FLUME STUDY ON
SALINITY INTRUSION IN ESTUARIES
XXIII

ANALYTICAL INVESTIGATIONS OF TWO-DIMENSIONAL EQUATIONS

DELFT HYDRAULICS LABORATORY
DELFT M 896-XXIII

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CONTENTS

1 Introduction

The principle that the salinity intrusion into a tidal river can be described to some degree of accuracy by a two-dimensional mathematical model provides the basis for several theoretical studies of this phenomenon. To verify this starting point the results of the systematic flume tests of the Delft Hydraulics Laboratory (see D.H.L.-publication nr. 72) are analysed as if they were satisfying the two-dimensional transport equations of mass and momentum approximated in a way as given in the equations 1 trough 4 in the next section. This investigation is justified by the fact that the fluid motion and salinity distribution of the flume tests are to a high degree two-dimensional as a result of the application of resistance bars in the flume. Considering the flume as a schematized hydraulic model of a prototype estuary the resistance bars were introduced to create the additional (extra) flow-resistance which is required in a distorted hydraulic model.
The investigation dealt with in this report shows that in the range of variation of the systematic tests the most upstream end of the salt intrusion cannot be described in an acceptable way by the above equations. The analytical considerations applied to the approximate equations reveals that unless the vertical diffusion coefficient at the upstream end of the salt intrusion tends to zero or infinite the solution of this problem formulation cannot exist. These unbounded values of the diffusion coefficient in any part of the flow region are considered here as non-realistic and practical consequencies of locally unbounded diffusion are not studied further.
It must be borne in mind however that an infinite diffusion coefficient of salt at the upstream end of the intrusion can be a direct consequence resulting from the applied approximations in the starting differential equations 1 through 4. In that case however the physical meaning of the approach is lost and a physical more realistic description of the local flow phenomena must be recommended.
The conclusions given above are derived under conditions restricted to the range of parameter values underlying the systematic flume tests. The question can be raised if these conclusions are still valid under more general conditions and especially in the prototype situation.
A decisive answer to this question cannot be given. Flow conditions without resistance bars are not considered and the mathematical methods used in this paper are hard to apply to those cases as well as to the three-dimen-

sional prototype situation. All that can be said is that the conclusions can be valid under less restrictive conditions.
It should be mentioned, however, that field data available for the partly mixed Rotterdam Waterway estuary reveal a more or less well mixed behaviour of this estuary at the tip of the salinity intrusion. This tip-behaviour was not found in the flume tests using resistance bars.

2 Formulation of the problem

The fluid motion and salinity distribution in a two-dimensional tidal flume are supposed to be described by the following equations:
the continuity equation for an incompressible fluid:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

the continuity equation for salt:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = \frac{\partial}{\partial y}\left(D_y \frac{\partial c}{\partial y}\right) \qquad (2)$$

where the time-averaged effect of turbulence is described by a diffusion term. Horizontal diffusive transport is neglected in comparison with the convective transport.
The equations of motion are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho_o}\frac{\partial p}{\partial x} + \frac{g}{C^2R}\, u/u/ = 0 \qquad (3)$$

$$\frac{\partial p}{\partial y} = -\rho g \qquad (4)$$

The friction term is expressed as a volume force brought about by resistance bars. Molecular and turbulent shear effects are supposed to be small compared with the last term in equation (3) and are neglected.
The Boussinesq-approximation has been applied i.e. the variation in density is neglected except in the gravity term.
In the equation of vertical motion the hydrostatic shallow water approximation is applied.

where
ad (1) u, v are velocity components in x, respectively y-direction
x, y coordinates directed horizontally in upstream direction respectively vertically upward

ad (2) c salinity

t time

D_y vertical diffusion coefficient

ad (3) p pressure

ρ density

g acceleration of gravity

C coefficient of De Chézy

R hydraulic radius of the flume

The relation between density and salinity is approximated by:

$$\rho = \rho_o \ (1 + \alpha c) \qquad (5)$$

where ρ_o is density of fresh water
and α is a given conversion factor

The resistance bars are vertically placed in a regular configuration on the bottom of the flume. The spacing between two bars in a row is a, the distance between the rows is denoted by b. So the number of bars per unit area is $\frac{1}{a.b}$. Let the drag coefficient of such a bar perpendicular to the flow direction being C_D and the bar thickness measured perpendicular to the mean stream denoted by d, then the relation between the Chézy coefficient and the uniformly distributed external volume force is:

$$\frac{g}{C^2R} = \frac{1}{2} \ \frac{C_D.d}{a.b} \qquad (6)$$

Substituting equation (5) into (2) gives:

$$\frac{\partial \rho}{\partial t} + u \ \frac{\partial \rho}{\partial x} + v \ \frac{\partial \rho}{\partial y} = \frac{\partial}{\partial y} \ (D_y \ \frac{\partial \rho}{\partial y}) \qquad (7)$$

Now we have four equations (1), (3), (4) and (7) for the following five unknowns:

$u \ (x, y, t)$

$v \ (x, y, t)$

$\rho \ (x, y, t)$

$p \ (x, y, t)$ and D_y

The Chézy coefficient and hydraulic radius are well defined and considered to be known.

The vertical diffusioncoefficient however may a priori not be assumed as a constant nor as a known function of local flow and salinity parameters. Some information on the value of D_y in the absence of resistance bars is available. In that case D_y is a decreasing function of the Richardson-number defined by:

$$Ri = \frac{-g\frac{\partial \rho}{\partial y}}{\rho(\frac{\partial u}{\partial y})^2}$$

In the case with resistance bars some assumption has to be made about the vertical diffusion. This is one of the main problems in connection with the study on salt intrusion. We will delay this assumption as long as possible waiting for an opportunity to base the assumption on results of the systematic flume tests.

Boundary conditions:

The inclination of the tidal flume is zero so the boundary conditions at the bottom are:

$$\text{at} \quad y = 0: \quad v = 0 \quad D_y \frac{\partial c}{\partial y} = 0$$

The water surface is denoted by:

$$y = \eta\ (x, t) + h_o$$

where h_o is a mean waterdepth.

$$\text{at} \quad y = h_o + \eta\ (x, t) : p = 0 : \quad D_y \frac{\partial c}{\partial y} = 0$$

$$\text{and} \qquad \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} - v = 0$$

The discharge control at the upstream end of the flume is operated such that the tidal motion in the flume coincides with that of a flume which would have a length L, while

$$\text{at} \quad x = L, \quad u\ (L,t)\ (h_o + \eta\ (L,t)\ ) = q_r$$

where q_r is a given constant river discharge
(note that u at x = L is independent of y)
The actual length of the flume exceeds the length of the zone with salinity intrusion
At x=L, c = 0

At the rivermouth:

$$x = 0$$

$$\eta\ (0,t) = A \cos 2\pi \frac{t}{T} + \text{const.}$$

where A is the tidal amplitude
and T the tidal period
Both are given constants

The required boundary conditions at the rivermouth with regard to the velocity and salinity distr bution depend on the final analytical model being used.
At this moment we assume that sufficient boundary conditions at the rivermouth are given and we postpone the formulation of conditions untill the moment we need them.
The starting point of assuming conditions at the mouth being known indeed seriously limits the predicting possibilities of the approach.
As a first step, however, the aim of this study is to get some insight into the susceptibility of the gross behaviour of salt intrusion of the various parameters.

Resumé: The four functions u, v, ρ and p are defined by the above set of approximated conservation laws and sufficient boundary conditions, provided the vertical diffusion coefficient is given. The approximations applied to the conservation laws are mostly based on the shallowness of the flume i.e., on the fact that the longitudinal length scale is large compared to the vertical length scale. The final solution depends on the special choice of boundary conditions at the rivermouth and on the choice of the vertical diffusion coefficient which have still to be made.
Note: Not all of the applied approximations are as convincing as they should be. Particularly the condition that the turbulence should not too much deviate from isotropic is a condition certainly not fullfilled in the case of a flow around resistance bars. As a result we shouls bear in mind the possibility that the horizontal turbulent transport mechanism is prematurely dropped out of the equations.

3 Application of asymptotic methods

Applying asymptotic methods to the system of partial differential equations leads to the construction of approximations of functions $\Phi(\vec{x}, \mu)$ of variable $\vec{x}$ and parameter μ for μ small, i.e. for $\mu \to 0$.

The particular functions Φ are u, v, ρ and p and the variable vector $\vec{x}$ has the components x, y, t. The choice of the small parameter μ is found by expressing the entire problem in suitable dimensionless coordinates.
Consider the system of differential equations again. Integrating equation (4) and using the boundary condition p = 0 on the free surface the pressure becomes

$$p = g \int_y^{h_o+\eta} \rho dy \tag{8}$$

Substitution of equation (8) in (3) eliminates the pressure.
The remaining equations repeated for convenience are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = \frac{\partial}{\partial y} \left(D_y \frac{\partial \rho}{\partial y}\right) \tag{7}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{g}{\rho_o} \int_y^{h_o+\eta} \frac{\partial \rho}{\partial x} dy + \frac{g\rho_{y=h_o+\eta}}{\rho_o} \frac{\partial \eta}{\partial x} +$$

$$+ \frac{g}{C^2 R} |u| u = 0 \tag{9}$$

We write these equations in dimensionless form.
Before doing so it is advisable to return for a moment to the non-approximated problem to see if dimensional analysis reveals any controlling parameter being lost in the simplified mathematical model. The flow- and salinity distribution in the physical flume model is controlled by the following parameters:

geometrical similarity parameters $\frac{L}{h_o}$; g/C^2

initial- and boundary condition A/h_o; $\Delta\rho/\rho o$; L/L_R ; $q_r / h_o\sqrt{gh_o}$

dynamical similarity $Re = \frac{q_r}{\nu}$; $Pr = \frac{\nu}{\gamma}$; $T\sqrt{\frac{g}{h_o}}$

where L_R is the resonance-length of the flume: $L_R = \frac{1}{4} T \sqrt{gh_o}$

and γ the molecular diffusioncoefficient of salt

Based on the results of the systematic tidal flume tests Rigter [1] stated that for the tests with resistance bars the Reynolds-number was large enough to neglect any effects due to variation in Re.

The Prandtl number as a controlling parameter will be rather unimportant as the molecular diffusioncoefficient is extremely small as compared to the coefficient of turbulent diffusion at this high Re-number flow.
Anyhow, as long as the physical model uses the same fluid and dissolved matter as in the prototype the Prandtl number will be nearly constant and can be omitted from the list of controlling parameters.
From these considerations it can be concluded that the Chézy coefficient (or drag coefficient on a resistance bar) is Reynolds independent, which is not contradictory to our assumption C = constant. Further, the relation between the diffusion coefficient D_y and the Prandtl number can be left out of consideration.
So, no essential features are lost at this moment in the simplified mathematical model.

4 The zeroth order outer approximation

Introduce the following dimensionless coordinates

$$\bar{x} = \frac{x}{L_R} \; ; \quad \bar{y} = \frac{y}{h_o} \quad ; \quad \bar{t} = \frac{t}{\frac{1}{4}T}$$

and the dimensionless variables:

$$\bar{u} = \frac{u}{\sqrt{\varepsilon g h_o}} \; ; \quad \bar{v} = \frac{L_R}{h_o} \frac{v}{\sqrt{\varepsilon g h_o}} \quad ; \quad \bar{\eta} = \frac{\eta}{\sqrt{\varepsilon h_o}}$$

and $\bar{\rho} = \dfrac{\rho - \rho_o}{\rho_z - \rho_o} = \dfrac{\rho - \rho_o}{\varepsilon \rho_o}$

where $\varepsilon = \dfrac{\rho_z - \rho_o}{\rho_o}$ $\quad \rho_o$ = density of fresh water

ρ_z = density of sea water

The variables may be expressed as a function of the controlling numbers and the dimensionless coordinates, so for instance:

$$\frac{u}{\sqrt{\varepsilon g h_o}} = F\left(\frac{x}{L_R}, \frac{y}{h_o}, \frac{t}{\frac{1}{4}T}, \frac{h_o}{L}, \frac{A}{h_o}, \frac{g}{C^2}, \frac{L}{L_R}, \frac{\Delta\rho}{\rho}, \frac{q_r}{h_o\sqrt{\varepsilon g h_o}}\right)$$

The hydrodynamic dimensionless equations become:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{10}$$

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} + \bar{u}\sqrt{\varepsilon}\frac{\partial \bar{\rho}}{\partial \bar{x}} + \bar{v}\sqrt{\varepsilon}\frac{\partial \bar{\rho}}{\partial \bar{y}} = \frac{L_R}{h_o} \cdot \frac{\partial}{\partial \bar{y}}\left(\frac{D_y}{h_o\sqrt{g h_o}} \frac{\partial \bar{\rho}}{\partial \bar{y}}\right) \tag{11}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\sqrt{\varepsilon}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\sqrt{\varepsilon}\frac{\partial \bar{u}}{\partial \bar{y}} + \sqrt{\varepsilon}\int_{\bar{y}}^{1+\sqrt{\varepsilon}\,\bar{\eta}} \frac{\partial \bar{\rho}}{\partial \bar{x}}\, d\bar{y} +$$

$$(1 + \varepsilon\bar{\rho}\,(h_o + \eta, t))\frac{\partial \bar{\eta}}{\partial \bar{x}} + \frac{g}{C^2}\frac{L_R}{R}\sqrt{\varepsilon}\,\bar{u}/\bar{u}/ = 0 \tag{12}$$

The basic parameters that enter the problem through the field equations are

$$\varepsilon = \frac{\rho_z - \rho_o}{\rho_o}$$

$$\frac{g L_R}{C^2 R} = \mu \qquad \text{and} \qquad \frac{L_R D_v}{h_o^2\sqrt{g h_o}} = \delta$$

The order of magnitude of these parameters have to be fixed in their mutual connection based on data from systematic flume test series. The controlling conditions in these test-series are systematically varied (one at a time) around a given reference test (T 3) corresponding to a measured prototype situation on the Rotterdam Waterway [2] . It will be obvious to consider test T 3 in this study too as a reference situation on which the order of magnitude estimates are based.

From test T 3 it appears that the dependent variables u, v, ρ and η are made dimensionless in such a way that their order of magnitude can be considered as unity.

The magnitude of ε is $O(10^{-2})$ hence small compared to unity.

The magnitude of μ is $O(10^{+1}) = O(\varepsilon^{-\frac{1}{2}})$

An estimate of the order of magnitude of δ is not easy to make for the following reason.

The vertical diffusion coefficient D_y is hitherto an unknown function of one or more of the controlling parameters already quoted and possibly also of the dimensionless coordinates. The value of D_y has been derived in a rather complicated way from the experimental data for a large amount of parameter values.

An estimate of the order of magnitude of a suitable chosen norm on δ has therefore to be made. However, due to the enormous effect of inaccuracies in the measurements on the so-derived D_y values this approach leads to unacceptable results (maximum $\rightarrow \infty$, mean tends to zero, L_2 or r.m.s.- norm tends to infinite).

An impression based on a L_2- norm (root mean square) after omitting D_y values outside a certain band width leads to $\delta \gtrsim O(\varepsilon^{\frac{1}{2}})$.

In view of the uncertainties around δ we will again delay a ruling on this point and see how far we can get without such information.

Introduce ε as the small parameter.

Note that the dependent variables are made dimensionless with the small parameter. So, if ε goes to zero, the discharge, tidal elevation, and friction-coefficient tends to zero in a prescibed mutual relationschip. As will be shown later on the salt intrusion in this formulation does not tend to zero if $\varepsilon \rightarrow 0$.

We assume an expansion of the following form

$$\bar{u} = u_o^{(o)} + \sqrt{\varepsilon}\, u_o^{(1)} + O(\varepsilon)$$

$$\frac{L_R}{h_o} \bar{v} = v_o^{(o)} + \sqrt{\varepsilon}\, v_o^{(1)} + O(\varepsilon)$$

$$\bar{\eta} = \eta_o^{(o)} + \sqrt{\varepsilon}\, \eta_o^{(1)} + O(\varepsilon) \qquad (12a)$$

$$\bar{\rho} = \rho_o^{(o)} + \sqrt{\varepsilon}\, \rho_o^{(1)} + O(\varepsilon)$$

The zeroth-order approximation equations become

$$\frac{\partial u_o^{(o)}}{\partial \bar{t}} + \frac{\partial \eta_o^{(o)}}{\partial \bar{x}} + \mu\sqrt{\varepsilon}\, u_o^{(o)} |u_o^{(o)}| = 0 \qquad (13)$$

and

$$\frac{\partial u_o^{(o)}}{\partial \bar{x}} + \frac{\partial v_o^{(o)}}{\partial \bar{y}} = 0 \qquad (14)$$

Boundary conditions:

For $\bar{x} = \frac{L}{L_R}$ the horizontal velocity is $\bar{u}_r = \frac{u_{river}}{\sqrt{\varepsilon g h_o}}$

From test T3 it appears that $\bar{u}_r = O(\sqrt{\varepsilon})$

Hence to the zeroth-order of approximation we have

$$u_o^{(o)} = 0 \quad \text{for } \bar{x} = \frac{L}{L_R} \qquad (15)$$

The equation of motion as well as the boundary condition for $u_o^{(o)}$ are independent of the vertical coordinate. So $u_o^{(o)}$ is independent of y and equation (14) can be integrated over the depth. Using the boundary conditions;

$v_o^{(o)} = 0$ at the bottom

and $v_o^{(o)} = \frac{\partial \eta_o^{(o)}}{\partial \bar{t}}$ at the watersurface , equation (14) integrated over the depth becomes

$$\frac{\partial u_o^{(o)}}{\partial \bar{x}} + \frac{\partial \eta_o^{(o)}}{\partial \bar{t}} = 0 \qquad (16)$$

With the boundary conditions (15) and

$$\eta_o^{(o)} = \frac{A}{\sqrt{\varepsilon}\, h_o} \cos \frac{\pi}{2} \bar{t} \qquad \text{for } \bar{x} = 0 \tag{17}$$

the equations (13) ans (16) can be solved.

In a study on salinity intrusion in estuaries Ippen and Harleman [3] calculated the tidal motion in a density homogeneous river.
The approach started with the introduction of a relation for the wave height valid for a damped cooscillating linear tidal wave.
Such a relation satisfies the equation of motion (13) only if the friction term is linearised, i.e. the approximation $u|u|$ is linear proportional to u has been applied. The factor of proportionality had to be determined empirically.
It is worthwile to evaluate the same approximation in this case of a tidal flow in a flume with resistance bars. Here the factor of proportionality can be related to known parameters. The relation is based on the approximation

$$u|u| \approx \frac{\pi}{4} u_{max}\, u \approx \frac{\pi^2}{8} u_{rms} . u$$

where u_{rms} is the root-mean-square value of u in the x - t domain, hence a constant

$$u_{rms} = \frac{1}{L} \int_0^L \{ \frac{1}{T} \int_0^T \sqrt{u^2}\, dt\} \, dx$$

The linearised friction term in equation (13) becomes

$$\mu u_o^{(o)} | u_o^{(o)} | \approx \mu \frac{\pi^2}{8} u_{rms}^{(o)} \quad u_o^{(o)}$$

The factor of proportionality $\mu \frac{\pi 2}{8} u_{rms}^{(o)}$ can be determined after $u_o^{(o)}$ has been solved as a function of this parameter. So $u_{rms}^{(o)}$ is given implicitly. After some straight forward algebraic manipulations the following analytic solution is obtained.

$$u_o^{(o)} = \frac{A}{\sqrt{\varepsilon h_o}} \sqrt{\frac{\cos 2\alpha}{2\cos 2\beta + 2\cosh 2\gamma}} \left[e^{-\gamma \frac{L_R}{L}(\bar{x} - \frac{L}{L_R})} \cos\left(\frac{\pi}{2}\bar{t} + \theta + \alpha - \beta\bar{x}\frac{L_R}{L} + \beta\right) - e^{\gamma \frac{L_R}{L}(\bar{x} - \frac{L}{L_R})} \cos\left(\frac{\pi}{2}\bar{t} + \theta + \alpha + \beta\bar{x}\frac{L_R}{L} - \beta\right)\right] \quad (18)$$

and

$$\eta_o^{(o)} = \frac{A}{\sqrt{\varepsilon h_o}} \frac{1}{\sqrt{2\cos 2\beta + 2\cosh 2\gamma}} \left[e^{-\gamma \frac{L_R}{L}(\bar{x} - \frac{L}{L_R})} \cos\left(\frac{\pi}{2}\bar{t} + \theta - \beta\bar{x}\frac{L_R}{L} + \beta\right) + e^{\gamma \frac{L_R}{L}(\bar{x} - \frac{L}{L_R})} \cos\left(\frac{\pi}{2}\bar{t} + \theta + \beta\bar{x}\frac{L_R}{L} - \beta\right)\right] \quad (19)$$

where θ is defined by $\operatorname{tg}\theta = \operatorname{tg}\beta \cdot \operatorname{tgh}\gamma$

with $\gamma = \frac{\pi}{2} \frac{L}{L_R} \frac{\sin\alpha}{\sqrt{\cos 2\alpha}}$ and $\beta = \frac{\pi}{2} \frac{L}{L_R} \frac{\cos\alpha}{\sqrt{\cos 2\alpha}}$

where α is implicitly given as

$$\operatorname{tg} 2\alpha = \mu \frac{\pi^2}{4} u_{rms}^{(o)} \sqrt{\varepsilon}$$

and

$$u_{rms}^{(o)} = \frac{L_R}{L} \int_0^{L/L_R} \left\{ \frac{1}{4} \int_0^4 \sqrt{u_o^{(o)2}}\, d\bar{t}\right\} d\bar{x}$$

Basically the solution is analogous to those of Ippen and Harleman.
The only difference is that whereas they need two coefficients both to determine empirically, the above result is based only on the Chézy-coefficient which can be considered as rather well-defined.
It is therefore possible to derive a relation between the two empirical constants from Ippen and Harleman and the Chézy coefficient. The theoretical result may be checked with the empirical relationship based on their experiments.
The elaboration is left to the reader.

To avoid any misunderstanding we recapitulate that the analytic solution cannot be considered as the zeroth order solution of the outer approximation equation (13)

It is a valuable practical approximation, but the accuracy does not increase with diminishing value of ε.
The higher harmonic contributions (terms in $\cos n \frac{\pi}{2} \bar{t}$) of the exact solution of the non-linear equation (13) are omitted without a definition of the order of accuracy of the approximation.

5 The first order outer approximation

In the first order approximation the continuity equation for the salt comes into play. Substituting the asymptotic expansions (12a) into the dimensionless differential equations (10), (11) and (12) and equating terms of the order of magnitude $\sqrt{\varepsilon}$ the following first order approximation equations are obtained.

$$\frac{\partial u_o^{(1)}}{\partial \bar{x}} + \frac{\partial v_o^{(1)}}{\partial \bar{y}} = 0 \qquad (20)$$

$$\frac{\partial u_o^{(o)}}{\partial \bar{t}} + u^{(o)} \frac{\partial u^{(o)}}{\partial \bar{x}} + v_o^{(o)} \frac{\partial u_o^{(o)}}{\partial \bar{y}} + \int_{\bar{y}}^{1} \frac{\partial \rho_o^{(o)}}{\partial \bar{x}} dy + \frac{\partial \eta_o^{(1)}}{\partial \bar{x}} +$$

$$2\mu\sqrt{\varepsilon}\, u_o^{(1)} |u_o^{(o)}| = 0 \qquad (21)$$

This procedure can only be applied to the continuity of salt after the question about the order of magnitude of the diffusionparameter δ in terms of ε is settled.

As mentioned already, we will try to proceed as far as possible without an estimate of δ. With this lack of information however we have to leave the attempt to solve part of the problem within the region of salt intrusion. Only integrated effects can be studied if enough boundary conditions are given.

Integration of equation (20) over $\bar{y}$ between bottom and water surface yields

$$\int_o^1 \frac{\partial u_o^{(1)}}{\partial \bar{x}} dy + \eta_o^{(o)} \frac{\partial u_o^{(o)}}{\partial \bar{x}} + \frac{\partial \eta_o^{(1)}}{\partial \bar{t}} + u_o^{(o)} \frac{\partial \eta_o^{(o)}}{\partial \bar{x}} = 0$$

or

$$\frac{\partial}{\partial \bar{x}} \left[\int_o^1 u_o^{(1)} dy + u_o^{(o)} \eta_o^{(o)} \right] = - \frac{\partial \eta_o^{(1)}}{\partial \bar{t}} \qquad (22)$$

The tidal motion is considered to be periodic with period T. Time averaging over one tidal period T yields

$$\frac{\partial \tilde{\eta}_o^{(1)}}{\partial \bar{t}} = 0 \qquad \left(\text{notation } \frac{1}{T} \int_{t_o}^{t_o + T} f dt = \tilde{f}\right)$$

or

$$\int_0^1 \widetilde{u_o^{(1)}}\, dy + \widetilde{u_o^{(o)}\, \eta_o^{(o)}} = \text{const.}$$

For $\bar{x} = \frac{L}{L_R}$ $\quad u_o^{(o)} = 0$ and $u_o^{(1)} = \bar{u}_{riv}$

so

$$\int_0^1 \widetilde{u_o^{(1)}}\, dy + \widetilde{u_o^{(o)}\, \eta_o^{(o)}} = \bar{u}_{riv} \tag{23}$$

Because $u_o^{(o)}$ is independent of y it follows from equation (21) that outside the region of saltintrusion, where $\frac{\partial \rho^{(o)}}{\partial x} = 0$, the first order approximation $u_o^{(1)}$ is as well independent of y.

Hence, outside the region of saltintrusion equation (23) becomes

$$\widetilde{u_o^{(1)}} = \bar{u}_{riv} - \widetilde{u_o^{(o)}\, \eta_o^{(o)}}$$

Integrating the equation of motion over x between 0 and x as well as over the waterdepth and taking the time average values we obtain

$$\widetilde{\eta_o^{(1)}}(x) - \widetilde{\eta_o^{(1)}}(o) + \tfrac{1}{2}\{ \widetilde{u_o^{(o)\,2}}(x) - \widetilde{u_o^{(o)\,2}}(o) \} + \int_0^1 \int_{\bar{y}}^1 \{ \widetilde{\rho_o^{(o)}}(\bar{x},\bar{y}) + \widetilde{\rho_o^{(o)}}(o,\bar{y}) \} d\bar{y} dy + 2\mu \sqrt{\varepsilon} \int_0^x | \widetilde{u_o^{(o)}} | \{ \bar{u}_{riv} - \widetilde{u_o^{(o)}\, \eta_o^{(o)}} \} dx = o \tag{24}$$

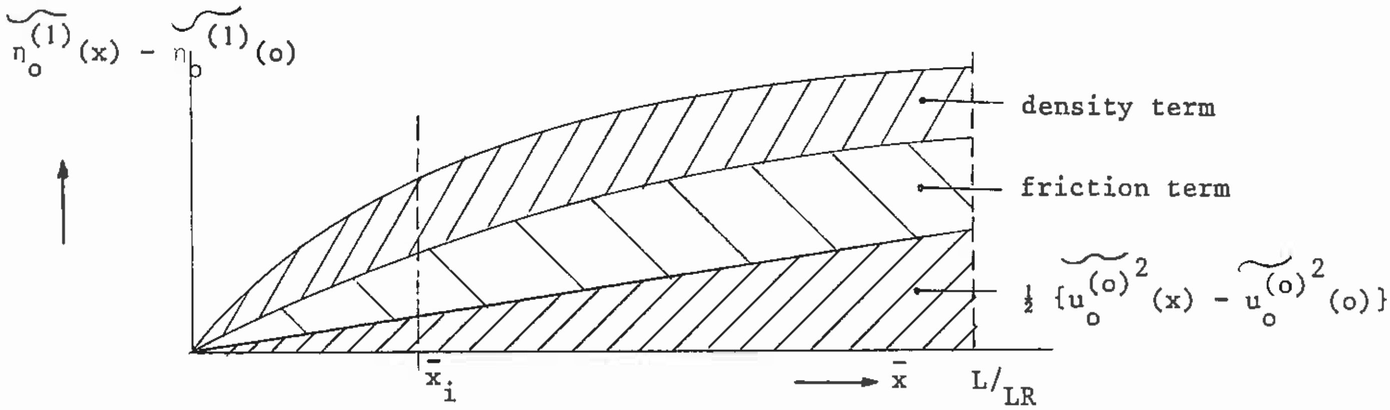
The last term contains a slight inaccuracy admitted here for the sake of simplicity, namely the approximation $u|u| = 2|\widetilde{u}|\, u$ similar to the one already mentioned earlier.

From equation (24) it follows that the mean water elevation

$$\widetilde{\eta_o^{(1)}}(\bar{x}) - \widetilde{\eta_o^{(1)}}(o)$$

can be calculated outside the region of salt intrusion, where $\rho_o^{(o)}(\bar{x}, \bar{y}) = 0$, using the zeroth order solution $u_o^{(o)}$ and the given boundary condition for the density distribution at x = 0

A sketch of the contributions of the various terms in (24) to the mean water elevation as a function of x is shown below



The region $0<\bar{x}< x_i$ is the region of salt intrusion, where not enough information is available to determine the mean water level.
Note that the slope of the waterlevel due to the tidal waves is an order of magnitude higher than the slope in waterlevel due to the discharge alone. The contribution of the latter does not appear in this order of approximation.
The mean-water level increase which has been found numerically by Stigter and Siemons, publ. no. 52 WL for one special case is compatible with equation (24) outside the region of salt intrusion.

6 Near field equations

Near the river mouth the density in the flume varies in x-direction over a distance equal to the length of the salt intrusion. In this region density currents occur.
Therefore to describe the flow and density phenomena near the river mouth, the longitudinal coordinate must be compared with a length scale of the order of the salt intrusion length L_i.

So $\bar{x}_i = \frac{x}{O(L_i)}$

From the systematic flume tests it appears that

$$\frac{L_i}{L_R} = O(\sqrt{\varepsilon})$$

So the x-coordinate will be stretched according to

$$\bar{x}_i = \frac{\bar{x}}{\sqrt{\varepsilon}}$$

Let us introduce now a coordinate system moving in the longitudinal direction with a representative velocity of the tidal motion.
We choose for this representative value the zeroth order outer approximation of the horizontal velocity $u_o^{(o)}(0,t)$ at the river mouth $x = 0$.
The idea of the approach is suggested by the well-known fact that the flow velocity in the region of salt intrusion is composed of two contributions, one contribution due to the tidal wave , the other due to density currents.

So $u(x,y,t) = u_{tide} + u_{density\ curr.} = u_t + u_d$

From the results of the systematic flume tests an estimate can be obtained of the magnitude of both contributions compared to each other.
Due to the use of resistance bars in the experiments the tidal velocities are nearly independent of y in contrast with the density currents and hence a distribution between both contributions can be made from the measured velocity profiles. It appeared that as a rule

$$u_d \ll u_t$$

So the density currents have an order of magnitude in between the discharge velocity and tidal velocity.

Note that the tidal velocity u_t actually varies as a function of x in the region of salt intrusion. However, the variable part is order $\sqrt{\varepsilon}$ because the region $\frac{L_i}{L_R} = 0\ (\sqrt{\varepsilon})$ and therefore can be incorporated into u_i defined as $u_i = u\ (x,y,t) - u_o^{(o)}\ (0,t)$

Denote the coordinates in the moving system by ξ, y and τ

$$\bar{x} = \sqrt{\varepsilon}\xi + \sqrt{\varepsilon} \int^{\tau} u_o^{(o)}\ (0,\bar{t})\ d\bar{t}$$

$$\bar{t} = \tau$$

$$\text{so } \frac{\partial}{\partial \xi} = \sqrt{\varepsilon} \frac{\partial}{\partial \bar{x}} \ ; \ \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \bar{t}} + \sqrt{\varepsilon}\, u_o^{(o)}\ (0,\tau)\ \frac{\partial}{\partial \bar{x}}$$

The differential equations (10), (11) and (12) transform in this stretched coordinate system into:

$$\frac{\partial \bar{u}_i}{\sqrt{\varepsilon}\partial \xi} + \frac{\partial \bar{v}_i}{\partial \bar{y}} = 0 \tag{25}$$

$$\frac{\partial \bar{\rho}}{\partial \tau} + \bar{u}_i \frac{\partial \bar{\rho}}{\partial \xi} + \bar{v}_i \sqrt{\varepsilon} \frac{\partial \bar{\rho}}{\partial \bar{y}} - \frac{\partial}{\partial \bar{y}} \left(\delta \frac{\partial \bar{\rho}}{\partial \bar{y}}\right) = 0 \tag{26}$$

$$\frac{\partial \bar{u}_i}{\partial \tau} + \bar{u}_i \frac{\partial \bar{u}_i}{\partial \xi} + \bar{v}_i \sqrt{\varepsilon} \frac{\partial \bar{u}_i}{\partial \bar{y}} + \int_{y}^{1+\sqrt{\varepsilon}\bar{\eta}} \frac{\partial \bar{\rho}}{\partial \xi} dy + \frac{\partial \eta_i}{\sqrt{\varepsilon}\partial \xi} + \tag{27}$$

$$\sqrt{\varepsilon}\bar{\rho}_s \frac{\partial \bar{\eta}}{\partial \xi} + \mu\sqrt{\varepsilon}\{(\bar{u}_i + u_o^{(o)})|\ \bar{u}_i + u_o^{(o)}| - u_o^{(o)}|\ u_0^{(o)}|\ \} = 0$$

where $\bar{u}_i = \bar{u}\ \ (\bar{x},\bar{y},\bar{t}) - u_o^{(o)}\ (0,\bar{t})$

$\bar{v}_i$, ρ_i

and

$\bar{\eta}_i = \bar{\eta} - \eta_o^{(o)}\ (0,\bar{t})$ are the inner variables

$\bar{\rho}_s$ is the dimensionless density at water surface.

Equation (27) is derived from the difference between equations (12) and (13).

7 The zeroth order inner approximation

In the development of the inner approximation the following asymptotic expansions are introduced

$$\bar{u}_i = u_i^{(o)} + \sqrt{\varepsilon}\, u_i^{(1)} + O(\varepsilon)$$

$$\sqrt{\varepsilon}\, \bar{v}_i = v_i^{(o)} + \sqrt{\varepsilon} v_i^{(o)} + O(\varepsilon) \qquad \text{(based on equation 25)}$$

$$\bar{\eta}_i = \eta_i^{(o)} + \sqrt{\varepsilon}\eta_i^{(1)} + O(\varepsilon)$$

$$\bar{\rho}_i = \rho_i^{(o)} + \sqrt{\varepsilon}\rho_i^{(1)} + O(\varepsilon)$$

From equations (27) it follows at once that

$$\frac{\partial \eta_i^{(o)}}{\partial \xi} = 0 \rightarrow \eta_i^{(o)} = \eta_i^{(o)}(t)$$

The zeroth order approximation equations become:

$$\frac{\partial u_i^{(o)}}{\partial \xi} + \frac{\partial v_i^{(o)}}{\partial \bar{y}} = 0$$

$$\frac{\partial u_i^{(o)}}{\partial \tau} + u_i^{(o)} \frac{\partial u_i^{(o)}}{\partial \xi} + v_i^{(o)} \frac{\partial u_i^{(o)}}{\partial \bar{y}} + \int_y^1 \frac{\partial \rho_i^{(o)}}{\partial \xi}\, dy + \frac{\partial \eta_i^{(1)}}{\partial \xi} + \mu\sqrt{\varepsilon}\, .$$

$$. \left[(u_i^{(o)} + u_o^{(o)}) \, |u_i^{(o)} + u_o^{(o)}| - u_o^{(o)} | \, u_o^{(o)} | \right] = 0$$

For the continuity equation of salt we have to introduce an order of magnitude estimate of ∂ in terms of ε.
As mentioned already, it is hard to derive a decisive answer to this question from the experimental flume test results.
Therefore we put $\delta = 0\ (\varepsilon^{k/2})$ and derive the approximation equation belonging to various integer values of k.
The main part of the study has been devoted to the question which choice of the diffusion coefficient has to be made in order to get differential-equations which can describe the main physical properties of the salt intrusion.
The extensive computations needed to judge the value of the applied approximations will not be reproduced for reasons to be described in the following chapter.

8 On the existence of a solution of the problem

The main part of this study has been devoted to the problem how general aspects of the salt intrusion depend on the specific choice of the diffusion coefficient being made.
The answer to this question turned out to be rather disappointing.
Provided that no essential mistakes have been made in the argumentation the final conclusion of this study is as follows:
The problem of salt intrusion in a tidal flume as formulated by the differentialequations (1), (7) and (9) with sufficient boundary conditions is not well-posed; a solution does not exist for practical values of the diffusion-coefficient.
This farreaching conclusion is a result of lengthy and cumbersome algebraic calculations.
However, the methods involved as well as the attention paid to elaboration cannot lay claim to the mathematical rigour which is needed for a proof of existence.
Therefore, instead of giving a tiresome explanation of the applied method we prefer to make the conclusion acceptable by physical arguments.
Due to the lack of stringency in the mathematical approach the following physical explanation lacks a rigid background and therefore must be considered with care.
Let us formulate the conclusion in more detail.
No solution exists for the set of differential-equations (1), (7) and (9) with sufficient boundary conditions periodically repeated in time for values of the diffusioncoefficient unequal to zero or infinity in some neighbourhood of the saltwedge tip.
The additional information means that a value zero nor infinite is considered as a practical value for a diffusioncoefficient.
The definition of the saltwedge tip is as follows:
The saltwedge tip is that region where $y = 0^+$ and x (t) follows from

$$\frac{d\,x\,(t)}{dt} = \lim_{t \to \infty} u(x(t),\ 0,\ t) \qquad \text{or}$$

$$x(t) - x(t_o) = \lim_{t \to \infty} \int_{to}^{t} u(x(t),\ 0,\ t)\ dt$$

where x_o and t_o are initial conditions chosen in such a way that

$x(t) \geq 0$ for all values of t.

Dependent on the choice of x_o, t_o the relation

$$x(t) - x(t_o) = \int_{t_o}^{t} u(x(t), 0, t)\, dt$$

describes the path of the infinitesimal particle (x_o, t_o) on the bottom of the flume inside or upstream the salt wedge.
The limit for $t \to \infty$ is independent of the initial conditions, and describes by definition the position and motion of the saltwedge tip. The allegation that $\lim_{t \to \infty} x(t)$ is independent of the initial conditions has still to be proved.
At this point, the restriction is made, that any solution of the mathematical model which is meaningless in a physical sense is disregarded.
This restriction implies that a distinct fluid particle moves or is at rest in the physical plane only, i.e. sources or sinks or movements in non-physical planes in the fluid domain under consideration are excluded.
Due to this restriction we may state that if a physical solution exist, the integral $\int_{t_o}^{t} u(x(t), 0, t)\, dt$ exists also for all values of t. The integral is unique because $y = 0^+$ is a streamline ($v(x,0,t) = 0$) and depends on the initial conditions only. (Fluid particles on a streamline do not pass each other in the physical plane). That the solution $\lim_{t \to \infty} x(t)$ if it exists is unique independent of the initial conditions will be amplified in the following:
From the formulation of the mathematical problem it can be derived that if $D_y \neq 0$ the time-mean velocity of a fluid particle over one tidal cycle on the bottom streamline is directed upstream as long as $c \neq 0$.
The derivation of this statement is omitted, only the physical arguments are given. Consider the equilibrium of time mean forces on a fluid particle on $y = 0^+$. The density force which is proportional to $\frac{\partial \rho}{\partial x}$ must be balanced by the friction force due to the density stream alone, i.e. a term proportionally to $\frac{\partial \tilde{u}}{\partial y}$ (in case of resistance bars). Because $\frac{\partial \rho}{\partial x} \leq 0$ while $D_y \neq o$, the vertical time mean velocity gradient $\frac{\partial \tilde{u}}{\partial y}$ is also ≤ 0. The continuity equation throught a vertical requires zero transport of salt during one tidal cycle. So $\tilde{u}$ bottom ≥ 0 if $c \geq 0$,hence a fluid particle on the bottom streamline coming from the sea moves to the front of the salt wedge. Consider now a fluid particle on the bottom streamline coming down from the riverside. Due to the fact that the horizontal diffusive salt transport has been neglected, the salt concentration of this fluid particle is zero and remains zero at least up till that moment it reaches some position x where somewhere on the vertical the concentration of salt is unequal zero.

If it can be shown that this somewhere is located on the bottom of the flume, than the position and motion of the salt wedgetip is defined.
Consider therefore the most upstream particle with salt concentration unequal zero. Again due to neglection of horizontal diffusion this special particle contains perhaps a small but still a finite amount of salt, unless the particle has been exposed to diffusion with sweet water during an infinite amount of time. (An infinite vertical diffusion coefficient is considered as unrealistic and therefore omitted in these considerations).
Hence there are two possibilities.

a) the most upstream end of the salt intrusion contains a finite amount of salt i.e. the density distrubution in x-direction is discountinuous
b) the most upstream end of the salt intrusion contains an amount of salt tending to zero below every fixed bound. In that case the upstream end is build up of always the same particles i.e. the upstream end is a stagnation point.

ad a) If the boundary of the upstream end is not a stagnation point the tangent to this boundary must be vertical. Due to the contact requirement between neighbouring fluid particles, the horizontal water velocity components on both sides of this boundary must be equal. Hence the frictionterm over the boundary is continuous and the discontinuity in pressure must be balanced by a discontinuity in acceleration. Again due to the requirement of contact, the pressure difference can be build up by vertical acceleration alone but these contributions have been neglected. Hence the only possible shape of the upstream end of the salt intrusion in this mathematical formulation is a stagnation point.

ad b) If the upstream end of the salt intrusion is a stagnation point, pressure distribution in x-direction can be continuous under certain conditions. The stagnation point is on the lowest possible situation i.e. on the bottom of the flume because the time mean horizontal velocities of the water particle on the bottom streamline within the saltwedgetip are maximum due to the non-decreasing value of the term

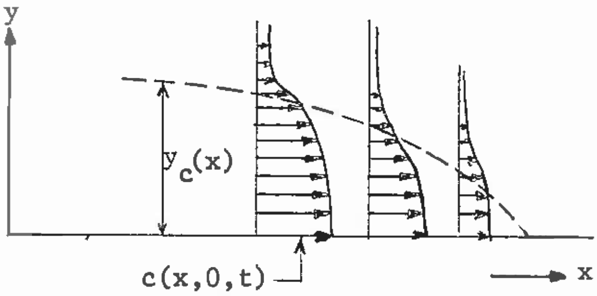
$$\int_h^y \frac{\partial \rho}{\partial x} \; dy \text{ with } y \quad \left(\frac{\partial \rho}{\partial x} \leq 0\right).$$

The position of the salt wedge tip as well as an indication of its shape is defined now. The angle between the tangent plane at the tip and the bottom must be equal or smaller than 90^o.

Consider next the continuity equation of salt

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{\partial}{\partial y} \left(D_y \frac{\partial c}{\partial y}\right)$$

The behaviour of c in the neighbourhood of the salt wedge tip on some moment is schematically shown in the next figure



$$\left(\frac{\partial c}{\partial y}\right)_{y=0} = 0$$

Denote $y_c(x)$ as some measure of the salt wedge thickness

f.i. $c(x,y_c(x), t) = \frac{1}{2}\, c(x,0,t)$ or else

$c(x,y_c(x), t) = \frac{1}{2}\, (c(x,0,t) + c(x,h_o,t)$

where $y_c \rightarrow 0$ at the tip

The singular behaviour of the right hand side of the continuity equation of salt at the wedge tip must be balanced by one or more of the other terms in this equation. Because the angle between the tangent at the salt wedge tip and the bottom is equal to or smaller than 90^o, the flow velocities u and v within the wedge tends to zero at a faster rate than $r^{\frac{1}{2}}$ where r is the distance from the point under consideration to the stagnation point. Equating the singular behaviour of the diffusion term to the remaining terms at the salt wedge tip leads to:
A solution of the stated salt intrusion problem can exist only if D_y tends to zero at a faster rate as the thickness of the salt wedge tends to zero.

The fulfilment of this condition leads to values of the vertical diffusion coefficient which are considered as unrealistic.
In the physical situation it is well-known that the diffusion attains a maximum at the front of the salt wedge contrary to the requirements imposed by this mathematical formulation. Values of the diffusion coefficient larger than every fixed bound however are omitted in the foregoing, because such values are again considered as physically unrealistic.

From a mathematical point of view, the singular behaviour of the diffusion coefficient in special points of the fluid domain is to be expected (and therefore realistic) when the approximations applied to the continuity equations of mass and momentum are based on overall properties of flow and salinity distributions, as has been done in this case. The physical interpretation of such a coefficient, largely defined by the mathematical approximation procedure, cannot be given. Therefore it was decided not to extent the investigation in this direction.

During this study several assumptions were tried on the value of D_y or its functional relatioship with the local Richardsonnumber. All assumptions excluded the value $D_y = 0$ except in the limit for $Ri \to \infty$. For the reasons as pointed out the computation procedures led to non-convergent series expansions of the unknown functions, in the region around the salt front at least as far as the expansions were carried out.

These computations are very tedious and are omitted on account of their slight practical value.

9 Some results of the inner approximation

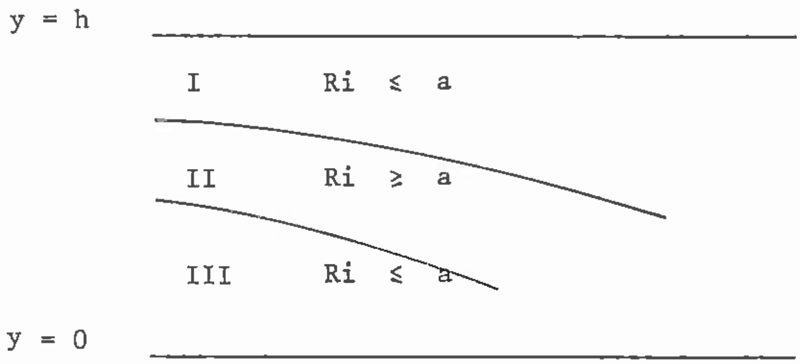
A short description will be given of those partial results emerging from the work done, which may have some practical value or may be of some importance for an extended study on this subject.

One of the assumptions about D_y has been:

$$D_y = D_o \quad \text{for } Ri \leq a$$

$$D_y = \frac{aD_o}{Ri} \quad \text{for } Ri \geq a \qquad \text{where a is a constant } O\ (1)$$

Only the time-mean differential equations over one tidal cycle in a with the tidal motion moving coordinate system has been considered, i.e. the coordinate system moves with the velocity $u_o^{(o)}$ (0,t) of the zeroth order outer approximation. The fluid domain containing the salt wedge is devided into 3 regions as shown in the next sketch based on the magnitude of the Ri-number. (All variables introduced from now on must be considered as time-mean values in the moving frame).



The boundary curves between the regions I and II and between II and III are defined by the condition Ri = a.
The order of magnitude of D_o is chosen from the systematic test series as:

$$\delta = \frac{L_R . D_y}{h_o^2 \sqrt{gh_o}} = O\ (1) \qquad \text{for } D_y \text{ attaining its maximum value } D_o$$

The friction coefficient as before:

$$\mu = \frac{gL_R}{C^2 R} = O\ (\varepsilon^{-\frac{1}{2}})$$

or in terms of inner dimensions

$$\frac{D_o}{u_r h_o} \quad \frac{L_i}{h_o} = 0\ (\frac{1}{\mu}) = 0\ (\varepsilon^{\frac{1}{2}})$$

The solution up to order ε has been calculated.
The relation of the solution u(x,y), v(x,y), c(x,y), Ri(x,y) with the vertical coordinate appeared to be:
in region III:

$$u_3\ (x,y) = u_{31}(x) + \frac{y}{h_o}\ u_{32}(x) + 0\ (\varepsilon^{\frac{1}{2}})$$

$$v_3(x,y) = \frac{y}{h_o}\ v_{31} + \frac{y^2}{h_o^2}\ v_{32}\ (x) + 0\ (\varepsilon^{\frac{1}{2}})$$

$$c_3(x,y) = c_{31}(x) + \varepsilon^{\frac{1}{2}}\ c_{32}(x) + \varepsilon\ \{c_{33}(x) + \frac{y^2}{h^2}\ c_{34}\ (x) + \frac{y^3}{h^3}\ c_{35}\ (x)\}$$

$$+\ 0\ (\ \varepsilon^{3/2})$$

$$Ri_3(x,y) = \frac{y}{h}\ R_{31}(x) + \frac{y^2}{h^2}\ R_{32}\ (x) + 0\ (\varepsilon^{\frac{1}{2}})$$

The upper boundary of region III is $y = y_a(x)$ and follows from $Ri_3\ (x,y_a) = a$.
In region I, the same differential-equations as in region III apply for the succesive order terms.
The solutions in region I is analogous reading $\frac{h-y}{h}$ instead of $\frac{y}{h}$. The lower boundary of region I is $y = y_b\ (x)$ and follows from $Ri_1\ (x,y_a) = a$.
In region II:
If solutions with a periodic character (standing wave solutions) are excluded, the remaining solution in this region is:

$$u_2(x,y) = u_{21}(x) + u_{22}\ \frac{y}{h} + 0\ (\varepsilon^{\frac{1}{2}})$$

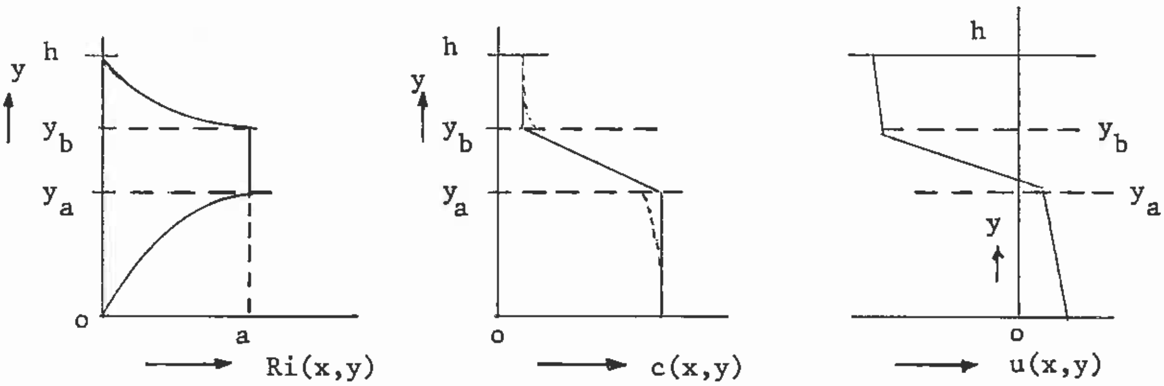
$$c_2(x,y) = c_{21}(x) + c_{22}\ \frac{y}{h} + 0\ (\varepsilon^{\frac{1}{2}})$$

where u_{22} and c_{22} are independent of x and y

$$v_2\ (x,y) = v_{21}(x) + v_{22}\ (x)\ \frac{y}{h} + 0\ (\varepsilon^{\frac{1}{2}})$$

$$Ri_2(x,y) = a + 0\ (\varepsilon^{\frac{1}{2}})$$

The next sketch shows graphically the obtained results for some value of x.



As mentioned already the various x-dependent functions showed a singular behaviour at the salt wedge front. The strength of the singularity increased with increasing order of perturbation in such a way that the power series in ε did not converge (at least as far as the expansions has been carried out). From these results however some still valid conclusions can be drawn.

1) The thickness of the shear-layer $y_b - y_a$ as calculated appeared to increase in seaward direction contrary to the results of the systematic test series.
2) The thickness of the shear-layer as calculated is much smaller than in the experiments unless a is order 10 instead of order unity.

Based on these observations it may be concluded that if the vertical diffusion coefficient is really only dependent on the Richardsonnumber in such a way that a = 0 (1) than vertical diffusion plays only a minor part along most of the salt intrusionlength. The most important mixing occurs in that case at the front of the salt wedge and the diluted mixture in the shear layer is mainly originating from that front region alone. The shear layer thickness decreases in downstream direction due to the converging streampattern.

The analytic considerations made in this study showed only the shortcomings of the mathematical description of the two-dimensional salt intrusion problem based on the concept of vertical salt transport of a diffusive type.
Indeed in most of the problems in nature where turbulence plays a part in the spreading of a substance the description by Fick law of diffusion is appropriate. In this special case however where mixing is predominant at the saltwedge-front where the flow pattern is complex but still more deterministic than stochastic such a conception like Fick second law is not necessary appropriate.
In fact the periodic breaking of internal waves into a fluid region which is continuously renewed leads to a vertical local time mean salt transport which is better described by a proportionality with the concentration itself than with the vertical concentration gradient.
Remark: The shortcomings in the mathematical description of the salt intrusion problem can be removed by adding a vertical, "diffusive" transport term proportional to c being dominant at the head of the salt intrusion to the already existing term

10 Some observations on the form and motion of the head of the density fronts in flume experiments.

Motion pictures made of the moving density front in the flume with resistance bars experiments showed head shapes with at least two different basic features dependent on the external conditions. In case of lock exchange flow and perhaps also in the case of salt intrusion under tidal conditions with high river discharge the shape of the head of the density current is shown in Fig. a.

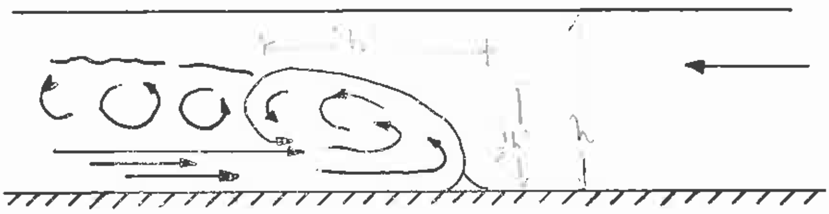


Fig.a

Under tidal conditions and river discharges usually met in the systematic flume tests, the head of the density front is shown in Fig. b.

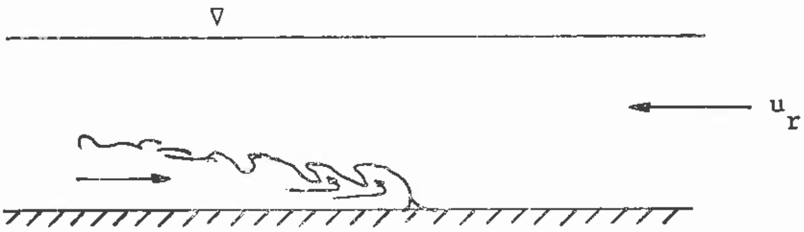


Fig.b

The difference in the flow pattern is apparent.

ad a) The height of the head in the lock exchange flow case is nearly equal to half the water depth. The front boundary is a sharp well-defined and smooth stable curve. After about a distance from the stagnation point comparable to the waterdepth a highly turbulent wake region develops bounded below by a stable layer containing salt water still moving to the front. The flow pattern as estimated from the motion pictures is as shown in the fig. a.

ad b) Under conditions usually met in the systematic test series, the height of the density front appeared to be much smaller. Unstable waves are travelling downstream along the surface of the salt wedge. The amplitude of this overtopping waves has the same order of magnitude as the thickness of the saltwedge. About three wave tops are clearly visible after which this pattern faded away by turbulence.

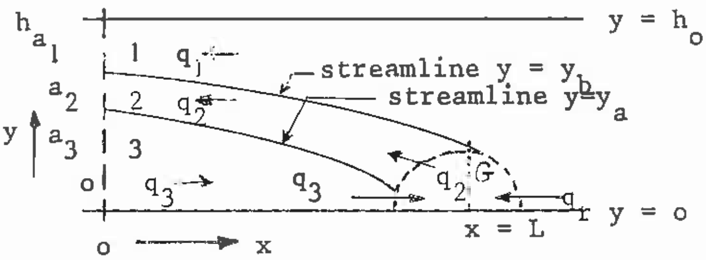
The no slip condition on the bottom of the flume leads to an overhang at the leading edge. This overhang however was small, apparently because bottom-friction is of minor importance compared to effect of the resistance bars.

Observations in literature made by Simpson (1969) [4] and Allen (1971) [5] have shown that the frontal flows are not two-dimensional and that the front of the density current is broken into regularly spaced lobes and bridges over clefts.
As was noted by Simpson the transverse spacing between the regular clefts was comparable to the vertical height of the overhang, which in turn appeared to be between 20 and 50 percent of the total head height of the density current. This shape could not be discoverd in our motion pictures. Due to the presence of resistance bars the overhang was small compared to these literature values. The lobes and clefts were if present in both cases not distinctable.

From these physical pictures it will be obvious that the vertical transport of salt by "turbulent" diffusion has nothing to do with the vertical concentration gradient at the head region of the salt wedge.
A vertical transport of salt on the time-mean interface proportional to the local concentration itself seems in this region more appropriate. The factor of proportionality will bear resemblence with the growing velocity of an unstable disturbance resulting from stability theories.

11 A proposed model of the salt intrusion problem

Based on the experience gained during this study the following three layer model of the salt intrusion problem is proposed.
Again time mean values are considered in a with the tidal motion moving coordinate system.



The fluid domain bounded by the bottom of the flume and the watersurface is divided into four regions. The regions denoted by 1, 2 and 3 in the above sketch are separated by the streamlines $y = y_a$ and $y = y_b$. Across these lines the normal time mean velocities are zero. The last region G contains the salt wedge tip. The horizontal dimension of region G is small compared to the length of the salt intrusion.
It is assumed that the diffusive type of vertical salt transport is valid everywhere outside the region G.
Within region G the laws describing the salt transport are not sufficiently known.
Therefore we have to recourse to assumptionson relations between integral in- and out-flow-quantities.
According to literature studies on the structure of the front of a density current f.i. Allen [5] an amount of fresh river water enters the front region with a rate proportional to the averageheight and the velocity of advance of the head relative to the river stream. The rate of mixing depends on the mean local internal Froudenumber based on the height and the relative density of the head of the salt wedge

$$\frac{u_r}{\sqrt{(\varepsilon g a)}_{head}}$$

So, the salt water discharge q_3 with the local mean concentration c_3 $(x = L_i)$ mixes with the fresh water discharge q_r to give a mixture $q_2 = q_3 + q_r$ entering the shear layer. The values q_3 and q_r are proportional to the local internal Froudenumber and have the same order of magnitude.

The horizontal dimension of G may be set equal to zero because it is small compared to the salt intrusion and because detailed information within G is missing.

The vertical dimension is

$$y_b(L_i) = O\left(\frac{u_r^2}{g\,\varepsilon L_i}\right)$$

Some assumptionsof minor importance have to be made on the concentration and velocity distribution of the water entering the intermediate layer.

As regards the diffusion outside the region G we observe that the systematic flume tests show a low degree of mixing of salt and fresh water, i.e. highly stratified conditions are prevailing.

So the order of magnitude of the diffusion coefficient in region 2 must be small compared to the values in region 1 and 3, because the vertical salt transport $vc - D\frac{\partial c}{\partial y}$ must be continuous crossing the boundary streamlines.

In fact, from the experimental results it appeared that in

region 1 $D \geq O\left(q_1 \frac{a_1}{L_i}\right)$

region 2 $D = O\left(q_2 \frac{a_2}{L_i}\right)$

region 3 $D \gg O\left(q_3 \frac{a_3}{L_i}\right)$

The meaning of the symbols used can be obtained from the foregoing sketch. From these order of magnitude estimations together with the equations of continuity of volume and momentum the essential features of the vertical density and velocity distributions can be obtained. Specific assumptions made within the scope of this proposed model appeared to affect the overall result only marginally.

12 An estimate of the salt intrusion length based on the proposed model

As an indication for this statement we give here an estimate of the salt intrusion length based on this model. For this purpose the following information is sufficient.

1[e]) The advective terms in the time mean equation of motion in the moving coordinate system can in general be neglected.
Together with the approximation $u|u| = \frac{4}{\pi} u_o u$ the equation of motion reduces to

$$\frac{g}{\rho_o} \frac{\partial \rho}{\partial x} = \frac{4}{\pi} \frac{g}{C^2 h_o} u_o \frac{\partial u}{\partial y}$$

2[e]) Only sufficiently stratified conditions as prevailing in the systematic test series are considered.
This restriction together with the equation of continuity leads to the following estimate of $\frac{\partial u}{\partial y}$ as a mean value over the salt intrusion length.

$$\left(\frac{\partial u}{\partial y}\right)_{mean} = \frac{2u_{riv}}{h_o}$$

So the mean slope of the depth-averaged salt concentration becomes

$$\left(\frac{\partial \rho}{\partial x}\right)_{mean} = \frac{8}{\pi} \frac{\rho_o}{C^2 h_o^2} u_o \; u_{riv}$$

or in dimensionless form

$$\left(\frac{\partial c/_{co}}{\partial x/_{h_o}}\right)_{mean} = \frac{8}{\pi} \frac{g}{C^2} F_o \; F_r$$

3[e]) Hitherto the effect on the vertical velocity gradient of the water circulation due to diffusion and mixing at the head of the salt wedge is not taken into account. A correction term must be added to $\left(\frac{\partial u}{\partial y}\right)_{mean}$ so

$$\left(\frac{\partial u}{\partial y}\right)_{mean} = \frac{2\,u_{riv}}{h_o} \; (1 + \alpha)$$

The correction term α must be small compared to unity and increases with increasing Froudenumber (Fr).

In the systematic test program the depth mean salt concentration as a function of x is evaluated for various time steps during one tidal cycle.

That part of the concentration distribution remote from the rivermouth is not directly influenced by the varying boundary conditions at the seaside, and can be used for comparison with the above relation after the longitudinal salt distributions are averaged over one tidal cycle in a coordinate system moving with the tidal motion. The agreement appears surprisingly good, despite the crudeness of the applied approximations.
The salt intrusion length can be determined now if an estimate could be made of the mean boundary conditions at the rivermouth. Because no special study has been made to this latter subject, we turn our attention directly to the experimental results. Figure 1 shows the measured values of $\frac{gL_i}{h_o C^2}$ plotted against $F_o\ F_r$.
Here L_i is the minimum salt intrusion length. The data corresponding to variation in tidal amplitude A, in Chézy coefficient C, in riverdischarge q_r in riverdepth h_o and in density difference $\Delta\rho$ are concentrated in one single curve.
As can be observed from the figure the line given by the equation

$$\frac{gL_i}{h_o C^2} = \frac{\pi}{16\ F_o\ F_r} - 1$$

fits these data quite well.
This empirical relation is backed by theoretical evidence. The factor $\frac{\pi}{16}$ implies that $(\frac{c}{c_o})_{mean}$ takes the value 0,5 at the mean mouth position in the moving coordinate system. It is curious that the integration constant should be a real constant. Based on physical reasoning a somewhat different relationship seems more appropiate namely:

$$\frac{gL_i}{h_o C^2} = \frac{\pi}{16\ F_o F_r} \cdot (1-\beta F_r) - \frac{g}{C^2} \cdot \sqrt{\frac{u_o\ T.B}{2\pi^2 h_o^2}}$$

where β is a constant > 0 such that $\beta\ F_r \ll 1$
and $\frac{u_o T}{2\pi}$ is equal to half the flood travel distance.
and B the width of the flume
(the two-dimensional distance $u_o T$ must be compared with the three-dimensional radius $\sqrt{\frac{u_o T.B}{\pi}}$)
The latter relation can fit the experimental data as well.
Only data of the salt intrusionlength as found in the flume tests with a flume length L = 0,884 L_R (1 $\pm$ 0 (10%) are used for comparison. By variation of the flume length, part of the driving force is contributed to hydrostatic pressure due to the mean slope of the watersurface. This effect is negligible for

$\frac{L}{L_R} \approx 0,9$, as can be calculated from the homogeous tidal theory.

$$\left(\frac{\partial \tilde{h}}{\partial x}\right)_{x=0} = 0 \qquad \text{for } \frac{L}{L_R} = 1 - 0\left(\frac{g}{C^2} F_o\right)$$

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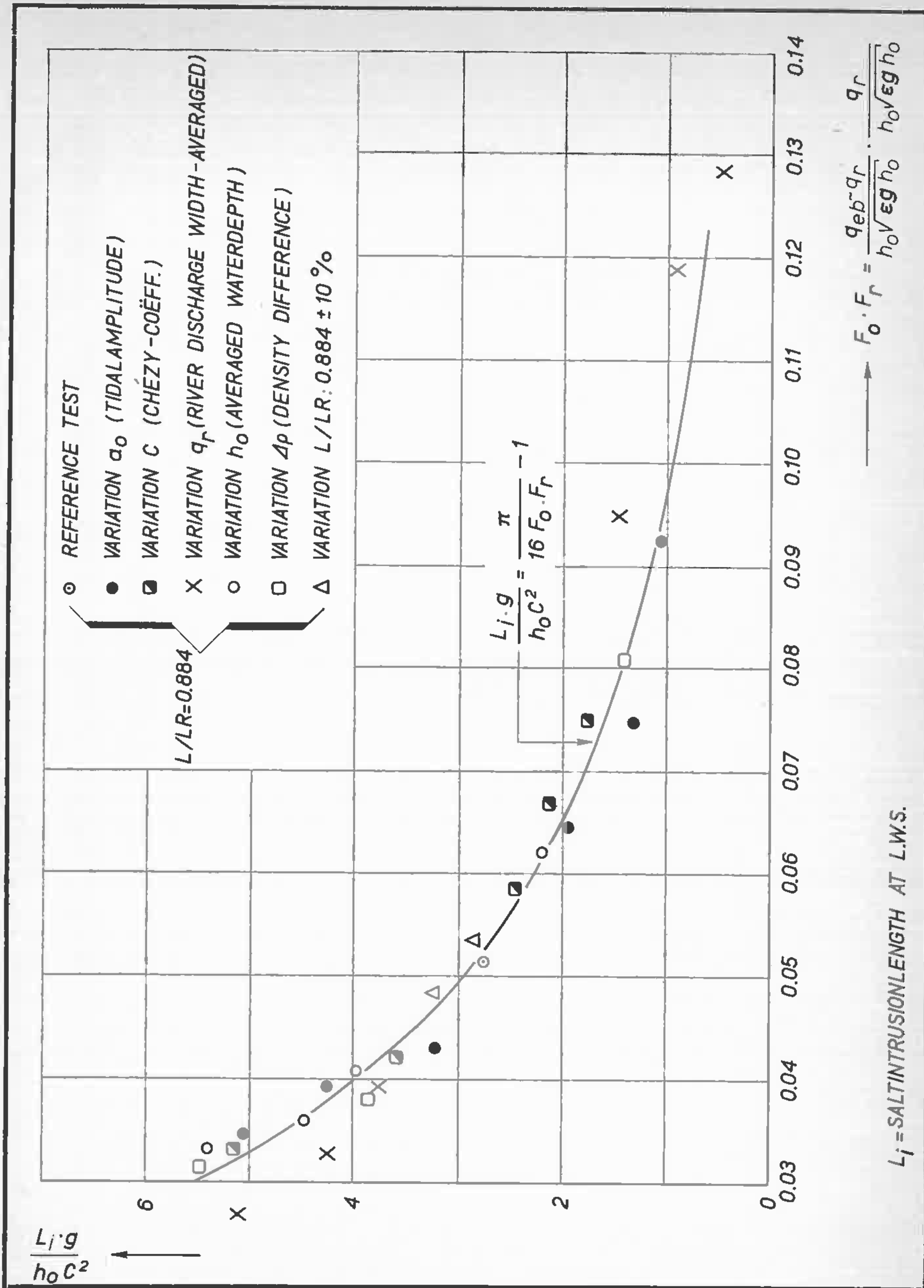

COMPARISON OF TEST RESULTS WITH EQ. 39

WK

A4

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FIG. 1