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Dartbord arrangement

(Engelse titel: Dartboard arrangement)

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“Dartbord arrangement”
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Motivation

In the research for this thesis we focused our attention on making the darts game as exciting as possible. Keeping in mind the motto “may the best man win”. If you play well, you should be rewarded, which also means that it should be unlikely for a lesser player to win. We have decided to focus all the attention on the arrangement of the numbers, and not attempt to optimize via changing the design of the dartboard.

The current dartboard makes it possible for a lesser player, or even a bad player to get a better result than what they really deserve by using an alternative strategy. This is because when you miss your target, but the neighbours of your target have a relatively high value, you haven’t lost that much. So the goal is to make it almost impossible for a bad darter to beat for example Raymond van Barneveld in a game of darts.

Even though it is possible to use an alternative strategy to improve your average score with the current dartboard, it is still a good dartboard. For the top male darters it is not possible to use an alternative strategy to improve their average score. They just have to try to score as high as possible. This is no longer true when you look at the top of the women dart league or the top of the amateur men dart league.

In this project we wish to find a dartboard design that makes alternative strategies impossible for competitive dart players. To find the best order of numbers on a dartboard we first need to define what a good dartboard is, what criteria a good dartboard must satisfy. We will first explain what others have done on this topic, and then we will try to expand their ideas.

In the ideal case we would find one ultimate dartboard. This will be quite a challenge considering that there are $19!/2 \approx 6.0823 \cdot 10^{16}$ different number combinations possible excluding multiples due to mirroring and rotations.

Acknowledgements:

Since the data used for the darters' statistics were not only mine, but also from Francis Hoenselaar and Raymond van Barneveld, I would like to thank them. I have to note that the average scores of Francis Hoenselaar have improved significantly, from the 64.5 % that I used in my thesis to a 74.6 % in a more recent match, which made it possible for her to win the Dart Masters in January 2009. She achieved this without using the alternative strategy of aiming at 19.

I would also like to thank my supervisors for staying patient with me during the long road that is called `writing my bachelor thesis'.

Abstract:

In this project we tried to find the best dartboard. To understand all aspects involved, the history and rules of the dart game should be taken into account. This helped us to find the right approach for this problem. We chose to only change the order of the numbers, not the layout of the dartboard. The different approaches from the literature are repeated in a reformulated way, following are our own investigations.

Part I consists of three criteria based on the literature. Criterion I is the most commonly cited in literature, which states that the sum of differences between two neighbouring numbers should be maximal. With this method still more than two hundred thousand dartboards are considered `best'. The next criterion is stricter; it states that the sum of the squared differences between neighbours should be maximal. This gives one `best' dartboard, namely the Squared dartboard, with the numbers [20 1 19 3 17 5 15 7 13 9 11 10 12 8 14 6 16 4 18 2] going round the board. Criterion III is based on the idea that all people; tall, small, left- or right-handed should have an equal chance of winning the game. Since tall people prefer the top and small people the bottom part of the dartboard, all numbers should be evenly spread over the dartboard, giving the dartboard some sort of balance.

Part II consists of three new criteria. We started by looking at an amateur dart player. On average he has a higher score if he would aim at the lower half of the dartboard. This means that an alternative strategy would make this player seem a bit better, and therefore would increase his chance of beating a slightly better player. In criterion IV, we try to prevent the possibility of using an alternative strategy. We test a number of dartboards to find within what kind of precision range an alternative strategy is not possible. The next criterion has to do with the finish of the game. At the end of the game the dart player has to throw a double. Each of the numbers on the dartboard multiplied by two could be the last ending dart of the game. But this part of the game is quite hard. Criterion V states that even and odd numbers should be interchanged. So that if the dart goes in a neighbouring section the player needs at least one more dart to finish. The last criterion VI is based on a well-balanced dartboard, giving horizontally and vertically talented people equal chances. For example the current standard dartboard gives an advantage to darters with a horizontal precision. Vertical precision should be rewarded in a similar way. With the use of simulations we compare different dartboards.

We tested a number of dartboards with respect to all of these criteria in order to find an overall best dartboard. The dartboard first introduced by Selkirk is the best, since it is consistent with almost all of the criteria mentioned above. The number order on the Selkirk dartboard is as follows [20 6 9 15 4 18 5 11 16 2 17 7 12 14 1 19 8 10 13 3]. But the now worldwide standard London dartboard is not so bad either.

As a last idea we tried to find an even better dartboard than the one designed by Selkirk. Since investigating all possible dartboard arrangements is not a time friendly option we use simulated annealing to look structurally for a better one. But even with more than 200 million attempts we did not succeed in finding a better number arrangement.

Contents

Introduction:

History:

Darts is a game that has been played for centuries. There is proof of the game being played going back to the 15th century for instance by Leonardo daVinci in 1497 [10]. The origin of darts is not well documented. It is generally accepted that in 1896 a carpenter named Brian Gamlin designed the current standard dartboard also known as the London dartboard [5]. There are quite a few other dartboards which are used only in a couple of pubs, a selection is shown below.

Figure 1: London

Figure 2: East-End

Figure 3: Manchester

Figure 4: Euro

Figure 5: Yorkshire

Figure 6: Quadro 240

Figure 7: Darto USA

Figure 8: Equalizer

Figure 9: ReMarkaBull

Figure 10: Casino 301

Figure 11: Old Fayre

Figure 12: Par-Darts Golf

In the rest of this report the London dartboard will be referred to as the current standard for a dartboard. The layout of this dartboard will be used as a model for finding the best dartboard arrangement. Only the number order will be modified to find a better dartboard arrangement.

Rules of the game:

Let us first consider the rules of the game. The dartboard itself is divided into 20 sections, and each section has a value between 1 and 20, see Figure 13. If a dart is thrown for example in sector 20, it gives a score of 20 unless it is thrown in the double ring which gives a double 20 score, so 40 points. The triple ring gives a score of three times the value for that section. In the middle of the dartboard the bull is located with value of 25, this is the small red circle. Inside that red circle is a green circle, also known as the inner bull or the Bulls-eye, it has value 50.

We analysed the well-known game 501. This game is played with two people or two teams; each player throws 3 darts in his turn. The points thrown are subtracted from 501 which is the starting point for each of the players. The last dart must be a double (thrown in the double ring). The person who first reaches exactly zero has won the leg. If your last dart is not a double, if you end up with score 1 or if you end up with a score below zero, the darts of your last turn are ignored. Usually a couple of legs are played, best of 3, best of 5, or best of 7.

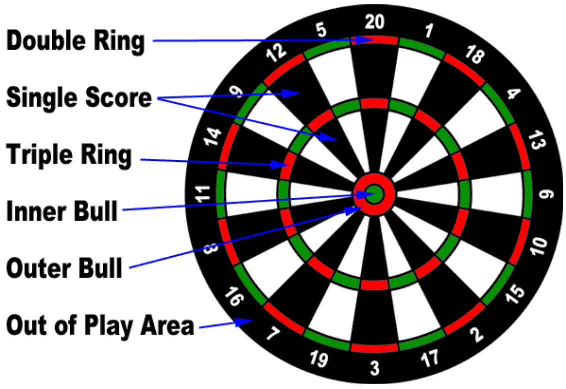

Figure 13: Scoring system of the London dartboard

Initially the strategy is to throw as many points as possible to get the starting value down as fast as possible, but keeping in mind that at the end of the game you need to have an even score. This leads us to the second part of the game, here the strategy changes. The idea is to get the score down to an even number, which is needed to end the game with a double. Preferably the last double is a double of an even number, for example double 20. This guarantees that if the double is missed and only a single is thrown, the next dart already gives "an out" in dart jargon. A single 20 in this case gives the opportunity of a double 10 with the next dart, to end the game.

Approach of my thesis:

The question that has kept us busy working on this bachelor project is a first in a line of problems: Is the London dartboard the best possible dartboard? This raises the next question: What is the best dartboard? What criteria would you use to define `the best'? First intuition says the dartboard has to have a few qualities such as it should punish bad shots, and reward good shots. Let the best man win. And there you have your next question: What are good shots?

What is playing good darts? In the first part of each leg, the goal of the darter is to throw as many points as possible, in practice this means trying to throw triple 20. At the end of the game the goal of the darter is to end with a double. This should all be done with as few darts as possible, to improve your chances of winning.

When looking at the game like this, it is easiest to end the game fast when all the high numbers are close to each other, in that case it would be easy to score high values. But this is not the approach we have taken in this thesis. We want to make sure that the best dart player wins. The best darter being the one with best scoring rate, or the least deviation when throwing a dart. For this darter to win, his scoring rate should be significantly higher than that of a person with a larger deviation.

In this thesis in part I we discuss criteria used in the literature to define the `optimal' dartboard. These will be reformulated. In part II some of our own ideas for finding the best dartboard are explained. In each of the chapters we discuss one of the criteria and compare them for 6 dartboards which in the literature are suggested to be the best. We will use the different criteria to grade the dartboards. The next page is a lose page showing these 6 dartboards as a reminder to use while reading the report.

Before we start explaining the criteria we need to introduce the dartboard itself. We will model a dartboard as a vector $D \in \mathbb{R}^{20}$, every component of the vector refers to a location on the dartboard. The first component, D_1, of vector D refers to the top section of the dartboard and the next components of the vector D refer to the other sections on the dartboard (with clockwise orientation). For the current standard dartboard this means that $D_1 = 20, D_2 = 1, D_3 = 18, \dots, D_{19} = 12, D_{20} = 5$ or

$$D^T = [20\ 1\ 18\ 4\ 13\ 6\ 10\ 15\ 2\ 17\ 3\ 19\ 7\ 16\ 8\ 11\ 14\ 9\ 12\ 5].$$

Note that we are arranging the numbers 1 till 20 on the dartboard, thus for every i we have that D_i is a natural number between 1 and 20, and if $i \neq j$ then $D_i \neq D_j$.

Definition*:*

A dartboard is a vector $D \in \mathbb{R}^{20}$, where for every i, $D_i \in \{1,2,\dots,20\}$,
and if $i \neq j$ then $D_i \neq D_j$. *This means that all 20 numbers are used around the dartboard.*

In the criteria we will discuss, we shall often consider neighbouring locations on the dartboard. Since D_{20} is a neighbour of D_1, it will be convenient to write D_0 instead of D_{20}. From now on, whenever we write D_i, where i is an index not between 1 and 20, we shall mean D_j, where $i \equiv j \bmod 20$ and $1 \leq j \leq 20$. So for example, $D_0 = D_{20}$, $D_{-1} = D_{19}$, etc.

Inlay sheet:

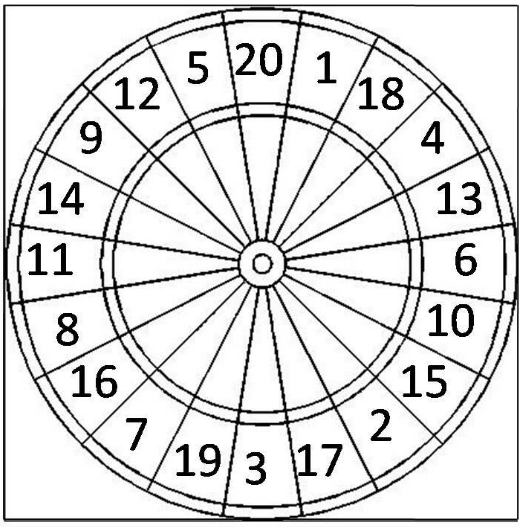

Figure 14: The London dartboard

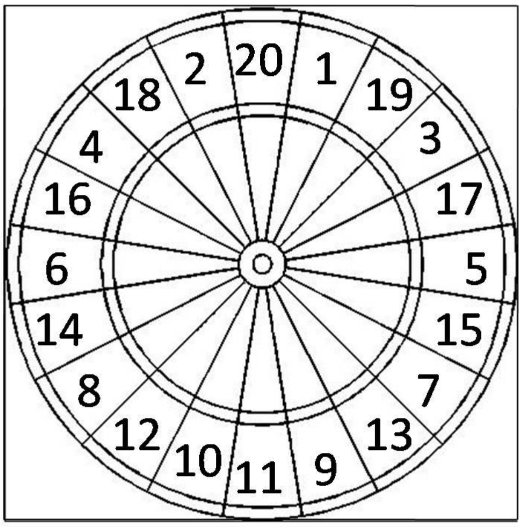

Figure 15: The Squared dartboard

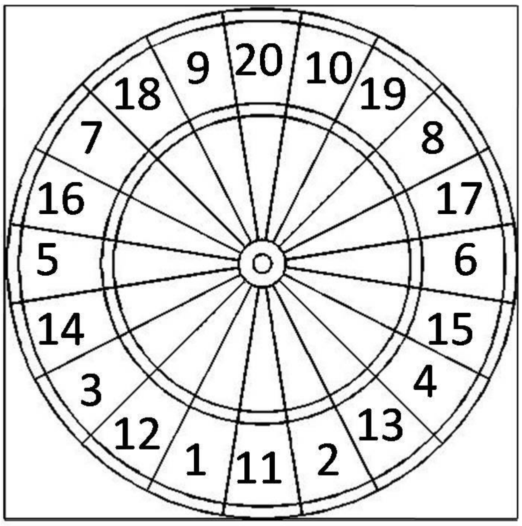

Figure 16: The Squared2 dartboard

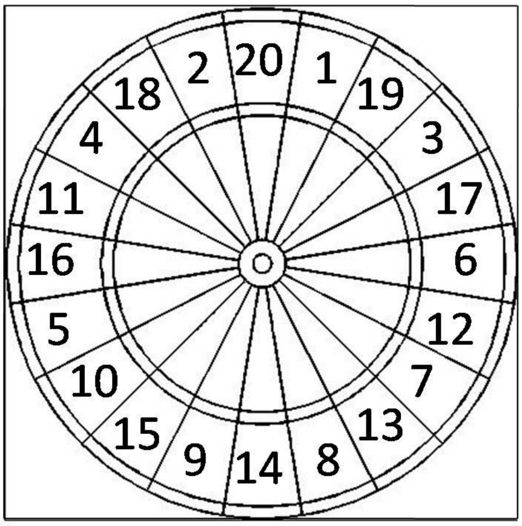

Figure 17: The Eiselt-Laporte dartboard

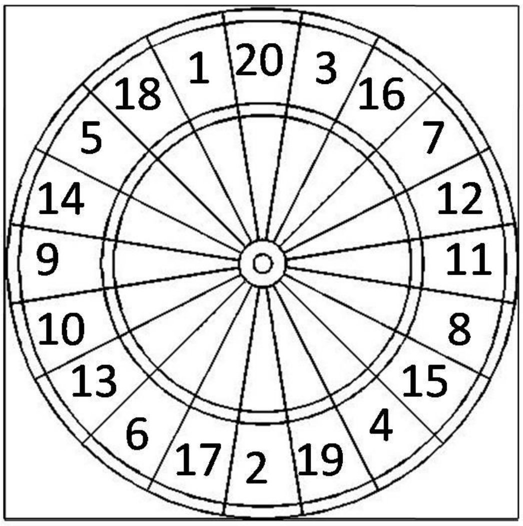

Figure 18: The Even-odd dartboard

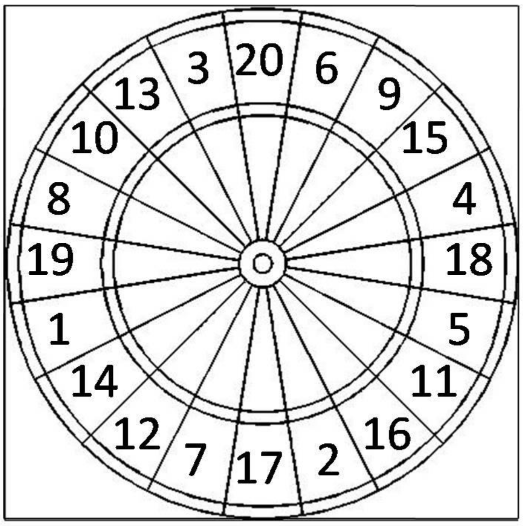

Figure 19: The Selkirk-III dartboard

PART I

Criterion I: Sum of the differences between neighbours

In this criterion we will consider neighbouring locations on the dartboard. The reason for looking at neighbours is that if we play bad darts we shall often miss the targeted section and hit one of the neighbouring sections. We want that hitting the target is awarded and hitting a neighbour is punished. If the difference between a high valued targeted section and its neighbours is large, missing the target is punished. In the first criterion the absolute differences between every two neighbours is added, and this sum is an indication of the quality of a dartboard: a higher sum corresponds to a better dartboard. If the difference between neighbouring numbers adds up to the maximum then it is an optimal dartboard. This definition of optimality was first introduced by Keith Selkirk [16].

Definition 1.1:

If D is a dartboard, then

$$S_1(D) = \sum_{i=1}^{20} |D_i - D_{i-1}|.$$

We call a dartboard D Selkirk-I-optimal if $S_1(D)$ is maximal.

Now we will determine the range of possible values for S_1. The next lemma will help explain that a Selkirk-I-optimal dartboard D has a maximal value S_1 of 200.

Lemma 1.2:

There is a dartboard D such that $S_1(D) = 200$, and for every dartboard D, $S_1(D) \leq 200$.
So a dartboard is Selkirk-I-optimal if and only if $S_1(D) = 200$.

Proof: For the dartboard D, *with*

$$D^T = [20\ 1\ 19\ 3\ 17\ 5\ 15\ 7\ 13\ 9\ 11\ 10\ 12\ 8\ 14\ 6\ 16\ 4\ 18\ 2],$$

we get $S_1(D) = 200$, this is the Squared dartboard (Figure 15 on the inlay paper).
Let $M = 10$ and let D be a dartboard. Using the triangular-inequality we have

$$S_1(D) = \sum_{i=1}^{20} |D_i - D_{i-1}| \leq \sum_{i=1}^{20} \{|D_i - M| + |M - D_{i-1}|\}$$
$$=^* 2 \sum_{i=1}^{20} |D_i - M| = 200$$

The equality in $=^*$ follows from the fact that all numbers between 1 and 20 are used around the dartboard, so in the sum $\sum_{i=1}^{20} \{|D_i - M| + |M - D_{i-1}|\}$ every number between 1 and 20 occurs exactly twice (once as D_i and once as D_{i-1}). ☐

For the London dartboard $S_1(D) = 198$, which is not maximal. So the London dartboard is not Selkirk-I-optimal. The value 200 can only be reached if you intertwine the lowest ten numbers ({1 2 3 4 5 6 7 8 9 10}) with the highest ten numbers ({11 12 13 14 15 16 17 18 19 20}) on the dartboard. There are $20 \cdot 9! \cdot 10! \approx 2.6336 \cdot 10^{13}$ different ways to intertwine the lowest ten and highest ten numbers. The London dartboard is not one of them; the numbers 11 and 14 are next to each other, as well as 6 and 10.

Theorem 1.3:

A dartboard D is Selkirk-I-optimal if and only if

for even indices $i, D_i \in L$ and for odd indices $i, D_i \in H$

or for even indices $i, D_i \in H$ and for odd indices $i, D_i \in L$

with $L = \{1,2, \dots ,10\}$ and $H = \{11,12, \dots ,20\}$

Proof: Let $M = 10$ and let D be a dartboard that satisfies the given hypothesis. Under the hypothesis on D, we have that M is between D_i and D_{i-1} for every $1 \leq i \leq 20$. This means that

$$|D_i - D_{i-1}| = |D_i - M| + |M - D_{i-1}|,$$

and therefore

$$S_1(D) = \sum_{i=1}^{20} |D_i - D_{i-1}| = \sum_{i=1}^{20} \{|D_i - M| + |M - D_{i-1}|\} = 200.$$

Now again let $M = 10$, but let D be a dartboard that does not satisfy the given hypothesis. Under this hypothesis on D, we know that there is at least one j, for which M is not between D_j and D_{j-1}. This means that for that j

$$|D_j - D_{j-1}| < |D_j - M| + |M - D_{j-1}|,$$

and therefore

$$S_1(D) = \sum_{i=1}^{20} |D_i - D_{i-1}| < \sum_{i=1}^{20} \{|D_i - M| + |M - D_{i-1}|\} = 200.$$

Joining the above statements we get: $S_1(D) = 200$ if and only if D satisfies the given hypothesis. ☐

After using this criterion in the search for the `best' dartboard, the number of possible ways to fill in the values on the dartboard is reduced by a big factor. The total number of different dartboard arrangements is $19!/2 = 6.0823 \cdot 10^{16}$. The number of different Selkirk-I-optimal dartboards is $9! \cdot 10!/2 = 6.5841 \cdot 10^{11}$.

This notion of optimality is not completely satisfying. For example on a Selkirk-I-optimal dartboard, the number 20 can still have the numbers 9 and 10 as neighbours, which is not really a punishment in the case of playing bad darts. When comparing this to the neighbours 1 and 5 of the number 20 on the current dartboard, this does not seem like a better dartboard.

When we look at the six dartboards on the inlay paper (page 12), the two squared dartboards are Selkirk-I-optimal $S_1(D) = 200$ (Figures 15 and 16), the London dartboard, Eiselt-Laporte dartboard, and even-odd dartboard all have $S_1(D) = 198$ (Figures 14, 17, and 18), and the Selkirk-III dartboard has $S_1(D) = 192$ (Figure 19). The Selkirk-III dartboard is based on criterion III, and is not Selkirk-I-optimal.

Criterion II: Sum of the squared differences between neighbours

The second criterion which is widely used in literature was also first introduced by Keith Selkirk [16] and later generalised and reformulated by other authors, see for example [11, 6, 9, 19, 20]. In this criterion the sum of the squared differences between neighbours is analysed. Because we square the differences, larger differences between neighbours have a greater influence on the quality of a dartboard than small differences. Thus Selkirk-II optimization guarantees even more than Selkirk-I optimization that high numbers have low neighbours, and thus that a bad throwing technique is punished.

Definition 2.1:

If D is a dartboard, then

$$S_2(D) = \sum_{i=1}^{20} |D_i - D_{i-1}|^2$$

We call a dartboard D Selkirk-II-optimal if $S_2(D)$ is maximal.

In a similar fashion as in criterion I the $S_2(D)$ value of a dartboard is proportional to the quality of that dartboard. A higher $S_2(D)$ value corresponds to a better dartboard.

The following lemma states the range of possible values for S_2. It will also state when a dartboard is Selkirk-II-optimal. For the proof behind this lemma read H.A. Eiselt and G. Laporte [11]. The general idea of the proof will be given after the lemma.

***Lemma 2.2**:*

There is a dartboard D such that $S_2(D) = 2642$, and for every dartboard D, $S_2(D) \leq 2642$. So a dartboard is Selkirk-II-optimal if and only if $S_1(D) = 2642$.

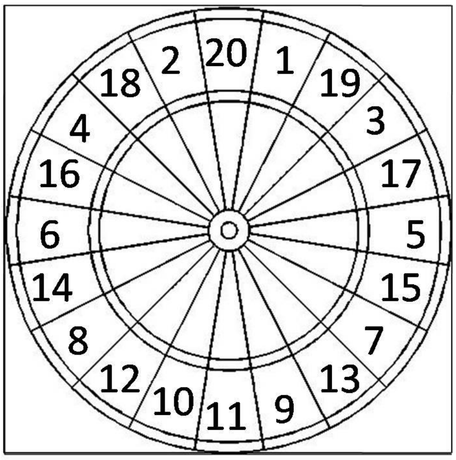

Figure 20: Dartboard with the maximum sum of the neighbours squared

This criterion guarantees small neighbours for the high numbers, which means that missing 20 or any other high number will be punished severely. H.A. Eiselt and G. Laporte [11] proved lemma 2.2 by modelling a dartboard as a travelling salesman problem. The neighbouring sections on the dartboard are simulated as neighbouring cities on the travelling salesman's route. The salesman has to travel to each city/section once and only once. The squared difference between the numbers of neighbouring sections represents the length of the travel path between them. Unlike the usual travelling salesman problem where the total travel path is minimized, we maximize the total travel path. H.A. Eiselt and G. Laporte [11] found out that $S_2(D) \leq 2642$, and they found at least one solution with $S_2(D) = 2642$. This solution is shown in Figure 20 and gives a maximal value. The solution is Selkirk-I-optimal as well as Selkirk-II-optimal. Using Eiselt and Laporte we can also find a Selkirk-I-optimal dartboard for which $S_2(D)$ is minimal, namely the Squared2 dartboard with $S_2(D) = 2018$ (Figure 16 on the inlay paper).

In comparison: the London dartboard has a value of $S_2(D) = 2478$, Figure 14 on your inlay paper. Figure 15 on this sheet is the same solution as shown above, with a value $S_2(D) = 2642$. Figure 16, the Squared2 dartboard, gives the solution $S_2(D) = 2018$ as mentioned above. Figures 17, 18, and 19 have respective values of $S_2(D) = 2498$, $S_2(D) = 2588$, and $S_2(D) = 2354$.

The values for $S_2(D)$ lay in the range of 74 till 2642. The value $S_2(D) = 74$ is calculated for the dartboard D with

$$D^T = [20\ 19\ 17\ 15\ 13\ 11\ 9\ 7\ 5\ 3\ 1\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 16\ 18],$$

which is not Selkirk-I-optimal.

Criterion III: Fair chances for the tall and small, left and right handed

It is said that for left-handed players it is easier to aim and hit at the left side of the dartboard, and for right-handed players on the right side. Also for tall people it is easier to hit a target at the top section of the board, and for small people it is easier to hit lower sections. Aiming at the left-hand side of the dartboard is also called `the married man's side' [5], because on the standard London dartboard, the average of the numbers on the left hand side is greater than the average on the right hand side (the average of the 5 left-most numbers is 11.6 and for the 5 right most numbers, the average is 9.6). However, for the better dart players both sides are not so interesting, since the individual numbers are not so high there. The high numbers are at the top and bottom part of the dartboard. In this chapter three sub-criteria will be discussed which are related to an even distribution of numbers on the board. First we look at a quadrant criterion, next we consider a centre of gravity criterion, and finally a graph criterion.

Quadrant criterion:

The quadrant method was first introduced by Selkirk in [16], and later used by Brown [3] and Lipscombe and Sangalli [18]. The method is based on the idea that the high and low numbers should be equally distributed over the dartboard; therefore each quadrant should contain some high and some low numbers. In this sub-criterion the sum of the values in all possible quadrants is added up. The differences between these 20 sums will be used as an indication of the quality of a dartboard; a small variation in these sums corresponds to a good dartboard. In the quadrant method of Selkirk, we always consider neighbouring section of 5 numbers. We can also consider groups of 3 neighbouring sections.

Definition 3.1:

If D is a dartboard, then

$$R^5_{max}(D) = \max_{1\le i\le 20}(D_{i-2} + D_{i-1} + D_i + D_{i+1} + D_{i+2})$$

$$R^5_{min}(D) = \min_{1\le i\le 20}(D_{i-2} + D_{i-1} + D_i + D_{i+1} + D_{i+2})$$

$$R^3_{max}(D) = \max_{1\le i\le 20}(D_{i-1} + D_i + D_{i+1})$$

$$R^3_{min}(D) = \min_{1\le i\le 20}(D_{i-1} + D_i + D_{i+1})$$

We call a dartboard D quadrant-5-optimal if $S^5_3(D) = R^5_{max}(D) - R^5_{min}(D)$ is minimal.

We call a dartboard D quadrant-3-optimal if $S^3_3(D) = R^3_{max}(D) - R^3_{min}(D)$ is minimal.

In a similar fashion by analysing groups of 7 and 9 sections, we can find dartboards that are quadrant-7-optimal and quadrant-9-opimal. Using this criterion we did not investigate what is the best dartboard. We do not know which values for $S^5_3(D)$ and $S^3_3(D)$ are the minimal values, thus we do not state which dartboard is quadrant-5-optimal or quadrant-3-optimal. We will use this criterion to grade the quality of the different dartboards with respect to each other, by comparing the different $S^5_3(D)$ and $S^3_3(D)$ values, the lower the value the better the dartboard is.

Definition 3.2:

We call a dartboard D *quadrant-5-better than dartboard* D' *if*

$$S_3^5(D) < S_3^5(D')$$

We call a dartboard D *quadrant-3-better than dartboard* D' *if*

$$S_3^3(D) < S_3^3(D')$$

In Table 1 the differences between the maximum and minimum of the sums of each five neighbouring numbers are shown for each of the dartboards described on the inlay paper. We also show the sum of three, seven, and nine neighbours. The maximum difference between the quadrants is the number we want to minimize. The sum of three numbers is easier to keep constant than the sum of five, the same holds for the sum of seven and nine. When looking at the sum of nine we are looking at the sum of almost the half of the dartboard, in that case the Even-odd dartboard gives the best results.

Table 1: Quadrant method minimum difference between sums of neighbours

	differences between the maximum and minimum of the sum of			
	three	five	seven	nine
London	20	20	18	18
Squared	17	15	13	11
Squared2	35	47	55	59
Eiselt-Laporte	17	15	15	16
Even-odd	15	11	7	3
Selkirk	11	3	11	19

When actually looking at quadrants, we have to look at the sum of five. The dartboard designed by Selkirk is in that case the best; you can see this as well by looking at the dartboard, the high numbers are nicely spread over the whole dartboard. Selkirk's dartboard is designed using the centre of gravity criterion which we will describe next.

Centre of gravity criterion

In this criterion we determine where the centre of the dartboard is pulled towards if you would imagine that each of the numbers pulls the centre towards itself with the strength of that number. So the number 20 pulls on the bull's-eye with strength 20. In vector terminology a vector of length 20 pulls straight up, see Figure 21. In this sub-criterion you add up all twenty vectors and the resulting vector is an indication of the quality of a dartboard: the smaller the vector combination the better the dartboard. For the vector combination to be small a dartboard needs to be in good balance. This method was first introduced by Selkirk [16].

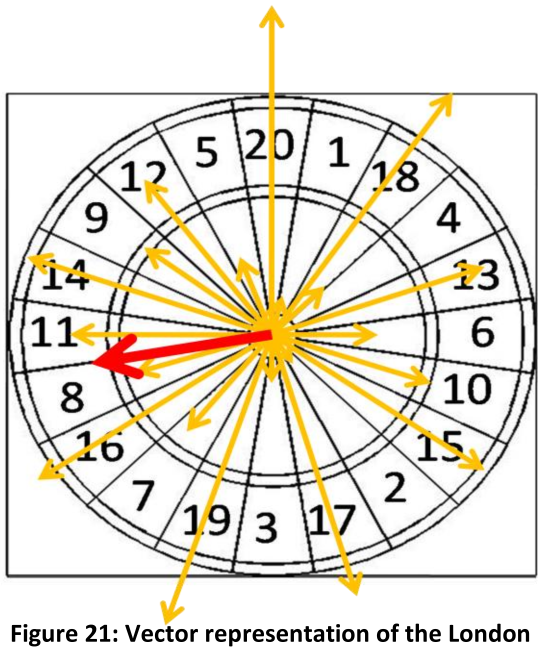

Figure 21: Vector representation of the London dartboard, in red the vector combination.

Definition 3.3:

If D is a dartboard, then

$$V(D) = \underline{d}_1 + \underline{d}_2 + \cdots + \underline{d}_{20},$$

where

$$\underline{d}_i = \left[D_i \cos\frac{2\pi(6-i)}{20} \qquad D_i \sin\frac{2\pi(6-i)}{20} \right]^T$$

the section vectors clockwise around the dartboard with $\underline{d}_1$ the top section.
A dartboard D is gravity-optimal if $\|V(D)\|$, the length of $V(D)$ is minimal.

We do not know the minimum length of $V(D)$, but it can be very close to 0, as demonstrated by the Selkirk dartboard which has $\|V(D)\| \approx 10^{-15}$. After the completion of this thesis we found out that when calculating $\|V(D)\|$ with Maple (instead of Excel or Matlab), we could find the exact value for the Selkirk dartboard, which is $\|V(D)\| = 0$. We will use this criterion to grade the quality of the different dartboards with respect to each other, by comparing the different $\|V(D)\|$ values of the different dartboards, the lower the value the better the dartboard is.

***Definition 3.4**:*

We call a dartboard D gravity-better than dartboard D' if and only if

$$\|V(D)\| < \|V(D')\|$$

In table 2 one can find the centre of gravity for the dartboards on the inlay paper. For the Selkirk dartboard an incredible small $V(D)$-vector, is calculated, the centre of gravity of the dartboard is only a length of $\sim 10^{-15}$ away from the real centre of the board. The different centres of gravity are calculated with Matlab. Matlab calculates using 15 decimals, this makes it impossible to reliable calculate any $\|V(D)\| < 10^{-15}$. Selkirk found this dartboard by spreading the numbers in a quadrant kind of way, not by proof or trying all combinations. Therefore there might be an even better number arrangement possible on the dartboard. Later in this thesis other arrangements are investigated using the definition of gravity-optimal.

The squared dartboard and even-odd dartboard give a rather good result. Unlike the London dartboard, which as discussed earlier has higher values on the left, you can see the centre is being pulled to the left as well as down. The Squared2 dartboard as can be expected has the worst results, all the high numbers are located very close together and all on the top of the dartboard.

Table 2: Centre of gravity of the six compared dartboards

	x-coordinate	y-coordinate	Vector length
London	-10.17	-2.34	10.44
Squared	-0.16	0.03	0.16
Squared2	6.31	39.86	40.36
Eiselt-Laporte	-2.08	-2.51	3.26
Even-odd	0.32	0.05	0.32
Selkirk	$\sim 10^{-15}$	$\sim 10^{-15}$	$\sim 10^{-15}$

Graph criterion

This criterion was first introduced in the dartboard design by Mark A.M. Lynch [13]. This criterion consists of two constraints. First of all, two adjacent numbers on the dartboard must have a difference of R or more and at the same time even and odd numbers must be interchanged. In criterion V we will isolate the second constraint, and thus specifically look at the influence of interchanging even and odd numbers. In the graph criterion if a dartboard has interchanging even and odd numbers, then R is an indication of the quality of the dartboard: a higher R corresponds to a better dartboard.

Definition 3.5:

If D is a dartboard where adjacent numbers have alternating parity (odd-even interchanged), then

$$R(D) = \min_{1 \le i \le 20} |D_i - D_{i+1}|$$

A dartboard $D$is graph-optimal if R is maximal.

The following lemma states the range of possible values for R, when we look at dartboards with alternating parity. It will also state when a dartboard is graph-optimal. For the complete proof of this lemma read Lynch [13]. A simplified version of the proof will be given after the lemma.

***Lemma 3.6**:*

There is a dartboard D where adjacent numbers have alternating parity such that $R(D) = 9$, and for every dartboard D where adjacent numbers have alternating parity $R(D) \le 9$.
So a dartboard is graph-optimal if and only if $R(D) = 9$.

Proof: For the Lynch dartboard D we get $R(D) = 9$, *with*

$$D^T = [20\ 11\ 2\ 13\ 4\ 15\ 6\ 17\ 8\ 19\ 10\ 1\ 12\ 3\ 14\ 5\ 16\ 7\ 18\ 9].$$

For a dartboard D, *with* $R(D) > 9$, we need $|D_i - D_{i+1}| > 9$ for all $1 \le i \le 20$. When we analyse possible neighbours for the number 11, these neighbours have to be at least ten higher or ten lower than number 11, leaving only the number 1 as a possible neighbour. Numbers 1 and 11 are both odd, and therefore not possible neighbours. This leaves us with no possible neighbours for the number 11, which is of course impossible. Thus all dartboards with alternating parity have $R(D) \le 9$. □

We now know that the Lynch dartboard is graph-optimal. The Lynch dartboard is the only dartboard that is graph-optimal, when ignoring multiples due to mirroring and rotation. We look at a dartboard D with neighbours with alternating parity and with $R(D) = 9$, and start with a number 20 in the top position

$$D^T = [20\ \dots\dots].$$

The neighbours of the number 11 can only be 2 and 20, resulting in a dartboard

$$D^T = [20\ 11\ 2\ \dots\dots].$$

The neighbours of the number 13 can only be 2 and 4, resulting in a dartboard

$$D^T = [20\ 11\ 2\ 13\ 4\ \dots\dots].$$

The neighbours of the number 15 can only be 4 and 6, resulting in a dartboard

$$D^T = [20\ 11\ 2\ 13\ 4\ 15\ 6\ \dots\dots].$$

When we continue filling in the numbers on the dartboard, we find a unique solution, when ignoring multiples due to mirroring and rotation. This dartboard is the Lynch dartboard shown in Figure 22.

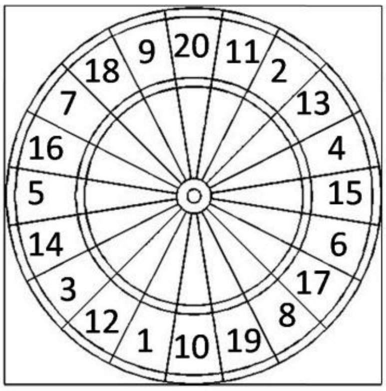

Figure 22: The Lynch dartboard

This solution looks a lot like the Squared2 solution (Figure 16), but then an even odd optimization of it. In Table 3 it can be seen what the R value for each of the dartboards is. The London dartboard has a minimal absolute difference between neighbours of 3. The Squared2 dartboard has $R = 9$, but is different from the Lynch dartboard only because it does not have alternating parity. The Lynch dartboard has not been compared in all the criteria; this will be done in the conclusion, where several extra dartboards will be used for comparison.

Table 3: graph-optimal constraints

	R: minimum difference between two neighbours	**number of alternating neighbours**
London	3	14
Squared	1	2
Squared2	9	18
Eiselt-Laporte	5	10
Even-odd	1	20
Selkirk	2	10
Lynch	9	20

PART II

Criterion IV: Excluding alternative strategies

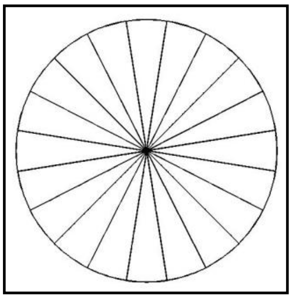
Figure 23: An empty board

In this part of the thesis, we will introduce three new criteria for analyzing dartboards. Criterion IV is inspired by Kohler [12], who investigated the results of aiming at certain parts of the dartboard. We are not only going to look at which result you will get from aiming at a part of the dartboard, but also when your strategy should change due to these results. For this criterion we only look at darters with some skills, this means that we exclude darters that miss their target by more than one section. When aiming at 20 they might miss, but at most by one section, so they could score 5, 20 or 1, when aiming for 20. We also neglect the triple ring, double ring, the bull and the bulls-eye. What is left is a dartboard with only simple pieces of a pie with their values, as is shown in Figure 23.

The normal strategy when playing darts is aiming for the highest value and wishing for the best result. But in some cases this is not the best strategy, for example for a mediocre darter. A mediocre darter has a relative low chance of hitting the target, and a relative high chance of hitting one of the neighbouring sections. This means that the neighbouring numbers influence the average return. In some cases it is recommended for this darter to use an alternative strategy, and aim at 19. This is because the number 19 has much higher neighbours, and when missing the target a lot of times but hitting the high neighbours this improves the average score. Note that for very bad darters the deviation is likely to be more than one section. This falls out of the reach of this criterion.

From now on we will refer to the `normal' strategy when we aim at the highest number (which is 20) and the `alternative' strategy when we aim at another number. In this criterion the presence of a better alternative strategy is an indication of the quality of a dartboard: dartboards with no alternative are preferred.

Let us find out what we know about this alternative strategy. First we simplify the situation even more than only leaving out the doubles, triples and bull and only take into account people with a deviation of at most one section on the board. We add that dart players have a symmetric deviation, so just as many darts are missed on the left as on the right, or top and bottom. We can say about the alternative strategy that if an alternative strategy is better than the normal strategy, then for each even worse player this alternative strategy is better than the normal strategy. Let's define p as the precision of a dart player, or the probability for a player to hit the target he or she aims for.

Definition 4.1:

If D is a dartboard, then the expected return when aiming at D_i is

$$E_i^p = \left(\frac{1-p}{2}\right) \cdot D_{i-1} + p \cdot D_i + \left(\frac{1-p}{2}\right) \cdot D_{i+1},$$

with p the chance of hitting a target while aiming at it.

A dartboard D is alternative-strategy-optimal if for all p and for all $i \neq 20$: $E_i^p \leq E_{20}^p$.

Note that we make quite some assumptions in this definition, for one the probability of hitting the target is equal all round the dartboard; also the probability of hitting each neighbour is equal; and the target is missed by maximum one section. In reality for very small values for p the target is missed by more than one section. Using this definition we can also find an optimal dartboard

Lemma 4.2:

There is a dartboard D such that $E_i^p \leq E_{20}^p$ for all p and for all $i \neq 20$, so this dartboard is alternative-strategy-optimal.

Proof: The dartboard D, *with*

$$D^T = [20\ 19\ 17\ 15\ 13\ 11\ 9\ 7\ 5\ 3\ 1\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 16\ 18]$$

is alternative-strategy-optimal, it has $E_i^p \leq E_{20}^p$ for all p and for all $i \neq 20$. $\square$

This dartboard is alternative-strategy-optimal, however it does not really punish missing the target, or reward hitting the target. In the definition we assume that independent of p the target section is missed by maximum one section. This assumption is not realistic for low p-values. For example for a darter with $p = 0.1$, this darter misses the target in almost all cases, and will most likely miss the target by more than one section. Therefore we will now neglect the very bad darters, or in other words the very low p-values and we will focus on alternative-strategy-optimal dartboards for a certain p-values.

So let us look at a dartboard that is not alternative-strategy-optimal. For a certain p value the alternative strategy gives a better return than the normal strategy. This means that a mediocre darter can improve the expected return when he or she changes strategy. This increase in expected return decreases the difference between this darter and its opponent. This decreases the chances for the best darter to win. To give an indication of the quality of the dartboards that have an alternative strategy with a higher expected return than number 20, we look at the p value for which this alternative becomes a better alternative. We want the break-even point between the normal and alternative strategy to be as small as possible.

Definition 4.3:

If D is a dartboard, then the break-even point $p^*(D)$ is the maximum p value for which there is an $i \neq 20$, such that

$$E_i^p = E_{20}^p.$$

For the London dartboard the normal strategy is aiming for 20. For this dartboard the alternative strategy of aiming at 19 gives a higher expected return somewhere between 70% and 60% precision. When we compare aiming for 20 and the alternative aiming for 19, we can calculate the precise break-even point from

$$E_{19}^p = E_{20}^p,$$

this gives

$$\tfrac{1-p}{2} \cdot \mathbf{7} + p \cdot \mathbf{19} + \tfrac{1-p}{2} \cdot \mathbf{3} = \tfrac{1-p}{2} \cdot \mathbf{1} + p \cdot \mathbf{20} + \tfrac{1-p}{2} \cdot \mathbf{5},$$

which gives $p = {}^{2}/_{3}$. In Table 5 the break-even point 60% and 70%, so between $p = 0.6$ and $p = 0.7$. This confirms what we calculated earlier.

In figure 24 the expected return for the numbers 20, 19, 7, and 1 is plotted against the p value, for the London dartboard. The point where the alternative strategy of aiming at 19 gives a better result than aiming at 20 is clearly visual in the figure at $p = {}^{2}/_{3}$. This point is referred to as the 1st break-even point, when the normal strategy aiming for 20 gets the same results as alternative strategy aiming for 19. The 2nd break-even point is at $p = 0.51$ when the alternative strategy 7 takes over from aiming for 19. The 3rd and final break-even point is at $p = 0.20$, when aiming for 1 takes over.

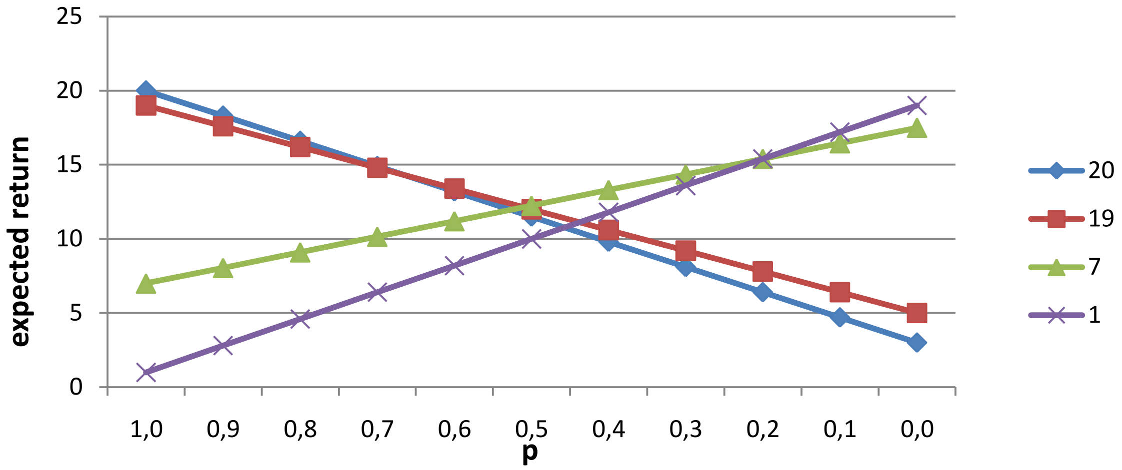

Figure 24: expected return when aiming for 20, 19, 7, and 1 for the London dartboard

We will use the break-even point to grade the quality of the different dartboards with respect to each other, by comparing the different $p^*(D)$ values for the first break-even points, the lower the value the better the dartboard is.

Lemma 4.4:

We call a dartboard D alternative-strategy-better than dartboard D' if and only if

$$p^*(D) < p^*(D').$$

In Table 5 the expected return is shown for the different dartboard sections of the London dartboard, based on a specific handicap of a player. In addition the expected return for Raymond van Barneveld (top Dutch dart player) and Francis Hoenselaar (top Dutch female dart player) are shown. These last expectations are based on repeatedly throwing at 20, and counting the number of hits and misses. This data is obtained by watching games of the Dutch open darts tournament of 2007 and literally counting every hit and miss when aiming for 20. In Table 4 the precision measurement are shown for Raymond, Francis and their opponents during the Dutch open. Because there is not enough data for throwing at any of the other numbers we only use the data from aiming at 20. For Francis Hoenselaar and Raymond van Barneveld we took the average of the left and right misses and used this precision to calculate the expected return for the other similarly shaped sections. We are allowed to do this because they punish horizontal deviation in a similar way.

Table 4: hit percentages when aiming at the number 20

Dart player:	**Hit percentage**
Raymond van Barneveld	91.6%
Mervyn King	92.8%
Francis Hoenselaar	64.4%
Carol Forwood	81.0%

The first column in Table 5 is the London dartboard rolled out clockwise. On top of each column is the percentage of hits, indicating how good the player is. Going down we see the expected value when aiming at the section shown in the left column. The gray highlighted areas are where the expected values are the highest in each column. The columns for Raymond van Barneveld and Francis Hoenselaar contain only information on the sections with a similar shape as section 20, this is because we only have data (Table 4) to base the calculations on for this kind of sections.

For Raymond van Barneveld it is advisable to use the normal strategy and aim at the number 20. For Francis Hoenselaar it is advisable to aim at the number 19 instead of at the number 20. For Francis this means an expected value of 14.0. By aiming at 19 instead of 20 Francis increases her expected return with 0.1. This means she could decrease the difference between her and her opponent and thereby improve her chances of beating a stronger opponent, who couldn't or doesn't take advantage of this alternative strategy. This is not desirable, because as stated before, the best dart player should have the biggest chance of winning.

Table 5: Expected returns per deviation, column 1 is the London dartboard, columns 2 till 9 are the expected returns when hitting at a section of the dartboard with a certain precisions, columns 10 and 11 are expected returns for Raymond and Francis based on Table 4 for sections with a similar shape as section 20, in grey the highest expected return.

Missed to the left and right:									**Raymond van Barneveld**	**Francis Hoenselaar**
Hits:		**↳ 10%**	**20%**	**30%**	**40%**	**50%**	**60%**	**66,6%**		
	↳ 100%	**90%**	**80%**	**70%**	**60%**	**50%**	**40%**	**33,3%**	**91.6%**	**64.4%**
20	20.0	18.3	16.6	14.9	13.2	11.5	9.8	8.7	18.6	13.9
1	1.0	2.8	4.6	6.4	8.2	10.0	11.8	13.0	2.5	7.4
18	18.0	16.5	14.9	13.4	11.8	10.3	8.7	7.7		
4	4.0	5.2	6.3	7.5	8.6	9.8	10.9	11.7		
13	13.0	12.2	11.4	10.6	9.8	9.0	8.2	7.7		
6	6.0	6.6	7.1	7.7	8.2	8.8	9.3	9.7		
10	10.0	10.1	10.1	10.2	10.2	10.3	10.3	10.3		
15	15.0	14.1	13.2	12.3	11.4	10.5	9.6	9.0		
2	2.0	3.4	4.8	6.2	7.6	9.0	10.4	11.3		
17	17.0	15.6	14.1	12.7	11.2	9.8	8.3	7.3	15.8	11.8
3	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.0	4.3	8.3
19	19.0	17.6	16.2	14.8	13.4	12.0	10.6	9.7	17.8	14.0
7	7.0	8.1	9.1	10.2	11.2	12.3	13.3	14.0		
16	16.0	15.2	14.3	13.5	12.6	11.8	10.9	10.3		
8	8.0	8.6	9.1	9.7	10.2	10.8	11.3	11.7		
11	11.0	11.0	11.0	11.0	11.0	11.0	11.0	11.0		
14	14.0	13.6	13.2	12.8	12.4	12.0	11.6	11.3		
9	9.0	9.4	9.8	10.2	10.6	11.0	11.4	11.7		
12	12.0	11.5	11.0	10.5	10.0	9.5	9.0	8.7		
5	5.0	6.1	7.2	8.3	9.4	10.5	11.6	12.3	5.9	8.9
Max:	**20.0**	**18.3**	**16.6**	**14.9**	**13.4**	**12.3**	**13.3**	**14.0**	**18.6**	**14.0**

Table 6: 1st break-even points for the six dartboards

	normal strategy	1st alternative strategy	break-even point
London	20	19	66.7%
squared	20	10	50.0%
squared2	20	10	50.0%
Eiselt-Laporte	20	16	61.9%
even-odd	20	19	50.0%
Selkirk	20	8	45.5%

For the six dartboards given on the inlay paper the break-even points are given in the Table above. It can be deduced from this Table that the London dartboard is not that good in comparison with the others. The break-even point is at a hit percentage of 66.7%. Which means that Francis Hoenselaar would be better off playing the alternative strategy, which would make her disadvantage with respect to for example Carol Forwood (one of England's best female dart players) smaller. For amateur men alternative strategies could be used to the players' advantages as well, to get a better average score. For the male top dart players the design differences between these dartboards have no influence. With an average scoring rate of over 90% this could be expected of course.

Note that the Selkirk dartboard looses the normal strategy to aiming at 8, which is a low value compared to the 20.

Dartboards with very low break-even points are in practise equally good as alternative-strategy-optimal dartboards, since people with such a deviation often miss their target by more than one section. This is not taken into account in the calculations though. For example let us look at the expected returns for Petra Donkers (true amateur, with an error of more than one section), see table 7 for her expected return when aiming at the London dartboard. She is advised to aim at 19. This is because the number 19 has much higher neighbours than 20. For Petra aiming at 20 gives the same result as aiming at 7, 11 or 14. The data for Petra is obtained by many attempts to hit numbers all around the dartboard and then averaging the precision.

Table 7: Expected return for Petra Donkers, with $p \approx 0.34$

Section	20	1	18	4	13	6	10	15	2	17	3	19	7	16	8	11	14	9	12	5
Expected return	11.3	9.8	10.8	9.0	9.7	9.2	9.7	10.2	9.2	10.6	10.0	12.2	11.3	12.0	11.0	11.3	11.3	10.6	10.9	9.9

Criterion V: An even or odd ending to the game

Up to this point we have only been looking at the first part of the game. In the beginning of a leg, the players try to score as many points as possible. To reach 501 points, they have to go down as fast as possible initially. Now we will take a short detour and look at the end of the game, where the players have to end with a double. This means that the last dart should go in the outer ring of the dartboard. This adds a whole new dimension to the game.

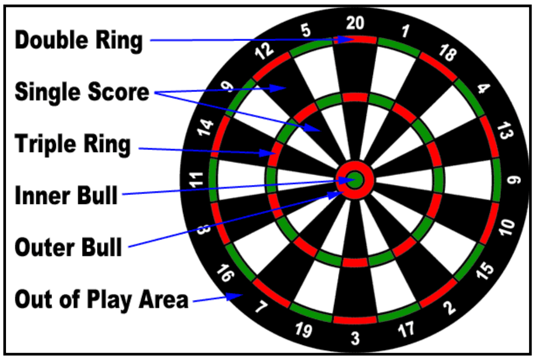

Figure 25: London dartboard with the rings

Having to end the game with a double should be taken into account when considering different dartboard orderings. The technique darters use at the end of the game is as follows. First they try to end with a double opportunity, this means that they have to get to an even score, like 40, 38, 36, … , 6, 4, or 2. They often try to reach an even number which is by itself the double of an even number, for example 40, 36, 32, … , 12, 8, or 4. The advantage is that if they would accidentally miss the double ring and hit a single score, they have the possibility of finishing with their next dart. Missing the double and hitting a single happens quite often. For example, if they have a finish option with 40 (a double 20) and they miss and hit a single 20 they would still have a finish of 20 (a double 10), if again they miss and hit a single they would still have a finish of 10 (a double 5). While if they have a finish of 38 (a double 19) and they hit a single 19, they end up with 19, an odd number which leaves no opportunity to finish immediately. This means that players prefer to end the game with a `safe' double, such as double 20 or double 16.

Our goal is to make the game as difficult as possible, in which case the chances for the best player to win will increase. One way of making the game more difficult is by spreading the `safe' endings for the game evenly around the dartboard. The idea of spreading the numbers evenly around the dartboard is already discussed in criterion III. Another way of looking at the spreading is by interchanging odd and even sections, so that when throwing the last double, and not just missing, but missing by a full dartboard section, there is a serious penalty. In practice this happens quite a few times during the top male tournaments.

Definition 5.1

If D is a dartboard, then $EO(D)$ is defined as the number of times even and odd numbers are not interchanged on the dartboard.

So

$$EO(D) = \#\{i{:}\, D_i \text{ and } D_{i+1} \text{ have the same parity}\}.$$

A dartboard D is finish-optimal if $EO(D) = 0$.

Figure 18 on the inlay paper or Figure 26, is the Even-odd dartboard, keeping in mind that the odd and even numbers should be interchanged. It is not possible to create a finish-optimal dartboard that is also Selkirk-I-optimal. This is because finish-optimal means that for all $i\ even{:}\, D_i\ even$ or $i\ even{:}\, D_i\ odd$ and Selkirk-I-optimal means that for all $i\ even{:}\, D_i \in L$ or $i\ even{:}\, D_i \in H$, (with $L = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $H = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$). Since L and H are not completely even and not completely odd there is no arrangement that is both finish-optimal and Selkirk-I-optimal. The even-odd dartboard is one of many dartboards with $EO(D) = 0$. The Even-odd dartboard is not Selkirk-II-optimal, but has the highest $S_2(D) = 2588$ value of all finish-optimal dartboards (this solution is found using a direct search [15]).

The Even-odd dartboard seems like an overall optimal dartboard. In all parts of the game the darter is challenged to a difficult game of darts. But maybe this is even more strict than necessary. As mentioned before dart players tend to get a nice finish, a finish with a small risk. These `safe' finishes are the ones that should definitely be closed in between two odd numbers, for the others it is less important.

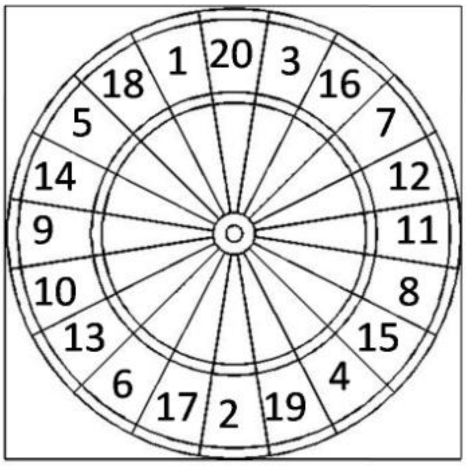

Figure 26: The even-odd optimized dartboard

There are in total

$$2 \cdot 10! \cdot 10! = 2.6336 \cdot 10^{13}$$

finish-optimal dartboard arrangements. With the first $10!$ is the number of possibilities to spread the even numbers, the second $10!$ is the number of possibilities to spread the odd numbers and $2 \cdot$ because we can start with the even or with the odd number. This is significantly less than the original

$$20! = 2.4329 \cdot 10^{18}$$

possible ways to arrange the numbers on the dartboard. There is the same amount of Selkirk-I-optimal dartboards, as finish-optimal dartboards. But there is no dartboard that is both finish-optimal and Selkirk-I-optimal.

Finally we will consider dartboards that are as close as possible to finish-optimal and Sekirk-I-optimal, we can do this if we accept two violations of the even-odd alternating rule or the high-low alternating rule. We will divide the numbers into four groups, $G_1 = Even \cap High$, $G_2 = Odd \cap Low$, $G_3 = Even \cap Low$, and $G_4 = Odd \cap High$. The sections D_1 up to D_{10} will be filled with groups G_1 and G_2 alternating, or groups G_3 and G_4 alternating, and the sections D_{11} up to D_{20} will be filled respectively with the groups G_3 and G_4 alternating, or groups G_1 and G_2 alternating. If we arrange the numbers in this manner, we will have two violations of the alternating rule between the neighbouring sections D_1 and D_{20}, and between D_{10} and D_{11}. When we allow at these two locations a violation of the even and odd, or the high and low alternating rule, then there are

$$2 \cdot 4 \cdot 5! \cdot 5! \cdot 5! \cdot 5! = 1658.9 \cdot 10^6$$

possible arrangements left. With $4 \cdot$ relating to the four groups we can use for the section D_1. And with the first $5! \cdot 5!$ relating to the number of possibilities to spread the sections D_1 up to D_{10}, and the second $5! \cdot 5!$ relating to the number of possibilities to spread the sections D_{11} up to D_{20}. We add $2 \cdot$ because we can have a violation of the even-odd alternating rule, or the high-low alternating rule.

In Table 8 the six dartboards are graded using criterion V. The Table shows the number of violations of the even-odd interchange rule. This number of violations is always even, since for every two even numbers next to each other two odd numbers have to be next to each other.

Table 8: Even-odd numbers for the six dartboards

	Number of times even and odd are not interchanged
London	6
Squared	18
Squared2	2
Eiselt-Laporte	10
Even-odd	0
Selkirk	10

Criterion VI: Horizontal and vertical talents

This criterion is inspired by the density approach to a dartboard by Deane [7] in combination with the standard deviation variations of darters used in the Dutch finale of the Alympiade 2006 [8]. In this last criterion the horizontal and vertical deviation is separately taken into account. On the London dartboard the high numbers are at the top and at the bottom, which means that people with good horizontal precision and bad vertical precision have an advantage over people with bad horizontal precision but good vertical precision. That is not ideal. By modelling a dartboard and simulating expectations of horizontally skilled and vertically skilled dart players, we want to see what kind of advantages or disadvantages they have on the different dartboards.

We will represent the darters and its deviation by two independent stochastic variables, X_H for the horizontal deviation and X_V for the vertical deviation. We will represent a horizontally talented darter with a small horizontal and a big vertical deviation $\sigma_H = 10$ and $\sigma_V = 20$ and a vertically talented darter with $\sigma_H = 20$ and $\sigma_V = 10$.

The dartboard is modelled on a square grid, which is divided up into 341 by 341 grid cells, where each grid cell has 1 mm spacing in both directions.

We want to find the expected return for each grid cell on the dartboard. We determine for each grid cell (i, j) the expected return E_{ij}, by modelling each dart that is thrown at that grid cell (i, j) as $(i + X_V, j + X_H)$, with $X_V \sim \mathcal{N}(0, \sigma_V)$ and $X_H \sim \mathcal{N}(0, \sigma_H)$.

In the criterion the number of grid cells with a high expected return for horizontally and vertically skilled darters are an indication of the quality of a dartboard; a similar number of grid cells corresponds to a good dartboard. Multiples due to mirroring and rotations influence this criterion; this was not the case in the previous criteria. Note that for all the dartboards on the inlay paper the top section is number 20. This top section is an easier aim for horizontally talented darters, when comparing to the vertically talented darters, because of the shape of the section. In the same way all sections on the top and bottom of the dartboard are easier aims for the horizontally talented darter, and all sections on the left and right are easier aims for the vertically talented darter.

Definition 6.1

If D is a dartboard, divided up into i x j cells of size 1mm x 1mm, then for a horizontally talented darter, with $\sigma_H = 10$ and $\sigma_V = 20$ we get:

$$A_H(D) = \#cells\ with\ E_{ij} \geq 15$$

and for a vertically talented darter, with $\sigma_H = 20$ and $\sigma_V = 10$ we get:

$$A_V(D) = \#cells\ with\ E_{ij} \geq 15.$$

A dartboard D is talent-optimal if and only if $S_6(D) = |A_H(D) - A_V(D)|$ is minimal.

Using this criterion we did not investigate what is the best dartboard. We do not know which value for $S_6(D)$ is the minimal value, thus we do not state which dartboard is talent-optimal. We will use this criterion to grade the quality of the different dartboards with respect to each other, by comparing the different $S_6(D)$ values, the lower the value the better the dartboard is.

***Definition 6.2**:*

We call a dartboard D talent-better than dartboard D' if and only if

$$S_6(D) < S_6(D')$$

We will estimate the E_{ij}, this is needed to be able to determine the size of $A_H(D)$ and $A_V(D)$. This is done by simulating many darts being thrown at the dartboard with the different horizontal and vertical deviations, and averaging the return. The modelling and simulating is done with Matlab, the code is shown in appendix A and B. For every grid cell the expected return is estimated by throwing 500 darts and averaging the return. For the plotting we used a smoothing operating to get a more readable image. In the Figures 27 - 32 the sigma in horizontal and vertical direction are the same $\sigma = 15$. For the Figure 33 – 38 $\sigma_H = 10$ and $\sigma_V = 20$, this represents the dart player who is more horizontally talented. And in the Figure 39 – 44 the vertically talented dart players expectations are shown, with $\sigma_H = 20$ and $\sigma_V = 10$. In all the Figures the contour lines are at expected values [0 2.5 5 7.5 10 12.5 15 17.5 20] and the colour scale is the same. So the Figures can be compared to each other, without minding scale.

The fact that horizontal and vertical deviation should be punished equally hard is used when we look at the number of grid cells with an expectation bigger or equal to 15. In Table 9 the size of the area in mm^2 is given for which the expected return in higher than 15, for the symmetrical, horizontal and vertical skilled player. The column on the right shows the difference in area size between the horizontally and vertically skilled, the smaller this number the more fair this dartboard is.

Table 9: For each of the dartboards the size of the area where the expected return for aiming at that point is higher than 15, for the symmetrical, horizontally and vertically talented, and the difference between horizontal and vertical talent (the smaller this number, the more equal the challenge)

	$A_{symmetrical}(D)$ σ_H=15 σ_V=15 [mm^2]	$A_H(D)$ σ_H =10 σ_V =20 [mm^2]	$A_V(D)$ σ_H =20 σ_V =10 [mm^2]	$S_6(D) = \lvert A_H(D) - A_V(D)\rvert$ [mm^2]
London	15106	16091	13770	2321
Squared	13427	12961	14819	1858
Squared2	19991	18929	20560	1631
Eiselt-Laporte	15408	15265	15031	234
Even-odd	14102	16469	11367	5102
Selkirk	16180	15904	15839	65

A low number in the right column of Table 9 means that the amount of red on the Figures 33 – 38 should be just as large as the amount of red on the Figures 39 – 44, because we chose the contour lines the same. There is a clearly visual advantage for horizontally talented darters when we look at red areas in the figures on the next pages. This advantage is less clear in Table 9 were we look at an expected return of 15 and higher, this 15 or higher in the figures on the next pages is represented by the colours light orange, dark orange and red. Perhaps looking only at $E_i \geq 17.5$ or $E_i \geq 20$ would have been better for the comparison of the horizontally and vertically talented darters.

We have to keep remembering that it is natural that horizontal darters have some advantage playing the dartboards on the inlay paper because the number 20 is located in a for them preferred location. But the locations of the other high numbers {19 18 17} are in most cases also easier to hit with horizontal talent, than with a vertical talent. Only in the Selkirk solution there is not that much of a difference between the two. This is logical, since the Selkirk dartboard is designed to have an even spread of high numbers all round the dartboard. And in a lesser extend this is true for the Eiselt-Laporte dartboard. In conclusion we say that the Selkirk and Eiselt-Laporte dartboard give a talent-better result than the others.

Symmetrical skilled dart players; $\sigma_H=\sigma_V=15$ [mm^2]

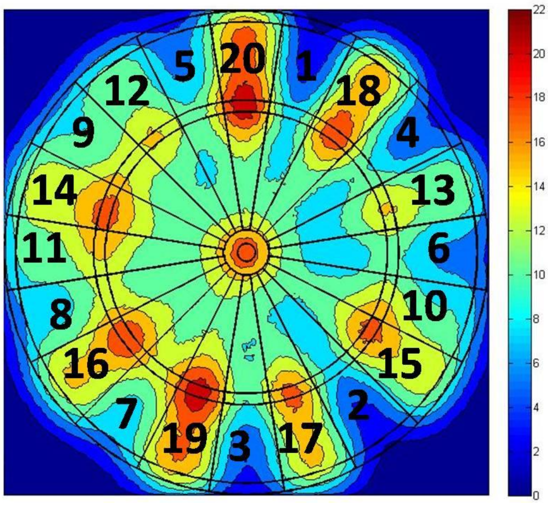

Figure 27: Expected return on the London board for symmetrical skilled players

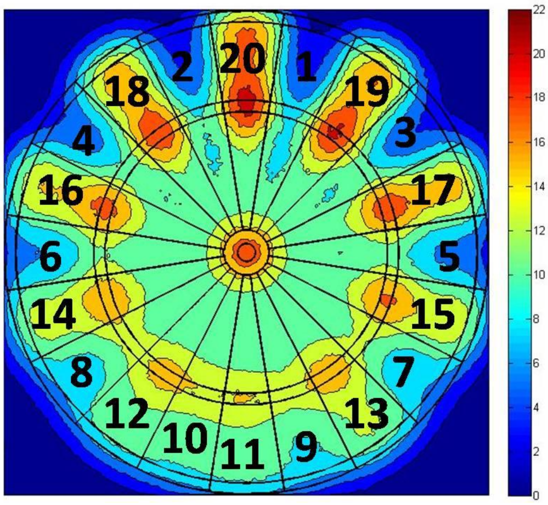

Figure 28: Expected return on the Squared board for symmetrical skilled players

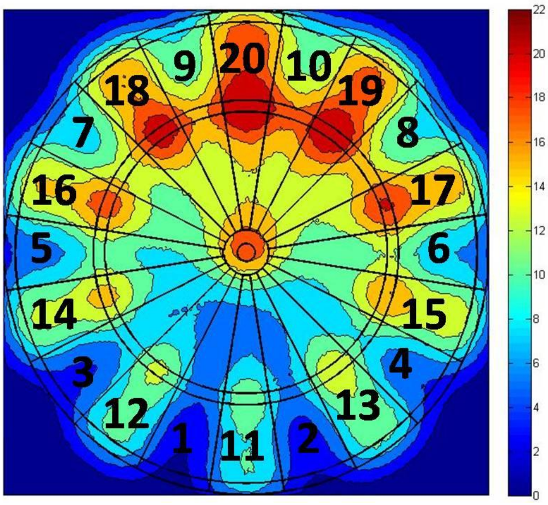

Figure 29: Expected return on the Squared2 board for symmetrical skilled players

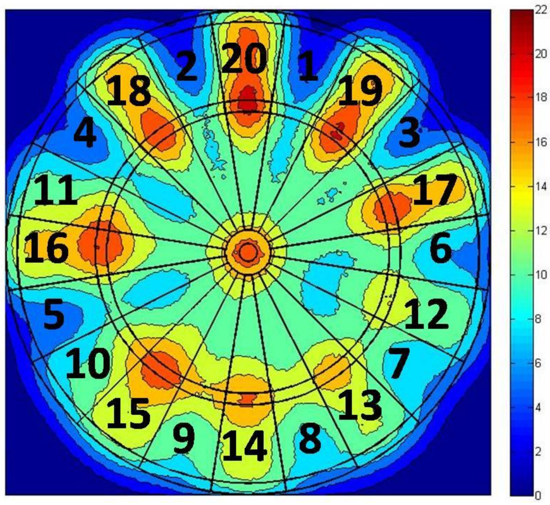

Figure 30: Expected return on the Eiselt-Laporte board for symmetrical skilled players

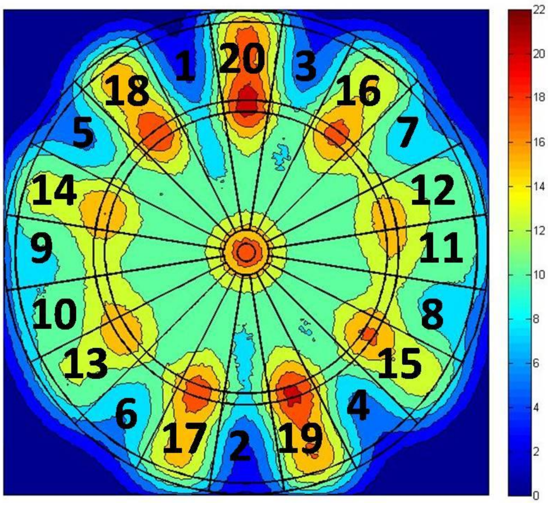

Figure 31: Expected return on the Even-odd board for symmetrical skilled players

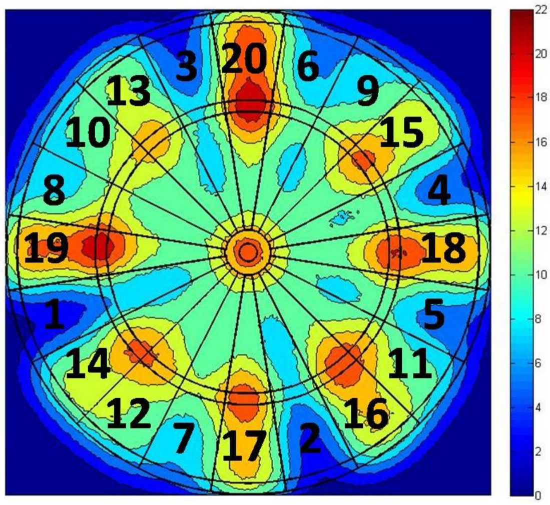

Figure 32: Expected return on the Selkirk board for symmetrical skilled players

Horizontally skilled dart players; σ_H =10 and σ_V =20 [mm^2]

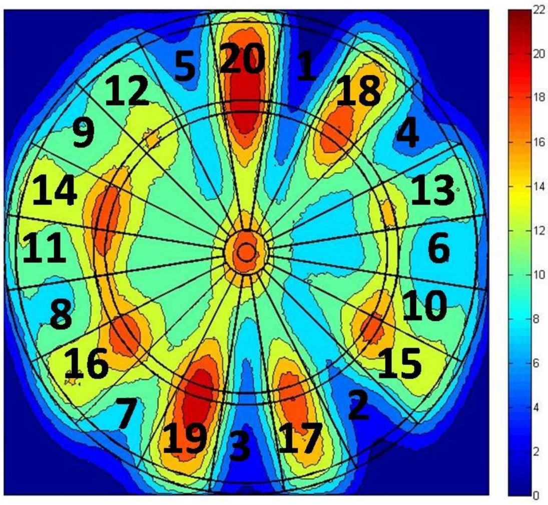

Figure 33: Expected return on the London board for horizontally skilled players

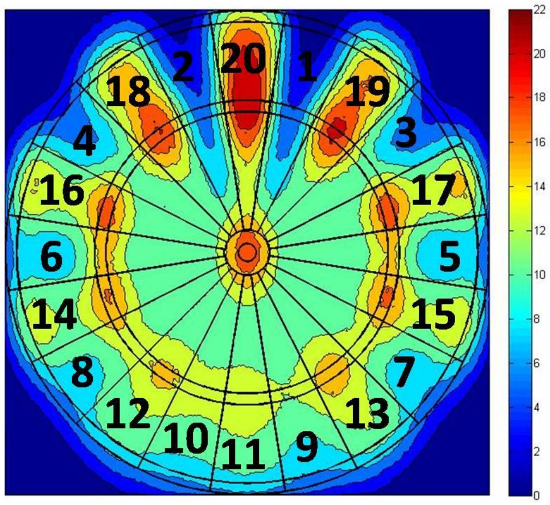

Figure 34: Expected return on the Squared board for horizontally skilled players

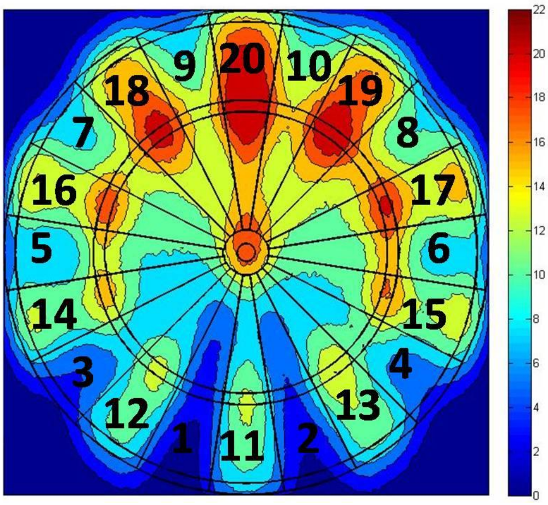

Figure 35: Expected return on the Squared2 for horizontally skilled players

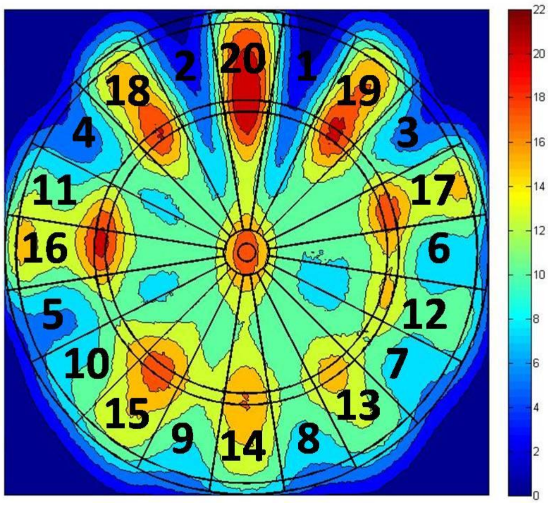

Figure 36: Expected return on the Eiselt-Laporte for horizontally skilled players

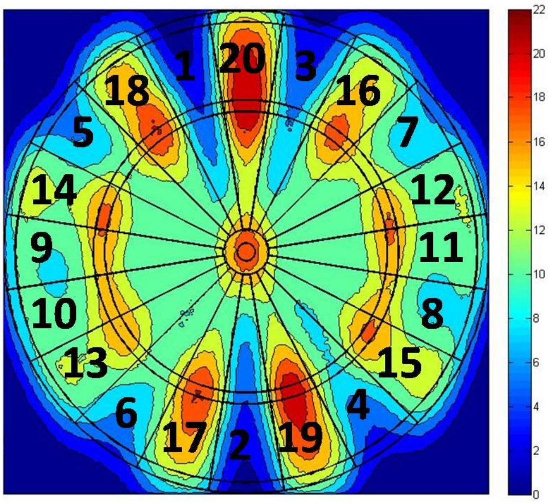

Figure 37: Expected return on the Even-odd for horizontally skilled players

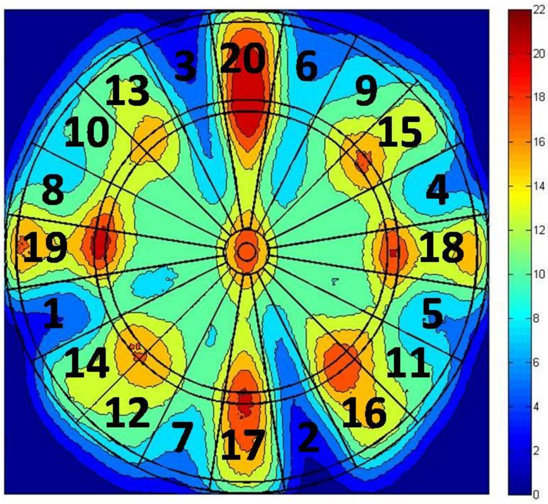

Figure 38: Expected return on the Selkirk for horizontally skilled players

Vertically skilled dart players; σ_H =20 and σ_V =10 [mm^2]

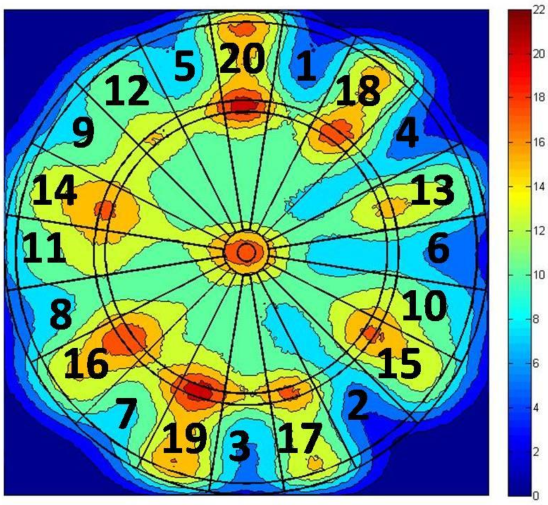

Figure 39: Expected return on the London for vertically skilled players

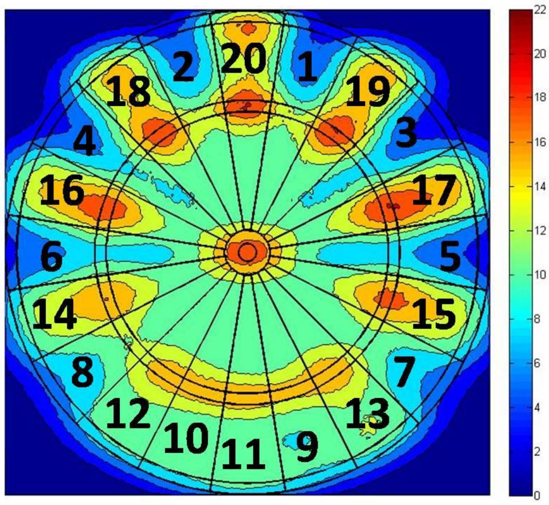

Figure 40: Expected return on the Squared for vertically skilled players

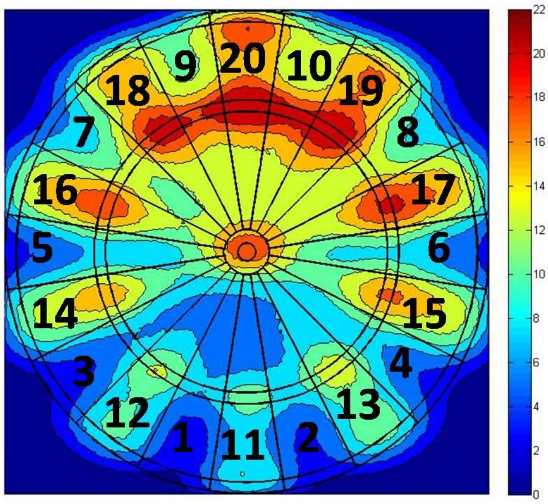

Figure 41: Expected return on the Squared2 for vertically skilled players

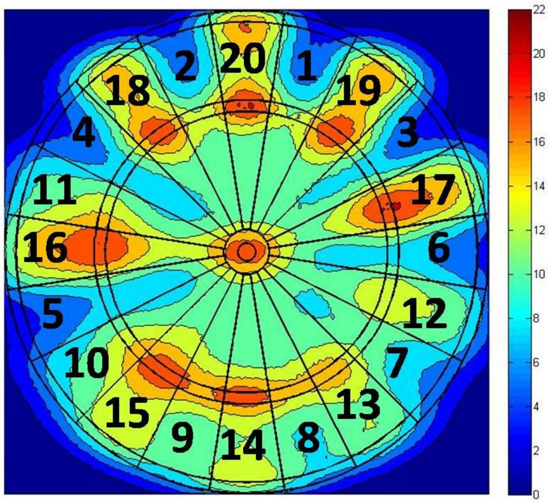

Figure 42: Expected return on the Eiselt-Laporte for vertically skilled players

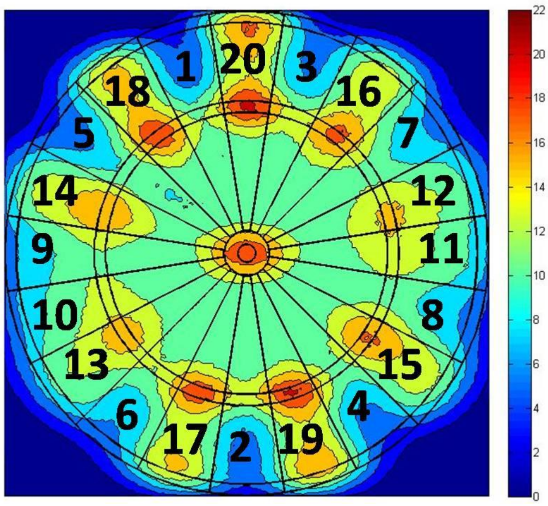

Figure 43: Expected return on the Even-odd for vertically skilled players

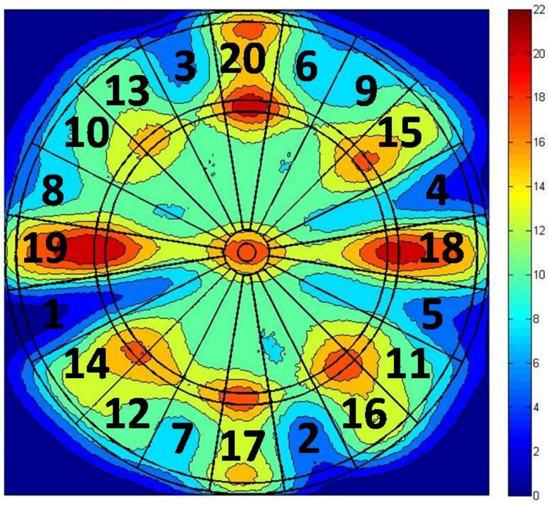

Figure 44: Expected return on the Selkirk for vertically skilled players

Simulated annealing:

Up to this point we have limited ourselves to only a few dartboards and tested our criteria only on these boards. We did this because there are $19!/2 = 6.0823 \cdot 10^{16}$ different number combinations possible excluding multiples due to mirroring and rotations. Finding the best dartboard for some of the criteria would simply take too long. In this chapter we introduce the method of simulated annealing (SA), which makes it possible to structurally consider more different number arrangements on the dartboard.

The original use of the simulated annealing technique is minimization of energy in the cooling of liquefied materials. When you cool a material slowly and controlled you can control the crystallisation of that material. To make sure you do not get stuck in a local minimum sometimes you need to increase the temperature, and start cooling again. In mathematics this technique has proven to be useful in solving travelling salesman problems by S. Kirkpatrick, C.D. Gelatta and Jr. M.P. Vecchi [15] and Cemy [4].

We applied simulated annealing using the criterion of the centre of gravity method, as described on page 19. In this criterion we defined a dartboard as optimal, when the vector combination of all twenty numbers adds up to a small vector. So we minimize the distance between the centre of the dartboard and the centre of gravity of the dartboard. We have called this distance

$$G(D) = \|V(D)\|.$$

Selkirk [16] found by cleverly spreading out the numbers on the dartboard an arrangement with $G(D) = \sim 10^{-15}$.

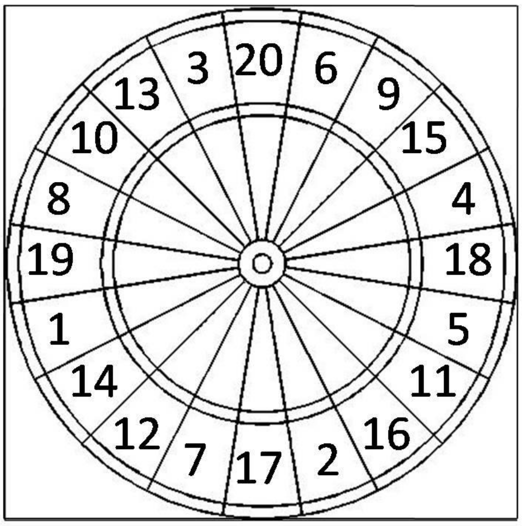

Figure 45: The Selkirk dartboard

The Matlab code we used is shown in Appendices E, F and G, but we will explain here the general steps of the algorithm. We start by explaining the original algorithm, and later we will explain the changes made for a dartboard application. Simulated annealing is an iterative process; we start by taking a possible solution as the current solution. We compare the current solution with a randomly chosen nearby solution. If this nearby solution is better than the current one, we accept it as the new current solution. If not, then depending on how close to the best solution it is, how much worse the solution is, and on chance, we still accept the solution. We choose the nearby solution by picking two random numbers on the dartboard and interchanging them. Just as suggested in the original algorithm, we start by interchanging more than one pair of numbers, and decrease the number of pairs we switch when getting closer to G-value of 0.

The decision on whether or not to accept a worse solution, as said before depends on how close to the zero vector solution we are, how much worse it is, and on chance. Worse solutions are allowed using the test:

$$R(0,1) < e^{-\Delta D/T},$$

where $\Delta D = G(D)_{new} - G(D)_{old}$ is the change in the $G(D)$ value, T is the `synthetic temperature' in Kelvin. For the synthetic temperature we take $T = \alpha \cdot G(D)$, where $\alpha > 0$ is a scaling factor used to determine how strict we want the algorithm to work. And $R(0,1)$ is a random number in the interval [0,1]. This test insures that better solutions (where $1 < e^{-\Delta D/T}$) are always accepted, and worse solutions (where $0 < e^{-\Delta D/T} < 1$) are in some cases accepted.

In the test phase we tried out many variations of the algorithm. We found out that a big α did not give good results; the algorithm did not tend to converge. A small α also gave bad results; the algorithm did not have the strength to get out of a local minimum. We decided to change the algorithm slightly keeping the general workings of the algorithm the same, and only change the test function that determines whether or not to accept a worse solution. We always keep accepting a better arrangement, and accept a worse arrangement if and only if we cannot find a better solution and it gives a less than 10 % setback in the $G(D)$ value. We are in a local minimum if we cannot find a better arrangement in 25 attempts, with an attempt being each time we investigate a neighbouring arrangement. Then we try to get out of the local minimum by accepting a setback of at most 10 % in the $G(D)$ value. After failing 50 attempts to try to get out of the local minimum while accepting a small setback, we can officially say we are stuck in the local minimum. This is the moment we accept a big setback, to start the algorithm all over again. This change in the algorithm makes it possible that one of the local minima was the global minimum; therefore in the end we look at the minimum of all the local minima.

The reason for changing the well proven simulated annealing algorithm is that we found it hard to find an α wherefore the algorithm converged to local minima, and at the same time did not get stuck in these local minima. For example the local minima at $G(D) = 0.005$, with number arrangement [4 5 2 18 14 12 19 10 13 6 3 1 9 20 15 17 7 11 8 16], could not be improved by interchanging one pair of numbers on the dartboard. In fact the smallest setback from this number arrangement is a factor 60 times worse and the biggest setback is a factor 6909 times worse. On average interchanging two numbers sets $G(D)$ back by a factor 1793 times the old $G(D)$. We found after roughly thirty experiments containing millions of iterations, that an α that makes such a setback possible would give local minima at $G(D) \approx 25$, which is not good enough. Thus for large α we do not get out of (local) minima, and for small α we do not reach any (local) minima.

A physical explanation for the need to change the simulated annealing algorithm is that our local minima are very deep and close to the zero vector solution.

Here we summarize the algorithm with the changes we made. We took as an initial arrangement [1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20] and calculated the distance $G(D)$. We chose a nearby solution by picking two random numbers on the dartboard and interchanging them. Just as suggested in the original algorithm, we start by interchanging more than one pair of numbers, and decrease the number of pairs we switch when getting closer to $G(D)$ = 0. We chose to implement this by always first trying to improve the arrangement by interchanging three pairs, if we do not find an improvement we try to determine if we are in a local minimum. We do this first by trying 25 times to interchange 3 pairs of numbers and finding an improvement, if we still have not found this we try 25 times by interchanging two pairs, and if we still have not found a better solution we try with another 25 attempts with interchanging one pair. If after all these attempts, we do not improve the dartboard

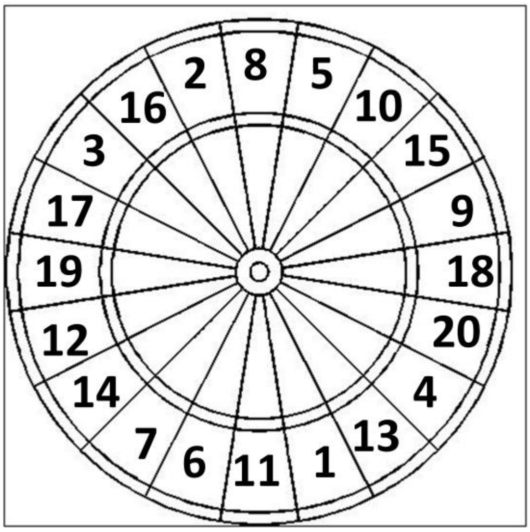

Figure 46: The best dartboard after optimizing with the simulated annealing algorithm

arrangement we state that we are stuck in a local minimum. And at this point we start to accept a 10 % worse arrangement.

In the end we want to find the $G(D)$ closest to zero, without having to check all $19!/2 = 6.0823 \cdot 10^{16}$ possible dartboard arrangements. The value zero was not found with the simulated annealing algorithm. After iterating well over 200 million times we found thousands of solutions with a $G(D) < 0.01$, but only a hand full with $G(D) < 0.0001$. The best result found by the algorithm is shown in Figure 46, which has $G(D) = 2.2 \cdot 10^{-14}$, and has its centre of gravity at x-coordinate $-1.9 \cdot 10^{-14}$ and y-coordinate $1.1 \cdot 10^{-14}$. When taking into account the rounding off to 15 digits in Matlab this solution could be even better than Selkirks dartboard. We can state both are comparable when looking at the centre of gravity criteria. When looking at the other criteria, for example the quadrant method, which also tries to achieve an equal spread of the numbers, the SA dartboard is one of the worst. Also criteria I, II and III give far worse results than any of the other dartboards. Therefore we do not use this arrangement in the comparison in the final conclusion.

For comparison the values of the other dartboards are shown in Table 10.

Table 10: Centre of gravity of the six dartboards and the with simulated annealing optimized dartboard

	x-coordinate	y-coordinate	Vector length
London	-10.17	-2.34	10.44
Squared	-0.16	0.03	0.16
Squared2	6.31	39.86	40.36
Eiselt-Laporte	-2.08	-2.51	3.26
Even-odd	0.32	0.05	0.32
Selkirk	$2.1 \cdot 10^{-15}$	$7.1 \cdot 10^{-15}$	$7.4 \cdot 10^{-15}$
Simulated annealing	$-1.9 \cdot 10^{-14}$	$1.1 \cdot 10^{-14}$	$2.2 \cdot 10^{-14}$

A suggestion for future work would be to try the simulated annealing algorithm on the other criteria, preferably criteria that do not mind the 15 digits round off of Matlab. Also the method for choosing a nearby solution would be worth an investigation. There are many other possible methods for choosing a nearby solution, such as only interchanging two neighbours.

After the completion of this thesis, calculations have also been made with Maple. From these calculations it follows that for the Selkirk dartboard D and also the one found with the simulated annealing algorithm, actually have $\|V(D)\| = 0$. So both these dartboards are gravity optimal. This shows that our algorithm produces good results, as we have actually found a new gravity optimal dartboard. This suggests some questions for future work. For example, it is interesting to find out whether we can prove that the Selkirk dartboard is gravity optimal without using a CAS like Maple or Matlab. Also, it would be interesting to find other gravity optimal dartboards and also to find a characterization of such dartboards.

Conclusion:

The six criteria discussed in the previous chapters have some overlap in finding the best dartboard. Even though they are based on quite different ideas and they use quite different mathematical methods (from algebra, vector analysis, graph theory, numerics up to statistics). The different methods don’t give a consistent solution to the problem of finding the optimal dartboard. In the Table below the six dartboards which are referred to throughout the whole report are shown together with a number of extra dartboards (Appendix C shows the design of these extra dartboards). The Table shows the six criteria with sub-criteria all together. The dark gray fields are the ones with the best results; the light gray fields are also quite good; the white fields are bad.

Criterion I is where the sum of the differences between neighbouring numbers on the dartboard are added up to get a number which is as high as possible. Criterion II, where similar to criterion I the sum of the differences, but this time squared, get summed up to an as high as possible number. In Criterion III several methods to evaluate a nice spreading of the numbers over the dartboard were used. In Criterion IV the possibility of an alternative strategy is seen as a disadvantage of a dartboard. Criterion V tries to make sure that even numbers are not neighbouring, and the same for the odd numbers. And Criterion VI is the final criterion in the Table, where the spreading of the numbers is being investigated as in Criterion III, but then by looking at the horizontal versus the vertical deviation of the dart player.

Table 11: Combining all the different criteria, the dark gray areas are conforming the theory, the lighter gray have values not for off the criteria.

	Criteria I (sum neighbours)	**Criteria II** (sum^2 neighbours)	**Criteria III** (left- right-handed)				**Criteria IV** (alt. strategy)	**Criteria V** (even-odd)	**Criteria VI** (Horizontal-vertical)
			max. diff. quadrant 5	max. diff. section of 3	vector mass centre	graph theory			
London	198	2478	20	20	10,44	3	66,7	14	2321
Squared	200	2642	15	17	0,16	1	50	2	1858
Squared2	200	2018	47	35	40,36	9	50	18	1631
Eiselt-Laporte	198	2498	15	17	3,26	5	61,9	10	234
Even-odd	198	2588	11	15	0,32	1	50	20	5102
Selkirk	192	2354	3	11	$7{,}4 \cdot 10^{-15}$	2	45,5	10	65
Manchester	200	2526	16	19	3,88	1	60	14	255
Lynch	198	1980	35	31	10,12	9	45	20	1423
Simple	38	380	75	51	63,92	1	90	20	196
Selkirk2	200	2402	18	18	0,49	1	51,1	10	489
Eiselt-Laporte 10	198	2182	31	25	19,45	8	52,2	10	817
Eiselt-Laporte 20	198	2540	15	17	0,51	4	60	8	749

Because each of the methods is graded differently it is hard to choose which dartboard is the best. The now standard London dartboard is not good, but also not bad. The simple dartboard scores worst. And the Selkirk dartboard is `best' according to most criteria. Interesting is the different results for the spreading criterion, the first three give quite similar results, but the fourth one is quite different. The graph method gives almost the opposite result to the quadrant and vector method. With criteria III and VI being completely my own work I would have to weigh their results the heaviest. And therefore I conclude that the Selkirk dartboard is the best when trying to make the dartboard challenging, keeping in mind that the best man should win independent of the more horizontal or vertical dart talent of the player.

In addition we tried to find the `best' dartboard by simulated annealing using the centre of gravity method. Even though we found good results, it is hard to say how good the results really are, due to the rounding off in 15 digits by Matlab. It would be interesting to see what results simulated annealing would find for the other criteria.

Attachment A: Matlab code to calculate the expected value per grid cell

```
function [expectation] = numericaldartboard(xx,yy,sigma_x,sigma_y)
% we aim at grid point (xx,yy)
  % with a horizontal deviation sigma_x and a vertical deviation sigma_y

attempts_x = random('Normal',0,sigma_x,1,500);
  % normal distribution to determine the horizontal deviations from xx
attempts_y = random('Normal',0,sigma_y,1,500);
  % normal distribution to determine the vertical deviations from yy

  % we now know where each of the 500 darts landed, written in to vectors
  % next step is to determine in which section of the dartboard that is

m1=11; m2=14; m3=9; m4=12; m5=5; m6=20; m7=1; m8=18; m9=4; m10=13;
m11=6; m12=10; m13=15; m14=2; m15=17; m16=3; m17=19; m18=7; m19=16 ;m20=8;
  % the value for each sector, clockwise starting on the left

som=0;  % here the points of all 500 attempts are added up

for ii=1:500  % for each of the 500 attempts
    x = attempts_x(1,ii) + xx;
    y = attempts_y(1,ii) + yy;  % convert the deviations and the aim to ...
      % coordinates in a system where the bull's-eye has coordinates (0,0)
    [theta,r] = cart2pol(x,y);  % convert to polar coordinates

      % are we in one of the rings:
    n = 1;  % in the single rings
    if  (170 > r) && (r > 162), n = 2;  % in the double ring
    elseif  (107 > r) && (r > 99), n = 3;  % in the triple ring
    end

    if  (r > 170), % no points, the dart has missed the dartboard

    elseif (r < 6.35), som = som + 50;  % dart in bull's-eye
    elseif (r < 15.9), som = som + 25;  % dart in outer bull

      % we now look at each sector whether the dart has hit it,
      % and we add one, two or three times the points,
      % depending on single, double or triple ring
    elseif ((pi) >theta) && (theta >(pi*19/20)),          som = som + n*m1;
    elseif ((pi* 19/20) >theta) && (theta> (pi* 17/20)), som = som + n*m2;
    elseif ((pi* 17/20) >theta) && (theta> (pi* 15/20)), som = som + n*m3;
    elseif ((pi* 15/20) >theta) && (theta> (pi* 13/20)), som = som + n*m4;
    elseif ((pi* 13/20) >theta) && (theta> (pi* 11/20)), som = som + n*m5;
    elseif ((pi* 11/20) >theta) && (theta> (pi*  9/20)), som = som + n*m6;
    elseif ((pi*  9/20) >theta) && (theta> (pi*  7/20)), som = som + n*m7;
    elseif ((pi*  7/20) >theta) && (theta> (pi*  5/20)), som = som + n*m8;
    elseif ((pi*  5/20) >theta) && (theta> (pi*  3/20)), som = som + n*m9;
    elseif ((pi*  3/20) >theta) && (theta> (pi*  1/20)), som = som + n*m10;
    elseif ((pi*  1/20) >theta) && (theta> (pi* -1/20)), som = som + n*m11;
    elseif ((pi* -1/20) >theta) && (theta> (pi* -3/20)), som = som + n*m12;
    elseif ((pi* -3/20) >theta) && (theta> (pi* -5/20)), som = som + n*m13;
```

```
    elseif ((pi* -5/20) >theta) && (theta> (pi* -7/20)), som = som + n*m14;
    elseif ((pi* -7/20) >theta) && (theta> (pi* -9/20)), som = som + n*m15;
    elseif ((pi* -9/20) >theta) && (theta> (pi*-11/20)), som = som + n*m16;
    elseif ((pi*-11/20) >theta) && (theta> (pi*-13/20)), som = som + n*m17;
    elseif ((pi*-13/20) >theta) && (theta> (pi*-15/20)), som = som + n*m18;
    elseif ((pi*-15/20) >theta) && (theta> (pi*-17/20)), som = som + n*m19;
    elseif ((pi*-17/20) >theta) && (theta> (pi*-19/20)), som = som + n*m20;
    elseif ((pi*-19/20) >theta) && (theta> (-pi)),       som = som + n*m1;
    end
end

expectation = som/500;  % returned is an average result over 500 attempts
```

Attachment B: Matlab code to go through the grid and plot

```
% This is the main program for the Monte Carlo experiment. We divide the
% dartboard in a matrix of 1 mm x 1 mm cells. We aim 500 darts at each
% cell, and throw them with a random deviation. From these results we
% calculate an expected return when aiming for a cell with a certain
% deviation.

clear all
sigma_x = 20; sigma_y = 10;     % horizontal and vertical deviation [mm]
how_long = 116281;              % a processing countdown, one step per cell
expectations = zeros(341,341); % the still empty expectations matrix

for xx = -170:170  % for each grid point in horizontal direction
    for yy = -170:170  % for each grid point in vertical direction
        expectations(yy+171,xx+171) = ...
numericaldartboard(xx,yy,sigma_x,sigma_y);
          % calculate average result over 500 attempts (see Attachment A)
        how_long = how_long - 1
    end
end

% next step is to visualise the results
Figure  % create a new figure

F = [.05 .1 .05; .1 .4 .1; .05 .1 .05];
ZC = conv2(expectations,F,'same');  % to make a smooth image
ZC2 = conv2(ZC,F,'same');  % to make an even smoother image
[C,h]=contourf(ZC2);
set(h,'LevelList',[0 2.5 5 7.5 10 12.5 15 17.5 20])  % fixed contour lines
  % to make comparison between different dartboards easier
colorbar
caxis([0 22])
axis off

hold on  % we need to add the dartboard lines

  % the circles for the triple, double, bull and bull's-eye ring
rr=[170 162 107 99 15.9 6.35];
for k=1:1:6
    count = 1;
    for jj = 0:0.01*pi:2*pi
        p(count) = rr(k)* cos(jj) + 171;
        q(count) = rr(k)* sin(jj) + 171;
        count=count+1;
    end
    plot(p,q,'k', 'LineWidth',2)
    hold on
end

  % the lines for each of the sectors
rho=[pi*1/20 pi*3/20 pi*5/20 pi*7/20 pi*9/20 pi*11/20 pi*13/20 pi*15/20 ...
    pi*17/20 pi*19/20 pi*21/20 pi*23/20 pi*25/20 pi*27/20 pi*29/20 ...
    pi*31/20 pi*33/20 pi*35/20 pi*37/20 pi*39/20];
```

```
for kk = 1:1:20
    count = 1;
    for rrr = 15.9:0.1:170
        pp(count) = rrr * cos(rho(kk)) + 171;
        qq(count) = rrr * sin(rho(kk)) + 171;
        count = count + 1;
    end
    plot(pp,qq,'k','LineWidth',2)
    hold on
end
```

Attachment C: The six other dartboards used for comparison.

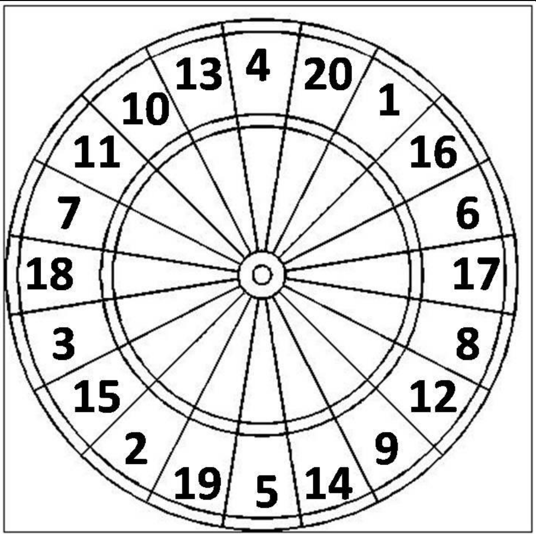

Figure 47: The Manchester dartboard

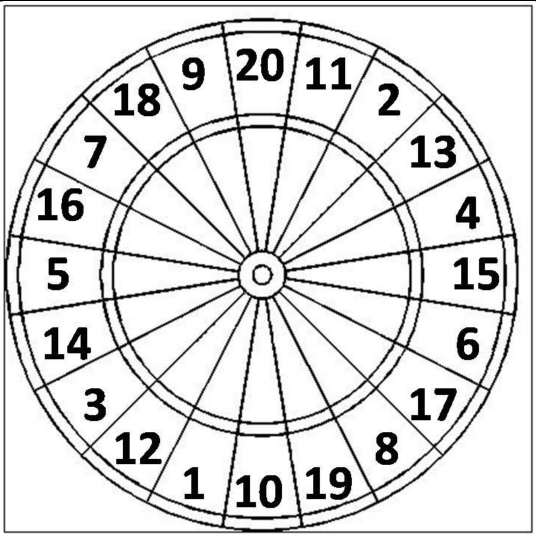

Figure 48: The Lynch dartboard

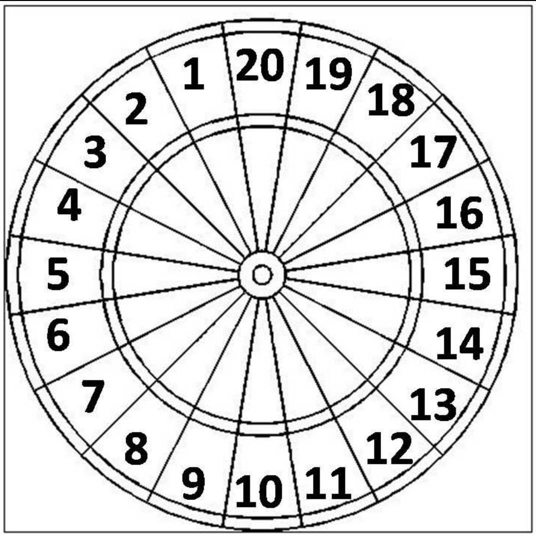

Figure 49: The Simple dartboard

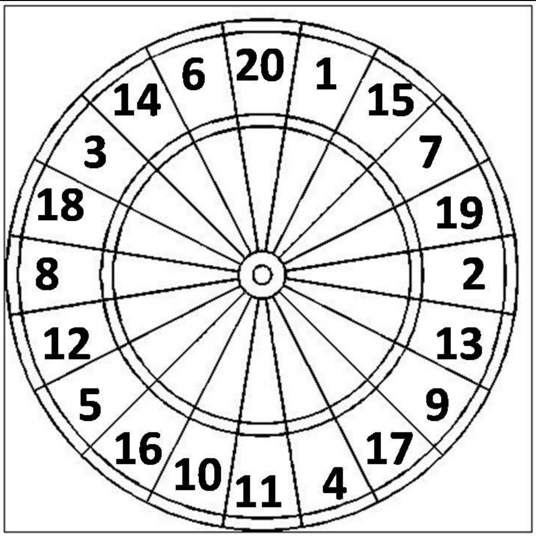

Figure 50: The Selkirk2 dartboard

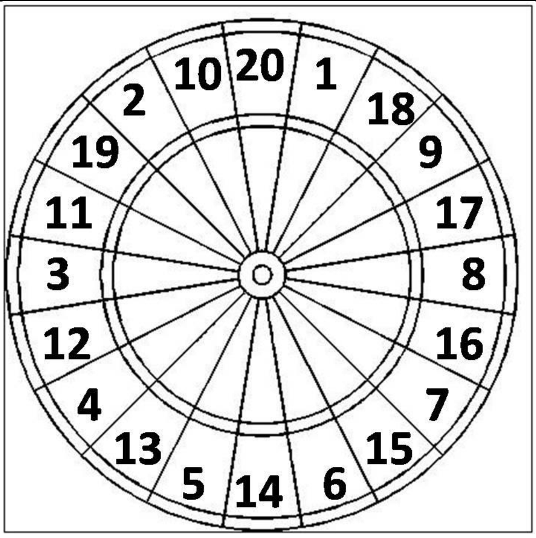

Figure 51: The Eiselt-Laporte 10 dartboard

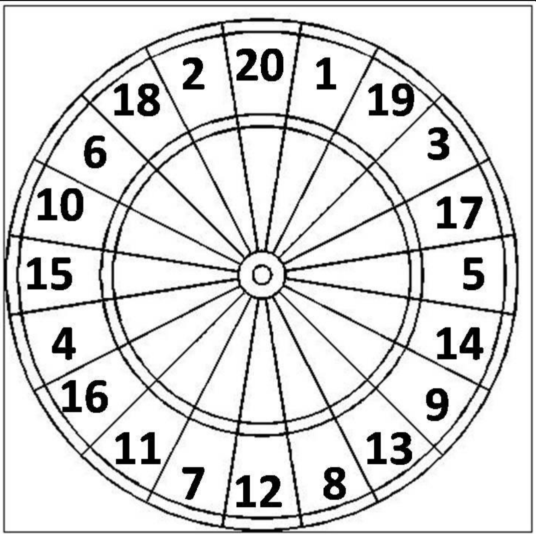

Figure 52: The Eiselt-Laporte 20 dartboard

Attachment D: Simulation results

Simulation results for decreasing precision or increasing sigma in the normal distributed probability function which describes the here displayed estimated values, calculate for the London dartboard.

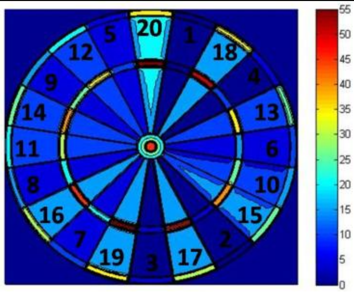
Figure 53: sigma = 1 mm

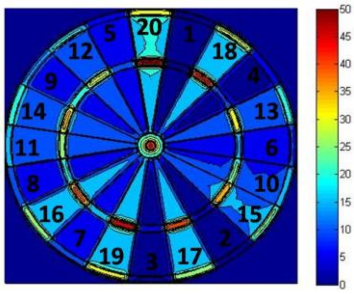
Figure 54: sigma = 3 mm

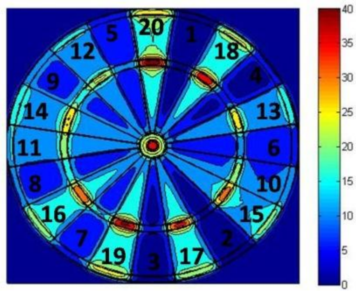
Figure 55: sigma = 5 mm

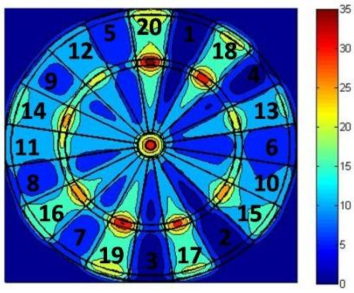
Figure 56: sigma = 7 mm

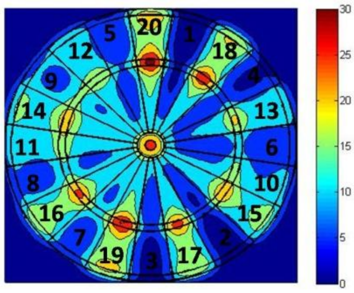
Figure 57: sigma = 9 mm

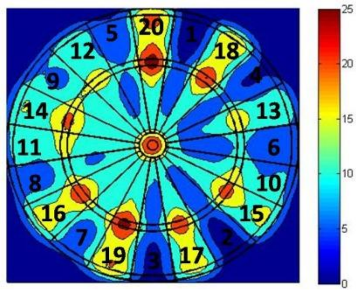
Figure 58: sigma = 11 mm

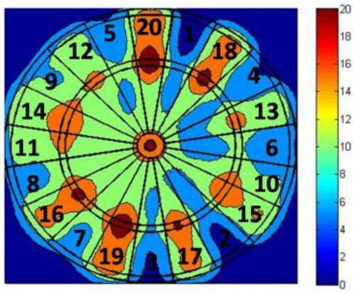
Figure 59: sigma = 13 mm

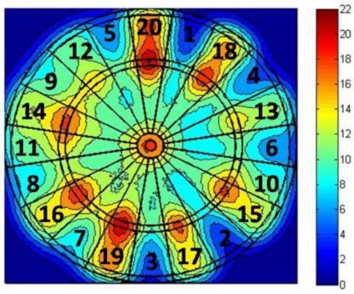
Figure 60: sigma = 15 mm

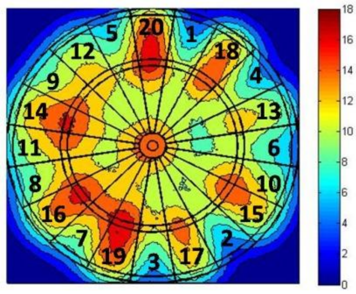
Figure 61: sigma = 20 mm

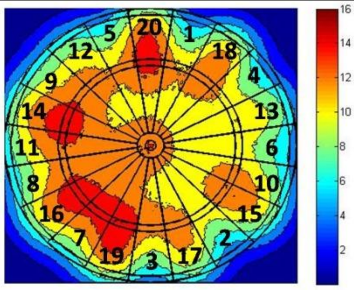
Figure 62: sigma = 25 mm

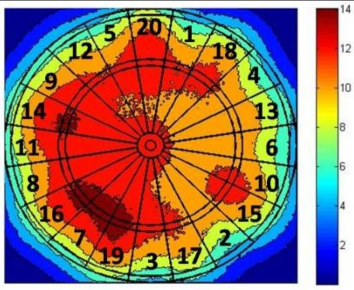
Figure 63: sigma = 30 mm

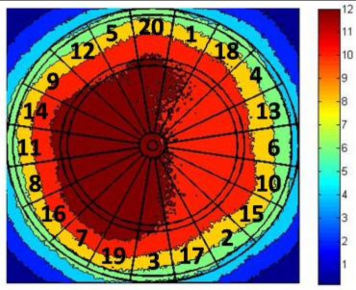
Figure 64: sigma = 40 mm

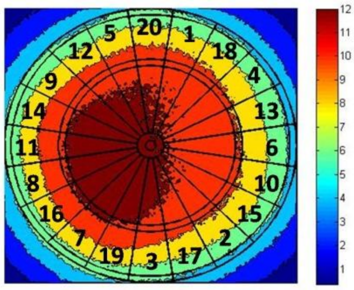
Figure 65: sigma = 50 mm

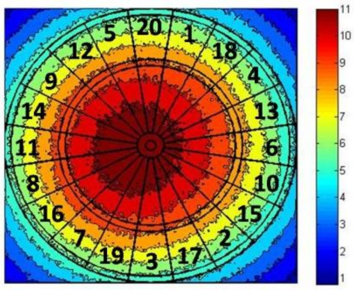
Figure 66: sigma = 75 mm

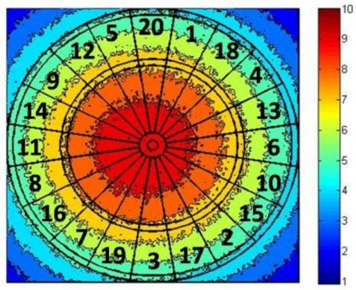
Figure 67: sigma = 99 mm

Attachment E: Matlab function to calculate centre of gravity

```
function g = gravity(inputarrangement)

% The input argument 'inputarrangement' is an array of size 20, with the
% numbers on the dartboard for each location, clockwise from starting at
% the top. Calculated is the distance of the centre of gravity of the
% dartboard to the actual centre of the dartboard. Returned is an array g
% with the horizontal, vertical and real distance.

% For each position on the dartboard the horizontal and vertical element is
% calculated in the cosine and sinus part of the equation, these are
% multiplied by the number of that position as taken from the input array.
% This way separately the horizontal and vertical distance is calculated.
horizontal_distance = cos(2*pi* 5/20) * inputarrangement( 1) ...
                    + cos(2*pi* 4/20) * inputarrangement( 2) ...
                    + cos(2*pi* 3/20) * inputarrangement( 3) ...
                    + cos(2*pi* 2/20) * inputarrangement( 4) ...
                    + cos(2*pi* 1/20) * inputarrangement( 5) ...
                    + cos(2*pi* 0/20) * inputarrangement( 6) ...
                    + cos(2*pi*19/20) * inputarrangement( 7) ...
                    + cos(2*pi*18/20) * inputarrangement( 8) ...
                    + cos(2*pi*17/20) * inputarrangement( 9) ...
                    + cos(2*pi*16/20) * inputarrangement(10) ...
                    + cos(2*pi*15/20) * inputarrangement(11) ...
                    + cos(2*pi*14/20) * inputarrangement(12) ...
                    + cos(2*pi*13/20) * inputarrangement(13) ...
                    + cos(2*pi*12/20) * inputarrangement(14) ...
                    + cos(2*pi*11/20) * inputarrangement(15) ...
                    + cos(2*pi*10/20) * inputarrangement(16) ...
                    + cos(2*pi* 9/20) * inputarrangement(17) ...
                    + cos(2*pi* 8/20) * inputarrangement(18) ...
                    + cos(2*pi* 7/20) * inputarrangement(19) ...
                    + cos(2*pi* 6/20) * inputarrangement(20);

vertical_distance = sin(2*pi* 5/20) * inputarrangement( 1) ...
                  + sin(2*pi* 4/20) * inputarrangement( 2) ...
                  + sin(2*pi* 3/20) * inputarrangement( 3) ...
                  + sin(2*pi* 2/20) * inputarrangement( 4) ...
                  + sin(2*pi* 1/20) * inputarrangement( 5) ...
                  + sin(2*pi* 0/20) * inputarrangement( 6) ...
                  + sin(2*pi*19/20) * inputarrangement( 7) ...
                  + sin(2*pi*18/20) * inputarrangement( 8) ...
                  + sin(2*pi*17/20) * inputarrangement( 9) ...
                  + sin(2*pi*16/20) * inputarrangement(10) ...
                  + sin(2*pi*15/20) * inputarrangement(11) ...
                  + sin(2*pi*14/20) * inputarrangement(12) ...
                  + sin(2*pi*13/20) * inputarrangement(13) ...
                  + sin(2*pi*12/20) * inputarrangement(14) ...
                  + sin(2*pi*11/20) * inputarrangement(15) ...
                  + sin(2*pi*10/20) * inputarrangement(16) ...
                  + sin(2*pi* 9/20) * inputarrangement(17) ...
                  + sin(2*pi* 8/20) * inputarrangement(18) ...
                  + sin(2*pi* 7/20) * inputarrangement(19) ...
                  + sin(2*pi* 6/20) * inputarrangement(20);

% From the horizontal and vertical part the real distance is calculated.
real_distance = sqrt(horizontal_distance ^2 + vertical_distance ^2);

g = [horizontal_distance, vertical_distance, real_distance];
```

Attachment F: Matlab function simulated annealing

```
function s = swaparrangement(current_arrangement,n)

% The input argument 'current_arrangement' is an array of size n, with the
% numbers on the dartboard for each location, clockwise from starting at
% the top. The second input is n, the number of swaps. In this function n
% times two numbers on the dartboard will be interchanged. Returned is the
% new arrangement.

% Two random positions on the dartboard are taken, with the random integer
% function. The numbers on these positions are switched. This is done n
% times.
% We do not check whether the two random numbers are different, so in some
% cases twice the same random integer will be taken in nothing will change.

s = current_arrangement;

for i = 1 : n
  sector_1 = randi(20,1);
  sector_2 = randi(20,1);

  temp        = s(sector_1);
  s(sector_1) = s(sector_2);
  s(sector_2) = temp;
end
```

Attachment G: Matlab code simulated annealing

```
% This is the main program for simulated annealing adapted to the dartboard
% optimalisation. We try to optimize the dartboard by minimizing the
% distance between the centre of the dartboard and the centre of gravity of
% the dartboard. The idea is that we cannot test all possible dartboard
% arrangements; there are simply too many possibilities. With simulated
% annealing we test one arrangement, interchange two, or more numbers and
% see if it gives a better arrangement. If it does we accept it, if not we
% ignore it and try again by interchanging to improve the arrangement. This
% will give you a controlled path going down to a minimum distance. In case
% we get stuck in a local minima the algorithm accepts a worse result, and
% starts over again. We safe every local minimum to determine the real
% minimum.

clear all
threshold  = 100000000;   % number of attempts to find a better arrangement
count = 1                 % to see how far in the program we are
test_local_min = 25;
% to determine if we are stuck we attempt to get out this many times
times_out_local_min = 0;  % number of times we get stuck in a local minimum

current_arrangement = [1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20];
% the start arrangement
best_arrangement = current_arrangement;
% for now this is the best arrangement
current_gravity  = gravity(current_arrangement);
% using the function shown in attachment E we determine the distance
% between center of gravity and centre of dartboard.
best_gravity     = current_gravity;
% for now this is the best gravity corresponding to the ‘best_arrangement’

while count < threshold
% We attempt to improve the arrangement ‘threshold’ times.
% First we try to improve the arrangement by interchanging 3 pairs of
% numbers.  If after ‘test_local_min’ times we don’t find a better
% arrangement we try to improve the arrangement by interchanging 2 pairs of
% numbers. If after ‘test_local_min’ times we don’t find a better
% arrangement we try to improve the arrangement by interchanging 1 pair of
% numbers. If after ‘test_local_min’ times we don’t find a better
% arrangement we know we are stuck in a local minimum and we accept a
% lesser result and try again.
  attempts3 = 0; attempts2 = 0; attempts1 = 0;  % reset the attempts count

  while attempts3 < test_local_min  % first we try by interchanging 3 pairs
    temp_arrangement = swaparrangement(current_arrangement, 3);
    % using the function shown in attachment F we interchange 3 times two
    % numbers and get back an array with the interchanged arrangement
    temp_gravity     = gravity(temp_arrangement);
    % we calculate the distance to the centre of gravity

    if temp_gravity(3) < current_gravity(3)
    % if we have an improvement we accept the new arrangement
      current_arrangement = temp_arrangement;   % new arrangement
      current_gravity     = temp_gravity;       % new gravity

      if temp_gravity(3) < best_gravity(3)
      % if the new arrangement is an improvement of the overall best
      % arrangement we take it as the new best arrangement
        best_arrangement = temp_arrangement;    % new best arrangement
        best_gravity     = temp_gravity;        % new best gravity
```

```
      end
      attempts3 = 100;
      % to get out of the while loop now that we have an improvement
    end
    attempts3 = attempts3 + 1;  % increase the number of attempts to get
    % out of the local minimum now that we do not have an improvement
    count = count + 1  % increase the general count of attempts
  end

  if attempts3 == test_local_min  % now we try by interchanging 2 pairs
  % the rest of the algorithm stays the same
    while attempts2 < test_local_min
      temp_arrangement = swaparrangement(current_arrangement, 2);
      temp_gravity     = gravity(temp_arrangement);
      if temp_gravity(3) < current_gravity(3)
        current_arrangement = temp_arrangement;
        current_gravity     = temp_gravity;
        if temp_gravity(3) < best_gravity(3)
          best_arrangement = temp_arrangement;
          best_gravity     = temp_gravity;
        end
        attempts2 = 100;
      end
      attempts2 = attempts2 + 1;
      count = count + 1
    end
  end

  if attempts2 == test_local_min  % now we try by interchanging 1 pair
  % the rest of the algorithm stays the same
    while attempts1 < test_local_min
      temp_arrangement = swaparrangement(current_arrangement, 1);
      temp_gravity     = gravity(temp_arrangement);
      if temp_gravity(3) < current_gravity(3)
        current_arrangement = temp_arrangement;
        current_gravity     = temp_gravity;
        if temp_gravity(3) < best_gravity(3)
          best_arrangement = temp_arrangement;
          best_gravity     = temp_gravity;
        end
        attempts1 = 100;
      end
      attempts1 = attempts1 + 1;
      count = count + 1
    end
  end

  if attempts1 == test_local_min
  % we have now tried three times 'test_local_min' times by interchanging
  % 3, 2 and 1 pair(s) of numbers in the arrangement to improve the
  % distance to the gravity point. And each time it did not improve. We can
  % now truly say we are stuck in a local minimum. And therefore now accept
  % our last attempt to improve and start all over again.
    current_arrangement = temp_arrangement;     % new arrangement
    current_gravity     = temp_gravity;         % new gravity
    times_out_local_min = times_out_local_min + 1;  % increase the count of
    % the number of time we got stuck and out of a local minimum.
  end
end
```

Figure list:

Table list:

Bibliography:

[1] P.J. Everson and A.P. Bassom. (1995). Optimal arrangements for a dartboard. *applied probability trust*.

[2] Aristidis, Argyri Eleni and Vlachos. (2009). A meta-heruistic algorithm is used for dartboard design. *International journal of management science and engineering management*, vol. 4, no. 3, pp 198-204.

[3] Brown, K. S. (posted February 1996). The dartboard sequence. *downloaded from the website* http://www.mathpages.com/home/kmath025.htm.

[4] Cerny, V. (1985, January 1). Thermodynamical approach to the traveling salesman problem: An efficient simulation algorithm. *Journal of optimization theory and applications*, 45(1), 41-51.

[5] Chaplin, P. (2007). *The official website of 'doctor darts' author and darts historian*. Retrieved April 30, 2010, from Patrick Chaplin PhD: http://www.patrickchaplin.com/Number.htm.

[6] Curtis, S. A. (2004, October 16). Darts and hoopla board design. Information processing letters, 92(1), 53-56.

[7] Deane, G. T. (2002). *Graphical models for high dimensional density estimation.* Australian National University, Department of Mathematics.

[8] Freudenthal Instituut voor didactiek van Wiskunde en Natuurwetenschappen. (n.d.). *Let's play darts; finale opdracht voor de Alympiade 2006*. Garderen http://www.fi.uu.nl/alympiade/nl/opgaven/opgaven2005-2006/2005-2006finale.pdf.

[9] G.L. Cohen and E. Tonkes. (2001). Dartboard arrangements. *The electronic journal of combinatorics*, 8(2), 1-8.

[10] Gertz, S. J. (2010, March 19). *reader blogs*. Retrieved April 30, 2010, from Seattle Pi: http://blog.seattlepi.com/bookpatrol/archives/198649.asp

[11] H.A. Eiselt and Gilbert Laporte. (1991). A combinatorial optimization problem arising in dartboard design. *Journal Operational Research Society*, 42(2), 113-118.

[12] Kohler, D. (1982). Optimal strategies for the game of darts. *Journal of the Operational Research Society*, 33, 871-884.

[13] Lynch, M. A. (1997, February). Designing a dartboard - an application of graph modelling. *Teaching mathematics and its applications*, 16(2), 51-54.

[14] Peterson, I. (1997, May 19). *Ivars Peterson's MathTrek.* Retrieved Oct 2, 2010, from MAA (Mathematical Association of America): http://www.maa.org/mathland/mathland_5_19.html

[15] S.Kirkpatrick, C.D. Gelatta and Jr. M.P. Vecchi. (1983, May 13). Optimization by simulated annealing. *Science*, 220(4598), 671-680.

[16] Selkirk, K. (1976, October). Re-designing the dartboard. *The Mathematical Gazette*, 60(413), 171-178.

[17] Singmaster, D. (1980, April). Arranging a dartboard. *Bulletin of the institute of mathematics and its applications*, 16, 93-97.

[18] Trevor Lipscombe and Arturo Sangalli. (2000). The devil's dartboard. *Canadian mathematical society*, 215-217.

[19] Vladimir G. Deineko and Gerhard J. Woeginger. (1997). The algorithmics column: Some problems around travelling salesmen, dart boards, and Euro-coins. *Bulletin of the European association for theoretical computer science*, 60, 43-52.

[20] Ying-Jie Liao, Min-Zheng Shieh and Shi-Chun Tsai. (2007). Arranging numbers on circles to reach maximum total variations. *The electronic journal of combinatorics*, 14(47), 1-11.