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Cenedese, Carlo; Belgioioso, Giuseppe; Grammatico, Sergio; Cao, Ming

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Time-varying constrained proximal type dynamics in multi-agent network games

Carlo Cenedese Giuseppe Belgioioso Sergio Grammatico Ming Cao

Abstract— **In this paper, we study multi-agent network games subject to affine time-varying coupling constraints and a time-varying communication network. We focus on the class of games adopting proximal dynamics and study their convergence to a persistent equilibrium. The assumptions considered to solve the problem are discussed and motivated. We develop an iterative equilibrium seeking algorithm, using only local information, that converges to a special class of game equilibria. Its derivation is motivated by several examples, showing that the original game dynamics fail to converge. Finally, we apply the designed algorithm to solve a constrained consensus problem, illustrating the theoretical results.**

I. Introduction

Multi-agent decision making over networks: in multi-agent decision making over networks, all the decision makers, in short, *agents*, share their information only with a selected number of agents. In particular, the agents' state (or decision) is the result of a *local decision making* process, e.g. a constrained optimization problem, and a *distributed communication* with the neighboring agents, defined by the communication network. In many problems, the goal of the agents is reaching a collective equilibrium state, where no agent can benefit from changing its state. The local interaction between the agents is exploited in opinion dynamics to model the evolution of a population's collective opinion as an emerging phenomenon, see [1], [2], [3]. Another interesting consequence of the network structure is that the agents keep their own data private, exchanging information only with selected agents. This characteristic is of particular interest in, for example, traffic and information networks problems [4] or in the charging scheduling of electric vehicles [5]. This class of problems arises also in other applications, e.g., in smart grids [6] and sensor network [7].

Multi-agent optimization and multi-agent network games: in this work, we study a particular instance of the problem introduced above, namely a multi-agent network game, where the communication network and the constraints between the agents are both time-varying. Multi-agent network games arise from the well established field of distributed optimization and equilibrium seeking over networks. In the past years, several results were proposed for optimization problems subject to a time-varying communication network: in [8] the subgradients of the cost functions are bounded and the communication is described by a strongly connected sequence of directed graphs, while in [9] the cost functions are assumed to be continuously differentiable and a linearly convergent algorithm is designed under the assumption of a time-varying undirected communication network. Another approach, explored in [10], is to construct a game, whose emerging behavior solves the optimization problems. In this case, the cost functions are differentiable and the communication ruled by an undirected time-varying graph connected over time. The problem of noncooperative multi-agent games, subject to coupling constrains, was firstly studied in [11], under the assumptions of continuosly differentiable cost functions and no network structure. In the past years, several researchers provided results for games over networks, e.g., in [12], [13], [5] where the communication network is always assumed undirected. Moreover, some authors also focused on the class of noncooperative games over time-varying communication network, in particular on the unconstrained case. For example, in [14] differentiable and strictly convex cost functions with Lipschitz continuous gradient were considered, where the sequence of time-varying communication networks was repeatedly strongly connected, and the associated adjacency matrices doubly stochastic.

Paper contribution: a complete formulation of multi-agent network games, subject to proximal type dynamics, can be found in [6] where the unconstrained case is studied for a time-varying strongly connected communication network, described by a doubly stochastic adjacency matrix. In [15], [16], the condition on the double stochasticity of the adjacency matrix was relaxed. Notice that these types of games can also be rephrased as paracontracions; in this framework, the work in [17] provided convergence for repeatedly jointly connected digraphs. Iterative equilibrium seeking algorithms were developed for constrained multi-agent network games in [6], [16] and a static communication network.

In this work, we aim to address the problem of a constrained multi-agent network games subject to a time-varying communication network. In particular, we first discuss the convergence of the game and motivate the technical assumption needed to ensure the existence of an equilibrium, and then we develop an equilibrium seeking algorithm that achieves global convergence for the game at hand. The main difference with the work in [16] is the presence of

C. Cenedese and M. Cao are with the Jan C. Wilems Center for Systems and Control, ENTEG, Faculty of Science and Engineering, University of Groningen, The Netherlands ({c.cenedese, m.cao}@rug.nl) G.Belgioioso is with the Control System group, TU Eindhoven, The Netherlands (g.belgioioso@tue.nl). S. Grammatico is with the Delft Center for Systems and Control, TU Delft, The Netherlands (s.grammatico@tudelft.nl). The work of Cenedese and Cao was supported in part by the European Research Council (ERC-CoG-771687) and the Netherlands Organization for Scientific Research (NWO-vidi-14134), while the work of Belgioioso and Gramatico by the EU Project 'MatchIT' (82203), NWO under research project OMEGA (613.001.702) and P2P-TALES (647.003.003) and by the ERC under research project COSMOS (802348).

both time-varying communication network and time-varying constraints, and this generalization leads to several technical challenges, requiring a more involved convergence analysis.

Notation: the notation adopted along the paper is the same as the one in [5], while we refer to [18] for more properties of operators of class $\mathfrak{I}$.

II. Mathematical setup and problem formulation

A. Mathematical formulation

We consider N players (or agents) taking part in a game. A constrained network game is defined by three main components: the constraints each players has to satisfy, the cost functions to be minimized and the communication network.

The constraints can be divided in two types: local and coupling. At every time instant $k \in \mathbb{N}$, each agent $i \in \mathcal{N} := \{1, \dots, N\}$ adopts an action (or strategy) $x_i \in \mathbb{R}^n$ belonging to its *local feasible set* $\Omega_i \subset \mathbb{R}^n$, i.e., the collection of those strategies meeting its local constraints. We assume that this set is convex and closed.

Standing Assumption 1 (Convexity): For every $i \in \mathcal{N}$, the set $\Omega_i \subset \mathbb{R}^n$ is non-empty, compact and convex. □

The agents are also subject to M time-varying affine and separable *coupling constraints*, that generate an entanglement between the strategy chosen by player i and those of the others. For an agent $i \in \mathcal{N}$, at time instant $k \in \mathbb{N}$, the time-varying set of strategies satisfying the coupling constraints, given the other agents' strategies $\boldsymbol{x}_{-i}$, reads as

$$\mathcal{X}_i(\boldsymbol{x}_{-i}, k) := \left\{ y \in \mathbb{R}^n \,|\, C_i(k) y + \textstyle\sum_{\substack{j=1 \\ j \neq i}}^{N} C_j(k) x_j(k) \leq c(k) \right\}$$

where $C_j(k) \in \mathbb{R}^{M \times n}$ and $c(k) \in \mathbb{R}^M$.

In the following, we refer to the collective vector $\boldsymbol{x} := \mathrm{col}((x_i)_{i \in \mathcal{N}}) \in \mathbb{R}^{Nn}$ as the *strategy profile* of the game. All the strategies profiles that satisfy both the local and coupling constraints determine the *collective feasible decision set*, defined as $\boldsymbol{\mathcal{X}}(k) := \boldsymbol{\Omega} \cap \{\boldsymbol{x} \in \mathbb{R}^{Nn} | \boldsymbol{C}(k)\boldsymbol{x} \leq c(k)\}$, where $\boldsymbol{C}(k) := [C_1(k), \dots, C_N(k)] \in \mathbb{R}^{M \times Nn}$ and $\boldsymbol{\Omega} := \prod_{i=1}^N \Omega_i$.

Standing Assumption 2: For all $i \in \mathcal{N}$ and $k \in \mathbb{N}$, the *collective feasible decision set* $\boldsymbol{\mathcal{X}}(k)$ satisfies Slater's conditions. □

All the agents are assumed myopic and rational, thus each agent $i \in \mathcal{N}$ aims only at minimizing its local cost function $J_i(x_i, z)$. In this work, we assume that the cost function have the proximal structure, as defined next.

Standing Assumption 3 (Proximal cost functions): For all $i \in \mathcal{N}$, the function $J_i : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined as

$$J_i(x_i, z) := \bar{f}_i(x_i) + \tfrac{1}{2}\|x_i - z\|^2, \tag{1}$$

where the function $\bar{f}_i := f_i + \iota_{\Omega_i} : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex and lower semi-continuous. □

The cost function is composed of two parts, $\bar{f}_i$ is the local part and has a double role: describing the local objective of agent i, via f_i, and ensuring that the next strategy belongs to Ω_i, through the indicator function ι_{Ω_i}. The quadratic part of J_i works as a regularization term and penalizes the distance of the local strategy from z. It is also responsible for the strict-convexity of J_i, even though $\bar{f}_i$ is only lower semi-continuous, see [19, Th. 27.23].

Now we introduce the time-varying communication network adopted by the agents. At each time instant k, it is described by a strongly connected digraph, defined via the couple $(\mathcal{V}, A(k))$. The set $\mathcal{V}$ represents the nodes of the graph, i.e., $\mathcal{V} = \mathcal{N}$, so this set does not vary over time. The matrix $A(k)$ denotes the adjacency matrix of the digraph, at time k, where $a_{i,j}(k) := [A(k)]_{ij}$. For every $i, j \in \mathcal{N}$, $a_{i,j}(k) \in [0,1]$ is the weight that agent i assigns to the strategy of agent j. If $a_{i,j}(k) = 0$, then agent i does not communicate with agent j. The set of all the neighbors of agent i is defined as $\mathcal{N}_i(k) := \{j \,|\, a_{i,j}(k) > 0\}$. The following assumption formalizes the properties of the adjacency matrix required throughout this work.

Standing Assumption 4 (Row stochasticity and self-loops): At every time instant $k \in \mathbb{N}$, the communication graph is strongly connected. The matrix $A(k) = [a_{i,j}(k)]$ is row stochastic, i.e., $a_{i,j}(k) \geq 0$ for all $i, j \in \mathcal{N}$, and $\sum_{j=1}^N a_{i,j}(k) = 1$, for all $i \in \mathcal{N}$. Moreover, $A(k)$ has strictly-positive diagonal elements, i.e., $\min_{i \in \mathcal{N}} a_{i,i}(k) =: \underline{a}_k > 0$. □

For each agent $i \in \mathcal{N}$, the term z in (1) represents an aggregative quantity defined by

$$z := \textstyle\sum_{j=1}^N a_{i,j}(k) x_j(k)\,,$$

and hence it is the average of the neighbors' strategies, weighted via the adjacency matrix $A(k)$. So, the actual cost function of agent i at time k is $J_i(x_i, \sum_{j=1}^N a_{i,j}(k) x_j(k))$.

As mentioned before, the agents are considered rational, thus their only objective is to minimize their local cost function, while satisfying the local and coupling constraints. The dynamics describing this behavior are the *myopic best response dynamics*, defined, for each player $i \in \mathcal{N}$, as:

$$x_i(k+1) = \underset{y \in \mathcal{X}_i(\boldsymbol{x}_{-i}, k)}{\mathrm{argmin}} \; J_i\left(y, \textstyle\sum_{j=1}^N a_{i,j}(k) x_j(k)\right) \,. \tag{2}$$

The interaction of the N players, using dynamics in (2), can be naturally formalized as a *noncooperative network game*, defined at $k \in \mathbb{N}$, as

$$\forall i \in \mathcal{N} : \begin{cases} \underset{y \in \mathbb{R}^n}{\mathrm{argmin}} & f_i(y) + \frac{1}{2}\left\| y - \sum_{j=1}^N a_{i,j} x_j \right\|^2 \\ \quad \text{s.t.} & y \in \Omega_i \cap \mathcal{X}_i(\boldsymbol{x}_{-i}, k)\,, \end{cases} \tag{3}$$

where we omitted the time dependency of $a_{i,j}(k)$ and $x_j(k)$ to ease the notation.

B. Equilibrium concept and convergence

For the game in (3), the concept of equilibrium point is non trivial. A popular equilibrium notion for constrained game is the, so called, *generalized network equilibrium* (GNWE). Loosely speaking, a profile strategy $\hat{\boldsymbol{x}}$ is a GNWE of the game, if no player i can change its strategy to another *feasible* one while decreasing $J_i\left(\hat{x}_i, \sum_{j=1}^N a_{i,j}\hat{x}_j\right)$. Notice that, if A does not have self-loops, an GNWE boils down to a *generalized Nash equilibrium*, see [11].

This idea of equilibrium cannot be directly applied to (3) and in fact every variation in the communication network generates a different game, with its own set of GNWE. Therefore, the equilibria in which we are interested are those invariant to the changes in the communication; they take the name of *persistent* GNWE (p–GNWE).

Definition 1 (persistent GNWE): A collective vector $\bar{\boldsymbol{x}} = \mathrm{col}((\bar{x}_i)_{i\in\mathcal{N}})$ is a persistent GNWE (p–GNWE) for the game (3), if there exists some $\bar{k} > 0$, such that for all $i \in \mathcal{N}$,

$$\bar{x}_i = \bigcap_{k\geq\bar{k}} \underset{y\in\mathcal{X}(\bar{\boldsymbol{x}}_{-i},k)}{\operatorname{argmin}} \; J_i\left(y, \textstyle\sum_{j=1}^N a_{i,j}(k)\bar{x}_j\right) . \qquad (4)$$

□

We have defined both the game and the set of equilibria we are interested in. Let us now elaborate on the convergence properties of the game in (3), introducing three examples highlighting different aspects of these dynamics. In [16, Ex. 2] the authors show that the dynamics in (2) can fail to converge to an equilibrium point, even in the case of a static communication network, where the existence of a GNWE is guaranteed by [6, Prop. 4]. Then, in Example 1 we highlight that the existence of p–GNWE is not guaranteed. Finally, the last example shows a case where the game in (3) converges.

Example 1 (equilibirum existence): Consider a 2-player game without local or coupling constraints and scalar strategies. The communication network can vary between the two graphs described respectively by the adjacency matrices $A_1 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix}$. The cost functions of the agents are in the form of (1), where the local part is chosen as $\bar{f}_i(x_i) = \frac{1}{2}\|x_i - i\|^2$, for $i \in \{1, 2\}$. For each one of the communication networks, there exists only one equilibrium point of the game, i.e., $\boldsymbol{x}_{A_1} = [5/4\,,\, 7/4]^\top$ and $\boldsymbol{x}_{A_2} = [4/3\,,\, 11/6]^\top$, when respectively A_1 or A_2 is adopted. Therefore the set of p–GNWE of the game is empty, leading the dynamics to oscillate between $\boldsymbol{x}_{A_1}$ and $\boldsymbol{x}_{A_2}$. □

Example 2 (convergence): Once again, consider the a 2-player game, where for a player $i \in \{1, 2\}$ the local feasible set is $\Omega_i = [-1, 1]$ and $f_i(x_i) = 0$. The collective feasible decision set is defined as $\boldsymbol{\mathcal{X}}(k) := \{\boldsymbol{x} \in [-1, 1]^2 \,|\, m(k) \leq x_1 + x_2\}$ where $m(k) \in [-1, -0.25]$. We choose $A(k)$ satisfying Standing Assumption 4 and it is doubly stochastic, for every time instant $k \in \mathbb{N}$. If the strategy profile belongs to the consensus subspace $\mathcal{C}$, both agents achieve the minimum of their cost function, and therefore all those points are equilibria of the unconstrained game. Furthermore, for the set $\hat{\mathcal{C}} = \{u \in \mathbb{R}^2 \,|\, u = \alpha\mathbf{1}^\top,\ \alpha \in [-0.25, 1]\}$, it always holds that $\hat{\mathcal{C}} \subseteq \mathcal{C} \cap \boldsymbol{\mathcal{X}}$, and hence they are p-GNWE of the game. Assume that at $\bar{k} > 0$, $m(\bar{k}) = -0.25$, then, for all $k > \bar{k}$, the dynamics reduce to $\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k)$, therefore the profile strategy will converge to a point in $\hat{\mathcal{C}}$, i.e., to a p–GNWE of the game. □

C. Primal–dual characterization

As illustrated in [16, Ex. 2], the dynamics in (2) can fail to converge, to restore convergence we recast them as pseudo collaborative ones. The idea is that each player will minimize its own cost function, while at the same time coordinate with the others to satisfy the constraints. With this approach, we aim to achieve asymptotic fulfillment of the coupling constraints. As a first step, we dualize the dynamics introducing, for each player $i \in \mathcal{N}$, a dual variable $\lambda_i \in \mathbb{R}^M_{\geq 0}$. The arising problem is an *auxiliary (extended) network game*, see [20, Ch. 3]. The collective vector of the dual variables is denoted by $\boldsymbol{\lambda} := \mathrm{col}((\lambda_i)_{i\in\mathcal{N}})$. The equilibrium concept is adapted to this modification in the dynamics, so we define the *persistent Extended Network Equilibrium* (p–ENWE).

Definition 2 (persistent Extended Network Equilibrium): The pair $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\lambda}})$, is a p–ENWE for the game in (3) if there exists $\bar{k} > 0$ such that, for every $i \in \mathcal{N}$,

$$\begin{aligned} \overline{x}_i &= \bigcap_{k\geq\bar{k}} \underset{y\in\mathbb{R}^n}{\operatorname{argmin}} \; J_i\left(y, \textstyle\sum_{j=1}^N a_{i,j}(k)\overline{x}_j\right) + \overline{\lambda}_i^\top C_i(k) y, \\ \overline{\lambda}_i &= \bigcap_{k\geq\bar{k}} \underset{\xi\in\mathbb{R}^M_{\geq 0}}{\operatorname{argmin}} \; -\xi^\top (C(k)\overline{\boldsymbol{x}} - c(k)) \, . \end{aligned} \qquad (5)$$

□

In the following, we assume the presence of a *central coordinator* facilitating the synchronization between agents. The central coordinator broadcasts an auxiliary variable $\sigma \in \mathbb{R}^M$ to each agent i, that, in turn, uses this information to compute its local dual variable λ_i. Specifically, at every time instant k, the agent scales the received variable $\sigma(k)$, by a possibly time-varying factor $\alpha_i(k) \in [0, 1]$, attaining in this way its local dual variable, i.e., $\lambda_i(k) := \alpha_i(k)\sigma(k)$. The scaling factors α_i describe how the burden of satisfying the constraints are divided between the agents, hence $\sum_{i=1}^N \alpha_i = 1$. If $\alpha_i = 1/N$, for all $i \in \mathcal{N}$, then the effort to satisfy the couplying constraints is fairly splitted between the agents, this case is considered in several works, e.g., [5], [12], [21]. This class of problems was introduced for the first time in the seminal work by Rosen [22], where the author formulates the concept of *normalized equilibrium*. We adapt this idea for the problem at hand, defining the *persistent normalized extended network equilibrium* (pn-ENWE).

Definition 3 (persistent normalized-ENWE): The pair $(\overline{\boldsymbol{x}}, \overline{\sigma})$, is a pn–ENWE for the game in (3), if it exists $\bar{k} > 0$, such that for all $i \in \mathcal{N}$ it satisfies

$$\begin{aligned} \overline{x}_i &= \bigcap_{k\geq\bar{k}} \underset{y\in\mathbb{R}^n}{\operatorname{argmin}} \; J_i\left(y, \textstyle\sum_{j=1}^N a_{i,j}(k)\overline{x}_j\right) + \alpha_i(k)\,\overline{\sigma}^\top C_i(k) y, \\ \overline{\sigma} &= \bigcap_{k\geq\bar{k}} \underset{\varsigma\in\mathbb{R}^M_{\geq 0}}{\operatorname{argmin}} \; -\varsigma^\top (\boldsymbol{C}(k)\overline{\boldsymbol{x}} - c(k)), \end{aligned} \qquad (6)$$

with $\alpha_i(k) > 0$. □

The following lemma shows that a pn–ENWE is also a p–GNWE, and vice versa.

Lemma 1 (p–GNWE as fixed point): The following statements are equivalent:

(i) $\overline{\boldsymbol{x}}$ is a p–GNWE for the game in (3);
(ii) $\exists\overline{\sigma} \in \mathbb{R}^M$ and $\bar{k} > 0$ such that $\mathrm{col}(\overline{\boldsymbol{x}}, \overline{\sigma}) \in \mathcal{E}$, where $\mathcal{E}$ is the set of all the pn–GNWE of the game (3). □

We omit the demonstration of the lemma, since it is analogous to that in [6, Lem. 2].

This reformulation of the problem addresses the criticism highlighted in Example 1. In the following, we develop a distributed iterative algorithm converging to a p-GNWE of the original game in (3).

D. On the existence of persistent equilibria

We devote the remainder of the section to a more in depth analysis of the problem of the existence of a p–GNWE for the game in (3). In general, there is no guarantee that such an equilibrium exists. The literature is split on how to handle this problem. Namely, two possible assumptions can be adopted. The first one supposes *a priori* the existence of at least one p–GNWE in the game. It does not restrict the problem at hand, since it is a necessary condition to establish convergence. However, it may be difficult to be checked. This approach is the one chosen in this work and it is usually adopted when authors focus more on theoretical results, see [19, Cor. 5.19], [23, Prop. 3.1], [6, Ass. 3] and [16, Ass. 6].

Standing Assumption 5 (Existence of a pn-ENWE): The set of pn-ENWE of (3) is non-empty, hence $\mathcal{E} \neq \varnothing$. □

On the other hand, the second assumption considers only those games in which the N local cost functions share at least one common fixed point. This implies that at least one point in the consensus subspace is an equilibrium invariant to the change of the communication network. If, at the same time, this point is also feasible, then it is a p–GNWE of the game. This assumption is clearly stronger than the previous one. Nevertheless, it is easier to verify in practice, since it only requires the analysis of the cost functions of the agents, as shown in Example 2. Mainly for this reason, it is widely spread throughout the literature, where it is either implicitly verified as in [24] or explicitly required [17, Ass in Th. 2] .

III. Convergence result

Next, we propose the main result of this paper, an iterative and decentralized algorithm converging to a pn-GNWE of the game in (3). We call it TV–Prox–GNWE and it is reported in (7a)–(7d), while its complete derivation is described in the Appendix. To provide the bounds for the choices of the parameters in the algorithm, let us redefine the matrix $\boldsymbol{A}(k)$ via a diagonal matrix, an upper and a lower triangular matrix, i.e., $\boldsymbol{A}(k) = \boldsymbol{A}_{\text{ut}}(k) + \boldsymbol{A}_{\text{d}}(k) + \boldsymbol{A}_{\text{lt}}(k)$, where $\boldsymbol{A}_{\text{ut}}$ and $\boldsymbol{A}_{\text{lt}}$ always have zeros diagonal elements. For each time instant $k \in \mathbb{N}$, the parameters in TV–Prox–GNWE are set such that, the following inequalities hold:

$$\min_{i\in\mathcal{N}}(\delta_i^{-1} + a_{i,i}) \geq \|\boldsymbol{A} - \boldsymbol{A}_{\text{d}}\| + \|\boldsymbol{C}^\top - \boldsymbol{\Lambda}\boldsymbol{C}^\top\| \tag{8a}$$

$$\max_{i\in\mathcal{N}}(2q_i(\delta_i^{-1} + a_{i,i})) < R + \gamma^{-1} \tag{8b}$$

$$R := 2\|\boldsymbol{Q}\boldsymbol{A}_{\text{ut}} + \boldsymbol{A}\boldsymbol{Q}_{\text{lt}}\| + \|\boldsymbol{Q}(\boldsymbol{C}^\top - \boldsymbol{\Lambda}\boldsymbol{C}^\top)\|$$

$$\beta \geq \tfrac{1}{2}\|\boldsymbol{C} - \boldsymbol{C}\boldsymbol{\Lambda}\| \tag{8c}$$

$$\beta < \tfrac{1}{2}\big(\gamma^{-1} - \|\boldsymbol{C} - \boldsymbol{C}\boldsymbol{\Lambda}\|\big) \tag{8d}$$

where $\boldsymbol{\Lambda}(k) := \operatorname{diag}((\alpha_i(k))_{i\in\mathcal{N}}) \otimes I_n$ and $\boldsymbol{Q}(k) := \operatorname{diag}((q_i(k))_{i\in\mathcal{N}}) \otimes I_n$, with q_i being the i-th element of the left Perron-Frobenius eigenvector of $A(k)$. Also in this case, we omitted the time dependency of the matrices to ease the notation. The bounds in (8c) – (8d) implicitly lead to a condition on the maximum value of the step size γ, namely $\gamma \leq \frac{1}{2}\|\boldsymbol{C} - \boldsymbol{C}\boldsymbol{\Lambda}\|^{-1}$.

The main technical result of the paper is the following theorem, where we establish global convergence of the sequence generated by the TV–Prox–GNWE to a p-GNWE of the game in (3).

Theorem 1: For all $i \in \mathcal{N}$ and $k \in \mathbb{N}$, set $\alpha_i(k) = q_i(k)$, with $q_i(k)$ the i-th element of the left Perron–Frobenius eigenvector of $A(k)$, and choose $\delta_i(k)$, $\beta(k)$ and γ satisfying (8). For any initial condition, the sequence $(\boldsymbol{x}(k))_{k\in\mathbb{N}}$ generated by (7) converges to a p-GNWE of the game in (3). □

IV. Simulation

In this section, we adopt TV–Prox–GNWE to solve a problem of constrained consensus. We consider a game with $N = 15$ agents, where the strategy of every agent i is $x_i \in \mathbb{R}^5$, and its local feasible decision set is $\Omega_i \in [m_i,\, M_i]$, with m_i and M_i randomly drawn respectively from $[-100, -5]$ and $[5, 100]$. The local cost function is equal to $f_i(x_i) = \iota_{\Omega_i}(x_i)$. The adjacency matrices, descibing the communication network at every time instant k, are randomly generated and define digraphs of the type small-word, satisfying Standing Assumption 4. The coupling constraints are used to force the strategies towards the consensus subspace and are in the form $|x_i(k) - x_j(k)| \leq s(k)\mathbf{1}$, for every $i, j \in \mathcal{N}$, where $s(k) > 0$ and it is decreasing over time. Notice that in this case the *multiplier graph* is complete, see [12]. Finally, the parameters of the algorithm are chosen such that they always satisfy (8).

The trajectory of the profile strategy generated by TV–Prox–GNWE converges to the consensus subspace, this is shown in Fig. 1a, by means of the Laplacian matrix $\boldsymbol{L}$ of the multiplier graph. The initial strategy profile $\boldsymbol{x}(0)$ is randomly chosen in $\boldsymbol{\Omega}$. As expected from the result in Theorem 1, the constraints are satisfied asymptotically, see Fig. 1b.

V. Conclusion and outlook

In multi-agent network games, subject to time-varying coupling constraints and time-varying communication network, described by strongly connected digraphs, agents can fail to converge when they adopt proximal dynamics. Nevertheless, it is developed an iterative equilibrium seeking algorithms (TV–Prox–GNWE) that ensures the global convergence of the agents' strategies to an normalized equilibrium of the game, when it exists.

One of the most important open question in these type of problems regards the existence of an equilibrium point. This work can be improved with a new assumption for the equilibrium existence, which is general and easy to check.

Appendix

A. Algorithm derivation

1) Equilibria reformulation: the set of pn-ENWE, defined by the two equalities in (6), can be equivalently rephrased

$$\forall i \in \mathcal{N}: \quad \tilde{x}_i = \operatorname{prox}_{\frac{\delta_i(k)}{\delta_i(k)+1}\bar{f}_i}\left(\tfrac{\delta_i(k)}{\delta_i(k)+1}\left(\tfrac{1}{\delta_i(k)}x_i + \textstyle\sum_{j=1}^{N} a_{i,j}(k)x_j - \alpha_i(k)C_i^\top(k)\sigma\right)\right) \tag{7a}$$

$$\tilde{\sigma} = \operatorname{proj}_{\mathbb{R}^M_{\geq 0}}\left(\sigma + \tfrac{1}{\beta(k)}\left(C(k)\boldsymbol{x} - c(k)\right)\right) \tag{7b}$$

$$\forall i \in \mathcal{N}: \quad x_i^+ = x_i + \gamma(k)q_i(k)\left[\delta_i(k)(\tilde{x}_i - x_i) + \textstyle\sum_{j=1}^{N} a_{i,j}(k)(\tilde{x}_j - x_j) - \alpha_i(k)C_i^\top(k)\left(\tilde{\sigma} - \sigma\right)\right] \tag{7c}$$

$$\sigma^+ = \sigma + \gamma(k)\left[\beta(k)(\tilde{\sigma} - \sigma) + C(k)(\tilde{\boldsymbol{x}} - \boldsymbol{x})\right] \tag{7d}$$

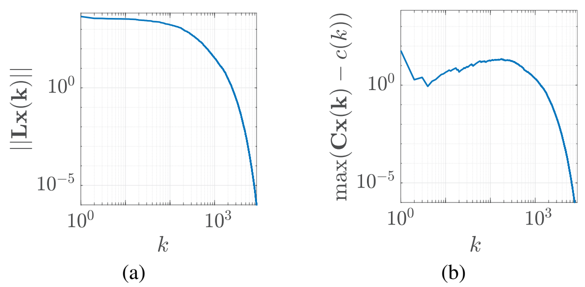

Fig. 1: (a) Convergence of the strategy profile $\boldsymbol{x}(k)$ to the consensus subspace. (b) Asymptotic satisfaction of the time-varying affine coupling constraints $\boldsymbol{C}\boldsymbol{x}(k) \leq c(k)$.

as the set of fixed points of a suitable mappings. First, we introduce the block-diagonal proximal operator

$$\operatorname{prox}_{\boldsymbol{f}}(\boldsymbol{z}) := \operatorname{col}\left((\operatorname{prox}_{f_i}(z_i))_{i\in\mathcal{N}}\right). \tag{9}$$

In (6), the first equality is equivalent to

$$\overline{\boldsymbol{x}} = \cap_{k>\bar{k}} \operatorname{prox}_{\boldsymbol{f}}(\boldsymbol{A}(k)\boldsymbol{x} - \boldsymbol{\Lambda}(k)\boldsymbol{C}^\top(k)\overline{\sigma}),$$

where $\boldsymbol{A}(k) := A(k) \otimes I_n$ and $\boldsymbol{\Lambda}(k) = \operatorname{diag}((\alpha_i(k))_{i\in\mathcal{N}}) \otimes I_n$. The second equality holds true if and only if $\overline{\sigma} = \operatorname{proj}_{\mathbb{R}^M}(\overline{\sigma} + \boldsymbol{C}(k)\overline{\boldsymbol{x}} - c(k))$.

In order to describe via operators these two relations, we define the static mappings

$$\boldsymbol{\mathcal{R}} := \operatorname{diag}(\operatorname{prox}_{\boldsymbol{f}}, \operatorname{proj}_{\mathbb{R}^M_{\geq 0}}) \tag{10}$$

and the time-varying affine one $\boldsymbol{\mathcal{G}}_k : \mathbb{R}^{nN+M} \to \mathbb{R}^{nN+M}$ as

$$\boldsymbol{\mathcal{G}}_k(\cdot) := \begin{bmatrix} \boldsymbol{A}(k) & -\boldsymbol{\Lambda}(k)\boldsymbol{C}^\top(k) \\ C(k) & I \end{bmatrix} \cdot - \begin{bmatrix} \boldsymbol{0} \\ c(k) \end{bmatrix}. \tag{11}$$

As a result, the dynamics of the game result equal to

$$\begin{bmatrix} \boldsymbol{x}(k+1) \\ \sigma(k+1) \end{bmatrix} = \boldsymbol{\mathcal{R}} \circ \boldsymbol{\mathcal{G}}_k\left(\begin{bmatrix} \boldsymbol{x}(k) \\ \sigma(k) \end{bmatrix}\right). \tag{12}$$

We exploit this new compact form to describe the set of pn–ENWE via the fixed points of $\boldsymbol{\mathcal{R}} \circ \boldsymbol{\mathcal{G}}_k$. In particular, by Definiton 3, a pair $(\bar{\boldsymbol{x}}, \bar{\sigma})$ is a pn–ENWE of the game in (3) if and only if $\operatorname{col}((\bar{\boldsymbol{x}}, \bar{\sigma})) \in \cap_{k>\bar{k}}\operatorname{fix}(\boldsymbol{\mathcal{R}} \circ \boldsymbol{\mathcal{G}}_k)$. Furthermore, from Lemma 1, we also know that a pn–ENWE is a p–GNWE of the original game. So, we focus on the design of an algorithm converging to the subset $\mathcal{E}$ for which we can take advantage of this new formulation.

We reformulate the fixed point seeking problem as a zero finding problem, see [19, Ch. 26].

Lemma 2 ([19, Prop. 26.1 (iv)]): Let $\boldsymbol{\mathcal{B}} := F \times N_{\mathbb{R}^M_{\geq 0}}$, with $F := \prod_{i=1}^{N} \partial \bar{f}_i$. Then, fix $(\boldsymbol{\mathcal{R}} \circ \boldsymbol{\mathcal{G}}_k) = \operatorname{zer}(\boldsymbol{\mathcal{A}}_k)$, where $\boldsymbol{\mathcal{A}}_k := \boldsymbol{\mathcal{B}} + \operatorname{Id} - \boldsymbol{\mathcal{G}}_k$. □

2) Modified proximal point algorithm: we describe the passages to develop the iterative algorithm solving the zero finding problem associated to the operator $\boldsymbol{\mathcal{A}}_k$, and, as a consequence, the original one of finding pn–ENWE of (3). We adopt a modified version of the *proximal point algorithm* (PPP) (see [19, Prop. 23.39] for its standard formulation). In particular, the update rule is a preconditioned version of the PPP algorithm proposed in [25, Eq. 4.18], after defining $\boldsymbol{\varpi} := \operatorname{col}(\boldsymbol{x}(k), \sigma(k))$ and $\boldsymbol{\varpi}^+ := \operatorname{col}(\boldsymbol{x}(k+1), \sigma(k+1))$, it can be rewritten as

$$\tilde{\boldsymbol{\varpi}} = \mathrm{J}_{\Phi^{-1}(k)\boldsymbol{\mathcal{A}}_k}\boldsymbol{\varpi} \tag{13a}$$

$$\boldsymbol{\varpi}^+ = \boldsymbol{\varpi} + \gamma(k)\overline{\boldsymbol{Q}}(k)\Phi(k)(\tilde{\boldsymbol{\varpi}} - \boldsymbol{\varpi}) \tag{13b}$$

where and $\gamma(k) > 0$ is the step–size of the algorithm and $\overline{\boldsymbol{Q}}(k) := \operatorname{diag}(\boldsymbol{Q}(k), I)$. The preconditioning matrix is chosen as

$$\Phi(k) := \begin{bmatrix} \boldsymbol{\delta}^{-1}(k) + \boldsymbol{A}(k) & -\boldsymbol{\Lambda}(k)C(k)^\top \\ C(k) & \beta(k)I_M \end{bmatrix} \tag{14}$$

where $\beta(k) \in \mathbb{R}_{>0}$ and $\boldsymbol{\delta}(k) := \operatorname{diag}((\delta_i(k))_{i\in\mathcal{N}}) \otimes I_n$. The *self-adjoint* and *skew symmetric* components are defined as $U(k) := (\Phi(k) + \Phi^\top(k))/2$ and $S(k) := (\Phi(k) - \Phi^\top(k))/2$. Due to the non symmetric preconditioning the resolvent operator takes the form

$$\mathrm{J}_{\Phi^{-1}(k)\boldsymbol{\mathcal{A}}_k} := \mathrm{J}_{U^{-1}(k)(\boldsymbol{\mathcal{A}}_k + S(k))}(\operatorname{Id} + U^{-1}(k)S(k)).$$

The parameters $\boldsymbol{\delta}(k)$ and $\beta(k)$ in the preconditioning have to be chosen such that $U(k) \succ 0$ and $\|\overline{\boldsymbol{Q}}(k)U(k)\| \leq \gamma^{-1}(k)$. The resulting bounds are reported in (8).

Using a reasoning akin to the one in [25, Proof of Th. 4.2], one can show that the set of fixed points of the mapping describing the update in (13a)–(13b) coincides with $\operatorname{zer}(\boldsymbol{\mathcal{A}}_k)$.

We focus on (13a) , so $\tilde{\boldsymbol{\varpi}} = \mathrm{J}_{U^{-1}(\boldsymbol{\mathcal{A}}+S)}(\operatorname{Id} + U^{-1}S)\boldsymbol{\varpi}$ leads to $\boldsymbol{0} \in \Phi(\tilde{\boldsymbol{\varpi}} - \boldsymbol{\varpi}) + \boldsymbol{\mathcal{A}}\tilde{\boldsymbol{\varpi}}$. Focusing on each row block, the update rules of $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\sigma}})$ are attained and, combining them with (13b), we complete the derivation of (7a) – (7d), see [26] for a step by step computation.

B. Convergence proof of TV–Prox–GNWE

In the following, we use the two scalars L_k and m_k, the former is the Lipschitz constant of $S(k)$ and the latter is such that $m_k\|x\|^2 < \langle U(k)x, x\rangle$, so $\|U^{-1}(k)\| \leq m_k^{-1}$. Next, we define $K(k) := \overline{\boldsymbol{Q}}_k U(k)$ and the scalars $\rho := m_k^{-1} L_k$, $q_m := \min_i [\overline{\boldsymbol{Q}}_k]_{ii}$ and $M_k \geq \|U(k)\|$. We always choose the normalized version of the left Perron Frobenius eigenvector $q(k)$ of $A(k)$, so $\max_i [\overline{\boldsymbol{Q}}_k]_{ii} \leq 1$. The proofs follow similar steps to the ones in [25, Prop.2.1 and 4.2].

Lemma 3: For all $k \in \mathbb{N}$, consider the time-varying operator

$$\mathcal{T}_\Phi := \mathrm{J}_{\Phi^{-1}\boldsymbol{\mathcal{A}}} + U^{-1} S(\mathrm{J}_{\Phi^{-1}\boldsymbol{\mathcal{A}}} - \mathrm{Id}) \,, \tag{15}$$

then the following hold:

(i) $\mathcal{T}_\Phi$is quasi-nonexpansive in the space $\mathcal{H}_K$,

(ii) if $L_k \leq m_k$, then $\mathrm{fix}(\mathcal{T}_\Phi) = \mathrm{zer}(\boldsymbol{\mathcal{A}})$. □

Proof: Refer to [16, Lem. 3]. ■

Proof of Theorem 1

Consider $\mathcal{T}_\Phi$, as in (15), then it holds

$$\overline{\boldsymbol{Q}} U(\mathrm{Id} - \mathcal{T}_\Phi)(x) = \overline{\boldsymbol{Q}} \Phi(x - \mathrm{J}_{\Phi^{-1}\boldsymbol{\mathcal{A}}}) \,. \tag{16}$$

Lemma 3 implies that $\mathcal{T}_\Phi$ is quasi-NE in $\mathcal{H}_K$, hence $\mathcal{S} := (\mathrm{Id} + \mathcal{T}_\Phi)/2$ belongs to $\mathfrak{I}$, [18, Prop. 2.2(v)]. From [25, Prop. 4.1] and (16), we define $\mathcal{W}_\Phi := \mathrm{Id} - \|K\|^{-1} K(\mathrm{Id} - \mathcal{S})$ belonging to $\mathfrak{I}$ in $\mathcal{H}$ and $\mathrm{fix}(\mathcal{W}_\Phi) = \mathrm{fix}(\mathcal{T}_\Phi) = \mathrm{zer}(\boldsymbol{\mathcal{A}})$, from Lemma 3. The update in (13) is equal to $\boldsymbol{\varpi}^+ = \boldsymbol{\varpi} + 2\gamma\|K\|(\mathcal{W}_\Phi \boldsymbol{\varpi} - \boldsymbol{\varpi})$, where $\gamma\|K\| < 1$ for all k, due to the choice of δ_i and β. Thus, from [18, Th. 4.2(ii) and 4.3], we have that $(\|\boldsymbol{\varpi} - \mathcal{W}_\Phi \boldsymbol{\varpi}\|^2)_{k\in\mathbb{N}}$ is summable and converges in $\mathcal{H}$ to an element $\overline{\boldsymbol{\varpi}} \in \mathcal{E}_c$, if and only if every sequential cluster point of sequence belong to $\mathcal{E}_c$.

Due to the choice of $\boldsymbol{\delta}$ and β in $\Phi(k)$, it holds

$$\|\boldsymbol{\varpi} - \mathcal{T}_\Phi \boldsymbol{\varpi}\|_K^2 \leq 4 m_k^{-1} M_k^2 \|\boldsymbol{\varpi} - \mathcal{W}_\Phi \boldsymbol{\varpi}\|^2 \to 0 \,. \tag{17}$$

Let us define $x_\varpi := \mathrm{J}_{\bar{\gamma}\mathcal{A}}(\varpi - \bar{\gamma} U^{-1} S \varpi)$, Moreover, from Lemma 3, it follows

$$\begin{aligned}(1-\rho^2)\|\varpi - x_\varpi\|_K &\leq \|\varpi - \varpi^*\|_K - \|\mathcal{T}_\Phi \varpi - \varpi\|_K \\ &\leq -\|\mathcal{T}_\Phi \varpi - \varpi\|_K - 2M_k \|\mathcal{T}_\Phi \varpi - \varpi\|_K \|\varpi - \varpi^*\|\end{aligned}$$

The above inequality leads to

$$\begin{aligned}(1-\rho^2) m_k q_m \|\varpi - x_\varpi\| \leq& -\|\mathcal{T}_\Phi \varpi - \varpi\|_K \\ &- 2M_k \|\mathcal{T}_\Phi \varpi - \varpi\|_K \|\varpi - \varpi^*\|\end{aligned} \tag{18}$$

The sequence $(\|\varpi(k) - \varpi^*\|)_{k\in\mathbb{N}}$ is bounded and from (17) and (18), we deduce that $\varpi - x_\varpi \to 0$. Notice also that

$$\|\overline{Q}\Phi(\varpi - x_\varpi)\| \leq (M_k + L)\|\varpi - x_\varpi\| \to 0 \,.$$

From the definition of x_ϖ, it follows $u_k := \overline{Q}\Phi(\varpi - x_\varpi) \in \boldsymbol{\mathcal{A}} x_\omega$. Thus, since $u_k \to 0$, we conclude that $x_\varpi \to \overline{x}_\varpi \in \mathrm{zer}(\boldsymbol{\mathcal{A}})$. ■

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