must be able to satisfy the constraints on the system. At each time step k , the problem is solved using the current state measurement x_k and new scenarios have to be generated at each time step $k \in \mathcal{T}$. Let $V_k|x_k$ denote the first step violation probability, i.e., the probability with which the first predicted state falls out of the set \mathcal{X} that are defined by the constraints.

$$V_k|x_k := P(Ax_k + Bu_k + Ew_k \notin \mathcal{X}|x_k) \quad (3-15)$$

The following theorem gives an explicit bound on the a priori probability of violation, and thereby deriving a bound on the number of scenarios [11].

Theorem 1. *Given a violation level $\alpha \in [0, 1]$, the prediction horizon N and the support rank $\rho = Nn_u$, then*

$$P^{N_s N}(V_k|x_k(\omega_k) > \alpha) = B(\alpha; N_s, \rho - 1) \quad (3-16)$$

where,

$$B(\alpha; N_s, \rho - 1) = \sum_{j=0}^{\rho-1} \binom{N_s}{j} \alpha^j (1 - \alpha)^{N_s - j} \quad (3-17)$$

N_s is selected such that $B(\alpha; N_s, \rho - 1) \leq \beta$, where β is a desired confidence probability level and $P^{N_s N}(V_k|x_k(\omega_k) > \alpha) \leq \beta$ holds.

However, this bound is rather conservative and may lead to over-satisfaction of chance constraints of the problem (3-8). Hence, a less conservative approach is used to obtain a bound on the expectation of the probability distribution by interpreting the chance constraints as a time average [12].

$$\mathbb{E}^{N_s N}[V_k|x_k] := \int_{\mathcal{W}^{N_s N}} V_k|x_k(\omega_k) dP^{N_s N} \quad (3-18)$$

where, the operator $\mathbb{E}^{N_s N}$ is the expectation operator on $(\mathcal{W}^{N_s N}, P^{N_s N})$. A reformulation via the indicator function $1 : \mathcal{W}^{N_s N} \rightarrow \{0, 1\}$ yields,

$$\mathbb{E}^{N_s N}[V_k | x_k] = \leq \int_{[0,1]} B(\alpha; N_s, \rho - 1) d\alpha \quad (3-19)$$

The integration can be replaced by a simple analytic formula,

$$\mathbb{E}^{N_s N}[V_k | x_k] \leq \frac{\rho}{N_s + 1} = \epsilon \quad (3-20)$$

where ϵ is a specified level. Let $M_k := \mathbf{1}_{\mathcal{X}^c}(x_{k+1})$ denote a random variable indicating that $x_{k+1} \notin \mathcal{X}$, i.e., $\mathbf{1}_{\mathcal{X}^c} : \mathbb{R}^{n_x} \rightarrow 0, 1$ is the indicator function on the complement \mathcal{X}^c of \mathcal{X} . At each time step k , there are a total of $D = (N_s N + 1)$ random variables, i.e., the scenarios and the disturbance $\{\Omega_k, w_k\} \in \mathcal{W}^{(N_s N + 1)} = \mathcal{W}^D$. Then, with the admissible sample complexity N_s , the expected time-average of closed-loop constraint violations remains below the specified level ϵ ,

$$\mathbb{E}^{ND} \left[\frac{1}{N} \sum_{k=0}^{N-1} M_k \right] \leq \epsilon \quad (3-21)$$

for any $N \in \mathbb{N}$ and the operator \mathbb{E}^{TD} is the expectation operator on $(\mathcal{W}^{TD}, P^{TD})$. To interpret the expected time-average of closed-loop violation in terms of the *CVaR* constraint, denote

$$\bar{g}(x, \eta, w) := \eta + \frac{1}{\alpha} [(g(x, w) - \eta)_+] \quad (3-22)$$

The *CVaR* constraint is then equivalent to

$$\mathbb{E}[\bar{g}(x, \eta, w)] \leq 0 \quad (3-23)$$

Let the indicator function for constraint violation $x_{k+1} \notin \mathcal{X}$ be given as,

$$M_k = \mathbf{1}_{\{x | \bar{g}(x, \eta, w) \geq 0\}}(x_{k+1}) \quad (3-24)$$

Then, given a sample complexity N_s and violation level ϵ , the expected time average of closed-loop constraint violation remains below the specified violation level,

$$\frac{1}{N} \sum_{k=0}^{N-1} CVaR_{1-\alpha}(g(x_k, w_k)) \leq \epsilon \quad (3-25)$$

Given the sample complexity N_s for the sample-average approximation of *CVaR*, the performance index of the tractable stochastic MPC strategy using *CVaR* optimization is now given as,

$$J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) = \bar{J}_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}, \mathbf{w}_{k \rightarrow k+N}) + \max \left\{ 0, \sum_{i=0}^{N-1} c CVaR_{1-\alpha}(g(x_{i|k}, w_{i|k})) \right\} \quad (3-26)$$

which is,

$$\begin{aligned} J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) = & \mathbb{E} \left[p(x_{N|k}) + \sum_{i=0}^{N-1} (q(x_{i|k}, u_{i|k})) \middle| x_{0|k} = x_k \right] \\ & + \max \left\{ 0, \sum_{i=0}^{N-1} c CVaR_{1-\alpha}(g(x_{i|k}, w_{i|k})) \right\} \end{aligned} \quad (3-27)$$

where, the penalty parameter c is chosen as a value greater than a Lagrange multiplier of the problem (2-10). As N_s number of scenarios are extracted, the sampled average approximation of the expected stage costs, terminal cost and $CVaR$ is given as,

$$J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) = \frac{1}{N_s} \sum_{j=1}^{N_s} \left(p(x_{N|k}^j) + \sum_{i=0}^{N-1} (q(x_{i|k}^j, u_{i|k}^j)) \right) + \max \left\{ 0, \sum_{i=0}^{N-1} c \left[\eta_i + \frac{1}{\alpha N_s} \sum_{j=1}^{N_s} ([g(x_{i|k}^j, w_{i|k}^j) - \eta_i]_+) \right] \right\} \quad (3-28)$$

As c takes higher values, the approximate unconstrained problem becomes increasingly accurate and closer to the original constrained problem. The stage costs $q(x_{i|k}, u_{i|k})$ and the terminal cost $p(x_{N|k})$ can be given in terms of weighted l_2 norms of the state and the control input to give a cumulative quadratic cost.

$$J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) = \mathbb{E} \left[\|x_{N|k}\|_P^2 + \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) \middle| x_{0|k} = x_k \right] + \max \left\{ 0, \sum_{i=0}^{N-1} c CVaR_{1-\alpha}(g(x_{i|k}, w_{i|k})) \right\} \quad (3-29)$$

The cost in terms of the l_2 norm of the output of the system z_k can be given as,

$$J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) = \mathbb{E} \left[\|x_{N|k}\|_P^2 + \sum_{i=0}^{N-1} (\|z_{i|k}\|^2) \middle| x_{0|k} = x_k \right] + \max \left\{ 0, \sum_{i=0}^{N-1} c CVaR_{1-\alpha}(g(x_{i|k}, w_{i|k})) \right\} \quad (3-30)$$

as z_k accounts for both the predicted state and control input.

The tractable stochastic MPC problem is now formulated as,

$$\begin{aligned} J_k^*(x_k) &= \min_{\boldsymbol{\eta}_{k \rightarrow k+N|k}, \mathbf{u}_{k \rightarrow k+N|k}} J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) \\ \text{s.t. } & x_{i+1|k} = Ax_{i|k} + Bu_{i|k} + Ew_{i|k}, \quad i = 0, \dots, N-1 \\ & u_{i|k} \in \mathcal{U}, \quad i = 0, \dots, N-1 \\ & x_{0|k} = x_k \end{aligned} \quad (3-31)$$

where, $\boldsymbol{\eta}_{k \rightarrow k+N|k} = \{\eta_{0|k}, \dots, \eta_{N-1|k}\}$ is the VaR at each time step. The first element of the optimal control input sequence, $u_{0|k}^*$, is applied to the system. The horizon is now shifted according to the receding horizon principle and the optimization problem is carried out again.

3-2 Closed-Loop Performance

The implementation of the controller is based on the receding horizon principle, where an optimization problem is solved at each time instant for a control horizon of length N . The model of the system as well as the performance index and optimization problem are time

invariant. The variable involved can be designed as functions of $i \in [0, N]$ rather than the current time k itself. To ensure that the optimization problem is feasible at all times, ensuring recursive feasibility, it is required that the initial state belongs to a predefined subset of the state space wherein all probabilistic constraints are satisfied and a sequence of control inputs exist such that all predicted states lie in this set. This feasibility set is given as,

$$\begin{aligned} \mathcal{X}_N = \{x_k : \exists \mathbf{u}_{k \rightarrow k+N|k} \text{ s.t. } P(g(x_{i|k}, w_{i|k}) \leq 0) \geq 1 - \alpha, i = 0, \dots, N - 1, k \geq 0 \\ \text{and } u_{i|k} \in \mathcal{U}\} \end{aligned} \quad (3-32)$$

Since the constraints are probabilistic, an admissible level of constraint violation is allowed. Given the chance constraints on the system as in 3-5, the admissible level of violation is given as,

$$P(g(x, w) \geq 0) \leq \alpha \quad (3-33)$$

An approximation of the chance constraints as multiple affine constraints by means of the scenario approach does not yield a problem that is feasible at all times. This is due to the new sampling of the state at every time instant that is affected by unbounded uncertainty that may not satisfy the finite affine constraints formed from sampled realizations of uncertainty. A way to overcome this is to allow the problem to become infeasible whenever necessary but penalize the violation to account for performance degradation. The problem formulation as given in (3-31) using the augmented cost with penalty function as given in (3-26) is an exact penalty method which gives the optimal solution through a single unconstrained minimization. Hence, the problem formulation in (3-31) equivalent to,

$$\begin{aligned} J_k^*(x_k) = \min_{\mathbf{u}_{k \rightarrow k+N|k}} \bar{J}_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) \\ \text{s.t. } x_{i+1|k} = Ax_{i|k} + Bu_{i|k} + Ew_{i|k}, i = 0, \dots, N - 1 \\ CVaR_{1-\alpha}(g(x_{i|k}, w_{i|k})) \leq 0, i = 0, \dots, N - 1 \\ u_{i|k} \in \mathcal{U}, i = 0, \dots, N - 1 \\ x_{0|k} = x_k \end{aligned} \quad (3-34)$$

which is feasible when the initial state x_k belongs to the following set.

$$\begin{aligned} \mathcal{X}_N = \{x_k : \exists \mathbf{u}_{k \rightarrow k+N|k} \text{ s.t. } CVaR_{1-\alpha}(g(x_{i|k}, w_{i|k})) \leq 0, i = 0, \dots, N - 1, k \geq 0 \\ \text{and } u_{i|k} \in \mathcal{U}\} \end{aligned} \quad (3-35)$$

3-3 Illustrative Example

Consider a 'double integrator' system of the form:

$$\begin{aligned} x_{k+1} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w_k \\ z_k &= \begin{pmatrix} 0 & 0 \\ 0.7 & 0 \\ 0 & 0.7 \end{pmatrix} x_k + \begin{pmatrix} 0.33 \\ 0 \\ 0 \end{pmatrix} u_k \end{aligned} \quad (3-36)$$

The input is constrained as,

$$-0.5 \leq u_k \leq 0.5, u_k \in \mathbb{R}, k \in \mathbb{N} \quad (3-37)$$

The disturbance is assumed to be a Gaussian distributed random variable with zero mean and variance 0.2.

$$w_k \in \mathcal{G}(0, 0.2), w_k \in \mathbb{R} \quad (3-38)$$

The state is parametrized as $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and the constraint on the state is given as

$$x_2 \geq 0 \quad (3-39)$$

It is assumed that the system has an initial state $x_0 = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$. In the presence of uncertainty, the task of the controller is to steer the state to the origin while satisfying the constraints on the system. The cost function used is of the form,

$$J_k(x_k, \mathbf{u}_{k \rightarrow k+N|k}) = \mathbb{E} \left[\|x_{N|k}\|_P^2 + \sum_{i=0}^{N-1} (\|z_{i|k}\|^2) + c \cdot \max\{0, CVaR_{1-\alpha}(g(x_{i|k}, w_{i|k}))\} \mid x_{0|k} = x_k \right] \quad (3-40)$$

When the predicted state violates the constraints, a non zero penalty is added to the cost function. This notifies the controller of constraint violation and a suitable control input is generated. The approximated matrices for the stage costs are,

$$Q = \begin{pmatrix} 0.7^2 & 0 \\ 0 & 0.7^2 \end{pmatrix}, R = (0.33^2) \quad (3-41)$$

and the weighting matrix for the terminal cost, by solving the discrete-time Riccati equation, is given as,

$$P = \begin{pmatrix} 1.6 & 0.9 \\ 0.6 & 1.33 \end{pmatrix} \quad (3-42)$$

By selecting the prediction horizon $N = 5$ and the expected average-over-time constraint violation level $\epsilon = 0.05$, the sample complexity is derived as $N_s = 99$ by employing the approach in [12]. The regular scenario approach as in [11] yields $N_s = 660$ and the sampled-approximation approach in [51] yields a value of $N_s = 2400$ approximately. This shows that the interpretation of probabilistic constraints as an average-in-time rather than point-wise is relatively more sample efficient compared to the other typical approaches. Two sets of simulation results are presented with 100 trials in each set. One set of simulations are performed as per standard deterministic MPC and the other as per the tractable stochastic MPC algorithm in (3-31). Fig. 3-1b shows the trajectory of x_2 on using deterministic MPC and Fig. 3-1a shows the trajectory of x_2 using tractable stochastic MPC.

Deterministic MPC is the standard approach of MPC and is included to compare its performance with that of stochastic MPC. The design of the controller for deterministic MPC is elaborated in [75]. When the state of the system is 'far' from the constraint boundary, both standard deterministic MPC and stochastic MPC perform similarly. In standard deterministic MPC, the disturbance is assumed to be equal to its mean value, zero, over the control

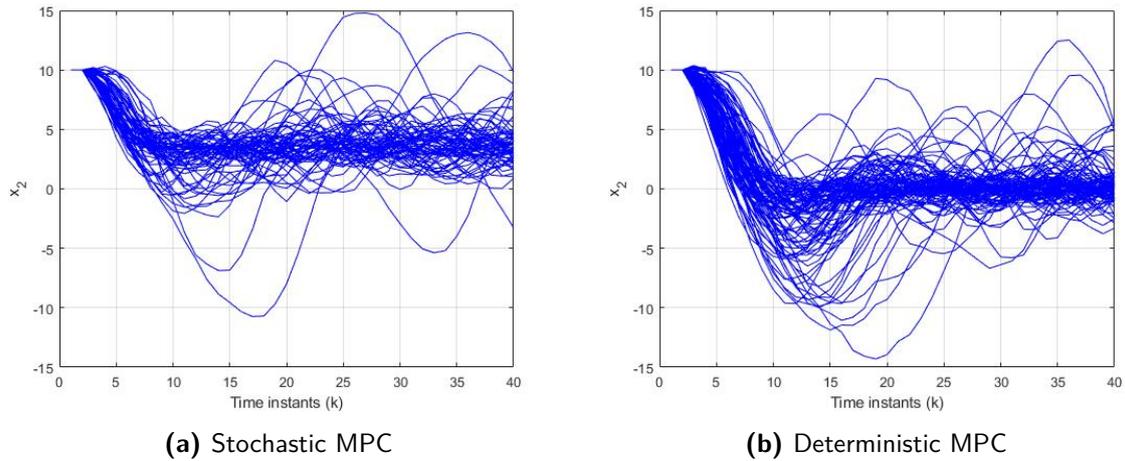


Figure 3-1: Trajectory of state x_2 , Constraint: $x_2 \geq 0$

horizon. Due to this assumption, the controller in deterministic MPC does not predict the possibility of a constraint violation, when the system is close to the constraint boundary. In reality, the possibility that the disturbance w is greater than zero is high at most time instants, and therefore, since the control input is unable to predict the disturbance, constraints are violated. The controller for SMPC, on the other hand, takes into account the possibility of constraint violation due to disturbance by using a number of sample disturbance values. Hence, the controller for SMPC provides a more realistic control input, as compared to deterministic MPC, to compensate for uncertainties. This is supported in the Fig. 3-2.

The estimate of the probability density of the state x_2 using both controllers, SMPC and deterministic MPC, shows that the probability of constraint violation at all times is significantly lower while applying SMPC. Furthermore, the performance of the controller of SMPC is determined by the penalty added to the cost in the case of constraint violations. As an exact penalty function is used, the performance of the controller must improve with an increase in the penalty parameter. With increase in the penalty parameter, more emphasis is laid on satisfaction of the chance constraints on the system by increasing the 'weight' of the 'risk' in case of constraint violation. This is supported by Fig. 3-3 and Fig. 3-4. Fig. 3-3 shows the gradual increase in the mean of x_2 with an increase of the penalty parameter and lower probability of constraint violation. Fig. 3-4 shows a monotonic increase in the median value of x_2 signifying higher constraint satisfaction, showing a monotonic increase in the performance of the algorithm 3-31 as is expected of penalty methods in optimization.

3-4 Areas of Application and Extensions

The crux of SMPC is the definition of chance constraints which utilize the probabilistic descriptions of uncertainty. Chance constraints enable the system to allow for an admissible level of closed-loop constraint violations in a probabilistic sense. *CVaR*, which is known as the tightest convex approximation of the chance constraint, has been adapted from the field of mathematical finance. MPC has a huge potential for application in this field for applications such as portfolio optimization and central bank operations. In terms of the

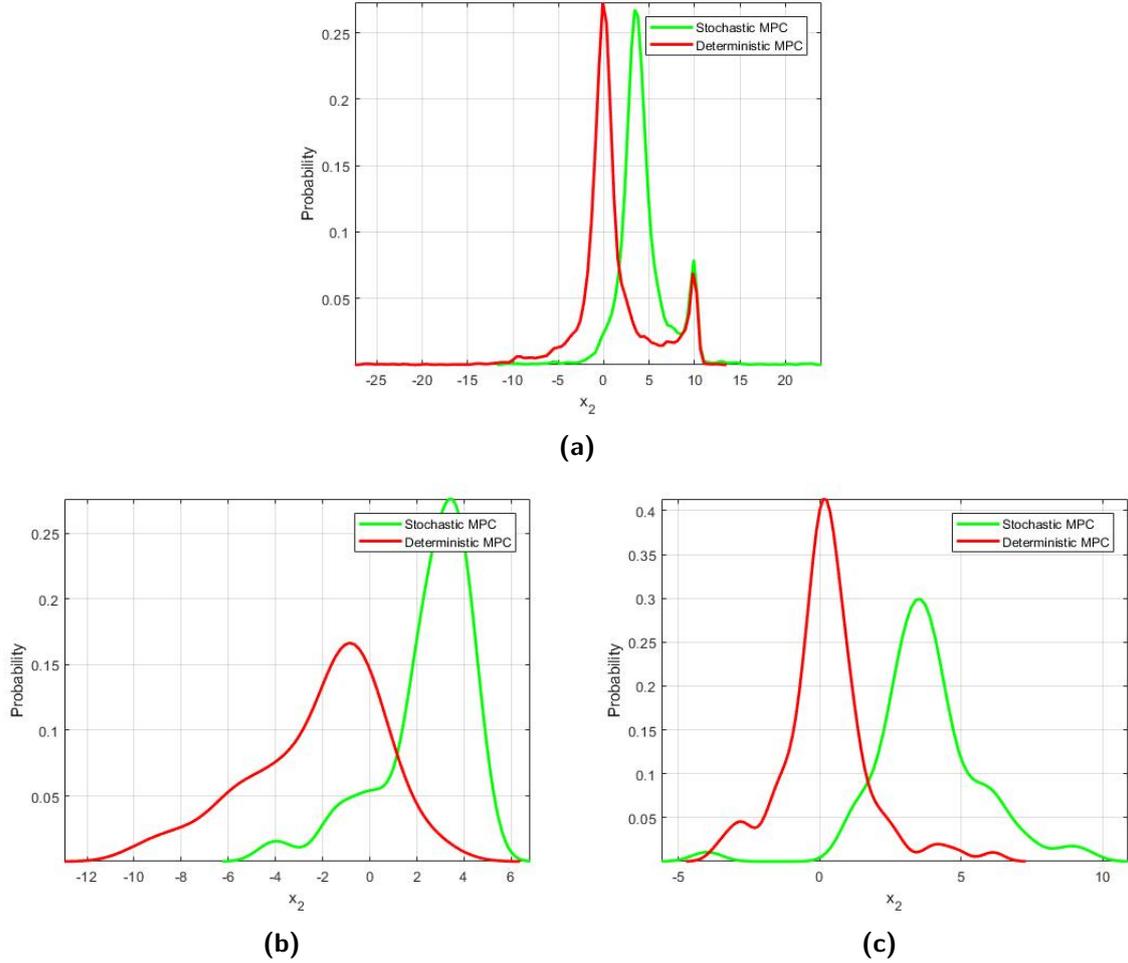


Figure 3-2: An estimate of the probability density of x_2 , 3-2a over 40 time instants, 3-2b at time instant $k = 11$ (overshoot region) and, 3-2c at time instant $k = 40$ (steady state region)

portfolio optimization problem, the $CVaR$ is the expected wealth at any given point in time conditional that the wealth is below the VaR level. $CVaR$ is used to represent constraints as convex functions of future asset allocation and an attempt is made to limit expected losses which are larger than the VaR . In multi-agent systems, joint constraints are used to impose probabilistic constraints on the entire system rather than on each agent. $CVaR$ can be used to represent the joint constraints, and its sampled approximation is beneficial when when probability descriptions of uncertainty are inaccurate. In this regard, the proposed formulation in this thesis can be applied to the portfolio optimization problem, and to obtain a centralized optimization problem for multi-agent systems.

3-4-1 Portfolio Optimization

For the portfolio optimization problem [76], the model and discrete-time wealth dynamics are given as follows. The returns of risky assets and the interest rate of the bank account are

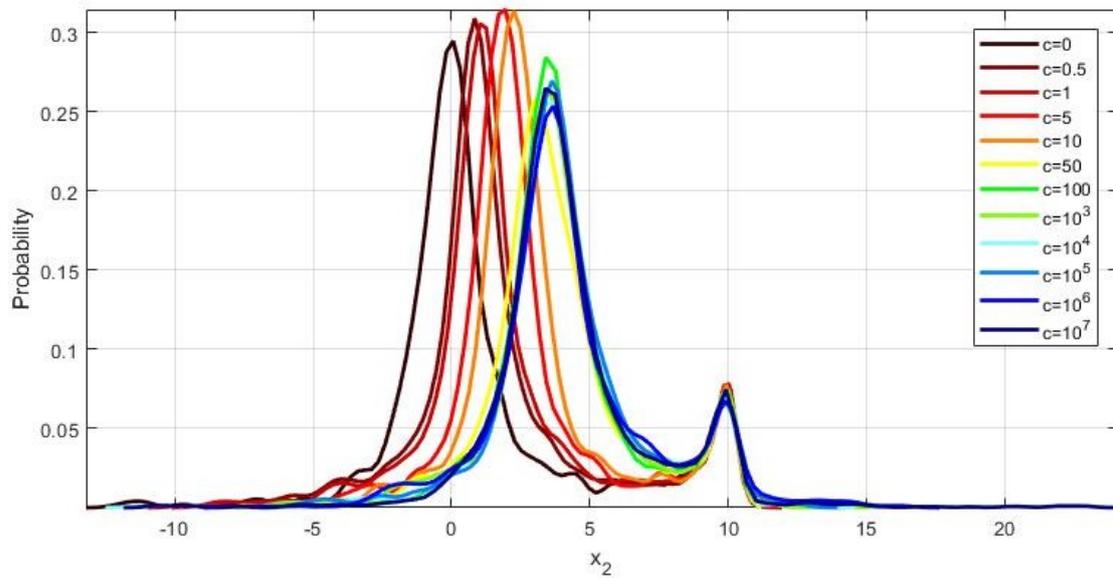


Figure 3-3: Estimate of probability density of x_2 over 40 time instants over 100 trials for different values of the penalty parameter.

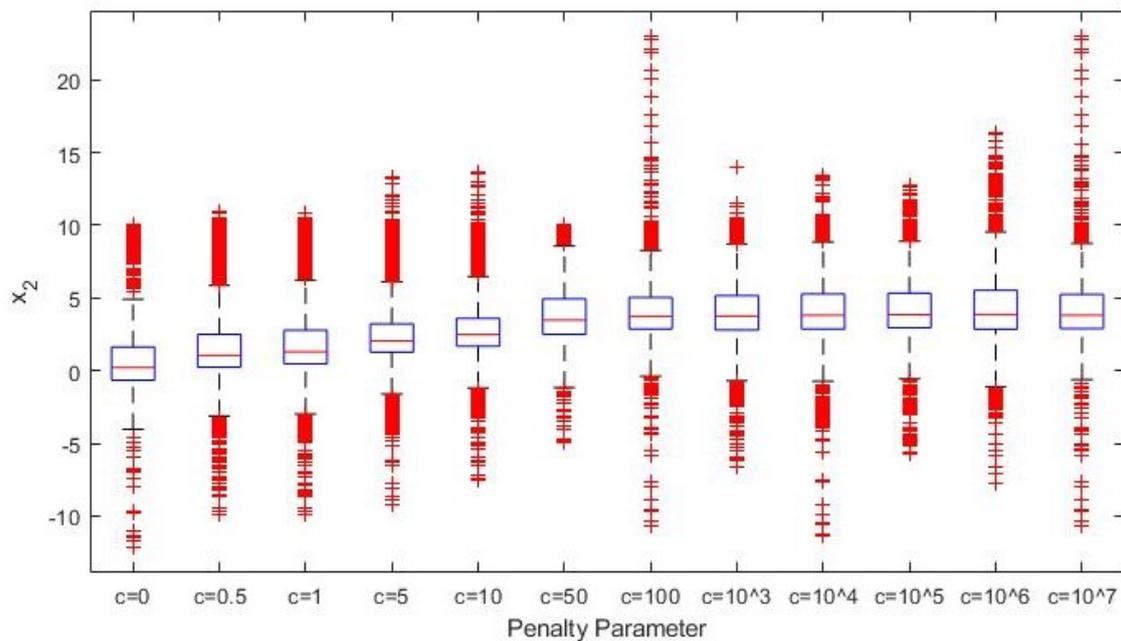


Figure 3-4: A Boxplot illustrating the variation of the median value of x_2 over 40 time instants over 100 trials for different values of the penalty parameter.

given by,

$$\begin{aligned} r_{t+1} &= Gx_t + g + \epsilon_t^r \\ r_{t+1}^B &= F_B x_t + h_b \end{aligned} \quad (3-43)$$

where $r_t \in \mathbb{R}^n$ is the vector of asset returns, $\mu_t(x_t) \in \mathbb{R}^n$ is the expected return of r_t , $x_t \in \mathbb{R}^m$ is the vector of factors and $\epsilon_t^r \in \mathbb{R}^n$ is a Gaussian White noise process with covariance matrix H . A risk free possibility to invest in a bank account is allowed with an interest rate $r_t^B(x_t)$. It is assumed that the factors are driven by stochastic processes,

$$x_{t+1} = \Theta_t(x_t) + \Psi_t(x_t)\epsilon_t^x \quad (3-44)$$

where, $\Theta_t(x_t) \in \mathbb{R}^m$, $\Psi_t(x_t) \in \mathbb{R}^{m \times k}$, and $\epsilon_t^x \in \mathbb{R}^k$ is a white noise process with unity covariance. Furthermore,

$$\begin{aligned} \Theta_t(x_t) &= Ax_t + b \\ \Psi_t(x_t) &= I \end{aligned} \quad (3-45)$$

where $A \in \mathbb{R}^{m \times m}$ and $b \in \mathbb{R}^m$. The portfolio return can be expressed as,

$$R_{t+1} = r_{t+1}^B + u_t^T(r_{t+1} - 1^T r_{t+1}^B) = F_B x_t + h_B + u_t^T(Fx_t + h) + u_t^T \epsilon_t^r \quad (3-46)$$

where $F = G - 1^T F_B$, $h = g - 1^T h_B$, $u_t = [u_{1t}, \dots, u_{nt}]^T$ and u_{jt} denotes the fraction of the portfolio invested in the j th risky investment. The wealth dynamics are given by,

$$\begin{aligned} W_{t+1} &= (1 + R_{t+1})W_t \\ w_{t+1} &= \ln(1 + R_{t+1}) + w_t \end{aligned} \quad (3-47)$$

where $W_t \in \mathbb{R}$ denotes the portfolio value at time t and $w_t = \ln W_t$. These nonlinear dynamics are replaced by a Taylor approximation and the wealth dynamics are now obtained as,

$$w_{t+1} = w_t + F_B x_t + h_B + u_t^T(Fx_t + h) - \frac{1}{2}u_t^T H u_t + u_t^T \epsilon_t^r \quad (3-48)$$

where $Var(R_{t+1}) = u_t^T H u_t$. The conditional mean m_{t+k}^w and variance V_{t+k}^w of the portfolio wealth and factor, as defined in [76] in terms of the returns, factors and investments, are used to define the objective of a portfolio optimization problem to balance expected returns and possible risks. State constraints for log-wealth values are given as,

$$P(w_{t+i} > L_{t+i}) \geq p_{t+i}, \quad i = 1, \dots, k \quad (3-49)$$

where L_{t+i} is the constraint level at time $t+i$ with a confidence probability of p_{t+i} . This is known as the *VaR* constraint. The *CVaR* constraint is given as,

$$\mathbb{E}[w_{t+i} \leq \eta(p_{t+i})] \geq \bar{L}_{t+i} \quad (3-50)$$

where $\eta(p_{t+i})$ denotes the *VaR* with confidence p_{t+i} and \bar{L}_{t+i} denotes the *CVaR*(p_{t+i}) constraint.

A simple portfolio optimization problem involves the maximization of a risk-averse objective function which balances the expected return and the possible risks. The mean-variance variance objective function is used for this purpose and is given by,

$$\max_{U_{t+k}} m_{t+k}^w + \frac{1}{2}\lambda V_{t+k}^w \quad (3-51)$$

where $\lambda \leq 1$ denotes the level of risk aversion and $U_{t+k} = [u_t, \dots, u_{t+k-1}]^T$. In the case when $\lambda = 1$, the *VaR* or *CVaR* constraint may be used. The *CVaR*(p_{t+i}) constraint is preferred due to its feature of convexity. The optimization problem is formulated as,

$$\begin{aligned} \max_{U_{t+k}} \quad & m_{t+k}^w + \frac{1}{2}V_{t+k}^w \\ \text{s.t.} \quad & F_u U_{t+k} \leq f \\ & \mathbb{E}[w_{t+i} \leq \eta(p_{t+i})] \geq \bar{L}_{t+i}, i = 1, \dots, k \end{aligned} \quad (3-52)$$

This optimization problem is known as the 'strategic asset allocation' problem describing a portfolio optimization problem with time-varying returns and objectives typical for long-term investments. Since *CVaR*(p_{t+i}) is difficult to estimate for all factors (states) with uncertainty, it can be used in an exact penalty to form an optimization problem without state constraints, given as,

$$\begin{aligned} \max_{U_{t+k}} \quad & m_{t+k}^w + \frac{1}{2}V_{t+k}^w + \sum_{i=1}^k c.CVaR(p_{t+i}) \\ \text{s.t.} \quad & F_u U_{t+k} \leq f \end{aligned} \quad (3-53)$$

SMPC is applied and the portfolio is optimized at each time instant upto k steps ahead. This is a straightforward application of the formulation developed in this thesis.

3-4-2 Joint Chance Constrained Multi Agent Systems

Consider multi-agent systems with state and control input constraints subject to unbounded stochastic uncertainty. Users of multi-agent systems would like to bound the probability of system failure rather than the probabilities of individual agents' failure. In this regard, *joint chance constraints* are imposed which limits the probability of having at least one agent failing to satisfy any of its state constraints. In such cases, agents are coupled through the joint chance constraints even if they are not coupled through state constraints. Consider the multi agent joint chance constraint FHOCP formulated as [54],

$$\begin{aligned} \min_{\mathbf{u}_{0 \rightarrow N}^{1:I}} \quad & \sum_{i=1}^I J^i(\mathbf{u}_{0 \rightarrow N}^i, \mathbf{x}_{0 \rightarrow N}^i) \\ \text{s.t.} \quad & x_{k+1}^i = A^i x_k^i + B^i u_k^i + w_k^i \\ & u_{min}^i \leq u_k^i \leq u_{max}^i \\ & P\left(\bigwedge_{i=1}^I g^i(\mathbf{x}_{0 \rightarrow N}^i, \mathbf{w}_{0 \rightarrow N}^i) \leq 0\right) \geq 1 - \alpha \\ & w_k^i \sim \mathcal{G}(0, \mathcal{Q}_w^i) \\ & i = 1, \dots, I, k = 0, \dots, N - 1 \end{aligned} \quad (3-54)$$

where, at time k , $x_k^i \in \mathbb{R}^{n_x}$ is the state of the i th agent, $u_k^i \in \mathbb{R}^{n_u}$ is the control input of the i th agent, $w_k^i \in \mathbb{R}^{n_w}$ is the additive disturbance on the i th agent, $\mathbf{u}_{0 \rightarrow N}^i = [u_0^{iT}, \dots, u_{N-1}^{iT}]^T$ and $\mathbf{x}_{0 \rightarrow N}^i = [x_0^{iT}, \dots, x_{N-1}^{iT}]^T$, and α risk bound of the system given by the user. The joint chance constraint in the formulation requires that the probability that all the state constraints of all

agents are satisfied must be more than $1 - \alpha$, where α is the upper bound of the probability of failure (risk bound). The joint constraint is hard to evaluate as it is difficult to compute an integral of multi-variable probability distribution over an arbitrary region. For ease of computation, the joint chance constraint is now approximated using the *CVaR* function as,

$$CVaR_{1-\alpha}(g(x, w)) = \min_{\eta \in \mathbb{R}} \left(\eta + \frac{1}{\alpha} \mathbb{E}[(g(x, w) - \eta)_+] \right) \quad (3-55)$$

where $g(x, w) = \max\{g^1(\mathbf{x}_{0 \rightarrow N}^1, \mathbf{w}_{0 \rightarrow N}^1), \dots, g^I(\mathbf{x}_{0 \rightarrow N}^I, \mathbf{w}_{0 \rightarrow N}^I)\}$ and $g^i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$, and $\eta \in \mathbb{R}$ is the Value at Risk of the system. The constraint is now given as,

$$CVaR_{1-\alpha}(g(\mathbf{x}, \mathbf{w})) \leq 0 \quad (3-56)$$

The multi-agent FHOCP using joint chance constraints is now reformulated as a centralized optimization problem, and is given as,

$$\begin{aligned} \min_{\mathbf{u}_{0 \rightarrow N}^{1:I}, \eta} \quad & \sum_{i=1}^I J^i(\mathbf{u}_{0 \rightarrow N}^i, \mathbf{x}_{0 \rightarrow N}^i) \\ \text{s.t.} \quad & x_{k+1}^i = A^i x_k^i + B^i u_k^i + w_k^i \\ & u_{min}^i \leq u_k^i \leq u_{max}^i \\ & CVaR_{1-\alpha}(g(\mathbf{x}, \mathbf{w})) \leq 0 \\ & w_k^i \sim \mathcal{G}(0, \mathcal{Q}_w^i) \\ & i = 1, \dots, I, k = 0, \dots, N - 1 \end{aligned} \quad (3-57)$$

The new decision variables introduced are $\eta^{1:I}$ which is the *VaR* for each agent and they represent the worst case threshold of risk bound for each agent. Using the exact penalty method, the centralized state unconstrained optimization problem is given as,

$$\begin{aligned} \min_{\mathbf{u}_{0 \rightarrow N}^{1:I}, \eta} \quad & \sum_{i=1}^I J^i(\mathbf{u}_{0 \rightarrow N}^i, \mathbf{x}_{0 \rightarrow N}^i) + c \max\{0, CVaR_{1-\alpha}(g(\mathbf{x}, \mathbf{w}))\} \\ \text{s.t.} \quad & x_{k+1}^i = A^i x_k^i + B^i u_k^i + w_k^i \\ & u_{min}^i \leq u_k^i \leq u_{max}^i \\ & w_k^i \sim \mathcal{G}(0, \mathcal{Q}_w^i) \\ & i = 1, \dots, I, k = 0, \dots, N - 1 \end{aligned} \quad (3-58)$$

In [54], the joint chance constraint is decomposed into individual constraints for each agent by determining the bound on the risk. The bound on the risk is determined using the inverse of the cumulative distribution of the univariate Gaussian distribution. This requires knowledge of the variance of the probability distribution. Inaccurate information of the probability distribution will lead to erroneous decomposition of the joint chance constraint. As the sampled approximation of *CVaR* does not require information about the probability distribution of uncertainty, the formulation in (3-58) is a relative improvement over the formulation in [54]. Furthermore, this centralized problem can be made decentralized with suitable definition of *VaR*, and hence the *CVaR* of each agent, to ensure that probability of system failure is within specified bounds.

Chapter 4

Conclusion

In this chapter, the proposed tractable S MPC strategy using $CVaR$ is summarized. Various aspects of the proposed strategy, with focus on sample complexity of the approximation of chance constraints and exact penalty method are discussed. Finally, this chapter concludes with possible future extensions based on this thesis work.

4-1 Summary and Discussion

Satisfaction of constraints in the stochastic setting has proven to be significant complication. In the presence of uncertainty affecting the system, hard constraints on the state are not satisfied. Employing the probability descriptions of uncertainty affecting the state, chance constraints are defined which allow constraint violation upto a certain admissible level. Since the feasible set of states defined by a chance constraint is usually non-convex, convex approximations such as the *scenario approach* and $CVaR$ can be applied. In the case of unbounded uncertainty, satisfaction of multiple affine constraints defined by the scenario approach is not guaranteed. Therefore, for practicality, the state of the system is allowed to become infeasible and violate constraints upto a certain level. To account for the performance degradation due to constraint violation, a penalty is added to the cost of the system at the infeasible states to form an augmented cost. To this end, incorporation of a penalty function in the cost gives rise to a penalty method for optimization. An attempt is made towards using an exact penalty method wherein, a single unconstrained minimization of the augmented cost gives the same optimal solution as the original constrained problem. A suitable choice for forming an exact penalty function is the $CVaR$ function which determines the risk faced by the system at an infeasible state. The $CVaR$, an expectation constraint, gives the expected loss when the worst case threshold for constraint violation is crossed. As evaluation of expectation constraints is difficult, a sampled-average approximation is used. A penalty parameter is used to weight the penalty function and is ideally chosen as a value greater than some Lagrange multiplier of the problem. With respect to the penalty function, the following technical considerations are discussed.

4-1-1 Sample Complexity

The *CVaR* constraint that is used to approximate the chance constraint is an expected value constraint. It gives the risk on the system when the worst case threshold is crossed. A sample-average approximation for *CVaR* is proposed in [51]. In the context of MPC, the scenario approach is used to determine sample complexity [11, 65]. However, these methods prove to be computationally taxing and lead to over-satisfaction of the constraints on the system. A sample efficient method is provided in [12] by interpreting the probabilistic constraint violations as the expected average-over-time of closed-loop constraint violations.

4-1-2 Penalty Parameter for Exact Penalty Function

For a penalty method to be exact, a single unconstrained minimization of the cost must yield the same optimal solution as the original constrained problem. The penalty parameter that weights the penalty function is chosen as a value greater than a Lagrange multiplier of the problem. A bound on the maximum magnitude on the Lagrange multiplier can be obtained as explained in [74]. This method uses the optimal value of the dual function of the cost, and this optimal value is not readily available. Hence, an ad hoc method of varying the penalty parameters to observe the effect of the change of parameters on the performance is used in the chosen *double integrator* example. As is expected of regular penalty methods, as the penalty parameter increases in value, the performance of the SMPC strategy observably improves.

4-2 Future Directions

This thesis presents a penalty method for SMPC using *CVaR* as the penalty function. The penalty parameter is usually a value greater than a Lagrange multiplier of the problem. This parameter is varied in an ad hoc manner to demonstrate its effects on performance. For future work, there is scope to determine how the penalty parameter must be selected and varied to evaluate performance of the SMPC strategy. Feasibility of the unconstrained problem using an exact penalty method would imply feasibility of the original state-constrained problem. However, convergence and other notions of stability must be explored in a rigorous manner for this formulation. Further, this formulation can be easily adapted to multi-agent systems to decompose joint chance constraints by using *CVaR*. It is easy to determine a centralized optimization scheme for the multi-agent system using the *CVaR* penalty method. Efforts can be taken in the direction of forming a decentralized scheme for multi-agent systems. This would lead to a tractable formulation for distributed SMPC.

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Glossary

List of Acronyms

MPC	Model Predictive Control
RMPC	Robust Model Predictive Control
SMPC	Stochastic Model Predictive Control
<i>CVaR</i>	Conditional Value at Risk
CPI	Controlled Positively Invariant
RHC	Receding Horizon Control
OCP	Optimal Control Problem
FHOCP	Finite Horizon Optimal Control Problem
MCPI	Maximal Controlled Positively Invariant
CCP	Chance Constrained Problem
SCMPC	Scenario-based SMPC
FHSCP	Finite Horizon Scenario Problem
<i>VaR</i>	Value at Risk