On Existence and Uniqueness Results for some Nonlinear Schrödinger equations

J.S. Eenhoorn

TU Delft

On Existence and Uniqueness Results for some Nonlinear Schrödinger equations

by

J.S. Eenhoorn

to obtain the degree of Bachelor of Science
at the Delft University of Technology,
submitted to the thesis committee on December 20, 2022.

Student number:	4099044	
Thesis committee:	dr. M.V. Gnann,	TU Delft, supervisor
	dr. ir. J.M. Thijssen,	TU Delft

TU Delft

Contents

1 Introduction **1**

2 Physics of NLS **7**

2.1 Derivation of the wave equation from Maxwell . . . 7
2.2 Validity of plane wave solutions . . . 7
2.3 Derivation of the Helmholtz equation . . . 8
2.4 Derivation of the Linear Schrödinger equation . . . 9
2.5 Polarisation field . . . 10
2.6 Implications of nonlinear polarisation . . . 11
2.7 Soliton solutions . . . 11

3 Existence of ground state **13**

3.1 Initial value problem and nonlinearity . . . 13
3.2 Definitions of solution sets . . . 13
3.3 Assumptions on f . . . 14
3.4 Main theorem . . . 14
3.5 Interval of definition . . . 15
3.6 Asymptotics of positive decreasing solutions . . . 17
3.7 P is non-empty and open . . . 19

4 Uniqueness of ground state **21**

4.1 Technical lemmata . . . 22
4.2 Proof of the main uniqueness theorem . . . 27
4.3 Derivation of the equation for radially symmetric solutions . . . 30

Bibliography **31**

1

Introduction

Current day applications of laser beams are laser cutting, lithography, laser spectography, laser ranging, laser pointers and more. The study of high power lasers is an important subfield of modern optics. These applications of laser technology depend on the coherence of the laser beam, which differentiates it from most other sources of light. Spatial coherence is crucial for focusing the beam, or to ensure collimation over distances. Laser beams also have high temporal coherence, which allows for narrow spectrums. Even though the beam never has one single frequency, we refer to beams with narrow spectrums as monochromatic or time-harmonic.

Laser beams are electromagnetic waves, whose behaviour in a medium (but in absence of external charges) is described by Maxwell's equations. In particular, laser beams are monochromatic and thin paraxial beams. For monochromatic light, the wave equation that follows from Maxwell's equations reduces to the linear Helmholtz equation. For thin paraxial beams, the amplitude of the electric field is described by the linear Schrödinger equation. When the propagation of the laser beam is weakly nonlinear, the beam could be self-focusing. This is the case for propagation in a nonlinear Kerr medium such as glass. In fact, if the self-focusing behaviour is sufficiently strong, the beam profile collapses. That is, the intensity at the beam center blows up for finite propagation distances. This behaviour can be predicted by the nonlinear Schrödinger equation (NLS) up to the initial stages of collapse. Therefore, we consider nonlinear forms of the aforementioned wave equations, in particular, the NLS equation.

The purpose of this thesis is to shed light on two mathematical papers from the 20th century that study an NLS equation called the soliton equation. Prior to the mathematical chapters, we derive the soliton equation from Maxwell's laws. The reader is expected to be familiar with electrodynamics, ordinary differential equations and basic proofs. As for the mathematical papers, the first paper, written in 1981 by Berestycki, Lions and Peletier, presents an existence result for ground state solutions to a family of equations

$$-\Delta u = g(u) \quad \text{in } \mathbb{R}^N \tag{1.1}$$

which for radial solutions $u(r)$ reduces to the ODE

$$u''(r) + \frac{N-1}{r}u'(r) = -g(u(r)) \quad \text{for } 0 < r < \infty. \tag{1.2}$$

Ground state solutions are radially symmetric, everywhere positive and the limit at infinity is zero. Physically, the ground state solution is the lowest-power nontrivial solution. The ground state solution represents the boundary between global existence of solutions and solutions blowing up in finite time (for energies lower than the ground state). For $N = 2$ and $g(u) = u^3 - u$, equation (1.2) is equivalent to equation (2.22) presented at the end of Chapter 2. From the derivation of the soliton equation, we know that $u(r)$ in equation (1.2) is the amplitude of the envelope of the electric-field.

The main proof method for existence of ground state solutions used in [1] is known as the shooting method, and categorises solutions for different initial conditions based on the qualitative behaviour. The second paper, written in 1972 by Coffman, gives a uniqueness result for ground state solutions to

$$\Delta u - u + u^3 = 0 \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

This problem is equivalent to equation (1.1) with $g(u) = u^3 - u$. More specifically, [3] provides an ODE uniqueness result for $w(r) = ru(r)$, where $u(r)$ solves equation (1.2)

$$w''(r) - w(r) + r^{-2}w(r)^3 = 0 \quad \text{for } 0 < r < \infty,$$

and demonstrates using variational methods how this implies uniqueness of ground state solutions to equation (1.3). We focus on ODE methods in the present work. The results presented in these mathematical papers contribute to the theory of self-focusing beams.

The proofs presented in the papers use standard methods for ordinary differential equations to show existence and uniqueness of ground state solutions, but the details are not apparent at a bachelor level. The contribution of this thesis is to expand the level of detail with which the analysis and proofs are presented. Where possible, we changed the notation from the papers to present the equations and proofs in a unified way.

In Chapter 2, we give a detailed derivation of the NLS based on [4, Chapter 1]. We start from Maxwell's laws and derive a vector wave equation in three dimensions in section 2.1. Then, we discuss plane wave solutions and time-harmonic solutions in section 2.2, which yields a scalar linear Helmholtz equation in section 2.3. For laser beams, most of the plane wave modes are axial, which motivates the paraxial approximation from which we obtain a linear Schrödinger equation in section 2.4. We study the polarisation field in section 2.5 and use the leading-order nonlinear term to obtain the NLS equation in section 2.6. Lastly, we rescale the coordinates and consider radial solutions, which yields a soliton equation in section 2.7. In the remaining two chapters, we study the existence and uniqueness of ground state solutions for this equation.

In Chapter 3, we present an existence proof for ground state solutions based on [1]. We discuss an initial value problem in section 3.1 of which the soliton equation that concluded Chapter 2 is a specific case. First, we categorise the solutions by their asymptotic behaviour in section 3.2. Next, we discuss assumptions on the initial value problem in section 3.3 and present the main theorem in section 3.4. The proof method is a shooting argument for the initial value and the proof depends on technical lemmata. For several lemmata, we provide additional proof details. We discuss the interval of definition for solutions in section 3.5, the asymptotic behaviour of positive decreasing solutions in section 3.6, and we show that the set of everywhere positive solutions is non-empty and open in section 3.7. To study the other lemmata requires knowledge of variational calculus, which could be the topic of further research.

In Chapter 4, we study [3] which proves the uniqueness of ground state solutions for the soliton equation presented at the end of Chapter 2. We focus on [3, Section 4], which uses ODE methods to establish the uniqueness of positive radially symmetric ground state solutions. In the other sections, the author discusses details on Sobolev spaces and the minimisation of a Rayleigh quotient associated with the general problem to show that the uniqueness of ground state solutions for the general problem follows from the uniqueness of ground state solutions for the radially symmetric case. The main theorem, similar to [1], uses a shooting argument for the initial value and requires four technical lemmata. We present more details on the proof of the main theorem in section 4.2 and prove one of these lemmata in section 4.1.

2

Physics of NLS

2.1. Derivation of the wave equation from Maxwell

Any electromagnetic wave is governed by Maxwell's laws. In this work, we work in absence of external charges or currents. Then Maxwell's laws for the electric field $\vec{\mathcal{E}}$, magnetic field $\vec{\mathcal{H}}$, induction electric field $\vec{\mathcal{D}}$ and induction magnetic field $\vec{\mathcal{B}}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \qquad (1.1.a) \qquad\qquad \nabla \cdot \vec{\mathcal{D}} = 0, \qquad (1.1.c)$$

$$\nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t}, \qquad (1.1.b) \qquad\qquad \nabla \cdot \vec{\mathcal{B}} = 0. \qquad (1.1.d)$$

The fields are in three-dimensional Cartesian coordinates, for example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in (x, y, z) coordinates. Besides considering no external charges or currents, we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) is given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \qquad (1.2.a) \qquad\qquad \vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}. \qquad (1.2.b)$$

The notation used here is from "The Nonlinear Schrödinger Equation" by G. Fibich [4, p. 3]. For more background on electrodynamics see "Introduction to Electrodynamics" by D.J. Griffiths [5]. This reference work also includes an introduction to the necessary vector calculus.

We use vector calculus and Maxwell's laws to rewrite the curl of the curl:

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) \overset{(1.1.a)}{=} \nabla \times \left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{\mathcal{B}}\right) \overset{\substack{(1.1.b)\\(1.2.a)}}{=} -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} \overset{(1.2.b)}{=} -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \text{ and}$$

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \nabla^2 \vec{\mathcal{E}} = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \Delta \vec{\mathcal{E}} \overset{\substack{(1.1.c)\\(1.2.b)}}{=} -\Delta \vec{\mathcal{E}}.$$

Combining these and using $\mu_0 \epsilon_0 = 1/c^2$, we arrive at the vector wave equation:

$$\Delta \vec{\mathcal{E}} = \frac{1}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2}. \qquad (2.3)$$

2.2. Validity of plane wave solutions

Stuyding the left and right hand sides of equation (2.3), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta \vec{\mathcal{E}} = \Delta \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathcal{E}_x}{\partial x^2} + \frac{\partial^2 \mathcal{E}_x}{\partial y^2} + \frac{\partial^2 \mathcal{E}_x}{\partial z^2} \\ \frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial y^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2} \\ \frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} + \frac{\partial^2 \mathcal{E}_z}{\partial z^2} \end{bmatrix} = \frac{1}{c^2} \begin{bmatrix} \frac{\partial^2 \mathcal{E}_x}{\partial t^2} \\ \frac{\partial^2 \mathcal{E}_y}{\partial t^2} \\ \frac{\partial^2 \mathcal{E}_z}{\partial t^2} \end{bmatrix}$$

$$\Delta \mathcal{E}_j = \sum_{l=1}^{3} \left[\frac{\partial^2 \mathcal{E}_j}{\partial x_l^2} \right] = \frac{1}{c^2} \frac{\partial^2 \mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z - \omega_0 t)}, \tag{2.4}$$

where k_0 is the wavenumber and ω_0 the frequency. These are so called plane wave solutions. The wavefronts have the simple geometry of an infinite plane at any z-value and the electric field is non-zero in the x and y directions. The wavefronts are spaced by the wavelength λ and the wavenumber k_0 is the reciprocal of the wavelength.

This plane wave travels in the positive z-direction for positive wavenumber k_0 and vice versa. Note that the solution does not depend on x or y. As a result, for a fixed z', the electric field $\mathcal{E}$ is constant in the (x, y, z')-plane.

We substitute (2.4) in equation (2.3). Note that only Δ_z will be non-zero:

$$\Delta \mathcal{E}_j = k_0^2 \cdot E_c e^{i(k_0 z - \omega_0 t)} = \frac{1}{c^2} \omega_0^2 \cdot E_c e^{i(k_0 z - \omega_0 t)}$$

yields the dispersion relation (2.5):

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{2.5}$$

For a general direction in (x, y, z)-coordinates, define the wavevector

$$\vec{k} = (k_x, k_y, k_z),$$

where $\left|\vec{k}^2\right| = k_0^2 = k_x^2 + k_y^2 + k_z^2$. This satisfies equation (2.3) when $\vec{k} \perp \vec{\mathcal{E}}$ and

$$\mathcal{E}_j = E_c e^{i(\vec{k} \cdot \vec{r} - \omega_0 t)}. \tag{2.6}$$

2.3. Derivation of the Helmholtz equation

We consider time-harmonic solutions to the scalar wave equation (2.3) of the form

$$\mathcal{E}_j(x, y, z, t) = e^{i\omega_0 t} E(x, y, z) + \text{c.c}, \tag{2.7}$$

which are continuous wave beam solutions as opposed to pulsed output beams. The continous beam has (approximately) constant power, whereas pulsed beams can reach higher peak powers. For more information on the operating principles of lasers, we refer to [7].

Substituting (2.7) in equation (2.3) and taking the derivatives leads to the expression

$$\Delta \left(e^{-i\omega_0 t} E \right) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(e^{-i\omega_0 t} E \right)$$
$$e^{-i\omega_0 t} \Delta E = \frac{1}{c^2} (-i\omega_0)^2 E e^{-i\omega_0 t},$$

where we can divide by $e^{-i\omega_0 t} \neq 0$ and use the dispersion relation (2.5) to arrive at the scalar linear Helmholtz equation for E

$$\Delta E(x, y, z) + k_0^2 E = 0. \tag{2.8}$$

As an example, equation (2.8) is solved by the general-direction plane waves (2.6), where

$$E = E_c e^{i(k_x x + k_y y + k_z z)}.$$

2.4. Derivation of the Linear Schrödinger equation

We write the incoming field $E_0^{\text{inc}}(x,y)$ as a sum of plane waves. Then the electric field $E(x,y,z)$ for non-zero z-value follows from propagation. This is the plane wave spectrum representation of the electromagnetic field and it is essential to Fourier optics. We have

$$E_0^{\text{inc}}(x,y) = \frac{1}{2\pi}\int_D E_c(k_x,k_y)\, e^{i(k_x x + k_y y)} \mathrm{d}k_x \mathrm{d}k_y, \quad \text{such that}$$
$$E(x,y,z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} E_c(k_x,k_y)\, e^{i(k_x x + k_y y + \sqrt{k_0^2 - k_x^2 - k_y^2}\, z)} \mathrm{d}k_x \mathrm{d}k_y,$$

where D denotes the (circular) laser input beam domain. For laser beams oriented in the z-direction, most of the plane wave modes are nearly parallel to the z-axis, which implies $k_z \approx k_0$. We define $k_\perp^2 = k_x^2 + k_y^2$, such that $k_0^2 = k_\perp^2 + k_z^2$. It is equivalent to $k_0 \approx k_z$ to say that $k_\perp \ll k_z$.

This motivates studying solutions of the form

$$E = e^{ik_0 z}\psi(x,y,z) \tag{2.9}$$

where $\psi(x,y,z)$ is an envelope (or amplitude) function. The envelope shape may vary over z, in contrast to soliton solutions, see (2.21).

Substituting (2.9) into the Helmholtz equation (2.8) yields

$$\psi_{zz}(x,y,z) + 2ik_0\psi_z + \Delta_\perp\psi = 0, \tag{2.10}$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ such that $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$. Basically, this is the Helmholtz equation for the envelope function $\psi(x,y,z)$. Remember that for lasers beams oriented in the z-direction, the wavenumber k_z dominates over $k_\perp$ such that $k_0 \approx k_z$. The envelope function $\psi(x,y,z)$ will vary slowly in z and curve even more slowly.

Claim: $|\psi_{zz}| \ll k_0|\psi_z|$ and $|\psi_{zz}| \ll \Delta_\perp\psi$.

To see this, we first show that $k_0 - k_z \ll 1$. We factor out k_0^2, take the square root on both sides and linearise the square root term of the right hand side:

$$k_z^2 = k_0^2 + k_\perp^2 = k_0^2\left(1 - \frac{k_\perp^2}{k_0^2}\right) \implies k_z = k_0\left(1 - \frac{k_\perp^2}{k_0^2}\right)^{\frac{1}{2}} \approx k_0\left(1 - \frac{1}{2}\frac{k_\perp^2}{k_0^2}\right).$$

Finally, we use $k_\perp \ll k_0$ to obtain the intermediate result:

$$k_0 - k_z \approx k_0 - k_0 + \frac{1}{2}\frac{k_\perp^2}{k_0} = \frac{1}{2}\frac{k_\perp^2}{k_0} \ll 1.$$

For the first statement of the claim, $|\psi_{zz}| \ll k_0|\psi_z|$, it is equivalent to show that the ratio of $|\psi_{zz}|$ over $k_0|\psi_z|$ is much smaller than 1. We calculate the ratio as follows:

$$\frac{[\psi_{zz}]}{[k_0\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_0 (k_0 - k_z) E_c} = \frac{k_0 - k_z}{k_0} = \frac{k_\perp}{k_0} \approx \frac{1}{2}\frac{k_\perp^2}{k_0}\cdot\frac{1}{k_0} \ll 1.$$

For the other statement of the claim, we calculate:

$$\frac{[\psi_{zz}]}{[\Delta_\perp\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_\perp^2 E_c} = \frac{(k_0 - k_z)^2}{k_\perp^2} \approx \frac{1}{k_\perp^2}\left(\frac{1}{2}\frac{k_\perp^2}{k_0}\right)^2 = \frac{1}{4}\frac{k_\perp^2}{k_0^2} \ll 1.$$

Using the approxations in equation (2.10) yields the linear Schrödinger equation:

$$2ik_0\psi_z + \Delta_\perp\psi = 0. \tag{2.11}$$

2.5. Polarisation field

Polarisation describes the influence of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}} + \vec{\mathcal{P}}.$$

In the following, we assume that the electric field is linearly polarised, such that

$$\vec{\mathcal{E}} = (\mathcal{E}, 0, 0), \ \vec{\mathcal{P}} = (\mathcal{P}, 0, 0), \ \vec{\mathcal{D}} = (\mathcal{D}, 0, 0),$$

Furthermore, we assume that $\mathcal{E}$ is the continuous wave electric field from (2.7). We write the Taylor expansion of the polarisation field $\mathcal{P} = c\mathcal{E}$ as:

$$\mathcal{P} = c_0 + c_1\mathcal{P} + c_2\mathcal{P}^2 + c_3\mathcal{P}^3 + c_4\mathcal{P}^4 + c_5\mathcal{P}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{2.12}$$

where the c_i are real for all i. Note that $c_0 = 0$ except in ferro-electric materials. The constants c_i are actually a function of the frequency ω_0. We rewrite $c_i = \epsilon_0 \chi^{(\mathrm{i})}(\omega_0)$, where $\chi^{(\mathrm{i})}$ is the i-th order susceptibility. Then equation (2.12) reads:

$$\mathcal{P} = \epsilon_0\chi^{(1)}\mathcal{E} + \epsilon_0\chi^{(2)}\mathcal{E}^2 + \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{2.13}$$

First we consider linear polarisation:

$$\mathcal{P}_{\mathrm{lin}} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}.$$

Then the induction electric field $\mathcal{D}$ is given by:

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\mathrm{lin}} = \epsilon_0\mathcal{E} + \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E} = \epsilon_0\mathcal{E}\left(1 + \chi^{(1)}(\omega_0)\right) = \epsilon_0 n_0^2(\omega_0)\mathcal{E},$$

where $n_0^2(\omega_0) := 1 + \chi^{(1)}(\omega_0)$ is the linear index of refraction (or refractive index) of the medium.

With this updated induction electric field $\mathcal{D} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}$, we can update the scalar wave equation and the Helmholtz equation. Only the dispersion relation is affected by considering linear polarisation:

$$k_0^2 = \frac{\omega_0^2}{c^2} n_0^2(\omega_0). \tag{2.14}$$

We now consider the nonlinear polarisation field $\mathcal{P}_{\mathrm{nl}}$ as the difference between the true polarisation and the linear approximation:

$$\mathcal{P} = \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}}.$$

In an isotropic medium, the relation between $\mathcal{P}$ and $\mathcal{E}$ should be same in all directions. Replacing $\mathcal{P}$ and $\mathcal{E}$ by $-\mathcal{P}$ and $-\mathcal{E}$ respectively,

$$\begin{aligned} -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}\left(-\mathcal{E}\right)^2 + \epsilon_0\chi^{(3)}\left(-\mathcal{E}\right)^3 + \epsilon_0\chi^{(4)}\left(-\mathcal{E}\right)^4 + \epsilon_0\chi^{(5)}\left(-\mathcal{E}\right)^5 + \mathcal{O}\left(\mathcal{P}^6\right) \\ -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}\mathcal{E}^2 - \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 - \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right), \end{aligned}$$

where we see that for the even exponents, the negative signs cancel. Hence, the even terms cannot contribute to $\mathcal{P}_{\mathrm{nl}}$ and we have only the odd terms:

$$\mathcal{P}_{\mathrm{nl}} = \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^7\right) \tag{2.15}$$

The leading-order term is called the Kerr nonlinearity:

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0\chi^{(3)}(\omega_0)\mathcal{E}^3. \tag{2.16}$$

2.6. Implications of nonlinear polarisation

Substituting the continuous wave electric field (2.7) into equation (2.16) yields

$$\mathcal{P}_{\text{nl}} \approx \epsilon_0 \chi^{(3)}(\omega_0)\mathcal{E}^3 = 3\epsilon_0 \chi^{(3)}(\omega_0)\,|E|^2\, E e^{i\omega_0 t} + \epsilon_0 \chi^{(3)}(\omega_0) E^3 e^{3i\omega_0 t} + \text{c.c.},$$

where the second term has a frequency of $3\omega_0$ (third harmonic). This has almost no contribution due to the phase-mismatch with the first harmonic. Hence, we approximate

$$\mathcal{P}_{\text{nl}} \approx 3\epsilon_0 \chi^{(3)}(\omega_0)\,|E|^2\, E e^{i\omega_0 t} + \text{c.c.} = 3\epsilon_0 \chi^{(3)}(\omega_0)\mathcal{E}|E|^2.$$

Then we simplify $\mathcal{P}_{\text{nl}}$ by defining

$$n_2 := \frac{3\chi^{(3)}}{4\epsilon_0 n_0},$$

so that we obtain the simplified expression

$$\mathcal{P}_{\text{nl}} = 4\epsilon_0 n_0 n_2\,|E|^2\,\mathcal{E}.$$

This allows us to write the induction electric field $\mathcal{D}$ as,

$$\mathcal{D} = \epsilon_0 \mathcal{E} + \mathcal{P}_{\text{lin}} + \mathcal{P}_{\text{nl}} = \epsilon_0 n^2 \mathcal{E},$$

where

$$n^2 = n_0^2\left(1 + \frac{4n_2}{n_0}\,|E|^2\right) = n_0^2 + 3\chi^{(3)}(\omega_0)\frac{1}{\epsilon_0}|E|^2.$$

For water, $n_2 \sim 10^{-22}$ which justifies neglecting nonlinear effects. With lasers, the nonlinear effect becomes more relevant, but is still weak. For a typical continuous wave laser with $|E| \sim 10^9$ V/m, we still have a weak nonlinearity, as $n_2|E|^2 \sim 10^{-4} \ll n_0 \approx 1.33$.

We update equation (2.8) to the scalar nonlinear Helmholtz equation (NLH):

$$\Delta E(x,y,z) + k^2 E = 0, \quad \text{where } k^2 = k_0^2\left(1 + \frac{4n_2}{n_0}|E|^2\right). \tag{2.17}$$

We write $E(x,y,z)$ as the product of the z-propagation and an envelope function $\psi(x,y,z)$:

$$E = e^{ik_0 z}\psi$$

and substitute in (2.17) to obtain:

$$\psi_{zz} + 2ik_0\psi_z + \Delta_\perp \psi + 4k_0^2 \frac{n_2}{n_0}|\psi|^2\,\psi = 0. \tag{2.18}$$

Just as in section 2.4, we apply the paraxial approximation, since for laser beams oriented in the z-direction, we have $|\psi_{zz}| \ll k_0|\psi_z|, |\psi_{zz}| \ll \Delta_\perp \psi$. We finally obtain the nonlinear Schrödinger equation (NLS):

$$2ik_0\psi_z(x,y,z) + \Delta_\perp \psi + k_0^2 \frac{4n_2}{n_0}|\psi|^2\,\psi = 0. \tag{2.19}$$

2.7. Soliton solutions

The NLS equation (2.19) can be written as a dimensionless equation. Starting from equation (2.18), we apply the rescaling of coordinates $(x,y,z) \to (\tilde{x},\tilde{y},\tilde{z})$ defined by:

$$\tilde{x} = \frac{x}{r_0} \quad \tilde{y} = \frac{y}{r_0} \quad \tilde{z} = \frac{z}{2L_{\text{diff}}},$$

where r_0 is the input beam width and L_{diff} is the diffraction length. We refer to chapter 2 of [4] for more information on the geometrical optics of lasers. There, we also find that $L_{\text{diff}} = k_0 \cdot r_0^2$. To rescale $\tilde{\psi}$, we define:

$$\tilde{\psi} = \frac{\psi}{E_c}, \quad \text{where } E_c := \max_{x,y}|\psi_0(x,y)|.$$

Through the rescaling we obtain the dimensionless NLH for $\tilde{\psi}$:

$$\frac{f^2}{4}\tilde{\psi}_{\tilde{z}\tilde{z}}(\tilde{z},\tilde{x},\tilde{y}) + i\tilde{\psi}_{\tilde{z}} + \Delta_\perp\tilde{\psi} + \nu\left|\tilde{\psi}\right|^2\tilde{\psi} = 0,$$

that depends on a nonparaxiality parameter f and a nonlinearity parameter ν:

$$f = \frac{1}{r_0 k_0} = \frac{r_0}{L_{\text{diff}}}, \quad \nu = r_0^2 k_0^2 \frac{4n_2}{n_0} E_c^2.$$

Here the approximation of paraxiality is valid for small $f \ll 1$ and this leads to the dimensionless NLS equation (2.20), where the tildes have been dropped for brevity.

$$i\psi_z(z,x,y) + \Delta_\perp\psi + \nu\left|\psi\right|^2\psi = 0. \tag{2.20}$$

Radial solitary-wave solutions to (2.20) were considered in [2] with ψ of the form:

$$\psi_\omega^{\text{solitary}}(r,z) = e^{i\omega z}R_\omega(r), \tag{2.21}$$

where ω is a real number and R_ω is the real solution of

$$-\omega R_\omega + \Delta_\perp R_\omega(r) + R_\omega^3 = 0.$$

This can be solved in general by, for example,

$$R_\omega(r) = \sqrt{\omega}R\left(\sqrt{\omega}r\right).$$

However, taking $\omega = 1$ leads to the simplest soliton equation

$$R''(r) + \frac{1}{r}R' - R + R^3 = 0, \quad 0 < r < \infty, \tag{2.22}$$

subject to initial condition $R'(0) = 0$ and integrability condition $\lim_{r\to\infty} R(r) = 0$. The (numerical) solution is known as the Townes profile, which is positive and monotonically decreasing in r.

3

Existence of ground state

3.1. Initial value problem and nonlinearity

In this chapter, we will study an existence proof for the initial value problem

$$-u''(r) - \frac{n-1}{r}u'(r) = f(u(r)) \quad \text{on } 0 < r < \infty, \tag{3.1}$$

satisfying two initial conditions and an integrability condition

$$\begin{cases} u(0) = \alpha, \\ u'(0) = 0, \\ \lim_{r\to\infty} u(r) = 0. \end{cases} \tag{3.2}$$

The existence proof will be based on [1], which generalises earlier results. For example, the uniqueness result [3], which was later generalised in [6] and forms the basis for Chapter 4 below.

The proof will be by a shooting method, where we categorise the solutions based on their asymptotic behaviour. Furthermore, solutions to the initial value problem (3.1) are also positive radial solutions to the more general problem

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n, \tag{3.3}$$

where $f(u)$ is a given nonlinear function. The partial differential equation (3.3) is relevant to many areas of mathematical physics.

The solutions $R(r)$ to equation (2.22) are solutions $u(r)$ to (3.1) with $n = 2$ and

$$f(u) = -u + u^3.$$

3.2. Definitions of solution sets

Definition 3.1. *A **ground state solution** is positive everywhere, strictly decreasing everywhere and has no finite zeroes. Yet, the solution should vanish in the limit as $r \to \infty$.*

We define the set G of ground state initial conditions as

$$G := \left\{ \alpha > 0 \;\middle|\; u(r,\alpha) > 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\}. \tag{3.4}$$

We consider two alternatives: either (i) the derivative vanishes or (ii) the solution vanishes. We define the set P of initial conditions with a vanishing derivative as

$$P := \left\{ \alpha > 0 \;\middle|\; \exists r_0 : u'(r_0,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \text{ for all } r \le r_0 \right\}. \tag{3.5}$$

We define the set N of initial conditions with a vanishing solution as

$$N := \left\{ \alpha > 0 \;\middle|\; \exists r_0 : u(r_0, \alpha) = 0 \text{ and } u'(r, \alpha) < 0 \text{ for all } r \leq r_0 \right\}. \tag{3.6}$$

We note that the sets P and N are disjoint by definition: either the derivative vanishes first, or the solution vanishes first.

We will show that the sets P and N are non-empty, and open. Then, there exist initial conditions that belong to neither P nor N. Solutions that belong to neither P nor N are *everywhere positive and decreasing*

$$\begin{cases} u(r, \alpha) > 0 & \text{for } r \geq 0, \text{ and} \\ u'(r, \alpha) < 0 & \text{for } r > 0. \end{cases} \tag{3.7}$$

Lastly, we will show that such solutions vanish in the limit as $r \to \infty$ under certain assumptions. This concludes the existence of elements in G.

3.3. Assumptions on f

We assume that $f : [0, \infty) \to \mathbb{R}$ is locally Lipschitz continuous and satisfies $f(0) = 0$. Additionally, we assume that hypotheses (H1)–(H5) are satisfied. Firstly,

$$f(\kappa) = 0, \text{ for some } \kappa > 0. \tag{H1}$$

Secondly, defining $F(t)$ as the integral of $f(t)$

$$F(t) := \int_0^t f(s)\,\mathrm{d}s, \tag{3.8}$$

there exists an initial condition $\alpha > 0$ such that $F(\alpha) > 0$. We define

$$\alpha_0 := \inf\{\alpha > 0 \mid F(\alpha) > 0\}. \tag{H2}$$

Thirdly, the right-derivative of $f(s)$ at κ is positive

$$f'(\kappa^+) = \lim_{s \downarrow \kappa} \frac{f(s) - f(\kappa)}{s - \kappa} > 0, \tag{H3}$$

and fourthly, we have

$$f(s) > 0 \quad \text{for } s \in (\kappa, \alpha_0]. \tag{H4}$$

We define

$$\lambda := \inf\{\, \alpha > \alpha_0 \mid f(\alpha) = 0 \,\}, \tag{3.9}$$

and note that $\alpha_0 < \lambda \leq \infty$. In the situation where $\lambda = \infty$, we assume

$$\lim_{s \to \infty} \frac{f(s)}{s^l} = 0, \quad \text{with } l < \frac{n+2}{n-2}. \tag{H5}$$

3.4. Main theorem

Theorem 3.1. *Let f be a locally Lipschitz continuous function on $\mathbb{R}_+ = [0, \infty)$ such that $f(0) = 0$ and f satisfies hypotheses $(H1) - (H5)$. Then there exists a number $\alpha \in (\alpha_0, \lambda)$ such that the solution $u(r, \alpha) \in C^2(\mathbb{R}_+)$ of the initial value problem*

$$\begin{cases} -u''(r) - \dfrac{n-1}{r} u'(r) = f(u(r)), \text{ for } r > 0, \\ u(0) = \alpha, \quad u'(0) = 0 \end{cases} \tag{3.10}$$

is an element of solution set G defined in (3.4)

$$G := \left\{ \alpha > 0 \;\middle|\; u(r, \alpha) > 0 \text{ and } u'(r, \alpha) < 0 \text{ for all } r > 0 \text{ and } \lim_{r \to \infty} u(r, \alpha) = 0 \right\}.$$

Proof. We will show in Lemma 3.1-3.3 that solutions to the differential problem (3.10) are defined for $0 < r < \infty$. Furthermore, by Lemma 3.4 solutions with $\alpha \notin (P \cup N)$ satisfy

$$\lim_{r\to\infty} u(r,\alpha) = 0.$$

Lastly, we will show that solution sets P and N are non-empty and open. In Lemma 3.5 we show that solution set P is non-empty and open. By similar argument, solution set N is open. For the argument that N is non-empty, we refer to "I_- is non-empty" in [1, p. 147].

In conclusion, G is non-empty. □

3.5. Interval of definition

Existence of local unique solutions is guaranteed by the Picard-Lindelöf theorem, see for example [8, Theorem. 2.2].

In these circumstances, boundedness of the solution $u(r,\alpha)$ is a sufficient condition for the solution to be defined on the maximal interval $[0,\infty)$. This is also called the *blow-up alternative*. Either (i) for some $r_0 > 0$ we have

$$\left|u(r_0,\alpha)\right| > M, \quad \text{for all } M > 0,$$

and the solution is defined on $[0, r_0)$. Or (ii) for some $M > 0$ we have

$$\left|u(r,\alpha)\right| \le M, \quad \text{for all } r \ge 0,$$

and the solution is defined for all $r \ge 0$.

Lemma 3.1. *For any initial condition $\alpha > 0$ and $r > 0$, we have the identity*

$$\frac{1}{2}\left[u'(r)\right]^2 + (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r)). \tag{3.11}$$

Proof. We multiply the IVP (3.1) by $-u'(r)$. Then we integrate from 0 to r to obtain

$$\int_0^r \left[u'(s)u''(s)\right] ds + \int_0^r \left[\frac{n-1}{s}[u'(s)]^2\right] \mathrm{d}s = -\int_0^r \left[u'(s)f(u(s))\right] \mathrm{d}s. \tag{3.12}$$

We use the chain rule simplify the first term in (3.12) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}[u'(r)^2] = 2u'(r)u''(r) \overset{(3.2)}{\Leftrightarrow} \frac{1}{2}[u'(r)]^2 = \int_0^r \left[u'(s)u''(s)\right] \mathrm{d}s.$$

Then, we rewrite the right-hand side of (3.12)

$$-\int_0^r \left[u'(s)f(u(s))\right] \mathrm{d}s = \int_r^0 \left[\tfrac{\mathrm{d}u}{\mathrm{d}s} f(u(s))\right] \mathrm{d}s$$

and use the fundamental theorem of calculus

$$\int_{u(r)}^{u(0)} f(u)\,\mathrm{d}u = F(u(0)) - F(u(r)).$$

Finally, using $u(0) = \alpha$, we have rewritten (3.12) as

$$\frac{1}{2}\left[u'(r)\right]^2 + (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r)). \qquad \square$$

In this section, we will derive an upper and a lower bound for $u(r,\alpha)$. Since the solution is initially decreasing, possibly the initial condition α is an upper bound.

Lemma 3.2. *Let $\alpha > \kappa$. Then $u(r,\alpha) \le u(0,\alpha) = \alpha \quad$ for $r \ge 0$.*

Proof. We suppose by contradiction that

$$\alpha < u(r_0,\alpha) < \lambda, \quad \text{for some } r_0 > 0. \tag{3.13}$$

By (H4) and (3.9), we have F non-decreasing on (κ,λ). Then,

$$F(\kappa) < F(\alpha) < F(u(r_0,\alpha)) < F(\lambda).$$

In particular, we have

$$F(\alpha) - F(u(r_0,\alpha)) < 0.$$

This contradicts Lemma 3.1, as the left-hand side is clearly non-negative. □

We will show that $u(r,\alpha)$ has a lower bound for $r < \infty$. Let r_0 be the first zero of $u(r,\alpha)$

$$r_0 := \inf\{\, r > 0 \mid u(r,\alpha) = 0 \,\}. \tag{3.14}$$

If $r_0 = \infty$, then we have $u(r,\alpha) > 0$ for all $r > 0$. When $r_0 < \infty$, we have the following bound on the derivative $u'(r,\alpha)$.

Lemma 3.3. *Suppose that there exists $r_0 > 0$ such that*

$$\begin{cases} u(r_0,\alpha) = 0 \\ u'(r_0,\alpha) < 0. \end{cases} \tag{3.15}$$

If we have $f(u) = 0$ for $u \le 0$, then for $r \ge r_0$ we have

$$u'(r,\alpha) = \left(\frac{r_0}{r}\right)^{n-1} u'(r_0,\alpha) \ge u'(r_0,\alpha). \tag{3.16}$$

Proof. For $u(r,\alpha) \le 0$ the IVP (3.1) reads

$$-u''(r,\alpha) - \frac{n-1}{r} u'(r,\alpha) = 0, \tag{3.17}$$

We solve (3.17) for $u' = u'(r,\alpha)$ and seperate the variables, resulting in

$$\frac{\mathrm{d}u'}{u'} = -\frac{n-1}{r}\,\mathrm{d}r.$$

We integrate the expression from r_0 to r and evaluate the limits

$$\ln u'\Big|_{r_0}^{r} = \left[(n-1)\ln r\right]_r^{r_0} \iff \ln u'(r) - \ln u'(r_0) = (n-1)\left[\ln r_0 - \ln r\right].$$

Then, we rewrite the expression to arrive at the desired result

$$\frac{u'(r)}{u'(r_0)} = \left(\frac{r_0}{r}\right)^{n-1} \iff u'(r,\alpha) = \left(\frac{r_0}{r}\right)^{n-1} u'(r_0,\alpha) \ge u'(r_0,\alpha).$$ □

In conclusion, the solution $u(r,\alpha)$ is bounded for bounded r. More specifically, in the case of everywhere positive solutions, we have

$$0 < u(r,\alpha) \le \alpha \quad \text{for all } r > 0.$$

Alternatively, for solutions with $u(r_0,\alpha) = 0$ and $u'(r_0,\alpha) < 0$ by Lemma 3.3 we have

$$u(r,\alpha) \ge \int_{r_0}^{r} \left(\frac{r_0}{s}\right)^{n-1} u'(r_0,\alpha)\,\mathrm{d}s > -\infty \quad \text{for } r > r_0, \tag{3.18}$$

such that for $n = 2$, we have

$$u(r,\alpha) \geq r_0 u'(r_0,\alpha)\left(\ln r - \ln r_0\right) \tag{3.19}$$

and for $n > 2$, we have

$$u(r,\alpha) \geq \frac{r_0^{n-1}u'(r_0,\alpha)}{2-n}\left(r^{2-n} - r_0^{2-n}\right). \tag{3.20}$$

3.6. Asymptotics of positive decreasing solutions

In this section, we will show that everywhere positive decreasing solutions $u(r,\alpha)$ vanish in the limit as $r \to \infty$.

Lemma 3.4. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a locally Lipschitz continuous function such that $f(0) = 0$. Let $u(r,\alpha_1)$ be a solution to initial value problem (3.1) with $\alpha_1 \in (0,\infty)$ such that*

$$\begin{cases} u(r,\alpha_1) > 0 & \text{for all } r \geq 0, \quad \text{and} \\ u'(r,\alpha_1) < 0 & \text{for all } r > 0. \end{cases} \tag{3.21}$$

Then the number $l := \lim\limits_{r\to\infty} u(r,\alpha_1)$ satisfies $f(l) = 0$.

If additionally, $f(u)$ satisfies (H3), then $l = 0$.

Proof step 1. By assumption (3.21) on $u(r,\alpha_1)$ and the monotone convergence theorem, we have $0 \leq l < \alpha_1$. Then $f(l) < f(\alpha_1)$. We consider the limit as $r \to \infty$ of the IVP (3.1)

$$\lim_{r\to\infty}\left[-u''(r,\alpha_1) - \frac{n-1}{r}u'(r,\alpha_1)\right] = f(l) < \infty. \tag{3.22}$$

We restate equation (3.11)

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2 + (n-1)\int_0^r \left[u'(s,\alpha_1)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha_1) - F(u(r,\alpha_1))$$

and note that the right hand side is finite. We write

$$(n-1)\int_0^r \left[u'(s,\alpha_1)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha_1) - F(u(r,\alpha_1)) - \frac{1}{2}\left[u'(r,\alpha_1)\right]^2$$

and note that the left hand side is increasing and bounded above. Hence,

$$\int_0^\infty u'(s,\alpha_1)^2 \frac{\mathrm{d}s}{s} < \infty.$$

We write

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2 = F(\alpha_1) - F(u(r,\alpha_1)) - (n-1)\int_0^r \left[u'(s,\alpha_1)\right]^2 \frac{\mathrm{d}s}{s}.$$

Then $\lim\limits_{r\to\infty} u'(r,\alpha_1)^2$ exists. Since $u'(r,\alpha_1) < 0$ and $u(r,\alpha_1)$ is bounded, we have

$$\lim_{r\to\infty} u'(r,\alpha_1) = 0. \tag{3.23}$$

Now, we return to equation (3.22) and use $\lim\limits_{r\to\infty} u'(r,\alpha_1) = 0$ to obtain

$$-\lim_{r\to\infty}\left[u''(r,\alpha_1)\right] = f(l).$$

We have (3.23) and hence, we have

$$\lim_{r\to\infty} u''(r,\alpha_1) = 0.$$

The desired result follows: $f(l) = 0$. □

Proof step 2. The nonlinearity $f(u)$ has more than one zero. Both $f(0) = 0$ and $f(\kappa) = 0$. We will show that under assumption (H3), only $l = 0$ satisfies the IVP (3.1).

Suppose to the contrary that $l = \kappa$. We will use the substitution

$$v(r) = r^{(1/2)(n-1)}\left[u(r,\alpha_1) - \kappa\right] \tag{3.24}$$

in equation (3.1) to obtain a differential equation in $v(r)$. In the remainder of the proof of this lemma, we will abbreviate $u(r,\alpha_1) = u(r)$. We note that $v(r) > 0$ by definition, since we have $u(r) \downarrow \kappa$.

We proceed to calculate the first derivative $v'(r)$

$$v'(r) = \frac{1}{2}(n-1)r^{(n-3)/2}\left[u(r) - \kappa\right] + r^{(n-1)/2}u'(r),$$

and the second derivative $v''(r)$, where we gather the terms by $u(r), u'(r)$ and $u''(r)$

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-5)/2}\left[u(r) - \kappa\right] + (n-1)r^{(n-3)/2}u'(r) + r^{(n-1)/2}u''(r). \tag{3.25}$$

We multiply the IVP (3.1) by $r^{(n-1)/2}$ to obtain

$$-r^{(n-1)/2}u''(r) - (n-1)r^{(n-1)/2}r^{-1}u'(r) = f(u(r))r^{(n-1)/2}. \tag{3.26}$$

We can use this to simplify (3.25) to

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-1)/2}r^{-2}\left[u(r) - \kappa\right] - f(u(r))r^{(n-1)/2}.$$

Now we factor out $v(r) = r^{(n-1)/2}\left[u(r) - \kappa\right]$ to obtain

$$v''(r) = r^{(n-1)/2}\left[u(r) - \kappa\right]\left\{\frac{1}{4}(n-1)(n-3)r^{-2} - \frac{f(u)}{u(r) - \kappa}\right\}.$$

Lastly, we multiply by -1 to obtain the exact expression from [1] as

$$-v''(r) = \left\{\frac{f(u)}{u(r) - \kappa} - \frac{(n-1)(n-3)}{4r^2}\right\}v. \tag{3.27}$$

In proof step 3, we will show that there exist $\omega > 0$ and $R_1 > 0$, such that

$$\frac{f(u)}{u(r) - \kappa} - \frac{(n-1)(n-3)}{4r^2} \geq \omega \quad \text{for all } r \geq R_1. \tag{3.28}$$

We have $v''(r) < 0$ for $r \geq R_1$, which implies by

$$v'(r) = v'(R_1) + \int_{R_1}^{r} v''(s)\,\mathrm{d}s$$

that

$$v'(r) \downarrow L \geq -\infty, \quad \text{as } r \to \infty.$$

Suppose that $L < 0$, then $v(r) \to -\infty$ as $r \to \infty$. However, by (3.24) we have $v > 0$.

Then $L \geq 0$. This implies $v'(r) \geq 0$ for $r \geq R_1$. But then $v(r) \geq v(R_1) > 0$ for $r \geq R_1$. By (3.28) and (3.27), we have

$$-v''(r) \geq \omega v(R_1) > 0,$$

such that $v'(r) \to -\infty$ as $r \to \infty$. This contradicts $L \geq 0$. Hence, we have $l = 0$. □

Proof step 3. The first term (3.28) is non-negative and decreasing by (H3). We will write

$$M(r) \coloneqq \frac{f(u)}{u(r)-\kappa} > 0, \tag{3.29}$$

and rewrite (3.28) to obtain

$$M(r) \geq \frac{(n-1)(n-3)}{4r^2} + \omega. \tag{3.30}$$

We choose $2\omega = \max_{r>0} M(r)$ and choose $R_1 > 0$ such that

$$\frac{(n-1)(n-3)}{4r^2} \leq \frac{1}{2}M(r) \quad \text{for } r \geq R_1.$$

□

3.7. P is non-empty and open

In this section we will show that P is non-empty and open. The proof that N is open is similar to the proof given for P. For the proof that N is non-empty, we refer to "I_- is non-empty" in [1, p. 147].

Lemma 3.5. *Solution set* P *as defined in* (3.5)

$$P \coloneqq \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u'(r_0,\alpha) = 0 \textit{ and } u(r,\alpha) > 0 \textit{ for all } r \leq r_0 \right\}$$

is non-empty and open.

Proof step 1. We will show that solution set P is non-empty. Let $\alpha \in (\kappa, \alpha_0]$. We refer to (H1) and (H2) for the definitions of κ and α_0.

First, we suppose by contradiction that $\alpha \in N$. By the definition of N in (3.6), there exists a number $r_0 > 0$ such that

$$\begin{cases} u(r_0,\alpha) = 0, \\ u'(r,\alpha) < 0 \quad \text{for } \; r \leq r_0. \end{cases} \tag{3.31}$$

We restate equation (3.11) from Lemma 3.1 for $r = r_0$ and use $F(u(r_0,\alpha)) = F(0) = 0$

$$\frac{1}{2}\left[u'(r_0,\alpha)\right]^2 + (n-1)\int_0^{r_0} u'(s,\alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha). \tag{3.32}$$

The left hand side of (3.32) is positive. For $\alpha \in (\kappa, \alpha_0]$, we have $F(\alpha) < 0$. Hence $\alpha \notin N$.

Next, we suppose that $\alpha \notin P$. Thus $\alpha \notin (P \cup N)$. We have the situation of (3.7)

$$\begin{cases} u(r,\alpha) > 0 \quad \text{for } r \geq 0, \text{ and} \\ u'(r,\alpha) < 0 \quad \text{for } r > 0, \end{cases}$$

which is the setting of Lemma 3.4. Thus, we have $l = 0$ and by equation (3.23), we have

$$\lim_{r\to\infty} u'(r,\alpha) = 0.$$

Then equation (3.32) evaluates to

$$(n-1)\int_0^{\infty} u'(s,\alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha) < 0,$$

but the left hand side is positive. We have $(\kappa, \alpha_0] \subset P$, since α was chosen arbitrarily. □

Proof step 2. We will show that P is open. Let $\alpha \in P$. There exists

$$r_0 := \inf\{\, r > 0 \mid u'(r,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \,\}$$

such that by the definition of P in (3.5)

$$\begin{cases} u(r,\alpha) > 0 & \text{for all } r \in [0, r_0], \\ u'(r,\alpha) < 0 & \text{for all } r \in (0, r_0). \end{cases} \tag{3.33a, 3.33b}$$

Evaluating the IVP (3.1) in r_0 yields

$$u''(r_0,\alpha) = -f(u(r_0,\alpha)).$$

Suppose that $u''(r_0,\alpha) = 0$. Then $-f(u(r_0,\alpha)) = 0$. The zeroes of $f(u)$ are $f(\kappa) = 0$ and $f(0) = 0$. Thus, $u(r_0,\alpha) = \kappa$ by (3.33a).

Then, the differential equation (3.1) with

$$\begin{cases} u(r_0,\alpha) = \kappa, \\ u'(r_0,\alpha) = 0, \\ u''(r_0,\alpha) = 0 \end{cases}$$

is solved by $u \equiv \kappa$, and by uniqueness of solutions this contradicts $u(0,\alpha) = \alpha > \kappa$.

Hence $u''(r_0,\alpha) \neq 0$. Since $u'(r,\alpha) < 0$ for $r < r_0$ and $u'(r_0,\alpha) = 0$, we have

$$u''(r_0,\alpha) > 0.$$

Then there exists $r_1 > r_0$, such that

$$u(r,\alpha) > u(r_0,\alpha) \quad \text{for all } r \in (r_0, r_1].$$

Since $u(r,\alpha)$ is pointwise continuous in α, we have

$$\forall \epsilon > 0 \,\exists\, \delta > 0 : \ |\alpha - \beta| < \delta \implies |u(r,\alpha) - u(r,\beta)| < \epsilon.$$

We define

$$\epsilon := \frac{1}{2}\big(u(r_1,\alpha) - u(r_0,\alpha)\big).$$

For $\delta_{r_0} > 0$ sufficiently small, we have

$$|u(r_0,\alpha) - u(r_0,\beta)| < \epsilon,$$

and for $\delta_{r_1} > 0$ sufficiently small, we have

$$|u(r_1,\alpha) - u(r_1,\beta)| < \epsilon.$$

Let $\delta = \min\{\delta_{r_0}, \delta_{r_1}\} > 0$. Then, for $|\alpha - \beta| < \delta$, we have

$$\begin{cases} u(r_1,\beta) > u(r_0,\beta) \\ \beta > u(r,\beta) > 0 & \text{for all } r \in (0, r_1]. \end{cases} \tag{3.34}$$

Thus $\beta \in P$ and P is open. $\square$

4

Uniqueness of ground state

In this chapter, we study the uniqueness of positive radially symmetric solutions to the equation

$$\Delta u - u + u^3 = 0 \quad \text{in } \mathbb{R}^3 \tag{4.1}$$

as presented in [3]. We note that in the radially symmetric case equation (4.1) reduces to

$$u''(r) + \frac{2}{r}u'(r) - u(r) + u(r)^3 = 0 \quad \text{for } 0 < r < \infty, \tag{4.2}$$

which is equal to equation (3.1) for $n = 3$ and $f(u) = u - u^3$. In Chapter 2 we derived a simple soliton equation (2.22) which is equal to equation (3.1) for $f(R) = R - R^3$, but for $n = 2$. In Chapter 3 we established the existence of ground state[1] solutions to equation (3.1) subject to the conditions (3.2). The solutions $u(r)$ to equation (4.2) are of the form $u(r) = r^{-1}w(r)$, where $w(r)$ solves

$$w''(r) - w(r) + r^{-2}w(r)^3 = 0 \quad \text{for } 0 < r < \infty. \tag{4.3}$$

The derivation of equation (4.3) is given in Section 4.3 below.

Our main focus is to shed light on Section 4 of [3], which uses ODE methods to establish the uniquess of positive radially symmetric solutions to equation (4.3) subject to the boundary conditions

$$0 < \lim_{r \downarrow 0} r^{-1}w(r) = \alpha < \infty \tag{4.4}$$

and

$$\lim_{r\to\infty} w(r) = 0. \tag{4.5}$$

Similar to Chapter 3, the proof method is a shooting argument for the initial value $u(0) = \alpha$ equivalent to equation (4.4). We write $w = w(r, \alpha)$ to denote the solution $w(r)$ to equation (4.3) satisfying equation (4.4). We now state the main uniqueness theorem.

Theorem 4.1. *There is at most one $\alpha > 0$ for which*

$$w(r, \alpha) > 0 \quad \textit{on } (0, \infty) \tag{4.6}$$

and

$$\lim_{r\to\infty} w(r, \alpha) = 0. \tag{4.7}$$

The proof is given in Section 4.2 below and requires several technical lemmata.

[1] See Definition 3.1

4.1. Technical lemmata

Lemma 4.1. *Let $w_\alpha = w_\alpha(r,\alpha) = \partial_\alpha w(r,\alpha) = \frac{\partial w(r,\alpha)}{\partial \alpha}$ and let $w'_\alpha = w'_\alpha(r,\alpha) = \partial_{r\alpha} w(r,\alpha) = \frac{\partial^2 w(r,\alpha)}{\partial r \partial \alpha}$. For each $\alpha > 0$ equation (4.3) has a unique solution $w = w(r,\alpha)$ which is of class C^2 in $r > 0$ and satisfies (4.4). The partial derivatives $w_\alpha(r,\alpha)$ and $w'_\alpha(r,\alpha)$ exist for all $r > 0$ and $\alpha > 0$. Furthermore, $w_\alpha = w_\alpha(r,\alpha)$ solves the regular initial value problem*

$$\begin{cases} w''_\alpha - w_\alpha + 3r^{-2}w^2 w_\alpha = 0 \quad \text{for } 0 < r < \infty, \\ w_\alpha(0,\alpha) = 0, \quad w'_\alpha(0,\alpha) = 1, \end{cases} \tag{4.8}$$

Proof. The proof follows similar ODE proof methods as given in e.g. [8, Chapter 2]. □

Lemma 4.2. *(i) If $\alpha > 0$ and $w(r,\alpha) > 0$ on $(0,r^*)$ with $w(r^*,\alpha) = 0$, then $w_\alpha(r^*,\alpha) < 0$.*

(ii) If $\alpha > 0$ and $w(r,\alpha)$ satisfies (4.6) and (4.7) then

$$\lim_{r\to\infty} e^{-r} w_\alpha(r,\alpha) < 0. \tag{4.9}$$

Proof. We refer to the proof of Lemma 4.2 on page 89 of [3]. □

Lemma 4.3. *Let $w(r,\alpha)$ satisfy the assumptions of Lemma 4.2. That is, either let $\alpha > 0$ and let $z(\alpha) > 0$ be minimal such that $w(z(\alpha),\alpha) = 0$ and $w(r,\alpha) > 0$ on $(0, z(\alpha))$; or let $\alpha > 0$ and let $w(r,\alpha)$ satisfy equations (4.6) and (4.7), for which we write $z(\alpha) = \infty$. Then $\alpha > \sqrt{2}$, there exists a unique $r_0 \in (0, z(\alpha))$ such that $w(r_0,\alpha) = r_0$ and $w'(r_0,\alpha) < 0$ holds for this r_0.*

Proof. First, we show that $\alpha > \sqrt{2}$ follows from the assumptions of Lemma 4.3. The function

$$u(r,\alpha) \coloneqq r^{-1} w(r,\alpha) \tag{4.10}$$

solves

$$u''(r,\alpha) + 2r^{-1}u'(r,\alpha) - u(r,\alpha) + u(r,\alpha)^3 = 0 \tag{4.11}$$

with the initial values

$$u(0,\alpha) = \alpha \quad \text{and} \quad u'(0,\alpha) = 0. \tag{4.12}$$

We define the function $\Phi(r)$ as

$$\Phi(r) = u'(r,\alpha)^2 + \frac{1}{2}u(r,\alpha)^4 - u(r,\alpha)^2 \tag{4.13}$$

and differentiate $\Phi(r)$ with respect to r to obtain

$$\Phi'(r) = 2u'(r,\alpha)u''(r,\alpha) + 2u(r,\alpha)^3 u'(r,\alpha) - 2u(r,\alpha)u'(r,\alpha). \tag{4.14}$$

We reorder the terms of equation (4.11) as

$$u''(r,\alpha) - u(r,\alpha) + u(r,\alpha)^3 = -2r^{-1}u'(r,\alpha) \tag{4.15}$$

and use this to rewrite $\Phi'(r)$ as

$$\Phi'(r) = 2u'(r,\alpha)\left(u''(r,\alpha) - u(r,\alpha) + u(r,\alpha)^3\right) = -4r^{-1}u'(r,\alpha)^2. \tag{4.16}$$

Suppose that $u'(r,\alpha) = 0$ on some interval $I \subset (0,\infty)$. Then $u(r,\alpha) = c$ on the interval I for some constant c. Suppose that $u(r,\alpha) = 0$ on the interval I. Then $u(r,\alpha) = 0$ solves equation (4.11) on the interval I and by uniqueness $u(r,\alpha) = 0$ on $(0,\infty)$. On the other hand, suppose that $c \neq 0$. Then from equation (4.11) it follows that on the interval I we have

$$u''(r,\alpha) - c + c^3 = 0. \tag{4.17}$$

However, $c \neq 0$ contradicts $u'(r,\alpha) = 0$ on the interval I. Hence, $u'(r,\alpha) \neq 0$ for all intervals of $(0,\infty)$ and $\Phi(r)$ is strictly decreasing. From equation (4.13), we see that

$$-u(r,\alpha)^2 < \Phi(r). \tag{4.18}$$

Let $r_0 \geq 0$ be such that $\Phi(r_0) \leq 0$. Since $\Phi(r)$ is strictly decreasing, we have $\Phi(r) < 0$ for $r \in (r_0,\infty)$. Thus, we have

$$-u(r,\alpha)^2 < 0 \quad \text{for } r \in (r_0,\infty). \tag{4.19}$$

From equation (4.19) it follows that $u(r,\alpha)^2 > 0$ for $r \in (r_0,\infty)$. Furthermore, from equation (4.18) it follows that $u(r,\alpha)^2$ is strictly increasing in r and

$$\liminf_{r\to\infty} u(r,\alpha)^2 > 0. \tag{4.20}$$

We calculate

$$\lim_{r\downarrow 0} \Phi(r) \overset{(4.13)}{=} \lim_{r\downarrow 0} u'(r,\alpha)^2 + \frac{1}{2}\lim_{r\downarrow 0} u(r,\alpha)^4 - \lim_{r\downarrow 0} u(r,\alpha)^2 \overset{(4.12)}{=} \frac{1}{2}\alpha^4 - \alpha^2. \tag{4.21}$$

We factor the right hand side as $\frac{1}{2}\alpha^2\left(\alpha^2 - 2\right)$ to see that the zeros of equation (4.21) are

$$\alpha = -\sqrt{2}, \quad \alpha = 0, \quad \alpha = \sqrt{2}. \tag{4.22}$$

From equations (4.21) and (4.22), we conclude that

$$\lim_{r\downarrow 0} \Phi(r) \leq 0 \quad \text{for } 0 < \alpha \leq \sqrt{2}. \tag{4.23}$$

From equation (4.19) with $r_0 = 0$, we see that $u(r,\alpha)^2 > 0$ on $(0,\infty)$ for the initial values $0 < \alpha \leq \sqrt{2}$. Since $u(0,\alpha) = \alpha > 0$, we have $u(r,\alpha) > 0$ on $(0,\infty)$ for the initial values $0 < \alpha \leq \sqrt{2}$. Next, we use

$$w(r,\alpha) \overset{(4.10)}{=} ru(r,\alpha) \tag{4.24}$$

to see that $w(r,\alpha) > 0$ on $(0,\infty)$ for the initial values $0 < \alpha \leq \sqrt{2}$. Lastly, we use equations (4.20) and (4.24) to see that $\lim_{r\to\infty} w(r,\alpha) > 0$. This contradicts both $w(z(\alpha),\alpha) = 0$ for $z(\alpha) < \infty$ and $\lim_{r\to\infty} w(r,\alpha) = 0$. In conclusion, in order to satisfy the conditions of Lemma 4.3, we must have $\alpha > \sqrt{2}$.

Let $\alpha > \sqrt{2}$ and let $r_0 \in \left(0, z(\alpha)\right)$ such that $w(r_0,\alpha) = r_0$. We show below that $w'(r_0,\alpha) \geq 0$ implies the existence of $r_1 > r_0$ minimal such that $w(r_1,\alpha) = r_1$. Hence, if there exists a unique $r_0 \in \left(0, z(\alpha)\right)$ such that $w(r_0,\alpha) = 0$, then $w'(r_0,\alpha) < 0$.

Let $r_0 \in \left(0, z(\alpha)\right)$ such that $w(r_0,\alpha) = r_0$. Suppose that $w'(r_0,\alpha) \geq 0$. We state equation (4.3) for $w(r) = w(r,\alpha)$ and reorder the terms as

$$w''(r,\alpha) = w(r,\alpha) - r^{-2}w(r,\alpha)^3. \tag{4.25}$$

We factor the right hand side of equation (4.25) as

$$w''(r,\alpha) = w(r,\alpha)\left(1 - r^{-2}w(r,\alpha)^2\right) = w(r,\alpha)\left(1 + r^{-1}w(r,\alpha)\right)\left(1 - r^{-1}w(r,\alpha)\right) \tag{4.26}$$

to see that the zeros of $w''(r,\alpha)$ are

$$w(r,\alpha) = 0 \ \vee \ w(r,\alpha) = r \ \vee \ w(r,\alpha) = -r. \tag{4.27}$$

From equation (4.26), we see that $w''(r,\alpha) > 0$ for $0 < w(r,\alpha) < r$ and $w''(r,\alpha) < 0$ for $w(r,\alpha) > r$.

To study the behaviour of $w(r,\alpha)$ for $r > r_0$, we consider three cases for $w'(r_0,\alpha)$:

$$\begin{cases} w'(r_0,\alpha) \in [0,1), \\ w'(r_0,\alpha) = 1, \\ w'(r_0,\alpha) > 1. \end{cases} \tag{4.28}$$

In all cases we use $w(r_0,\alpha) = r_0$ in equation (4.3) to see that $w''(r_0,\alpha) = 0$. Suppose that $w'(r_0,\alpha) = 1$, then the function $w(r,\alpha) = r$ solves equation (4.3) for $r \in (r_0,\infty)$ with the initial values

$$w(r_0,\alpha) = r_0 \quad \text{and} \quad w'(r_0,\alpha) = 1. \tag{4.29}$$

By uniqueness, we have $w(r,\alpha) = r$ for $r > 0$. However, this contradicts both $w(z(\alpha),\alpha) = 0$ for $z(\alpha) < \infty$ and $\lim_{r\to\infty} w(r,\alpha) = 0$. Hence, $w'(r_0,\alpha) \neq 1$.

On the other hand, suppose that $w'(r_0,\alpha) > 1$. We show that there exists $r_1 > r_0$ minimal such that $w(r_1,\alpha) = r_1$. Let $r_a > r_0$ with $r_a - r_0 > 0$ sufficiently small such that $w(r_a,\alpha) > r_a$ and $w'(r_a,\alpha) > 1$. Write $\delta \coloneqq w(r_a,\alpha) - r_a$. We write equation (4.26) as

$$w''(r,\alpha) = w(r,\alpha)\,\frac{r + w(r,\alpha)}{r}\,\frac{r - w(r,\alpha)}{r} \tag{4.30}$$

and note that for $r > r_a$ and $w(r,\alpha) - r \geq \delta$ we have

$$w(r,\alpha)\frac{r + w(r,\alpha)}{r} = w(r,\alpha) + \frac{w(r,\alpha)^2}{r} \geq 2r \quad \text{and} \quad \frac{r - w(r,\alpha)}{r} \leq -\frac{\delta}{r}. \tag{4.31}$$

We use equation (4.31) in equation (4.30) to see that for all $r > r_a$ and $w(r,\alpha) - r \geq \delta$ we have

$$w''(r,\alpha) \leq 2r\,\frac{-\delta}{r} = -2\delta. \tag{4.32}$$

Furthermore, we see that for $r > r_a$ and $w(r,\alpha) - r \geq \delta$ we have

$$\begin{aligned} w(r,\alpha) &\leq w(r_a,\alpha) + w'(r_a,\alpha)\left(r - r_a\right) - \int_{r_a}^{r}\int_{r_a}^{s} 2\delta\,\mathrm{d}t\,\mathrm{d}s \\ &= w(r_a,\alpha) + w'(r_a,\alpha)\left(r - r_a\right) - 2\delta\int_{r_a}^{r}\left(s - r_a\right)\mathrm{d}s \\ &= w(r_a,\alpha) + w'(r_a,\alpha)\left(r - r_a\right) - \delta\left(r - r_a\right)^2. \end{aligned} \tag{4.33}$$

Hence, there exists $r_b > r_a$ minimal such that $w(r_b,\alpha) = r_b + \delta$.

If $w'(r,\alpha) > 1$ for all $r \in (r_a, r_b)$, then

$$\begin{aligned} w(r_b,\alpha) &= w(r_a,\alpha) + \int_{r_a}^{r_b} w'(r,\alpha)\,\mathrm{d}r \\ &> w(r_a,\alpha) + \int_{r_a}^{r_b} 1\,\mathrm{d}r \\ &= r_a + \delta + r_b - r_a = r_b + \delta. \end{aligned} \tag{4.34}$$

This contradicts $w(r_b,\alpha) = r_b + \delta$. Hence, there exists $r_c \in (r_a, r_b)$ such that

$$w'(r_c,\alpha) = 1. \tag{4.35}$$

Thus, we have

$$w'(r_b,\alpha) = w'(r_c,\alpha) + \int_{r_c}^{r_b} w''(r,\alpha)\,\mathrm{d}r \overset{(4.35)}{=} 1 + \int_{r_c}^{r_b} w''(r,\alpha)\,\mathrm{d}r \overset{(4.32)}{<} 1. \tag{4.36}$$

Hence, for $r \geq r_b$ and $w(r,\alpha) > r$ we have

$$\begin{aligned} w(r,\alpha) &= w(r_b,\alpha) + w'(r_b,\alpha)\left(r - r_b\right) + \int_{r_b}^{r}\int_{r_b}^{s} w''(t,\alpha)\,\mathrm{d}t\,\mathrm{d}s \\ &\overset{(4.32)}{\leq} w(r_b,\alpha) + w'(r_b,\alpha)\left(r - r_b\right). \end{aligned} \tag{4.37}$$

From equation (4.37), we solve for the value $r - r_b$ such that $w(r,\alpha) \le r$

$$\begin{aligned} r &= w(r_b,\alpha) + w'(r_b,\alpha)\left(r - r_b\right) \\ r - r_b &= w(r_b,\alpha) - r_b + w'(r_b,\alpha)\left(r - r_b\right) \\ \left(r - r_b\right)\left(1 - w'(r_b,\alpha)\right) &= w(r_b,\alpha) - r_b \\ r - r_b &= \frac{w(r_b,\alpha) - r_b}{1 - w'(r_b,\alpha)} \\ r - r_b &= \frac{\delta}{1 - w'(r_b,\alpha)}. \end{aligned} \tag{4.38}$$

Thus, there exists $r_1 \in \left(r_b, r_b + \frac{\delta}{1-w'(r_b,\alpha)}\right]$ such that $w(r_1,\alpha) = r_1$.

For the remaining case, we also show that there exists $r_1 > r_0$ minimal such that $w(r_1,\alpha) = r_1$. Suppose that $0 \le w'(r_0,\alpha) < 1$. Let $r_a > r_0$ with $r_a - r_0$ sufficiently small such that $w(r_a,\alpha) < r_a$ and $0 \le w'(r_a,\alpha) < 1$. Write $\delta := r_a - w(r_a,\alpha)$. We restate equation (4.30)

$$w''(r,\alpha) = w(r,\alpha)\,\frac{r + w(r,\alpha)}{r}\,\frac{r - w(r,\alpha)}{r}.$$

For $r > r_a$ and $r - w(r,\alpha) \ge \delta$ we have $w''(r,\alpha) > 0$. Since $w(r_0,\alpha) = r_0$ and $0 \le w'(r_0,\alpha) < 1$, we have $w(r,\alpha) \ge r_0$ for $r > r_a$ and $r - w(r,\alpha) \ge \delta$. Thus, for $r > r_a$ and $r - w(r,\alpha) \ge \delta$ we have

$$w(r,\alpha)\frac{r + w(r,\alpha)}{r} \ge r_0\,\frac{r + r_0}{r} \ge r_0 \quad \text{and} \quad \frac{r - w(r,\alpha)}{r} \ge \frac{\delta}{r}. \tag{4.39}$$

We use equation (4.39) in equation (4.30) to see that for all $r > r_a$ and $r - w(r,\alpha) \ge \delta$ we have

$$w''(r,\alpha) \ge \frac{\delta r_0}{r}. \tag{4.40}$$

Furthermore, we see that for $r > r_a$ and $r - w(r,\alpha) \ge \delta$ we have

$$\begin{aligned} w(r,\alpha) &\ge w(r_a,\alpha) + w'(r_a,\alpha)\left(r - r_a\right) + \int_{r_a}^{r}\int_{r_a}^{s} \frac{\delta r_0}{t}\,\mathrm{d}t\,\mathrm{d}s \\ &= w(r_a,\alpha) + w'(r_a,\alpha)\left(r - r_a\right) + \delta r_0 \int_{r_a}^{r} \ln\left(\frac{s}{r_a}\right)\mathrm{d}s \\ &= w(r_a,\alpha) + w'(r_a,\alpha)\left(r - r_a\right) + \delta r_0\left(r \ln\frac{r}{r_a} - r + r_a\right). \end{aligned} \tag{4.41}$$

Hence, there exists $r_b > r_a$ minimal such that $w(r_b,\alpha) = r_b - \delta$.

If $0 \le w'(r,\alpha) < 1$ for all $r \in \left(r_a, r_b\right)$, then

$$\begin{aligned} w(r_b,\alpha) &= w(r_a,\alpha) + \int_{r_a}^{r_b} w'(r,\alpha)\,\mathrm{d}r \\ &< w(r_a,\alpha) + \int_{r_a}^{r_b} 1\,\mathrm{d}r \\ &= r_a - \delta + r_b - r_a = r_b - \delta. \end{aligned} \tag{4.42}$$

This contradicts $w(r_b,\alpha) = r_b - \delta$. Hence, there exists $r_c \in \left(r_a, r_b\right)$ such that

$$w'(r_c,\alpha) = 1. \tag{4.43}$$

Thus, we have

$$w'(r_b,\alpha) = w'(r_c,\alpha) + \int_{r_c}^{r_b} w''(r,\alpha)\,\mathrm{d}r \overset{(4.43)}{=} 1 + \int_{r_c}^{r_b} w''(r,\alpha)\,\mathrm{d}r \overset{(4.40)}{>} 1. \tag{4.44}$$

Hence, for $r \geq r_b$ and $w(r,\alpha) < r$ we have

$$\begin{aligned} w(r,\alpha) &= w(r_b,\alpha) + w'(r_b,\alpha)\left(r - r_b\right) + \int_{r_b}^{r}\int_{r_b}^{s} w''(t,\alpha)\,\mathrm{d}t\,\mathrm{d}s \\ &\overset{(4.40)}{\geq} w(r_b,\alpha) + w'(r_b,\alpha)\left(r - r_b\right). \end{aligned} \tag{4.45}$$

Similar to equation (4.38), we solve equation (4.45) for the value $r - r_b$ such that $w(r,\alpha) \geq r$

$$r - r_b = \frac{\delta}{1 - w'(r_b,\alpha)}. \tag{4.46}$$

Thus, there exists $r_1 \in \left(r_b, r_b + \dfrac{\delta}{1 - w'(r_b,\alpha)}\right]$ such that $w(r_1,\alpha) = r_1$.

In conclusion, for $r_0 \in \big(0, z(\alpha)\big)$ such that $w(r_0,\alpha) = r_0$, the assumption $w'(r_0,\alpha) \geq 0$ implies the existence of $r_1 \in (r_0, z(\alpha))$ minimal such that $w(r_1,\alpha) = r_1$. Thus, if there exists a *unique* $r_0 \in \big(0, z(\alpha)\big)$ such that $w(r_0,\alpha) = r_0$, then $w'(r_0,\alpha) < 0$. To complete the proof of Lemma 4.3 we show the uniqueness of $r_0 \in \big(0, z(\alpha)\big)$ such that $w(r_0,\alpha) = r_0$.

Suppose by contradiction that there exist $0 < r_0 < r_1 < z(\alpha)$ such that $w(r_0,\alpha) = r_0$ and $w(r_1,\alpha) = r_1$ and $w(r,\alpha) \neq r$ on $\big(r_0, r_1\big)$. From equation (4.10) it follows that $u(r_0,\alpha) = u(r_1,\alpha) = 1$. Since

$$u(0,\alpha) = \alpha > \sqrt{2} > 1, \tag{4.47}$$

we have

$$0 < u(r,\alpha) < 1 \quad \text{for } r \in \big(r_0, r_1\big). \tag{4.48}$$

Thus, there exists $r_2 \in (r_0, r_1)$ such that $u'(r_2,\alpha) = 0$. From the definition of $\Phi(r)$ in (4.13) we see that

$$\Phi(r_2) = \frac{1}{2}u(r_2,\alpha)^4 - u(r_2,\alpha)^2. \tag{4.49}$$

We use equation (4.48) in equation (4.49) to see that $\Phi(r_2) < -\frac{1}{2}$. Since $\Phi(r)$ is strictly decreasing by equation (4.16), we have

$$\Phi(r) < -\frac{1}{2} \quad \text{on } \big(r_2, z(\alpha)\big). \tag{4.50}$$

From equation (4.19) with $r_0 = r_2$ it follows that $u(r,\alpha)^2 > 0$ on $\big(r_2, \infty\big)$. By equation (4.48), we have $u(r_2,\alpha) > 0$. Thus, $u(r,\alpha) > 0$ on $\big(r_2, \infty\big)$. By the definition of $w(r,\alpha)$ in equation (4.24), we have $w(r,\alpha) > 0$ on on $\big(r_2, \infty\big)$. This contradicts $w(z(\alpha),\alpha) = 0$ for $z(\alpha) < \infty$ and $\lim_{r\to\infty} w(r,\alpha) = 0$. Hence, there exists precisely one $0 < r_0 < z(\alpha)$ such that $w(r_0) = r_0$, which implies $w'(r_0,\alpha) < 0$. This completes the proof of Lemma 4.3. □

For $w(r,\alpha)$ as in Lemma 4.3 we have $\alpha > \sqrt{2}$. Since $u(0,\alpha) = \alpha$ and $w(r,\alpha) \overset{(4.24)}{=} ru(r,\alpha)$, we have $w(0,\alpha) = 0$. From equation (4.4) it follows that $w'(0,\alpha) = \alpha$. Thus, there exist minimal positive values $r = a,\ r = b$ and $r = c$ such that

$$w'(a,\alpha) = 1, \quad w'(b,\alpha) = 0 \quad \text{and} \quad w(c,\alpha) = r. \tag{4.51}$$

The following properties follow from Lemma 4.3:

- We have $0 < a < b < c < z(\alpha)$.
- We have $w(r,\alpha) > r$ and $w''(r,\alpha) < 0$ on $(0,c)$.
- We have $0 < w(r,\alpha) < r$ and $w''(r,\alpha) > 0$ on $\big(c, z(\alpha)\big)$.

The final lemma is required in the proof of Lemma 4.2.

Lemma 4.4. *Let y_1 denote the least positive zero of $w_\alpha = w_\alpha(r,\alpha)$. Then $a < y_1 < b$.*

Proof. We refer to the proof of Lemma 4.4 on page 88 of [3]. □

4.2. Proof of the main uniqueness theorem

Proof of Theorem 4.1. We assume Lemma 4.2. Let N be the set of initial values $\alpha > 0$ such that $w(r,\alpha) = 0$ for some $r > 0$. Let G be the set of initial values $\alpha > 0$ such that $w(r,\alpha)$ satisfies (4.6) and (4.7). Let $z(\alpha)$ be the least positive value $r > 0$ for which $w(r,\alpha) = 0$. We write $z(\alpha) = \infty$ for $\alpha \in G$. See Figure 4.1 below.

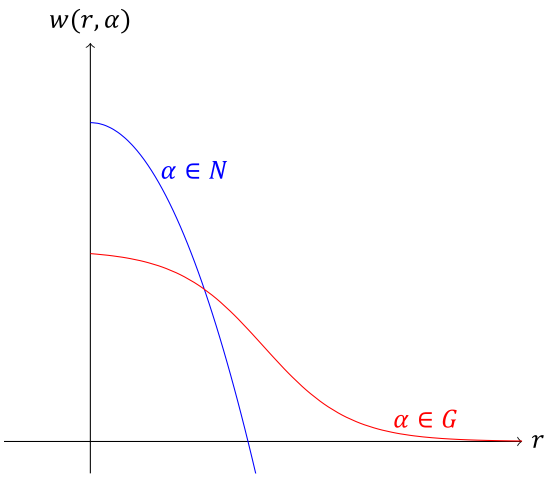

Figure 4.1: Sketch of solutions $w(r,\alpha)$ for $\alpha \in N$ and $\alpha \in G$

Proof step (i)

Since $w(r,\alpha) > 0$ on $(0, z(\alpha))$, we have $w'(z(\alpha),\alpha) \leq 0$. However, by equation (4.3) we have $w''(z(\alpha),\alpha) = 0$. Thereby $w'(z(\alpha),\alpha) = 0$ would yield the trivial solution $w = 0$ everywhere. Hence $w'(z(\alpha),\alpha) < 0$ must hold. By Lemma 4.2(i), we have $w_\alpha(z(\alpha),\alpha) < 0$. Thus, by the implicit function theorem and Lemma 4.1, $z(\alpha)$ is differentiable with respect to α on N and we have

$$w_\alpha(z(\alpha),\alpha) + w'(z(\alpha),\alpha)\frac{\mathrm{d}z(\alpha)}{\mathrm{d}\alpha} = 0, \tag{4.52}$$

which implies that

$$\frac{\mathrm{d}z(\alpha)}{\mathrm{d}\alpha} < 0 \quad \text{on } N. \tag{4.53}$$

Therefore, $z(\alpha)$ is monotonically decreasing for $\alpha \in N$. If N is non-empty, then N is of the form (α^*, ∞) for some $\alpha^* > 0$.

We next show that if there exists $\alpha_1 \in G$, then N is non-empty and α_1 is the left endpoint of N. This implies Theorem 4.1. To see this, suppose that there exists $0 < \alpha_2 < \alpha_1$ such that $\alpha_2 \in G$. Then $(\alpha_2, \infty) = N$ by the same argument, which contradicts $\alpha_1 \in G$. Thus, G contains at most one point.

Proof step (ii)

Let $\alpha_1 \in G$. Let $\alpha_2 > \alpha_1$ with $\alpha_2 - \alpha_1$ sufficiently small and suppose by contradiction that

$$w(r,\alpha_2) > 0 \quad \text{on } (0,\infty). \tag{4.54}$$

Since $\alpha_1 \in G$, there exists $r_0 > 0$ such that by equation (4.7)

$$3r^{-2}w(r,\alpha_1)^2 < \frac{1}{2} \quad \text{for } r \geq r_0, \tag{4.55}$$

and by equation (4.9)

$$w_\alpha(r_0,\alpha_1) < 0 \quad \text{and } w'_\alpha(r_0,\alpha_1) < 0. \tag{4.56}$$

For $\alpha_2 > \alpha_1$ and $\alpha_2 - \alpha_1$ sufficiently small, by equation (4.56) we have

$$w(r_0, \alpha_2) < w(r_0, \alpha_1) \quad \text{and} \quad w'(r_0, \alpha_2) < w'(r_0, \alpha_1). \tag{4.57}$$

We set

$$v(r) = w(r, \alpha_1) - w(r, \alpha_2), \tag{4.58}$$

so that $v = v(r)$ satisfies

$$v'' - v + r^{-2}\left(w(r, \alpha_1)^2 + w(r, \alpha_1)w(r, \alpha_2) + w(r, \alpha_2)^2\right) v = 0. \tag{4.59}$$

Let $r_1 > r_0$ be such that $\left[r_0, r_1\right)$ is the maximal interval such that we have the ordering

$$0 < w(r, \alpha_2) < w(r, \alpha_1) \quad \text{on } \left[r_0, r_1\right). \tag{4.60}$$

We will show that $r_1 = \infty$ must hold. Suppose by contradiction that $r_1 < \infty$. From equations (4.55) and (4.60) it follows that

$$\frac{1}{2} > 3r^{-2}w(r, \alpha_1)^2 > r^{-2}\left(w(r, \alpha_1)^2 + w(r, \alpha_1)w(r, \alpha_2) + w(r, \alpha_2)^2\right) \quad \text{on } \left[r_0, r_1\right), \tag{4.61}$$

which we use in equation (4.59) to conclude that

$$v'' > \frac{1}{2}v \quad \text{on } \left[r_0, r_1\right). \tag{4.62}$$

Furthermore, from equations (4.58) and (4.60) it follows that v is positive on $\left[r_0, r_1\right)$. Thus, v is convex on $\left[r_0, r_1\right)$ by equation (4.62). Moreover, from equation (4.57), $v'(r_0) > 0$ so that v is increasing on $\left[r_0, r_1\right)$. Thus equation (4.60) holds at $r = r_1$. This contradicts with the requirement that $\left[r_0, r_1\right)$ is the maximal interval on which we have the ordering (4.60). Hence, $r_1 = \infty$, equation (4.60) holds on $\left[r_0, \infty\right)$ and v is increasing on $\left[r_0, \infty\right)$.

The inequality (4.60) on $\left[r_0, \infty\right)$ and equation (4.7) imply that

$$\lim_{r \to \infty}\left(w(r, \alpha_1)^2 + w(r, \alpha_1)w(r, \alpha_2) + w(r, \alpha_2)^2\right) = 0. \tag{4.63}$$

Using this fact and the monotone character of v, we conclude from asymptotic integration of (4.59) *v grows exponentially as $r \to \infty$.* We solve a differential equation that demonstrates exponential growth as $r \to \infty$. Next, we use Taylor series expansions in an ordering argument and in a contradiction argument, from which exponential growth of $v(r)$ as $r \to \infty$ is immediate. Let $\Psi(r)$ be the solution to the differential equation

$$\Psi'' = \frac{1}{4}\Psi, \tag{4.64}$$

where

$$\Psi(r_0) = v(r_0) \quad \text{and} \quad \Psi'(r_0) = v'(r_0). \tag{4.65}$$

Solutions to equation (4.64) are of the form

$$\Psi\left(r\right) = ae^{\frac{1}{2}r} + be^{-\frac{1}{2}r}, \tag{4.66}$$

from which we calculate

$$\Psi'\left(r\right) = \frac{1}{2}ae^{\frac{1}{2}r} - \frac{1}{2}be^{-\frac{1}{2}r}. \tag{4.67}$$

We use $r = r_0$ in both equations (4.66) and (4.67) to obtain the system of equations

$$\begin{cases} a + b = \Psi(r_0) = v(r_0) \\ \frac{1}{2}a - \frac{1}{2}b = \Psi'(r_0) = v'(r_0). \end{cases} \tag{4.68}$$

As presented earlier, by equations (4.57) and (4.60), we have $v(r_0) > 0$ and $v'(r_0) > 0$. We use this result in equation (4.68) to see that $2a = v(r_0) + 2v'(r_0) > 0$. Thus,

$$\Psi(r) \sim e^{\frac{1}{2}r} \quad \text{as } r \to \infty. \tag{4.69}$$

Next, we use a Taylor series expansion of $\Psi(r)$ to write

$$\Psi(r) = \Psi(r_0) + \Psi'(r_0)\left(r - r_0\right) + \frac{\Psi''(r_0)}{2!}\left(r - r_0\right)^2 + \mathcal{O}\left(\left(r - r_0\right)^3\right). \tag{4.70}$$

We use the initial values for $\Psi(r_0)$ and $\Psi'(r_0)$ in equation (4.70) and use equation (4.64) to obtain

$$\Psi(r) = v(r_0) + v'(r_0)\left(r - r_0\right) + \frac{1}{8}v(r_0)\left(r - r_0\right)^2 + \mathcal{O}\left(\left(r - r_0\right)^3\right). \tag{4.71}$$

We expand $v(r)$ around $r = r_0$ to obtain

$$v(r) = v(r_0) + v'(r_0)\left(r - r_0\right) + \frac{1}{2}v''(r_0)\left(r - r_0\right)^2 + \mathcal{O}\left(\left(r - r_0\right)^3\right). \tag{4.72}$$

We use equation (4.62) in equation (4.72) to see that

$$v(r) > v(r_0) + v'(r_0)\left(r - r_0\right) + \frac{1}{4}v(r_0)\left(r - r_0\right)^2 + \mathcal{O}\left(\left(r - r_0\right)^3\right). \tag{4.73}$$

We compare equations (4.71) and (4.73) to see that $v(r) > \Psi(r)$ on some interval $\left(r_0, r_1\right)$. Let r_1 be maximal. Similarly, we expand $\Psi'(r)$ and $v'(r)$ around $r = r_0$

$$\begin{cases} \Psi'(r) = v'(r_0) + \frac{1}{4}v(r_0)\left(r - r_0\right) + \mathcal{O}\left(\left(r - r_0\right)^2\right) \\ v'(r) > v'(r_0) + \frac{1}{2}v(r_0)\left(r - r_0\right) + \mathcal{O}\left(\left(r - r_0\right)^2\right). \end{cases} \tag{4.74}$$

Hence, $v'(r) > \Psi'(r)$ on some interval $\left(r_0, r^*\right)$. Lastly, we expand $\Psi''(r)$ and $v''(r)$ around $r = r_0$

$$\begin{cases} \Psi''(r) = \frac{1}{4}v(r_0) + \mathcal{O}\left(\left(r - r_0\right)\right) \\ v''(r) > \frac{1}{2}v(r_0) + \mathcal{O}\left(\left(r - r_0\right)\right). \end{cases} \tag{4.75}$$

Hence, $v''(r) > \Psi''(r)$ on $\left[r_0, r_1\right)$. We will show that $r_1 = \infty$. Suppose by contradiction that $r_1 < \infty$. Then $v(r_1) = \Psi(r_1)$, which requires $v'(r_2) < \Psi'(r_2)$ for some $r_0 < r_2 < r_1$. We have $v'(r) > \Psi'(r)$ for $r_0 < r < r^*$. Thus, we must have $r^* < r_2 < r_1$. This implies that there exists some $r_3 < r_2$ such that $v''(r_3) = \Psi''(r_3)$. However, we have $v''(r) > \Psi''(r)$ on $\left[r_0, r_1\right)$. This contradiction shows that $r_1 = \infty$ and $v(r) > \Psi(r)$ holds on $\left(r_0, \infty\right)$. In conclusion, we have shown that $\Psi(r) \sim e^{\frac{1}{2}r}$ and that $v(r) > \Psi(r)$ on $\left[r_0, \infty\right)$. Thus, $v(r)$ grows exponentially as $r \to \infty$.

We now finish the proof by contradiction started with equation (4.54). Since $\alpha_1 \in G$, we have (4.7)

$$\lim_{r \to \infty} w(r, \alpha_1) = 0.$$

Furthermore, we have shown that under the assumption (4.54), we have $v(r) \to \infty$ as $r \to \infty$. Thus $w(r, \alpha_2) > 0$ on $(0, \infty)$ cannot hold by equation (4.58). Hence, $w(r, \alpha_2) = 0$ for some $r < \infty$ and for all $\alpha_2 > \alpha_1$ we have $\alpha_2 \in N$. This completes the proof of Theorem 4.1. □

4.3. Derivation of the equation for radially symmetric solutions

As presented in Chapter 3, positive radially symmetric solutions $u(x)$ with $x \in \mathbb{R}^n$ to equation (3.3) are solutions $u(r)$ for $r > 0$ to the ODE (3.1), restated here for $n = 3$ as

$$u'' + \frac{2}{r}u' - u + u^3 = 0. \tag{4.76}$$

We calculate the first and second derivative of the expression $u(r) = r^{-1}w(r)$

$$\begin{cases} u'(r) = -r^{-2}w(r) + r^{-1}w'(r) \\ u''(r) = 2r^{-3}w(r) - 2r^{-2}w'(r) + r^{-1}w''(r). \end{cases} \tag{4.77}$$

and substitute these derivatives in equation (4.2) to obtain

$$\begin{aligned} u''(r) + \frac{2}{r} - u(r) + u^3(r) &= 2r^{-3}w(r) - 2r^{-2}w'(r) + r^{-1}w''(r) \\ &\quad + \frac{2}{r}\left(-r^{-2}w(r) + r^{-1}w'(r)\right) - r^{-1}w(r) + r^{-3}w^3(r) = 0, \end{aligned} \tag{4.78}$$

which is simplified to

$$r^{-1}\left(w'' - w + r^{-2}w^3\right) = 0. \tag{4.79}$$

In conclusion, since $r > 0$, we have derived equation (4.3).

Bibliography

[1] H. BERESTYCKI, P. L. LIONS, and L. A. PELETIER. An ode approach to the existence of positive solutions for semilinear problems in rn. *Indiana University Mathematics Journal*, 30(1):141–157, 1981. ISSN 00222518, 19435258. URL http://www.jstor.org/stable/24893128.

[2] R. Y. Chiao, E. Garmire, and C. H. Townes. Self-trapping of optical beams. *Phys. Rev. Lett.*, 13: 479–482, Oct 1964. doi: 10.1103/PhysRevLett.13.479. URL https://link.aps.org/doi/10.1103/PhysRevLett.13.479.

[3] Charles V. Coffman. Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions. *Arch. Rational Mech. Anal.*, 46:81–95, 1972. ISSN 0003-9527. doi: 10.1007/BF00250684. URL https://doi.org/10.1007/BF00250684.

[4] Gadi Fibich. *The nonlinear Schrödinger equation*, volume 192 of *Applied Mathematical Sciences*. Springer, Cham, 2015. ISBN 978-3-319-12747-7; 978-3-319-12748-4. doi: 10.1007/978-3-319-12748-4. URL https://doi.org/10.1007/978-3-319-12748-4. Singular solutions and optical collapse.

[5] D.J. Griffiths. *Introduction to Electrodynamics*. Pearson international edition. Prentice Hall, 1999. ISBN 9780139199608. URL https://books.google.nl/books?id=x0akQgAACAAJ.

[6] Man Kam Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.*, 105(3):243–266, 1989. ISSN 0003-9527. doi: 10.1007/BF00251502. URL https://doi.org/10.1007/BF00251502.

[7] A.E. Siegman. *Lasers*. University Science Books, 1986. ISBN 9780935702118. URL https://books.google.de/books?id=1BZVwUZLTkAC.

[8] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*. Graduate studies in mathematics. American Mathematical Society, 2012. ISBN 9780821883280. URL https://books.google.nl/books?id=FZ0CAQAAQBAJ.