Modelling Fluid-structure Interaction in Offshore Photovoltaics

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by

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Preface

With this bachelor thesis, I present my investigation into analytical methods for studying fluid-structure interaction in large-scale offshore floating photovoltaics. I wrote this report as a part of the Final Bachelor project for graduating an applied mathematics bachelor degree.

While writing this report I assumed the readers had a basic knowledge about mathematical topics such as solving Partial Differential Equations with separation of variables and Fourier transforms. Readers do not need to be familiar with perturbation analysis, as all the steps in chapter 3 will be explained in detail.

Firstly, I would like to thank my supervisor Wim van Horssen for his guidance during my research. I would also like to thank him for introducing me to such an interesting topic.

Furthermore, I would like to thank Martin van Gijzen and Peter Wellens for participating in the graduation committee.

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Mieke Daemen
Delft, July 2022

Abstract

The main aim of the research presented in this report is investigating analytical methods to model fluid-structure interaction in large-scale offshore floating photovoltaics. The model that was attempted to be solved analytically is based on a model presented by Pengpeng Xu (2022).

The dimensions in the equations were removed. Applying a perturbation method yielded hierarchic partial differential equations by introducing the wave amplitude divided by the depth of the ocean as a small perturbation parameter. The analytical solution of the first order problem was found by applying separation of variables and by using a Fourier transform. For certain classes of problems it is shown in this report that it is possible to analytically solve a model for fluid-structure interaction in offshore solar farms for various initial conditions.

Contents

Nomenclature

Abbreviations

Abbreviation	Definition
EBVK	Euler Bernoulli-von Karmann
LOFPV	Large-scale offshore floating photovoltaics
FSI	Fluid-structure interaction

Symbols

Symbol	Definition	Unit
b	Beam width	[m]
c_1	External damping coefficient	[kg/(m^{2}s)]
c_2	Internal damping coefficient	[kg m /s]
d	Beam thickness (in z-direction)	[m]
E	Young's modulus	[(kg) / (ms^2)]
g	Acceleration of gravity	[m/s^2]
h	Depth of the ocean	[m]
I	Inertial moment	[m^4]
L	Length of the beam	[m]
p	Water pressure	[N/m^2]
q_w	External distributed load	[N/m]
S	Cross-section area	[m^2]
t	Time	[s]
V	Velocity	[m/s]
w	Transverse displacement	[m]
w_{max}	maximal transverse displacement	[m]
x	X coordinate	[m]
z	Z coordinate	[m]
η	Free surface elevation	[m]
ν	Poisson's ratio	[-]
ρ_w	Density (water)	[kg/m^3]
ρ_s	Material Density (beam)	[kg/m^3]
ϕ	Fluid velocity potential	[m^2/s]

1 Introduction

Solar photovoltaics are expected to be the largest source of electricity by 2040 (IEA, 2020). Generating large quantities of solar energy puts a heavy strain on land usage (Xu, 2022). Large-scale offshore floating photovoltaics provides a solution to that problem (figure 1.1, Ocean Sun, 2019). Photovoltaic panels are placed on a plate in the ocean. In figure 1.2 one can observe the detailed configuration of a LOFPV (Ocean Sun, 2019). Modelling vibrations and forces in such a plate is crucial for the construction of safe and reliable offshore solar farms.

Figure 1.1: LOFPV (Ocean Sun, 2019)

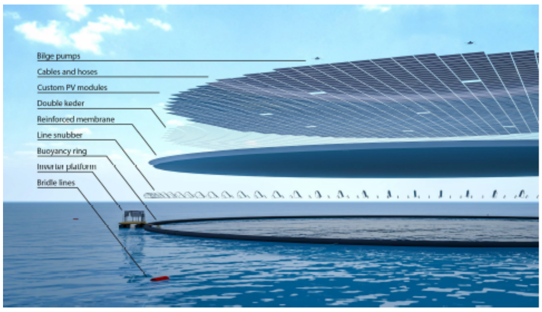


Figure 1.2: Detailed Construction of LOFPV (Ocean Sun, 2019)

The purpose of this report is examining analytical methods for modelling fluid-structure interaction (FSI) in large-scale offshore floating photovoltaics (LOFPV). Xu (2022) models FSI in LOFPV by representing the floating plate as a nonlinear Euler-Bernoulli-von Kármann (EBVK) beam coupled to the water in the ocean. The model of Xu is used and slightly changed in our report. To find an analytical approach for representing FSI in LOFPV, we investigate different ways to solve Xu's (2022) model. A perturbation method is applied to deal with weak nonlinearities. With separation of variables and a Fourier transform one can solve the linear model.

This Thesis is organised as follows. In Chapter 2 one can find a Theoretical Model for depicting the semi-nonlinear fluid-structure interaction of large scale offshore floating photovoltaics. Chapter 3 presents the nondimensionalized model and contains details about dividing the model into different order equations by using a perturbation method. In Chapter 4 the analytical solution of the first order model is derived. Finally, in chapter 5 the conclusions of the report are presented.

2

Model for Large-scale Offshore Floating Photovoltaics

In this chapter one can find the representation of waves in a model for large-scale offshore floating photovoltaics (LOFPV). The floating plate is modeled as a nonlinear Euler Bernoulli-van Karmann beam (EBVK). The water potential in the ocean is modelled and the EBVK is linked to this model as a boundary condition.

In section 2.1 one can find some key assumptions for modelling waves in a beam. Subsequently section 2.2 contains the model beam. In particular in 2.1.2 it is shown how an equation of motion for a vibrating beam can be obtained. Furthermore in section 2.1.3 the motion of water is explained with linear potential theory. The derivations in section 2.1.4 connect the boundary conditions of the water potential theory with the beam equation.

2.1. Assumptions

The transverse vibrations of the membrane is much bigger than the vertical displacement of the beam. Therefore only the impact of the waves is used in our models. We make the following assumptions in our model (Xu and Wellens, 2022), (Xu, 2022):

- Because the structural length is large compared to the deflection of the beam, an infinite domain is chosen. Therefore there are boundary conditions on x at infinity. That means $\frac{\partial^n w}{\partial x^n}$ converges to zero for every non-negative integer n as $|x| \to \infty$.This is true because, the total amount of energy in the beam has to be finite.;
- One also assumes that the ocean has a uniform depth h;
- A LFPV is held in place by an anchor and its movement can change as a result of wave, current and wind loads. These global motions are neglected in our model;
- The floating membrane is impermeable;
- Both the PV panels and marine growth on the bottom of the planes can impact the stresses in the membrane. But these effects are so small that they are not part of the model.

Figure 2.1 (Xu, 2022) shows a 2D model of LOFPV on the sea. Thus the movements in the LOFPV can be depicted as a EVBK beam that only takes transverse deflections into account.

2.2. Model

2.2.1. Equation of Motion

The equation of motion EOM (Xu, 2022):

$$\rho_s S \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} - \frac{3}{2} SE \left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^2 w}{\partial x^2} + c_1 b \frac{\partial w}{\partial t} + c_2 b \frac{\partial^5 w}{\partial t \partial x^4} = q_w. \tag{2.1}$$

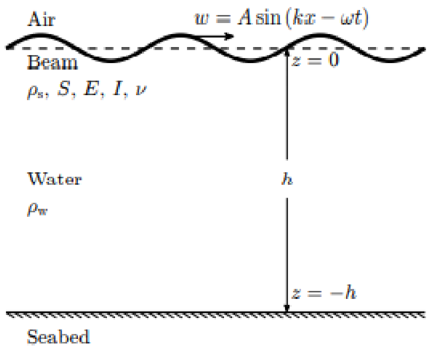


Figure 2.1: A picture of waves in the LOFPV Xu, 2022

The variable t represents the time in seconds. The variable x represents the place on the horizontal axis. The transverse displacement of the beam is denoted by w. The function w depends on the variables t and x. Also, E is Young's modulus and ρ_s is the density of the material. The beam thickness is d and b stands for the the width of the beam. The cross-section area is $S = bd$. The Poisson's ratio is represented by ν. The inertial moment is given by $I = \frac{bd^3}{12(1-\nu^2)}$. The viscous and structural damping coefficients are given by c_1 and c_2, respectively. The external distributed load has a constant value q_w and is uniformly distributed in the x-direction.

In appendix B the derivation of equation (2.1) can be found.

2.2.2. Potential Theory for Water

The water has a constant density ρ_w. The depth of the ocean is uniform and is equal to h. The Velocity potential $\phi(x, z, t)$ represents the invicid, irrotational and incompressible flow as (Xu and Wellens, 2022):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{2.2}$$

Since there is no flow of water at the seabed of the ocean, one can obtain the following boundary condition.

$$\frac{\partial \phi}{\partial z}\Big|_{z=-h} = 0. \tag{2.3}$$

Let $\eta(x, t)$ be the free surface elevation. At the free surface one obtains the following boundary condition:

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}\Big|_{z=\eta}. \tag{2.4}$$

Xu chooses to set the boundary conditions at free surface elevation at $z = 0$ (Xu, 2022). To improve this model the boundary conditions at the surface of the ocean is taken at $z = \eta$. The free surface of the ocean is not found at $z = 0$, but at $z = \eta$.

With the Bernoulli equation the kinematic boundary condition is obtained (Xu and Wellens, 2022).

$$p + \rho_w \frac{\partial \phi}{\partial t}\Big|_{z=\eta} + \rho_w g \eta = 0. \tag{2.5}$$

2.2.3. FSI Equations

The water models and the EOM of the beam are coupled at the boundary conditions. At $z = \eta$, substitute $w = \eta$ and $q_w = pb$. Now the equations at the free surface are :

$$\frac{\partial w}{\partial t} = \frac{\partial \phi}{\partial z}\Big|_{z=w}, \quad (2.6)$$

$$\rho_s d\frac{\partial^2 w}{\partial t^2} + \frac{Ed^3}{12(1-\nu^2)}\frac{\partial^4 w}{\partial x^4} - \frac{3Ed}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} + \rho_w g w + \rho_w \frac{\partial \phi}{\partial t}\Big|_{z=w} + c_1\frac{\partial w}{\partial t} + c_2\frac{\partial^5 w}{\partial t \partial x^4} = 0. \quad (2.7)$$

Here q_w has the unit $[N]$ because the pressure $p[N/m]$, is multiplied with beam width b. The variables S and I are changed to $S = bd$ and $I = \frac{bd^3}{12(1-\nu^2)}$. Also η is replaced by w.

3

Removing Dimensions & Nonlinearity from FSE-equations

Before one wishes to solve the FSI equations in section 2.2.3, it is a good idea to nondimensionalize the equations. We also look for ways deal with nonlinearity. In order to remove nonlinearity in these equations one needs to apply a perturbation method.

In subsection 3.1 one can find an explanation of the Buckingham PI Theorem. Subsequently subsection 3.2 contains the steps that remove the dimensions from the FSI equations. Furthermore in subsection 3.2 we also split the FSI equations into different hierarchical differential equations by applying the perturbation method.

3.1. Buckingham PI Theorem

The Buckingham Π Theorem is a valuable concept for making equations dimensionless. This theorem roughly states that an equation with n physical variables expressed in k physical dimensions, can be reduced to to an equation with $n - k$ dimensionless variables. From this theorem one can obtain a method for deriving groups of non dimensional variables from given dimensional parameters in an equation.

In subsection 3.1.1. the Buckingham Π theorem is stated and explained with a simple example. Furthermore in subsection 3.1.2 one can find the application of the Buckingham Π theorem to the FSI-equations from equation 2.7.

3.1.1. Buckingham PI Theorem explained

Statement

Let there exist n number of physical variables such as depth, density force and k number of units such as kilograms or seconds in an equation. Then the equation can be reformulated such that there exists $p = n - k$ number of dimensionless Pi groups (Torczynski, 1988).

An equation of the form $f(x_1, \ldots, x_n) = 0$, with independent physical parameters $x_1, ..x_n$ can be restated to an equation with $\pi_1, ...\pi_p$ independent dimensionless parameters. These new independent variables are called Pi-Groups. The new dimensionless equation has the form: $g(\pi_1, \ldots, \pi_p) = 0$. Every Pi-group is a product of powers of the dimensional parameters (Torczynski, 1988):

$$\pi_i = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}.$$

Example

The acceleration of an object is be computed by dividing distance by time twice:

$$a = \frac{d}{t^2} = \text{acceleration}(d, t).$$

The distance is given by d and the time by t. The variable a corresponds to the acceleration. In this equation there exist 3 physical variables. These 3 physical variables are expressed in 2 units: seconds and meters. Therefore there exists 3-2 = 1 dimensionless quantity. Let π be a dimensionless Π-Group.

$$\pi = a^x d^y t^z.$$

Because π does not have a dimension:

$$s^0 m^0 = m^x s^{-2x} m^y s^z.$$

To find the expression of π the following linear system needs to be solved:

$$x + y = 0,$$
$$-2x + z = 0.$$

A solution to this system is: $x = 1$, $y = -1$, $z = 2$.

$$\pi = a^1 d^{-1} t^2.$$

Now the equation for the acceleration can be rewritten as:

$$\pi = a^1 d^{-1} t^2 = 1,$$
$$\pi - 1 = 0.$$

3.1.2. Application to FSI-equations

In the equations 2.7 there exist 14 independent variables and 3 units (s, m, kg). Thus these equations can be restated as an equation with 11 dimensionless parameters (pi-groups).

First we list the dimensional parameters (in eq 2.7). Those are: x, z, t, ϕ, w, ρ_s, ρ_w, d, E, g, h, L, w_{max}, c_1 and c_2. These dimensional parameters have units kg,m,s. Subsequently we choose ρ_s, ρ_w, d, E, g, h, L, w_{max}, c_1 and c_2 as a subset of parameters which can be used to nondimensionalize x, z, t, ϕ and w.

The first π-group can be written as:

$$\pi_1 = x z^{a_1} t^{a_2} \phi^{a_3} w^{a_4} \rho_s^{a_5} \rho_w^{a_6} d^{a_7} E^{a_8} g^{a_9} h^{a_{10}} L^{a_{11}} w_{max}^{a_{12}} c_1^{a_{13}} c_2^{a_{14}}.$$

Next the equation is given in terms of its dimensions:

$$kg^0 m^0 s^0 = m^1 m^{a_1} s^{a_2} (m^2 s^{-1})^{a_3} m^{a_4} (kgm^{-3})^{a_5} (kgm^{-3})^{a_6} m^{a_7} (kgm^{-1}s^{-2})^{a_8} (ms^{-2})^{a_9}$$
$$m^{a_{10}} m^{a_{11}} m^{a_{12}} (kgm^{-2}s^{-1})^{a_{13}} (kgms^{-1})^{a_{14}}.$$

The product of all these units must be equal to the product of kg, m and s to a zero power. Next, the dimensions are separated and we try to find the exponent that obtains the right solution.

$$m : 1 + a_1 + 2a_3 + a_4 - 3a_5 - 3a_6 + a_7 - a_8 + a_9 + a_{10} + a_{11} + a_{12} - 2a_{13} + a_{14} = 0,$$
$$kg : a_5 + a_6 + a_8 + a_{13} + a_{14} = 0,$$
$$s : a_2 - a_3 - 2a_8 - 2a_9 - a_{13} - a_{14} = 0.$$

This linear system has an infinite number of solutions. Examples of possible solutions are:

$$a_{11} = -1, \quad \pi_1 = xL^{-1}.$$
$$a_5 = \frac{1}{6}, \quad a_8 = \frac{1}{6}, \quad a_{14} = -\frac{1}{3}, \quad \pi_1 = x\rho_s^{\frac{1}{6}} E^{\frac{1}{6}} c_2^{-\frac{1}{3}}.$$

To nondimensionalize z, t, ϕ and w, one can apply the same buckingham-π method to regroup those variables into nondimensional π-groups. There are also an infinite number of ways to make z, t, ϕ and w nondimensional.

Using the same method to nondimensionalize x, the variable z can for example be nondimensionalized in the following number of ways:

$$\pi_2 = zh^{-1},$$
$$\pi_2 = z\rho_s^{\frac{1}{6}}E^{\frac{1}{6}}c_2^{-\frac{1}{3}}.$$

When we continue to nondimensionalize other variables, the options for t are:

$$\pi_3 = tg^{\frac{1}{2}}d^{\frac{1}{2}},$$
$$\pi_3 = t\rho_s^{-\frac{1}{3}}E^{\frac{2}{3}}c_2^{-\frac{1}{3}}.$$

Subsequently one can remove the dimensions in the variable w:

$$\pi_4 = ww_{max}^{-1},$$
$$\pi_4 = wh^{-1},$$
$$\pi_4 = w\rho_w^{\frac{1}{6}}E^{-\frac{1}{3}}.$$

Finally the velocity potential, ϕ, can for example be restated as:

$$\pi_5 = \phi\rho_w^{\frac{2}{3}}E^{-\frac{1}{3}}c_2^{-\frac{1}{3}}.$$

3.2. Nondimensionalisation

First the boundary conditions are made linear. Secondly the dimensions in the FSI-equations are removed. After removing dimensions the size of different terms is examined with a perturbation method. The higher order and lower order terms of are separated into different equations. This yields a series of hierarchically linear partial differential equations that one can solve.

3.2.1. Removing Non-linearity in the Kinematic Boundary Condition

One can observe that the boundary conditions are nonlinear in the equation 2.6. To remove this nonlinearity the variable $\xi(x,z,t) = \frac{z-w(x,t)}{h+w(x,t)}$ is substituted in the function ϕ, such that $\phi(x,z,t) = \hat{\phi}(x,\xi,t)$. The domain of ξ is $-1 < \xi < 0$, since the domain of z is $-h < z < w(x,t)$. Substituting $\hat{\phi}$ in the boundary conditions for $z = w$ yields a linear kinematic boundary condition:

$$\frac{\partial w}{\partial t} = \hat{\phi}_\xi \xi_z = \hat{\phi}_\xi \frac{-w}{h+w}, \quad \xi = 0.$$

Moreover substituting $\hat{\phi}$ in equation 2.7 yields:

$$\rho_s d\frac{\partial^2 w}{\partial t^2} + \frac{Ed^3}{12(1-\nu^2)}\frac{\partial^4 w}{\partial x^4} - \frac{3Ed}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} + \rho_w g w + \rho_w[\frac{\partial\hat{\phi}}{\partial t} + \frac{\partial\hat{\phi}}{\partial\xi}\frac{-w_t}{h+w}]\Big|_{\xi=0} + c_1\frac{\partial w}{\partial t} + c_2\frac{\partial^5 w}{\partial t\partial x^4}.$$

3.2.2. Nondimensionalizing Variables

In this sub section one can see how the system in section 2.2.3 are nondimensionalized.

The non-dimensional variables $\bar{t}$, $\bar{x}$, $\bar{z}$, $\bar{\phi}$ and $\bar{w}$ are introduced.

$$\bar{t} = \frac{t}{t_c}, \tag{3.1}$$
$$\bar{x} = \frac{x}{d}, \tag{3.2}$$
$$\bar{z} = \frac{z}{h}, \tag{3.3}$$
$$\hat{\phi}(x,\xi,t) = \phi(x,z,t), \tag{3.4}$$
$$\xi = \frac{z-w}{h+w} = \frac{\bar{z} - \bar{w}\frac{w_{max}}{h}}{1+\bar{w}\frac{w_{max}}{h}}, \tag{3.5}$$
$$\bar{\phi} = \frac{\hat{\phi}}{\phi_c}, \tag{3.6}$$
$$\bar{w} = \frac{w}{w_{max}}. \tag{3.7}$$

With this normalisation the new domain of the model is $-\infty \leq \bar{x} \leq \infty$, $-1 \leq \xi \leq 0$ and $-1 \leq \bar{w} \leq 1$. The values of t_c and ϕ_c are chosen by judicious guessing in paragraph 3.2.3.
This yields the following FSI-equation:

$$\bar{w}_{\bar{t}\bar{t}} + \frac{t_c^2 E}{d^2 12(1-\nu^2)\rho_s}\bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} - \frac{3Ew_{max}^2 t_c^2}{2\rho_s d^4}(\bar{w}_{\bar{x}})^2 w^*_{\bar{x}\bar{x}} + \frac{\rho_w g t_c^2}{\rho_s d}\bar{w} + \frac{\rho_w \phi_c t_c}{\rho_s w_{max} d}\left[\bar{\phi}_{\bar{t}} + \bar{\phi}_\xi \xi_{\bar{t}}\right] \tag{3.8}$$

$$+\frac{c_1 t_c}{\rho_s d}\bar{w}_{\bar{t}} + \frac{t_c c_2}{\rho_s d^5}\bar{w}_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad \xi = 0. \tag{3.9}$$

Substituting $\hat{\phi}(x, \xi(x,t), t) = \phi(x,z,t)$ in the water potential equation yields an equation with nonlinear terms:

$$\begin{aligned}\phi_{xx} + \phi_{zz} &= \hat{\phi}_{xx} + \hat{\phi}_{x\xi}\xi_x + (\hat{\phi}_{\xi x} + \hat{\phi}_{\xi\xi}\xi_x)\xi_x + \hat{\phi}_\xi \xi_{xx} + \phi_{\xi\xi}\xi_z^2 \\ &= \hat{\phi}_{xx} + 2\hat{\phi}_{x\xi}\xi_x + \hat{\phi}_{\xi\xi}\xi_x^2 + \hat{\phi}_\xi \xi_{xx} + \hat{\phi_{\xi\xi}}\xi_z^2 \\ = \frac{\phi_c}{d^2}\bar{\phi}_{\bar{x}\bar{x}} + \frac{2\phi_c}{d^2}\bar{\phi}_{\bar{x}\xi}\xi_{\bar{x}} &+ \frac{\phi_c}{d^2}\bar{\phi}_{\xi\xi}\xi_{\bar{x}}^2 + \frac{\phi_c}{d^2}\bar{\phi}_\xi \xi_{\bar{x}\bar{x}} + \frac{\phi_c}{h^2}\bar{\phi_{\xi\xi}}\xi_{\bar{z}}^2 = 0.\end{aligned}$$

The derivatives for $\xi(x,t)$ are:

$$\begin{aligned}\xi_{\bar{x}} &= \frac{-\bar{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\bar{w}\frac{w_{max}}{h})} - \frac{(\bar{z} - \bar{w}\frac{w_{max}}{h})\bar{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\bar{w}\frac{w_{max}}{h})^2}, \\ \xi_{\bar{x}\bar{x}} &= -\frac{\bar{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{1+\bar{w}\frac{w_{max}}{h}} + \frac{2(\bar{w}_{\bar{x}}\frac{w_{max}}{h})^2}{(1+\bar{w}\frac{w_{max}}{h})^2} + \\ \frac{2(\bar{z} - \bar{w}\frac{w_{max}}{h})\bar{w}_{\bar{x}}^2(\frac{w_{max}}{h})^2}{(1+\bar{w}\frac{w_{max}}{h})^3} &- \frac{(\bar{z} - \bar{w}\frac{w_{max}}{h})\bar{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{(1+\bar{w}\frac{w_{max}}{h})^2}, \\ \xi_{\bar{z}} &= \frac{1}{1+\bar{w}\frac{w_{max}}{h}}, \\ \xi_{\bar{z}\bar{z}} &= 0, \\ \xi_{\bar{t}} &= \frac{-\bar{w}_{\bar{t}}\frac{w_{max}}{h}}{(1+\bar{w}\frac{w_{max}}{h})} - \frac{(\bar{z} - \bar{w}\frac{w_{max}}{h})\bar{w}_{\bar{t}}\frac{w_{max}}{h}}{(1+\bar{w}\frac{w_{max}}{h})^2}.\end{aligned}$$

The boundary equation at the free surface elevation is given at $z = w$ which is equivalent to $\xi = 0$. When $z = w$, then $\bar{z} = \bar{w}\frac{w_{max}}{h}$. The value of the function $\xi_{\bar{t}}$ evaluated at $\bar{z} = \bar{w}\frac{w_{max}}{h}$ is : $\xi_{\bar{t}} = \frac{-\bar{w}_{\bar{t}}\frac{w_{max}}{h}}{(1+\bar{w}\frac{w_{max}}{h})}$.
Substituting the values of $\xi_{\bar{t}}$ at $z = w$ in the equation (3.9) the boundary condition yields:

$$\bar{w}_{\bar{t}\bar{t}} + \frac{t_c^2 E}{d^2 12(1-\nu^2)\rho_s}\bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} - \frac{3Ew_{max}^2 t_c^2}{2\rho_s d^4}(\bar{w}_{\bar{x}})^2 w^*_{\bar{x}\bar{x}} + \frac{\rho_w g t_c^2}{\rho_s d}\bar{w} + \frac{\rho_w \phi_c t_c}{\rho_s w_{max} d} \tag{3.10}$$

$$\left[\bar{\phi}_{\bar{t}} + \bar{\phi}_\xi \frac{-\bar{w}_{\bar{t}}\frac{w_{max}}{h}}{(1+\bar{w}\frac{w_{max}}{h})}\right] + \frac{c_1 t_c}{\rho_s d}\bar{w}_{\bar{t}} + \frac{t_c c_2}{\rho_s d^5}\bar{w}_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad \xi = 0. \tag{3.11}$$

Substituting the values of $\bar{t}$, ξ, $\bar{\phi}$ and $\bar{w}$ in the kinematic boundary condition yields:

$$\begin{aligned}\bar{w}_{\bar{t}} &= \frac{t_c \phi_c}{h w_{max}}\bar{\phi}_\xi \xi_{\bar{z}}, \quad \xi = 0. \Rightarrow \\ \bar{w}_{\bar{t}} &= \frac{t_c \phi_c}{h w_{max}}\bar{\phi}_\xi \frac{1}{1+\bar{w}\frac{w_{max}}{h}}, \quad \xi = 0.\end{aligned}$$

The boundary condition at the bottom of the ocean is restated as:

$$\begin{aligned}\frac{\phi_c}{h}\bar{\phi}_\xi \xi_{\bar{z}} &= 0, \quad \xi = -1. \implies \\ \frac{\phi_c}{h}\frac{1}{1+\bar{w}\frac{w_{max}}{h}}\bar{\phi}_\xi &= 0, \quad \xi = -1. \implies \\ \bar{\phi}_\xi &= 0, \quad \xi = -1.\end{aligned}$$

3.2.3. Simplifying FSI-equations

Choose ε as a very small and dimensionless variable. Substitute $\bar{w} = \varepsilon\tilde{w}$ in the equations.

The boundary condition at $\xi = 0$ can be restated as:

$$\tilde{w}_{\bar{t}\bar{t}} + \frac{t_c^2 E}{d^2 12(1-\nu^2)\rho_s}\tilde{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} - \varepsilon^2 \frac{3Ew_{max}^2 t_c^2}{2\rho_s d^4}(\tilde{w}_{\bar{x}})^2\tilde{w}_{\bar{x}\bar{x}} + \frac{\rho_w g t_c^2}{\rho_s d}\tilde{w} + \tag{3.12}$$

$$\frac{\rho_w \phi_c t_c}{\rho_s w_{max} d\varepsilon}\left[\bar{\phi}_{\bar{t}} + \bar{\phi}_\xi \frac{-\varepsilon\tilde{w}_{\bar{t}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})}\right] + \frac{c_1 t_c}{\rho_s d}\tilde{w}_{\bar{t}} + \frac{t_c c_2}{\rho_s d^5}\tilde{w}_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad \xi = 0. \tag{3.13}$$

Replacing $\bar{w} = \varepsilon\tilde{w}$ in the equation for velocity potential yields:

$$\begin{aligned}\phi_{xx} + \phi_{zz} &= \hat{\phi}_{xx} + \hat{\phi}_{x\xi}\xi_x + (\hat{\phi}_{\xi x} + \hat{\phi}_{\xi\xi}\xi_x)\xi_x + \hat{\phi}_\xi \xi_{xx} + \phi_{\xi\xi}\xi_z^2 \\ &= \hat{\phi}_{xx} + 2\hat{\phi}_{x\xi}\xi_x + \hat{\phi}_{\xi\xi}\xi_x^2 + \hat{\phi}_\xi\xi_{xx} + \hat{\phi_{\xi\xi}}\xi_z^2 \\ = \frac{\phi_c}{d^2}\bar{\phi}_{\bar{x}\bar{x}} + \frac{2\phi_c}{d^2}\bar{\phi}_{\bar{x}\xi}\xi_{\bar{x}} + \frac{\phi_c}{d^2}\bar{\phi}_{\xi\xi}\xi_{\bar{x}}^2 + \frac{\phi_c}{d^2}\bar{\phi}_\xi\xi_{\bar{x}\bar{x}} &+ \frac{\phi_c}{h^2}\bar{\phi_{\xi\xi}}\xi_{\bar{z}}^2 = 0, \implies \\ \bar{\phi}_{\bar{x}\bar{x}} + 2\bar{\phi}_{\bar{x}\xi}\xi_{\bar{x}} + \bar{\phi}_{\xi\xi}\xi_{\bar{x}}^2 + \bar{\phi}_\xi\xi_{\bar{x}\bar{x}} &+ \frac{d^2}{h^2}\bar{\phi_{\xi\xi}}\xi_{\bar{z}}^2 = 0.\end{aligned}$$

the derivatives for ξ are:

$$\begin{aligned}\xi_{\bar{x}} &= \frac{-\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})} - \frac{(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}, \\ \xi_{\bar{x}\bar{x}} &= -\frac{\varepsilon\tilde{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{1+\varepsilon\tilde{w}\frac{w_{max}}{h}} + \frac{2(\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h})^2}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2} + \\ \frac{2(\bar{z}-\tilde{\varepsilon w}\frac{w_{max}}{h})\varepsilon^2\tilde{w}_{\bar{x}}^2(\frac{w_{max}}{h})^2}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^3} &- \frac{(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon\tilde{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}, \\ \xi_{\bar{z}} &= \frac{1}{1+\varepsilon\tilde{w}\frac{w_{max}}{h}}.\end{aligned}$$

Furthermore the derivatives are substituted in the equation for velocity potential:

$$\begin{aligned}&\bar{\phi}_{\bar{x}\bar{x}} + 2\bar{\phi}_{\bar{x}\xi}\left(\frac{-\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})} - \frac{(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}\right) + \bar{\phi}_{\xi\xi}\left(\frac{-\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})} - \frac{(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}\right)^2 \\ &+\bar{\phi}_\xi(\frac{-\varepsilon\tilde{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{1+\varepsilon\tilde{w}\frac{w_{max}}{h}} + \frac{2(\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h})^2}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2} + \\ &\frac{2(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon^2\tilde{w}_{\bar{x}}^2(\frac{w_{max}}{h})^2}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^3} - \frac{(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon\tilde{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}) + \frac{d^2}{h^2}\bar{\phi}_{\xi\xi}\frac{1}{\left(1+\varepsilon\tilde{w}\frac{w_{max}}{h}\right)^2} = 0.\end{aligned}$$

The boundary condition at the surface, $\xi = 0$ can be reformulated as :

$$\tilde{w}_{\bar{t}} = \frac{t_c\phi_c}{hw_{max}\varepsilon}\bar{\phi}_\xi\frac{1}{1+\varepsilon\tilde{w}\frac{w_{max}}{h}}, \quad \xi = 0.$$

3.2.4. Finding Constants

With judicious guessing the values for variables t_c and ϕ_c are found:

$$t_c^2 = \frac{d^2\rho_s}{E}, \quad \phi_c = \frac{w_{max}d\varepsilon}{t_c}. \tag{3.14}$$

Replacing the values ϕ_c and t_c in equation (3.13) yields:

$$\tilde{w}_{\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\tilde{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} - \varepsilon^2\frac{3w_{max}^2}{2d^2}(\tilde{w}_{\bar{x}})^2\tilde{w}_{\bar{x}\bar{x}} + \frac{\rho_w g d}{E}\tilde{w} + \frac{\rho_w}{\rho_s}\left[\bar{\phi}_{\bar{t}} + \bar{\phi}_\xi\frac{-\varepsilon\tilde{w}_{\bar{t}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})}\right] \tag{3.15}$$

$$+\frac{c_1}{\sqrt{E\rho_s}}\tilde{w}_{\bar{t}} + \frac{c_2}{d^4\sqrt{E\rho_s}}\tilde{w}_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad \xi = 0. \tag{3.16}$$

After we replace ϕ_c and t_c in the kinetic boundary condition at the surface:

$$\tilde{w}_{\bar{t}} = \frac{d}{h}\bar{\phi}_\xi \frac{1}{1+\varepsilon\tilde{w}\frac{w_{max}}{h}}, \quad \xi = 0. \Rightarrow$$
$$(1+\varepsilon\tilde{w}\frac{w_{max}}{h})\tilde{w}_{\bar{t}} = \frac{d}{h}\bar{\phi}_\xi \quad \xi = 0.$$

3.2.5. Hierarchical Differential Equations

In both the water potential and vibrations of the beam the equations are nonlinear in the previous sections. Therefore both the unknown functions $\bar{w}$ and $\bar{\phi}$ are expanded accordingly:

$$\tilde{w} = w_0 + \varepsilon w_1 + \varepsilon^2 w_2, \quad \bar{\phi} = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2. \tag{3.17}$$

We choose $\varepsilon^2 = (\frac{w_{max}}{h})^2$, since it is assumed that the deflection of the beam is very small compared to the depth of the ocean. The depth of a Norwegian Fjord is for example $h = 200m$ and we assume $w_{max} = 0.02m - 0.2m$. Thus $\varepsilon \approx 10^{-4} - 10^{-5}$.

Furthermore the value of $\varepsilon = \frac{\rho_w g d}{E}$. For a metal pontoon (for VLFS) the constants have the following values: $\rho_w = 1.025 \cdot 10^3$, $g = 9.81$, $d = 2.0 \cdot 10^{-2}$ and $E = 1.6416 \cdot 10^6$ (Xu, 2022, p.65). Now we find that: $\varepsilon \approx 1.225 \cdot 10^{-4}$.

First Order Equations

In this subsection the $\mathcal{O}(\varepsilon^0)$-order components of the nondimensionalized FSI-equation are collected.

Because $\xi_{\bar{x}}$ and $\xi_{\bar{x}x}$ are of at least order $\mathcal{O}(\varepsilon^2)$, we can extract the $\mathcal{O}(\varepsilon^0)$ order equations for velocity potential flow:

$$\bar{\phi}_{\bar{x}\bar{x}} + \frac{d^2}{h^2}\bar{\phi}_{\xi\xi}\xi_{\bar{z}}^2 = 0, \Rightarrow$$
$$\bar{\phi}_{\bar{x}\bar{x}} + \frac{d^2}{h^2}\bar{\phi}_{\xi\xi}\frac{1}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2} = 0, \Rightarrow$$
$$(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2\bar{\phi}_{\bar{x}\bar{x}} = -\frac{d^2}{h^2}\bar{\phi}_{\xi\xi}, \Rightarrow$$
$$(1+\varepsilon^2(w_0+\varepsilon w_1+\varepsilon^2 w_2))^2\bar{\phi}_{\bar{x}\bar{x}} = -\frac{d^2}{h^2}\bar{\phi}_{\xi\xi}, \Rightarrow$$
$$\phi_{0\bar{x}\bar{x}} = -\frac{d^2}{h^2}\phi_{0\xi\xi}.$$

The $\mathcal{O}(\varepsilon^0)$-order boundary conditions are:

$$\phi_{0\xi} = 0, \quad \xi = -1.$$
$$\tilde{w}_{\bar{t}} = \frac{d}{h}\bar{\phi}_\xi, \quad \xi = 0, \Rightarrow$$
$$w_{0\bar{t}} = \frac{d}{h}\phi_{0\xi}, \quad \xi = 0.$$

The $\mathcal{O}(\varepsilon^0)$-order component of the Euler-Bernoulli equation at $\xi = 0$ is :

$$\tilde{w}_{\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\tilde{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{\rho_w}{\rho_s}[\bar{\phi}_{\bar{t}}] \tag{3.18}$$
$$+\frac{c_1 L^2\sqrt{\rho_s}}{\rho_s d^2\sqrt{E}}\tilde{w}_{\bar{t}} + \frac{c_2\sqrt{\rho_s}}{\rho_s d^2 L^2\sqrt{E}}\tilde{w}_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad \xi = 0. \tag{3.19}$$
$$w_{0\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}w_{0\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{\rho_w}{\rho_s}[\phi_{0\bar{t}}] \tag{3.20}$$
$$+\frac{c_1 L^2\sqrt{\rho_s}}{\rho_s d^2\sqrt{E}}w_{0\bar{t}} + \frac{c_2\sqrt{\rho_s}}{\rho_s d^2 L^2\sqrt{E}}w_{0\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad \xi = 0. \tag{3.21}$$

Furthermore new damping coefficients $\tilde{c}_1$ and $\tilde{c}_2$ are introduced such that: $c_1 = \varepsilon^2\tilde{c}_1$ and $c_2 = \varepsilon^2\tilde{c}_2$. This approach is valid because the damping coefficients are often very small. In addition one can not find the precise values of the hybrid damping coefficients beforehand (Xu, 2022). Hybrid damping is defined as (structural damping exluded): when energy is dissipated by flowing waves, damping occurs(Xu, 2022). Substituting $\tilde{c}_1$ and $\tilde{c}_2$ in the normalized Bernoulli-Euler equation (3.21) and collecting the $\mathcal{O}(\varepsilon^0)$-terms yields:

$$w_{0\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)} w_{0\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{\rho_w}{\rho_s}\left[\phi_{0\bar{t}}\right] = 0, \quad \xi = 0. \tag{3.22}$$

For ϕ_0 one can find the first separate equation:

$$\phi_{0\xi\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)} \phi_{0\bar{x}\bar{x}\bar{x}\bar{x}\xi} + \frac{\rho_w h}{\rho_s d}\phi_{0\bar{t}\bar{t}} = 0, \quad \xi = 0. \tag{3.23}$$

In appendix C one can find the computations that were used to find the equation for ϕ_0.

For w_0 we can not find a separate equation:

$$\begin{aligned}
w_{0\bar{t}\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)} w_{0\bar{x}\bar{x}\bar{x}\bar{x}\bar{t}} + \frac{\rho_w}{\rho_s}\left[\phi_{0\bar{t}\bar{t}}\right] = 0, \quad \xi = 0, \Rightarrow \\
w_{0\bar{t}\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)} w_{0\bar{x}\bar{x}\bar{x}\bar{x}\bar{t}} - \frac{d}{h}\phi_{0\xi\bar{t}\bar{t}} - \frac{d}{h12(1-\nu^2)}\phi_{0\bar{x}\bar{x}\bar{x}\bar{x}\xi} = 0, \quad \xi = 0, \Rightarrow \\
w_{0\bar{t}\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)} w_{0\bar{x}\bar{x}\bar{x}\bar{x}\bar{t}} - w_{0\bar{t}\bar{t}\bar{t}} - \frac{1}{12(1-\nu^2)} w_{0\bar{x}\bar{x}\bar{x}\bar{x}\bar{t}} = 0, \quad \xi = 0, \Rightarrow \\
0 = 0.
\end{aligned}$$

Second Order Equations

In this subsection one can see how the $\mathcal{O}(\varepsilon)$-order terms from the FSI-equations have been extracted. Because $\xi_{\bar{x}}$ and $\xi_{\bar{x}x}$ are of at least order $\mathcal{O}(\varepsilon^2)$, $\xi_{\bar{x}}^2$ is at least of order $\mathcal{O}(\varepsilon^4)$, removing $\mathcal{O}(\varepsilon^2)$-order and higher order components from the equation for water velocity potential yields:

$$\bar{\phi}_{\bar{x}\bar{x}} + \frac{d^2}{h^2}\bar{\phi}_{\xi\xi}\frac{1}{(1+\varepsilon^2\tilde{w})^2} = 0.$$

Moreover multiplying with the previous equation with$(1+\varepsilon^2\tilde{w})^2$ and removing $\mathcal{O}(\varepsilon^2)$-order terms yields:

$$\bar{\phi}_{\bar{x}\bar{x}} + \frac{d^2}{h^2}\bar{\phi}_{\xi\xi} = 0.$$

Replacing $\bar{\phi}$ and $\tilde{w}$ with w_0,w_1,w_2, ϕ_0, ϕ_1, ϕ_2 and extracting the $\mathcal{O}(\varepsilon)$-order terms yields:

$$\phi_{1\bar{x}\bar{x}} + \frac{d^2}{h^2}\phi_{1\xi\xi} = 0. \tag{3.24}$$

The boundary condition at the surface, $\xi = 0$:

$$(1+\varepsilon^2(w_0+\varepsilon w_1+\varepsilon^2 w_2))(w_{0\bar{t}}+\varepsilon w_{1\bar{t}}+\varepsilon^2 w_{0\bar{t}}) = \frac{d}{h}(\phi_{0\xi}+\varepsilon\phi_{1\xi}+\phi_{2,\xi}), \quad \xi = 0.$$

The $\mathcal{O}(\varepsilon)$ component is collected at the kinematic boundary condition:

$$w_{1\bar{t}} = \frac{d}{h}\phi_{1\xi}, \quad \xi = 0. \tag{3.25}$$

The$\mathcal{O}(\varepsilon)$-order boundary condition on the ocean floor:

$$\phi_{1\xi} = 0, \quad \xi = -1. \tag{3.26}$$

Since $\frac{1}{1+\varepsilon^2\tilde{w}}$ can be written as a geometric series:

$$\frac{1}{1+\varepsilon^2\tilde{w}} = \sum_{n=0}^{\infty}(-\varepsilon^2\tilde{w})^n = 1-\varepsilon^2\tilde{w}+\mathcal{O}((\varepsilon^2\tilde{w})^2),$$

the Bernoulli-Euler boundary condition at $\xi = 0$ is:

$$\tilde{w}_{\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\tilde{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} - \varepsilon^2\frac{3w_{max}^2}{2d^2}(\tilde{w}_{\bar{x}})^2\tilde{w}_{\bar{x}\bar{x}} + \varepsilon\tilde{w} + \frac{\rho_w}{\rho_s}\left[\bar{\phi}_{\bar{t}} + \bar{\phi}_{\xi}\frac{-\varepsilon^2\tilde{w}_{\bar{t}}}{(1+\varepsilon^2\tilde{w})}\right]$$
$$+\frac{\varepsilon^2\tilde{c}_1 L^2\sqrt{\rho_s}}{\rho_s d^2\sqrt{E}}\tilde{w}_{\bar{t}} + \frac{\varepsilon^2\tilde{c}_2\sqrt{\rho_s}}{\rho_s d^2 L^2\sqrt{E}}\tilde{w}_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \Rightarrow$$
$$\tilde{w}_{\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\tilde{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} - \varepsilon^2\frac{3w_{max}^2}{2d^2}(\tilde{w}_{\bar{x}})^2\tilde{w}_{\bar{x}\bar{x}} + \varepsilon\tilde{w} + \frac{\rho_w}{\rho_s}\left[\bar{\phi}_{\bar{t}} + \bar{\phi}_{\xi}(-\varepsilon^2\tilde{w}_{\bar{t}})(1-\varepsilon^2\tilde{w}+\mathcal{O}((\varepsilon^2\tilde{w})^2))\right]$$
$$+\frac{\varepsilon^2\tilde{c}_1 L^2\sqrt{\rho_s}}{\rho_s d^2\sqrt{E}}\tilde{w}_{\bar{t}} + \frac{\varepsilon^2\tilde{c}_2\sqrt{\rho_s}}{\rho_s d^2 L^2\sqrt{E}}\tilde{w}_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0.$$

After one substitutes $\tilde{w} = w_0 + \varepsilon w_1 + \varepsilon^2 w_2$ and $\bar{\phi} = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2$ and extracts the $\mathcal{O}(\varepsilon)$ order terms, one obtains the following equation:

$$w_{1\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}w_{1\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{\rho_w}{\rho_s}\left[\phi_{1\bar{t}}\right] = -w_0, \quad \xi = 0$$

For ϕ_1 one can find a separate non-homogeneous partial differential equation:

$$\phi_{1\xi\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\phi_{1\bar{x}\bar{x}\bar{x}\bar{x}\xi} + \frac{\rho_w h}{\rho_s d}\phi_{1\bar{t}\bar{t}} = -w_{0,\bar{t}}, \quad \xi = 0. \tag{3.27}$$

In appendix C one can find the steps that were used to derive the partial differential equation for ϕ_1.

Third Order Equations

In this subsection the $\mathcal{O}(\varepsilon^2)$ order equation is found. The functions $\xi_{\bar{x}}$ and $\xi_{\bar{x}\bar{x}}$ are of at least order $\mathcal{O}(\varepsilon^2)$. The function $\xi_{\bar{x}}^2$ is of order $\mathcal{O}(\varepsilon^4)$.

Removing $\mathcal{O}(\varepsilon^4)$-terms from the differential equation for velocity potential yields:

$$\bar{\phi}_{\bar{x}\bar{x}} + 2\bar{\phi}_{\bar{x}\xi}\left(\frac{-\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})} - \frac{(\bar{z})\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}\right)$$
$$+\bar{\phi}_{\xi}\left(\frac{-\varepsilon\tilde{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{1+\varepsilon\tilde{w}\frac{w_{max}}{h}} + -\frac{\bar{z}\varepsilon\tilde{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}\right) + \frac{d^2}{h^2}\bar{\phi}_{\xi\xi}\frac{1}{\left(1+\varepsilon\tilde{w}\frac{w_{max}}{h}\right)^2} = 0.$$

Multiplying the previous equation with $(1+\varepsilon^2 w)^2$ and removing $\mathcal{O}(\varepsilon^4)$-order terms yields:

$$\bar{\phi}_{\bar{x}\bar{x}}(1+2\varepsilon^2 w) + 2\bar{\phi}_{\bar{x}\xi}(-\varepsilon^2\tilde{w}_{\bar{x}} - \varepsilon^2\bar{z}\tilde{w}_{\bar{x}}) + \bar{\phi}_{\xi}\varepsilon^2\tilde{w}_{\bar{x}\bar{x}}(-1-\bar{z}) + \frac{d^2}{h^2}\bar{\phi}_{\xi\xi} = 0.$$

Substituting ϕ_0, ϕ_1, ϕ_2, w_0, w_1, w_2 and $\bar{z} = \xi(1+\varepsilon^2\tilde{w}) + \varepsilon^2\tilde{w}$ in the previous equation and collecting the higher order terms yields:

$$\phi_{2\bar{x}\bar{x}} + 2w_0\phi_{0\bar{x}\bar{x}} + 2\phi_{0\bar{x}\xi}w_{0\bar{x}}(-1-\xi) + \phi_{0\xi}w_{0\bar{x}\bar{x}}(-1-\xi) + \frac{d^2}{h^2}\phi_{2\xi\xi} = 0.$$

The $\mathcal{O}(\varepsilon^2)$-order boundary conditions are:

$$w_{2\bar{t}} + w_0 w_{0\bar{t}} = \frac{d}{h}\phi_{2\xi}, \quad \xi = 0.$$
$$\phi_{2,\xi} = 0, \quad \xi = -1.$$

The Euler Bernoulli-boundary condition can be written as:

$$\tilde{w}_{\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\tilde{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} - \varepsilon^2\frac{3w_{max}^2}{2d^2}(\tilde{w}_{\bar{x}})^2\tilde{w}_{\bar{x}\bar{x}} + \varepsilon\tilde{w} + \frac{\rho_w}{\rho_s}\left[\bar{\phi}_{\bar{t}} + \bar{\phi}_{\xi}(-\varepsilon^2\tilde{w}_{\bar{t}})(1-\varepsilon^2\tilde{w} + \mathcal{O}((\varepsilon^2\tilde{w})^2))\right]$$
$$+\frac{\varepsilon^2\tilde{c}_1 L^2\sqrt{\rho_s}}{\rho_s d^2\sqrt{E}}\tilde{w}_{\bar{t}} + \frac{\varepsilon^2\tilde{c}_2\sqrt{\rho_s}}{\rho_s d^2 L^2\sqrt{E}}\tilde{w}_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad \xi = 0.$$

Collecting the $\mathcal{O}(\varepsilon^2)$-order components of the Euler-Bernoulli equation yields:

$$w_{2\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}w_{2\bar{x}\bar{x}\bar{x}\bar{x}} - \frac{3w_{max}^2}{2d^2}(w_{0\bar{x}})^2 w_{0\bar{x}\bar{x}} + w_1 + \frac{\rho_w}{\rho_s}\left[\phi_{2\bar{t}} + \phi_{0\xi}(-w_{0\bar{t}})\right]$$
$$+\frac{\tilde{c}_1 L^2\sqrt{\rho_s}}{\rho_s d^2\sqrt{E}}w_{0\bar{t}} + \frac{\tilde{c}_2\sqrt{\rho_s}}{\rho_s d^2 L^2\sqrt{E}}w_{0\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad \xi = 0.$$

4

Solving the FSI-Equations

In chapter 3 the main FSI-equations were separated into different order problems. In this chapter the $\mathcal{O}(\varepsilon^0)$-order boundary value is solved analytically. In section 4.1 one can find the derivation for the analytical solution with a Fourier transform. Section 4.2 contains a few examples for different initial conditions.

4.1. Solving the First Order Problem

In this chapter one can see how it is possible to find the first order solution of the normalized FSI-equations from the previous chapter 3

The boundary value problem for the first order variable ϕ_0 :

$$\phi_{0\bar{x}\bar{x}}(\bar{x},\xi,\bar{t}) = -\frac{d^2}{h^2}\phi_{0\xi\xi}(\bar{x},\xi,\bar{t}), \tag{4.1}$$

$$\phi_{0\xi}(\bar{x},\xi,\bar{t}) = 0, \quad \xi = -1, \tag{4.2}$$

$$\phi_{0\xi\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\phi_{0\bar{x}\bar{x}\bar{x}\bar{x}\xi} + \frac{\rho_w h}{\rho_s d}\phi_{0\bar{t}\bar{t}} = 0, \quad \xi = 0. \tag{4.3}$$

To solve this equation separation of variables is applied. In section 4.1.2 a Fourier transform is applied to solve the first order FSI-equations.

4.1.1. Separation of Variables

We apply separation of variables: $\phi_0(\bar{x},\xi,\bar{t}) = \hat{\phi}(\bar{x},\bar{t})h(\xi)$. Substituting both $\hat{\phi}$ and $h(\xi)$ in equation (4.3) yields:

$$\hat{\phi}_{\bar{x}\bar{x}}(\bar{x},\bar{t})h(\xi) = \frac{-d^2}{h^2}\hat{\phi}(\bar{x},\bar{t})h''(\xi), \tag{4.4}$$

$$h'(-1) = 0, \tag{4.5}$$

$$\hat{\phi}_{\bar{t}\bar{t}}h'(0) + \frac{1}{12(1-\nu^2)}\hat{\phi}_{\bar{x}\bar{x}\bar{x}\bar{x}}h'(0) + \frac{\rho_w h}{\rho_s d}\hat{\phi}_{\bar{t}\bar{t}}h(0) = 0. \tag{4.6}$$

Let $\frac{h(0)}{h'(0)} = \tilde{K}$. We also introduce the variable c: $c^2 = \frac{1}{12(1-\nu^2)\left(1+\frac{\rho_w h\tilde{K}}{\rho_s d}\right)}$. We assume $c^2 > 0$. The beam equation (4.6) can be restated as:

$$\hat{\phi}_{\bar{t}\bar{t}} + c^2\hat{\phi}_{\bar{x}\bar{x}\bar{x}\bar{x}} = 0. \tag{4.7}$$

Furthermore we assume that for the variable $\hat{\phi}(\bar{x},\bar{t})$ two initial conditions exist:

$$\hat{\phi}(\bar{x},0) = f(\bar{x}), \quad \hat{\phi}_{\bar{t}}(\bar{x},0) = g(\bar{x}). \tag{4.8}$$

4.1.2. Fourier Transform

Subsequently the Fourier transform of equations (4.4-4.8) with respect to the spatial variable $\bar{x}$ is taken. The Fourier transform of variable $\hat{\phi}(\bar{x},\bar{t})$ is given by $U = U(\omega,\bar{t})$.

$$-\omega^2 U h(\xi) = -\frac{d^2}{h^2} U h''(\xi), \Rightarrow h''(\xi) = \frac{h^2\omega^2}{d^2} h(\xi), \tag{4.9}$$

$$h'(-1) = 0, \tag{4.10}$$

$$U_{\bar{t}\bar{t}} + c^2\omega^4 U = 0, \tag{4.11}$$

$$U(\omega,0) = \hat{f}(\omega), \quad U_{\bar{t}}(\omega,0) = \hat{g}(\omega). \tag{4.12}$$

The functions $\hat{f}(\omega)$ and $\hat{g}(\omega)$ are respectively the Fourier transforms of $f(\bar{x})$ and $g(\bar{x})$.
The equation (4.11) is an ordinary differential equation and can easily be solved. The solution is:

$$U(\omega,\bar{t}) = \hat{f}(\omega)\cos(c\omega^2\bar{t}) + \hat{g}(\omega)\frac{\sin(c\omega^2\bar{t})}{c\omega^2}. \tag{4.13}$$

According to Guenther and Lee, 1988 (p. 203-204), the Fourier inverse of (4.13) is :

$$\hat{\phi}(\bar{x},\bar{t}) = \int_{-\infty}^{\infty} \left[K(y-\bar{x},\bar{t})f(y) + L(y-\bar{x},\bar{t})g(y)\right]\, dy, \tag{4.14}$$

where the functions K and L are:

$$K(\bar{x},\bar{t}) = \frac{1}{\sqrt{4\pi c\bar{t}}}\sin\left(\frac{\bar{x}^2}{4ct} + \frac{\pi}{4}\right),$$

$$L(\bar{x},\bar{t}) = \frac{1}{\pi c}\left\{\frac{\pi\bar{x}}{2}\left[S\left(\frac{\bar{x}^2}{4c\bar{t}}\right) + C\left(\frac{\bar{x}^2}{4c\bar{t}}\right)\right] + \sqrt{\pi c\bar{t}}\sin\left(\frac{\bar{x}^2}{4c\bar{t}} + \frac{\pi}{4}\right)\right\}.$$

The functions $S(z)$ and $C(z)$ are (Guenther and Lee, 1988):

$$C(z) = \frac{1}{\sqrt{2\pi}}\int_0^z s^{-1/2}\cos s\, ds,$$

$$S(z) = \frac{1}{\sqrt{2\pi}}\int_0^z s^{-1/2}\sin s\, ds.$$

The steps that are necessary to obtain the solution (4.14) can be found in appendix D.
The function $h(\xi)$ is obtained by solving equation (4.9):

$$h(\xi) = c_1 e^{\frac{h\omega}{d}(\xi+1)} + c_2 e^{-\frac{h\omega}{d}(\xi+1)},$$

$$h(\xi) = c_1 e^{-|\omega|(\xi+1)\frac{h}{d}},$$

$$h'(-1) = -c_1|\omega|\frac{h}{d}e^{-|\omega|(0)\frac{h}{d}} = 0, \Rightarrow c_1 = 0$$

For real values of ω the solution is $h(\xi) = 0$ and the solution $\phi_0 = 0$ is trivial. Hence the values of ω are not purely real. To find a function $h(\xi)$ we investigate solutions of equation (4.9) for complex values of ω.

Purely Imaginary Values for Omega

If one assumes that ω is purely imaginary and can be written as $\omega = i\tilde{\omega}$. Such that $\tilde{\omega}$ is real. Replacing ω with $i\tilde{\omega}$ in equation (4.11) will ultimately yield the same solution $\hat{\phi}(\bar{x},\bar{t})$ in equation (4.14).

Now one can solve the following ODE to find $h(\xi)$:

$$h''(\xi) = \frac{h^2(i\tilde{\omega})^2}{d^2}h(\xi), \tag{4.15}$$

$$h'(-1) = 0. \tag{4.16}$$

The ODE (4.15), satisfying the boundary condition (4.16) can be solved by:

$$h(\xi) = c_1 \cos\left(\frac{h\tilde{\omega}}{d}(\xi+1)\right) = c_1 \frac{e^{i\frac{h\tilde{\omega}}{d}(\xi+1)} + e^{-i\frac{h\tilde{\omega}}{d}(\xi+1)}}{2}. \tag{4.17}$$

The inverse Fourier transform of function (4.17) is (Haberman, 2014, p.483):

$$h(\xi) = \pi c_1 \left(\delta\left(\bar{x} - \frac{h}{d}(\xi+1)\right) + \delta\left(\bar{x} + \frac{h}{d}(\xi+1)\right)\right). \tag{4.18}$$

The value of $\frac{h(0)}{h'(0)}$ depends on ω. This means that the constant c in equation (4.7) depends on ω. In this report the value of $\frac{h(0)}{h'(0)}$ is estimated.

$$h(0) = c_1 \cos(\frac{h\tilde{\omega}}{d}), \quad h'(0) = -\frac{c_1 h\tilde{\omega}}{d}\sin(\frac{h\tilde{\omega}}{d}), \tag{4.19}$$

$$\frac{h(0)}{h'(0)} \approx -\frac{\cos(\frac{h\tilde{\omega}}{d})}{\frac{h\tilde{\omega}}{d}\sin(\frac{h\tilde{\omega}}{d})}. \tag{4.20}$$

We have not found the inverse Fourier transform of $\frac{\cos(\frac{h\tilde{\omega}}{d})}{\frac{h\tilde{\omega}}{d}\sin(\frac{h\tilde{\omega}}{d})}$. At $\tilde{\omega} = 1$, the value of $\tilde{K}$ is equal to $-\frac{\cos(\frac{h}{d})}{\frac{h}{d}\sin(\frac{h}{d})}$. In our model we choose to compute ϕ_0 with this value of $\tilde{K}$. Studying the dependence of $\frac{h(0)}{h'(0)}$ on ω and the impact of this dependence on the constant c and the final solution is recommended for further research.

Complex Values for Omega

If one assumes that ω is a complex number with both a real and an imaginary part one can write $\omega = \omega_1 + i\omega_2$.The values of ω_1 and ω_2 are both real.

For these values of ω one can solve the following ODE to find $h(\xi)$:

$$h''(\xi) = \frac{h^2(\omega_1 + i\omega_2)^2}{d^2} \tag{4.21}$$

$$h(\xi), \tag{4.22}$$

$$h'(-1) = 0. \tag{4.23}$$

The ODE (4.23) with boundary condition (4.22) have the solution:

$$h(\xi) = e^{(\omega_1 + i\omega_2)\frac{h}{d}(\xi+1)}, \tag{4.24}$$

$$h(\xi) = e^{-|\omega_1|\frac{h}{d}(\xi+1)}(c_1 \cos(\omega_2 \frac{h}{d}(\xi+1)) + c_2 \sin(\omega_2 \frac{h}{d}(\xi+1)) \tag{4.25}$$

$$h'(-1) = -|\omega_1|\frac{h}{d}c_1 + \omega_2 \frac{h}{d}c_2 = 0, \Rightarrow c_2 = c_1 \frac{|\omega_1|}{\omega_2}. \tag{4.26}$$

$$h(\xi) = c_1 e^{-|\omega_1|\frac{h}{d}(\xi+1)}(\cos(\omega_2 \frac{h}{d}(\xi+1)) + \frac{|\omega_1|}{\omega_2}\sin(\omega_2 \frac{h}{d}(\xi+1))). \tag{4.27}$$

The Fourier inverse of $e^{-|\omega_1|\frac{h}{d}(\xi+1)}$ is: $\frac{2\frac{h}{d}(\xi+1)}{\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2}$. The Fourier inverse of $\cos(\omega_2 \frac{h}{d}(\xi+1)$ is $2\pi\frac{\delta(\bar{x} - \frac{h}{d}(\xi+1)) + \delta(\bar{x} + \frac{h}{d}(\xi+1))}{2}$. Fourier inverse of $\frac{1}{\omega_2}\sin(\omega_2 \frac{h}{d}(\xi+1)$ is:

$$\begin{cases} 0 & |\bar{x}| > \frac{h}{d}(\xi+1), \\ \pi & |\bar{x}| < \frac{h}{d}(\xi+1). \end{cases} \tag{4.28}$$

The function $F'(\xi) = |\omega_1|e^{-|\omega_1|\frac{h}{d}(\xi+1)}$ has the anti derivative $F(\xi) = \frac{d}{h}e^{-|\omega_1|\frac{h}{d}(\xi+1)}$.The Fourier inverse of F is $F(\xi) = \frac{2(\xi+1)}{\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2}$ and $F'(\xi) = \frac{2(\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2) - 4\frac{h^2}{d^2}(\xi+1)^2}{(\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2)^2} = \frac{2(\bar{x}^2 - \frac{h^2}{d^2}(\xi+1)^2)}{(\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2)^2}$.

After applying an inverse Fourier transform:

$$h(\xi) = c_1 \frac{2\frac{h}{d}(\xi+1)}{\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2} 2\pi(\frac{\delta(\bar{x} - \frac{h}{d}(\xi+1)) + \delta(\bar{x} + \frac{h}{d}(\xi+1))}{2}) + c_1 \tag{4.29}$$

$$\begin{cases} 0 & |\bar{x}| > \frac{h}{d}(\xi+1), \\ \pi\frac{2(\bar{x}^2 - \frac{h^2}{d^2}(\xi+1)^2)}{(\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2)^2} & |\bar{x}| < \frac{h}{d}(\xi+1). \end{cases} \tag{4.30}$$

Final Solution

We can find ϕ_0 by taking a convolution integral (Haberman, 2014, p.483). In case ω is purely imaginary:

$$\phi_0(\bar{x},\xi,\bar{t}) = c_1\pi \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} [K(y-\tilde{x},\bar{t})f(y) + L(y-\tilde{x},\bar{t})g(y)]\, dy \right) (\delta(\tilde{x} - \bar{x} - \frac{h}{d}(\xi+1)) + \delta(\tilde{x} - \bar{x} + \frac{h}{d}(\xi+1)))\, d\tilde{x} =$$
$$c_1\pi \int_{-\infty}^{\infty} \left[K(y - (\bar{x} - \frac{h}{d}(\xi+1)),\bar{t})f(y) + L(y - (\bar{x} - \frac{h}{d}(\xi+1),\bar{t})g(y) \right] dy +$$
$$c_1\pi \int_{-\infty}^{\infty} \left[K(y - (\bar{x} + \frac{h}{d}(\xi+1)),\bar{t})f(y) + L(y - (\bar{x} + \frac{h}{d}(\xi+1),\bar{t})g(y) \right] dy.$$

For the other case, where: $\omega = \omega_1 + i\omega_2$:

$$\phi_0(\bar{x},\xi,\bar{t}) = c_1 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} [K(y-\tilde{x},\bar{t})f(y) + L(y-\tilde{x},\bar{t})g(y)]\, dy \right) \frac{2\pi\frac{h}{d}(\xi+1)}{((\tilde{x}-\bar{x})^2 + \frac{h^2}{d^2}(\xi+1)^2}$$
$$(\delta(\tilde{x} - \bar{x} - \frac{h}{d}(\xi+1)) + \delta(\tilde{x} - \bar{x} + \frac{h}{d}(\xi+1)))\, d\tilde{x}$$
$$+c_1 \int_{-\frac{h}{d}(\xi+1)}^{\frac{h}{d}(\xi+1)} \left(\int_{-\infty}^{\infty} [K(y-\tilde{x},\bar{t})f(y) + L(y-\tilde{x},\bar{t})g(y)]\, dy \right) \pi\frac{2(\bar{x}^2 - \frac{h^2}{d^2}(\xi+1)^2)}{(\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2)^2}\, d\tilde{x} =$$
$$c_1 \left(\int_{-\infty}^{\infty} \left[K(y - (\bar{x} - \frac{h}{d}(\xi+1)),\bar{t})f(y) + L(y - (\bar{x} - \frac{h}{d}(\xi+1)),\bar{t})g(y) \right] dy \right) \frac{\pi}{\frac{h}{d}(\xi+1)} +$$
$$c_q \left(\int_{-\infty}^{\infty} \left[K(y - (\bar{x} + \frac{h}{d}(\xi+1)),\bar{t})f(y) + L(y - (\bar{x} + \frac{h}{d}(\xi+1)),\bar{t})g(y) \right] dy \right) \frac{\pi}{\frac{h}{d}(\xi+1)} +$$
$$c_1 \int_{-\frac{h}{d}(\xi+1)}^{\frac{h}{d}(\xi+1)} \left(\int_{-\infty}^{\infty} [K(y-\tilde{x},\bar{t})f(y) + L(y-\tilde{x},\bar{t})g(y)]\, dy \right) \pi\frac{2(\bar{x}^2 - \frac{h^2}{d^2}(\xi+1)^2)}{(\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2)^2}\, d\tilde{x}.$$

The initial conditions are when ω is purely imaginary:

$$\phi_0(\bar{x},\xi,0) = 2\pi \int_{-\infty}^{\infty} f(\tilde{x})\frac{c_1}{2}(\delta(\tilde{x} - \bar{x} - \frac{h}{d}(\xi+1)) + \delta(\tilde{x} - \bar{x} + \frac{h}{d}(\xi+1)))\, d\tilde{x} =$$
$$c_1\pi(f(\bar{x} - \frac{h}{d}(\xi+1)) + f(\bar{x} - \frac{h}{d}(\xi+1))),$$
$$\phi_{0\bar{t}}(\bar{x},\xi,0) = 2\pi \int_{-\infty}^{\infty} g(\tilde{x})\frac{c_1}{2}(\delta(\tilde{x} - \bar{x} - \frac{h}{d}(\xi+1)) + \delta(\tilde{x} - \bar{x} + \frac{h}{d}(\xi+1)))\, d\tilde{x} =$$
$$c_1\pi g(\bar{x} - \frac{h}{d}(\xi+1)) + g(\bar{x} - \frac{h}{d}(\xi+1)).$$

The initial conditions are for complex values of ω equal to:

$$\phi_0(\bar{x},\xi,0) = c_1 \int_{-\infty}^{\infty} f(\tilde{x})\frac{2\pi\frac{h}{d}(\xi+1)}{((\tilde{x}-\bar{x})^2 + \frac{h^2}{d^2}(\xi+1)^2}(\delta(\tilde{x} - \bar{x} - \frac{h}{d}(\xi+1)) + \delta(\tilde{x} - \bar{x} + \frac{h}{d}(\xi+1)))d\tilde{x} +$$
$$c_1 \int_{-\frac{h}{d}(\xi+1)}^{\frac{h}{d}(\xi+1)} f(\tilde{x})\pi\frac{2(\bar{x}^2 - \frac{h^2}{d^2}(\xi+1)^2)}{(\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2)^2}\, d\tilde{x},$$
$$\phi_{0\bar{t}}(\bar{x},\xi,0) = c_1 \int_{-\infty}^{\infty} g(\tilde{x})\frac{2\pi\frac{h}{d}(\xi+1)}{((\tilde{x}-\bar{x})^2 + \frac{h^2}{d^2}(\xi+1)^2}(\delta(\tilde{x} - \bar{x} - \frac{h}{d}(\xi+1)) + \delta(\tilde{x} - \bar{x} + \frac{h}{d}(\xi+1)))d\tilde{x} +$$
$$c_1 \int_{-\frac{h}{d}(\xi+1)}^{\frac{h}{d}(\xi+1)} g(\tilde{x})\pi\frac{2(\bar{x}^2 - \frac{h^2}{d^2}(\xi+1)^2)}{(\bar{x}^2 + \frac{h^2}{d^2}(\xi+1)^2)^2}\, d\tilde{x}.$$

4.2. Example: Vibrations in a Metal Beam

In this section of behaviour of the functions ϕ_0 and $w_{0\bar{t}}$ on a metal beam for different initial conditions is shown. First the behaviour of a standing wave on a metal plate is predicted. Secondly a source-term is added to the standing wave.

4.2.1. Modelling a Standing Wave in a Metal Beam

The following functions and constants are chosen for modelling a standing wave in a metal beam: $g = 0$, $c_1 = 1$ and $f = \sin(\bar{x})1_{[-80000,80000]}$. We choose to define f one large domain instead of an infinite domain. This is done because Maple is used to compute the functions of ϕ_0 and $w_{0\bar{t}}$ and computing the values of ϕ_0 an infinite domain is extremely time-consuming. In this example we use the solution ϕ_0 in equation 4.31.

The initial conditions on the metal plate are:

$$\phi_0(\bar{x},\xi,0) = \pi\left(f\left(\bar{x}-\frac{h}{d}(\xi+1)\right)+f\left(\bar{x}-\frac{h}{d}(\xi+1)\right)\right),$$
$$\phi(\bar{x},-1,0) = 2\pi f(\bar{x}),$$
$$\phi_{0\bar{t}}(\bar{x},\xi,0) = 0.$$

The water velocity potential is:

$$\phi_0(\bar{x},\xi,\bar{t}) = \pi\int_{-80000}^{80000}\left[K(y-(\bar{x}-\frac{h}{d}(\xi+1)),\bar{t})\sin(y)\right]dy + \pi\int_{-80000}^{80000}\left[K(y-(\bar{x}+\frac{h}{d}(\xi+1)),\bar{t})\sin(y)\right]dy. \tag{4.31}$$

With the kinematic boundary condition, one can obtain the function for the speed of the deflections of the beam:

$$w_{0\bar{t}} = \frac{d}{h}\phi_{0\xi}(\bar{x},0,\bar{t}) = \tag{4.32}$$
$$\pi\int_{-80000}^{80000}\frac{\partial}{\partial\xi}\left[K(y-(\bar{x}-\frac{h}{d}(\xi+1)),\bar{t})\sin(y)\right]\bigg|_{\xi=0}dy+ \tag{4.33}$$
$$\pi\int_{-80000}^{80000}\frac{\partial}{\partial\xi}\left[K(y-(\bar{x}+\frac{h}{d}(\xi+1)),\bar{t})\sin(y)\right]\bigg|_{\xi=0}dy. \tag{4.34}$$

For the computations we model waves in a metal beam. The constants have the following values: $\rho_s = 6.0\times10^2$, $d = 2.0\times10^{-2}$, $\nu = 3.0\times10^{-1}$, $\rho_w = 1.025\times10^3$ and $h = 8.0\times10^{-1}$ (Xu, 2022, p. 65). The constant c has for this case the value:

$$c = \sqrt{\frac{1}{12(1-\nu^2)\left(1+\frac{\rho_w h\left(-\frac{d\cos\left(\frac{h}{d}\right)}{h\sin\left(\frac{h}{d}\right)}\right)}{\rho_s d}\right)}} \approx 0.1787906717. \tag{4.35}$$

Furthermore the values of function $w_{0\bar{t}}$ are shown in figure 4.1. The speed of the beam is modelled on a domain of $-4000 < \bar{x} < 4000$. Because of $\bar{x} = \frac{x}{d} = \frac{x}{0.02}$, the domain of the function in the real world is: $-80m < x < 80m$. In addition the nondimensionalised values of $\bar{t}$ are shown in the plot for $0 < \bar{t} < 10000$. Because $\bar{t} = \frac{t}{t_c}$ and $t_c \approx 0.0003824s$, the values of t are $0s < t < 3.83s$. In figure 4.1 one can see 3.5 standing waves in 5.74 seconds. Thus the period of the wave can be approximated by $T = \frac{3.83}{7.5} \approx 0.509s$. In figure 4.2 the function $\phi_0(\bar{x},\xi,\bar{t})$ is shown for different values of $\bar{t}$. One can notice that the amplitude of the waves for different values of $\bar{t}$ is different. However, there is no displacement of the waves in the $\bar{x}$- or ξ-direction.

In appendix A one can find the Maple-code that was used to produce the figures 4.1 and 4.2.

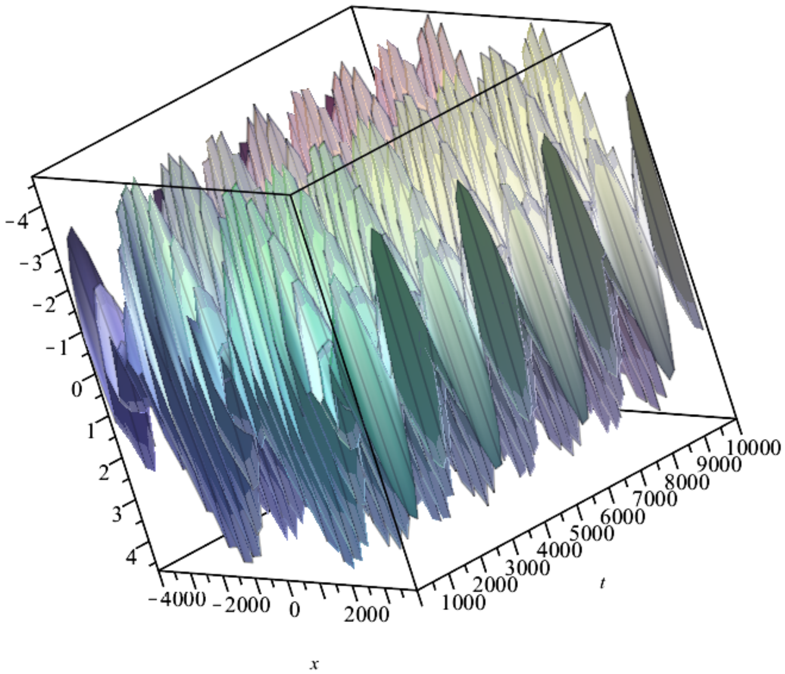
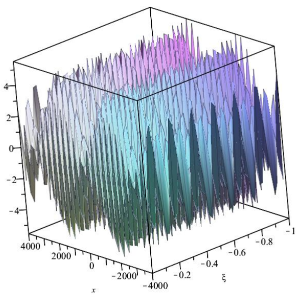
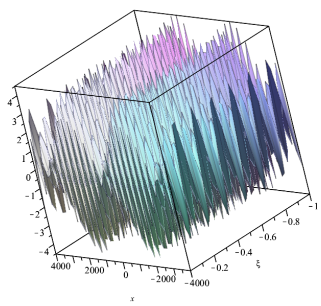


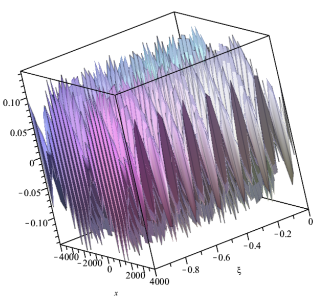
Figure 4.1: The function $w_{0\bar{t}}$ (On a metal plate)



(a) ϕ_0 at $\bar{t} = 0.001$



(b) ϕ_0 at $\bar{t} = 1000$



(c) ϕ_0 at $\bar{t} = 15000$

Figure 4.2: Water Velocity Potential: ϕ_0

4.2.2. Modelling a Standing Wave in a Metal Beam with an Added Source

The following functions and constants are chosen for modelling a standing wave in a plate: $g = 0$, $c_1 = 1$ and $f = \sin(\bar{x})1_{[-80000,80000]} + 201_{[0,5]}$. Here the function f from the previous subsection is applied again and a source is added between $\bar{x} = 0$ and $\bar{x} = 5$.

The initial conditions on the metal plate are:

$$\phi_0(\bar{x}, \xi, 0) = \pi(f(\bar{x} - \frac{h}{d}(\xi + 1)) + f(\bar{x} - \frac{h}{d}(\xi + 1))),$$
$$\phi(\bar{x}, -1, 0) = 2\pi f(\bar{x}) = \sin(\bar{x})1_{[-80000,80000]} + 201_{[0,5]},$$
$$\phi_{0\bar{t}}(\bar{x}, \xi, 0) = 0.$$

From these equations one can notice that we are adding a source-term to the bottom of the ocean ($\xi = -1$). This source can be caused by for example an explosion on the bottom of the ocean.

The water velocity potential is:

$$\phi_0(\bar{x}, \xi, \bar{t}) = \pi \int_{-80000}^{80000} \left[K(y - (\bar{x} - \frac{h}{d}(\xi + 1)), \bar{t}) \sin(y)\right] dy + 20\pi \int_0^5 \left[K(y - (\bar{x} - \frac{h}{d}(\xi + 1)), \bar{t})\right] dy \tag{4.36}$$

$$+\pi \int_{-80000}^{80000} \left[K(y - (\bar{x} + \frac{h}{d}(\xi + 1)), \bar{t}) \sin(y)\right] dy + 20\pi \int_0^5 \left[K(y - (\bar{x} + \frac{h}{d}(\xi + 1)), \bar{t})\right] dy. \tag{4.37}$$

By applying the kinematic boundary condition, one can obtain the function for the speed of the deflections of the beam:

$$w_{0\bar{t}} = \frac{d}{h}\phi_{0\xi}(\bar{x}, 0, \bar{t}) = \tag{4.38}$$

$$\frac{d}{h}\pi \int_{-80000}^{80000} \frac{\partial}{\partial \xi} \left[K(y - (\bar{x} - \frac{h}{d}(\xi + 1)), \bar{t}) \sin(y)\right]\Bigg|_{\xi=0} dy+ \tag{4.39}$$

$$\frac{d}{h}20\pi \int_0^5 \frac{\partial}{\partial \xi} \left[K(y - (\bar{x} - \frac{h}{d}(\xi + 1)), \bar{t})\right]\Bigg|_{\xi=0} dy+ \tag{4.40}$$

$$\frac{d}{h}\pi \int_{-80000}^{80000} \frac{\partial}{\partial \xi} \left[K(y - (\bar{x} + \frac{h}{d}(\xi + 1)), \bar{t}) \sin(y)\right]\Bigg|_{\xi=0} dy+ \tag{4.41}$$

$$\frac{d}{h}20\pi \int_0^5 \frac{\partial}{\partial \xi} \left[K(y - (\bar{x} + \frac{h}{d}(\xi + 1)), \bar{t})\right]\Bigg|_{\xi=0} dy. \tag{4.42}$$

Furthermore the values of function $w_{0\bar{t}}$ are plotted in figure 4.3. The values of $\bar{t}$ are shown in the plot for $0 < \bar{t} < 10000$. Because $\bar{t} = \frac{t}{t_c}$ and $t_c \approx 0.0003824$, the values of t are $0s < t < 3.83$. In figure 4.1 one can see 6 standing waves in 5.74 seconds. Thus the period of the wave can be estimated by $T = \frac{3.83}{6.5} \approx 0.589s$. The waves in figure 4.1 have a similar frequency compared to the waves in figure 4.3.

In figure 4.4 the values of function $\phi_0(\bar{x}, \xi, \bar{t})$ are presented for different values of $\bar{t}$. One can see that the source first spreads from $\xi = -1$ to $\xi = 0$ between figure 4.4a and 4.4b. Between $\bar{t} = 1000$ and $\bar{t} = 15000$ the increase in velocity potential between $\bar{x} = 0$ and $\bar{x} = 5$ decreases gradually.

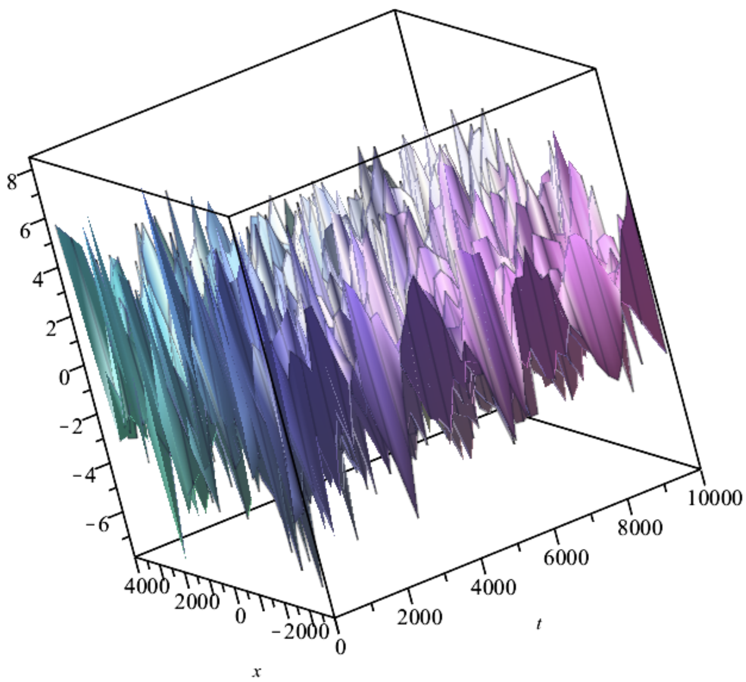
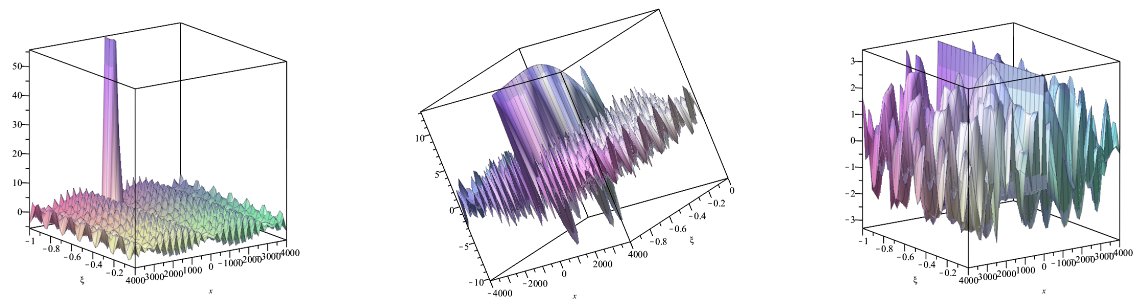


Figure 4.3: The function $w_{0\bar{t}}$ (On Metal Plate) with a source-term



(a) ϕ_0 at $\bar{t} = 0.001$ **(b)** ϕ_0 at $\bar{t} = 1000$ **(c)** ϕ_0 at $\bar{t} = 15000$

Figure 4.4: Water Velocity Potential ϕ_0 with source-term at $\xi = -1$, beween $\bar{x} = 0$ and $\bar{x} = 5$.

5

Conclusion

5.1. Examining Analytical Solutions

The purpose of this report is to investigate analytical methods for modelling fluid structure interaction in large-scale offshore floating photovoltaics. To do so, we looked at nonlinear FSI equations. Furthermore these equations were nondimensionalized. With a perturbation method the non-linearity was dealt with. Different order equations were found by collecting the different order terms. The perturbation constant ε was equal to the amplitude of w_{max} divided by the depth of the ocean, h. By applying separation of variables and a Fourier transform we tried to examine analytical solutions to the $\mathcal{O}(1)$-order equations.

The equations which we tied to solve are (Xu, 2022):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0‘, \quad -\infty < x < \infty, \quad -h < z < w, \tag{5.1}$$

$$\frac{\partial \phi}{\partial z}\Big|_{z=-h} = 0, \quad -h < z < w, \tag{5.2}$$

$$\frac{\partial w}{\partial t} = \frac{\partial \phi}{\partial z}\Big|_{z=w}, \quad 0 < t < \infty, \quad -h < z < w, \tag{5.3}$$

$$\rho_s d \frac{\partial^2 w}{\partial t^2} + \frac{Ed^3}{12(1-\nu^2)} \frac{\partial^4 w}{\partial x^4} - \frac{3Ed}{2}\left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial^2 w}{\partial x^2} + \rho_w g w + \rho_w \frac{\partial \phi}{\partial t}\Big|_{z=w} + c_1 \frac{\partial w}{\partial t} + c_2 \frac{\partial^5 w}{\partial t \partial x^4} = 0. \tag{5.4}$$

The function $\phi(x,z,t)$ denotes the water velocity potential. The vertical deflection of the beam is expressed by the function $w(x,t)$. To get rid of the non-linearity at the boundary conditions placed at $z = w$, the variable z is replaced with $\xi = \frac{z-w}{h+w}$. Next the equations are nondimensionalized. After applying a perturbation method with a small dimensionless constant ε we can collect the first order-terms.

The $\mathcal{O}(\varepsilon^0)$-order equations expressed in ϕ_0 are:

$$\phi_{0\bar{x}\bar{x}}(\bar{x},\xi,\bar{t}) = -\frac{d^2}{h^2}\phi_{0\xi\xi}(\bar{x},\xi,\bar{t}), \tag{5.5}$$

$$\phi_{0\xi}(\bar{x},\xi,\bar{t}) = 0, \quad \xi = -1, \tag{5.6}$$

$$\phi_{0\xi\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\phi_{0\bar{x}\bar{x}\bar{x}\bar{x}\xi} + \frac{\rho_w h}{\rho_s d}\phi_{0\bar{t}\bar{t}} = 0, \quad \xi = 0, \tag{5.7}$$

$$w_{0\bar{t}} = \frac{d}{h}\phi_\xi, \quad \xi = 0. \tag{5.8}$$

in order to find a solution separation of variables is applied. In equation (5.7) we substitute $\phi_0(\bar{x},\xi,\bar{t}) = \hat{\phi}(\bar{x},\bar{t})h(\xi)$ and obtain a beam equation. By applying a Fourier transform with respect to $\bar{x}$ to (5.5 to 5.7), a solution can be found. The function of $\hat{\phi}$ with initial conditions:

$$\hat{\phi}(\bar{x},0) = f(\bar{x}), \quad \hat{\phi}_{\bar{t}}(\bar{x},0) = g(\bar{x}),$$

is equal to:

$$\hat{\phi}(\bar{x},\bar{t}) = \int_{-\infty}^{\infty} \left[K(y-\bar{x},\bar{t})f(y) + L(y-\bar{x},\bar{t})g(y)\right] dy, \tag{5.9}$$

where the functions K and L are:

$$K(\bar{x},\bar{t}) = \frac{1}{\sqrt{4\pi c\bar{t}}}\sin\left(\frac{\bar{x}^2}{4ct}+\frac{\pi}{4}\right),$$

$$L(\bar{x},\bar{t}) = \frac{1}{\pi c}\left\{\frac{\pi\bar{x}}{2}\left[S\left(\frac{\bar{x}^2}{4c\bar{t}}\right)+C\left(\frac{\bar{x}^2}{4c\bar{t}}\right)\right]+\sqrt{\pi c\bar{t}}\sin\left(\frac{\bar{x}^2}{4c\bar{t}}+\frac{\pi}{4}\right)\right\}.$$

The functions C and S are respectively cosine and sine Fresnel integrals. The constant c is equal to $c^2 = \frac{1}{12(1-\nu^2)\left(1+\frac{\rho_w h \frac{h(0)}{h'(0)}}{\rho_s d}\right)}$. To find the function $h(\xi)$ the following equation has to be solved:

$$h''(\xi) = \frac{h^2\omega^2}{d^2}h(\xi), \quad h'(-1) = 0. \tag{5.10}$$

If ω is purely real this ordinary differential equation cannot be solved. Also if the values of ω are complex values with a nonzero real there does exist a solution for equations 5.10. Furthermore if ω has purely imaginary values, one can also find a solution for $h(\xi)$. Applying an inverse Fourier transform to solution $h(\xi)$ for purely imaginary values of ω with respect to ω yields:

$$h(\xi) = \pi c_1\left(\delta(\bar{x}-\frac{h}{d}(\xi+1))+\delta(\bar{x}+\frac{h}{d}(\xi+1))\right).$$

The analytical solution of (5.5- 5.7) is:

$$\phi_0(\bar{x},\xi,\bar{t}) = c_1\pi\int_{-\infty}^{\infty}\left[K(y-(\bar{x}-\frac{h}{d}(\xi+1)),\bar{t})f(y)+L(y-(\bar{x}-\frac{h}{d}(\xi+1),\bar{t})g(y)\right] dy+ \tag{5.11}$$

$$c_1\pi\int_{-\infty}^{\infty}\left[K(y-(\bar{x}+\frac{h}{d}(\xi+1)),\bar{t})f(y)+L(y-(\bar{x}+\frac{h}{d}(\xi+1),\bar{t})g(y)\right] dy. \tag{5.12}$$

The initial conditions are:

$$\phi_0(\bar{x},\xi,0) = c_1\pi(f(\bar{x}+\frac{h}{d}(\xi+1))+f(\bar{x}-\frac{h}{d}(\xi+1))),$$

$$\phi_{0\bar{t}}(\bar{x},\xi,0) = c_1\pi(g(\bar{x}+\frac{h}{d}(\xi+1))+g(\bar{x}-\frac{h}{d}(\xi+1))).$$

5.2. Recommendations for Further Research

In this section two ideas are presented that could be explored for further research. First one can study the Impact of Dispersion by solving a higher order equation. This higher order equation is found by collecting the $\mathcal{O}(\varepsilon^1)$-terms after implementing a perturbation method. Secondly the value of the variable c depends on ω. After applying both separation of variables and a Fourier transform to the equations (5.5-5.7) one can observe that c does depend on ω. In our research we have chosen c as a constant. Can one find a solution ϕ_0 if c depends on ω?

5.2.1. The Impact of Dispersion

With the perturbation method one can also find the $\mathcal{O}(\varepsilon^1)$-terms in the FSI-equations. The nonhomogeneous $\mathcal{O}(\varepsilon^1)$-order problem is:

$$\phi_{1\bar{x}\bar{x}} + \frac{d^2}{h^2}\phi_{1\xi\xi} = 0, \tag{5.13}$$

$$\phi_{1\xi} = 0, \quad \xi = -1, \tag{5.14}$$

$$\phi_{1\xi\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\phi_{1\bar{x}\bar{x}\bar{x}\bar{x}\xi} + \frac{\rho_w h}{\rho_s d}\phi_{1\bar{t}\bar{t}} = -w_{0,\bar{t}}, \quad \xi = 0, \tag{5.15}$$

$$w_{1\bar{t}} = \frac{d}{h}\phi_{1\xi}, \quad \xi = 0. \tag{5.16}$$

The equation (5.15) is nonhomogeneous because the $g\rho_w w$-term in equation (5.4) becomes a $\mathcal{O}(\varepsilon^1)$-term after nondimensionalizing the FSI-equations. This $g\rho_w w$-component represents the dispersion of the water. Collecting the $\mathcal{O}(\varepsilon^1)$-terms after applying the perturbation method yields the nonhomogeneous equation (5.15). For further research we recommend solving the $\mathcal{O}(\varepsilon^1)$-problem. By obtaining the solution one can study the impact of dispersion for fluid-structure interaction in LOFPV.

5.2.2. Constant c

The main solution in (5.12) depends on the constant c. This constant c depends on the values of h(0) and h'(0):

$$c = \sqrt{\frac{1}{12(1-\nu^2)\left(1+\frac{\rho_w h \frac{h(0)}{h'(0)}}{\rho_s d}\right)}} \tag{5.17}$$

To find this constant we solve (5.10) for purely imaginary values of ω and evaluate the solution in $\xi = 0$. This yields:

$$\frac{h(0)}{h'(0)} = -\frac{d \cos \frac{h\omega}{d}}{h\omega \sin \frac{h\omega}{d}}.$$

Xu (2022, p.61) solves FSI-equations with an eigenvalue of $\omega = 1$. Therefore we choose substitute the value of $\frac{h(0)}{h'(0)}$ at $\omega = 1$ in (5.17):

$$\frac{h(0)}{h'(0)} \approx -\frac{d \cos \frac{h}{d}}{h \sin \frac{h}{d}},$$

$$c \approx \sqrt{\frac{1}{12(1-\nu^2)\left(1-\frac{\rho_w}{\rho_s}\frac{\cos\left(\frac{h}{d}\right)}{\sin\left(\frac{h}{d}\right)}\right)}}.$$

For further research we recommend examining the impact of the ω-dependency on c and the final solution ϕ_0 by trying to compute the values of c for different values of ω.

Another challenging problem for further research could be examining whether analytical solutions exist of the $\mathcal{O}(\varepsilon^0)$-order equations (5.5-5.7) for values of c depending on ω.

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A

Maple Computations

The code in figure A.1 and A.2 was used to produce the plots for water velocity potential and speed of beam deflection in chapter 4.

$$> f := \frac{\pi}{\sqrt{4\cdot\pi\cdot c\cdot t}}\int_{-80000}^{80000}\sin\left(\frac{\left(y-x-\frac{h\cdot(\mathrm{xi}+1)}{d}\right)^2}{4\cdot c\cdot t}+\frac{4}{\pi}\right)\cdot\sin(y)\,dy+\frac{\pi}{\sqrt{4\cdot c\cdot t\cdot\pi}}\int_{-80000}^{80000}\sin\left(\frac{\left(y-x+\frac{h\cdot(\mathrm{xi}+1)}{d}\right)^2}{4\cdot c\cdot t}+\frac{4}{\pi}\right)\cdot\sin(y)\,dy;$$

$$> g := \frac{\partial}{\partial \mathrm{xi}}(f);$$

$$> l := eval(g, [d=0.02, h=0.8, \xi=0, c=0.1787906717]);$$

```
>
> plot3d(l*(0.02/0.8),t =10 ..10000, x= -4000 ..4000)
```

Figure A.1: Code for figures of $w_{0\bar{t}}$

$$> f := \frac{\pi}{\sqrt{4\cdot\pi\cdot c\cdot t}}\int_{-80000}^{80000}\sin\left(\frac{\left(y-x-\frac{h\cdot(\mathrm{xi}+1)}{d}\right)^2}{4\cdot c\cdot t}+\frac{4}{\pi}\right)\cdot\sin(y)\,dy+\frac{\pi}{\sqrt{4\cdot c\cdot t\cdot\pi}}\int_{-80000}^{80000}\sin\left(\frac{\left(y-x+\frac{h\cdot(\mathrm{xi}+1)}{d}\right)^2}{4\cdot c\cdot t}+\frac{4}{\pi}\right)\cdot\sin(y)\,dy;$$

```
> g := subs({d = 0.02, h = 0.8, c = 0.1787906717 }, f);
> plot3d(eval(g, t = 15000), x =-4000 ..4000, xi =-1 ..0)
> plots[animate](plot3d, [g, x =-4000 ..4000, xi = -1 ..0], t = 1 ..5000, frames = 50);
> plot3d(eval(g, t = 1000), x =-4000 ..4000, xi =-1 ..0)
> plot3d(eval(g, t = 0.001), x =-4000 ..4000, xi =-1 ..0)
```

Figure A.2: Code for figures of ϕ_0

B

Modelling Transverse Vibrations in a Beam

In this appendix the beam equation from section 2.2.1 is derived from physics.

The transverse vibrations are modelled for a homogeneous beam which can have fixed length L. In our paper an infinite length is chosen. One can think of a nonbended beam as composed of many horizontal fibers. When a beam is bent, a portion of the fibers will be compressed and another portion of the fibers will be pulled out.

The deflection is modelled with the x and z coordinates. In this model the beam is only subject to small, transverse vibrations. The beam will only move in a vertical direction. In this model the transverse vibrations of the beam are represented by the movement of the neutral axis. In figure B.1 one can see the neutral axis (Guenther& Lee, 1988, p.195). The unbent beam fibre at $(x, 0, 0)$ on the neutral axis will after bending be $(x, 0, w(x, t))$ at time t. The variable $w(x, t)$ represents the bending of the beam.

One assumes that the density of the beam ρ_s stays constant over time. Because composition of the beam material is equal everywhere on the beam, the density does not depend on x. Thus ρ_s is constant.

To derive a differential equation of the vibrating beam, a part of the beam is considered between x and $x + \Delta x$. With Newton's second law:

$$\frac{d}{dt}\int_x^{x+\Delta x} S\rho_s w_t(\xi, t)d\xi = \sum \text{vertical forces.} \quad \text{(B.1)}$$

The vertical forces are:

$$\int_x^{x+\Delta x} b(-c_1 w_t(\xi, t) - c_2 w_{txxxx}(\xi, t) + q_w)d\xi. \quad \text{(B.2)}$$

Here $-c_1 w_t(\xi, t)$ represents viscous damping and $c_2 w_{txxxx}$ corresponds to the structural damping of the material. Both the upthrust of the water and the gravitational forces cancel each other out and are therefore not included in this model. The load in the one dimensional x-direction on the structure is uniform and equal to q_w.

F is the shear. The surface forces are:

$$F(x + \Delta x, t) - F(x, t) = \int_x^{x+\Delta x} F_x(\xi, t)d\xi. \quad \text{(B.3)}$$

After substituting, one can obtain the equation:

$$S\rho_s w_{tt}(x, t) = F_x(x, t) + S[-c_1 w_t(x, t) - c_2 w_{txxxx} + q_w]. \quad \text{(B.4)}$$

Let $M(x, t)$ be the bending moment in the cross section at x as a result of the beam to the right of x. Let $F(x + \Delta x, t)$ be the positive upward shear force and $M(x + \Delta x, t)$ be the positive bending moment

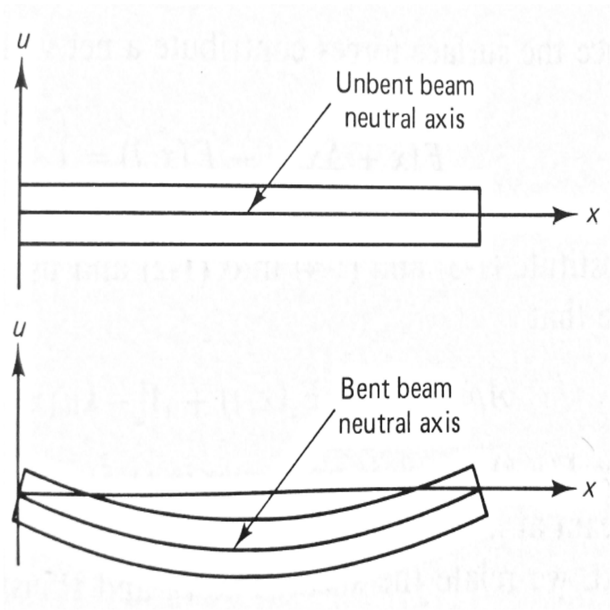

Figure B.1: Movement in a beam (Guenther & Lee, 1988, p.193)

acting on the area $(x, x + \Delta x)$ at the point $x + \Delta x$. The part beam at the left of the cross section will exert a shear $F(x, t)$. Then we find:

$$M(x + \Delta x, t) + F(x, t)\Delta x = M(x, t). \tag{B.5}$$

As Δx goes to zero:

$$\frac{\partial M}{\partial x} = -F. \tag{B.6}$$

Geometric reasoning (Guenther & Lee,1988, p.194) is used to describe M in terms of w. In figure B.2 one can observe the bending of the beam (Guenther and Lee, 1988,p.195). At z units above the neutral axis the fiber has length $(R-z)\Delta\theta$. This fiber is compressed by $z\Delta\theta$ units. The unstrained fiber has length $\Delta s = \Delta x = R\Delta\theta$. Therefore the strain in the fiber is: $z\frac{\Delta\theta}{\Delta x} = \frac{z}{R}$. By Hooke's law the force on the small area from z to $z + \Delta z$ and from x to $x + \Delta x$ is equal to: $F_e = \frac{E}{1-\nu^2}(\frac{z}{R})b\Delta z$. The bending moment M is caused by F_e:

$$|M| = \frac{Eb}{R(1-\nu^2)}\int_{-h/2}^{h/2} z\,dz = \frac{EI}{R}, \tag{B.7}$$

where I represents:

$$I = \frac{bh^3}{12(1-\nu^2)}. \tag{B.8}$$

If the limit of Δx approaches 0, the radius R converges to the radius of the curvature of the neutral axis. Also $\frac{1}{R}$ is the curvature of the neutral axis at x (Guenther and Lee, 1988, p. 195):

$$\frac{1}{R} = \frac{|w_{xx}|}{(1 + w_x^2)^{3/2}} \approx |w_{xx}|. \tag{B.9}$$

For small vibrations one can obtain: $|M| = EI|w_{xx}|$. If M is positive, then the neutral axis bends downward. Thus w is concave and $w_{xx} > 0$. If M is negative, then the neutral axis bends upward. In that case w is convex and $w_{xx} < 0$. Thus $M = EIu_{xx}$ is true.

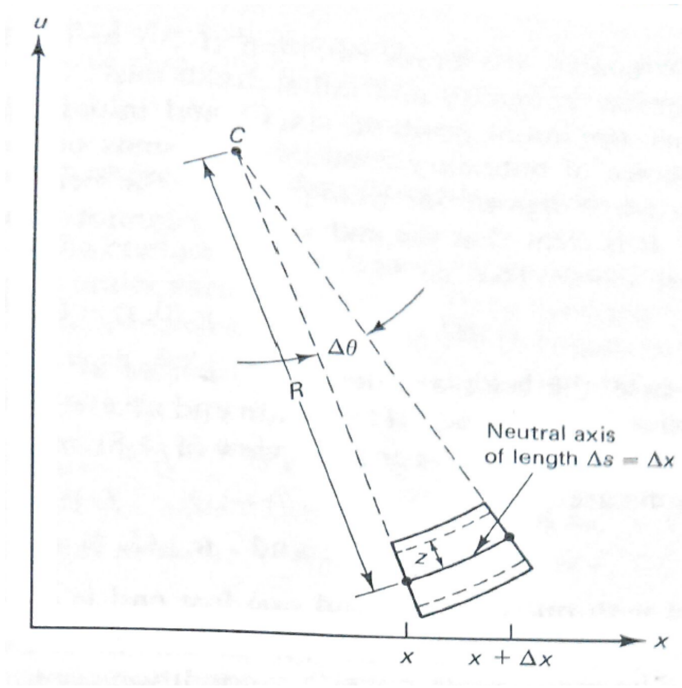

Figure B.2: Bending of a Beam (Guenther & Lee, 1988, p.195)

$$M = EIw_{xx}, \tag{B.10}$$

$$F = -EIw_{xxx}. \tag{B.11}$$

Substituting (B.11)in (B.4) yields:

$$\rho_x w_{tt} + \frac{EI}{S} w_{xxxx} = -c_1 w_t - c_2 w_{txxxx} + q_w. \tag{B.12}$$

Boertjens (2022) formulates an equation for the beam by deriving the Hamiltonian integral. First the work performed to bend the beam can be computed with the following integral :

$$\mathscr{A} = \frac{1}{2} ES \int_0^L [u_x + \frac{1}{2} w_x^2]^2 \, dx + \frac{1}{2} EI \int_0^L w_{xx}^2 \, dx. \tag{B.13}$$

Here u is the horizontal displacement of the beam and w is the vertical displacement of the beam. Furthermore the kinetic energy of the beam is given by (Boertjens, 2000):

$$E_k = \frac{\mu}{2} \int_0^L [u_t^2 + w_t^2] \, dx. \tag{B.14}$$

The mass of the beam per unit length is given by $\mu = S\rho_w$ The Hamiltonian integral is given by:

$$\mathscr{F} = \frac{1}{2} \int_{t_1}^{t_2} \int_0^L \left\{ ES[u_x + \frac{1}{2} w_x^2]^2 + EIw_{xx}^2 - \mu[u_t^2 + w_t^2] \right\} \, dx \, dt. \tag{B.15}$$

Hamilton's principle asserts that the variation of $\mathscr{F}$ is equal to 0. After computing the variation of $\mathscr{F}$, one can derive the Euler equations. Then the beam equation is:

$$\rho_s S w_{tt} + EIw_{xxxx} - SE \frac{\partial}{\partial x} [w_x (u_x + \frac{1}{2} w_x^2)] = 0. \tag{B.16}$$

Because the horizontal deflection u is ignored in our model, we can say that $u_x = 0$. Therefore the beam equation becomes:

$$\rho_s S w_{tt} + EI w_{xxxx} - SE\frac{3}{2}\left(w_x\right)^2 w_{xx} = 0. \tag{B.17}$$

If one adds the load and the damping from equation (B.12) to equation (B.17) one can find the equation shown in section 2.2.1:

$$\rho_s S w_{tt} + EI w_{xxxx} - \frac{3}{2} SE\left(w_x\right)^2 w_{xx} + c_1 b w_t + c_2 b w_{txxxx} = q_w. \tag{B.18}$$

C

Computations of the Equations

C.1. Finding Equations for phi

In this section the main goal is to extract a homogeneous PDE for ϕ_0 and a non-homogeneous PDE for ϕ_1 using respectively the $\mathcal{O}(\varepsilon^0)$-order differential equations and $\mathcal{O}(\varepsilon^1)$-order differential equations from chapter 3.

C.1.1. Differential Equation for phi0

The first order Bernoulli-Euler equation is:

$$w_{0\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)} w_{0\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{\rho_w}{\rho_s} [\phi_{0\bar{t}}] = 0, \quad \xi = 0. \tag{C.1}$$

Next we differentiate kinematic boundary condition with respect to $\bar{t}$ and eliminate ε^2-order components. The function $\xi_{\bar{x}}$ has order $\mathcal{O}(\varepsilon^2)$:

$$w_{0\bar{t}\bar{t}} = \frac{d}{h}(\phi_{0\xi\bar{t}} + \phi_{0\xi\xi}\xi_{\bar{x}}), \quad \xi = 0, \tag{C.2}$$

$$w_{0\bar{t}\bar{t}} = \frac{d}{h}\phi_{0\xi\bar{t}}, \quad \xi = 0. \tag{C.3}$$

Substituting the values of (C.3) in equation (C.1) yields:

$$\frac{d}{h}\phi_{0\xi\bar{t}} + \frac{1}{12(1-\nu^2)} w_{0\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{\rho_w}{\rho_s} [\phi_{0\bar{t}}] = 0, \quad \xi = 0.$$

Differentiating to $\bar{t}$ and removing higher order terms yields:

$$\frac{d}{h}\phi_{0\xi\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)} w_{0\bar{x}\bar{x}\bar{x}\bar{x}\bar{t}} + \frac{\rho_w}{\rho_s} [\phi_{0\bar{t}\bar{t}}] = 0, \quad \xi = 0. \tag{C.4}$$

Substituting the four times differentiated (to $\bar{x}$) kinematic boundary condition in (C.4) and removing terms of orderε and ε^2 yields:

$$\frac{d}{h}\phi_{0\xi\bar{t}\bar{t}} + \frac{d}{h12(1-\nu^2)} \phi_{0\bar{x}\bar{x}\bar{x}\bar{x}\xi} + \frac{\rho_w}{\rho_s} [\phi_{0\bar{t}\bar{t}}] = 0, \quad \xi = 0. \tag{C.5}$$

C.1.2. Differential Equation for phi1

The second order Bernoulli-Euler equation is:

$$w_{1\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)} w_{1\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{\rho_w}{\rho_s} [\phi_{1\bar{t}}] = -w_0, \quad \xi = 0. \tag{C.6}$$

Next we differentiate kinematic boundary condition with respect to $\bar{t}$ and eliminate higher order components. The function $\xi_{\bar{x}}$ has order $\mathcal{O}(\varepsilon^2)$:

$$w_{1\bar{t}\bar{t}} = \frac{d}{h}(\phi_{1\xi\bar{t}} + \phi_{1\xi\xi}\xi_{\bar{x}}), \quad \xi = 0, \tag{C.7}$$

$$w_{1\bar{t}\bar{t}} = \frac{d}{h}\phi_{1\xi\bar{t}}, \quad \xi = 0. \tag{C.8}$$

Substituting the values of (C.6) in equation (C.8) yields:

$$\frac{d}{h}\phi_{1\xi\bar{t}} + \frac{1}{12(1-\nu^2)}w_{1\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{\rho_w}{\rho_s}\left[\phi_{1\bar{t}}\right] = -w_0, \quad \xi = 0.$$

Differentiating to $\bar{t}$ and removing higher order terms yields:

$$\frac{d}{h}\phi_{1\xi\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}w_{1\bar{x}\bar{x}\bar{x}\bar{x}\bar{t}} + \frac{\rho_w}{\rho_s}\left[\phi_{1\bar{t}\bar{t}}\right] = -w_{0\bar{t}}, \quad \xi = 0. \tag{C.9}$$

Differentiating the second order kinematic boundary condition four times with respect to $\bar{x}$, removing higher-order terms and substituting in (C.9) yields:

$$\frac{d}{h}\phi_{1\xi\bar{t}\bar{t}} + \frac{d}{h12(1-\nu^2)}\phi_{1\bar{x}\bar{x}\bar{x}\bar{x}\xi} + \frac{\rho_w}{\rho_s}\left[\phi_{1\bar{t}\bar{t}}\right] = -w_{0\bar{t}}, \quad \xi = 0. \tag{C.10}$$

D
Solving a Beam Equation on an Infinite Domain

In section 4.1. we try to solve a model for a variable $\hat{\phi}$ that is similar to a beam equation. In this appendix one can see how this model is solved with a Fourier transform.

The model in subsection 4.1. is equal to:

$$\hat{\phi}_{\bar{t}\bar{t}} + c^2\hat{\phi}_{\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad -\infty < \bar{x} < \infty, \quad t > 0, \tag{D.1}$$

$$\hat{\phi}(\bar{x}, 0) = f(\bar{x}), \quad \hat{\phi}_{\bar{t}}(\bar{x}, 0) = g(\bar{x}), \quad -\infty < \bar{x} < \infty, \tag{D.2}$$

where $c > 0$. To this initial value problem a Fourier transform is applied with respect to $\bar{x}$:

$$U_{\bar{t}\bar{t}} + c^2\omega^4 U = 0,$$
$$U(\omega, 0) = \hat{f}(\omega), \quad U_{\bar{t}}(\omega, 0) = \hat{g}(\omega),$$

where $U = U(\omega, \bar{t}$ is the Fourier transform of $\hat{\phi}(\bar{x}, \bar{t})$. After applying the Fourier transform we can solve an ordinary differential equation. The solution is:

$$U(\omega, \bar{t}) = \hat{f}(\omega)\cos(c\omega^2\bar{t}) + \hat{g}(\omega)\frac{\sin(c\omega^2\bar{t})}{c\omega^2}. \tag{D.3}$$

The solution $\hat{\phi}$ is acquired by implementing an inverse Fourier transform:

$$\hat{\phi}(\bar{x}, \bar{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\bar{x}}\left[\hat{f}(\omega)\cos(c\omega^2\bar{t}) + \hat{g}(\omega)\frac{\sin(c\omega^2\bar{t})}{c\omega^2}\right] d\omega. \tag{D.4}$$

The equation (D.4) must be considered as a formal solution. We check whether the formal solution solves the original problem. To verify this solution, we first need a proposition:

Proposition

A proposition in (Guenther and Lee, 1988, p. 85) is used in our derivations. Let $h(x, y)$ and $h_x(x, y)$ be continuous for $-\infty \le a < x < b \le \infty$ and $-\infty < y < \infty$. The following equality holds for $x \in (a, b)$:

$$\frac{d}{dx}\int_{-\infty}^{\infty} h(x, y)\, dy = \int_{-\infty}^{\infty} h_x(x, y)\, dy.$$

This equality holds if the integral on the is convergent and the integral on the right is uniformly convergent on each closed and bounded subinterval of (a, b).

Secondly we verify if the formal solution satisfies the initial conditions :

$$\hat{\phi}(\bar{x},0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\bar{x}}\hat{f}(\omega)\,d\omega.$$
$$\hat{\phi}_{\bar{t}}(\bar{x},0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\bar{x}}\hat{g}(\omega)\,d\omega.$$

With the Proposition the differentiation under the integral sign is justified. Thus the formal solution satisfies the initial conditions.

Likewise we check whether the formal solution satisfies the original PDE (D.1). The proposition allows $\hat{\phi}$ to be differentiated twice with respect to $\bar{t}$ under the integral.

$$\hat{\phi}_{\bar{t}\bar{t}} = \frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\bar{x}}\left[c^2\omega^4\hat{f}(\omega)\cos(c\omega^2\bar{t}) + c\omega^2\hat{g}(\omega)\sin(c\omega^2\bar{t})\right]\,d\omega.$$

Finally we can also use the proposition to differentiate under the integral times four times with respect to $\bar{x}$.

$$\hat{\phi}_{\bar{x}\bar{x}\bar{x}\bar{x}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^4 e^{-i\omega\bar{x}}\left[\hat{f}(\omega)\cos(c\omega^2\bar{t}) + c\omega^2\hat{g}(\omega)\frac{\sin(c\omega^2\bar{t})}{c\omega^2}\right]\,d\omega.$$

Thus our the formal solution (D.4) satisfies $\hat{\phi}_{\bar{t}\bar{t}} + c^2\hat{\phi}_{\bar{x}\bar{x}\bar{x}\bar{x}} = 0$.

We would like to express the formal solution directly in terms of $f(\bar{x})$ and $g(\bar{x})$. To achieve this we first consider the integral:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\bar{x}}\hat{f}(\omega)\cos\left(c\omega^2\bar{t}\right)d\omega =$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} e^{i\omega(y-\bar{x})}\cos(c\omega^2\bar{t})\,d\omega\,dy.$$

According to Guenther and Lee, 1988, this integral can be rewritten with by applying a coordinate and an inverse Fourier transform (Guenther and Lee, 1988, p.204):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\bar{x}}\hat{f}(\omega)\cos\left(c\omega^2\bar{t}\right)d\omega =$$
$$\frac{1}{2\pi}\cdot 2\cdot\frac{1}{4}\left(\frac{2\pi}{c\bar{t}}\right)^{1/2}\int_{-\infty}^{\infty}\left(\cos\left(\frac{(y-\bar{x})^2}{4c\bar{t}}\right) + \sin\left(\frac{(y-\bar{x})^2}{4c\bar{t}}\right)\right)f(y)\,dy =$$
$$\frac{1}{\sqrt{4c\bar{t}}}\int_{-\infty}^{\infty} f(y)\sin\left(\frac{(y-x)^2}{4c\bar{t}} + \frac{\pi}{4}\right)\,dy.$$

Furthermore the second part of the integral from the formal solution can be expressed as (Guenther and Lee, 1988, p.204):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\bar{x}}\hat{g}(\omega)\frac{\sin\left(c\omega^2\bar{t}\right)}{c\omega^2}\,d\omega =$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} e^{i\omega(y-\bar{x})}\frac{\sin(c\omega^2\bar{t})}{c\omega^2}\,d\omega\,dy =$$
$$\frac{1}{\pi c}\int_{-\infty}^{\infty}\left\{\frac{\pi(y-\bar{x})}{2}\left[S\left(\frac{(y-\bar{x})^2}{4c\bar{t}}\right) + C\left(\frac{(y-\bar{x})^2}{4c\bar{t}}\right)\right] + \sqrt{\pi c\bar{t}}\sin\left(\frac{(y-\bar{x})^2}{4c\bar{t}} + \frac{\pi}{4}\right)\right\}g(y)\,dy.$$

Where the functions $C(z)$ and $S(z)$ are the Fresnel integrals:

$$C(z) = \frac{1}{\sqrt{2\pi}}\int_0^z s^{-1/2}\cos s\,ds,\quad S(z) = \frac{1}{\sqrt{2\pi}}\int_0^z s^{-1/2}\sin s\,ds.$$

Subsequently the functions $K(\bar{x},\bar{t})$ and $L(\bar{x},\bar{t})$ are chosen such that:

$$K(\bar{x},\bar{t}) = \frac{1}{4\pi c\bar{t}} \sin\left(\frac{\bar{x}^2}{4at} + \frac{\pi}{4}\right),$$

$$L(\bar{x},\bar{t}) = \frac{1}{\pi c}\left\{\frac{\pi\bar{x}}{2}\left[S\left(\frac{\bar{x}^2}{4c\bar{t}}\right) + C\left(\frac{\bar{x}^2}{4c\bar{t}}\right)\right] + \sqrt{\pi c\bar{t}} \sin\left(\frac{\bar{x}^2}{4c\bar{t}} + \frac{\pi}{4}\right)\right\}.$$

Finally one can state the solution of the PDE (D.1) in terms of its initial conditions:

$$\hat{\phi}(\bar{x},\bar{t}) = \int_{-\infty}^{\infty} \left[K(y-\bar{x},\bar{t})f(y) + L(y-\bar{x},\bar{t})g(y)\right]\, dy.$$

From this solution in its final form we can see that it is important for the functions f and g to decay fast when $\bar{x}$ approaches infinity.

E

How Not To Solve The Problem

The main goal of this project is to find a solution for the FSI-equations in chapter 2. This appendix contains methods to solve this equation that do not work. In the first section of this appendix one can observe how a 2-dimensional beam equation with a dispersion competent cannot be solved with a Fourier transform. In the second section of this appendix one can observe how only using separation of variables is an impractical way to solve the beam-equations.

E.1. Beam Equation with Dispersion-terms

The FSI-equations in chapter 2 are nondimensionalized with different constants (compared to the normalization from chapter 3). When we apply a perturbation method, a different first order equation is obtained with a dispersion component.

E.1.1. Nondimensionalising equations

The FSI equations are nondimensionalized:

$$\begin{aligned} \bar{x} &= \frac{x}{L}, \quad \bar{z} = \frac{z}{h}, \\ \hat{\phi}(x,\xi,t) = \phi(x,z,t), \quad \xi &= \frac{z-w}{h+w} = \frac{\bar{z} - \bar{w}\frac{w_{max}}{h}}{1+\bar{w}\frac{w_{max}}{h}}, \\ \bar{\phi} &= \frac{\hat{\phi}}{\phi_c}, \quad \bar{w} = \frac{w}{w_{max}}, \quad \bar{w} = \varepsilon\tilde{w}, \\ t_c^2 &= \frac{L^4\rho_s}{Ed^2}, \quad \phi_c = \frac{w_{max}d\varepsilon}{t_c}. \end{aligned}$$

With this normalisation the domain of the model is $-\infty \leq \bar{x} \leq \infty$, $-1 \leq \xi \leq 0$ and $-1 \leq \bar{w} \leq 1$. The equation for water velocity potential:

$$\begin{aligned} &\bar{\phi}_{\bar{x}\bar{x}} + 2\bar{\phi}_{\bar{x}\xi}\left(\frac{-\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})} - \frac{(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}\right) + \bar{\phi}_{\xi\xi}\left(\frac{-\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})} - \frac{(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}\right)^2 \\ &+\bar{\phi}_{\xi}\left(\frac{-\varepsilon\tilde{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{1+\varepsilon\tilde{w}\frac{w_{max}}{h}} + \frac{2(\varepsilon\tilde{w}_{\bar{x}}\frac{w_{max}}{h})^2}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2} + \right. \\ &\left.\frac{2(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon^2\tilde{w}_{\bar{x}}^2(\frac{w_{max}}{h})^2}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^3} - \frac{(\bar{z}-\varepsilon\tilde{w}\frac{w_{max}}{h})\varepsilon\tilde{w}_{\bar{x}\bar{x}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})^2}\right) + \frac{L^2}{h^2}\bar{\phi}_{\xi\xi}\frac{1}{\left(1+\varepsilon\tilde{w}\frac{w_{max}}{h}\right)^2} = 0. \end{aligned}$$

The boundary condition at the bottom of the ocean:

$$\bar{\phi} = 0, \quad \xi = -1.$$

The FSI equations are:

$$\tilde{w}_{\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\tilde{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} - \varepsilon^2\frac{3w_{max}^2}{2d^2}(\tilde{w}_{\bar{x}})^2\tilde{w}_{\bar{x}\bar{x}} + \frac{\rho_w g L^4}{\rho_s E d^2}\tilde{w} + \frac{\rho_w}{\rho_s}\left[\bar{\phi}_{\bar{t}} + \bar{\phi}_{\xi}\frac{-\varepsilon\tilde{w}_{\bar{t}}\frac{w_{max}}{h}}{(1+\varepsilon\tilde{w}\frac{w_{max}}{h})}\right]$$
$$+\frac{c_1 L^2}{d^2\sqrt{E\rho_s}}\tilde{w}_{\bar{t}} + \frac{c_2}{d^2 L^2\sqrt{E\rho_s}}\tilde{w}_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \quad \xi = 0.$$

The boundary condition at the surface is:

$$\tilde{w}_{\bar{t}} = \frac{L}{h}\bar{\phi}_{\xi}\frac{1}{1+\varepsilon\tilde{w}\frac{w_{max}}{h}}, \quad \xi = 0, \implies$$
$$(1+\varepsilon\tilde{w}\frac{w_{max}}{h})\tilde{w}_{\bar{t}} = \frac{d}{h}\bar{\phi}_{\xi} \quad \xi = 0.$$

E.1.2. Finding the First Order Problem

Let $\varepsilon = \frac{w_max}{h}$, since the depth of the ocean is great compared to the maximum amplitude of the waves. We also split both $\bar{\phi}$ and $\tilde{w}$ in a $\mathcal{O}(\varepsilon^0)$- and a $\mathcal{O}(\varepsilon^2)$-order component:

$$\tilde{w} = w_0 + \varepsilon^2 w_1, \quad \bar{\phi} = \phi_0 + \varepsilon^2\phi_1.$$

The constants c_1 and c_2 are of order $\mathcal{O}(\varepsilon^2)$:

$$c_1 = \varepsilon^2\tilde{c}_1, \quad c_2 = \varepsilon^2\tilde{c}_2.$$

After substituting the values for $\tilde{w}$, $\bar{\phi}$, c_1 and c_2 we collect the $\mathcal{O}(\varepsilon^0)$-terms of the equations. The following Partial Differential Equations can be obtained:

$$\phi_{0\bar{x}\bar{x}} = -\frac{L^2}{h^2}\phi_{0\xi\xi}, \quad -1 < \xi < 0 \quad -\infty < \bar{x} < \infty, \tag{E.1}$$
$$w_{0,\bar{t}} = \frac{d}{h}\bar{\phi}_{\xi}, \quad \xi = 0, \tag{E.2}$$
$$\phi_{0\xi} = 0, \quad \xi = -1, \tag{E.3}$$
$$w_{0\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}w_{0\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{\rho_w g L^4}{\rho_s E d^2}w_0 + \frac{\rho_w}{\rho_s}\left[\phi_{0\bar{t}}\right] = 0, \quad \xi = 0. \tag{E.4}$$

For ϕ_0 the following equation can be found:

$$\phi_{0\xi\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\phi_{0\bar{x}\bar{x}\bar{x}\bar{x}\xi} + \frac{\rho_w g L^4}{\rho_s E d^2}\phi_{0\xi} + \frac{\rho_w h}{\rho_s d}\phi_{0\bar{t}\bar{t}} = 0, \quad \xi = 0. \tag{E.5}$$

E.1.3. Solving the First Order Problem

Separation Of Variables

First we apply separation of variables: $\phi_0(\bar{x},\xi,\bar{t}) = \hat{\phi}(\bar{x},\bar{t})h(\xi)$. Substituting ϕ_0 in (E.1-E.4) yields:

$$\hat{\phi}_{\bar{x}\bar{x}}(\bar{x},\bar{t})h(\xi) = \frac{-d^2}{h^2}\hat{\phi}(\bar{x},\bar{t})h''(\xi) \tag{E.6}$$
$$h'(-1) = 0, \tag{E.7}$$
$$\hat{\phi}_{\bar{t}\bar{t}}h'(0) + \frac{1}{12(1-\nu^2)}\hat{\phi}_{\bar{x}\bar{x}\bar{x}\bar{x}}h'(0) + \frac{\rho_w g L^4}{\rho_s E d^2}\hat{\phi}h'(0) + \frac{\rho_w h}{\rho_s d}\hat{\phi}_{\bar{t}\bar{t}}h(0) = 0. \tag{E.8}$$

Fourier & Laplace transforms

A Fourier transform is applied to the previous equations in the $\bar{x}$-direction. Let $U(\omega,t)$ be the Fourier transform of $\hat{\phi}(\bar{x},\bar{t})$:

$$\mathcal{F}(\hat{\phi}_{\bar{x}\bar{x}}(\bar{x},\bar{t})h(\xi)) = \mathcal{F}(\frac{-L^2}{h^2}\hat{\phi}(\bar{x},\bar{t})h''(\xi)), \Rightarrow$$
$$(i\omega)^2 U h(\xi) = \frac{-L^2}{h^2}Uh''(\xi), \Rightarrow \frac{h''(\xi)}{h(\xi)} = \frac{-(i\omega)^2 h^2}{L^2} = \frac{\omega^2 h^2}{L^2}.$$

To the boundary condition at $\xi = 0$ a Fourier transformation is applied:

$$U_{\bar{t}\bar{t}} + \frac{\omega^4}{12(1-\nu^2)}U + \frac{\rho_w g L^4}{Ed^3}U + \frac{\rho_w h}{\rho_s d}U_{\bar{t}\bar{t}}\frac{h(0)}{h'(0)} = 0.$$

In the $\bar{t}$ variable we apply a Laplace transformation to the equation. Let $W(\omega, s)$ be the Laplace transformation of $U(\omega, \bar{t})$.

$$[s^2W - sU(\omega,0) - U_{\bar{t}}(\omega,0)] + \frac{\omega^4}{12(1-\nu^2)}W + \frac{\rho_w g L^4}{Ed^3}W + \frac{\rho_w h}{\rho_s d}[s^2W - sU(\omega,0) - U_{\bar{t}}(\omega,0)]\frac{h(0)}{h'(0)} = 0, \implies$$

$$W(\omega, s) = \frac{sU(\omega,0) + U_{\bar{t}}(\omega,0)}{s^2 + \frac{\frac{\omega^4}{12(1-\nu^2)} + \frac{\rho_w g L^4}{Ed^3}}{1 + \frac{\rho_w h h(0)}{\rho_s d h'(0)}}} = \frac{sU(\omega,0) - U_{\bar{t}}(\omega,0)}{s^2 + q(\omega^4)}.$$

From Haberman (2014, p.612), the inverse Laplace transformation of $W(\omega, s)$ can be obtained (Haberman, 2014) :

$$U(\omega, t) = U(\omega, 0)\cos(\sqrt{q(\omega^4)}\bar{t}) + U_t(\omega, 0)\frac{1}{\sqrt{q(\omega^4)}}\sin(\sqrt{q(\omega^4)}\bar{t}). \tag{E.9}$$

For both the functions $\cos(\sqrt{q(\omega^2)}\bar{t})$ and $\frac{1}{\sqrt{q(\omega^4)}}\sin(\sqrt{q(\omega^4)}\bar{t})$ we could not find a Fourier inverse. For a beam equation without the dispersion term: $\frac{\rho_w g L^4}{\rho_s E d^2}\hat{\phi}h'(0)$ one could solve the problem by computing the Fourier Inverse of $\cos(\omega^2 c\bar{t})$and $\frac{1}{w^2 c}\sin(\omega^2 c\bar{t})$. Where $c^2 = \frac{\frac{\omega^4}{12(1-\nu^2)}}{1+\frac{\rho_w h h(0)}{\rho_s d h'(0)}}$. The dispersion term can not be removed with a perturbation method, because the dispersion term is quite large:

$$\frac{\rho_w g L^4}{\rho_s E d^2} \approx 10^8 - 10^{13}$$

. Therefore in chapter 3 we nondimensionalize the equations differently and apply the perturbation method to deal the dispersion-term from the first order equation.

E.1.4. Separation of Variables & Fourier Transform

Separation of Variables

First we apply separation of variables: $\phi_0(\bar{x}, \xi, \bar{t}) = \hat{\phi}(\bar{x}, \bar{t})h(\xi)$. Substituting ϕ_0 in (E.1-E.4) yields:

$$\begin{aligned}
\hat{\phi}_{\bar{x}\bar{x}}(\bar{x},\bar{t})h(\xi) &= \frac{-d^2}{h^2}\hat{\phi}(\bar{x},\bar{t})h''(\xi), \Rightarrow \\
\frac{\hat{\phi}_{\bar{x}\bar{x}}}{\hat{\phi}} &= -\frac{d^2}{h^2}\frac{h''(\xi)}{h(\xi)} = -\lambda, \Rightarrow \\
\hat{\phi}_{\bar{x}\bar{x}} &= -\lambda\hat{\phi}, \quad \frac{d^2}{h^2}h'' = \lambda h \\
h'(-1) &= 0, \\
\hat{\phi}_{\bar{t}\bar{t}}h'(0) + \frac{1}{12(1-\nu^2)}\hat{\phi}_{\bar{x}\bar{x}\bar{x}\bar{x}}h'(0) + \frac{\rho_w g L^4}{\rho_s E d^2}\hat{\phi}h'(0) + \frac{\rho_w h}{\rho_s d}\hat{\phi}_{\bar{t}\bar{t}}h(0) &= 0.
\end{aligned}$$

Because $\hat{\phi}_{\bar{x}\bar{x}} = -\lambda\hat{\phi}$, we know that $\hat{\phi}_{\bar{x}\bar{x}\bar{x}\bar{x}} = -\lambda\hat{\phi}_{\bar{x}\bar{x}} = \lambda^2\hat{\phi}$. Therefore we replace $\hat{\phi}_{\bar{x}\bar{x}\bar{x}\bar{x}}$ with $\lambda^2\hat{\phi}$ in our equations:

$$\hat{\phi}_{\bar{t}\bar{t}}h'(0) + \frac{\lambda^2}{12(1-\nu^2)}\hat{\phi}h'(0) + \frac{\rho_w g L^4}{Ed^3}\hat{\phi}h'(0) + \frac{\rho_w h}{\rho_s d}\hat{\phi}_{\bar{t}\bar{t}}h(0) = 0, \implies \tag{E.10}$$

$$\frac{\hat{\phi}_{\bar{t}\bar{t}}}{\hat{\phi}} = -\frac{\frac{\rho_w g L^4}{Ed^3} + \frac{\lambda^2}{12(1-\nu^2)}}{1 + \frac{\rho_w h}{\rho_s d}\frac{h(0)}{h'(0)}} = -\frac{\frac{\rho_w g L^4 h'(0)}{Ed^3} + \frac{\lambda^2 h'(0)}{12(1-\nu^2)}}{h'(0) + \frac{\rho_w h}{\rho_s d}h(0)} = -\mu(\lambda). \tag{E.11}$$

For different values of λ there exist different solutions: If $\lambda = 0$, the solution has the form: $\hat{\phi} = a(\bar{t}) + b(\bar{t})\bar{x}$. If $\bar{x} \to \infty$ or $\bar{x} \to -\infty$ then $\hat{\phi}$ converges to zero. Therefore we can set the values of b to zero: $b(\bar{t}) = 0$.

Moreover if $\lambda > 0$ is positive, then the solution is equal to $\hat{\phi} = c_1(\bar{t})\cos(\sqrt{\lambda}\bar{x}) + c_2(\bar{t})\sin(\sqrt{\lambda}\bar{x})$.

Furthermore if $\lambda < 0$, we set $\lambda = -s$. The solution is equal to $\hat{\phi} = c_3(\bar{t})e^{\sqrt{s}x} + c_4(\bar{t})e^{-\sqrt{s}x}$. If $\bar{x} \to \infty$ or $\bar{x} \to -\infty$, the function of $\hat{\phi}$ converges to zero. Thus $c_3(\bar{t}) = 0$ if $\bar{x} > 0$ and $c_4(\bar{t}) = 0$ if $\bar{x} < 0$. Therefore the velocity potential is: $\hat{\phi}(\bar{x}, \bar{t}) = \tilde{c}(\bar{t})e^{-\sqrt{s}|\bar{x}|} = \tilde{c}(\bar{t})e^{-\sqrt{-\lambda}|\bar{x}|}$.

If one assumes $h'(0) + \frac{\rho_w h h(0)}{\rho_s d} > 0$, then $\mu > 0$. Therefore the solution of $\hat{\phi}$ has the form $\hat{\phi} = a(\bar{x})\cos(\sqrt{\mu}\bar{t}) + b(\bar{x})\sin(\sqrt{\mu}\bar{t})$.

If one assumes $h'(0) = -\frac{\rho_w h h(0)}{\rho_s d}$, the solution $\hat{\phi}$ becomes equal to zero and is trivial. Hence $h'(0)$ is not equal to $-\frac{\rho_w h h(0)}{\rho_s d}$.

The following wave-equation can be obtained for $\lambda > 0$:

$$\hat{\phi}_{\bar{t}\bar{t}} - \frac{\mu(\lambda)}{\lambda}\hat{\phi}_{\bar{x}\bar{x}} = 0, \quad c^2 = \frac{\mu(\lambda)}{\lambda}. \tag{E.12}$$

Whenever λ is zero:

$$\hat{\phi}_{\bar{x}\bar{x}} = 0. \tag{E.13}$$

If $\lambda < 0$ one substitutes a variable s such that: $s = -\lambda$.

$$\hat{\phi}_{\bar{t}\bar{t}} + \frac{\mu(s)}{s}\hat{\phi}_{\bar{x}\bar{x}} = 0, \quad c^2 = \frac{\mu(s)}{s}. \tag{E.14}$$

Substituting $\hat{\phi}_{\bar{x}\bar{x}} = -\mu(\lambda)\hat{\phi}$ into equation E.11 and dividing by $\hat{\phi}$ yields:

$$\begin{aligned}
-\left(\frac{\frac{\rho_w g L^4 h'(0)}{Ed^3} + \frac{\lambda^2 h'(0)}{12(1-\nu^2)}}{h'(0) + \frac{\rho_w h}{\rho_s d}h(0)}\right)h'(0) + \frac{\lambda^2 h'(0)}{12(1-\nu^2)} + \frac{\rho_w g L^4 h'(0)}{Ed^3} - \frac{\rho_w h}{\rho_s d}\left(\frac{\frac{\rho_w g L^4 h'(0)}{Ed^3} + \frac{\lambda^2 h'(0)}{12(1-\nu^2)}}{h'(0) + \frac{\rho_w h}{\rho_s d}h(0)}\right)h(0) &= 0, \Rightarrow \\
-\left(\frac{1}{h'(0) + \frac{\rho_w h}{\rho_s d}h(0)}\right)h'(0) + 1 - \frac{\rho_w h}{\rho_s d}\left(\frac{1}{h'(0) + \frac{\rho_w h}{\rho_s d}h(0)}\right)h(0) &= 0, \Rightarrow \\
-h'(0) + h'(0) + \frac{\rho_w h}{\rho_s d}h(0) - \frac{\rho_w h}{\rho_s d}h(0) &= 0, \Rightarrow \\
0 &= 0.
\end{aligned}$$

Therefore $h(0)$ or $h'(0)$ can have any value in order to satisfy the boundary conditions.

The ODE one needs to solve:

$$h''(\xi) = \frac{h^2\lambda}{L^2}h(\xi) \tag{E.15}$$

Whenever $\lambda = 0$, the solution is $h(\xi) = c_1$.

In case that $\lambda > 0$:

$$\begin{aligned}
h(\xi) &= c_1\cosh(\sqrt{\lambda}\frac{h}{L}(\xi+1)) + c_2\sinh(\sqrt{\lambda}\frac{h}{L}(\xi+1)). \\
h'(-1) = c_1\sqrt{\lambda}\frac{h}{L}\sinh(\sqrt{\lambda}\frac{h}{L}(0)) + c_2\sqrt{\lambda}\frac{h}{L}\cosh(\sqrt{\lambda}\frac{h}{L}(0)) &= 0 \implies c_2 = 0 \\
h(0) = c_1\cosh(\sqrt{\lambda}\frac{h}{L}) &= 0, \implies \\
e^{\sqrt{\lambda}\frac{h}{L}} = -e^{-\sqrt{\lambda}\frac{h}{L}}, \implies e^{2\sqrt{\lambda}\frac{h}{L}} &= -1, \implies \lambda < 0 \\
h'(0) &= 0 \implies \lambda < 0.
\end{aligned}$$

Also supposing that $\lambda < 0$, the following solution can be found:

$$h(\xi) = c_1 \cos(\sqrt{-\lambda}\frac{h}{L}(\xi+1)) + c_2 \sin(\sqrt{-\lambda}\frac{h}{L}(\xi+1))$$
$$h'(-1) = -c_1\sqrt{-\lambda}\frac{h}{L}\sin(0) + c_2\sqrt{-\lambda}\frac{h}{L}\cos(0) = 0, \implies c_2 = 0.$$
$$h(\xi) = c_1 \cos(\sqrt{-\lambda}\frac{h}{L}(\xi+1)).$$

The value of λ has to be a real number. The following proof is given:

$$L(h) = \frac{L^2}{h^2}h'' = \lambda h \implies \frac{L^2}{h^2}\overline{h''} = \overline{\lambda h}. \tag{E.16}$$
$$\tag{E.17}$$

Let $h_1 = h$ and $h_2 = \overline{h}$ be the eigenfunctions to the corresponding eigenvalues $\lambda_1 = \lambda$ and $\lambda_2 = \overline{\lambda}$. Therefore one can obtain the following equality:

$$(\lambda_1 - \lambda_2)\int_{-1}^{0} h_1 h_2 \, d\xi = 0$$
$$\text{if } [h_1(0)h_2'(0) - h_1'(0)h_2(0)] - [h_1(-1)h_2'(-1) + h_1'(-1)h_2(-1)] =$$
$$[h_1(0)h_2'(0) - h_1'(0)h_2(0)] = 0, \Rightarrow$$
$$\frac{\overline{h'(0)}}{\overline{h(0)}} = \frac{h'(0)}{h(0)}$$
.

This implies that $\lambda = \overline{\lambda}$, since both $h(0)$ and $h'(0)$ are real numbers. Therefore λ is a real number.

Fourier Transform
Let two Initial conditions exist:

$$\hat{\phi}(\bar{x}, 0) = f(\bar{x}), \quad \hat{\phi}_{\bar{t}}(\bar{x}, 0) = g(\bar{x})$$

For $\lambda > 0$ we solve the wave equation in (E.12) Let $U(\omega, \bar{t})$ be the Fourier transform of $\hat{\phi}$. Apply a Fourier transform to the equation (E.12). Taking the Fourier transform of the wave equation yields:

$$U_{\bar{t}\bar{t}} = -c^2\omega^2 U.$$

The Fourier transforms of the initial conditions are:

$$U(\omega, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})e^{i\omega\bar{x}} \, dx$$
$$U_{\bar{t}}(\omega, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})e^{i\omega\bar{x}} \, dx.$$

The solution is:

$$U(\omega, \bar{t}) = A(\omega)\cos(c\omega\bar{t}) + B(\omega)\sin(c\omega\bar{t})$$
$$, A(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})e^{i\omega\bar{x}} \, d\bar{x}$$
$$B(w) = \frac{1}{2\pi c\omega}\int_{-\infty}^{\infty} g(\bar{x})e^{i\omega\bar{x}} \, d\bar{x}.$$

After using the inverse Fourier transform:

$$\hat{\phi}(\bar{x},\bar{t}) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{i\omega\bar{x}}\, d\bar{x} \cos(c\omega\bar{t}) e^{-i\omega\bar{x}}\, d\omega + \int_{-\infty}^{\infty} \frac{1}{2\pi c\omega} \int_{-\infty}^{\infty} g(\bar{x}) e^{i\omega\bar{x}}\, d\bar{x} \sin(c\omega t) e^{-i\omega x}\, d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-y)\pi(\delta(y-ct)+\delta(y+ct))\, dy + \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x-y)h(x)$$
$$h(x) = 1 \text{ if } |x| < ct$$
$$\hat{\phi}(\bar{x},\bar{t}) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy.$$

For $\lambda = 0$ the solution of equation (E.13) is :

$$\hat{\phi}(\bar{x},\bar{t}) = a(t).$$

Let $U(\omega,\bar{t})$ be the Fourier transform of $\hat{\phi}$. Apply a Fourier transform to the equation (E.14). Taking the Fourier transform of the wave equation in $\bar{x}$ yields:

$$U_{\bar{t}\bar{t}} = c^2\omega^2 U.$$

The solution is:

$$U(\omega,\bar{t}) = C_1(\omega)e^{-c\omega\bar{t}} + C_2(\omega)e^{c\omega\bar{t}},$$
$$U(\omega,\bar{t}) = D_1(\omega)e^{-c|\omega|\bar{t}}, \quad D_1(\omega) = U(\omega,0).$$

The Fourier inverse of $e^{-c|w|t}$ is $\frac{2ct}{x^2+c^2t^2}$. The solution for equation (E.14) is: $\hat{\phi} = \int_{-\infty}^{\infty}(\frac{2ct}{y^2+c^2t^2})f(x-y)\, dy$.

The final solution is equal to :

$$\phi_0(\bar{x},\xi,\bar{t}) = \int_{0^+}^{\infty} c(\lambda)\left(\frac{f(x-\sqrt{\frac{\mu(\lambda)}{\lambda}}t)+f(x+\sqrt{\frac{\mu(\lambda)}{\lambda}}t)}{2} + \frac{1}{2\sqrt{\frac{\mu(\lambda)}{\lambda}}}\int_{x-\sqrt{\frac{\mu(\lambda)}{\lambda}}t}^{x+\sqrt{\frac{\mu(\lambda)}{\lambda}}t} g(y)\, dy\right)\cosh(\sqrt{\lambda}\frac{h}{L}(\xi+1))\, d\lambda \tag{E.18}$$

$$+\int_{0^+}^{\infty} d(s)\left(\int_{-\infty}^{\infty}\left(\frac{2\sqrt{\frac{\mu(s)}{s}}t}{y^2+\frac{\mu(s)}{s}t^2}\right)f(x-y)\, dy\right)\cos(\sqrt{s}\frac{h}{L}(\xi+1))\, ds. \tag{E.19}$$

Finding the functions $c(\lambda)$ and $d(s)$ is quite complicated.

E.2. Separation of Variables

From chapter 3 we obtain the first order problem. In this section we attempt to solve this first order problem with separation of variables.

The first order PDE is:

$$\phi_{0\bar{x}\bar{x}}(\bar{x},\xi,\bar{t}) = -\frac{d^2}{h^2}\phi_{0\xi\xi}(\bar{x},\xi,\bar{t}), \quad -1<\xi<0 \quad -\infty<\bar{x}<\infty,$$
$$\phi_{0\xi}(\bar{x},\xi,\bar{t}) = 0, \quad \xi = -1,$$
$$\phi_{0\xi\bar{t}\bar{t}} + \frac{1}{12(1-\nu^2)}\phi_{0\bar{x}\bar{x}\bar{x}\bar{x}\xi} + \frac{\rho_w h}{\rho_s d}\phi_{0\bar{t}\bar{t}} = 0, \quad \xi = 0.$$

First we apply separation of variables: $\phi_0(\bar{x},\xi,\bar{t}) = T(\bar{t})\Phi(\bar{x},\xi)$.

The boundary-value problem can now be restated as:

$$\Phi_{\bar{x}\bar{x}} = \frac{-d^2}{h^2}\Phi_{\xi\xi}, \quad -1 < \xi < 0, \quad -\infty < \bar{x} < \infty, \tag{E.20}$$

$$\Phi(\bar{x}, -1) = 0, \tag{E.21}$$

$$\frac{T''(\bar{t})}{T(\bar{t})}\frac{\Phi_\xi}{\Phi} + \frac{1}{12(1-\nu^2)}\frac{\Phi_{\bar{x}\bar{x}\bar{x}\bar{x}\xi}}{\Phi} + \frac{\rho_w h}{\rho_s d}\frac{T''}{T} = 0, \Longrightarrow \tag{E.22}$$

$$\frac{T''}{T} = \frac{-\Phi_{\bar{x}\bar{x}\bar{x}\bar{x}\xi}}{12(1-\nu^2)\Phi\left(\frac{\Phi_\xi}{\Phi} + \frac{\rho_w h}{\rho_s d}\right)} = -\lambda. \tag{E.23}$$

We apply separation of variables again: $\Phi(\bar{x}, \xi) = H(\bar{x})G(\xi)$. Substituting H and G in (E.20 - E.23) yields:

$$\frac{H''(\bar{x})}{H(\bar{x})} = -\frac{d^2}{h^2}\frac{G''(\xi)}{G(\xi)} = -\mu, \Rightarrow H'''' = -\mu H'' = \mu^2 H, \tag{E.24}$$

$$G(-1) = 0, \tag{E.25}$$

$$\lambda = -\frac{-\mu^2 H G'(0)}{12(1-\nu^2)HG(0)\left(\frac{HG'(0)}{HG(0)} + \frac{\rho_w h}{\rho_s d}\right)} = -\frac{-\mu^2 G'(0)}{12(1-\nu^2)G(0)\left(\frac{G'(0)}{G(0)} + \frac{\rho_w h}{\rho_s d}\right)}. \tag{E.26}$$

$$\tag{E.27}$$

The function of G has the form: $G(\xi) = c_1 \cosh(\frac{h\sqrt{\mu}}{d}(\xi+1))$ for $\mu > 0$. Substituting $G(\xi)$ in the boundary value problem yields:

$$G(0) = \frac{1}{2}c_1(e^{\frac{h\sqrt{\mu}}{d}} + e^{-\frac{h\sqrt{\mu}}{d}}), \quad G'(0) = \frac{1}{2}\frac{h\sqrt{\mu}}{d}c_1(e^{\frac{h\sqrt{\mu}}{d}} - e^{-\frac{h\sqrt{\mu}}{d}}), \tag{E.28}$$

$$\frac{G'(0)}{G(0)} = \frac{h\sqrt{\mu}}{d}\frac{e^{2\frac{h\sqrt{\mu}}{d}}+1}{e^{2\frac{h\sqrt{\mu}}{d}}-1}, \tag{E.29}$$

$$\lambda = \frac{\mu^2}{12(1-\nu^2)\left(\frac{h\sqrt{\mu}}{d}\frac{e^{2\frac{h\sqrt{\mu}}{d}}+1}{e^{2\frac{h\sqrt{\mu}}{d}}-1} + \frac{\rho_w h}{\rho_s d}\right)}\frac{h\sqrt{\mu}}{d}\frac{e^{2\frac{h\sqrt{\mu}}{d}}+1}{e^{2\frac{h\sqrt{\mu}}{d}}-1}. \tag{E.30}$$

The function H has the shape $H(\bar{x}) = c_2 \sin(\sqrt{\mu}\bar{x}) + c_3\cos(\sqrt{\mu}\bar{x})$ for $\mu > 0$. The function $T(\bar{x}) = c_4 \sin(\sqrt{\lambda}\bar{t}) + c_5\cos(\sqrt{\lambda}\bar{t})$.

When one applies superposition to the solution for positive values of μ:

$$\phi_0 = \int_0^\infty \cosh\left(\frac{h\sqrt{\mu}}{d}(\xi+1)\right)(c_2(\mu)\sin(\sqrt{\mu}\bar{x}) + c_3(\mu)\cos(\sqrt{\mu}\bar{x}) \tag{E.31}$$

$$(c_4(\mu)\sin\left(\sqrt{\left(\frac{\mu^2}{12(1-\nu^2)\left(\frac{h\sqrt{\mu}}{d}\frac{e^{2\frac{h\sqrt{\mu}}{d}}+1}{e^{2\frac{h\sqrt{\mu}}{d}}-1} + \frac{\rho_w h}{\rho_s d}\right)}\frac{h\sqrt{\mu}}{d}\frac{e^{2\frac{h\sqrt{\mu}}{d}}+1}{e^{2\frac{h\sqrt{\mu}}{d}}-1}\right)}\bar{t}\right) + \tag{E.32}$$

$$c_5(\mu)\cos\left(\sqrt{\left(\frac{\mu^2}{12(1-\nu^2)\left(\frac{h\sqrt{\mu}}{d}\frac{e^{2\frac{h\sqrt{\mu}}{d}}+1}{e^{2\frac{h\sqrt{\mu}}{d}}-1} + \frac{\rho_w h}{\rho_s d}\right)}\frac{h\sqrt{\mu}}{d}\frac{e^{2\frac{h\sqrt{\mu}}{d}}+1}{e^{2\frac{h\sqrt{\mu}}{d}}-1}\right)}\bar{t}\right))\,d\mu. \tag{E.33}$$

Here the c_2,c_3, c_4 and c_5 are unknown functions of μ. Because λ depends on μ as a function with an exponential component, we cannot solve the superposition integral.