Degree-based approximations for network reliability polynomials

Van Mieghem, Piet; Liu, Xinhan

DOI
10.1093/comnet/cnaf001

Publication date
2025

Document Version
Final published version

Published in
Journal of Complex Networks

Citation (APA)
Van Mieghem, P., & Liu, X. (2025). Degree-based approximations for network reliability polynomials. *Journal of Complex Networks*, *13*(2), Article cnaf001. https://doi.org/10.1093/comnet/cnaf001

Journal of Complex Networks, 2025, **2**, cnaf001
https://doi.org/10.1093/comnet/cnaf001
Advance access publication 28 March 2025
Article

OXFORD

Degree-based approximations for network reliability polynomials

Piet Van Mieghem * and Xinhan Liu

Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, 2600 GA Delft, The Netherlands

*Corresponding author. Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, PO Box 5031, 2600 GA Delft, The Netherlands. E-mail: p.f.a.vanmieghem@tudelft.nl

ABSTRACT

Two approximations for network reliability polynomials, only based upon the knowledge of the degree vector of the graph, are compared: the first-order approximation by Brown *et al.* and our stochastic approximation. Our method is an extension of the connectivity probability of Erdős–Rènyi random graphs. Both approximations are shown to upper bound the actual reliability polynomial and are increasingly accurate for dense and large graphs. Moreover, the first-order approximation is always sharper or at least as good as the stochastic approximation, whereas the stochastic approximation is computationally easier. Our stochastic approximation (2.2) can determine the critical operational probability under which the graph is disconnected almost surely for any graph and an approximation for the number F_j of sets of j links whose removal retains the graph G connected, which is helpfull because the exact computation of F_j is NP-hard.

KEYWORDS: network robustness; node failure; probabilistic graph; reliability polynomial.

1. INTRODUCTION

The reliability of a system or network assesses its ability to remain operational after the failure of some components [1]. In 1956, Moore and Shannon [2] proposed a probabilistic model for network reliability, where nodes were considered to be perfectly reliable, while links had a certain probability of failure. Colbourn [3] categorized network reliability into three types: two-terminal reliability, all-terminal reliability, and k-terminal reliability. The computation of the all-terminal reliability was proven to be NP-hard [4, 5]. To speed up the computation, reduction techniques that utilize principles such as the factoring theorem and mincuts have been proposed [6–10]. For series-parallel graphs, the all-terminal reliability can be computed in linear time [11, 12]. Despite the availability of these techniques, calculating the network reliability for large networks remains challenging, as recently overviewed by Brown *et al.* [13]. Monte Carlo methods offer accurate estimates of the network reliability, but suffer from extensive computations [14]. Besides determining the exact value of the network reliability, many researchers have provided upper and lower bounds [15–18].

In this article, we first introduce in Section 2 a stochastic approximation (2.2) that (i) upper bounds the network reliability, (ii) demonstrates high accuracy in estimating the reliability polynomial for large and dense graphs, and (iii) allows to deduce an approximation of the critical operational probability p_N^* under which the graph is disconnected almost surely for

Received: 15 March 2024. **Editorial decision:** 26 December 2024. **Accepted:** 24 January 2025

any graph of size N. Section 2.3 exemplifies the stochastic method and compares it analytically with exactly known reliability polynomials of a few graphs. Section 3 extends the performance analysis of the stochastic approximation, benchmarked by precise Monte Carlo simulations. In addition, an approximation is found in Appendix B.2 for the number F_j of sets of j links whose removal keeps the graph G connected. The exact computation of F_j is NP-hard [3–5]. Section 4 explains the first-order approximation of Brown *et al.* [19], derives some properties and compares the first-order approximation with the stochastic approximation. A key result are the inequalities in (4.5) that relates the first-order approximation, our new stochastic approximation and the all-terminal reliability. Moreover, the first-order approximation is shown to be always at least as accurate as the stochastic approximation. Section 5 summarizes the paper.

2. RANDOM GRAPH ANALOGY AND EXTENSION

A graph $G(\mathcal{N},\mathcal{L})$, in short G, is composed of a set $\mathcal{N}$ of $N=|\mathcal{N}|$ nodes and a set $\mathcal{L}$ of $L=|\mathcal{L}|$ links. An undirected and unweighted graph with N nodes can be represented by a $N\times N$ symmetric adjacency matrix A. The element a_{ij} of the adjacency matrix A equals $a_{ij}=1$ if there exists a link between node i and j, else $a_{ij}=0$. Because self-loops do not affect the network reliability, we exclude self-loops, implying that A has zero diagonal elements, i.e. $a_{jj}=0$ for $1\leq j\leq N$. We call a graph simple [20] if it is undirected without self-loops.

The network (or all-terminal) reliability is defined as the probability that a network remains connected if each link is operational with probability p, independent of any other link in the graph G. We map the network reliability in a given, simple and undirected graph G to a stochastic setting, where a random graph $\widehat{G}$ is considered. The operational activity of a link is transferred to a probabilitistic *link existence* setting: each link in G is transformed to a Bernoulli random variable with mean p in the random graph $\widehat{G}$. In other words, the link $\widehat{a}_{ij}=1_{\{\text{link between node } i \text{ and } j\}}$ in $\widehat{G}$ is an indicator, implying that the link exists and $\widehat{a}_{ij}=1$ if the link is operational, else $\widehat{a}_{ij}=0$ and the link does not exist, i.e. fails or does not operate. The probability p that the link between node i and j is operational in G means then that $\Pr\left[\widehat{a}_{ij}=1\right]=p$. Since any link in the random graph $\widehat{G}$ is a Bernoulli random variable, the powerful property of Bernoulli random variables that probability equals expectation, i.e. $\Pr\left[\widehat{a}_{ij}=1\right]=E\left[\widehat{a}_{ij}\right]$, holds. The adjacency matrix of $\widehat{G}$ then equals $\widehat{A}=A_{G_p(N)}\circ A$, where $A_{G_p(N)}$ is the random adjacency matrix of the Erdős–Rènyi (ER) random graph $G_p(N)$ on N nodes with link density p and A is the adjacency matrix of the given graph G, while $\circ$ denotes the Hadamard product [20]. The relation between the degree of a node in the given graph G and in its random companion $\widehat{G}$ is derived in Appendix A.

The corresponding reliability polynomial is defined [3] as the probability that the random graph $\widehat{G}$ is connected, given the same operational probability p for all links,

$$rel_G(p)=\Pr\left[\widehat{G}\text{ is connected}\right] \tag{2.1}$$

which is a function of the graph topology (i.e. adjacency matrix A) of the given graph G and of the operational probability p of each link in G. Let $\Pr[D=k]$ be the probability that a randomly chosen node in the graph G has degree k. The probability generating function (pgf) [21] of the degree D in the original graph G is $\varphi_D(z)=E\left[z^D\right]=\sum_{j=0}^{N-1}\Pr[D=j]\,z^j$. Our main result here is an approximation $\overline{rel_G(p)}$ for the reliability polynomial $rel_G(p)$ for large size N of the graph G,

$$rel_G(p)\simeq\overline{rel_G(p)}=\left(1-\varphi_D(1-p)\right)^N \tag{2.2}$$

which is derived below in (2.4) and further compared with simulations in (2.4) Section 3. Our new approximation $\overline{rel_G(p)}$ in (2.2) expresses the reliability polynomial of a graph G in terms of the degree distribution of G. The reliability polynomial of a graph G can be expressed [3] as

$$rel_G(p) = \sum_{j=0}^{L-N+1} F_j (1-p)^j p^{L-j} \tag{2.3}$$

where F_j is the number of sets of j links whose removal leaves G connected. We deduce from (2.2) analytic approximations for the coefficients F_j of the reliability polynomial (2.3) in Appendix B.2 and list a few instances of F_j in explicit analytic form.

2.1 Connectivity and degree in the random graph $\widehat{G}$

We extend here the analysis of the connectivity in ER random graphs in [21, Section 15.7.5], by reviewing the interesting relation between the connectivity of a graph, a global property, and the degree D of an arbitrary node, a local property. The implication $\{G \text{ is connected}\} \Longrightarrow \{D_{\min} \geq 1\}$, where the minimum degree is $D_{\min} = \min_{\text{all nodes } \in G} D$, is always true. The opposite implication is not always true, because a network can consist of separate, disconnected clusters containing nodes each with minimum degree larger than 1. For large size N of the graph G and a certain link density p_N which depends on N, the implication $\{D_{\min} \geq 1\} \Longrightarrow \left\{G_p(N) \text{ is connected}\right\}$ is almost surely (a.s.) correct for ER random graphs. Thus, for a large ER random graph $G_p(N)$, the equivalence $\left\{G_p(N) \text{ is connected}\right\} \Leftrightarrow \{D_{\min} \geq 1\}$ holds almost surely such that

$$\Pr\left[G_p(N) \text{ is connected}\right] = \Pr\left[D_{\min} \geq 1\right] + o(1),$$

meaning that the difference between the left- and right-hand sides vanishes as $N \to \infty$. The equivalence is rigorously proved in the book [22, Section 5.3] of van der Hofstad, with references to earlier work, and the error $o(1)$ can be sharpened to $O\left(\frac{\log^b N}{N}\right)$ for some $b > 0$ by [22, Equation 5.3.21]. Our main *hypothesis* here is that we assume, in our random graph companion $\widehat{G}$ of the given graph G, the validity of

$$\Pr\left[\widehat{G} \text{ is connected}\right] = \Pr\left[\widehat{D}_{\min} \geq 1\right] + o(1),$$

where $\widehat{D}_{\min} = \min_{\text{all nodes } \in \widehat{G}} \widehat{D}$.

The basic law of the degree [20], $\sum_{j=1}^{N} d_j = 2L$, couples the degree d_j of nodes to the number of links L in any graph. For large N and $p < 1$ in $\widehat{G}$; however, this dependence is negligibly weak. Assuming independence, the minimum of independent random variables is [21, Section 6.6]

$$\Pr\left[\widehat{D}_{\min} \geq 1\right] \simeq \left(\Pr\left[\widehat{D} \geq 1\right]\right)^N = \left(1 - \Pr\left[\widehat{D} = 0\right]\right)^N$$

Invoking (A.3) in Appendix A yields

$$\Pr\left[\widehat{D}_{\min} \geq 1\right] \simeq \left(1 - \varphi_D(1-p)\right)^N$$

Our *hypothesis* then leads, for large N, to the approximation of the network reliability

$$\Pr\left[\widehat{G} \text{ is connected}\right] = \left(1 - \varphi_D(1-p)\right)^N + o(1) \tag{2.4}$$

which, in combination with the definition (2.1) of a reliability polynomial $rel_G(p)$, demonstrates our key approximation $\overline{rel_G(p)} = \left(1 - \varphi_D(1-p)\right)^N$ in (2.2). If $p = 1$, then $\varphi_D(1-p) = 0$ in a connected graph G and $\Pr\left[\widehat{G} \text{ is connected}\right] = 1$ almost surely, while if $p = 0$, then $\varphi_D(1-p) = 1$ and $\Pr\left[\widehat{G} \text{ is connected}\right] = 0$ almost surely.

Since the event $\{\widehat{D}_{\min} \geq 1\}$ includes the event $\{\widehat{G} \text{ is connected}\}$ (but not vice versa as explained above), it holds that $\Pr[\widehat{D}_{\min} \geq 1] \geq \Pr[\widehat{G} \text{ is connected}]$. Equivalently, the order term $o(1)$ in (2.4) is non-negative. Consequently, our asymptotic approximation (2.2) upper bounds the actual reliability polynomial, i.e.

$$rel_G(p) \leq \overline{rel_G(p)} \tag{2.5}$$

and that the accuracy of (2.2) increases with size N. The theory of ER graphs then guarantees that equality in (2.2), i.e. $rel_G(p) = \overline{rel_G(p)}$, holds in the limit for graph sizes $N \to \infty$. Numerical computations confirm, for the graphs computed, the upper bound (2.5) and the increasing accuracy of the asymptotic approximation (2.2) for large graph sizes N.

It is thus quite remarkable that only the degree distribution of a sufficiently large graph G seems sufficient to conclude about the connectivity of the companion random graph $\widehat{G}$, in which failures occur independently per link and with a same Bernoulli distribution with mean p!

2.2 Asymptotic analysis for large N of $\Pr[\widehat{G} \text{ is connected}]$ in (2.4)

We write $\Pr[\widehat{G} \text{ is connected}]$ in (2.4) in terms of p_N, expressing that p is a function of N, and omit the order $o(1)$,

$$\Pr[\widehat{G} \text{ is connected}] \simeq \exp(N \log(1 - \varphi_D(1 - p_N)))$$

Invoking the Taylor series of $\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$ for $|x| < 1$ yields

$$\log(1 - \varphi_D(1 - p_N)) = -\varphi_D(1 - p_N) - \sum_{j=2}^{\infty} \frac{(\varphi_D(1 - p_N))^j}{j}$$

and

$$\Pr[\widehat{G} \text{ is connected}] \simeq e^{-N\varphi_D(1-p_N)} \exp\left(-N \sum_{j=2}^{\infty} \frac{(\varphi_D(1 - p_N))^j}{j}\right)$$

If we denote $c_N \triangleq N\varphi_D(1 - p_N)$, then

$$N \sum_{j=2}^{\infty} \frac{(\varphi_D(1 - p_N))^j}{j} = \sum_{j=2}^{\infty} \frac{c_N^j}{jN^{j-1}}$$

can be made arbitrarily small for large N, provided we choose $c_N = cN^{\beta}$ with $\beta < \frac{1}{2}$, where c is a positive real and constant number. Thus, for large N, we have that

$$\Pr[\widehat{G} \text{ is connected}] = e^{-cN^{\beta}}\left(1 + O\left(N^{2\beta-1}\right)\right)$$

which tends to 0 for $0 < \beta < \frac{1}{2}$ and to 1 for $\beta < 0$. Hence, the critical exponent where a sharp transition occurs is $\beta = 0$. In that case, $c_N = c$ and $N\varphi_D(1 - p_N) = c$, which implies, assuming that the inverse function $z = \varphi_D^{-1}(w)$ of $w = \varphi_D(z)$ exists, that the critical value of the operational probability is

$$p_N^* = 1 - \varphi_D^{-1}\left(\frac{c}{N}\right) \tag{2.6}$$

In summary, for $N \to \infty$, the network is operational if the operational link probability p exceeds p_N^*,

$$\Pr\left[\widehat{G} \text{ is connected}\right] \longrightarrow \begin{cases} 0 & \text{if } p < p_N^* \\ 1 & \text{if } p > p_N^* \end{cases}$$

For a complete graph, $\varphi_D(z) = E\left[z^D\right] = z^{N-1}$ and the inverse function $\varphi_D^{-1}(z) = z^{\frac{1}{N-1}}$. Introduced into the critical operational link probability (2.6) yields

$$p^*_{N;K_N} = 1 - \left(\frac{c}{N}\right)^{\frac{1}{N-1}} = 1 - \exp\left(\frac{\log\frac{c}{N}}{N-1}\right) = \frac{\log N}{N} + O\left(\frac{\log c}{N}\right)$$

which, indeed, agrees with ER random graph theory [21, Section 15.7.4-5] and $p_c \sim \frac{\log N}{N}$ is the famous critical link density in $G_p(N)$ for large N. Our method thus generalizes the phase transition in the growth process of ER graphs (see e.g. [23]) to the critical operational probability p_N^* in (2.6) for any given graph G and sufficiently large N. For operational probabilities $p < p_N^*$, the graph G does not function anymore almost surely. In other words, the operational regime for any given graph G stretches from $p = p_N^*$ in (2.6) up to $p = 1$.

2.3 Examples of the stochastic approximation (2.2)

2.3.1 Complete graph

The random graph $\widehat{G}$ associated to the complete graph K_N is precisely the Erdős-Rènyi random graph $G_p(N)$ on N nodes with link existence probability p for each link. The definition (2.1) of the reliability polynomial then reduces to

$$rel_{K_N}(p) = \Pr\left[G_p(N) \text{ is connected}\right]$$

which has been computed in the classical paper by Erdős and Rènyi [24]. If $G_r(N, L)$ is the Erdős-Rènyi random graph on N nodes with precisely L links, they proved that

$$\lim_{N\to\infty} \Pr\left[G_r\left(N, \left[\frac{1}{2}N\log N + xN\right]\right) \text{ is connected}\right] = e^{-e^{-2x}} \tag{2.7}$$

Ignoring the integral part $[.]$ operator in (2.7) and eliminating x using the number of links $L = \frac{1}{2}N\log N + xN$ gives, for large N,

$$\Pr[G_r(N, L) \text{ is connected}] \sim e^{-N e^{-\frac{2L}{N}}} \tag{2.8}$$

which should be compared with the exact result,

$$\Pr[G_r(N, L) \text{ is connected}] = \frac{C(N, L)}{\binom{\binom{N}{2}}{L}} \tag{2.9}$$

where the number of connected random graphs $C(N, L)$ in the class $G_r(N, L)$ has been intensively studied. Gilbert [25] has derived a recursion for $C(N, L)$, which is computed for small N up to $N = 7$ in [21, Section 15.7.3]. Hence, if the operational probability equals a rational number $p = \frac{L}{\binom{N}{2}}$, then the reliability polynomial $rel_{K_N}(p)$ is exactly computable by (2.9). The goodness of the Erdős-Rènyi asymptotic (2.8) is assessed in [21, Fig. 15.7 on p. 381]. If we replace the average degree $\frac{2L}{N}$ by $p(N-1)$, then we find from (2.8), for large N, that

$$rel_{K_N}(p) \simeq e^{-N e^{-p(N-1)}} \tag{2.10}$$

The major interest of (2.10) is that any reliability polynomial $rel_G(p)$ in a graph with N nodes and operational probability p is upper bounded by $rel_{K_N}(p)$.

2.3.2 Crown graph

The crown graph Cr_N is a complete bipartite graph $K_{N-2,2}$ on N nodes, with an extra link between the two nodes in the smallest set. The reliability polynomial of the crown graph Cr_N is [26]

$$rel_{Cr_N}(p) = p^{N-2}\left((2-p)^{N-2} - 2^{N-2}(1-p)^{N-1}\right)$$

Each node in the largest set of N nodes has degree 2 and the degree of the two nodes in the smallest set is $N-1$. The degree distribution of the crown graph Cr_N is $\Pr[D=2] = \frac{N-2}{N}$ and $\Pr[D=N-1] = \frac{2}{N}$. The pgf is $\varphi_D(p) = \Pr[D=2]p^2 + \Pr[D=N-1]p^{N-1} = \frac{N-2}{N}p^2 + \frac{2}{N}p^{N-1}$. The approximation (2.2) is

$$\overline{rel_{Cr_N}(p)} \approx \left(1 - \frac{N-2}{N}(1-p)^2 - \frac{2}{N}(1-p)^{N-1}\right)^N$$

For large N, the approximation for the crown graph Cr_N is about

$$\begin{aligned}
\overline{rel_{Cr_N}(p)} &\approx \left(1-(1-p)^2 + \frac{2}{N}(1-p)^2 - \frac{2}{N}(1-p)^{N-1}\right)^N \\
&= \left(p(2-p) + \frac{2}{N}(1-p)^2 - \frac{2}{N}(1-p)^{N-1}\right)^N \\
&= p^N(2-p)^N\left(1 + \frac{2}{N}\frac{(1-p)^2}{p(2-p)} - \frac{2}{N}\frac{(1-p)^{N-1}}{p(2-p)}\right)^N \\
&\approx p^N(2-p)^N e^{\frac{2(1-p)^2}{p(2-p)}}
\end{aligned}$$

The general upper bound (2.5) then illustrates that $\overline{rel_{Cr_N}(p)} \geq rel_{Cr_N}(p)$, implying, for large N, the non-trivial inequality

$$p^{N-2}\left((2-p)^{N-2} - 2^{N-2}(1-p)^{N-1}\right) \leq p^N(2-p)^N e^{\frac{2(1-p)^2}{p(2-p)}}$$

2.3.3 Cycle graph

The reliability polynomial for the cycle C_N on N nodes is

$$rel_{C_N}(p) = p^N + Np^{N-1}(1-p)$$

The pgf of the degree of the cycle C_N, which is a regular graph with degree $D = 2$, equals $\varphi_D(p) = E\left[p^D\right] = p^2$ and approximation (2.2) is

$$\overline{rel_{C_N}(p)} = \left(1-(1-p)^2\right)^N = p^N(2-p)^N$$

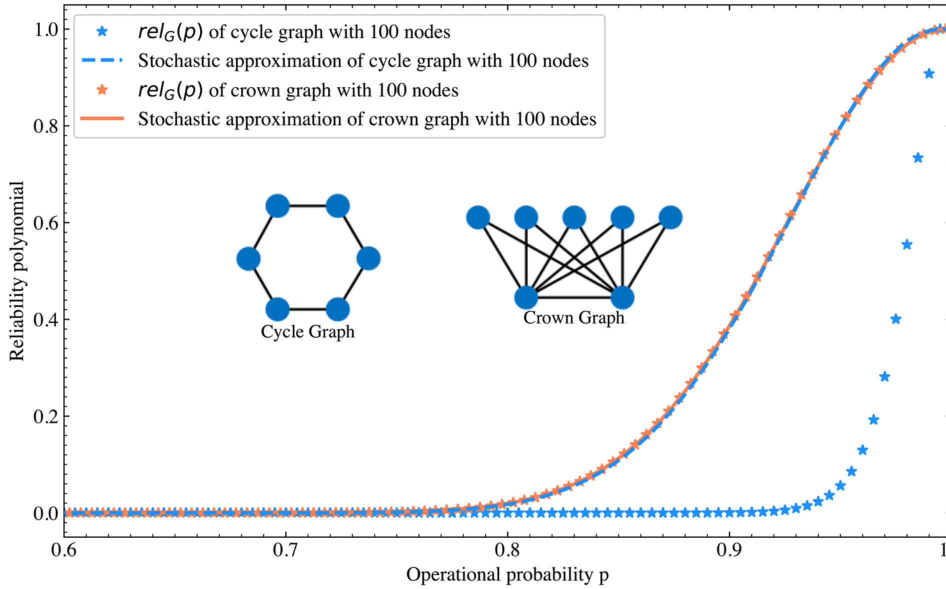

Figure 1. The reliability polynomials $rel_G(p)$ and stochastic approximations for a cycle graph on $N = 100$ nodes and a crown graph on $N = 100$ nodes.
Alt text: This figure illustrates the reliability polynomials of a cycle graph and a crown graph, both containing 100 nodes. The plot compares the exact reliability functions with their stochastic approximations across different values of p.

which is, after expansion of the binomial,

$$\overline{rel_{C_N}(p)} = p^N \sum_{k=0}^{N} \binom{N}{k} (1-p)^k = p^N + Np^N (1-p) + p^N \sum_{k=2}^{N} \binom{N}{k} (1-p)^k$$
$$= rel_{C_N}(p) + p^N \sum_{k=2}^{N} \binom{N}{k} (1-p)^k$$

Since all terms in the binomial expansion are non-negative, we clearly observe that our approximation (2.2) is an upper bound, i.e. $\overline{rel_{C_N}(p)} > rel_{C_N}(p)$. Additionally, for small p as well as large operational probabilities $p \to 1$, our approximation (2.2) is increasingly accurate. The explicit error term $p^N \sum_{k=2}^{N} \binom{N}{k} (1-p)^k$ allows us to compute the operational probability p^* that maximizes the error of the stochastic approximation.

Figure 1 illustrates that the stochastic approximation closely matches the reliability polynomial $rel_{Cr_N}(p)$ of the crown graph, but deviates from the reliability polynomial $rel_{C_N}(p)$ of the cycle graph. Since the stochastic approximation relies solely on the degree distribution of the graph rather than the specific connectivity patterns, two graphs with very similar degree distributions will have similar stochastic approximations. Both the cycle graph C_{100} and the crown graph Cr_{100} have a large number of nodes with a degree of two, leading to similar degree distributions. Consequently, Fig. 1 shows that their stochastic approximations are alike. However, due to the influence of specific connections, the actual reliability polynomials of the two graphs differ significantly.

2.4 Subgraph properties of reliability polynomials

A homogeneous ER graph, where each link has the same probability p to exist, tends to a regular graph when the graph size N tends to infinity and p is kept constant [21, p. 40]. The tight connection between the stochastic approximation (2.2) and homogeneous ER graphs suggests that the largest difference $\overline{rel_G(p)} - rel_G(p)$ occurs in dense graphs with low edge connectivity [21, p. 366-368]. (The edge connectivity equals the minimum number of links whose removal disconnects the graph.) A typical example are two complete graphs connected by a single link. Here, we explore two dense subgraphs connected by a few links.

Consider two disconnected subgraphs G_1 on n nodes and G_2 on the remaining $m = N - n$ nodes of a graph G. The adjacency matrix A of G is written in block matrix form

$$A = \begin{bmatrix} (A_{G_1})_{n\times n} & B_{n\times m} \\ (B^T)_{m\times n} & (A_{G_2})_{m\times m} \end{bmatrix}$$

where the matrix B specifies the links between the subgraph G_1 and G_1, which we call the cutset C. In terms of the all-one vector $u = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$, the cutset C consists of $L_C = (u^T)_{1\times m} B_{n\times m} u_{m\times 1}$ links, each with one node in G_1 and the other node in G_2. The two subgraphs G_1 and G_2 are disconnected if and only if all links in the cutset C are removed. The event that the corresponding random graph $\widehat{G}$ is connected consists now of three events

$$\left\{\widehat{G} \text{ is connected}\right\} \supseteq \left\{\widehat{G}_1 \text{ is connected}\right\} \cap \left\{\widehat{G}_2 \text{ is connected}\right\} \cap \left\{\widehat{C} \text{ connects } \widehat{G}_1 \text{ and } \widehat{G}_2\right\} \quad (2.11)$$

where the event $\left\{\widehat{C} \text{ connects } \widehat{G}_1 \text{ and } \widehat{G}_2\right\}$ means that at least one link in the cutset $\widehat{C}$ must be operational, given that both subgraphs $\widehat{G}_1$ and $\widehat{G}_2$ are connected. However, the graph $\widehat{G}$ can still be connected if $\widehat{G}_1$ or/and $\widehat{G}_2$ are disconnected, as e.g. in bipartite graphs where G_1 and G_2 are empty graphs. More generally, suppose that G_1 itself consists of two disconnected subgraphs S_1 and S_2, but each subgraph apart is connected and the cutset C contains a link from a node in S_1 to a node in G_2 and a link from a node in S_2 to a node in G_2. Then, there exists a path between each pair of nodes in G and G is connected, while G_1 is not, which explains the included sign $\supseteq$ in the event sets in (2.11). Since the links in the subgraphs $\widehat{G}_1$, $\widehat{G}_2$ and $\widehat{C}$ are all different, the events are mutually exclusive [21, p. 9], which implies that

$$\Pr\left[\widehat{G} \text{ is connected}\right] \geq \Pr\left[\widehat{G}_1 \text{ is connected}\right] \Pr\left[\widehat{G}_2 \text{ is connected}\right] \Pr\left[\widehat{C} \text{ connects } \widehat{G}_1 \text{ and } \widehat{G}_2\right] \quad (2.12)$$

Given that $\left\{\widehat{G}_1 \text{ is connected}\right\} \cap \left\{\widehat{G}_2 \text{ is connected}\right\}$, the event that *at least* one link in the cutset is operational is equivalent to the event that not all links in the cutset $\widehat{C}$ may fail and includes the event $\left\{\widehat{C} \text{ connects } \widehat{G}_1 \text{ and } \widehat{G}_2\right\}$

$$\Pr\left[\widehat{C} \text{ connects } \widehat{G}_1 \text{ and } \widehat{G}_2\right] = 1 - (1-p)^{L_C}$$

If the cutset contains L_C links, then the definition (2.1) of the reliability polynomial and (2.12) leads to

$$rel_G(p) \geq \left(1 - (1-p)^{L_C}\right) \Pr\left[\widehat{G}_1 \text{ is connected}\right] \Pr\left[\widehat{G}_2 \text{ is connected}\right]$$

which reduces for each partition of the graph G into two non-overlapping subgraphs G_1 and G_2 connected by $L_c = 1$ link to an equality

$$rel_G(p) = p \Pr\left[\widehat{G}_1 \text{ is connected}\right] \Pr\left[\widehat{G}_2 \text{ is connected}\right] \quad (2.13)$$

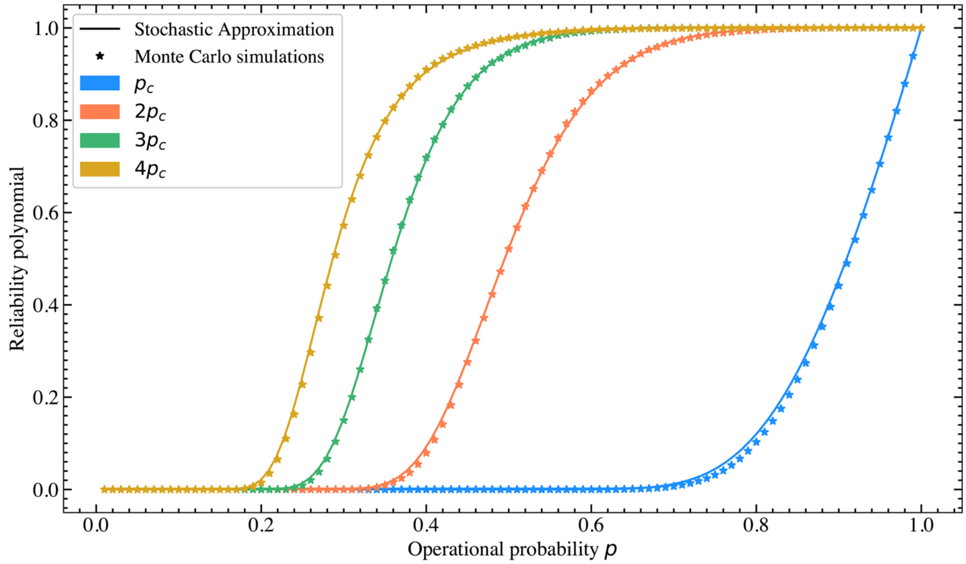

Figure 2. The stochastic approximation (2.2) and Monte Carlo simulations for the class of ER graphs $G_p(N)$ with $N = 200$ and critical link density $p_c \sim \frac{\log N}{N} = 0.0265$.
Alt text: This figure presents the stochastic approximation and Monte Carlo simulations for Erdős–Rényi (ER) graphs with 200 nodes. The plot compares the reliability polynomial for different link densities, with a critical density value of approximately 0.0265, calculated using the formula Pc Log N/N.

Figure 6 below illustrates that the stochastic approximation applied to the entire graph G is considerably worse than applied to each dense subgraph G_1 and G_2 only connected by one link in (2.13).

In general, $m + 1$ disconnected subgraphs $\{S_1, S_2, \ldots, S_{m+1}\}$ are connected by precisely m links forming a connected graph $\widehat{G}$, provided each $\widehat{S}_j$ of the individual subgraphs is itself connected, resulting in

$$\Pr\left[\widehat{G} \text{ is connected}\right] = p^m \prod_{j=1}^{m+1} \Pr\left[\widehat{S}_j \text{ is connected}\right]$$

Indeed, each subgraph can be represented in a hierarchical structure (see e.g. [27]) by a node and $m + 1$ nodes can always be connected by a spanning tree consisting of m links.

3. PERFORMANCE ANALYSIS OF THE STOCHASTIC APPROXIMATION (2.4)

The accuracy of our approximation (2.2) is compared with Monte Carlo simulations, based on a variation of the method in [19], in terms of link density $p_L = \frac{E[L]}{\binom{N}{2}}$, where $E[L]$ is the average number of links. We regard the Monte Carlo simulations as precise enough to act as a benchmark for the correct reliability polynomial $rel_G(p)$.

Figures 2 and 4 demonstrate that the stochastic approximation (2.2) is accurate for the ER random graphs $G_{p_L}(N)$ with link density p_L above the disconnectivity threshold p_c as well as for Barbasi–Albert graphs.

Figure 3 demonstrates, indeed, that the approximation (2.2) becomes increasingly accurate for graphs with large size N. Random geometric graphs are compared in Fig. 5 and indicate that the stochastic approximation (2.2) is slightly less accurate for low link density.

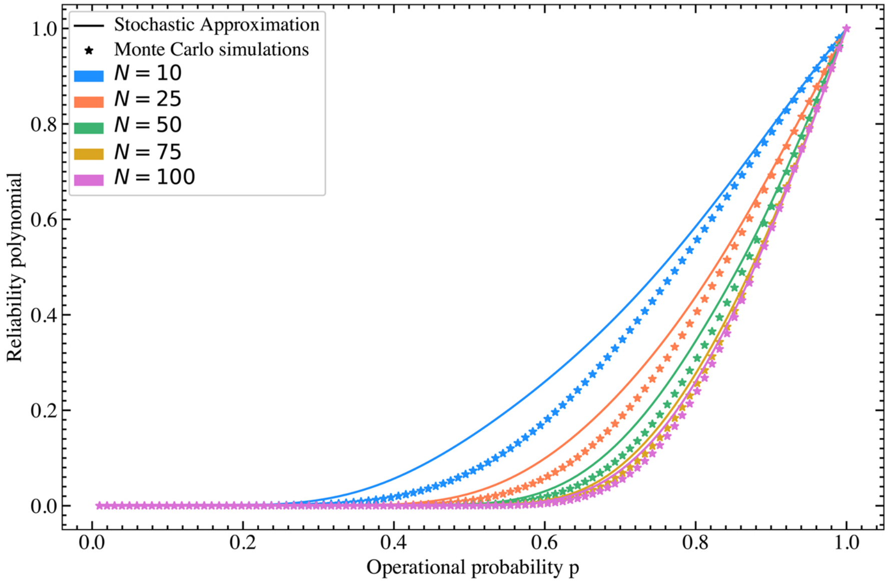

Figure 3. The stochastic approximation (2.2) and Monte Carlo simulations for the class of ER graphs $G_{p_L}(N)$ with different sizes of $N = 10, 25, 50, 75$ and 100 and critical link density $p_c \sim \frac{\log N}{N}$.
Alt text: This figure shows the stochastic approximation and Monte Carlo simulations for Erdős–Rényi graphs of different sizes, ranging from 10 to 100 nodes. The plot demonstrates the reliability polynomial behavior under varying network scales and critical link densities.

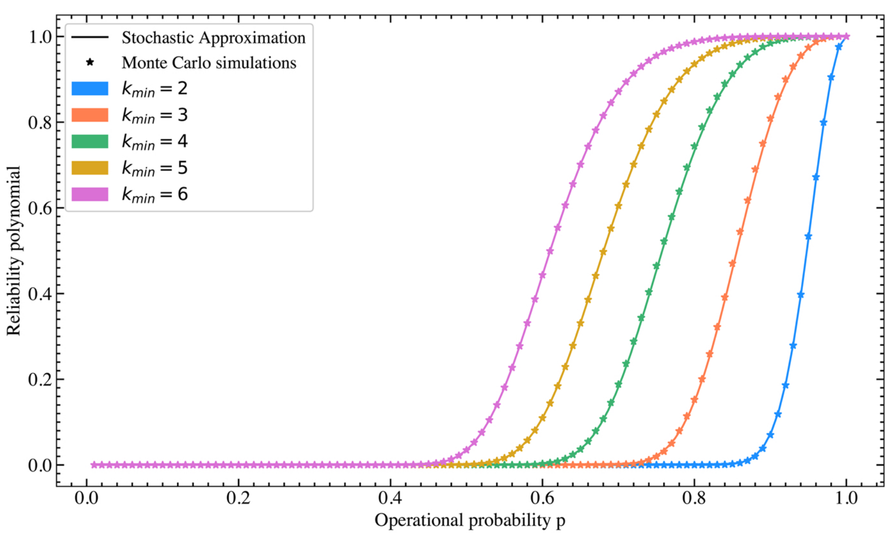

Figure 4. The stochastic approximation (2.2) and Monte Carlo simulations for the class of Barabasi-Albert graphs with $N = 500$ and different link densities $p_{BA}(k_{\min})$ as function of the minimum degree $k_{\min}$: $p_{BA}(2) = 0.00798$, $p_{BA}(3) = 0.01195$, $p_{BA}(4) = 0.0159$, $p_{BA}(5) = 0.01984$, and $p_{BA}(6) = 0.02375$.
Alt text: This figure presents the stochastic approximation and Monte Carlo simulations for Barabási-Albert graphs with 500 nodes. The figure compares reliability polynomial estimates under varying minimum degree values, ranging from 2 to 6, and corresponding link densities.

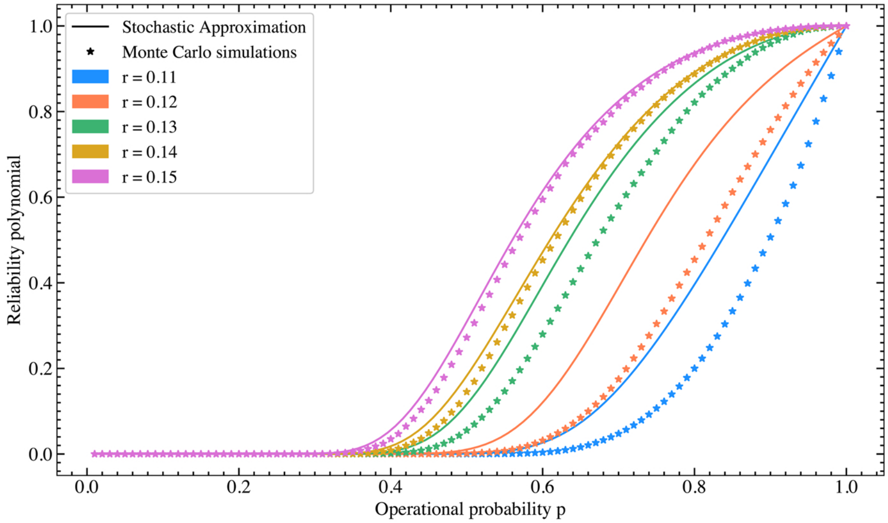

Figure 5. The stochastic approximation (2.2) and Monte Carlo simulations for the class of random geometric graphs with $N = 200$ for various link connectivity radius $r = 0.11, 0.12, 0.13, 0.14$ and 0.15 with corresponding link density $p_L = 0.04266, 0.0506, 0.05346, 0.06517$ and 0.07839.
Alt text: This figure presents the stochastic approximation and Monte Carlo simulations for random geometric graphs with 200 nodes. It compares the reliability polynomial estimates for different connectivity radii (r values) and their corresponding link densities.

For two ER graphs with $N = 30$ nodes that are connected by $L_C = 1$ link, Fig. 6 compares, for different link density p_L the stochastic approximation (2.2) for the entire graph with formula (2.13) that takes the substructure into account and where each subgraph is computed by (2.2). The latter computation via (2.13) is almost the same as Monte Carlo simulations; their difference on the plot is not visible.

Figures 7 and 8 assess the stochastic approximation (2.2) for lattices in two and three dimensions.

We observe that, in all simulated graphs, the approximate formula (2.2) *upper bounds* the exact reliability polynomial, as anticipated in $\widehat{G}$ Section 2.1.

4. COMPARISON OF THE STOCHASTIC APPROXIMATION AND THE FIRST-ORDER APPROXIMATION

Brown *et al.* [19] have introduced their so-called first-order approximation

$$(R_1)_G(p) = \prod_{i=1}^{N} \left(1 - (1-p)^{d_i}\right) \tag{4.1}$$

which reflects the probability that none of the nodes in the graph G has inoperational links. Indeed, all links incident to node i fail with probability $f_i = (1-p)^{d_i}$ and the complement, i.e. not all links incident to node i fail, has probability $1 - f_i$. Given that the original graph G is connected, the event $\mathscr{E}$ ={none of the nodes in the random graph $\widehat{G}$ has all links inoperational} is a sufficient, but not necessary condition for the event $\{\widehat{G}$ is connected$\}$, which implies that $\Pr[\mathscr{E}] \geq \Pr[\widehat{G}$ is connected$]$. Hence, the definition of the reliability polynomial in (2.1) indicates that

$$(R_1)_G(p) \geq rel_G(p) \tag{4.2}$$

Both the first-order approximation $(R_1)_G(p)$ in (4.1) and the stochastic approximation $\overline{rel_G(p)}$

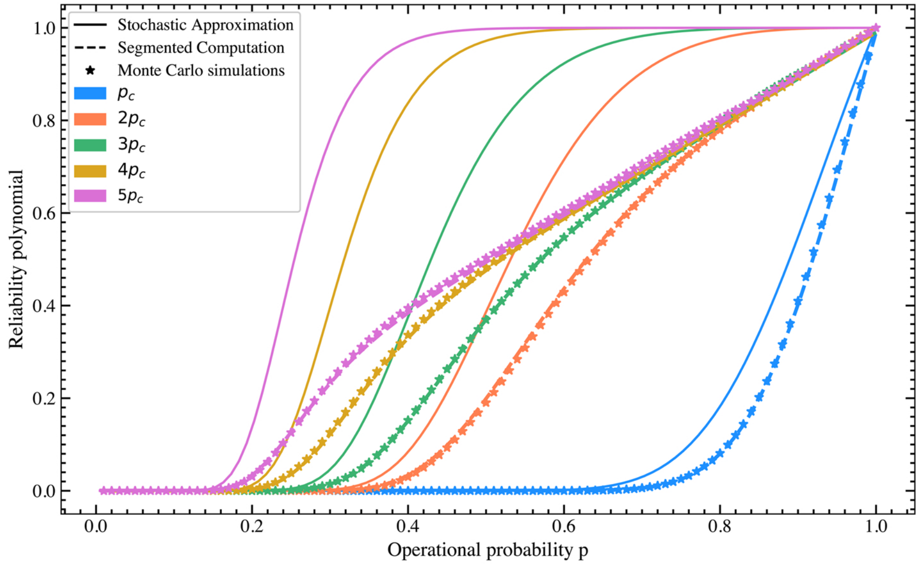

Figure 6. Two ER graphs with $N = 30$ nodes, connected by $L_C = 1$ link, are, for different link density $p_L = kp_c$ with $k = \{1, 2, 3, 4, 5\}$, computed by the stochastic approximation (2.2) for the entire graph and by formula (2.13), which is close to exact.
Alt text: This figure presents two Erdős–Rényi (ER) graphs, each with 30 nodes, connected by a single link. The reliability polynomial approximations for the entire graph are compared under different link densities, showing results from the stochastic approximation and an analytical formula.

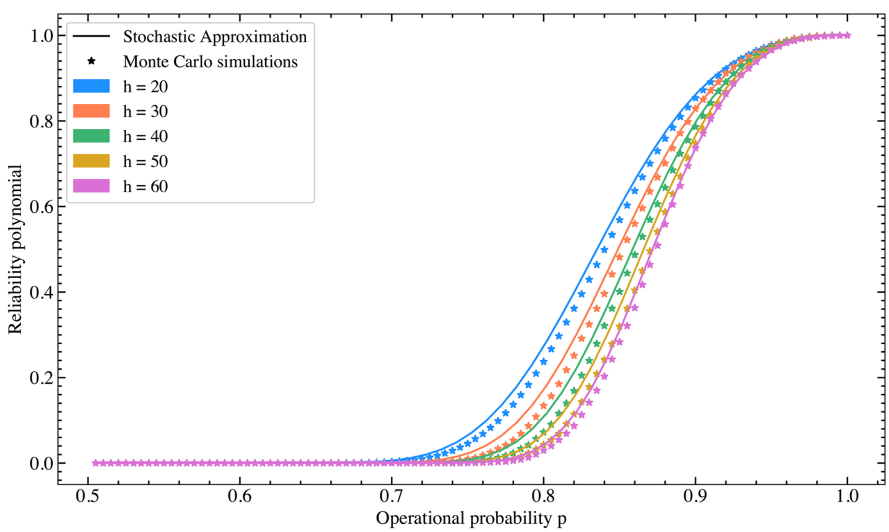

Figure 7. The stochastic approximation (2.2) and Monte Carlo simulations for 2D-lattice width 20 and different height $h = 20, 30, 40, 50$ and corresponding link density $p = 0.00952, 0.00639, 0.00481, 0.00386$ and 0.00322.
Alt text: This figure shows the stochastic approximation and Monte Carlo simulations for 2D lattice graphs with a fixed width of 20 and varying heights. The graph compares reliability polynomial estimates for different link densities corresponding to heights ranging from 20 to 50.

in (2.2) are larger than the actual reliability polynomial $rel_G(p)$. Moreover, both approximations solely use the degree vector or degree distribution to assess the connectivity of the random graph $\widehat{G}$.

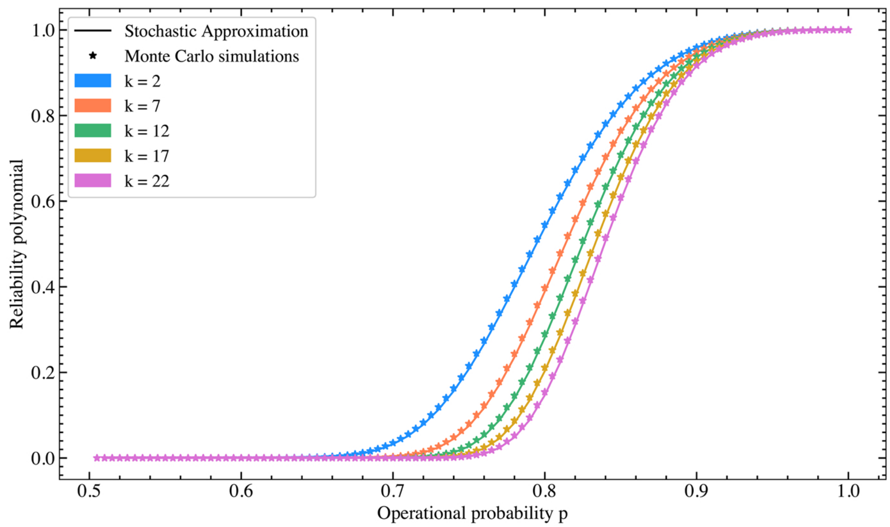

Figure 8. The stochastic approximation (2.2) and Monte Carlo simulations for 3D-lattice with width 20 and length 20 for different height $k = 2, 7, 12, 17, 22$ and corresponding link density $p = 0.006, 0.00197, 0.00117, 0.00084$ and 0.00064.
Alt text: This figure presents the stochastic approximation and Monte Carlo simulations for 3D lattice graphs with fixed width and length of 20. The plot compares the reliability polynomial estimates for varying graphs with fixed width and length of 20. The plot compares the reliability polynomial estimates for varying.

In the sequel, we present an interesting relation between the first-order approximation (4.1) and the stochastic approximation (2.2).

4.1 Arithmetic-geometric mean inequality

The arithmetic mean over all nodes of the probability $1 - f_i$ is

$$P_{AM} = \frac{1}{N}\sum_{i=1}^{N}(1-f_i) = \frac{1}{N}\sum_{i=1}^{N}\left(1-(1-p)^{d_i}\right)$$

which we rewrite as a sum over the nodal degrees by denoting n_j as the number of nodes with degree j and realizing that $\Pr[D = j] = \frac{n_j}{N}$,

$$P_{AM} = \frac{1}{N}\sum_{j=0}^{N-1} n_j\left(1-(1-p)^{j}\right) = \sum_{j=0}^{N-1}\Pr[D=j]\left(1-(1-p)^{j}\right)$$

After simplification with $\sum_{j=0}^{N-1}\Pr[D = j] = 1$ and (A.3), we arrive at

$$P_{AM} = 1 - \varphi_D(1-p) \tag{4.3}$$

The corresponding geometric mean is

$$P_{GM} = \sqrt[N]{\prod_{i=1}^{N}(1-f_i)} = \sqrt[N]{\prod_{i=1}^{N}(1-(1-p)^{d_i})} \tag{4.4}$$

Comparison of (4.3) and (4.4) with the definitions (2.2) and (4.1) leads to the interesting observation

$$(P_{AM})^N = \overline{rel_G\left(p\right)} \text{ and } (P_{GM})^N = (R_1)_G\left(p\right)$$

Since for any set of positive numbers $\{b_1, ., b_N\}$, the arithmetic mean is always greater than or equal to the geometric mean [28], with equality if and only if all $b_i = b$ are equal, the first-order approximation $(R_1)_G(p)$ is smaller than or equal to stochastic approximation $\overline{rel_G\left(p\right)}$, resulting in the key inequality

$$rel_G(p) \leq (R_1)_G\left(p\right) \leq \overline{rel_G\left(p\right)} \tag{4.5}$$

In other words, the first-order approximation $(R_1)_G(p)$ in (4.1) is always at least as accurate as the stochastic approximations $\overline{rel_G\left(p\right)}$ in approximating reliability polynomials $rel_G(p)$ in (2.1). Only if all nodes in the network G have the same degree, thus only for regular graphs, it holds that $(R_1)_G(p) = \overline{rel_G\left(p\right)}$.

4.2 Probabilistic proof of the inequality (4.5)

With $\Pr[D = j] = \frac{n_j}{N}$ and $q = 1 - p$, we rewrite the first-order approximation $(R_1)_G(p)$ in (4.1) as

$$(R_1)_G\left(p\right) = \prod_{i=1}^{N}(1 - q^{d_i}) = \prod_{d=0}^{N-1}(1 - q^d)^{n_d} = \prod_{d=0}^{N-1}(1 - q^d)^{N\Pr[D=d]}$$

After taking the logarithm of both sides,

$$\log\left[(R_1)_G\left(p\right)\right] = N\sum_{d=0}^{N-1}\Pr\left[D = d\right]\log(1 - q^d) \tag{4.6}$$

The definition [21, Equation (2.12) on p. 12] of the expectation E shows that

$$\log\left[(R_1)_G\left(p\right)\right] = N.E\left[\log\left(1 - q^D\right)\right] \tag{4.7}$$

or as

$$(R_1)_G\left(p\right) = \left(e^{E\left[\log\left(1-(1-p)^D\right)\right]}\right)^N \tag{4.8}$$

If $g(x)$ is a convex function on an interval $x \in [a, b]$, then Jensen's inequality [21, Equation (5.7) on p. 101] for a random variable $X \in [a, b]$ is $g(E[X]) \leq E[g(X)]$. Since $-\log x$ is convex, Jensen's inequality states that $E[\log(X)] \leq \log(E[X])$ from which

$$E\left[\log\left(1 - q^D\right)\right] \leq \log\left(E\left[1 - q^D\right]\right) = \log\left(1 - E\left[q^D\right]\right) = \log\left(1 - \varphi_D\left(q\right)\right)$$

Finally, (4.7) is upper bounded by

$$\log\left[(R_1)_G\left(p\right)\right] = N.E\left[\log\left(1 - q^D\right)\right] \leq N\log\left(1 - \varphi_D\left(q\right)\right)$$

which is equivalent, with the stochastic approximation (2.2) and $q = 1 - p$, to inequality (4.5).

By introducing the Taylor series $\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$, convergent for all $|z| < 1$, into (4.6), we obtain

$$\log\left[(R_1)_G(p)\right] = -N \sum_{d=0}^{N-1} \Pr[D=d] \sum_{k=1}^{\infty} \frac{q^{dk}}{k}$$
$$= -N \sum_{k=1}^{\infty} \frac{1}{k} \sum_{d=0}^{N-1} \Pr[D=d] \left(q^k\right)^d$$

With the definition of the pgf $\varphi_D(z) = E\left[z^D\right] = \sum_{j=0}^{N-1} \Pr[D=j]\, z^j$ of the degree D in the original graph G, another form of the logarithm of the first-order approximation in terms of the pgf $\varphi_D(z)$ is

$$\log\left[(R_1)_G(p)\right] = -N \sum_{k=1}^{\infty} \frac{\varphi_D\left((1-p)^k\right)}{k} \tag{4.9}$$

while the logarithm of the stochastic approximation (2.2) is

$$\log\left[\overline{rel_G(p)}\right] = N \log\left(1 - \varphi_D(1-p)\right) = -N \sum_{k=1}^{\infty} \frac{\left(\varphi_D(1-p)\right)^k}{k}$$

Jensen's inequality for the convex function $g(x) = x^k$ for integers k and positive real x, i.e.

$$\varphi_D\left(q^k\right) = E\left[q^{kD}\right] \geq \left(E\left[q^D\right]\right)^k = \left(\varphi_D(q)\right)^k$$

again demonstrates the inequality (4.5).

By invoking the harmonic, geometric and arithmetic mean inequality (C.1), we deduce a lower bound $-\log\left(E\left[\frac{1}{1-q^D}\right]\right) \leq E\left[\log\left(1-q^D\right)\right]$ for (4.7) in Appendix C. However, Monte Carlo simulations indicate that $rel_G(p) \not\leq \left(E\left[\frac{1}{1-(1-p)^D}\right]\right)^{-N}$, meaning that the lower bound does not lead to a tighter upper bound for the reliability polynomial $rel_G(p)$.

4.3 Comparison of stochastic and first-order approximation

Figures 9 and 10 show the comparison of first-order approximation $(R_1)_G(p)$, stochastic approximation $\overline{rel_G(p)}$ and Monte Carlo simulations on two real-world networks in the Network Repository [29]. Figure 9 shows that when the variance of the degree distribution in the network is large, the first-order approximation is significantly smaller than the stochastic approximation. If the variance var$[D]$ of the degree distribution in the network is relatively small, as in Fig. 10, then there is no significant difference between the first-order approximation $(R_1)_G(p)$ and stochastic approximation $\overline{rel_G(p)}$.

Another advantage of the first-order approximation $(R_1)_G(p)$ in (4.1) is the easy extension to heterogeneous reliability polynomials, where each link l between node i and j has operational probability $p_l = p_{ij}$,

$$(R_1)_G\left(\{p_{ij}\}_{1\leq i,j\leq N}\right) = \prod_{i=1}^{N} \left(1 - \prod_{l=1}^{d_i} (1-p_{il})\right)$$

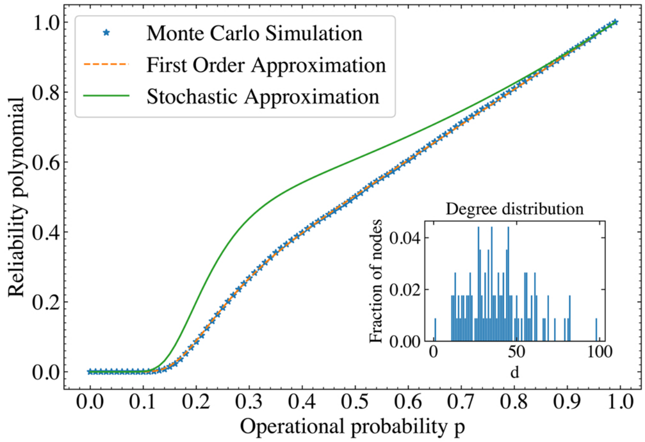

Figure 9. The stochastic approximation, first-order approximation and Monte Carlo simulations for Infect-hyper Network [29].
Alt text: This figure compares the stochastic approximation, first-order approximation, and Monte Carlo simulations for the Infect-hyper network. The plot illustrates the differences in reliability estimation methods based on the network's structure and characteristics.

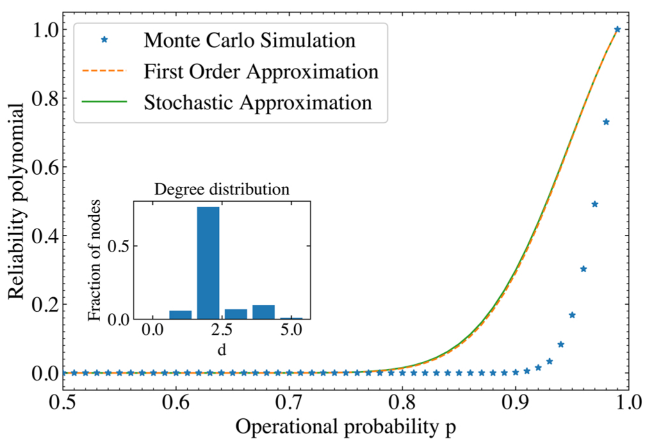

Figure 10. The stochastic approximation, first-order approximation and Monte Carlo simulations for Singapore MRT network [29].
Alt text: This figure presents the stochastic approximation, first-order approximation, and Monte Carlo simulations for the Singapore MRT network. The plot demonstrates the reliability estimation methods applied to the public transportation system's connectivity.

The computational complexity C_{stoch} of the stochastic approximation $\overline{rel_G\left(p\right)} = e^{N\log(1-\varphi_D(1-p))}$ requires the computation of $\varphi_D(1-p)$, which equals $C_{\text{stoch}} = O\left(\mathcal{N}_D\right)$, where $\mathcal{N}_D$ is the number of different degrees in the graph. The computational complexity $C_{\text{first-order}}$ of the first-order approximation $(R_1)_G(p)$ in (4.8) requires the computation of the average over N nodes and equals $C_{\text{first-order}} = O(N)$.

If the network has a very large number N of nodes, but only a few distinct degrees $\mathcal{N}_D$, then the stochastic approximation $\overline{rel_G\left(p\right)}$ becomes particularly efficient. Therefore, in networks where $\mathcal{N}_D \ll N$, the stochastic approximation requires significantly fewer computational resources, resulting in a substantial reduction in computation time and in an increased efficiency. This computational advantage becomes even more pronounced as the size N of the network grows.

5. CONCLUSION

We have presented and evaluated the accuracy of a simple, approximate formula (2.2) for reliability polynomials, which can be regarded as an extension of Erdős–Rènyi random graphs. The stochastic approximation is compared with the recently proposed first-order approximation by Brown *et al.* [19]. We show that the first-order approximation is always more or at least as accurate as the stochastic approximation. Both approximations are increasingly accurate for large graphs N and seems to perform better for graphs with a large link density p_L. Just for such dense and large graphs, computations of the reliability polynomials are demanding. Hence, both degree-based approximations fill a gap in current reliability theory. At last, our stochastic approximation provides an estimate for the critical operational probability p_N^* in (2.6) and for the number F_j of sets of j links whose removal retains the graph G connected in (B.11).

ACKNOWLEDGEMENTS

We are very grateful to reviewers, whose excellent comments led to the inclusion of Probabilistic proof of the inequality (4.5) section and Appendix C.

FUNDING

This research has been funded by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 101019718).

A. Degree of a random node in $\widehat{G}$

The degree $\widehat{D}$ of a randomly chosen node in $\widehat{G}$ follows from the law of total probability [21, p. 23],

$$\Pr\left[\widehat{D}=k\right]=\sum_{j=0}^{N-1}\Pr\left[\widehat{D}=k\middle|\,D=j\right]\Pr\left[D=j\right]$$

where D is the degree of random node in G. The conditional probability is

$$\Pr\left[\widehat{D}=k\middle|\,D=j\right]=\binom{j}{k}p^k\left(1-p\right)^{j-k}$$

because a node in G with degree j has j links, that each operate properly with probability p and the corresponding node in $\widehat{G}$ has j independent Bernoulli links, whose sum is a binomial random variable [21, Section 3.1.2]. Since G is connected, the minimum degree $d_{\min}(G) \geq 1$ and the degree $\widehat{D}$ of a randomly chosen node in $\widehat{G}$ has the distribution

$$\Pr\left[\widehat{D}=k\right]=\sum_{j=d_{\min}(G)}^{N-1}\Pr\left[D=j\right]\binom{j}{k}p^k\left(1-p\right)^{j-k} \tag{A.1}$$

We can rewrite (A.1) as

$$\Pr\left[\widehat{D}=k\right]=\frac{p^k}{k!}\sum_{j=0}^{N-1}\Pr\left[D=j\right]\frac{j!}{(j-k)!}\left(1-p\right)^{j-k}$$

Since $\frac{d^k}{dz^k} z^j = \frac{j!}{(j-k)!} z^{j-k}$, we have

$$\Pr\left[\widehat{D}=k\right] = \frac{p^k}{k!} \sum_{j=0}^{N-1} \Pr\left[D=j\right] \left.\frac{d^k}{dz^k} z^j\right|_{z=1-p} = \frac{p^k}{k!} \frac{d^k}{dz^k} \left.\left(\sum_{j=0}^{N-1} \Pr\left[D=j\right] z^j\right)\right|_{z=1-p}$$

Finally, we arrive at

$$\Pr\left[\widehat{D}=k\right] = \frac{p^k}{k!} \left.\frac{d^k \varphi_D(z)}{dz^k}\right|_{z=1-p} \tag{A.2}$$

Clearly, $\varphi_D(z) = 1$ and $|\varphi_D(z)| \leq 1$ for $|z| \leq 1$; also $\varphi_D(0) = \Pr[D=0] = 0$, which is zero in a connected graph G. In particular, (A.1) and (A.2) reduce for $k = 0$ to

$$\Pr\left[\widehat{D}=0\right] = \sum_{j=0}^{N-1} (1-p)^j \Pr\left[D=j\right] = \varphi_D(1-p) \tag{A.3}$$

where $\Pr[D=j] = 0$ for $0 < j < j_{\min} = D_{\min}$.

Furthermore, the pgf of the degree $\widehat{D}$ in $\widehat{G}$ follows from (A.2) with $\Pr\left[\widehat{D}=j\right] = 0$ for $j \geq N$ as

$$\varphi_{\widehat{D}}(z) = E\left[z^{\widehat{D}}\right] = \sum_{j=0}^{N-1} \Pr\left[\widehat{D}=j\right] z^j = \sum_{j=0}^{\infty} \frac{1}{j!} \left.\frac{d^j \varphi_D(z)}{dz^j}\right|_{z=1-p} (pz)^j$$

After invoking Taylor's theorem $f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} \left.\frac{d^j f(z)}{dz^j}\right|_{z=z_0} (z-z_0)^j$, we find that pgf $\varphi_{\widehat{D}}(z) = E\left[z^{\widehat{D}}\right]$ of the degree $\widehat{D}$ in $\widehat{G}$ is

$$\varphi_{\widehat{D}}(z) = \varphi_D(1-p+pz)$$

where the pgf of a link in $\widehat{G}$ is $\varphi_{\widehat{a}}(z) = 1-p+pz$. Thus, it holds that $\varphi_{\widehat{D}}(z) = \varphi_D(\varphi_{\widehat{a}}(z))$, which is an instance of the general formula in [21, (2.79) on p. 31].

B. The approximation (2.2) in polynomial form

B.1 Polynomial forms of (2.3) and (2.2)

We write (2.3) as a pure polynomial in p by using Newton's binomium,

$$rel_G(p) = p^L \sum_{j=0}^{L-N+1} F_j \left(\frac{1}{p}-1\right)^j = p^L \sum_{j=0}^{L-N+1} F_j \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} p^{-k}$$

$$= p^L \sum_{j=0}^{L-N+1} \sum_{k=0}^{j} F_j \binom{j}{k} (-1)^{j-k} p^{-k}$$

Reversing the sums yields

$$rel_G(p) = \sum_{k=0}^{L-N+1} \sum_{j=k}^{L-N+1} F_j \binom{j}{k} (-1)^{j-k} p^{L-k}$$

Let $l = L - k$, then

$$rel_G(p) = \sum_{l=N-1}^{L} \sum_{j=L-l}^{L-N+1} F_j \binom{j}{L-l} (-1)^{j-L-l} p^l$$

Subsequently, let $m = L - j$ to finally obtain,

$$rel_G(p) = \sum_{l=N-1}^{L} r_l p^l \tag{B.1}$$

where the polynomial coefficient r_l equals

$$r_l = \sum_{m=N-1}^{l} (-1)^{m-l} \binom{L-m}{L-l} F_{L-m} \tag{B.2}$$

The alternative polynomial in $q = 1 - p$ follows similarly as

$$rel_G(p) = \sum_{j=0}^{L-N+1} F_j q^j (1-q)^{L-j} = \sum_{j=0}^{L-N+1} F_j q^j \sum_{k=0}^{L-j} \binom{L-j}{k} (-1)^k q^k$$

$$= \sum_{j=0}^{L-N+1} \sum_{k=0}^{L-j} \binom{L-j}{k} F_j (-1)^k q^{k+j}$$

Let $l = k + j$, then $0 \le l \le L$ and $0 \le j = l - k \le L - N + 1$ implies that $0 \le k \le l$, so that

$$rel_G(p) = \sum_{l=0}^{L} \sum_{k=0}^{l} \binom{L-(l-k)}{k} F_{l-k} (-1)^k q^l$$

Hence, the reliability polynomial in terms of $q = 1 - p$ is

$$rel_G(1-q) = \sum_{l=0}^{L} v_l q^l \tag{B.3}$$

with the coefficient

$$v_l = \sum_{k=0}^{l} (-1)^k \binom{L-l+k}{k} F_{l-k} = \sum_{n=0}^{l} (-1)^{l-n} \binom{L-n}{l-n} F_n \tag{B.4}$$

In particular, for $p = 1$, we know that $rel_G(p) = 1$. Thus, for $q = 0$ in (B.3), it follows that $1 = rel_G(1) = v_0 = F_0$.

Similarly, we write our approximation (2.2) as a polynomial in $q = 1 - p$

$$\overline{rel_G(p)} = (1 - \varphi_D(q))^N = \left(1 - \sum_{k=0}^{N-1} \Pr[D = k] q^k\right)^N$$

The integer power of a Taylor series $f(z) = \sum_{k=0}^{\infty} f_k z^k$ follows as $f^N(z) = \left(\sum_{k=0}^{\infty} f_k z^k\right)^N = \sum_{j_1=0}^{\infty} \cdots \sum_{j_N=0}^{\infty} \prod_{i=1}^{N} f_{j_i} z^{\sum_{i=1}^{N} j_i}$. After letting $m = \sum_{i=1}^{N} j_i$ with $j_i \geq 0$ for each $1 \leq i \leq N$, we obtain

$$f^N(z) = \sum_{m=0}^{\infty} \left(\sum_{\sum_{i=1}^{N} j_i = m; j_i \geq 0} \prod_{i=1}^{N} f_{j_i} \right) z^m \tag{B.5}$$

Applying (B.5) shows, with $f_0 = 1 - \Pr[D = 0] = 1$ and $f_k = -\Pr[D = k]$, that

$$\overline{rel_G(p)} = 1 + \sum_{m=1}^{\infty} c_m q^m \tag{B.6}$$

where $c_0 = 1$ and the coefficient c_m for $m > 0$,

$$c_m = \sum_{\sum_{i=1}^{N} j_i = m; j_i \geq 0} \prod_{i=1}^{N} f_{j_i} = \sum_{j=0}^{N} \binom{N}{j} s[j, m] \tag{B.7}$$

where the characteristic coefficients, first defined in [30],

$$s[k, m] = \sum_{\sum_{i=1}^{k} j_i = m; j_i > 0} \prod_{i=1}^{k} f_{j_i} \tag{B.8}$$

are efficiently computed by the recursion

$$\begin{aligned} s[1, m] &= f_m \\ s[k, m] &= \sum_{j=k}^{m} f_{m-j+1}\, s[k-1, j-1] \qquad (k > 1) \\ &= \sum_{j=1}^{m-k+1} f_j\, s[k-1, m-j] \qquad (k > 1) \end{aligned} \tag{B.9}$$

B.2 Estimates for the coefficients F_j in (2.3)

Equating corresponding powers in (B.3) and (B.6) yields $v_m \approx c_m$, for $m \geq 1$,

$$\sum_{n=0}^{m} (-1)^{m-n} \binom{L-n}{m-n} F_n \approx c_m$$

Using $\binom{L-n}{m-n} = \frac{(L-n)!}{(m-n)!(L-m)!} = \binom{L-n}{L-m}$ and with $k = L - n$, we have

$$\sum_{k=L-m}^{L} (-1)^{L-m+k} \binom{k}{L-m} F_{L-k} \approx c_m$$

For an arbitrary q, applying the second binomial inverse pair [31, chap. 2]

$$f_n = \sum_{k=n}^{q} \binom{k}{n} g_k \Leftrightarrow g_n = \sum_{k=n}^{q} \binom{k}{n} (-1)^{k+n} f_k \tag{B.10}$$

yields (with $n = L - m$ and $q = L$, it holds that $\sum_{k=n}^{L} \binom{k}{n} (-1)^{n+k} F_{L-k} \approx c_{L-n}$; next, let $f_k = F_{L-k}$ and $g_n = c_{L-n}$ in (B.10))

$$F_{L-n} \approx \sum_{k=n}^{L} \binom{k}{n} c_{L-k} = \sum_{k=0}^{L-n} \binom{k+n}{n} c_{L-n-k}$$

Finally, with $m = L - n$, we arrive at

$$F_m \approx \sum_{k=0}^{m} \binom{k+L-m}{k} c_{m-k} \tag{B.11}$$

In other words, we found an approximation (B.11) for the coefficients F_m of the reliability polynomial in (2.3), whose exact computation was proven to be an NP-complete problem in [4, 5] and [3].

We compute the approximation (B.11) of F_m for the first few values of m. For $m = 1$, (B.11) becomes $F_1 \approx \sum_{k=0}^{1} \binom{k+L-1}{k} c_{1-k} = c_1 + Lc_0 = c_1 + L$ and $c_1 = \sum_{j=0}^{N} \binom{N}{j} s[j,1] = Ns[1,1] = Nf_1 = -N \Pr[D = 1]$. To shorten the notation, we denote $d_k = \Pr[D = k]$ and present the list:

$$\begin{aligned}
F_1 &\approx L - Nd_1 \\
F_2 &\approx \binom{L}{2} - d_1(L-1)N + d_1^2 \binom{N}{2} - d_2 N \\
F_3 &\approx \binom{L}{3} - d_1 \binom{L-1}{2} N + (L-2)\left(d_1^2 \binom{N}{2} - d_2 N\right) - d_1^3 \binom{N}{3} + 2d_2 d_2 \binom{N}{2} - d_3 N \\
F_4 &\approx \binom{L}{4} - d_1 \binom{L}{3} N + \binom{L-2}{2} \left(d_1^2 \binom{N}{2} - d_2 N\right) \\
&\quad + (L-3)\left(-d_1^3 \binom{N}{3} + 2d_1 d \binom{N}{2} - d_3 N\right) \\
&\quad + d_1^4 \binom{N}{4} - 3d_1^2 d_2 \binom{N}{3} + \left(d_2^2 + 2d_1 d_3\right) \binom{N}{2} - d_4 N
\end{aligned}$$

With current mathematical symbolic programs, such as Mathematica, we can explicitly compute (B.11) for any desired, finite value of m.

C. Harmonic, geometric and arithmetic mean inequality

For positive real numbers $a_1, a_{2,}, \ldots, a_n$, the harmonic, geometric and arithmetic mean inequality is [21, p. 99]

$$\frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}} \leq \sqrt[n]{\prod_{k=1}^{n} a_k} \leq \frac{1}{n} \sum_{k=1}^{n} a_k \tag{C.1}$$

with equality only if all a_j are equal.

The harmonic mean corresponding to P_{AM} in (4.3) is

$$P_{HM} = \frac{N}{\sum_{i=1}^{N}\frac{1}{1-f_i}} = \frac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{1-(1-p)^{d_i}}} = \frac{1}{\frac{1}{N}\sum_{d=0}^{N-1} n_d \frac{1}{1-(1-p)^{d}}}$$
$$= \frac{1}{\sum_{d=0}^{N-1}\Pr[D=d]\frac{1}{1-(1-p)^{d}}}$$

which is rewritten in terms of the expectation operator as

$$P_{HM} = \frac{1}{E\left[\frac{1}{1-(1-p)^{D}}\right]}$$

Alternatively, invoking the geometric series yields an expression terms of the pgf $\varphi_D(z)$ of the degree,

$$P_{HM} = \frac{1}{\sum_{d=0}^{N-1}\Pr[D=d]\frac{1}{1-(1-p)^{d}}} = \frac{1}{\sum_{d=0}^{N-1}\Pr[D=d]\sum_{k=0}^{\infty}(1-p)^{dk}}$$
$$= \frac{1}{\sum_{k=0}^{\infty}\sum_{d=0}^{N-1}\Pr[D=d]\left((1-p)^{k}\right)^{d}} = \frac{1}{\sum_{k=0}^{\infty}\varphi_D\left((1-p)^{k}\right)}$$
$$= \frac{1}{1+\sum_{k=1}^{\infty}\varphi_D\left((1-p)^{k}\right)}$$

Since $P_{HM} \leq P_{GM} \leq P_{AM}$ by (C.1) and we find, with $P_{GM}^{N} = (R_1)_G(p)$ and $P_{AM}^{N} = \overline{rel_G(p)}$, that

$$\frac{1}{\left(E\left[\frac{1}{1-(1-p)^{D}}\right]\right)^{N}} \leq [(R_1)_G(p)] \leq \overline{rel_G(p)}$$

Taking the logarithms,

$$-N\log\left(E\left[\frac{1}{1-(1-p)^{D}}\right]\right) \leq \log[(R_1)_G(p)] \leq \log\left[\overline{rel_G(p)}\right]$$

then translates, with the expectation forms (4.7) of $(R_1)_G(p)$ and $\overline{rel_G(p)}$ and with $q = 1-p$, to

$$-\log\left(E\left[\frac{1}{1-q^{D}}\right]\right) \leq E\left[\log\left(1-q^{D}\right)\right] \leq \log\left(1-E\left[q^{D}\right]\right)$$

If we replace the random variable $X = 1-q^{D}$, then we find the harmonic, geometric and arithmetic expectation inequality for any non-negative random variable X:

$$-\log\left(E\left[X^{-1}\right]\right) \leq E[\log(X)] \leq \log(E[X]) \qquad \text{(C.2)}$$

In summary, the harmonic approximation would be a sharper approximation than $(R_1)_G(p)$ for the reliability polynomial $rel_G(p)$, if the inequality

$$rel_G(p) \leq \left(E\left[\frac{1}{1-(1-p)^D} \right] \right)^{-N} \tag{C.3}$$

would hold. Unfortunately, Monte Carlo simulations indicate that the harmonic inequality (C.3) is violated.

D. Additional example of the stochastic approximation

D.1 Tree graphs

The reliability polynomial for any tree graph T with N nodes and $N-1$ links is $rel_T(p) = p^{N-1}$. Although the stochastic approximation (2.2) for any tree T is more complicated than the exact, simple $rel_T(p) = p^{N-1}$, the main purpose here is to compare the goodness of the stochastic approximation (2.2) with exact results.

In contrast to the actual reliability polynomial $rel_T(p)$, which is the same for all tree graphs on N nodes, the stochastic approximation (2.2) of the reliability polynomial of a tree graph with N nodes varies with the degree distribution D.

Consider two tree graphs T_1 and T_2 on N nodes, where the only difference between the degree distributions is that the degrees of node i and j of T_1 are d_i and d_j, while the corresponding degrees in T_2 are d_i-1 and d_j+1, and all the other nodes have the same degree. Anticipating the analysis in Section 4, where we show that $1-\varphi_D(1-p) = \frac{\sum_{i=1}^{N}(1-(1-p)^{d_i})}{N}$, the difference

$$\begin{aligned}\Delta &= \left(1-\varphi_{D_{T_1}}(1-p)\right) - \left(1-\varphi_{D_{T_2}}(1-p)\right) \\ &= \frac{p}{N}\left((1-p)^{d_i-1} - (1-p)^{d_j}\right)\end{aligned}$$

from which we conclude that $\Delta \geq 0$ if $d_i \geq d_j+1$, because the function $f(x) = x^k$, where k is a positive integer, is monotonically increasing with x in the domain [0, 1]. Hence, $\overline{rel_{T_1}(p)} \geq \overline{rel_{T_2}(p)}$ only if $d_i \geq d_j+1$. We define a tree graph *descending restructuring*, which disconnects a branch of a node i with degree d_i in a tree graph and connects that branch to another node j with degree d_j, where $d_i \geq d_j$. The *ascending restructuring* is defined as the inverse of the descending restructuring. The argument indicates that the stochastic approximation $\overline{rel_{T'}}$ of a tree T' after an ascending restructuring is smaller than or equal than $\overline{rel_T}$.

A path graph P_N with N nodes can be reconstructed into any other tree graph with N nodes by multiple ascending restructurings. Thus, the stochastic approximation $\overline{rel_{P_N}}$ of path graph P_N with N nodes is larger than $\overline{rel_T}$ in any other tree graph with N nodes. A star graph $K_{1,N-1}$ with N nodes can also be reconstructed into any other tree graph with N nodes by multiple descending restructurings. Hence, the stochastic approximation $\overline{rel_{K_{1,N-1}}}$ of star graph with N nodes is smaller than $\overline{rel_T}$ in any other tree graph with N nodes. In conclusion, it holds for any $0 \leq p \leq 1$ that

$$\overline{rel_{K_{1,N-1}}(p)} \leq \overline{rel_T(p)} \leq \overline{rel_{P_N}(p)} \tag{D.1}$$

The stochastic approximation of a star graph with N nodes and a path graph with N nodes are respectively.

$$\overline{rel_{K_{1,N-1}}(p)} = \left(1 - \frac{N-1}{N}(1-p) - \frac{1}{N}(1-p)^{N-1}\right)^N \tag{D.2}$$

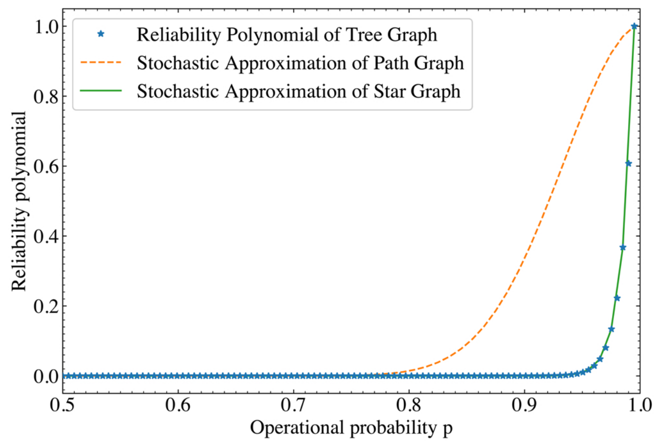

Figure A1. Comparison of the stochastic approximation $\overline{rel_{K_{1,N-1}}(p)}$ for the star and $\overline{rel_{P_N}(p)}$ for the path graph with $N = 100$ and the actual reliability polynomial $rel_T(p)$.
Alt text: This figure compares the stochastic approximation of the reliability polynomial for a star graph and a path graph, each with 100 nodes. The plot also includes the actual reliability polynomial for the tree graph to highlight the differences in approximation accuracy.

and

$$\overline{rel_{P_N}(p)} = \left(1 - \frac{2}{N}(1-p) - \frac{N-2}{N}(1-p)^2\right)^N \tag{D.3}$$

while the actual reliability polynomial of any tree graph with N nodes is $rel_T(p) = p^{N-1}$.

Dividing $\overline{rel_{K_{1,N-1}}(p)}$ by $rel_T(p)$ yields

$$\frac{\overline{rel_{K_{1,N-1}}(p)}}{rel_T(p)} = p\left(\frac{1 - \frac{N-1}{N}(1-p) - \frac{1}{N}(1-p)^{N-1}}{p}\right)^N$$

For large N, we have

$$\frac{\overline{rel_{K_{1,N-1}}(p)}}{rel_T(p)} \approx p\left(\frac{1 - \frac{N-1}{N}(1-p)}{p}\right)^N = p\left(\frac{1-p+Np}{Np}\right)^N$$
$$= p\left(1 + \frac{1-p}{Np}\right)^N = p\left(\left(1 + \frac{1-p}{Np}\right)^{\frac{Np}{1-p}}\right)^{\frac{1-p}{p}} \approx pe^{\frac{1-p}{p}}$$

Similarly, we find that

$$\frac{\overline{rel_{P_N}(p)}}{rel_T(p)} = p\left(\frac{1 - \frac{2}{N}(1-p) - \frac{N-2}{N}(1-p)^2}{p}\right)^N$$
$$= p\left(\frac{2N - 2 - (N-2)p}{N}\right)^N \approx p(2-p)^N$$

In summary, we arrive at

$$pe^{\frac{1-p}{p}}rel_T(p) \leq \overline{rel_T(p)} \leq p\left(2-p\right)^N rel_T(p)$$

Figure A1 shows that stochastic approximation performs best for a star graph $K_{1,N-1}$ and worst for a path graph P_N.

REFERENCES

1. Pérez-Rosés H. Sixty years of network reliability. *Math Comput Sci* 2018;**12**:275–93.
2. Moore EF, Shannon CE. Reliable circuits using less reliable relays. *J Franklin Inst* 1956;**262**:191–208.
3. Colbourn CJ. *The Combinatorics of Network Reliability*. Oxford, UK: Oxford University Press, 1987.
4. Scott Provan J, Ball MO. The complexity of counting cuts and of computing the probability that a graph is connected. *SIAM J Comput* 1983;**12**:777–88.
5. Ball MO. Complexity of network reliability computations. *Networks* 1980;**10**:153–65.
6. Colbourn CJ. *The Combinatorics of Network Reliability*. New York: Oxford University Press, Inc., 1987.
7. Carlier J, Lucet C. A decomposition algorithm for network reliability evaluation. *Discret Appl Math* 1996;**65**:141–56.
8. Silva J, Gomes T, Tipper D, Martins L, Kounev V. An algorithm for computing all-terminal reliability bounds. In: *2014 6th International IEEE Workshop on Reliable Networks Design and Modeling (RNDM)*, Barcelona, 2014, 76–83.
9. Silva J, Gomes T, Tipper D, Martins L, Kounev V. An effective algorithm for computing all-terminal reliability bounds. *Networks* 2015;**66**:282–295.
10. Ball MO. Computational complexity of network reliability analysis: an overview. *IEEE Trans Reliab* 1986;**35**:230–9.
11. Satyanarayana A, Wood RK. A linear-time algorithm for computing K-terminal reliability in series-parallel networks. *SIAM J Comput* 1985;**14**:818–32.
12. Shooman AM. Algorithms for network reliability and connection availability analysis. In: *Proceedings of IEEE Electro/International 1995*, Boston MA, 1995, 309–33.
13. Brown JI, Colbourn CJ, Cox D, Graves C, Mol L. Network reliability: heading out on the highway. *Networks* 2021;**77**:146–60.
14. Fishman GS. A Monte Carlo sampling plan for estimating network reliability. *Oper Res* 1986;**34**:581–94.
15. Nelson AC, Batts JR, Beadles RL. A computer program for approximating system reliability. *IEEE Trans Reliab* 1970;**19**:61–5.
16. Shier DR. *Network Reliability and Algebraic Structures*. Oxford, UK: Clarendon Press, 1991.
17. Esary JD, Proschan F. A reliability bound for systems of maintained, interdependent components. *J Am Stat Assoc* 1970;**65**:329–38.
18. Sebastio S, Trivedi KS, Wang D, Yin X. Fast computation of bounds for two-terminal network reliability. *Eur J Oper Res* 2014;**238**:810–23.
19. Brown JI, Kolokolnikov T, Kooij RE. New approximations for network reliability. *Networks* 2024;**84**:1–13. doi: 10.1002/net.22215.
20. Van Mieghem P. *Graph Spectra for Complex Networks*, 2nd edn. Cambridge, UK: Cambridge University Press, 2023.
21. Van Mieghem P. *Performance Analysis of Complex Networks and Systems*. Cambridge, UK: Cambridge University Press, 2014.
22. van der Hofstad R. *Random Graphs and Complex Networks. Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge, UK: Cambridge University Press, 2016. Notes at www.win.tue/rhofstad/notesrgcn2023.pdf.
23. Janson S, Knuth DE, Luczak T, Pittel B. The birth of the giant component. *Random Struct Algorithms* 1993;**4**:233–358.
24. Erdős P, Rényi A. On random graphs. *Publ Math Debrecen* 1959;**6**:290–7.
25. Gilbert EN. Enumeration of labelled graphs. *Can J Math* 1956;**8**:405–11.
26. Zhou W, Janic M, Kooij RE, Kuipers FA. On the availability of networks. In: *Proceedings of Broadband 2007*, BroadBand Europe, Antwerpen. December, 2007, 1–6.
27. Van Mieghem P. Topology information condensation in hierarchical networks. *Computer Networks* 1999;**31**:2115–37.
28. Hardy GH, Littlewood JE, Polya G. *Inequalities*, 2nd edn. Cambridge, UK: Cambridge University Press, 1999.

29. Rossi RA, Ahmed NK. The network data repository with interactive graph analytics and visualization. In: Proceedings of the AAAI Conference on Artificial Intelligence, Austin, Texas, 2015.
30. Van Mieghem P. The asymptotic behaviour of queueing systems: Large deviations theory and dominant pole approximation. *Queueing Syst* 1996;**23**:27–55.
31. Riordan J. *Combinatorial Identities*. New York: John Wiley & Sons, 1968.