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ORIGINAL PAPER

Generalized fractional operators do not preserve periodicity

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Abstract

This work allows proving that the action of fractional derivatives and fractional integrals on periodic functions does not preserve the periodicity of any period. This result is proved not only for one type of fractional operator but also for the wide class of generalized fractional operators based on the Sonine condition, a class that encompasses the majority of the fractional operators commonly used. Moreover, for several specific fractional operators, we provide explicit representations of the derivatives and integrals of the sine function, showing that they are composed of a local periodic term and a non-local term, which is the cause of the loss of periodicity.

Keywords Fractional calculus · Generalized fractional derivatives · Generalized fractional integrals · Periodic function · Special functions

Mathematics Subject Classification 26A33 · 44A35 · 33E12

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1 Introduction

Fractional derivatives are nowadays used in the modeling of many physical phenomena related, for example, to diffusion [18, 40], viscoelasticity and relaxation [12, 15, 31], wave behavior [23, 29], frequency change [36, 37], and so on. The description of processes showing some kind of persistent memory, non-locality, or anomalous behavior is one of the main reasons for introducing fractional derivatives in a variety of models.

A peculiar characteristic of fractional-order operators is that most of the properties of classical integer-order operators no longer hold.

One such property is the preservation of periodicity. Periodicity is well-known to play an important role in the Fourier transform and in various contexts where wave behavior is observed. For instance, this includes applications in medicine, such as ultrasonography [39], and navigation systems such as the Global Positioning System (GPS) [42], among others. Periodic boundary conditions are often employed to solve integro-differential equations using methods such as integral transforms [9, 10, 33] or Green's functions [6, 32, 35], which are widely applied in both classical and quantum mechanics.

Some years ago, Tavazoei [43] showed that fractional derivatives of periodic functions with a given period cannot be periodic functions with the same period. Subsequently, the non-existence of periodic solutions to fractional differential equations was proved by Kaslik and Sivasundaram [19]. Similar results have also been obtained by different approaches in subsequent studies [4, 46]. These findings are concerned with the Caputo, Riemann-Liouville, and Grünwald-Letnikov definitions of fractional derivatives, which were among the few definitions widely used in practice at the time these papers were published.

In recent years, with the aim of describing specific phenomena with greater accuracy, a variety of different fractional-order operators have been introduced. Consequently, and in light of the introduction of some questionable operators [8], several authors have made efforts to develop a general theory to properly characterize new operators.

A general theory of fractional derivatives and fractional integrals was established in the seminal paper by Kochubei [21] and has been further developed by Hanyga, Kochubei, Luchko, and others [17, 22, 25–27, 30]. This theory provides the basis for defining a broad class of fractional derivatives and integrals based on convolution integrals with Sonine kernels, namely, pairs of kernels satisfying the Sonine condition.

It is natural to go beyond the analysis in [19, 43] and face the question of whether the loss of periodicity is common to all fractional derivatives and integrals, defined within the framework of this more general theory. The aim of this work is therefore to further explore operators built on Sonine pairs and, in particular, to show that their action on periodic functions always leads to non-periodic results.

Therefore, after reviewing the main information about general fractional integrals and derivatives in section 2, we provide in Section 3 the main results concerning the non-periodicity, of any period, of the general fractional integral and derivative of a periodic function. Hence, to provide some explanatory examples, for some generalized pairs of operators (namely, Caputo derivative with Riemann-Liouville integral, Prabhakar derivative and integral, distributed-order derivative and integral) in Section 4 we study in detail the action on the sine function (the analysis can be easily extended to other periodic functions) and we provide a representation in terms of a local and periodic term plus a non-local term which is the source for the lack of periodicity. Some concluding remarks, together with a discussion about future investigations, are given in Section 5. An Appendix at the end of the paper collects definitions and results about some special functions which are used throughout the paper.

2 General fractional integrals and derivatives: a brief review

Before introducing the main results of this work it is useful to briefly review the theory of general fractional integrals and derivatives. Throughout this section, we will refer to [21, 22, 24–28, 30] as main references, without further mentioning them.

Definition 1 Two functions $h, k \in L^1(0, +\infty)$ are said to be Sonine kernels, and they form a Sonine pair, if they satisfy the Sonine condition

$$\int_0^t h(t-\xi)k(\xi)\,\mathrm{d}\xi = \int_0^t h(\xi)k(t-\xi)\,\mathrm{d}\xi = 1, \quad \forall t > 0. \tag{2.1}$$

Well-known examples of Sonine pairs satisfying (2.1) are power kernels defining the usual Riemann-Liouville integral and the Caputo derivative of order $0 < \alpha < 1$ (their characterization as Sonine kernels will be presented later).

Definition 2 Let $h, k \in L^1(0, +\infty)$ be a Sonine pair. The general fractional integral (GFI), and the corresponding general fractional derivative (GFD) regularized in the Caputo-style, are defined respectively as

$$\big(\mathbb{I}_{(h)} f\big)(t) = \int_0^t h(t-\xi) f(\xi)\,\mathrm{d}\xi \tag{2.2}$$

$$\big({}^{\mathrm{C}}\mathbb{D}_{(k)} f\big)(t) = \int_0^t k(t-\xi)\partial_\xi f(\xi)\,\mathrm{d}\xi, \tag{2.3}$$

where $f(t)$ belongs to the domain of the corresponding operator.

Among Sonine kernels $k(t)$, those that adhere to the following assumptions are of particular interest in defining operators that can be interpreted as fractional derivatives:

H1: the LT $K(s)$ of $k(t)$ exists for any $s > 0$, i.e.

$$K(s) = \mathcal{L}\Big(k(t)\,;\, s\Big) = \int_0^{+\infty} \mathrm{e}^{-st} k(t)\,\mathrm{d}t, \quad \forall s > 0;$$

H2: $K(s) \to 0$ and $sK(s) \to +\infty$ as $s \to +\infty$;
H3: $K(s) \to +\infty$ and $sK(s) \to 0$ as $s \to 0$.

A consequence of assumptions **H1**, **H2** and **H3** is that the kernel $k(t)$ must be a weakly singular function. Moreover, these assumptions ensure the existence of a unique solution for Cauchy problems with GFD [21], and thus we adopt them throughout the paper.

Remark 1 In [21] a fourth assumption was also imposed, requiring $K(s)$ to be a Stieltjes function. This assumption has physical importance and ensures that the function $k(t)$ is a (complete monotone) Sonine kernel and that there exists its associated Sonine kernel $h(t)$ that is also complete monotone.

Whenever h and k are Sonine pairs, the corresponding operators satisfy a kind of generalized fundamental theorem of calculus

$$\big({}^{\mathrm{C}}\mathbb{D}_{(k)}\big(\mathbb{I}_{(h)} f\big)\big)(t) = f(t), \quad \big(\mathbb{I}_{(h)}\big({}^{\mathrm{C}}\mathbb{D}_{(k)} f\big)\big)(t) = f(t) - f(0),$$

that justifies naming ${}^{\mathrm{C}}\mathbb{D}_{(k)}$ as a derivative associated with the integral $\mathbb{I}_{(h)}$.

Remark 2 Usually, GFDs are defined in terms of derivatives of Riemann-Liouville type $\big(\mathbb{D}_{(k)} f\big) = \partial_t \int_0^t k(t-\xi) f(\xi)\,\mathrm{d}\xi$ and hence regularized according to $\big(\mathbb{D}_{(k)} f\big)(t) - k(t)f(0)$ which, under suitable assumptions for f, is equivalent to $\big({}^{\mathrm{C}}\mathbb{D}_{(k)} f\big)(t)$. For simplicity and in view of their major applications, we prefer to introduce regularized derivatives directly in the form (2.3).

One of the main tasks to define GFIs and GFDs is to characterize Sonine kernels h and k since finding functions satisfying (2.1) is not easy. From a practical point of view this task is simplified in the Laplace transform (LT) domain where the Sonine condition (2.1) becomes $H(s)K(s) = 1/s$, with $H(s)$ and $K(s)$, respectively, denoting the LTs of $h(t)$ and $k(t)$.

The extension of generalized integrals and derivatives to higher orders requires a Sonine condition more general than (2.1). To this end we must preliminarily introduce the function spaces [28]

$$\begin{aligned}
&C_{-1}(0,\infty) := \{f : f(t) = t^p f_1(t),\ t > 0,\ p > -1,\ f_1 \in C[0,\infty)\},\\
&C_{-1,0}(0,\infty) := \{f : f(t) = t^p f_1(t),\ t > 0,\ p \in (-1,0),\ f_1 \in C[0,\infty),\ f_1(0) \neq 0\}.
\end{aligned}$$

Definition 3 Let $m \in \mathbb{N}$. Two functions $h, k_m : (0,\infty) \to \mathbb{R}$ such that $h \in C_{-1}(0,\infty)$ and $k_m \in C_{-1,0}(0,\infty)$ are said to be Sonine functions of order m, and they form a Sonine pair of order m, if they satisfy the generalized Sonine condition

$$\int_0^t h(t-\xi)k_m(\xi)\,\mathrm{d}\xi = \int_0^t h(\xi)k_m(t-\xi)\,\mathrm{d}\xi = \frac{t^{m-1}}{(m-1)!}, \quad \forall t > 0. \tag{2.4}$$

We must observe that the term "of order m" for Sonine functions and Sonine pairs is not the same used in the literature where, instead, a specific space denoted as $\mathcal{L}_m$ is introduced to characterize kernels [24]. We have used a different terminology, together with the notation h and k_m, in order to avoid a rigid formalism which is not strictly necessary for this work.

In the LT domain, the generalized Sonine condition (2.4) reads

$$H(s)K_m(s) = \frac{1}{s^m}. \tag{2.5}$$

Based on Theorem 3 from [24], we may consider $k_m = \{1\}^{m-1} * k$, where k is a Sonine function in the sense of Definition 1. Here $\{1\}$ is the function identically equal to 1 for $t \geq 0$, and $\{1\}^{m-1}$ denotes its $(m-1)$-fold convolution with itself, namely $\{1\}^{m-1}(t) = t^{m-2}/(m-2)!$, and due to the associativity of convolution, we have:

$$k_m(t) = \big(\{1\}^{m-1} * k\big)(t) = \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-2}} k(\tau_{m-1})\, \mathrm{d}\tau_{m-1} \cdots \mathrm{d}\tau_2\, \mathrm{d}\tau_1.$$

Additionally, it is important to note that the Sonine pair of order m of the function k_m is, in fact, the Sonine pair of order 1 of the function k. Indeed, if h is the Sonine pair of order 1 of k then $k*h = \{1\}$, and hence $k_m*h = (\{1\}^{m-1}*k)*h = \{1\}^{m-1}*\{1\} = \{1\}^m$, so h is the Sonine pair of order m of k_m.

Thanks to the generalized Sonine condition (2.4) it is now possible to introduce generalized operators of arbitrary order.

Definition 4 Let h, k_m be a Sonine pair of order m. The GFI, and the corresponding GFD regularized in the Caputo-style, are defined respectively as

$$\big(\mathbb{I}_{(h)} f\big)(t) = \int_0^t h(t-\xi) f(\xi)\, \mathrm{d}\xi \tag{2.6}$$

$$\big({}^{\mathrm{C}}\mathbb{D}^m_{(k)} f\big)(t) = \int_0^t k_m(t-\xi)\partial_\xi^{(m)} f(\xi)\, \mathrm{d}\xi, \tag{2.7}$$

where $f(t)$ belongs to the domain of the corresponding operator.

We prefer to explicitly mention the order m in the notation ${}^{\mathrm{C}}\mathbb{D}^m_{(k)}$ of the derivative since the m-th order derivative of the function is involved. The same is not necessary for the notation $\mathbb{I}_{(h)}$ of the integral.

In addition, generalized operators (2.6) and (2.7) allow the realization of a generalized fundamental theorem of calculus in the form

$$\big({}^{\mathrm{C}}\mathbb{D}^m_{(k)}\big(\mathbb{I}_{(h)} f\big)\big)(t) = f(t), \quad \big(\mathbb{I}_{(h)}\big({}^{\mathrm{C}}\mathbb{D}^m_{(k)} f\big)\big)(t) = f(t) - \sum_{j=0}^{m-1} \frac{t^j}{j!} f^{(j)}(0).$$

In the following, we recall a few examples related to some of the most used fractional operators that can be framed in the above theory of GFIs and GFDs.

2.1 Riemann-Liouville integral and Caputo derivative

The most known (and maybe the most used) operators in fractional calculus are the Riemann-Liouville integral and the Caputo derivative of order $\alpha > 0$ [20] whose corresponding kernels are the power functions

$$h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad k_m(t) = \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)},$$

where $m \in \mathbb{N}$ such that $\alpha \in (m-1, m)$. Their LTs are

$$H(s) = \mathcal{L}\Big(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\,;\, s\Big) = \frac{1}{s^{\alpha}}, \quad K_m(s) = \mathcal{L}\Big(\frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)}\,;\, s\Big) = \frac{1}{s^{m-\alpha}},$$

which clearly satisfy the generalized Sonine condition (2.5). When referring to these specific operators, we will use the more traditional notations

$$\big(I^{\alpha} f\big)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi,$$
$$\big({}^{\mathrm{C}}D^{\alpha} f\big)(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} \partial_{\xi}^{(m)} f(\xi)\,\mathrm{d}\xi.$$

To lighten the notation, here and in the remainder, we omit the initial point in the symbols for the operators, and the initial point must always be intended placed at $t = 0$.

2.2 Prabhakar integrals and derivatives

The next example concerns the so-called Prabhakar integral and derivative [13], a couple of operators based on the Prabhakar function (A.6). Given $\alpha, \gamma, \lambda > 0$, and $\beta \in (m-1, m)$, with $m \in \mathbb{N}$, operators of Prabhakar type are defined by means of the following kernels

$$h(t) = t^{\beta-1} E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha}), \quad k_m(t) = t^{m-\beta-1} E^{-\gamma}_{\alpha,m-\beta}(-\lambda t^{\alpha}),$$

with corresponding LT

$$H(s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}+\lambda)^{\gamma}} = \frac{1}{s^{\beta}}\Big(1+\frac{\lambda}{s^{\alpha}}\Big)^{-\gamma}, \quad K_m(s) = \frac{s^{-\alpha\gamma-m+\beta}}{(s^{\alpha}+\lambda)^{-\gamma}} = \frac{1}{s^{m-\beta}}\Big(1+\frac{\lambda}{s^{\alpha}}\Big)^{\gamma}$$

(note that when $\lambda = 0$ or $\gamma = 0$ these are the kernels of the classical Riemann-Liouville integral and Caputo derivative of order β). Prabhakar integral and derivative will be denoted in this paper respectively as

$$\big({}^{\mathrm{C}}\mathcal{D}^{\beta}_{\alpha,\gamma,\lambda} f\big)(t) = \int_0^t (t-\xi)^{m-\beta-1} E^{-\gamma}_{\alpha,m-\beta}(-\lambda(t-\xi)^{\alpha}) f^{(m)}(\xi)\,\mathrm{d}\xi,$$
$$\big(\mathcal{I}^{\beta}_{\alpha,\gamma,\lambda} f\big)(t) = \int_0^t (t-\xi)^{\beta-1} E^{\gamma}_{\alpha,\beta}(-\lambda(t-\xi)^{\alpha}) f(\xi)\,\mathrm{d}\xi$$

2.3 Integrals and derivatives of distributed fractional order

The last example is given by the derivative of distributed order

$$\left({}^{\mathrm{D}}D_0^m f\right)(t) = \int_{m-1}^{m} \left({}^{\mathrm{C}}D^{\alpha} f\right)(t)\,\mathrm{d}\alpha = \int_0^t k_D(t-\xi) f^{(m)}(\xi)\,\mathrm{d}\xi$$

with the kernel k_D given by

$$k_D(t) = \int_{m-1}^{m} \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)}\,\mathrm{d}\alpha = \int_0^1 \frac{t^{\beta-1}}{\Gamma(\beta)}\,\mathrm{d}\beta$$

and where, to obtain the last formulation of $k_D(t)$, we made the change of variable $\beta = m - \alpha$. The LT of $k_D(t)$ is hence

$$K_D(s) = \mathcal{L}\Big(k_D(t)\,;\, s\Big) = \frac{s-1}{s\ln s} \tag{2.8}$$

and hence one readily obtain the kernel for the corresponding integral

$$H_D(s) = \frac{\ln s}{s-1} \quad \text{and} \quad h_D(t) = -\mathrm{e}^t \mathrm{Ei}(-t).$$

3 Main results

In this section we present the main results of the paper showing that the GFD, as well as the GFI, of a periodic function cannot preserve periodicity.

We first remind the following standard result together with a sketch of the proof.

Lemma 1 *Let $n \in \mathbb{N}$ and $T > 0$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a non-constant T-periodic function that belongs to $C^n(\mathbb{R})$. Then for any $j \in \{0, 1, 2, \ldots, n\}$, the j-th derivative $f^{(j)}(t)$ is also a non-constant T-periodic function.*

Proof Starting from the fact that f is T-periodic and repeatedly differentiating both sides of the equation $f(t+T) = f(t)$ with respect to t, relying on the chain rule, we conclude that all derivatives up to order n are T-periodic. If one of the derivatives were constant, the function would reduce to a polynomial, which would contradict the periodicity assumption. Hence, all derivatives are non-constant. □

In what follows, we present the main result of this section.

Theorem 1 *Assume that k is a Sonine function (of order 1), whose Laplace transform K satisfies conditions **H1-H3**. Then, the convolution of k with a non-constant bounded periodic function is not a periodic function.*

Proof We proceed by reductio ad absurdum. Let us assume that the non-constant function f is T-periodic, with $T > 0$, and that the convolution $g(t) := (k * f)(t)$ is a T'-periodic function, with $T' > 0$. Applying the Laplace transform, we obtain:

$$G(s) = K(s)F(s) \tag{3.1}$$

where G, K and F are respectively, the Laplace transforms of the functions g, k and f.

As the function f and g are T-periodic and T'-periodic, respectively, we have:

$$F(s) = \frac{\int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t}{1 - \mathrm{e}^{-sT}} \quad \text{and} \quad G(s) = \frac{\int_0^{T'} \mathrm{e}^{-st} g(t)\,\mathrm{d}t}{1 - \mathrm{e}^{-sT'}},$$

and therefore, equation (3.1) becomes:

$$L(s) \int_0^{T'} \mathrm{e}^{-st} g(t)\,\mathrm{d}t = K(s) \int_0^{T} \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \tag{3.2}$$

where

$$L(s) = \frac{1 - \mathrm{e}^{-sT}}{1 - \mathrm{e}^{-sT'}} \to \frac{T}{T'} \quad \text{as } s \to 0_+.$$

Let us consider the Sonine pair h of the kernel k, and denote by H its Laplace transform. Hence, $K(s)H(s) = s^{-1}$, and equality (3.2) becomes:

$$sH(s)L(s) \int_0^{T'} \mathrm{e}^{-st} g(t)\,\mathrm{d}t = \int_0^{T} \mathrm{e}^{-st} f(t)\,\mathrm{d}t. \tag{3.3}$$

Based on condition **H3**, we have

$$\lim_{s\to 0_+} sH(s) = \lim_{s\to 0_+} \frac{1}{K(s)} = 0.$$

Therefore, taking the limit as $s \to 0_+$ in (3.3), it follows that

$$\int_0^T f(t)\,\mathrm{d}t = 0,$$

and hence, equation (3.3) can be rewritten as

$$sH(s)L(s) \int_0^{T'} \mathrm{e}^{-st} g(t)\,\mathrm{d}t = \int_0^{T} (\mathrm{e}^{-st} - 1) f(t)\,\mathrm{d}t.$$

Again, using $K(s)H(s) = s^{-1}$, it follows that

$$L(s)\int_0^{T'} \mathrm{e}^{-st}\, g(t)\,\mathrm{d}t = sK(s)\int_0^{T} \frac{\mathrm{e}^{-st}-1}{s} f(t)\,\mathrm{d}t,$$

and taking the limit as $s \to 0_+$ we get

$$\int_0^{T'} g(t)\,\mathrm{d}t = 0.$$

We will now prove by mathematical induction that for any $n \in \mathbb{Z}^+$, we have

$$\int_0^{T} t^n f(t)\,\mathrm{d}t = 0 \quad \text{and} \quad \int_0^{T'} t^n g(t)\,\mathrm{d}t = 0. \tag{3.4}$$

Indeed, (3.4) is true for $n = 0$, as seen above. Now let us consider $n \in \mathbb{Z}^+$, and let us assume that (3.4) is true for any $p \leq n$. Let us denote by $P_n(x) = \sum_{p=0}^{n} \frac{x^p}{p!}$ the n-th order Taylor polynomial of e^x centered at 0. Therefore, based on the induction hypothesis, we can see that

$$\int_0^{T'} \mathrm{e}^{-st}\, g(t)\,\mathrm{d}t = \int_0^{T'} \Big(\mathrm{e}^{-st} - P_n(-st)\Big) g(t)\,\mathrm{d}t,$$

and a similar equality holds for f as well. Hence, we can write (3.3) as

$$sH(s)L(s)\int_0^{T'} \Big(\mathrm{e}^{-st} - P_n(-st)\Big) g(t)\,\mathrm{d}t = \int_0^{T} \Big(\mathrm{e}^{-st} - P_n(-st)\Big) f(t)\,\mathrm{d}t$$

and dividing by s^{n+1}, we obtain:

$$sH(s)L(s)\int_0^{T'} \frac{\mathrm{e}^{-st} - P_n(-st)}{s^{n+1}} g(t)\,\mathrm{d}t = \int_0^{T} \frac{\mathrm{e}^{-st} - P_n(-st)}{s^{n+1}} f(t)\,\mathrm{d}t. \tag{3.5}$$

Since $sH(s) \to 0$ as $s \to 0_+$ and

$$\lim_{s\to 0_+} \frac{\mathrm{e}^{-st} - P_n(-st)}{s^{n+1}} = \frac{(-1)^{n+1}}{(n+1)!} t^{n+1},$$

taking the limit as $s \to 0_+$ in (3.5) leads to

$$\int_0^{T} t^{n+1} f(t)\,\mathrm{d}t = 0.$$

This means that, making use of $K(s)H(s) = s^{-1}$, we can rewrite equation (3.5) as

$$L(s)\int_0^{T'} \frac{\mathrm{e}^{-st} - P_n(-st)}{s^{n+1}} g(t)\,\mathrm{d}t = sK(s)\int_0^{T} \frac{\mathrm{e}^{-st} - P_{n+1}(-st)}{s^{n+2}} f(t)\,\mathrm{d}t,$$

and yet again, taking the limit at $s \to 0_+$, we finally obtain

$$\int_0^{T'} t^{n+1} g(t)\,\mathrm{d}t = 0.$$

Therefore, (3.4) is true for any $n \in \mathbb{Z}^+$, and hence $f = g = 0$, which contradicts the hypothesis of the theorem. This completes the proof. □

Corollary 1 *Assume that k is a Sonine function of order 1, whose Laplace transform satisfies conditions **H1-H3**. Then, the convolution of the Sonine pair h of the function k with a non-constant bounded periodic function is not a periodic function.*

Proof It is easy to see that if the Laplace transform of the Sonine function k satisfies conditions **H1-H3**, so does the Laplace transform of its Sonine pair h. Hence, the proof follows as a direct consequence of Theorem 1. □

Corollary 2 *Assume that $k_m = \{1\}^{m-1} * k$, where k is a Sonine function (of order 1), satisfying conditions **H1-H3**. Then, the convolution of k_m with a non-constant bounded periodic function is not a periodic function.*

Proof Let us assume that f is a non-constant bounded periodic function such that $g_m = k_m * f$ is periodic. It can be easily seen that

$$g_m = k_m * f = (\{1\}^{m-1} * k) * f = \{1\}^{m-1} * (k * f),$$

and hence,

$$(k * f)(t) = (\delta^{(m-1)} * g_m)(t) = g_m^{(m-1)}(t).$$

As k is a Sonine function of order 1 satisfying the hypotheses of Theorem 1, it is clear that $k * f$ cannot be periodic, and hence, we have reached a contradiction. □

Corollary 3 *Consider the Sonine kernel $k_m = \{1\}^{m-1} * k$, where k is a Sonine function of order 1, which satisfies conditions **H1-H3**. Let h be its Sonine pair.*

(a) The action of the GFD with Sonine kernel k_m, on a non-constant periodic function $f(t)$ with a bounded derivative of order m, is not a periodic function.
(b) The action of the GFI with Sonine kernel h, on a non-constant bounded periodic function is not a periodic function.

It is useful to mention that Corollary 3 holds for Caputo, Riemann-Liouville and Grünwald-Letnikov definitions which are just specific cases. With respect to the one presented in [19, 43], the outcome of Corollary 3 is more general not only because it applies to a wider range operators, but also because it excludes periodicity of any period of derivatives of periodic functions.

4 Some examples with the sin function

As illustrative examples we study in more detail the action of different fractional derivatives, all of which are particular cases of the generalized fractional derivative (2.7), on the sine function. A similar reasoning can be applied for the cosine function or any other combination of these and other periodic functions.

We start by observing that, for any integer order $m = 0, 1, \ldots,$ derivatives of $\sin t$ can be represented as

$$\frac{d^m}{dt^m} \sin t = \sin\left(\frac{m\pi}{2} + t\right) = \left(\cos \frac{m\pi}{2}\right) \sin t + \left(\sin \frac{m\pi}{2}\right) \cos t, \tag{4.1}$$

and the corresponding LTs are

$$\mathcal{L}\left(\frac{d^m}{dt^m} \sin t\,;\, s\right) = \left(\cos \frac{m\pi}{2}\right) \frac{1}{s^2+1} + \left(\sin \frac{m\pi}{2}\right) \frac{s}{s^2+1}. \tag{4.2}$$

The action on this function of a GFD (2.7) with kernel $k_m(t)$, namely

$$\begin{aligned}\left({}^{\mathrm{C}}\mathbb{D}^m_{(k)} sin\right)(t) &= \int_0^t k_m(t-\xi) \frac{d^m}{d\xi^m} \sin \xi \, d\xi \\ &= \left(\cos \frac{m\pi}{2}\right) \int_0^t k_m(t-\xi) \sin \xi \, d\xi + \left(\sin \frac{m\pi}{2}\right) \int_0^t k_m(t-\xi) \cos \xi \, d\xi,\end{aligned}$$

can be therefore more conveniently represented in the LT domain

$$\mathcal{L}\left(\left({}^{\mathrm{C}}\mathbb{D}^m_{(k)} \sin\right)(t)\,;\, s\right) = \left(\cos \frac{m\pi}{2}\right) \frac{K_m(s)}{s^2+1} + \left(\sin \frac{m\pi}{2}\right) \frac{s K_m(s)}{s^2+1}, \tag{4.3}$$

and, after inversion of the LT, one obtains

$$\left({}^{\mathrm{C}}\mathbb{D}^m_{(k)} \sin\right)(t) = \cos \frac{m\pi}{2} \mathcal{L}^{-1}\left(\frac{K_m(s)}{s^2+1}\,;\, t\right) + \sin \frac{m\pi}{2} \mathcal{L}^{-1}\left(\frac{s K_m(s)}{s^2+1}\,;\, t\right). \tag{4.4}$$

To simplify the above representation, we will employ the following formulas, which hold for any integer m

$$\sin \frac{m\pi}{2} = \begin{cases} 0 & m \text{ even} \\ (-1)^{\frac{m-1}{2}} & m \text{ odd} \end{cases}, \quad \cos \frac{m\pi}{2} = \begin{cases} (-1)^{\frac{m}{2}} & m \text{ even} \\ 0 & m \text{ odd} \end{cases}. \tag{4.5}$$

The same reasoning allows us to describe the action of the GFI (2.6) as

$$\left(\mathbb{I}_{(h)} \sin\right)(t) = \mathcal{L}^{-1}\left(\frac{H(s)}{s^2+1}\,;\, t\right).$$

A more detailed analysis can be provided only after characterizing the kernels $k_m(t)$ and $h(t)$ and the corresponding operators.

4.1 Caputo fractional derivative of the sin function

Since the LT of the kernel of the Caputo fractional derivative is $K_m(s) = s^{\alpha-m}$, by using (4.4) we obtain

$$\begin{aligned}\left({}^{\mathrm{C}}D^{\alpha}\sin\right)(t) &= \left(\cos\frac{m\pi}{2}\right)\mathcal{L}^{-1}\left(\frac{s^{\alpha-m}}{s^2+1}\,;\,t\right) + \left(\sin\frac{m\pi}{2}\right)\mathcal{L}^{-1}\left(\frac{s^{\alpha-m+1}}{s^2+1}\,;\,t\right)\\ &= \left(\cos\frac{m\pi}{2}\right)t^{1+m-\alpha}E_{2,2+m-\alpha}(-t^2) + \left(\sin\frac{m\pi}{2}\right)t^{m-\alpha}E_{2,1+m-\alpha}(-t^2),\end{aligned}$$

where we have exploited Eq. (A.2) for the LT of the ML function defined in (A.1). In view of Proposition 1 we can represent these instances of the ML function thanks to Eq. (A.3) to obtain (after using elementary trigonometric identities)

$$\begin{aligned}&\left({}^{\mathrm{C}}D^{\alpha}\sin\right)(t) =\\ &\quad= \left(\cos\frac{m\pi}{2}\right)\cos\left(t+(\alpha-m)\frac{\pi}{2}-\frac{\pi}{2}\right) + \left(\sin\frac{m\pi}{2}\right)\cos\left(t+(\alpha-m)\frac{\pi}{2}\right)\\ &\qquad+ \left(\cos\frac{m\pi}{2}\right)\varphi_{2,2+m-\alpha}(t) + \left(\sin\frac{m\pi}{2}\right)\varphi_{2,1+m-\alpha}(t)\\ &\quad= \sin\left(t+\alpha\frac{\pi}{2}\right) + \left(\cos\frac{m\pi}{2}\right)\varphi_{2,2+m-\alpha}(t) + \left(\sin\frac{m\pi}{2}\right)\varphi_{2,1+m-\alpha}(t)\end{aligned}$$

where functions $\varphi_{2,\beta}(t)$ are defined by (A.4). Therefore, by using (4.5) we have

$$\left({}^{\mathrm{C}}D^{\alpha}\sin\right)(t) = \begin{cases}\sin\left(t+\frac{\alpha\pi}{2}\right) + (-1)^{\frac{m}{2}}\varphi_{2,2+m-\alpha}(t) & m \text{ even}\\ \sin\left(t+\frac{\alpha\pi}{2}\right) + (-1)^{\frac{m-1}{2}}\varphi_{2,1+m-\alpha}(t) & m \text{ odd}\end{cases}. \tag{4.6}$$

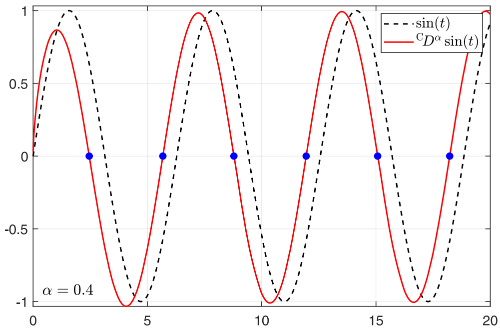
A similar reasoning allows to compute the RL integral of the sin function in the form

$$\left(I_0^{\alpha}\sin\right)(t) = \sin\left(t-\frac{\alpha\pi}{2}\right)+\varphi_{2,2+\alpha}(t). \tag{4.7}$$

The non-periodic character of (4.6) and (4.7) can be inferred from the nature of the functions $\varphi_{2,\beta}$. Indeed, from Eq. (A.4) we notice that $\varphi_{2,\beta}$ is the inverse LT of a positive function and, in view of the Bernstein theorem [41, Theorem 1.4], it is completely monotone, and hence it is positive and monotonically decreases for $t \geq 0$, thus preventing $\left({}^{\mathrm{C}}D^{\alpha}\sin\right)(t)$ and $\left(I_0^{\alpha}\sin\right)(t)$ from being periodic. Moreover, from (A.4) one immediately observes that, as expected, the terms $\varphi_{2,2+m-\alpha}(t)$ and $\varphi_{2,1+m-\alpha}(t)$ in (4.6) vanish when α is an integer, namely when $m = \alpha$.

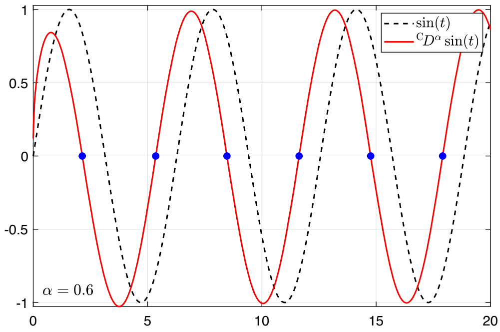
Figures 1 and 2 show for $\alpha = 0.4$ and $\alpha = 0.6$, respectively, the comparison between ${}^{\mathrm{C}}D^{\alpha}\sin(t)$ and $\sin(t)$. To better highlight non-periodicity of ${}^{\mathrm{C}}D^{\alpha}\sin(t)$, their first zeros z_k (evaluated numerically) are presented in the nearby tables, together with distances between consecutive zeros. The nonuniform distribution of its zeros confirms the lack of periodicity of ${}^{\mathrm{C}}D^{\alpha}\sin(t)$.

Remark 3 The representations (4.6) and (4.7) for $\left({}^{\mathrm{C}}D^{\alpha}\sin\right)(t)$ and $\left(I_0^{\alpha}\sin\right)(t)$ are also interesting for a different reason. They show that the fractional derivative and



z_k	$z_k - z_{k-1}$
2.4550	
5.6767	3.222
8.7841	3.107
11.9461	3.162
15.0738	3.128
18.2258	3.152

Fig. 1 Comparison between ${}^{C}D^{\alpha}\sin(t)$ (red solid line) and $\sin(t)$ (gray dashed line) for $\alpha = 0.4$ ($m = 1$) and first few zeros of ${}^{C}D^{\alpha}\sin(t)$ with their distance.



z_k	$z_k - z_{k-1}$
2.1444	
5.3573	3.213
8.4739	3.117
11.6291	3.155
14.7619	3.133
17.9098	3.148

Fig. 2 Comparison between ${}^{C}D^{\alpha}\sin(t)$ (red solid line) and $\sin(t)$ (gray dashed line) for $\alpha = 0.6$ ($m = 1$) and first few zeros of ${}^{C}D^{\alpha}\sin(t)$ with their distance.

integral of $\sin t$ are made by a local term (which, in some sense, follows the same rules of integer-order derivation and integration) plus an extra term which takes into account the nonlocality of the operator. As we will see, this feature is common to other operators.

4.2 Prabhakar derivative of the sin function

To evaluate the action of the Prabhakar derivative it is convenient to expand the LT of the kernel by means of the binomial series

$$K_m(s) = \frac{1}{s^{m-\beta}}\left(1 + \frac{\lambda}{s^{\alpha}}\right)^{\gamma} = \sum_{r=0}^{\infty}\binom{\gamma}{r}\lambda^r s^{\beta-m-\alpha r} \tag{4.8}$$

where we assumed $|s^\alpha| > \lambda$. The substitution of Eq. (4.8) into Eq. (4.4) gives

$$\left({}^{\mathrm{C}}\mathcal{D}^{\beta}_{\alpha,\gamma,\lambda} \sin\right)(t) = \sum_{r=0}^{\infty} \binom{\gamma}{r} \lambda^r \left(\cos \frac{m\pi}{2}\right) \mathcal{L}^{-1}\left[\frac{s^{\beta-m-\alpha r}}{1+s^2}; t\right] + \sum_{r=0}^{\infty} \binom{\gamma}{r} \lambda^r \left(\sin \frac{m\pi}{2}\right) \mathcal{L}^{-1}\left[\frac{s^{1+\beta-m-\alpha r}}{1+s^2}; t\right],$$

and the inversion of the LTs, by means of Eq. (A.2), allows to represent the above derivative in terms of ML functions (A.1) according to

$$\left({}^{\mathrm{C}}\mathcal{D}^{\beta}_{\alpha,\gamma,\lambda} \sin\right)(t) = \sum_{r=0}^{\infty} \binom{\gamma}{r} \lambda^r t^{m+\alpha r-\beta+1} \left(\cos \frac{m\pi}{2}\right) E_{2,2+m+\alpha r-\beta}(-t^2) + \sum_{r=0}^{\infty} \binom{\gamma}{r} \lambda^r t^{m+\alpha r-\beta} \left(\sin \frac{m\pi}{2}\right) E_{2,1+m+\alpha r-\beta}(-t^2),$$

Applying Proposition 1, and after exploiting the relationship $\cos(x - \pi/2) = \sin x$, together with the standard rules for the angle additions of sin, we obtain

$$\left({}^{\mathrm{C}}\mathcal{D}^{\beta}_{\alpha,\gamma,\lambda} \sin\right)(t) = \sum_{r=0}^{\infty} \binom{\gamma}{r} \lambda^r \left[\sin\left(t + (\beta - \alpha r)\frac{\pi}{2}\right) + \psi_r(t)\right], \tag{4.9}$$

where for shortness we wrote

$$\psi_r(t) = \left(\cos \frac{m\pi}{2}\right) \varphi_{2,2+m+\alpha r-\beta}(t) + \left(\sin \frac{m\pi}{2}\right) \varphi_{2,1+m+\alpha r-\beta}(t)$$

(note that $\psi_r(t)$ depends on α, β, γ, and m as well). We can use again the relationships (4.5) to write

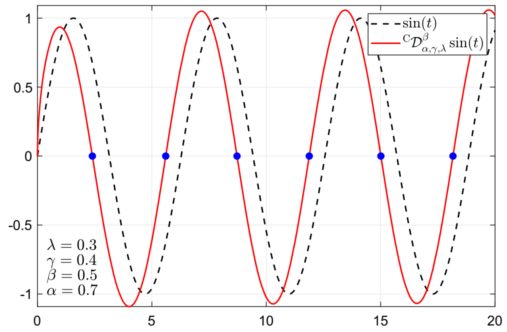
$$\psi_r(t) = \begin{cases} (-1)^{\frac{m}{2}} \varphi_{2,2+m+\alpha r-\beta}(t) & m \text{ even} \\ (-1)^{\frac{m-1}{2}} \varphi_{2,1+m+\alpha r-\beta}(t) & m \text{ odd} \end{cases}$$

and verify that $\psi_r(t)$ is a positive and monotonically decreasing function.

The plot for a selection of parameters α, β, γ and λ is presented in Figure 3 where the absence of periodicity can be easily verified. In this case, we also tabulated the first few zeros of ${}^{\mathrm{C}}\mathcal{D}^{\beta}_{\alpha,\gamma,\lambda} \sin t$, together with their distances, to provide further evidence of nonperiodicity of ${}^{\mathrm{C}}\mathcal{D}^{\beta}_{\alpha,\gamma,\lambda} \sin t$.

Following a similar reasoning, it is possible to evaluate the Prabhakar integral of $\sin t$ as

$$\left(\mathcal{I}^{\beta}_{\alpha,\gamma,\lambda} \sin\right)(t) = \sum_{r=0}^{\infty} \binom{-\gamma}{r} \lambda^r \left[t^{1+\beta+\alpha r} \sin\left(t - (\beta + \alpha r)\frac{\pi}{2}\right) + \varphi_{2,2+\beta+\alpha r}(t)\right].$$



z_k	$z_k - z_{k-1}$
2.4006	
5.6071	3.206
8.7225	3.115
11.8797	3.157
15.0104	3.131
18.1608	3.150

Fig. 3 Comparison between ${}^{\mathrm{C}}\mathcal{D}^{\beta}_{\alpha,\gamma,\lambda} \sin t$ (red solid line) and $\sin t$ (gray dashed line) for $\alpha = 0.7$, $\beta = 0.5$ ($m = 1$), $\gamma = 0.4$, $\lambda = 0.3$ and first few zeros of ${}^{\mathrm{C}}\mathcal{D}^{\beta}_{\alpha,\gamma,\lambda} \sin t$ with their distance.

4.3 Distributed order derivative of the sin function

Similarly to the previous example, we now consider the action on $\sin t$ of the distributed-order derivative described in section 2.3

$$
{}^{\mathrm{D}}D^m f(t) = \int_0^t k_D(t-\xi) f^{(m)}(\xi)\,\mathrm{d}\xi, \quad k_D(t) = \int_0^1 \frac{t^{\beta-1}}{\Gamma(\beta)}\,\mathrm{d}\beta.
$$

By using in Eq. (4.4) the representation (2.8) of the LT of the kernel $k_D(t)$ we readily obtain

$$
\begin{aligned}
\mathcal{L}\left[\left({}^{\mathrm{D}}D^m \sin\right)(t); s\right] &= \left(\sin\frac{m\pi}{2}\right)\frac{s-1}{s\ln s}\frac{s}{1+s^2} + \left(\cos\frac{m\pi}{2}\right)\frac{s-1}{s\ln s}\frac{1}{1+s^2} \\
&= \left[\left(\sin\frac{m\pi}{2}\right)\left(\frac{1}{s\ln s} - \frac{1}{s^2\ln s}\right) + \left(\cos\frac{m\pi}{2}\right)\left(\frac{1}{s^2\ln s} - \frac{1}{s^3\ln s}\right)\right]\frac{1}{1+\frac{1}{s^2}}.
\end{aligned}
$$

Now, taking $|s^2| > 1$ we can use the series form of $(1 + 1/s^2)^{-1}$ to rewrite the above representation of the LT of $\left({}^{\mathrm{D}}D^m \sin\right)(t)$ in the form

$$
\begin{aligned}
\mathcal{L}\left[\left({}^{\mathrm{D}}D^m \sin\right)(t); s\right] = \sum_{r=0}^{\infty}(-1)^r \Bigg\{ &\left(\sin\frac{m\pi}{2}\right)\frac{1}{s^{1+2r}\ln s} - \left(\cos\frac{m\pi}{2}\right)\frac{1}{s^{3+2r}\ln s} \\
&+ \left[\left(\cos\frac{m\pi}{2}\right) - \left(\sin\frac{m\pi}{2}\right)\right]\frac{1}{s^{2+2r}\ln s}\Bigg\}
\end{aligned}
$$

and, thanks to (A.8) it is possible to perform the inversion of the Laplace transform as

$$\left({}^{\mathrm{D}}D^{m}\sin\right)(t) = \left(\sin\frac{m\pi}{2}\right)\sum_{r=0}^{\infty}(-1)^{r}\nu(t,2r) - \left(\cos\frac{m\pi}{2}\right)\sum_{r=0}^{\infty}(-1)^{r}\nu(t,2+2r)$$
$$+\left[\left(\cos\frac{m\pi}{2}\right) - \left(\sin\frac{m\pi}{2}\right)\right]\sum_{r=0}^{\infty}(-1)^{r}\nu(t,1+2r)$$

where $\nu(t,\alpha)$ is a Volterra function (see Subsection A.4). The above series can be reformulated in terms of Volterra-Prabhakar functions $\epsilon_{\alpha,\,p}^{\gamma}(\lambda;t)$ since, in view of Eq. (A.10), it is

$$\sum_{r=0}^{\infty}(-1)^{r}\nu(t,2r+p) = \epsilon_{2,p}^{1}(1;t) =: \epsilon_{2,p}(t).$$

Hence, we are able to rewrite the distributed-order derivative of $\sin t$ as

$$\left({}^{\mathrm{D}}D^{m}\sin\right)(t) = \sin\frac{m\pi}{2}\epsilon_{2,0}(t) + \left[\cos\frac{m\pi}{2} - \sin\frac{m\pi}{2}\right]\epsilon_{2,1}(t) - \cos\frac{m\pi}{2}\epsilon_{2,2}(t).$$

A simpler formulation of $\epsilon_{2,p}(t)$ can be provided thanks to Eq. (A.11). We note the presence of an exponential term in $\epsilon_{2,p}(t)$ which however does not depend on p and hence disappears when multiplied by $\sin m\pi/2$ and $\cos m\pi/2$ and summed up in $\left({}^{\mathrm{D}}D\sin\right)(t)$. Elementary manipulations hence lead to

$$\left({}^{\mathrm{D}}D^{m}\sin\right)(t) = \frac{2}{\pi}\left[\sin\left(t+\frac{m\pi}{2}\right) - \cos\left(t+\frac{m\pi}{2}\right)\right] + \left(\sin\frac{m\pi}{2}\right)\phi_{2,0}(t) +$$
$$\left[\left(\cos\frac{m\pi}{2}\right) - \left(\sin\frac{m\pi}{2}\right)\right]\phi_{2,1}(t) - \left(\cos\frac{m\pi}{2}\right)\phi_{2,2}(t).$$

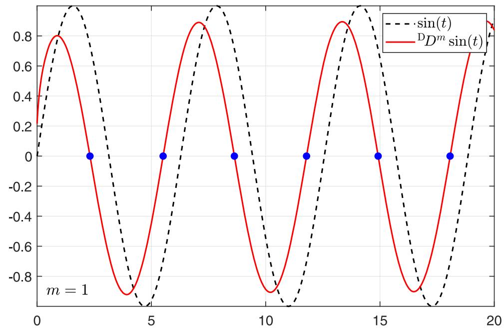
Moreover, we observe that

$$\sin\left(t+\frac{m\pi}{2}\right) = \begin{cases}(-1)^{\frac{m}{2}}\sin t & m \text{ even}\\ (-1)^{\frac{m-1}{2}}\cos t & m \text{ odd}\end{cases},$$
$$\cos\left(t+\frac{m\pi}{2}\right) = \begin{cases}(-1)^{\frac{m}{2}}\cos t & m \text{ even}\\ -(-1)^{\frac{m-1}{2}}\sin t & m \text{ odd}\end{cases}$$

and hence, after exploiting (4.5), we obtain

$$\left({}^{\mathrm{D}}D^{m}\sin\right)(t) = \begin{cases}(-1)^{\frac{m}{2}}\left[\frac{2}{\pi}\sin t - \frac{2}{\pi}\cos t + \left(\phi_{2,1}(t) - \phi_{2,2}(t)\right)\right] & m \text{ even}\\ (-1)^{\frac{m-1}{2}}\left[\frac{2}{\pi}\cos t + \frac{2}{\pi}\sin t - \left(\phi_{2,1}(t) - \phi_{2,0}(t)\right)\right] & m \text{ odd}\end{cases}$$

In Figure 4 we show the distributed order derivative $\left({}^{\mathrm{D}}D^{m}\sin\right)(t)$ for $m = 1$ and the location of its first zeros together with their distances; also in this case the distribution of zeros is not uniform.



z_k	$z_k - z_{k-1}$
2.3131	
5.5133	3.200
8.6304	3.117
11.7866	3.156
14.9187	3.132
18.0669	3.148

Fig. 4 Comparison between $^{\mathrm{D}}D^m \sin(t)$ (red solid line) and $\sin(t)$ (gray dashed line) for $m = 1$ and first few zeros of $^{\mathrm{D}}D^m \sin(t)$ with their distance.

From Eq. (A.12) it is easy to observe that

$$\phi_{2,1}(t) - \phi_{2,2}(t) = \int_0^\infty \frac{\mathrm{e}^{-rt}(1+r)}{r(r^2+1)\big[\log(r)^2+\pi^2\big]}\,\mathrm{d}r$$

and

$$\phi_{2,1}(t) - \phi_{2,0}(t) = \int_0^\infty \frac{\mathrm{e}^{-rt}(1+r)}{(r^2+1)\big[\log(r)^2+\pi^2\big]}\,\mathrm{d}r$$

and therefore by the Bernstein theorem they are completely monotone functions, which explains the loss of the periodicity of $^{\mathrm{D}}D^m \sin(t)$.

5 Conclusions and discussion

In this work we have shown that fractional derivatives and integrals do not preserve periodicity, and this property is shared by a wide class of operators, namely by GFDs and GFIs defined by means of Sonine kernels.

Moreover, we have derived an explicit representation of the action of some of these operators on the sine function. We observed that the resulting function consists of a periodic term plus a non-local term, which accounts for the loss of periodicity. Similar results readily extend to the cosine function and, consequently, to any periodic function expressed in terms of its Fourier series.

In light of the growing interest in GFDs and GFIs, a future study will aim to investigate the conditions under which fractional differential equations, particularly boundary value problems, may or may not admit periodic solutions.

We think that this investigation will contribute to deepening the knowledge of a wide class of fractional operators and facilitating their use in modeling different phenomena.

A Appendix: Special functions

The action of different types of fractional derivatives on some elementary functions can be expressed in terms of special functions. To facilitate the reading of this paper, in this Appendix, we review the main information about some special functions employed throughout this work.

All the special functions that we are going to introduce depend on one or more parameters. Although, in general, complex values are allowed for the parameters, for simplicity, we confine here just to real parameters, which are of major interest for our analysis.

Moreover, since most of the analysis relies on complex-plane formulas for the inversion of the LT, and the presence of real powers and other functions with complex arguments generally leads to multi-valued functions, a branch-cut on the real negative axis will always be considered to make them single-valued functions.

A.1 The Mittag-Leffler function

The Mittag-Leffler (ML) function is known to have a pivotal role in fractional calculus. It is indeed the eigenfunction for fractional derivatives. This function is not only involved in the analysis and solution of FDEs but it is also useful to represent fractional derivatives of some elementary functions.

The two-parameter ML function is defined by its series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad z \in \mathbb{C}, \tag{A.1}$$

and it is an entire function of order $1/\alpha$. We refer to the recent monograph [14] for a review of the main properties of the ML function. The LT transform can be evaluated for the special case of the ML function

$$\mathcal{L}\Big(t^{\beta-1} E_{\alpha,\beta}(t^{\alpha}\lambda)\,;\, s\Big) = \frac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}, \quad t > 0, \quad \lambda \in \mathbb{C}, \quad |s| > |\lambda|^{1/\alpha}. \tag{A.2}$$

Of special interest in this work (in particular to discuss examples of derivatives of the sine function) is the ML function with first parameter $\alpha = 2$ for which the following result holds.

Proposition 1 *Let $\beta \in \mathbb{R}$ and $t > 0$. Then*

$$t^{\beta-1} E_{2,\beta}(-t^2) = \cos\Big(t + (1-\beta)\frac{\pi}{2}\Big) + \varphi_{2,\beta}(t), \tag{A.3}$$

where

$$\varphi_{2,\beta}(t) = -\frac{\sin((2-\beta)\pi)}{\pi} \int_0^{\infty} \mathrm{e}^{-rt} \frac{r^{2-\beta}}{r^2+1}\,\mathrm{d}r. \tag{A.4}$$

Proof We start from the formula of the LT

$$\mathcal{L}\Big(t^{\beta-1}E_{2,\beta}(-t^2)\,;\ s\Big) = \frac{s^{2-\beta}}{s^2+1}$$

which allows to represent the function thanks to the formula for the inversion of its LT

$$t^{\beta-1}E_{2,\beta}(-t^2) = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}} \mathrm{e}^{st}\frac{s^{2-\beta}}{s^2+1}\,\mathrm{d}s,$$

with $\mathcal{C}$ a contour encompassing both poles $s^\star = \pm\mathrm{i}$. A residue subtraction allows to obtain

$$t^{\beta-1}E_{2,\beta}(-t^2) = \mathrm{Res}\Big(\mathrm{e}^{st}\frac{s^{2-\beta}}{s^2+1}, +\mathrm{i}\Big) + \mathrm{Res}\Big(\mathrm{e}^{st}\frac{s^{2-\beta}}{s^2+1}, -\mathrm{i}\Big) + \varphi_{2,\beta}(t),$$

where

$$\varphi_{2,\beta}(t) = \frac{1}{2\pi\mathrm{i}}\int_{\bar{\mathcal{C}}} \mathrm{e}^{st}\frac{s^{2-\beta}}{s^2+1}\,\mathrm{d}s$$

and now $\bar{\mathcal{C}}$ is a contour crossing the real axis in any point on $(0, +\infty)$ and the imaginary axis in any points on the intervals $(0, +\mathrm{i})$ and $(0, -\mathrm{i})$. Residues are easy to compute

$$\mathrm{Res}\Big(\mathrm{e}^{st}\frac{s^{2-\beta}}{s^2+1}, \pm\mathrm{i}\Big) = \Big[\frac{s^{2-\beta}}{2s}\Big|_{s=\pm\mathrm{i}}\Big] = \frac{1}{2}\mathrm{e}^{\pm\mathrm{i}t}(\pm\mathrm{i})^{1-\beta},$$

and hence

$$\begin{aligned}\mathrm{Res}\Big(\mathrm{e}^{st}\frac{s^{2-\beta}}{s^2+1}, +\mathrm{i}\Big) + \mathrm{Res}\Big(\mathrm{e}^{st}\frac{s^{2-\beta}}{s^2+1}, -\mathrm{i}\Big) &= \\ &= \frac{1}{2}\mathrm{e}^{\mathrm{i}(t+(1-\beta)\frac{\pi}{2})} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}(t+(1-\beta)\frac{\pi}{2})} = \cos\Big(t+(1-\beta)\frac{\pi}{2}\Big).\end{aligned}$$

Since the contour defining $\varphi_{2,\beta}(t)$ does not include any pole of the integrand, the application of the Titchmarsh inversion formula [44] leads to

$$\varphi_{2,\alpha}(t) = \int_0^\infty \mathrm{e}^{-rt}K_{2,\alpha}(r)\,\mathrm{d}r, \qquad \text{(A.5)}$$

where

$$K_{2,\alpha}(r) = -\frac{1}{\pi}\,\mathfrak{Im}\Big[\frac{s^{2-\beta}}{s^2+1}\Big|_{s=r\mathrm{e}^{+\mathrm{i}\pi}}\Big] = -\frac{\sin((2-\beta)\pi)}{\pi}\frac{r^{2-\beta}}{(r^2+1)},$$

and from which the proof immediately follows. □

A.2 The Prabhakar or three-parameter ML function

The Prabhakar function, introduced in [38], is an extension to three parameters of the ML function. For $\alpha > 0$, $\beta, \gamma \in \mathbb{R}$ and $z \in \mathbb{C}$, it is defined according to

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\alpha k + \beta)}, \tag{A.6}$$

where $(\gamma)_k = \Gamma(\gamma+k)/\Gamma(\gamma) = \gamma(\gamma+1)\ldots(\gamma+k-1)$ is the rising factorial, often denoted as the Pochammer symbol.

A.3 The Volterra function

Volterra functions were introduced by the Italian mathematician Vito Volterra to represent solutions of integral equations of convolution type with a logarithmic kernel [45]. Following the notations introduced in [7], Volterra functions are defined in terms of the definite integral (see also [1, 5])

$$\mu(t, \beta, \alpha) = \frac{1}{\Gamma(1+\beta)} \int_0^{\infty} \frac{t^{u+\alpha}\, u^{\beta}}{\Gamma(u+\alpha+1)} \, \mathrm{d}u, \quad \beta > -1, \quad t > 0, \tag{A.7}$$

and special notations are adopted to denote particular instances of parameters

$$\begin{aligned} \alpha = \beta = 0 : &\qquad \nu(t) = \mu(t, 0, 0), \\ \alpha \neq 0,\ \beta = 0 : &\qquad \nu(t, \alpha) = \mu(t, 0, \alpha), \\ \alpha = 0,\ \beta \neq 0 : &\qquad \mu(t, \beta) = \mu(t, \beta, 0). \end{aligned}$$

The Laplace transform of the Volterra functions $\mu(t, \beta, \alpha)$ is given by [5]

$$\mathcal{L}\Big(\mu(t, \beta, \alpha)\,;\, s\Big) = \frac{1}{s^{\alpha+1} \ln^{\beta+1} s}, \quad \alpha > -1, \quad \beta > -1, \quad \Re(s) > 1,$$

which in the special case $\beta = 0$ becomes

$$\mathcal{L}\Big(\nu(t, \alpha)\,;\, s\Big) = \frac{1}{s^{\alpha+1} \ln s}, \quad \alpha > -1, \quad \Re(s) > 1, \tag{A.8}$$

and for more properties of this family of functions we refer to [2, 3, 5, 11, 34].

A.4 The Volterra-Prabhakar function

The Volterra-Prabhakar function has been recently introduced in [16] in view of its role in solution and analysis of FDEs of distributed order. It is defined by integration of one of the parameters of the Prabhakar function according to

$$\epsilon^{\gamma}_{\alpha,\, p}(\lambda; t) = \int_0^\infty t^{u+p} E^{\gamma}_{\alpha,u+p+1}(-\lambda t^{\alpha})\, \mathrm{d}u, \tag{A.9}$$

and one can easily check that when $\gamma = 0$ or $\lambda = 0$ this function corresponds to the Volterra function $\nu(t, p)$. More generally, as shown in [16], $\epsilon^{\gamma}_{\alpha,\, p}(\lambda; t)$ can be written in terms of series of Volterra's functions since

$$\begin{aligned}\epsilon^{\gamma}_{\alpha,\, p}(\lambda; t) &= \int_0^\infty \sum_{r=0}^{\infty} \frac{(\gamma)_r (-\lambda)^r t^{\alpha r+u+p}}{r!\Gamma(\alpha r + u + p + 1)}\, \mathrm{d}u \\ &= \sum_{r=0}^{\infty} \frac{(-\lambda)^r (\gamma)_r}{r!} \int_0^\infty \frac{t^{\alpha r+u+p}}{\Gamma(\alpha r + u + p + 1)}\, \mathrm{d}u\end{aligned}$$

and hence

$$\epsilon^{\gamma}_{\alpha,\, p}(\lambda; t) = \sum_{r=0}^{\infty} \frac{(-\lambda)^r (\gamma)_r}{r!} \nu(t, \alpha r + p), \tag{A.10}$$

thus motivating the name Volterra-Prabhakar given to this function.

The Laplace transform of the Volterra-Prabhakar function has been evaluated in [16] and it is given by

$$\mathcal{L}\Big(\epsilon^{\gamma}_{\alpha,\, p}(\lambda; t)\, ;\; s\Big) = \frac{s^{\alpha\gamma-p-1}}{\big(s^{\alpha}+\lambda\big)^{\gamma} \ln s}.$$

Some special instances of the Volterra-Prabhakar function are of interest for this work. This is the case when $\alpha = 2$, $\gamma = 1$, $\lambda = 1$, since the function

$$\epsilon_{2,p}(t) := \epsilon^{1}_{2,p}(1; t).$$

is involved in the evaluation of the distributed derivative of the sine function (see section 4.3). In this perspective, it is useful to consider the following result.

Proposition 2 *Let $p \in \mathbb{N}$ and $t \in \mathbb{R}$. Then*

$$\epsilon_{2,p}(t) = (-1)^{p} \frac{2}{\pi} \sin\Big(t + p\frac{\pi}{2}\Big) + \frac{1}{2}\mathrm{e}^{t} + \phi_{2,p}(t), \tag{A.11}$$

where

$$\phi_{2,p}(t) = (-1)^{1-p} \int_0^\infty \frac{\mathrm{e}^{-rt} r^{1-p}}{(r^2+1)\big[\log(r)^2 + \pi^2\big]}\, \mathrm{d}r. \tag{A.12}$$

Proof The LT of $\epsilon_{2,p}(t)$ is

$$\hat{\epsilon}_{2,p}(s) = \mathcal{L}\Big(\epsilon_{2,p}(t)\, ;\; s\Big) = \frac{s^{1-p}}{(s^2+1)\ln s}$$

which, thanks to the formula for the inversion of the Laplace transform, allows to represent the function as

$$\epsilon_{2,p}(t) = \frac{1}{2\pi i}\int_{\mathcal{C}} \mathrm{e}^{st}\frac{s^{1-p}}{(s^2+1)\ln s}ds,$$

where $\mathcal{C}$ is any contour encompassing all the poles of $\widehat{\epsilon}_{2,p}(s)$ which are located at $s^\star = \pm\mathrm{i}$ and $s^\star = 1$. Therefore, by residue subtraction we can write

$$\epsilon_{2,p}(t) = \mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), +\mathrm{i}\big) + \mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), -\mathrm{i}\big) + \mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), 1\big) + \phi_{2,p}(t)$$

where

$$\phi_{2,p}(t) = \frac{1}{2\pi i}\int_{\bar{\mathcal{C}}} \mathrm{e}^{st}\frac{s^{1-p}}{(s^2+1)\ln s}ds$$

and $\bar{\mathcal{C}}$ is any complex contour crossing the real axis in any point in the interval (0, 1) and the imaginary axis in any points on the intervals (0, +i) and (0, –i). It is simple to compute

$$\mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), +\mathrm{i}\big) = \frac{\mathrm{i}^{1-p}}{2\mathrm{i}\ln\mathrm{i}}\mathrm{e}^{+\mathrm{i}t}, \quad \mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), -\mathrm{i}\big) = \frac{(-\mathrm{i})^{1-p}}{(-2\mathrm{i})\ln(-\mathrm{i})}\mathrm{e}^{-\mathrm{i}t}$$

and since $\ln(\pm\mathrm{i}) = \pm\mathrm{i}\pi/2$ we can evaluate

$$\mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), +\mathrm{i}\big) = -\frac{\mathrm{i}^{1-p}}{\pi}\mathrm{e}^{+\mathrm{i}t}, \quad \mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), -\mathrm{i}\big) = -\frac{(-\mathrm{i})^{1-p}}{\pi}\mathrm{e}^{-\mathrm{i}t}.$$

Therefore, by means of elementary manipulations, one obtains

$$\begin{aligned}
\mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), +\mathrm{i}\big) + \mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), -\mathrm{i}\big) &= -\frac{\mathrm{i}^{1-p}}{\pi}\mathrm{e}^{+\mathrm{i}t} - \frac{(-\mathrm{i})^{1-p}}{\pi}\mathrm{e}^{-\mathrm{i}t} \\
&= -\frac{1}{\pi}\left[\mathrm{e}^{\mathrm{i}\left(t+(1-p)\frac{\pi}{2}\right)} + \mathrm{e}^{-\mathrm{i}\left(t+(1-p)\frac{\pi}{2}\right)}\right] = -\frac{2}{\pi}\cos\Big(t+(1-p)\frac{\pi}{2}\Big) \\
&= (-1)^p\frac{2}{\pi}\sin\Big(t+p\frac{\pi}{2}\Big)
\end{aligned}$$

and, moreover, it is simple to evaluate

$$\mathrm{Res}\big(\mathrm{e}^{st}\hat{\epsilon}_{2,p}(s), 1\big) = \frac{1}{2}\mathrm{e}^{t}.$$

Since the path $\bar{\mathcal{C}}$ does not include any pole of the integrand in $\phi_{2,p}(t)$, we can deform it in a Hankel path starting at $-\infty$ along the lower negative real axis, encircling a sufficiently small circle $|s| = \varepsilon$ in the positive sense and returning to $+\infty$ along the upper negative real axis. Thanks to Titchmarsh inversion formula (e.g., see [44]), it is

$$\phi_{2,p}(t) = \int_0^{\infty} \mathrm{e}^{-rt} K_{2,p}(r) dr \tag{A.13}$$

where

$$K_{2,p}(r) = -\frac{1}{\pi} \Im\mathfrak{m}\Big[\frac{s^{1-p}}{(s^2+1)\ln s} \Big|_{s=r\mathrm{e}^{+\mathrm{i}\pi}} \Big]$$

and standard manipulations allows to show that

$$K_{2,p}(r) = \frac{(-1)^{1-p} r^{1-p}}{(r^2+1)\big[\log(r)^2 + \pi^2\big]}$$

from which the proof immediately follows. □

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Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

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