

Angular acceleration measurement and estimation on helicopters

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Master of Science Thesis



Angular acceleration measurement and estimation on helicopters

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Abstract

Controllers currently used by aerospace vehicles such as helicopters and aircraft are able to reject almost completely the disturbances introduced by wind in their attitude and position. Once attenuated, wind gusts just create some random, low amplitude oscillations in the vehicle, whose more severe consequence in common flight scenarios is a reduction in crew and passengers comfort. However, for certain high-precision operations, these oscillations can put the mission at risk. This is the case for rescue helicopters, which need to hover over precise spots in the sea or mountains to allow personnel to assist injured people in difficult-access areas. In many occasions, the accuracy with which helicopters are able to hold a desired position in adverse weather conditions is not enough to safely perform a rescue mission.

The disturbance rejection performance of helicopters can be improved by providing their controllers with angular acceleration information of the vehicle. Although angular acceleration sensors are available on the market, their current cost and weight prevent their widespread use.

This master thesis covers the design and testing of a novel estimation method for angular acceleration in aerospace vehicles based on linear acceleration measurements. The proposed estimator is a Stationary Kalman Filter with Modified Input (MISKF), which outperforms current approaches found in industry in terms of accuracy and time delay, while being lightweight and Linear-Time-Invariant (LTI). This observer relies only on gyroscopes and linear accelerometers, which are usually present in most vehicles. In addition to this, it does not require the dynamic model of the system in which it is used, nor the accurate position of its Center of Gravity (C.G.). These advantages make the MISKF applicable in a broad range of aerospace vehicles, including helicopters. The proposed observer is compared to other angular estimation methods such as speed differentiators using real flight data, showing a faster convergence and lower noise amplification, as well as to other observers like an Extended Kalman Filter (EKF), leading to similar performance while being more compact and lightweight.

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Chapter 1

Introduction

This master's thesis is focused on an estimation problem. However, a brief overview of the control problem for which the estimations are required is now presented, since it will provide the necessary motivation for the desired characteristics of the estimations.

Nowadays, rotorcraft perform several in-flight operations that require high precision maneuvering close to obstacles. In many cases, due to bad weather conditions such as strong wind, heavy rain, or snow, the performance of the available controllers is insufficient to complete the demanded maneuvers and the operations have to be aborted. For instance, in rescue helicopters, a long rope is commonly used to reach the location of injured people. Stretchers can be attached to the rope to carry the person if necessary, who is then pulled up to the helicopter. While this operation is taking place, the helicopter must remain as steady as possible, compensating not only external disturbances such as wind, but also the changes in weight distribution due to the people hanging from the rope, the movement of operators inside the helicopter cabin, etc.

Wind gusts can induce oscillations in the cabin of the rotorcraft, and although the helicopter's autopilot is able to attenuate them, they can get highly amplified along the rope, leading to a significant displacement of its end tip, where the operators or injured people hang from. This situation poses a high risk, but could be mitigated by improving the disturbance rejection performance of the helicopter autopilot.

Another example is air-to-air refuelling of aircraft, where a boom is used to transport the fuel, which suffers similar effects than the rope of the previous case, and thus wind gusts cause many unsuccessful refuelling attempts.

Therefore, in order to meet the demands of manoeuvrability in high precision operations it is necessary to improve the disturbance rejection performance of the current autopilots. However, if the dynamic model used by the autopilot does not include wind gusts dynamics, this leads to a lack of information in the controller and thus reduces its performance when trying to reject the gusts. As a countermeasure, it has been proposed that, in addition to the angular velocity, linear acceleration and magnetic heading provided by the sensors, the autopilot also uses the angular acceleration of the helicopter. The angular acceleration is

proportional to the torques applied to the helicopter, including the rotor torque (which is known) and the external torques, mainly those created by wind gusts (which are not known). In this way, the angular acceleration can provide the controller with information of the wind gusts although they are not included in the dynamic model of the autopilot. However, it is not possible to obtain a direct measurement of the angular acceleration of helicopters because the high cost and weight of current angular acceleration sensors makes it unrealistic to incorporate them in a generalized way. For that reason, a master thesis has been launched to analyse the options to estimate the angular accelerations in helicopters.

The control problem for which the angular acceleration estimations are required has been presented. Based on this description one can determine several characteristics expected from the estimations. Since the main objective is to reject the effect of external disturbances faster than current controllers can do, the estimations should have a very low time delay. In the case of the rescue helicopters, this would allow to rapidly compensate the rotations that wind gusts induce in the helicopter cabin, before they can get amplified along the rope. Currently, controllers reject the induced rotations mainly using the angular velocity measurements provided by the gyroscopes. In order to reject the induced rotations faster, one possible option is to increase the gain applied to the derivative of the angular velocity, leading to a high reactive response.

However, there are some constraints which limit the weight that the derivative term can have in the controller action. The gyroscope measurements are always noisy to some extent, and the fact of differentiating them along with the noise, in addition to multiplying them by a high gain, can lead to severe overshoots in the helicopter response, unintentionally injecting the noise in the helicopter's attitude response, which would be perceivable to the passengers and strain the structural integrity of the fuselage. On the other hand, it is common practice to filter the angular velocity measurements to reduce their level of noise. After that, they can be differentiated and weighted much more before causing noise amplification. However, filtering a measurement takes time, and forces the controller to act later, i.e., it introduces time delay. This can become counterproductive for the task of rapidly rejecting disturbances, since the controller can respond in a reactive way (because it works with filtered measurements), but it responds later than with raw data. In general, it is necessary to reach a balance between a tolerable level of noise in the filtered measurements and an effective reactivity of the controller. This trade-off, along with several others, will be a key aspect along this master thesis.

This leads to a second requirement for the angular acceleration estimations. In addition to a low associated time delay, it is necessary that they are smoother than the signal currently achievable by filtering and differentiating the angular velocity measurements.

At this point, the expected characteristics of the angular acceleration estimations have been outlined. Before introducing the proposed estimation methods, a general background on the framework used in aerospace systems is presented. Chapter 2 covers the most relevant reference frames used to describe orientation of helicopters in flight, as well as the transformations between the frames. Moreover, an introduction to the generally available sensors and their working principles is presented. The chapter concludes with the introduction of the kinematic and dynamic equations of motion of helicopters when considered as a rigid body.

Chapter 3 describes the limitations of the angular acceleration estimation methods currently used, and analyses the advantages and constraints of using linear acceleration sensors for this goal. Moreover, it motivates the need for an observer and presents an overview of the

proposed observer design procedure. Chapters 4 and 5 address deeply the observer design and testing outlined in Chapter 3. Finally, Chapter 6 concludes the thesis with a summary of the most relevant aspects analysed during this research and the potential improvements that can be incorporated in the future.

Helicopter and sensors modelling

2-1 Reference frame and transformations

In flight dynamics, reference frames are used to express dynamic and kinematic variables (such as orientation, position, etc.) in different coordinate systems. The need for multiple reference frames arises from the fact that one frame could suit better to describe the desired variable than another.

For instance, using the Earth as a reference frame is useful to describe the displacement of a helicopter from one city to another. However, in order to describe the response of a rotorcraft to a wind gust, its location in the Earth is not relevant. Instead, the use of a reference frame aligned with the vehicle attitude and velocity relative to the wind greatly simplifies the analysis. For this reason, the former reference frame is used for navigation, while the latter is used to study the helicopter response to aerodynamic forces.

Nevertheless, the evolution of some states of the vehicle which are usually expressed in one reference frame can influence other states usually analyzed in a different reference frame. For example, a wind gust hitting the helicopter (analyzed using a reference frame suited for describing aerodynamic forces) can cause the helicopter to stop flying straight North and start flying North-East (what is relevant for navigation, and studied with the Earth reference frame). Therefore, some transformations are required to translate the variables expressed in one reference frame, such as attitude, to other reference frames where the desired variables are expressed, like the orientation towards North. These transformations are covered in Section 2-1-2 later in this chapter.

2-1-1 Reference frames

In this document only the reference frames and their corresponding transformations that are more relevant for the analyzed topic are introduced. More precisely, the required reference frames are those concerning the helicopter rotational mechanics and not those which define the vehicle aerodynamic behaviour. This is due to the fact that the sensors considered for the

thesis are gyroscopes and linear accelerometers, which do not provide wind speed information (expressed in an aerodynamic reference frame), as a Pitot tube does. The reader is referred to (Etkin, 1972, Ch.4) [1] for a complete description of the reference frames and transformations used for flight dynamics analysis.

Earth-Centered Inertial Reference Frame F_I

The Earth-Centered Inertial Reference Frame $\{F_I\} = (X_I, Y_I, Z_I)$ is a right-handed orthogonal axis-system, centered at the Earth's center of mass. The Z_I -axis coincides with the spin-axis of the Earth, which goes through the geographic North Pole. The X_I -axis passes through the equator at the location where the ecliptic and the equator cross: the vernal equinox, as shown in Figure 2-1. Finally, the Y_I -axis passes through the equatorial plane and is perpendicular to the X_I, Z_I -axis to complete the coordinate axes system.

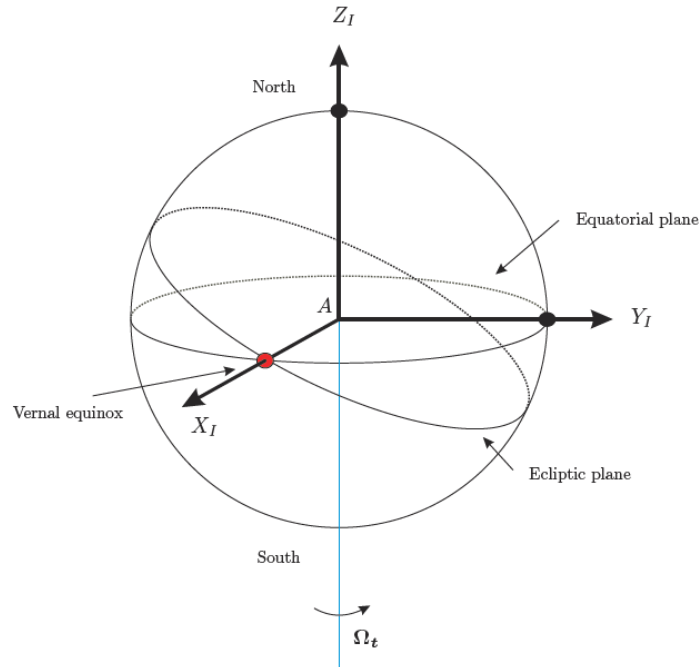


Figure 2-1: Definition of the Earth-Centered Inertial reference frame F_I (Mulder and De Visser, 2013 [2]).

This reference frame is commonly used for guidance and navigation, since it allows to track the translational movement of the aircraft with respect to a fix reference. Moreover, it will be relevant for deriving the translational and rotational equations of motion in the next chapter.

Earth-centered Earth-fixed Reference Frame F_C

The Earth-centered Earth-fixed Reference Frame $\{F_C\} = (X_C, Y_C, Z_C)$ is a right-handed orthogonal axis-system, centered at the Earth's center of mass, shown in Figure 2-2. The Z_C -axis coincides with the spin-axis of the Earth, as well as the Z_I -axis. The X_C -axis is

fixed to the Greenwich meridian, and thus Earth-centered, Earth-fixed reference frame (F_C) rotates around Z_C with the angular velocity of the Earth. This implies that F_C is not an inertial reference frame, in contrast to Earth-centered Inertial reference frame (F_I). Finally, the Y_C -axis passes through the equatorial plane and is perpendicular to the X_C , Z_C -axis to complete the coordinate axes system.

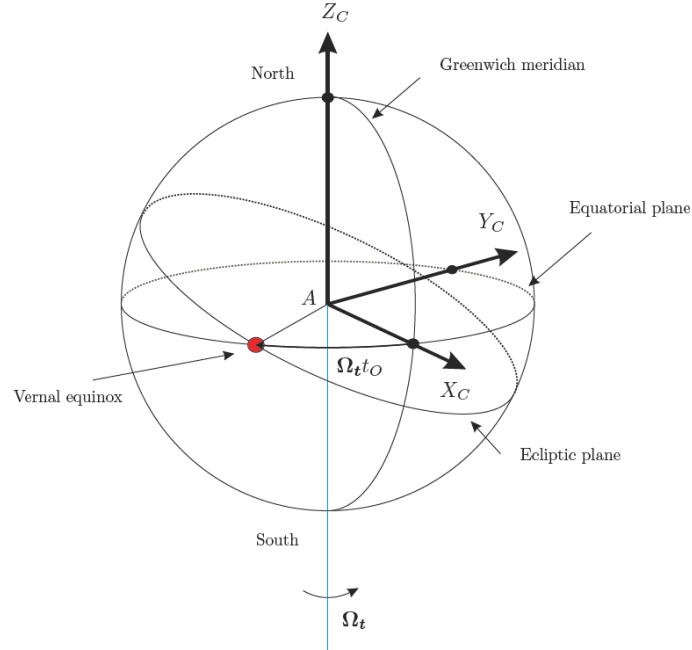


Figure 2-2: Definition of the Earth-centered Earth-fixed reference frame F_C (Mulder and De Visser, 2013 [2]).

Vehicle-carried normal Earth reference frame F_E

While the previous reference frame is inertial, the Vehicle-carried normal Earth reference frame $\{F_E\} = (X_E, Y_E, Z_E)$ moves with the body and its origin the center of gravity of the vehicle. The Z_E -axis points to Earth's center (see Figure 2-3), the X_E -axis points towards the true North, as seen from the vehicle Center of Gravity (C.G.), and the Y_E -axis to the East, completing a right-handed coordinate axes system. For this reason, this reference frame is also usually called the North-East-Down (NED) reference frame.

However, the Earth is actually an ellipsoid, and thus the Z_E -axis does not perfectly coincide at all times with the gravitational attraction vector g , since g points to the real Earth's center of gravity that slightly differs from that of a perfect sphere. Nevertheless, for many applications concerning small distances, this reference frame can be used under two assumptions by defining its center on a plane that is tangent to the local gravity vector g of the Earth.

The first assumption is based on the fact that the curvature of the Earth is very small when analyzing the helicopter behaviour for small distances and thus assumes that the orientation of this reference frame remains practically unchanged with respect to the surface of the Earth. The second one considers that the rotation of the helicopter (and thus of its local

horizontal plane) is of a much higher frequency than the rotation of the Earth, what makes the Coriolis and centripetal accelerations of the reference frame negligible. However, it should be highlighted that this assumptions cannot be applied to guidance or navigation systems for flying over large distances.

In this document the assumption of the flat, non-rotating Earth is considered as valid, since the evolution of the helicopter dynamics is fast in comparison to the Earth's rotation, and the analysis is not used for long-term or long distance calculations.

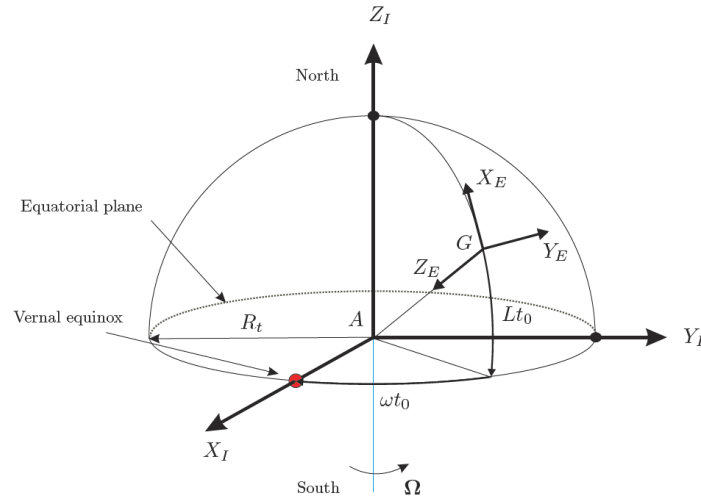


Figure 2-3: Definition of the Vehicle-carried normal Earth reference frame F_E (Mulder and De Visser, 2013 [2]).

Body-fixed reference frame F_b

The Body-fixed reference frame $\{F_b\} = (X_b, Y_b, Z_b)$ is a right-handed orthogonal axis-system with the origin at the body center of mass. If the gravity field is considered as constant then this point coincides with the center of gravity of the aircraft. For this thesis, the considered changes in height in which the angular estimations should be obtained do not require a varying gravity field and thus g is considered as a constant.

In general, a vertical plane of symmetry of the vehicle is defined based on its geometry (to a good approximation), in which the x-axis and the z-axis are contained. The X_b -axis points towards the front of the fuselage and is aligned with its centerline, while the Z_b -axis points down, perpendicular to X_b in the symmetry plane, as shown in Figure 2-4. Finally, the Y_b -axis is defined to obtain a right-handed reference system.



Figure 2-4: Definition of Body-fixed reference frame F_b [3]

Therefore, both Vehicle-carried normal Earth reference frame (F_E) and Body-fixed reference frame (F_b) move with the vehicle and their origin is the vehicle centre of gravity. However, while the axes of F_b are always aligned with the vehicle, those of F_E remain fixed pointing towards the North, East, and Down directions. In other words, if initially F_b and F_E coincide, and the vehicle changes its pointing direction 3 degrees to the right, the axes Z_b and Z_E will still coincide, while X_b will be rotated 3 degrees clockwise with respect to X_E (which still points to the true North), as well as Y_b with respect to Y_E . The angles between both reference frames are called the Euler Angles, and are more deeply covered in the next section.

This reference frame is also used to define the body linear and rotational velocities, as shown in Table 2-1.

Table 2-1: Linear and rotational velocities in F_b

Reference frame axis	Linear speed	Rotational speed
X_b	u	Roll rate p
Y_b	v	Pitch rate q
Z_b	w	Yaw rate r

2-1-2 Transformations

As previously explained, the analysis of different phenomena is usually performed in the most suitable reference frame for every specific case, leading to different kinematic and dynamic variables expressed in different coordinate systems. In order to express a given variable, such as attitude, in a reference frame different from the one it currently is, a transformation is used. These transformations allow to calculate the coordinates of a vector in different coordinate systems that are related through the rotation of one of their axis.

Transformation from F_E to F_b : Euler angles

The transformation from the Vehicle-carried normal Earth reference frame F_E to the Body-fixed reference frame F_b is done through three consecutive rotations by the angles ϕ , θ and ψ ,

called the Euler Angles. For the rotation sequence, two intermediate reference systems will be defined, namely $F_{E'}$ and $F_{E''}$:

$$F_E \rightarrow F_{E'} \rightarrow F_{E''} \rightarrow F_b \quad (2-1)$$

The positive rotation direction of the angles are in accordance with the definition of right-hand reference systems. In Figure 2-5 a negative yaw angle ($-\psi$) is taken to enhance the graphical representation. The three rotations in sequence are:

- Rotation ψ : yaw angle around the $Z_E = Z_{E'}$ -axis

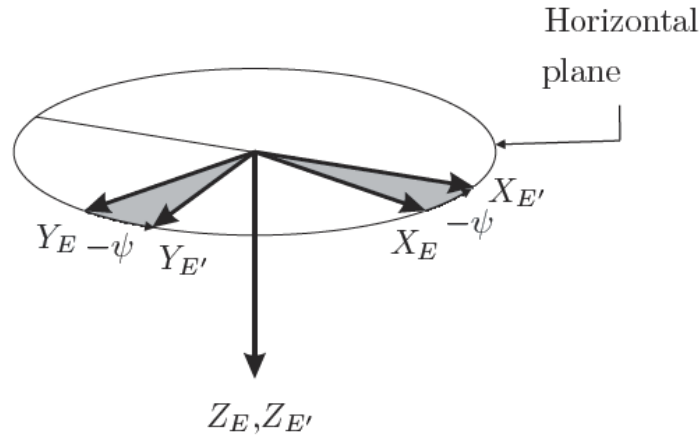


Figure 2-5: First rotation: ψ degrees around the $Z_E = Z_{E'}$ -axis (Mulder and De Visser, 2013 [2]).

$$F_{E'} = R_{E'E}(\psi) F_E \quad (2-2)$$

$$\begin{bmatrix} X'_{E'} \\ Y'_{E'} \\ Z'_{E'} \end{bmatrix} = R_{E'E}(\psi) \begin{bmatrix} X_E \\ Y_E \\ Z_E \end{bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_E \\ Y_E \\ Z_E \end{bmatrix} \quad (2-3)$$

- Rotation θ : pitch angle around the $Y_{E'} = Y_{E''}$ -axis

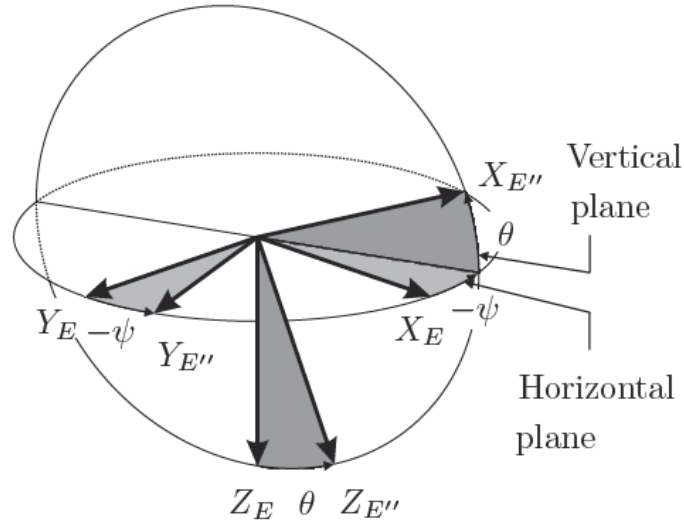


Figure 2-6: Second rotation: θ degrees around the $Y_{E'} = Y_{E''}$ -axis (Mulder and De Visser, 2013 [2]).

$$F_{E''} = R_{E''E'}(\theta) F_E' \quad (2-4)$$

$$\begin{bmatrix} X_E'' \\ Y_E'' \\ Z_E'' \end{bmatrix} = R_{E''E'}(\theta) \begin{bmatrix} X_E' \\ Y_E' \\ Z_E' \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} X_E' \\ Y_E' \\ Z_E' \end{bmatrix} \quad (2-5)$$

- Rotation ϕ : roll angle around the $X_{E''} = X_b$ -axis

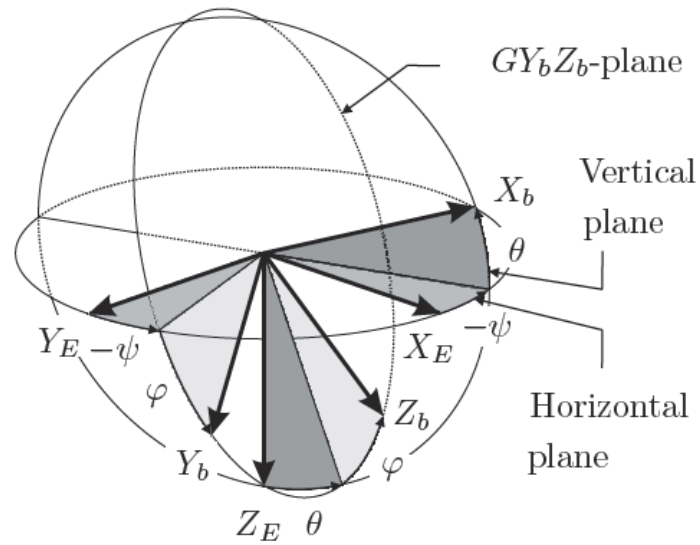


Figure 2-7: Third rotation: ϕ degrees around the $X_{E''} = X_b$ -axis (Mulder and De Visser, 2013 [2]).

$$F_b = R_{bE''}(\phi) F_E'' \quad (2-6)$$

$$\begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = R_{bE''}(\phi) \begin{bmatrix} X_E'' \\ Y_E'' \\ Z_E'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} X_E'' \\ Y_E'' \\ Z_E'' \end{bmatrix} \quad (2-7)$$

The complete rotation from F_E to F_b , as shown in Figure 2-8 can be directly calculated by multiplying the three independent transformation matrices $R_{bE''}$, $R_{E''E'}$, and $R_{E'E}$ to obtain R_{bE} , as shown in Equation 2-8.

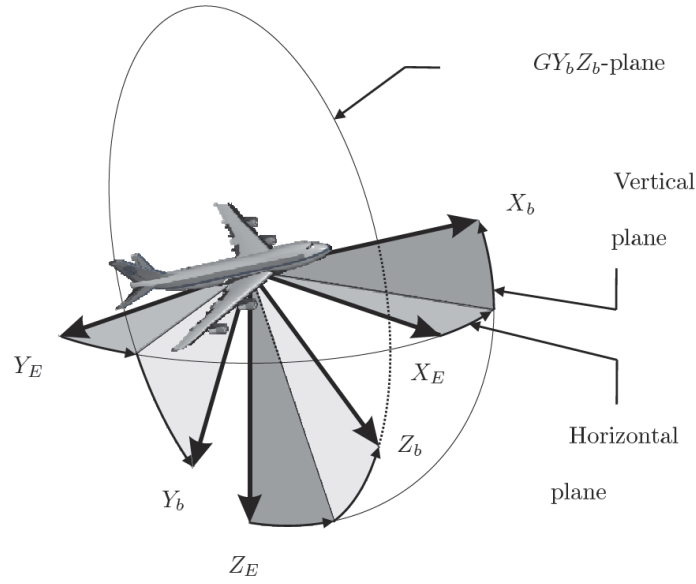


Figure 2-8: Definition of Euler Angles (*Mulder and De Visser, 2013 [2]*).

$$F_b = R_{bE} F_E \quad (2-8)$$

where

$$R_{bE} = R_{bE''}(\phi) R_{E''E'}(\theta) R_{E'E}(\psi) =$$

$$= \begin{bmatrix} \cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\ \left(\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi \right) & \left(\sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi \right) & \sin \phi \cos \theta \\ \left(\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi \right) & \left(\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi \right) & \cos \phi \cos \theta \end{bmatrix}$$

It should be remarked that the order in which the transformation matrices are multiplied is crucial, since each reference frame that is rotated in the three-rotations-sequence depends on the rotation performed by the previous transformation matrix. This highlights the relevance of specifying the order in which the rotations are performed. As previously stated, the convention in flight dynamics is to first perform the rotation of the yaw angle ψ about the Z -axis, then a rotation of the pitch angle θ about the intermediate $Y_{E'} = Y_{E''}$ -axis and finally a rotation of the roll angle ϕ about the intermediate $X_{E''} = X_b$ -axis.

2-2 Description of available sensors

Helicopters have a wide variety of sensors such as Pitot tubes, GPS antennas, radars, etc. For the estimation problem analyzed in this master thesis, the most relevant sensors are gyroscopes and linear accelerometers. Gyroscopes provide angular speed measurements, which are nowadays used by the autopilot to control the helicopter's attitude and angular speed.

In most of the cases, linear accelerometers are used to measure the longitudinal, vertical, and lateral acceleration of helicopters. In general, they are placed at the C.G., or are compensated to show an approximate value of the linear accelerations at the C.G. This information is crucial for many maneuvers during navigation, such as ascend, descend or altitude hold. However, if placed out of the center of gravity, linear accelerometers include relevant information about the helicopter angular accelerations, as will be explained in this section.

2-2-1 Gyroscopes description and working principles

Gyroscopes can be divided into two groups depending on their working principle: mechanical or optical. Mechanical gyros use the conservation of angular momentum to detect any out-of-plane rotation and calculate the aircraft attitude and angular speed. There are several types of mechanical gyroscopes based on this principle, like classical mechanical gyroscopes, which have a high-speed-spinning disc mounted on a frame called gimbal that allows free rotation in any axis by a ball bearing mechanism. If the disc has enough mass and speed, and the friction with the frame is sufficiently low, conservation of angular momentum leads to rigidity in space and gyroscopic precession of the disc. Rigidity in space stands for the tendency of the disc to maintain its spinning axis although the gimbal to which it is attached rotates, and is used to measure changes in aircraft attitude.

On the other hand, gyroscopic precession is the effect that makes the gyroscope feel any force acting on it as it was 90 degrees out of phase in the direction of the rotation. For example, if the disc of a gyroscope was rotating in clock-wise sense and an upwards vertical force was applied to the axis of rotation, the gyroscope would feel a horizontal force to the right (since a vector pointing upwards that is rotated 90 degrees in the clockwise direction yields a vector pointing right), causing a yawing motion in the gyroscope. Since the gimbal is attached to the aircraft frame, when the aircraft rotates, a force acts on the gimbal, which makes the disc rotate in the direction predicted by the gyroscopic precession. A higher force creates a higher precession and thus the angular speed of the aircraft can be obtained from the direction in which the disc precesses.

While there are many configurations of classical gyroscopes, such as gas-bearing gyroscopes, which substitute the ball bearings by a gas system to handle the disc in order to reduce friction, all of them are based on the same principles previously explained. However, due to their big dimensions and limited accuracy, they are generally not used in modern helicopters.

Micro-electromechanical System (MEMS) gyroscopes have replaced the classical ones, since they are smaller, lighter and more accurate. Instead of having a spinning disc, MEMS gyroscopes have a small oscillating piezoelectric device, which plane of oscillation is also maintained due to the conservation of angular momentum. In this case, the attitude of the device is measured by the electrical signals that it produces due to the piezoelectric effect, and

the calculation of the corresponding aircraft attitude and angular speed is performed in a similar fashion as with the classical one.

Finally, the other type of gyroscope that has outperformed mechanical gyroscopes is the optical gyroscope, that can be also sub-divided into Ring Laser and Fiber optic gyros, although their working principle is the same: the Sagnac effect. Optical gyroscopes make two beams of light travel in opposite senses through two circular paths that are attached to the aircraft. They are based on the fact that light takes longer to complete the journey if the path is rotating in the same direction as the light is traveling than when the path is rotating in the opposite direction. The rotation of the circular path makes the distance that light must travel longer or shorter, defining an effective path length. Light wavelength compresses or expands as the loop changes its effective length, which also causes a change in the frequencies. By examining the difference in the frequencies of the two counterrotating beams of light, the rate at which the path is rotating (and thus the aircraft angular rate) can be measured.

In conclusion, gyroscopes provide angular speed measurements in the body reference frame F_b . Moreover, under the assumption that the aircraft behaves as a rigid body, the angular velocity can be considered the same along all the fuselage and thus the gyroscopes would provide the same measurements independently of their position. In reality, the fuselage is elastic, and the gyroscopes' measurements vary slightly depending on their location.

2-2-2 Linear accelerometers description and working principles

Linear accelerometers used in the aerospace industry are based on the response of a small mass that is linked by a flexible material to a frame, that is fixed to the aircraft as shown in Figure 2-9. When the aircraft accelerates, the frame is accelerated by the same amount, while the inner mass is not (due to Newton's First Law), which creates a difference in the distances from the mass to the top part, and the from the mass to the lower part. The acceleration of the frame, and thus that of the aircraft, can be obtained from the difference in the distances, which can be measured with several devices such as strain gauges, capacitive sensors, etc.

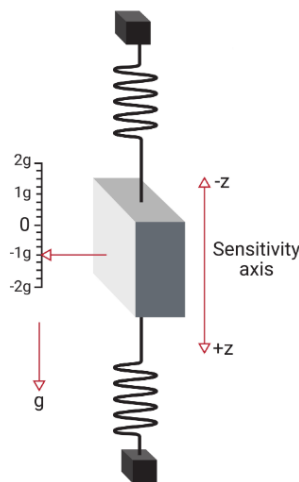


Figure 2-9: Linear accelerometer scheme. The black top and lower parts represent the frame, which is fixed to the aircraft, the grey mass is linked to the frame by springs (*Vectornav*[4]).

Every Inertial Measurement Unit (IMU) usually includes three orthogonal linear accelerometers, and each of them is aligned with one axis of the vehicle body X_b, Y_b, Z_b . The set of three linear accelerometers is usually referred to as triaxial linear accelerometer, and has the same orientation than F_b , shown in Figure 2-4.



Figure 2-10: Definition of Body-fixed reference frame F_b [3]

In contrast with gyroscopes, the measurements provided by linear accelerometers are highly dependent on their location. The linear acceleration measured by a tri-axial linear accelerometer placed at a distance x, y, z from the center of gravity of a rigid body is derived in Appendix A-2 and has the following expression in the body frame axes X_b, Y_b, Z_b :

$$\begin{aligned} a_x &= \dot{u} - rv + qw - x(q^2 + r^2) + y(pq - \dot{r}) + z(pr + \dot{q}) + g \sin \theta \\ a_y &= \dot{v} - pw + ru - y(p^2 + r^2) + z(qr - \dot{p}) + x(pq + \dot{r}) - g \cos \theta \sin \phi \\ a_z &= \dot{w} - qu + pv - z(p^2 + q^2) + x(pr - \dot{q}) + y(qr + \dot{p}) - g \cos \theta \cos \phi \end{aligned} \quad (2-9)$$

where u, v, w and $\dot{u}, \dot{v}, \dot{w}$ are the aircraft linear velocities and accelerations relative to Earth, p, q, r are its angular speeds, ϕ, θ, ψ are the Euler Angles, which define the attitude of the aircraft, and $g = 9.81$ is the gravitational acceleration of the Earth.

These equations can be expressed in vectorial form as follows:

$$a = \dot{V} + \omega \times V + (\omega^2 + \dot{\omega}) \times d + g_{(\Theta)} \quad (2-10)$$

where

$$\begin{aligned} d &= \begin{bmatrix} x_{COG} - x_{sensor} \\ y_{COG} - y_{sensor} \\ z_{COG} - z_{sensor} \end{bmatrix} \\ g_{(\Theta)} &= g \begin{bmatrix} \sin(\theta) \\ -\cos(\theta)\sin(\phi) \\ -\cos(\theta)\cos(\phi) \end{bmatrix} \end{aligned} \quad (2-11)$$

It is important to notice that when the helicopter is on ground, i.e., its linear velocities, linear accelerations, Euler angles and angular velocities are zero, a linear accelerometer would

measure $[0 \ 0 \ -g]^T$. This is more evident when looking at Figure 2-9. The mass is hanging down only subjected to the acceleration of gravity and thus the lower spring is compressed and the upper one is stretched. The same measurement is obtained when the aircraft is in hover. The linear accelerometers read zero acceleration when the frame moves with the acceleration of gravity but in the opposite sense, this is, when it is in free fall. Likewise, when it is accelerating upwards the sensors would read a larger negative acceleration, and when accelerating downwards they would measure smaller negative accelerations or even positive if it descends faster than in free fall.

As a side note, it can be highlighted that in these equations appear the angular accelerations $\dot{p}, \dot{q}, \dot{r}$, which are the goal of the master thesis. However, they can not be analytically obtained by inverting the previous equations, as will be more deeply explained in Section 3-2.

2-3 Kinematic and dynamic rigid-body equations of the helicopter

This chapter gives an overview of the variables and parameters involved in the modeling process of a helicopter. This process is followed in Chapter 5 in a simplified way to model the helicopter as a rigid body under the effect of wind gusts. The first section of this chapter shows the interconnections between the drivers of the helicopter motion (rotors, wind gusts, gravity, etc.) and its states. Sections 4.2 and 4.3 introduce the dynamic and kinematic equations used for flight simulation, while in Section 4.4 it is explained how to merge these equations with the sensors measurements.

2-3-1 Modelling of the helicopter

The general way to model the behaviour of a rigid body under which some forces are acting is to use the dynamic equations to obtain the body acceleration, which is later introduced in the kinematic equations to obtain the displacement of the body.

In rotating systems under the force of gravity, the expression of the forces acting on the body are not straightforward to obtain, since depending on the body orientation different components of the gravity will influence the body acceleration in every axis. In other words, if the body z -axis is aligned with the gravity vector the components of the gravity force in the x and y -axis will be zero. However, if the helicopter is tilted forward, the gravity force will have a component both in the z and the x -axes.

For this reason, the rotational dynamic equations of the body are first considered, in order to calculate its angular acceleration in the Body-fixed reference frame F_b . As depicted in Figure 2-11, the angular acceleration is then integrated to obtain the angular velocity in F_b , which is translated into the Earth reference frame F_E using the rotational kinematic equations. The angular velocity is integrated to obtain the helicopter attitudes (Euler angles) ϕ, θ, ψ . With this information, the components of the gravity vector acting on the three different axis of the body can be calculated, finally allowing to write all the forces acting on the system. At this point, the translational dynamic equations can be applied to obtain the body linear acceleration and later its linear velocities and position through the linear kinematic equations.

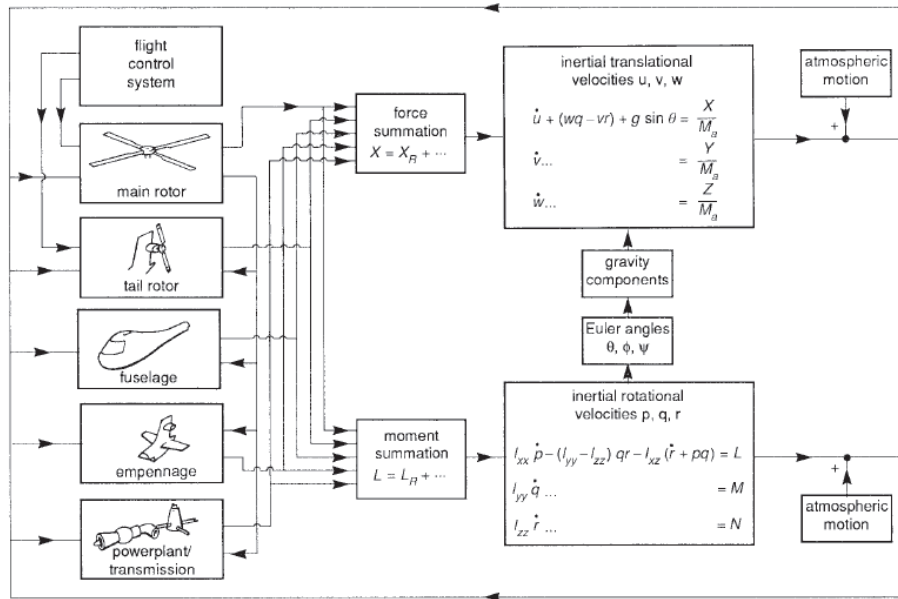


Figure 2-11: The helicopter simulation model (*Padfield, 2007* [5]).

There are two crucial aspects about the simulation model that should be taken into account. The first one is that the rotational dynamic equations are expressed in F_b since the attitude of the helicopter with respect to the Earth gravity field does not influence its rotational dynamics. This is due to the fact that gravity acts directly on the helicopter C.G., hence it does not generate torque. In contrast, this does not apply for the forces acting on the body, for which the relative attitude with respect to the Earth (and thus with the gravity vector) plays an important role.

The second crucial aspect is that in order to calculate the Euler angles via the rotational kinematic equations, the attitude of the helicopter with respect to the F_E reference frame has to be known. This will become more evident in Section 2-3-3 where the rotational kinematic equations are shown.

2-3-2 Rotational dynamic equations of a 6-DOF rigid-body

As introduced in the previous section, the rotational dynamic equations of the body are first considered, in order to calculate its angular acceleration in Body-fixed reference frame F_b . The helicopter model is generally considered symmetric in the X_b, Z_b -plane, although this is not completely accurate due to the presence of the tail rotor. However, the tail rotor inertia is small in comparison to that of the fuselage, and thus the simulations assuming this symmetry are commonly used.

The rotational equations of motion of a 6-Degrees of Freedom (DoF) rigid body which is symmetric in the X, Z -plane can be expressed in scalar form as follows:

$$\begin{aligned}
\dot{p} &= \frac{J_{zz}}{J^*} M_x + \frac{J_{xz}}{J^*} M_z + \frac{(J_{xx} - J_{yy} + J_{zz})J_{xz}}{J^*} pq + \frac{(J_{yy} - J_{zz})J_{zz} - J_{xz}^2}{J^*} qr \\
\dot{q} &= \frac{M_y}{J_{yy}} + \frac{J_{xz}}{J_{yy}} (r^2 - p^2) + \frac{J_{zz} - J_{xx}}{J_{yy}} pr \\
\dot{r} &= \frac{J_{xz}}{J^*} M_x + \frac{J_{xx}}{J^*} M_z + \frac{(J_{xx} - J_{yy})J_{xx} + J_{xz}^2}{J^*} pq + \frac{(-J_{xx} + J_{yy} - J_{zz})J_{xz}}{J^*} qr
\end{aligned} \tag{2-12}$$

where $J^* = J_{xx}J_{zz} - J_{xz}^2$ and $\sum T_{CG} = [M_x \ M_y \ M_z]^T$ are the net torques in the X_b , Y_b , and Z_b -axis, respectively.

The angular acceleration can be integrated to obtain the angular velocity $\omega = [p, q, r]^T$ using different integration methods, such as Range-Kutta or Tustin integration schemes. It should be remarked that an accurate dynamic model of the helicopter, as well as precise knowledge of the torques and forces which act upon it are necessary to obtain a representative simulation of the system.

2-3-3 Rotational kinematic equations in Euler angles

The kinematic equations relate the angular velocities in F_b (p, q, r) to the angular velocities in F_E ($\dot{\phi}, \dot{\theta}, \dot{\psi}$).

It should be highlighted that the rotation matrices presented in Chapter 2-1 provide the relation between angles (not angular velocities) in different reference frames. Taking into account that the angular velocities corresponding to the sequential rotations from F_b to F_E reference frames can be added up, Appendix A-1-1 shows that the rotational kinematic equations can be expressed as in Eq. 2-13:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \frac{\sin \phi}{\cos \theta} & \frac{\cos \phi}{\cos \theta} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{2-13}$$

The integration scheme applied to obtain the velocities from the accelerations can be used again to the variables $\dot{\phi}, \dot{\theta}, \dot{\psi}$ in order to obtain the Euler Angles ϕ, θ, ψ . These angles represent the attitude of the helicopter with respect to the Earth, and thus allow to express the gravity vector in body reference frame, as explained in the next section.

2-3-4 Translational dynamic equations of a 6-DOF rigid-body

The derivation of the translational dynamic equations in F_E can be found in Appendix A-1-2 and their expression is:

$$\begin{aligned}
\dot{u} &= rv - qw + \frac{X}{M_a} - g \sin \theta \\
\dot{v} &= pw - ru + \frac{Y}{M_a} + g \cos \theta \sin \phi \\
\dot{w} &= qu - pv + \frac{Z}{M_a} + g \cos \theta \cos \phi
\end{aligned} \tag{2-14}$$

where M_a is the mass of the helicopter and X, Y, Z are the net forces in the X_b, Y_b , and Z_b -axis, respectively.

The components of the gravity vector in every axis depend on the orientation of the body. It can be observed that if the helicopter is oriented parallel to the Earth surface ($\phi = \theta = 0$), the gravity vector only acts on the z -axis, while if the helicopter is tilted forward ($\phi \neq 0$) then there exist components of the gravity vector both in the x and z -axes.

2-3-5 Sensors-integrated equations of motion

The analysis performed in the previous sections does not consider any sensors in the helicopter model. In this section, the models of a 3-axis linear accelerometer and 3-axis gyroscopes are integrated into the helicopter model. As a first step for the derivation of the equations, the sensors are considered ideal, i.e., they do not include bias nor noise, and the accelerometers are assumed to be located at the centre of gravity of the helicopter. These assumptions will be discussed later in this section.

When the gyroscopes models are considered ideal, the angular velocity measurements that they provide p_m, q_m, r_m correspond to the exact angular velocity of the vehicle p, q, r . On the other hand, linear accelerometers measure the translational acceleration due to applied forces, excluding gravity. Therefore, ideal linear accelerometers would provide the values for the terms $X/M_h, Y/M_h, Z/M_h$ that appear in Equations 2-14 . With this information, the set of differential equations which describe the helicopter motion when using gyroscopes and linear accelerometers placed at the C.G. can be expressed as follows:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \frac{\sin \phi}{\cos \theta} & \frac{\cos \phi}{\cos \theta} \end{bmatrix} \begin{bmatrix} p_m \\ q_m \\ r_m \end{bmatrix} \quad (2-15)$$

$$\begin{aligned} \dot{u} &= r_m v - q_m w + a_{x_m} - g \sin \theta \\ \dot{v} &= p_m w - r_m u + a_{y_m} + g \cos \theta \sin \phi \\ \dot{w} &= q_m u - p_m v + a_{z_m} + g \cos \theta \cos \phi \end{aligned} \quad (2-16)$$

or in vectorial form:

$$\dot{\Theta} = \mathbf{B}\omega_m$$

$$\dot{V} = -\omega_m \times V + a_m - g(\Theta)$$

where p_m, q_m, r_m are the angular velocity measurements provided by the gyroscopes and $a_{x_m}, a_{y_m}, a_{z_m}$ are the linear accelerations read by the linear accelerometers.

At this point, the required equations have been derived assuming ideal sensor characteristics. However, both gyroscopes and linear accelerometers usually introduce bias and noise to some extent in their measurements. For this reason, the real sensor equations are expressed as follows:

$$\begin{aligned} p_m &= p + w_{g,x} \\ q_m &= q + w_{g,y} \\ r_m &= r + w_{g,z} \end{aligned} \quad (2-17)$$

$$\begin{aligned} a_{x_m} &= a_x + w_{a,x} \\ a_{y_m} &= a_y + w_{a,y} \\ a_{z_m} &= a_z + w_{a,z} \end{aligned} \quad (2-18)$$

where the subscript m indicates that the variable is a measurement, and $w_{g,i}$ and $w_{a,i}$ are the gyroscopes and the linear accelerometers noise, which are a combination of zero-mean noise and bias.

The new sensor equations can be introduced in Eqs.2-15 and 2-16 to obtain the following expressions:

$$\dot{\Theta} = \mathbf{B}(\omega_m - w_b) \quad (2-19)$$

$$\dot{V} = -(\omega_m - w_g) \times V + a_m - w_a - g(\Theta) \quad (2-20)$$

Equations 2-16 and 2-20 have been derived under the assumption that the sensors are placed in the C.G. of the helicopter. The relative position of the gyroscopes with respect to the C.G. does not affect their measurements. However, that is not the case for the linear accelerometers. For linear accelerometers mounted at a location different from the C.G., rotational motion results in a tangential and a centripetal acceleration that are added to the acceleration of the C.G. that is measured by the accelerometers. Therefore, in that case the accelerations read by the linear accelerometers would have the expression of Equations 2-10, shown in the previous chapter, and the dynamics Equations 2-16 would become:

$$\dot{V} = -(\omega_m - w_b) \times V - ((\omega_m - w_b)^2 + \dot{\omega}) \times d + a_m - w_a - g(\Theta) \quad (2-21)$$

where

$$d = \begin{bmatrix} x_{COG} - x_{sensor} \\ y_{COG} - y_{sensor} \\ z_{COG} - z_{sensor} \end{bmatrix} \quad (2-22)$$

$$g(\Theta) = g \begin{bmatrix} \sin(\theta) \\ \cos(\theta)\sin(\phi) \\ \cos(\theta)\cos(\phi) \end{bmatrix}$$

One can observe that the angular acceleration $\dot{\omega} = [\dot{p} \ \dot{q} \ \dot{r}]^T$ is required to calculate $\dot{V}$. In simulation, $\dot{\omega}$ can be calculated off-line using the rotational equations of motion as explained in Section 2-3-2, as long as an accurate dynamic model of the system and the applied forces are available.

In real time operation this approach is not usually taken, either due to the high computational weight of the rotorcraft dynamic model, or to the uncertainty in some of its parameters. In contrast, the angular acceleration is usually estimated in real time using sensors measurements, generally the angular speed, as will be explained in the next chapter.

Chapter 3

Core topics and goals of the thesis

The main gap found in literature is the potential use of linear acceleration sensors to estimate the angular acceleration of aerospace vehicles. Although some authors have approached the topic, the assumptions they have taken do not apply in general for helicopters and other aerospace vehicles. For instance, *Jiali et al.* [6] proposed an angular acceleration estimation method for a highly-maneuverable aircraft based on its dynamic model, assuming all the relevant forces and torques can be known, while *Carlsson and Hendebby* [7] designed an Extended Kalman Filter (EKF) to observe the angular acceleration of a vehicle assuming that the position of its Center of Gravity (C.G.) does not vary during flight.

Although it is common to take these assumptions for simple system such as drones, it is in general not possible for helicopters found in industry. The former assumption is not applicable to helicopters since it is extremely difficult to calculate their aerodynamic forces and torques in real time, what prevents to use them in a dynamic model. On the other hand, the position of the C.G. is usually shifted due to low-frequency factors, such as the consumption of the fuel during the flight, and to high-frequency ones, like the movement of the crew inside the cabin, or the loading and unloading of rescue personnel and equipment. For these reasons, the approaches used in the aerospace industry to estimate angular acceleration are usually based on angular speed differentiation.

The main goal of this master thesis is the design and testing of an estimation method for angular acceleration which improves the performance of current approaches used in industry. The performance of the estimation method is assessed based on several criteria, introduced in Section 3-3, while the background, advantages, and limitations of differentiation methods are presented hereafter.

3-1 Angular acceleration estimation by angular speed differentiation

3-1-1 Euler-Backwards differentiator

The Euler backwards differentiator is a discrete-time Linear-Time-Invariant (LTI) system that outputs the approximated derivative of a variable given two consecutive samples of that variable. For the case of the angular speed ω , sampled at a frequency $f_s = 1/t_s$, the angular acceleration obtained using the Euler backwards differentiator $\hat{\omega}$ is calculated as

$$\hat{\omega}_k = \frac{\omega_k - \omega_{k-1}}{t_s} \quad (3-1)$$

where t_s is the sampling period of the angular speed measurements.

The complex operator z can be used to obtain the expression for the the angular acceleration obtained using the Euler backwards differentiator $\hat{\omega}_E$ in the Z -domain:

$$\hat{\omega}_E = \frac{\omega - \omega z^{-1}}{t_s} = \frac{\omega(1 - z^{-1})}{t_s} = \omega \frac{z - 1}{z t_s} \quad (3-2)$$

Finally, the Euler backwards differentiator can be formulated as a discrete-time transfer function D_{Euler} that relates the angular speed input and the angular acceleration output:

$$D_{Euler} = \frac{\hat{\omega}_E}{\omega} = \frac{z - 1}{z t_s} \quad (3-3)$$

3-1-2 Lanczos differentiator

The discrete Lanczos derivative corresponds to the slope of a line which is the optimal fit for a set of points in the least-squares sense. Figure 3-1 shows an example of the line that best fits five points, which slope corresponds to the discrete Lanczos derivative at P .

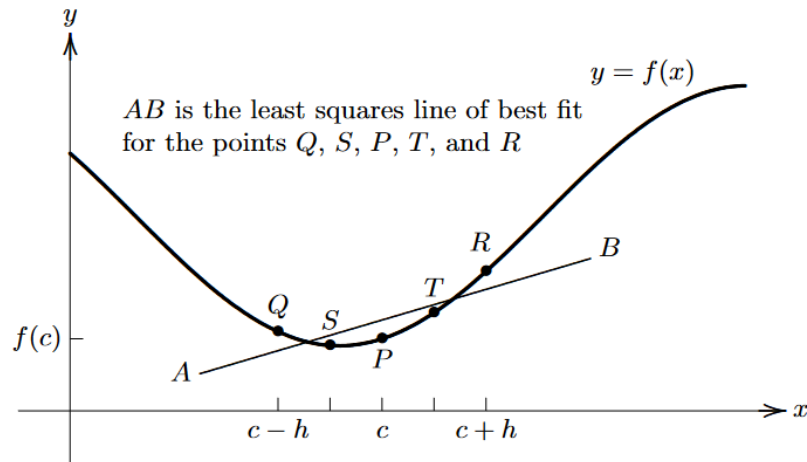


Figure 3-1: Lanczos derivative with five points. *Gordon, 2009 [8].*

The concept can be generalized for a continuous set of points which form a curve. However, since the goal in this section is to find the derivative of a sampled signal, the discussion is restricted to the discrete version of the Lanczos derivative. For a set of $2n$ points symmetrically placed around a point n and with a step size h between each sample, the discrete Lanczos derivative at n coincides with the slope m_c of the best fit in least squares sense for the $2n$ points:

$$m_c = \frac{\sum_{k=-n}^n \frac{k}{n} h f(c + \frac{k}{n} h)}{\sum_{k=-n}^n (\frac{k}{n} h)^2} \quad (3-4)$$

where $f(c + \frac{k}{n} h)$ are the values of the $2n$ samples around n .

The minimum amount of samples for computing a Lanczos derivative is three ($2n + 1 = 3$), for which the Lanczos derivative at point $n = 1$ is

$$m_c = \frac{f(c + h) - f(c - h)}{2h} \quad (3-5)$$

In real-time operation, the term $f(c + h)$ is not available, and therefore a causal version of the previous expression shall be used:

$$m_{c-h} = \frac{f(c) - f(c - 2h)}{2h} \quad (3-6)$$

For real-time angular acceleration estimation, the sampled signal is the angular velocity ω_m , and the step size is the sampling period Δt , which yields

$$\hat{\omega}_L = \frac{\omega_m(t) - \omega_m(t - 2\Delta t)}{2\Delta t} \quad (3-7)$$

The causal Lanczos differentiator using three samples introduces a time delay of one sampling period, as can be observed by comparing Eqs.3-5 and 3-6. A higher number of samples reduces the impact of random noise in the differentiation, at the expense of a higher time delay, what allows the Lanczos differentiator to balance the impact of these two factors.

In the Z -domain, the expression for the angular acceleration estimate $\hat{\omega}_L$ calculated with the Lanczos differentiator is as follows:

$$\hat{\omega}_L = \frac{\omega - \omega z^{-2}}{2t_s} = \frac{\omega(1 - z^{-2})}{2t_s} = \omega \frac{z^2 - 1}{2z^2 t_s} \quad (3-8)$$

Finally, the Lanczos differentiator can be formulated as a discrete-time transfer function $D_{Lanczos}$ that relates the angular speed input and the angular acceleration output:

$$D_{Lanczos} = \frac{\hat{\omega}_L}{\omega} = \frac{z^2 - 1}{2z^2 t_s} \quad (3-9)$$

3-1-3 Limitations of angular acceleration estimation by angular speed differentiation

The transfer functions of the Euler and the Lanczos differentiator can be discretized with the sampling rate of the available processor, and then applied to the angular speed measurements provided by the gyroscopes to obtain the angular acceleration estimations.

In order to address the accuracy of the estimations, they are compared to a non-causal derivative of the angular speed, which is calculated by taking into account several past and future samples of the signal in order to avoid noise amplification. This signal is considered from now onwards the real angular acceleration of the helicopter, or equivalently, an ideal estimation. The estimations provided by an Euler differentiator, and two Lanczos differentiators (with $n = 1$ and $n = 5$, respectively) are compared to the ideal angular acceleration in Figure 3-2.

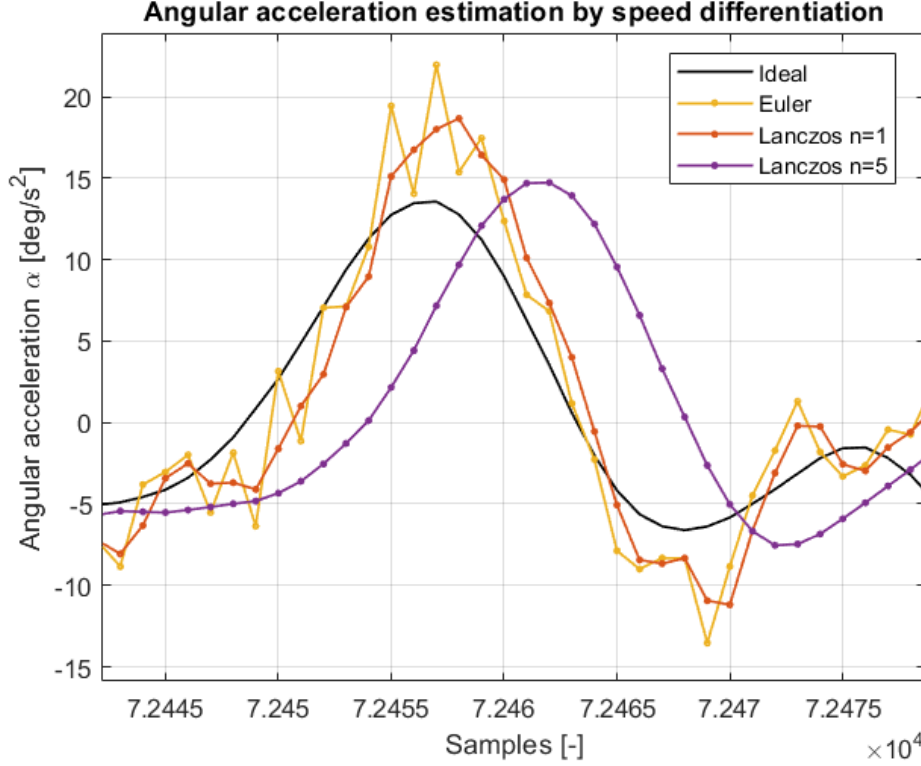


Figure 3-2: Comparison of the ideal angular acceleration and the Euler and Lanczos estimations.

It can be observed that the Euler estimations introduce a remarkable noise amplification, while the Lanczos differentiator with $n = 5$ provides an angular acceleration estimation with lower noise but a higher time delay (5 sampling periods). The Lanczos differentiator with $n = 1$ provides an intermediate balance, with an associated time delay of one sampling period and lower noise amplification than the Euler differentiator. For that reason, this estimator is chosen as the benchmark for this master thesis, and will be compared to the proposed estimation methods.

Figure 3-3 shows the angular acceleration estimations calculated with Lanczos differentiator with $n = 1$ using real flight data. The metric chosen to define the error of the angular acceleration estimations is the normalized standard deviation:

$$\sigma = \frac{\sqrt{\frac{1}{N-1} \sum_{i=1}^N (\alpha_i - \hat{\alpha}_i)^2}}{\sqrt{\frac{1}{N-1} \sum_{i=1}^N \alpha_i^2}} \quad (3-10)$$

where N is the number of samples of the angular acceleration signal and α is its ideal estimate.

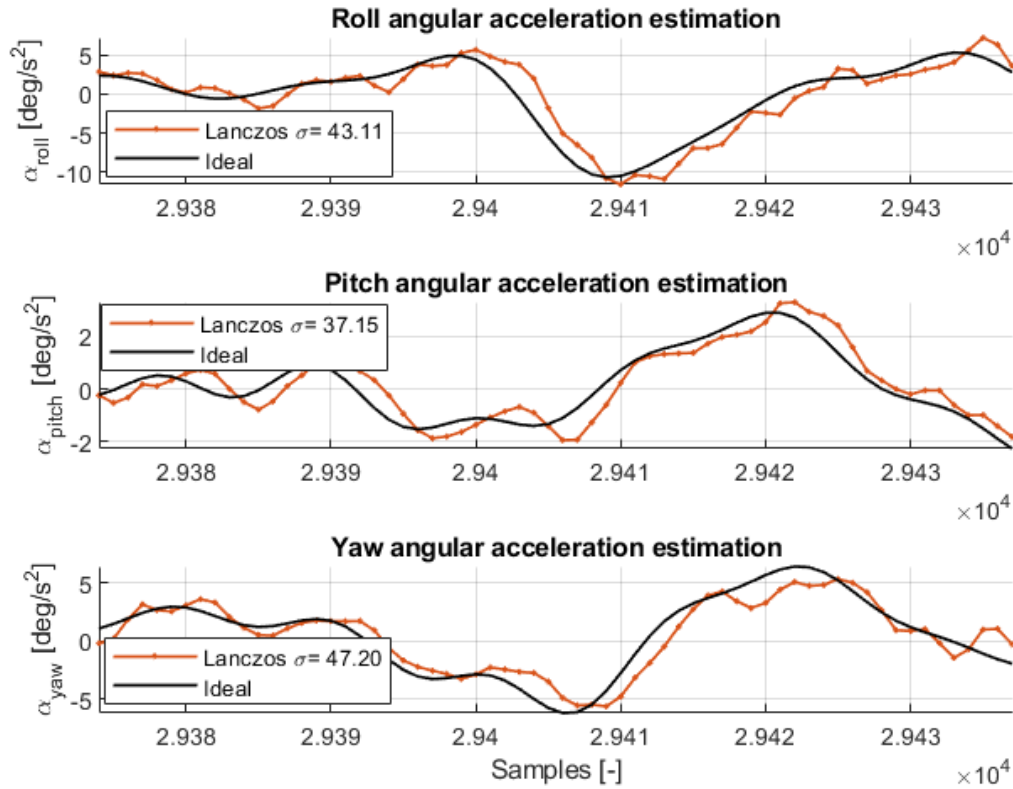


Figure 3-3: Angular acceleration estimation obtained with the Lanczos differentiator.

The benchmark solution shows the best accuracy in the pitch axis. For that reason, and to ease the visualization, henceforth the comparison of the proposed estimation methods with the benchmark and the ideal estimates are plotted only in the pitch axis, although their associated time delay and estimation errors in all axes can be found in Table 3-1.

Table 3-1: Estimators performance

	Lanczos n=1
Time delay [samples]	1
Error [%]	$\sigma_x=43.11$
	$\sigma_y=37.15$
	$\sigma_z=47.20$

It is important to remark that in the aerospace industry it is common practice to filter the estimated angular accelerations to reduce their noise. For instance, *Simplicio*[9] showed the smooth characteristics of the estimations obtained by finite-differentiation with a first-order Low-Pass-Filter (LPF), while *Bacon et al.* [10] computed the angular accelerations by taking

the time derivative of the angular velocity and applying a second-order filter. However, these techniques significantly increase the time delay of the angular acceleration estimates. For the topic analyzed in this master thesis, a low delay is crucial, and thus a different approach is followed. The linear acceleration sensors are used to obtain information about the angular acceleration of the helicopter, as will be explained in the next section.

3-2 Angular acceleration estimation using linear accelerometers

Linear acceleration sensors are present in most rotorcraft since they are included in every Inertial Measurement Unit (IMU). However, these sensors are usually only used to estimate the attitude of the helicopter and the linear velocities and linear accelerations. The main objective of this section is to analyse how the linear accelerometers can be used for angular acceleration estimation of helicopters.

A sketch of a linear accelerometer is shown in Figure 3-4. In a simplified way, this kind of sensor is essentially a mass (grey box in the figure) linked by two springs to a solid frame.

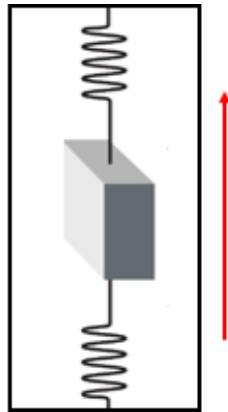


Figure 3-4: Sketch of a linear accelerometer

The working principle of linear accelerometers is based on the principle of inertia. If the frame is accelerated upwards as indicated by the red arrow, the tips of the springs attached to the frame will also accelerate, but the grey mass will tend to keep in the same position. This will lead to a contraction of the lower spring and an elongation in the upper spring, which the sensor uses to estimate the acceleration that the frame is subjected to. This is an example of an accelerometer for vertical acceleration. IMUs are composed of 3-axis accelerometers, which are placed orthogonally to measure accelerations in three dimensions.

The main idea of the proposed method is shown in Figure 3-5. If a linear accelerometer is located at a distance R from the C.G. of a body that has an angular acceleration α , the linear acceleration a measured by the sensor is

$$a = R\alpha \quad (3-11)$$

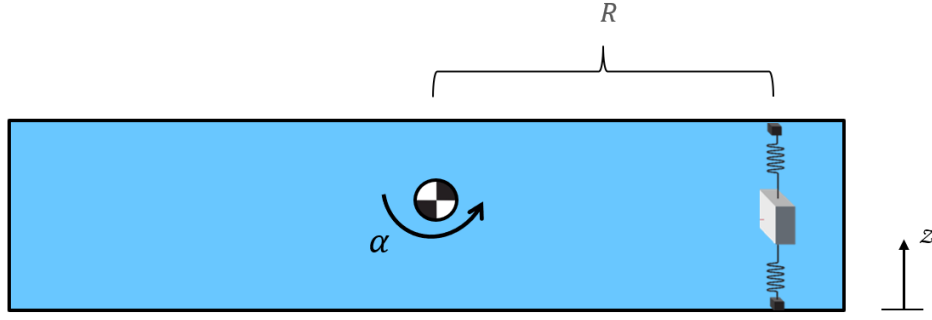


Figure 3-5: Estimation of angular acceleration in 1-D.

In this simple case, the angular acceleration of the body can be obtained by isolating α in the previous equation:

$$\alpha = \frac{a}{R} \quad (3-12)$$

If gravity is taken into account, and it is considered that, in addition to rotate, the body can also accelerate in the z -axis, the linear acceleration a has the expression:

$$a = \dot{V}_z \cos \theta + R\alpha + g \cos \theta \quad (3-13)$$

This equation shows how different types of motion and even forces have an identical effect in the measurement of the sensor. In other words, if only the measurement of the linear accelerometer was available, it would not be possible to distinguish if the body is either accelerating upwards, rotating, or subjected to gravity. However, if the vertical acceleration the body is known and it is assumed that the body is located at ground level (thus it is subjected to gravity), the angular acceleration can be obtained by isolating α in the same way than before:

$$\alpha = \frac{a - \dot{V}_z \cos \theta - g \cos \theta}{R} \quad (3-14)$$

where θ is the orientation of the rigid body with respect to the horizontal plane.

As explained in Section 2-2-2, linear accelerometers are usually grouped in orthogonal triads forming a tri-axial accelerometer. Moreover, they are aligned with the axes of the body, instead of the generic axis z used in the example of Figure 3-5. If instead of restricting the motion to 1-D as in the example, one considers the 3-D motion of a triaxial linear accelerometer aligned with the Body-fixed reference frame (F_b), the measured linear acceleration a_x, a_y, a_z has the expression obtained in Appendix A-2:

$$\begin{aligned} a_x &= \dot{u} - rv + qw - x(q^2 + r^2) + y(pq - \dot{r}) + z(pr + \dot{q}) + g \sin \theta \\ a_y &= \dot{v} - pw + ru - y(p^2 + r^2) + z(qr - \dot{p}) + x(pq + \dot{r}) - g \cos \theta \sin \phi \\ a_z &= \dot{w} - qu + pv - z(p^2 + q^2) + x(pr - \dot{q}) + y(qr + \dot{p}) - g \cos \theta \cos \phi \end{aligned} \quad (3-15)$$

where (u, v, w) and $(\dot{u}, \dot{v}, \dot{w})$ are the aircraft linear velocities and linear accelerations relative to the Earth, respectively; (p, q, r) and $(\dot{p}, \dot{q}, \dot{r})$ are the angular velocities and angular accelerations in F_b , respectively; (ϕ, θ, ψ) are the Euler Angles; and (x, y, z) is the distance from the sensors to the C.G. expressed in F_b .

Unlike in expressions 3-11 or 3-13, the angular acceleration $\alpha = [\dot{p} \ \dot{q} \ \dot{r}]^T$ in the equation 3-15 cannot be isolated since the set of equations is rank deficient. This is more evident if one uses the set of equations 3-15 in vectorial form:

$$a = \dot{V} + \Omega V + \dot{\Omega} d + \Omega^2 d - R_{bE} \bar{g}_{(E)} \quad (3-16)$$

where $a = [a_x, a_y, a_z]^T$, $V = [u, v, w]^T$, $\dot{V} = [\dot{u}, \dot{v}, \dot{w}]^T$ and

$$d = \begin{bmatrix} x_{C.G.} - x_{sensor} \\ y_{C.G.} - y_{sensor} \\ z_{C.G.} - z_{sensor} \end{bmatrix}$$

$$e_B = \begin{bmatrix} \sin(\theta) \\ -\cos(\theta)\sin(\phi) \\ -\cos(\theta)\cos(\phi) \end{bmatrix} \quad (3-17)$$

$$\dot{\Omega} = \begin{bmatrix} 0 & -\dot{r} & \dot{q} \\ \dot{r} & 0 & -\dot{p} \\ -\dot{q} & \dot{p} & 0 \end{bmatrix} \quad \Omega^2 = \begin{bmatrix} -(q^2 + r^2) & pq & pr \\ pq & -(p^2 + r^2) & qr \\ pr & qr & -(p^2 + q^2) \end{bmatrix}$$

Taking into account that for the cross-product multiplication of two vectors $b = [b_1 \ b_2 \ b_3]^T$, $c = [c_1 \ c_2 \ c_3]^T$ the following identities hold

$$c \times b = -b \times c \quad c \times b = \tilde{c} b = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} b \quad (3-18)$$

one can isolate $\dot{\Omega}$ in 3-16 as follows:

$$a = \dot{V} + \Omega V + \dot{\Omega} d + \Omega^2 d - R_{bE} \bar{g}$$

$$a = \dot{V} + \Omega V - \tilde{d} \dot{\Omega} + \Omega^2 d - R_{bE} \bar{g} \quad (3-19)$$

$$\tilde{d} \dot{\Omega} = -a + \dot{V} + \Omega V + \Omega^2 d - R_{bE} \bar{g}$$

$$\dot{\Omega} = \tilde{d}^{-1} [-a + \dot{V} + \Omega V + \Omega^2 d - R_{bE} \bar{g}]$$

Since $\tilde{d}$ is rank-deficient, there is not an unique solution for the Equation 3-16 and thus it can be concluded the angular acceleration can not be obtained with only one triaxial linear accelerometer. The physical reason for this fact lies in the structure of linear accelerometers. If these sensors are rotated around their own center of gravity, then the springs are neither

compressed nor stretched, and thus no linear acceleration is induced. Therefore, there always exists a line around which the body can have an angular acceleration that will not induce a linear acceleration in the sensor, as shown in the simplified Figure 3-6.

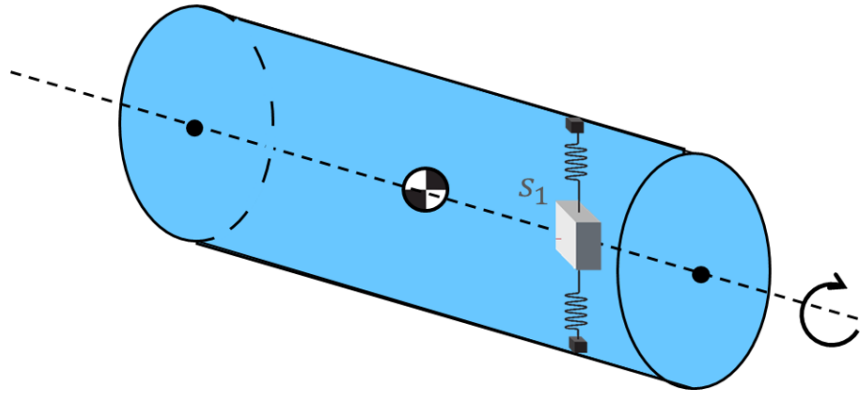


Figure 3-6: Blind axis of rotation with one linear accelerometer.

The line around which the body can rotate without inducing a linear acceleration in the sensor is called the blind axis of rotation, and is the line that connects the C.G. of the body and the C.G. of the linear accelerometer.

It should be remarked that in Figure 3-6 only one linear accelerometer is shown to simplify the visualization. However, in reality it would usually be assembled together with two more linear accelerometers forming an orthogonal triad which share the same origin and C.G. It is important to notice that also in that case, the blind axis problem persists.

In addition to this issue, there is another challenge to estimate angular acceleration using linear accelerometers. The distance d from the C.G. to the sensor is assumed to be known when using Equation 3-16. However, in many aerospace vehicles, the position of the C.G. is not known with high precision (due to manufacturing tolerances) and it varies during flight. For instance, during the high precision operation mentioned in the introduction, one operator and an injured person could be hanged from a rope attached to the helicopter. This fact, in addition to the possible movement of the crew inside the helicopter cabin, implies continuous changes in the C.G. which limit the direct application of Equation 3-16.

The rank deficiency problem previously introduced and the uncertainty of the C.G. location are the main challenges for estimating angular acceleration using linear accelerometers. These limitations are tackled in Section 4-1-1 of Chapter 4.

3-3 Performance criteria

The main goal of this master thesis is to improve the angular acceleration estimations that can be obtained by already available methods. The improvement can be quantified using the following three metrics.

- Time delay: This is the most relevant aspect since the estimations should be used to avoid any displacement of the helicopter due to external disturbances, for what a later response of the controller would imply a higher deviation from the desired position. In this regard, it is desired that the new estimator introduces, on average, a delay lower than one sample, since that is the delay introduced by the Lanczos differentiator.
- Noise amplification: Since the main characteristic to be improved is the reactivity of the estimations, having a similar level of noise amplification than the benchmark estimator is considered acceptable.
- Computational weight: The complexity of the algorithms increases the workload required for crucial aspects in the industry, such as real-time execution, testing, and implementation. In this regard, linear estimators are preferred over estimation methods which include online matrix inversion operations or nonlinearities.

In summary, minimizing the time delay of the estimations is prioritized over reducing their noise level, while the estimator complexity should be limited as much as possible. As concluded in Section 3-1, the Lanczos differentiator is used as benchmark, and its performance -quantified by the three criteria previously introduced- will be compared to the estimators proposed in the next section.

Angular acceleration estimation methods

This chapter shows and tests the method introduced by *Parsa et al.*[11] to overcome the challenges of angular acceleration estimation using linear accelerometers. In addition to this, other estimators are proposed, as well as several procedures to analyze the approximations on which the estimators are based on, in order to test if they are suitable for the estimation problem of this master thesis.

4-1 Angular acceleration estimation by Least Squares

As shown in the previous chapter, there are two limitations when using linear accelerometers to estimate the angular acceleration of a vehicle. The first one is that it is not possible to invert the model equations (Eqs.3-15) to isolate the angular acceleration due to the rank deficiency of the lever arm matrix $\tilde{d}$. The second limitation is the uncertainty in the location of the Center of Gravity (C.G.) of the vehicle, which can vary during flight.

4-1-1 Rigid body assumption

The rank deficiency problem can be explained by the existence of a blind axis, as shown in Figure 3-6. If the vehicle is assumed to be a rigid body, one can use a second tri-axial linear accelerometer S_2 placed in a position d_2 outside the blind axis of the first sensor S_1 to solve the rank deficiency problem. This configuration implies that every component of the angular acceleration which cannot be measured by one linear accelerometer is measurable by the other one, as shown in Figure 4-1.

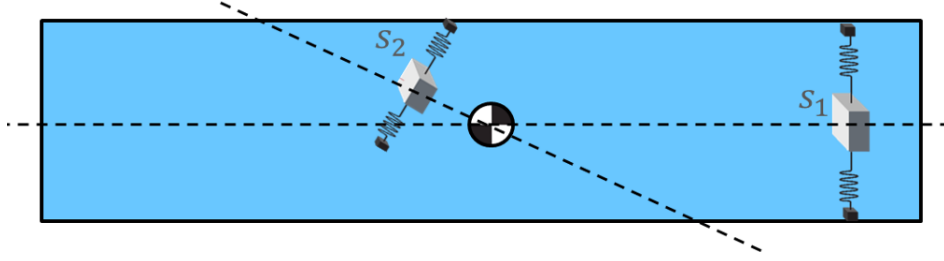


Figure 4-1: No common blind axis of rotation with two linear accelerometers.

This leads to a system of two vector equations, one from each linear accelerometer, and each with rank 2, as shown in Eqs. 4-1. In total, there are four independent scalar equations and three unknowns $\dot{p}$, $\dot{q}$, $\dot{r}$, and thus the angular acceleration can be calculated in a least squares sense.

$$\begin{aligned} a_1 &= \dot{V} + \Omega V + \dot{\Omega} d_1 + \Omega^2 d_1 - R_{bE} \bar{g} \\ a_2 &= \dot{V} + \Omega V + \dot{\Omega} d_2 + \Omega^2 d_2 - R_{bE} \bar{g} \end{aligned} \quad (4-1)$$

One requirement for using linear accelerometers for angular acceleration estimation has been presented: at least two linear accelerometers are necessary.

Under the rigid body assumption, the attitude, the angular velocity and angular acceleration is the same at any point of the body. This implies that, independently of the position of the sensors, the linear acceleration measured by S_1 and S_2 will be determined by the same gravity component $R_{bE} \bar{g}$, angular velocity Ω and angular acceleration $\dot{\Omega}$. Moreover, since $V, \dot{V}$ are the linear velocity and acceleration of the C.G. of the vehicle, the only difference between Eqs. 4-1 are the terms d_1, d_2 . Therefore, if both equations are subtracted, the following expression is obtained:

$$\Delta a_{2,1} = (\Omega^2 + \dot{\Omega}) \Delta d_{2,1} \quad (4-2)$$

where

$$\begin{aligned} \Delta a_{2,1} &= a_2 - a_1 = \begin{bmatrix} a_{x_2} - a_{x_1} \\ a_{y_2} - a_{y_1} \\ a_{z_2} - a_{z_1} \end{bmatrix} \\ \Delta d_{2,1} &= \begin{bmatrix} (x_{sensor_2} - x_{C.G.}) - (x_{sensor_1} - x_{C.G.}) \\ (y_{sensor_2} - y_{C.G.}) - (y_{sensor_1} - y_{C.G.}) \\ (z_{sensor_2} - z_{C.G.}) - (z_{sensor_1} - z_{C.G.}) \end{bmatrix} = \begin{bmatrix} x_{sensor_2} - x_{sensor_1} \\ y_{sensor_2} - y_{sensor_1} \\ z_{sensor_2} - z_{sensor_1} \end{bmatrix} = \begin{bmatrix} \Delta x_{2,1} \\ \Delta y_{2,1} \\ \Delta z_{2,1} \end{bmatrix} \end{aligned} \quad (4-3)$$

It can be observed that the position of the C.G. is cancelled out, and thus the impact of the C.G. location uncertainty can be avoided.

However, the term $\Delta d_{2,1}$ is rank deficient, which leads to the blind axis problem previously described, but this time in the line that connects S_1 and S_2 . The solution proposed by *Parsa*

et al. [11] is to use a third linear accelerometer placed out of the blind axis, as shown in Figure 4-2.

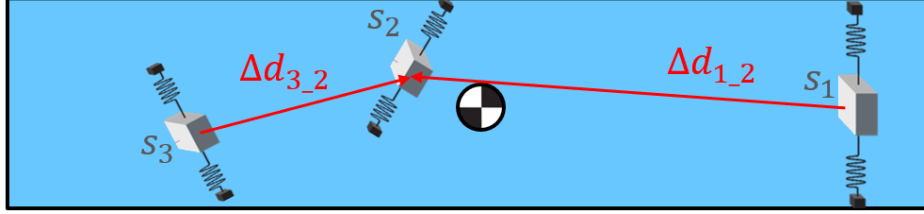


Figure 4-2: Distribution of three 3-axis linear accelerometers for angular acceleration estimation without C.G. location availability.

This sensors distribution leads to the following set of equations

$$\begin{aligned}\Delta a_{2,1} &= (\Omega^2 + \dot{\Omega}) \Delta d_{2,1} \\ \Delta a_{3,2} &= (\Omega^2 + \dot{\Omega}) \Delta d_{3,2}\end{aligned}\tag{4-4}$$

where

$$\begin{aligned}\Delta a_{3,2} &= a_3 - a_2 = \begin{bmatrix} a_{x_3} - a_{x_2} \\ a_{y_3} - a_{y_2} \\ a_{z_3} - a_{z_2} \end{bmatrix} \\ \Delta d_{3,2} &= \begin{bmatrix} (x_{sensor3} - x_{C.G.}) - (x_{sensor2} - x_{C.G.}) \\ (y_{sensor3} - y_{C.G.}) - (y_{sensor2} - y_{C.G.}) \\ (z_{sensor3} - z_{C.G.}) - (z_{sensor2} - z_{C.G.}) \end{bmatrix} = \begin{bmatrix} x_{sensor3} - x_{sensor2} \\ y_{sensor3} - y_{sensor2} \\ x_{sensor3} - z_{sensor2} \end{bmatrix} = \begin{bmatrix} \Delta x_{3,2} \\ \Delta y_{3,2} \\ \Delta z_{3,2} \end{bmatrix}\end{aligned}\tag{4-5}$$

The system is composed of six equations, four of which are linearly independent, and therefore the system is overdetermined. This implies that, in contrast to the previous cases, a solution for $\dot{\Omega}$ can now be calculated, for instance, by Least Squares (LS).

This approach provides several advantages. First, the expressions do not involve the C.G. and thus its variations during the flight will not affect the estimations. In addition to this, one can notice that while in the approach based on two linear accelerometers and a known C.G. position the Eqs.4-1 depend on $\dot{V}$ and $g(e_B)$, in the current approach these terms do not appear (Eqs.4-4).

This aspect is a major benefit because aerospace vehicles do not include sensors which can directly measure $\dot{V}$, nor the Euler angles required for term e_B . In general, these variables are estimated by fusing measurements from other sensors, what implies that they are not available for use at the same time than the variables which are directly measured (such as the angular velocity ω and the linear acceleration a). The process to estimate $\dot{V}$, and e_B requires several iterations (depending on the method used), and thus a time delay would be introduced in the estimation of α if those variables were required. However, since Eqs. 4-4 only include variables which are directly measured by sensors, no processing of other signals

is necessary and thus the estimations of the angular acceleration using this method can be obtained as fast as the processor can run.

The third advantage of this method is that it does not require the dynamic model of the system whose angular acceleration is to be estimated. In contrast to the approach of *Jiali et al.*[6], this estimation method is applicable to all systems which behave as a rigid body to a reasonable extent, depending on the desired accuracy of the estimations. Moreover, since the required sensors are lightweight and widespread used, the range of vehicles in which this approach can be applied is remarkably broad.

The last advantage lies in the fact that, for a given noise variance in the linear accelerometers, the Signal-to-noise Ratio (SNR) of the linear acceleration difference Δa , increases with the square of the distance between the sensors, as shown by *Villeneuve et al.*[12]. Therefore, the separation between the linear accelerometers is a new parameter to take into account in the classical trade-off between noise attenuation and time delay. While in other approaches it is necessary to use several samples to filter out noise (what introduces time delay), in the LS approach it is possible to spread out the linear acceleration sensors to reduce the noise level in the angular acceleration estimations without introducing any time delay.

Nevertheless, increasing the distance between the sensors imply some drawbacks, such as the additional weight due to the wires to the new location, as well as constraints, such as the availability of points in the fuselage with low influence of vibrations or oscillations. For instance, one could think of placing the linear accelerometers at the wingtips of an aircraft to better estimate the roll acceleration, or at the end of the tail boom of a helicopter to improve the pitch acceleration estimation. However, these are relatively elastic parts, and thus sensors placed there would measure additional linear accelerations due to their oscillations. These aspects have to be included in the trade-off along with the time delay and the noise level.

In summary, angular acceleration estimation using linear accelerometers is not possible with just one sensor. If two triads of sensors are available and the position of the C.G. is known, the angular acceleration can be estimated, although the estimations are slower than when three triaxial linear accelerometers are available, in which case it is not necessary to know the location of the C.G., and the estimations of α have the lowest possible time delay for a given hardware.

Each Inertial Measurement Unit (IMU) is composed of a triaxial linear accelerometer and a gyroscope, and since the helicopter used for this master thesis includes three IMUs, it is possible to test the estimation approach based on least squares regression. The next sections show the method used to test the rigid body assumption, as well as the estimation results obtained using LS.

As a side note, it is important to highlight that it is possible to estimate the angular acceleration of a body with less than nine linear accelerometers (three triaxial linear accelerometers) if the sensors are not grouped in triads. The first works on this matter were performed by *Mertz* [13], *Schuler* [14], and *Padgaonkar* [15] for 2D and 3D angular acceleration estimation. Based on their work, *Anantha et al.* [16] estimated the angular acceleration of a boat by placing seven linear accelerometers on a rigid cross-arm structure. *Chen et al.* [17] proved that only six linear accelerometers are required to estimate the angular acceleration of a vehicle if they are distributed in a cube shape and each sensor is oriented along the diagonal of each face of the cube. This approach was applied by *Tan and Park* [18] to design Inertial Navigation

Systems (INS) based on angular acceleration estimation. Other sensors distributions can be found in literature, which vary depending on the application, such as spherical [19], cylindrical [20], or triangular [21], [22].

The advantage of using six linear accelerometers instead of nine, as in the approach proposed in this thesis, is remarkable. However, most of the sensors which measure linear acceleration and are available in industry are included as triaxial accelerometers inside IMUs. Moreover, the requirement of imposing a cubic shape (as well as spherical, cylindrical, etc.) for the linear accelerometers distribution instead of a compact triad would significantly increase the size and weight of IMUs. For these reasons, the use of these approaches is not widely spread, and thus the focus of this thesis is kept on the triaxial linear accelerometers.

4-1-2 Accuracy of the rigid body assumption

The helicopter has an elastic fuselage, which is excited by the presence of rotating parts like the main rotor, as well as by the modes of vibration at the structural natural frequencies of specific parts of the rotorcraft, like the tail boom. These factors lead to small and continuous deformations of the fuselage, which are captured by the sensors. Therefore, depending on the position of the IMUs, the angular velocity measured by each gyroscope can be different, what violates the core of the rigid body approximation. In the same way, the linear accelerometers might provide measurements which do not comply with the values expected in a rigid body.

In order to analyze if the helicopter can be considered a rigid body to a reasonable extent, a non-causal Low-Pass-Filter (LPF) is applied to the angular velocity and the linear acceleration measurements to remove their noise. The non-causal low-pass filtering is performed by taking into account several past and future samples of the signal, and provides a smooth signal without introducing time delay (in contrast to a causal LPF).

The filtered measurement of the linear acceleration difference Δa , which appears at the left side of Equation 4-2 (hereafter repeated) can be compared with the right hand side of the equation, which is calculated with the filtered angular velocity measurements and the ideal angular acceleration estimate. Since there is not noise in the measurements, both sides of the equation should match for a rigid body.

$$\Delta a_{2,1 \text{ LPF}} = (\Omega_{LPF}^2 + \dot{\Omega}_{LPF}) \Delta d_{2,1} \quad (4-6)$$

Therefore, the accelerations which are measured by the linear accelerometers but are not predicted from the calculation performed with the filtered angular speed and angular accelerations, can be claimed to be due to elasticity effects.

The comparison is performed for the x-axis of the sensors S_1 and S_2 . It is shown in Figure 4-3, where the filtered values of $\Delta a_{2,1}$ corresponding to the left part of Equation 4-2 are depicted with a solid line, and the values calculated with the expression of the right part are represented with a dashed line.

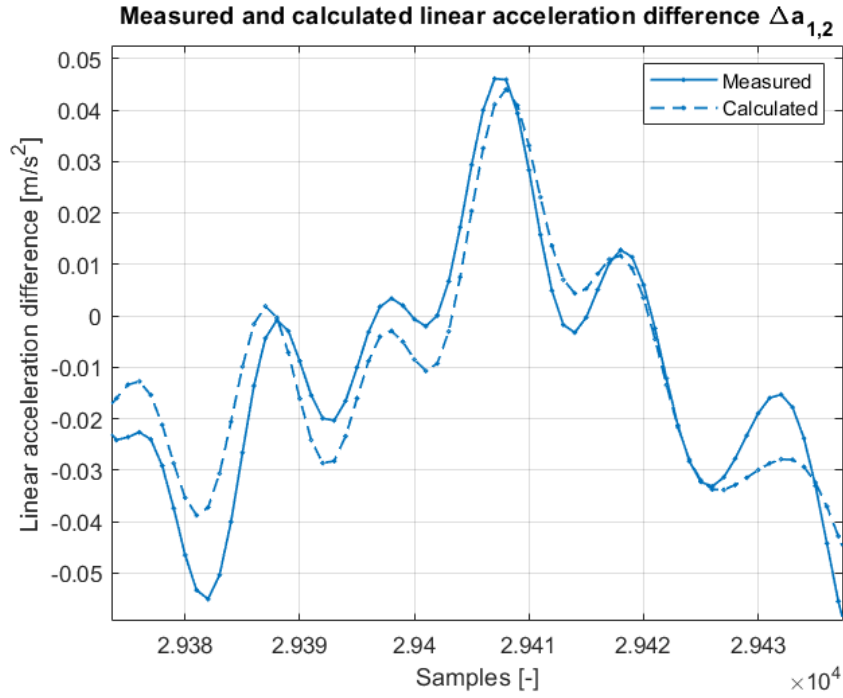


Figure 4-3: Comparison of the measured and calculated linear acceleration differences in the x-axis between S_1 , S_2 using filtered data.

The comparison shows that, although the overall trend of the measured and the calculated linear acceleration difference coincide, there are relative errors which reach more than 50 %, for instance in the first samples of the graph, and thus the rigid body approximation is not precise in this case. However, this fact does not imply that it has to be completely disregarded, as will be explained later in this chapter.

4-1-3 Estimation results using Least Squares

Following the approach proposed by *Parsa et al.* [11] and based on the set of Equations 4-4:

$$\begin{aligned}\Delta a_{2,1} &= (\Omega^2 + \dot{\Omega})\Delta d_{2,1} \\ \Delta a_{3,2} &= (\Omega^2 + \dot{\Omega})\Delta d_{3,2}\end{aligned}\tag{4-7}$$

one can define the matrices $R = [\Delta d_{2,1} \quad \Delta d_{3,2}]$ and $A_r = [\Delta a_{2,1} \quad \Delta a_{3,2}]$ to write the estimation problem

$$\dot{\Omega}R = A_r - \Omega^2 R = Y\tag{4-8}$$

which has the least-squares solution

$$\hat{\Omega} = YR^T(RR^T)^{-1}\tag{4-9}$$

Before estimating the angular acceleration using the noisy measurement data, it is considered useful to use the previous filtered data to calculate the helicopter angular acceleration in the least-squares sense. The estimated angular acceleration in the pitch axis using least squares

is shown in Figure 4-4, along with the ideal estimate and the angular acceleration obtained by applying the Lanczos differentiator with $n = 1$ to the same filtered angular speed data.

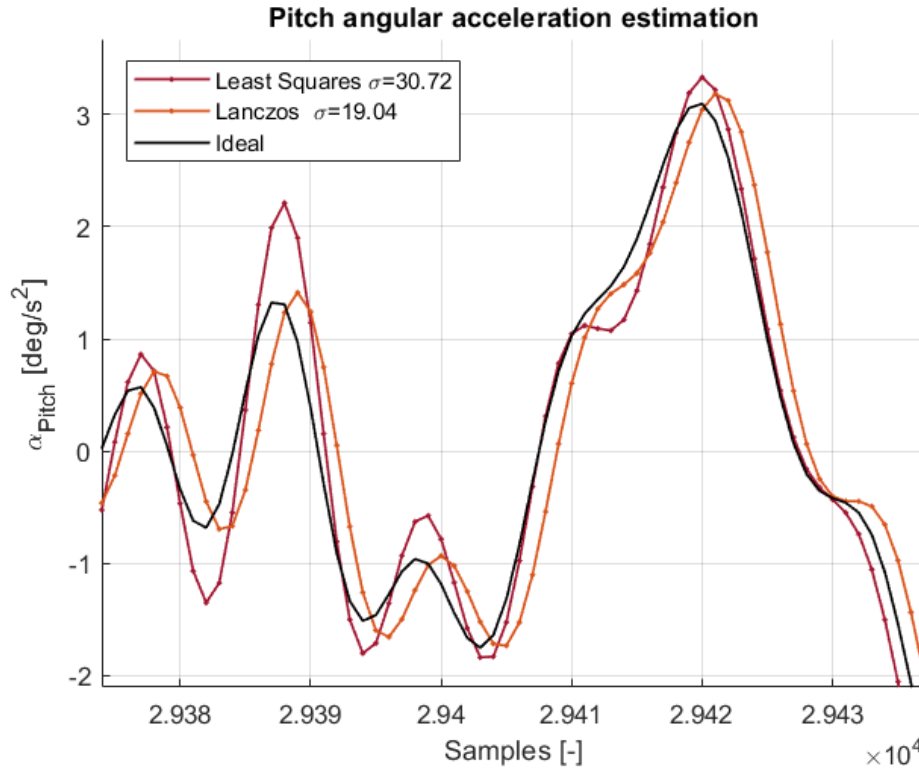


Figure 4-4: Angular acceleration estimation obtained with the Lanczos differentiator and by Least Squares using filtered data.

Some relevant conclusions can be drawn from this analysis. The first one is that combining the measurements of the linear accelerometers and the gyroscopes to estimate the angular acceleration using LS regression does not improve the estimation accuracy compared to just using the gyroscopes data. The reason for the accuracy decrease can be explained by the fact that the LS method is based on the rigid body assumption, which is not precise for the helicopter used in this master thesis, as well as for most rotorcraft one can find in industry.

However, one can notice that the estimations obtained using LS do not show any time delay with respect to the ideal angular acceleration estimate. This aspect is highly advantageous for the purpose of this master thesis since it matches the high reactivity linked to the first performance criterion introduced in the previous section. In addition to this, the LS algorithm is remarkably lightweight, and thus it is evaluated positively with respect to the third performance criterion.

It remains to evaluate the performance of the LS in terms of the second metric, which compares the level of noise of the proposed method with the benchmark solution. Unfiltered data is used to calculate the angular acceleration in the least-squares sense, which is shown in Figure 4-5, together with the ideal pitch acceleration estimate and the estimations obtained with the Lanczos differentiator using unfiltered data.

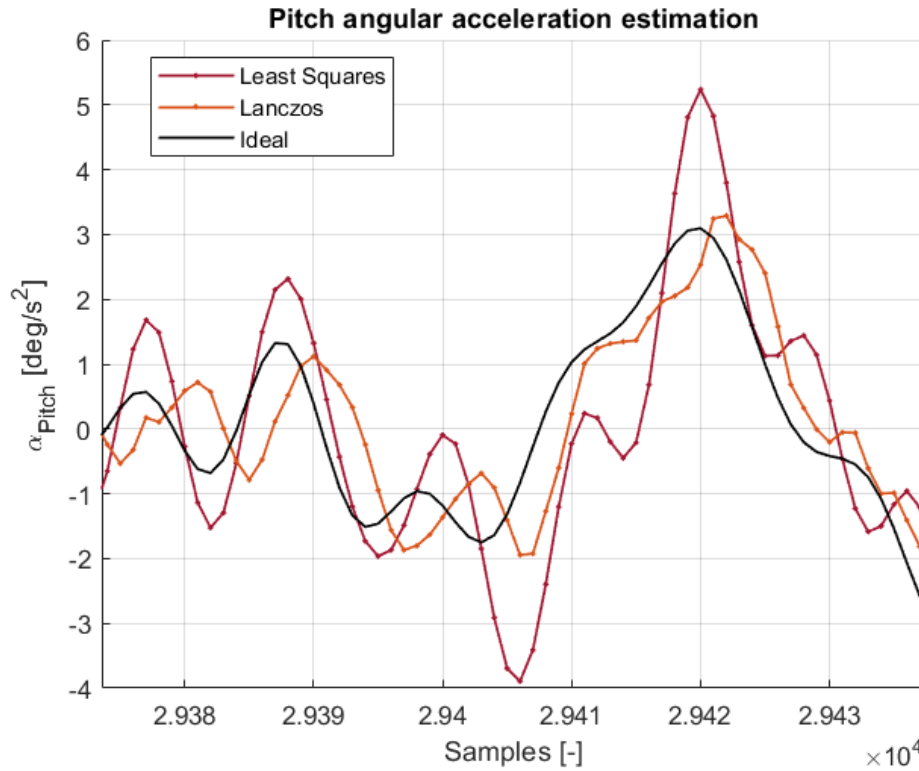


Figure 4-5: Angular acceleration estimation by Least Squares

The accuracy of the angular acceleration estimations obtained using LS is remarkably poorer than the current benchmark estimator for all axes, as indicated in Table 4-1. This low performance is partially due to the influence of the rigid body assumption, which already leads to severe estimation errors even when working with filtered data, as shown in the previous subsection.

In addition to this, it has to be taken into account that linear accelerometers usually have an associated noise level approximately one order of magnitude larger than gyroscopes, as it is the case in the sensors considered in this thesis. In contrast to the Lanczos differentiator, which moderates the noise level in the estimations at the cost of introducing some time delay, the LS estimation method can not balance these two factors, and thus the noise in the linear accelerometers is always transferred to the angular acceleration estimations.

Table 4-1: Estimators performance

	Lanczos n=1	Least Squares
Time delay [samples]	1	0
Error [%]	$\sigma_x=43.11$	$\sigma_x=65.72$
	$\sigma_y=37.15$	$\sigma_y=90.55$
	$\sigma_z=47.20$	$\sigma_z=55.78$

Although the LS method does not introduce any time delay and it is a lightweight algorithm, its estimation errors are excessively large in comparison to the Lanczos differentiator, and

thus it is concluded that it is not a suitable estimation method in this case. However, the advantages of its working principle based on using the linear accelerometers measurements can be exploited.

In the next section, state observers are introduced as a method to benefit from the high reactivity of the angular acceleration estimations obtained using linear accelerometers while limiting their noise content by fusing them with the gyroscopes measurements.

4-2 Angular acceleration estimation by State Observers

Angular acceleration estimation using state observers is frequently found in literature related to robotics. For instance, *Auger et al.* [23] designed a Luenberger observer to estimate the angular velocity and acceleration of an electric motor using a resolver-to-digital converter, *Han et al.* [24] used a Kalman Filter (KF) to observe the angular acceleration of the joints of a multiple-degree-of-freedom mechatronic system, and *Evren et al.* [25] developed an Extended Kalman Filter (EKF) to estimate the angular acceleration of a pan-tilt robotic platform by fusing gyroscopes, triaxial accelerometers and magnetometer data.

In the automotive industry, *Brunner et al.* [26] showed an improved road-vehicle interaction by estimating the vehicle angular acceleration using an EKF. In the aerospace field, *Carlsson and Hendebay* [7] designed a Discrete Lie-Group Extended Kalman Filter (D-LG-EKF) to increase the accuracy of the algorithms for INS. However, all the previous approaches rely on some assumptions which are not applicable to helicopters one can find in industry, as will be explained later in the next section.

4-2-1 System model

State observers fuse the information of one or several measurements of a system and the system dynamic model to estimate (some of) its states. Therefore, in contrast to other estimation methods like speed differentiation or Least Squares regression, observers require a system model:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + w_k \\ y_k &= g(x_k, u_k) + v_k \end{aligned} \quad (4-10)$$

where x is the state vector, u is the input vector, $f(x, u)$ is the state equation, $g(x, u)$ is the measurement equation, and w, v are the process and measurement noise. In this work, the system state-space model is expressed in discrete-time because the observers have to work with sampled data.

Since it is desired to estimate the angular acceleration of the helicopter, the need of a system model implies that a state equation for the angular acceleration is required, i.e., an equation of the form:

$$\alpha_{k+1} = f_\alpha(x_k, u_k) + w_{\alpha_k} \quad (4-11)$$

The angular acceleration of the helicopter depends on its inertia and the torques applied upon it. These can be generated by the thrust of the tail rotor, wind gusts, the aerodynamic

lift of the fuselage, the tilt of the main rotor, etc. Therefore, it could be possible to build a model of the derivative of the angular acceleration by differentiating the dynamics of all its components. However, the dynamic models of these phenomena are in general nonlinear and remarkably complex, and thus including them in the system model would significantly increase its computational weight. In order to keep the complexity of the observers as limited as possible, a new assumption is taken. The angular acceleration is considered constant, and the following state equation for the system model is used:

$$\begin{aligned}\omega_{k+1} &= \omega_k + t_s \alpha_k \\ \alpha_{k+1} &= \alpha_k + w_{\alpha, k}\end{aligned}\tag{4-12}$$

where

$$w_{\alpha} = \begin{bmatrix} w_{\alpha_x} & 0 & 0 \\ 0 & w_{\alpha_y} & 0 \\ 0 & 0 & w_{\alpha_z} \end{bmatrix}\tag{4-13}$$

is the noise in the dynamics of α , i.e., it quantifies the considered inaccuracy of the constant angular acceleration assumption.

In this system model, the jerk of the helicopter $\beta = \dot{\alpha}$ is set equal to zero, and thus the observer estimations would only converge to the real ones when the torques applied to the helicopter are constant. This is a strong simplification in comparison to the real helicopter dynamics, which are driven by time-varying forces and torques. However, it is also possible to extend the state vector to include higher derivatives of the acceleration and set them equal to zero.

In that sense, if the snap $s = \dot{\beta} = \ddot{\alpha}$ is introduced in the state-vector and set equal to zero, the estimation error would converge to zero both when the angular acceleration is constant and when it changes linearly. If the crackle $c = \dot{s} = \ddot{\beta}$ is set to zero, the estimation error converges to zero also for quadratic changes in the acceleration. The choice of the higher derivative included in the state vector is a design choice, and is analyzed in Section 5.

In order to write the measurement equation, the sensor models of the gyroscope and the linear accelerometers introduced in Section 2-2 are considered. The expressions of the linear acceleration measurements are subtracted in the same fashion than in the previous section in order to use the linear acceleration difference as a measurement for the observer. Taking into account the rigid body approximation, it is assumed that the angular speed measured by the gyroscope coincides, independently of the sensor position and except for the noise, with the angular velocity which appears in the state equation.

Therefore, the system model including the state and measurement equations can be formu-

lated as follows:

$$\begin{aligned}
 x_{k+1} = \begin{bmatrix} p \\ q \\ \omega \\ \alpha \end{bmatrix}_{k+1} &= \begin{bmatrix} p \\ q \\ r \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}_{k+1} = Ax_k + w_k = \begin{bmatrix} 1 & 0 & 0 & t_s & 0 & 0 \\ 0 & 1 & 0 & 0 & t_s & 0 \\ 0 & 0 & 1 & 0 & 0 & t_s \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}_k + w_k \\
 y_k = g(x_k) + v_k &= \begin{bmatrix} \omega \\ (\Omega^2 + \dot{\Omega})\Delta d_{2,1} \\ (\Omega^2 + \dot{\Omega})\Delta d_{3,2} \end{bmatrix}_k + v_k \\
 &= \begin{bmatrix} p \\ q \\ r \\ -\Delta x_{2,1}(q^2 + r^2) + \Delta y_{2,1}(pq - \dot{r}) + \Delta z_{2,1}(pr + \dot{q}) \\ \Delta x_{2,1}(pq + \dot{r}) - \Delta y_{2,1}(p^2 + r^2) + \Delta z_{2,1}(qr - \dot{p}) \\ \Delta x_{2,1}(pr - \dot{q}) + \Delta y_{2,1}(qr + \dot{p}) - \Delta z_{2,1}(p^2 + q^2) \\ -\Delta x_{3,2}(q^2 + r^2) + \Delta y_{3,2}(pq - \dot{r}) + \Delta z_{3,2}(pr + \dot{q}) \\ \Delta x_{3,2}(pq + \dot{r}) - \Delta y_{3,2}(p^2 + r^2) + \Delta z_{3,2}(qr - \dot{p}) \\ \Delta x_{3,2}(pr - \dot{q}) + \Delta y_{3,2}(qr + \dot{p}) - \Delta z_{3,2}(p^2 + q^2) \end{bmatrix}_k + v_k
 \end{aligned} \tag{4-14}$$

It can be noticed that three sets of linear accelerometers model equations have been used in the measurement equation of the proposed observer model. Since they are linearly dependent, one set is redundant when the measurements are noiseless. However, when there is measurement noise, including the three sets of sensors reduces the impact on the estimation error.

Since the observers that will be proposed in the next section for angular acceleration estimation are different types of Kalman Filters, the characteristics of the state and measurement noise are chosen to comply with the requirements for optimality of the Kalman Filters. As indicated in *Verhaegen and Verdult* [27], w_k and v_k are defined as samples of zero-mean normal distributions with variance Q and R , respectively. In addition to this, the state and measurement noise are assumed to be non-correlated.

$$\begin{aligned}
w &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ w_{\alpha_x} & 0 & 0 \\ 0 & w_{\alpha_y} & 0 \\ 0 & 0 & w_{\alpha_z} \end{bmatrix} & w_{\alpha} \sim \mathcal{N}(0, Q_{\alpha}) \\
v &= \begin{bmatrix} v_{\omega} \\ v_{\Delta a_{2,1}} \\ v_{\Delta a_{3,2}} \end{bmatrix} \\
v_{\omega} &= \begin{bmatrix} v_{\omega_x} & 0 & 0 \\ 0 & v_{\omega_y} & 0 \\ 0 & 0 & v_{\omega_z} \end{bmatrix} & \sim \mathcal{N}(0, R_{\omega}) \\
v_{\Delta a_{2,1}} &= \begin{bmatrix} v_{\Delta a_{x,2,1}} & 0 & 0 \\ 0 & v_{\Delta a_{y,2,1}} & 0 \\ 0 & 0 & v_{\Delta a_{z,2,1}} \end{bmatrix} & \sim \mathcal{N}(0, R_{\Delta a_{2,1}}) \\
v_{\Delta a_{3,2}} &= \begin{bmatrix} v_{\Delta a_{x,3,2}} & 0 & 0 \\ 0 & v_{\Delta a_{y,3,2}} & 0 \\ 0 & 0 & v_{\Delta a_{z,3,2}} \end{bmatrix} & \sim \mathcal{N}(0, R_{\Delta a_{3,2}})
\end{aligned} \tag{4-15}$$

Therefore, the system model is autonomous, with a linear state equation and a nonlinear measurement equation. Next section covers how this model is used to build the observers models, as well as the simplifications required for some of them.

At this point it is possible to point out the reasons why the approaches proposed by the authors shown at the beginning of the chapter are not applicable for the case of this master thesis. In order to estimate the angular acceleration of an electric motor, *Auger et al.* [23] used the same process equation as in (4-14), but included the position of the motor in the state vector. However, they used position sensors in the measurement equation, instead of linear accelerometers as in the approach proposed here. Therefore, their Luenberger observer was equivalent to a second order differentiator, which derived the position measurements of the motor to obtain its angular acceleration. Since the helicopter does not include angular attitude sensors, and the angular speed measurements are noisy, the method proposed by *Auger et al.* [23] would provide angular acceleration estimations with an excessive noise content.

Han et al. [24] used the same measurement equation than in (4-14). However, their goal was to estimate the angular acceleration of the joints of a mechatronic system, for which they know that the only applied forces are the actuator input and gravity. This fact allowed them to write the equations of motion of the system, and get a dynamic model for the state equation. In the case of a helicopter, it is not possible to obtain fast estimations while using its dynamic model due to its complexity.

In the case of *Evren et al., 2017* [25], *Carlsson and Hendebay* [7], and *Brunner et al.*[26], the state equation is the same than the one proposed here, and the measurement equation

also uses the gyroscopes and the linear accelerometers, but they do not follow the approach of subtracting the measurements of two linear accelerometers, and thus they are forced to assume that the position of the C.G. is known and constant. As previously discussed, this assumption is not applicable for the case of this master thesis neither for many aerospace vehicles, and thus the method proposed in this work is considered more widely applicable.

4-2-2 Proposed observers

State observers with linear -or linearized- dynamics can be expressed as follows

$$\begin{aligned}\hat{x}_{k+1} &= A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k) \\ \hat{y}_k &= C\hat{x}_k + Du_k\end{aligned}\tag{4-16}$$

where L is the observer gain, u is the observer input, $\hat{x}$ and $\hat{y}$ are the observer state vector and observer estimated measurement, respectively. The matrices A , B , C , D can be the same -although not necessarily- than the ones of the system model.

In the next subsections, four types of Kalman Filters are introduced, with different variations of the structure shown in Eqs. 4-16 and different procedures to design the observer gain L .

Extended Kalman Filter

The EKF linearizes the system model at every operating point, and thus it is in general well suited to deal with nonlinear systems. Moreover, it is able to adapt online the weight that each source of information has on the estimation in an optimal way. The EKF model can be written as follows:

$$\begin{aligned}\hat{x}_{k+1|k} &= A\hat{x}_{k|k} \\ P_{k+1|k} &= AP_{k|k}A^T + Q \\ H &= \left. \frac{dg(x_k)}{dx_k} \right|_{x_k=\hat{x}_{k|k}} \\ L_k &= P_{k|k-1}H \left(H(\hat{x}_{k|k-1}, u_k) P_{k|k-1} H(\hat{x}_{k|k-1}, u_k)^T + R \right)^{-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + L_k \left(y_k - g(\hat{x}_{k|k-1}) \right) \\ P_{k|k} &= P_{k|k-1} - P_{k|k-1}H^T \left(HP_{k|k-1}H^T + R \right)^{-1} HP_{k|k-1}\end{aligned}\tag{4-17}$$

where A and $g(x_k)$ are the same than in Eqs.4-14, $\hat{x}_{k|k-1}$ and $P_{k|k-1}$ are the *a priori* state estimation and covariance matrix, $\hat{x}_{k+1|k+1}$ and $P_{k+1|k+1}$ are their *a posteriori* values updated with the new measurement. The noise matrices Q , R can be obtained from the standard deviations E , F of the system process and measurement equations $Q = EE^T$, $R = FF^T$.

It should be highlighted that, although the Kalman Filters assume that the noise which affects the sensors is white and Gaussian, this is known to be far from reality, since the vibrations caused by the main rotor are usually visible in the frequency spectrum of the measurements. These frequency component are not noise introduced by the sensors but a physically real phenomenon due to the elasticity of the fuselage. However, since the goal is to estimate the angular acceleration of the helicopter C.G., they are here considered as noise in the measurement of the sensors.

Kalman Filter

Since the analyzed system is known to be nonlinear, it is reasonable to use the EKF for the estimation problem. However, one relevant criterion is the computational weight of the observer, which is high for the EKF due to the required calculation of the Jacobian matrix of the H matrix and a matrix inversion calculation for the observer gain L .

The KF uses a constant observer model, linearized around an operating point decided by the designer. In this case, the linearization point is chosen to be the origin $\omega = \alpha = 0$ and thus the measurement equation gets simplified in comparison to that of the EKF:

$$\begin{aligned}
 \hat{x}_{k+1|k} &= A\hat{x}_{k|k} \\
 P_{k+1|k} &= AP_{k|k}A^T + Q \\
 C &= \left. \frac{dg(x_k)}{dx_k} \right|_{x_k=0} \\
 L_k &= P_{k|k-1}C \left(\hat{x}_{k|k-1}, u_k \right)^T \left(C(\hat{x}_{k|k-1}, u_k) P_{k|k-1} C(\hat{x}_{k|k-1}, u_k)^T + R \right)^{-1} \\
 \hat{x}_{k|k} &= \hat{x}_{k|k-1} + L_k \left(y_k - C\hat{x}_{k|k-1} \right) \\
 P_{k|k} &= P_{k|k-1} - P_{k|k-1}C^T \left(CP_{k|k-1}C^T + R \right)^{-1} CP_{k|k-1}
 \end{aligned} \tag{4-18}$$

Since the angular velocity and acceleration of the helicopter during the flight will take values different from the states values chosen as linearization point, the observer model will deviate from the true system. The impact of this deviation depends on the values that the angular velocity can reach during flight, since Ω^2 is the only nonlinear term in the considered rigid model. The impact on the observed estimations due to the nonlinearities is isolated and analyzed in the next chapter, as explained in the previous section.

Stationary Kalman Filter with nonlinear output equation

The complexity of the KF is lower than that of the EKF, although it still requires a matrix inversion operation every cycle of the Flight Control Computer. In order to further reduce the computational weight of the estimator, a constant-gain observer is designed so that no matrix inversion is required to be performed online. The proposed observer gain is obtained in the same fashion than the Stationary Kalman Filter. However, the nonlinear equation of Δa is used as measurement equation, as well as in the model of the EKF.

The substantiation for this choice is that the observer gain can be designed off-line, what reduces its computational weight, while the system nonlinearities are still considered in the observer model, so that its performance far from the linearization point is not impacted.

The proposed observer is called Stationary Kalman Filter with Nonlinear Output (NOSKF), and its gain $L = PC^T(R + CPC^T)^{-1}$ is calculated using the Discrete-Algebraic Riccati Equation (DARE):

$$P = APA^T + Q - (APC^T) \times (R + CPC^T)^{-1} \times (APC^T)^T \tag{4-19}$$

where the Q, R matrices are obtained from the standard deviations E, F of the state and measurement equations $Q = EE^T, R = FF^T$.

The NOSKF is designed in discrete-time as follows:

$$\hat{x}_{(k|k)} = A\hat{x}_{(k|k-1)} + L(y_{(k)} - g(\hat{x}_{(k|k-1)})) \tag{4-20}$$

Stationary Kalman Filter with modified input

A novel observer is proposed for the estimation of angular acceleration under the constraints of low computational weight while providing faster convergence and better noise rejection than the current available option, the Lanczos differentiator.

As well as the KF and the EKF, the Stationary Kalman Filter with Modified Input (MISKF) is based on the approximation that the helicopter can be analyzed as a rigid body and on the constant angular acceleration assumption. However, in contrast to the previous observers, the MISKF uses a modified version of Equations 4-4 as measurement equation.

One can notice that the only nonlinear term Ω^2 in Eqs.4-4 is known since it is measured by the gyroscopes -in addition to some measurement noise. Therefore, Eqs.4-4 can be manipulated as follows

$$\begin{aligned}\Delta a_{2,1} &= (\Omega^2 + \dot{\Omega})\Delta d_{2,1} \\ \Delta a_{3,2} &= (\Omega^2 + \dot{\Omega})\Delta d_{3,2}\end{aligned}\quad (4-21)$$

$$\begin{aligned}\Delta a_{2,1} - \Omega^2\Delta d_{2,1} &= \dot{\Omega}\Delta d_{2,1} \\ \Delta a_{3,2} - \Omega^2\Delta d_{3,2} &= \dot{\Omega}\Delta d_{3,2}\end{aligned}\quad (4-22)$$

Using the properties of the cross-product matrices, one can rewrite the right hand side of the expressions:

$$\begin{aligned}\Delta a_{2,1} - \Omega^2\Delta d_{2,1} &= \dot{\omega} \times \Delta d_{2,1} = -\Delta d_{2,1} \times \dot{\omega} = -\Delta D_{2,1} \dot{\omega} \\ \Delta a_{3,2} - \Omega^2\Delta d_{3,2} &= \dot{\omega} \times \Delta d_{3,2} = -\Delta d_{3,2} \times \dot{\omega} = -\Delta D_{3,2} \dot{\omega}\end{aligned}\quad (4-23)$$

where

$$\Delta D = \begin{bmatrix} 0 & -\Delta z & \Delta y \\ \Delta z & 0 & -\Delta x \\ -\Delta y & \Delta x & 0 \end{bmatrix}\quad (4-24)$$

The left part of the equations are the modified measurements, which can be calculated every time instant since Δd is constant and ω is measured. They are grouped under the variables $\Delta \tilde{a}_{2,1}$ and $\Delta \tilde{a}_{3,2}$, respectively. Finally, taking into account that ΔD is skew-symmetric, the identity $-\Delta D = \Delta D^T$ holds, and the modified measurement equations can be expressed as

$$\begin{aligned}\Delta \tilde{a}_{2,1} &= \Delta D_{2,1}^T \dot{\omega} \\ \Delta \tilde{a}_{3,2} &= \Delta D_{3,2}^T \dot{\omega}\end{aligned}\quad (4-25)$$

In this way, it is possible to obtain a linear measurement equation for the system model:

$$y_k = \bar{C}x_k + \bar{v}(k) = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & \Delta D_{1,2} \\ 0_{3 \times 3} & \Delta D_{3,2} \end{bmatrix} \begin{bmatrix} \omega \\ \alpha \end{bmatrix}_k + \begin{bmatrix} v_\omega \\ \bar{v}_{\tilde{a}_{1,2}} \\ \bar{v}_{\tilde{a}_{3,2}} \end{bmatrix}_k\quad (4-26)$$

Based on the modified system model, the MISKF can be designed as follows:

$$\hat{x}_{(k|k)} = A\hat{x}_{(k|k-1)} + L(y_k - \bar{C}\hat{x}_{(k|k-1)})\quad (4-27)$$

The observer gain $L = P\bar{C}^T(\bar{R} + \bar{C}P\bar{C}^T)^{-1}$ can be calculated off-line using the DARE:

$$P = APA^T + Q - (AP\bar{C}^T) \times (\bar{R} + \bar{C}P\bar{C}^T)^{-1} \times (AP\bar{C}^T)^T\quad (4-28)$$

where the Q , $\bar{R}$ matrices are obtained from the standard deviations E , $\bar{F}$ of the state and measurement equations $Q = EE^T$, $\bar{R} = \bar{F}\bar{F}^T$.

4-2-3 Observers test procedure

The main difference between observers and the estimation methods previously introduced is that the former require a system model, as well as state and measurement noise matrices. These parameters allow observers to fit to different estimation problems, in contrast to the more rigid estimation methods such as speed differentiation or Least Squares. However, they also require further analysis, for instance, to test if the system model is accurate enough to describe the system dynamics, or to tune the noise matrices for the specific sensors and system characteristics.

The process to tune the noise matrices in the estimation problem considered in this master thesis is not straightforward, since one can not directly relate the values of the noise matrices to the estimation errors. This is due to the fact that the errors in the angular acceleration estimations can be caused by several factors in addition to an incorrect noise matrices tuning. One factor could be the assumption that the measurement noise is white when in reality it has relevant components of specific frequencies. Another possible factor could be the usage of an excessively simplistic system model in comparison to the real helicopter, or operating at a point which is too far from the linearization point.

Data recorded in flight can violate at the same time all the assumptions on which the observers are based (constant angular acceleration, rigid body approximation, Gaussian noise characteristics, operation in the neighborhood of the linearization point, etc.). For this reason, it is desired to isolate the different sources of estimation errors as much as possible, in order to ease the tuning process and to identify the main factors which affect the observers performance.

In Chapter 5, a simulation environment in which the helicopter is modeled as a rigid body is used to isolate the impact of the angular acceleration assumption. Later, the influence of measurement noise and the observer performance far from the linearization point are also analyzed.

Finally, after analyzing whether the constant angular acceleration assumption is considered acceptable and which observers provide accurate estimations in the simulation environment, their performance is analyzed with real flight data in Chapter 6.

Observer design for simulated flight data

This chapter shows the impact of three different factors on the performance of the proposed observers using a simulation environment. The first factor is the constant angular acceleration assumption on which all the observer models are based. The second one is the deviation of the operating point from the linearization point used to design the observers. The last factor is the presence of Gaussian white measurement noise in the data used for estimation.

The impact is analyzed in a step-wise approach, i.e., first the feasibility of the constant angular acceleration assumption is studied while the rest of the observer assumptions are respected. To do so, the helicopter and the available sensors are replicated in Simulink, that is used as simulation environment. However, the helicopter is modeled as a rigid body so that the rigid body assumption is fulfilled. In addition to this, the flight conditions are limited near the rest state (zero angular speed and zero angular acceleration) to ensure that the observers operate near their linearization point. All the helicopter states can be retrieved from the simulation, and the goal is to compare some of them, such as the angular acceleration and angular speed, with the estimations obtained with the observers. Once the observers are tuned in this ideal case, they are tested in a second scenario, in which the simulation is performed in flight conditions distant from the linearization point. Finally, the observers are tuned and tested with simulated measurements corrupted with Gaussian white noise.

5-1 Simulation model

The helicopter is modeled in Simulink as a rigid body that can move in six degrees of freedom, according to the Equations of motion 2-12 introduced in Section 2-3, hereafter repeated. In addition to this, the sensors are modeled using equations shown in Section 2-2.

$$\begin{aligned}
\dot{p} &= \frac{J_{zz}}{J^*} M_x + \frac{J_{xz}}{J^*} M_z + \frac{(J_{xx} - J_{yy} + J_{zz})J_{xz}}{J^*} pq + \frac{(J_{yy} - J_{zz})J_{zz} - J_{xz}^2}{J^*} qr \\
\dot{q} &= \frac{M_y}{J_{yy}} + \frac{J_{xz}}{J_{yy}} (r^2 - p^2) + \frac{J_{zz} - J_{xx}}{J_{yy}} pr \\
\dot{r} &= \frac{J_{xz}}{J^*} M_x + \frac{J_{xx}}{J^*} M_z + \frac{(J_{xx} - J_{yy})J_{xx} + J_{xz}^2}{J^*} pq + \frac{(-J_{xx} + J_{yy} - J_{zz})J_{xz}}{J^*} qr
\end{aligned}$$

where $J^* = J_{xx}J_{zz} - J_{xz}^2$ and $\sum T_{CG} = [M_x \ M_y \ M_z]^T$ are the net torques in the X_b , Y_b , and Z_b -axis, respectively.

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \frac{\sin \phi}{\cos \theta} & \frac{\cos \phi}{\cos \theta} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

The system model can then be described with the state-space formulation

$$\begin{aligned}
\dot{x} &= f(x, u) \\
y &= g(x)
\end{aligned} \tag{5-1}$$

where the state vector x includes the angular speed and the Euler angles $x = [\omega \ \Theta]^T$, where $\omega = [p \ q \ r]^T$ and $\Theta = [\phi \ \theta \ \psi]^T$, and the system outputs are the measurements obtained from the gyroscopes and the three tri-axial linear accelerometers

$$\begin{aligned}
\dot{\omega} &= J^{-1} [\sum T_{CG} - \omega \times J \omega] \\
\dot{\Theta} &= B_{(\Theta)} \omega
\end{aligned} \tag{5-2}$$

$$y = \begin{bmatrix} \omega \\ \dot{V} + \Omega V + \dot{\Omega} d_{S_1} + \Omega^2 d_{S_1} - R_{bE} g \\ \dot{V} + \Omega V + \dot{\Omega} d_{S_2} + \Omega^2 d_{S_2} - R_{bE} g \\ \dot{V} + \Omega V + \dot{\Omega} d_{S_3} + \Omega^2 d_{S_3} - R_{bE} g \end{bmatrix} \tag{5-3}$$

where J is the inertia matrix under the assumption that the helicopter is symmetric in the X_b, Z_b -plane, and Ω is the skew-symmetric matrix of ω :

$$J = \begin{bmatrix} J_{xx} & 0 & -J_{xz} \\ 0 & J_{yy} & 0 \\ -J_{xz} & 0 & J_{zz} \end{bmatrix} \tag{5-4}$$

$$\Omega = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \quad \Omega^2 = \begin{bmatrix} -(q^2 + r^2) & pq & pr \\ pq & -(p^2 + r^2) & qr \\ pr & qr & -(p^2 + q^2) \end{bmatrix} \tag{5-5}$$

The sensors are modelled as ideal in a first stage, i.e., the measurements of the gyroscopes are directly obtained from the angular speeds of the helicopter simulation, and the measurements of the linear accelerometers are computed based on Equation 4-2 using the simulation states. After testing the results obtained with ideal sensors, white noise is added to the measurements, as shown in Eq. 5-6.

$$y_n = \begin{bmatrix} \dot{V} + \Omega V + \dot{\Omega} d_{S_1} + \Omega^2 d_{S_1} - R_{bE} g \\ \dot{V} + \Omega V + \dot{\Omega} d_{S_2} + \Omega^2 d_{S_2} - R_{bE} g \\ \dot{V} + \Omega V + \dot{\Omega} d_{S_3} + \Omega^2 d_{S_3} - R_{bE} g \end{bmatrix} + \begin{bmatrix} w_{gyro} \\ w_{S_1} \\ w_{S_2} \\ w_{S_3} \end{bmatrix} \quad (5-6)$$

where $w_{gyro} = [w_{g,x} \ w_{g,y} \ w_{g,z}]^T$ and $w_{S_i} = [w_{S_i,x} \ w_{S_i,y} \ w_{S_i,z}]^T$ for $i = \{1, 2, 3\}$.

As explained when the observer models were introduced in Section 4-2-2, in addition to angular speed measurement ω , the Extended Kalman Filter (EKF), the Kalman Filter (KF), and the Stationary Kalman Filter with Nonlinear Output (NOSKF) use the difference of linear accelerations measurements Δa , while the Stationary Kalman Filter with Modified Input (MISKF) uses the modified version of the measurement $\Delta \tilde{a}$ (Eq. 4-25). Therefore, the output of the simulation model y , y_n are further manipulated to provide the observers with the required information.

5-2 Analysis of the model assumptions and observers tuning

The goal of this section is to address the influence of the constant angular acceleration assumption used in the observer model, as well as the impact due to measurement noise and due to operating far from the linearization point. In addition to this, it is desired to determine the weight that each measurement variable has in the estimations. This aspect can be analyzed through the observer gain L , which multiplies the error in the predicted output $y - \hat{y}$ to correct the observed states $\hat{\omega}, \hat{\alpha}$. A block structure can be identified in the matrix L :

$$L = \begin{bmatrix} L_{\hat{\omega}, e_{\omega}} & L_{\hat{\omega}, e_{\Delta a, 1}} & L_{\hat{\omega}, e_{\Delta a, 2}} \\ L_{\hat{\alpha}, e_{\omega}} & L_{\hat{\alpha}, e_{\Delta a, 1}} & L_{\hat{\alpha}, e_{\Delta a, 2}} \end{bmatrix} = \begin{bmatrix} L_{\hat{\omega}, e_{\omega}} & L_{\hat{\omega}, e_{\Delta a}} \\ L_{\hat{\alpha}, e_{\omega}} & L_{\hat{\alpha}, e_{\Delta a}} \end{bmatrix} \quad (5-7)$$

where $e_{\omega} = \omega - \hat{\omega}$ and $e_{\Delta a} = \Delta a - \Delta \hat{a}$ are the errors in the observer predicted output.

The terms of the first row indicate the influence of e_{ω} and $e_{\Delta a}$ on the estimation of $\hat{\omega}$, respectively, while the elements on the second row show the influence of e_{ω} and $e_{\Delta a}$ on the estimation of $\hat{\alpha}$. The two latter components are especially relevant to address the behaviour of the observer, since the gains grouped in $L_{\hat{\alpha}, e_{\Delta a}}$ reveal if the observer uses the information from the linear accelerometers to estimate the angular acceleration.

On the other hand, the term $L_{\hat{\alpha}, e_{\omega}}$ performs the estimation of the angular acceleration using the angular velocity, i.e., it is a pseudo-differentiator, which operation is similar to the one used by current differentiation methods and can be performed without the use of the linear accelerometers. For this reason, the components $L_{\hat{\alpha}, e_{\omega}}$, $L_{\hat{\alpha}, e_{\Delta a}}$ are shown for every configuration of the observers during the tuning process, in order to analyze whether they use the linear accelerometers information and thus if incorporating that measurements is useful for estimating the angular acceleration.

5-2-1 Ideal sensors and flight conditions near the linearization point

The main goal in this scenario is to isolate the estimation error so that it is only due to the assumption that the system has a constant angular acceleration. In order to test it,

a low-amplitude sinusoidal torque with a frequency of 3 *Hz* is applied in the roll axis, a high-amplitude sinusoidal torque is applied in the pitch axis, and three step input torques are applied in the yaw axis. The input torques lead to angular velocities between 10 and -10 *rad/s*, i.e., in the neighborhood of the rest state (zero angular speed) used as linearization point for the KF, the NOSKF, and the MISKF. In addition to this, the simulated measurements from the gyroscopes and the linear accelerometers are not corrupted by noise.

Since the angular accelerations of the body are different from zero, the first assumption of every observer model is known to be violated. Since the angular velocity remains near zero, the approximation taken by linearizing the measurement model in the KF remains valid. In addition to this, the measurement models of the EKF and the MISKF match with the system model, since the former observer is independent of the linearization point (since it linearizes the model at every operating point), and in the latter observer the modifications in the input measurements avoid the nonlinearities. However, the case of the NOSKF has to be closely analyzed, since the observer gain has been designed using the linearized observer model, while it uses the nonlinear measurement model equations. This aspect, as well as well as the performance of all the observers is covered hereafter.

Extended Kalman Filter

The EKF model shown in Eqs. 4-17 is used to observe the angular acceleration of the simulated rigid body and analyze the constant angular acceleration assumption. Since the measurements from the gyroscopes and the linear accelerometers are noiseless, their associated standard deviations are set very low:

$$F_{\omega} = \begin{bmatrix} 10^{-3} & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix} \quad F_{\Delta a_{2,1}} = F_{\Delta a_{3,2}} = \begin{bmatrix} 10^{-3} & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix} \quad (5-8)$$

The assumption of a constant angular acceleration is known to be incorrect, since the simulation is designed to generate angular accelerations in the range from 10 to $\pm 30 \text{ deg/s}^2$. Since that assumption is reflected in the observer model via the process equation of the angular acceleration α , its corresponding standard deviation matrix E_{α} is designed to indicate a low confidence on the dynamics of that state. On the other hand, the standard deviation of the angular speed is set to zero, since it is known to be exactly the integral of the angular acceleration, and thus its dynamic model is accurate:

$$E_{\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{\alpha} = \begin{bmatrix} 10^3 & 0 & 0 \\ 0 & 10^3 & 0 \\ 0 & 0 & 10^3 \end{bmatrix} \quad (5-9)$$

The angular speed and angular acceleration estimations obtained with the EKF are shown in Figures 5-1 and 5-2, respectively.

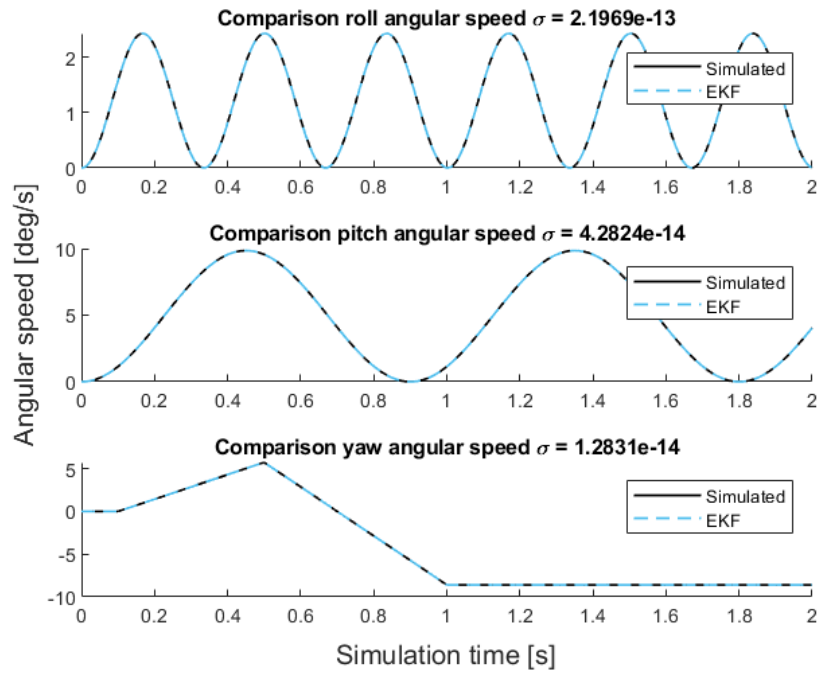


Figure 5-1: Comparison of simulated angular speed and Extended Kalman Filter angular speed estimations

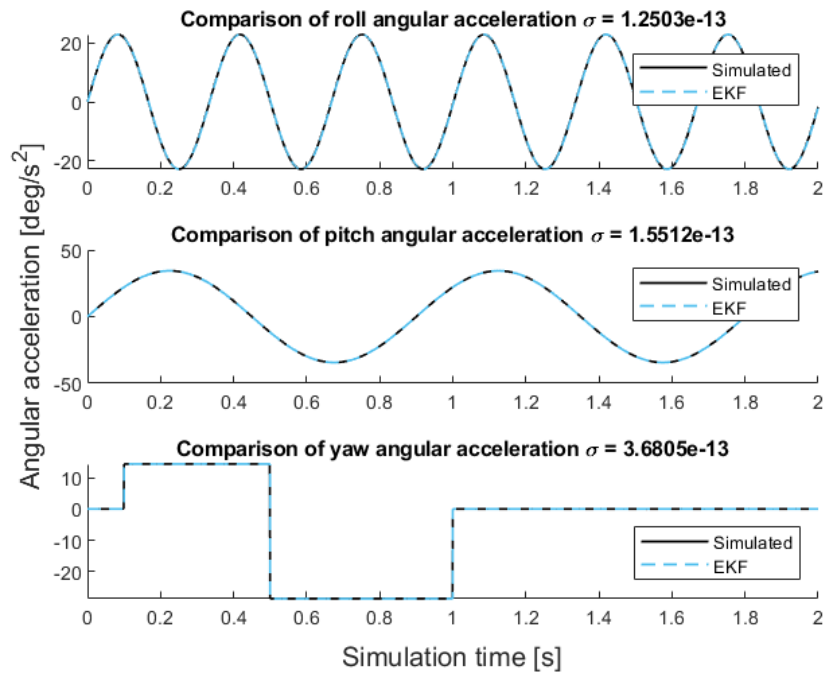


Figure 5-2: Comparison of simulated angular acceleration and Extended Kalman Filter angular acceleration estimations

Regarding the angular speeds, one can observe that they are limited to the range of 5 to 10 deg/s , i.e., in the neighborhood of the rest point. Since the measurement of ω does not include noise, the estimated angular speed match the measurement.

The angular acceleration estimations show a remarkable accuracy, even with standard deviation matrices not thoroughly tuned. The observer has proved to correctly deal with angular acceleration steps of 40 deg/s^2 , as can be observed at second 0.5 in the yaw angular acceleration in Figure 5-2. Therefore, the constant angular acceleration assumption is considered acceptable for the EKF in this scenario.

In order to analyze the weight that the angular speed and the linear accelerations difference have on the estimations, one can inspect the observer gain components $L_{\hat{\alpha},e_{\omega}}$, $L_{\hat{\alpha},e_{\Delta a_{2,1}}}$, and $L_{\hat{\alpha},e_{\Delta a_{3,2}}}$. However, it should be noticed that the gain of the EKF gets updated at every operating point of the system, which is constantly varying in this scenario, and thus the observer gain also changes continuously. Nevertheless, the observer components previously mentioned show the same order of magnitude along all the simulation, what allows to draw some conclusions on the relevance that ω and Δa have on the estimations. A sample of the observer components taken at time instant 0.6 seconds is shown hereafter:

$$L_{\hat{\alpha},e_{\omega}} = 10^{-6} \begin{bmatrix} -22.5 & 0.36 & -8.2 \\ 15.88 & 8.35 & -0.79 \\ 21.44 & 5.14 & -1.58 \end{bmatrix}$$

$$L_{\hat{\alpha},e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} 2.77 & 2.77 & -5.54 \\ -8.31 & -8.32 & -16.7 \\ 13.89 & 13.87 & 5.57 \end{bmatrix} \quad L_{\hat{\alpha},e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} 5.54 & 13.90 & 13.89 \\ -16.63 & -8.36 & -8.34 \\ -5.57 & 2.78 & 2.79 \end{bmatrix}$$

One can observe that the weight that the linear accelerations difference has on the angular acceleration estimations is four orders of magnitude higher than that of the angular speed measurement. This indicates that, at least for this first scenario, the availability of the linear accelerometers is useful for the estimation problem, and the EKF does not work as a pseudo-differentiator.

Kalman Filter

The KF model shown in Eqs. 4-18 is used to obtain angular acceleration estimations using the same the standard deviation matrices than in the EKF:

$$F_{\omega} = \begin{bmatrix} 10^{-3} & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix} \quad F_{\Delta a_{2,1}} = F_{\Delta a_{3,2}} = \begin{bmatrix} 10^{-3} & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix} \quad (5-10)$$

$$E_{\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{\alpha} = \begin{bmatrix} 10^3 & 0 & 0 \\ 0 & 10^3 & 0 \\ 0 & 0 & 10^3 \end{bmatrix} \quad (5-11)$$

In contrast to the EKF, the KF does not update its gain based on the system state, but just on the relative confidence on the system model and the available measurements. The observer gain after convergence has the following components which influence the angular acceleration estimations:

$$L_{\hat{\alpha}, e_{\omega}} = 10^{-18} \begin{bmatrix} 3.00 & -3.86 & 1.63 \\ 3.87 & 8.46 & -3.86 \\ 1.64 & -3.86 & 3.02 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} 2.78 & 2.78 & -5.56 \\ -8.32 & -8.32 & -16.68 \\ 13.88 & 13.88 & 5.56 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} 5.56 & 13.89 & 13.88 \\ -16.65 & -8.34 & -8.34 \\ -5.56 & 2.78 & 2.78 \end{bmatrix}$$

In a similar fashion than the EKF, the observer heavily relies on the linear accelerations difference Δa , while the angular speed measurement ω is essentially excluded of the estimation process.

The estimations of the angular speed coincide with the measured ω , and thus its comparison is here omitted for the sake of conciseness. The angular acceleration estimations obtained with the KF are shown in the Figure 5-3.

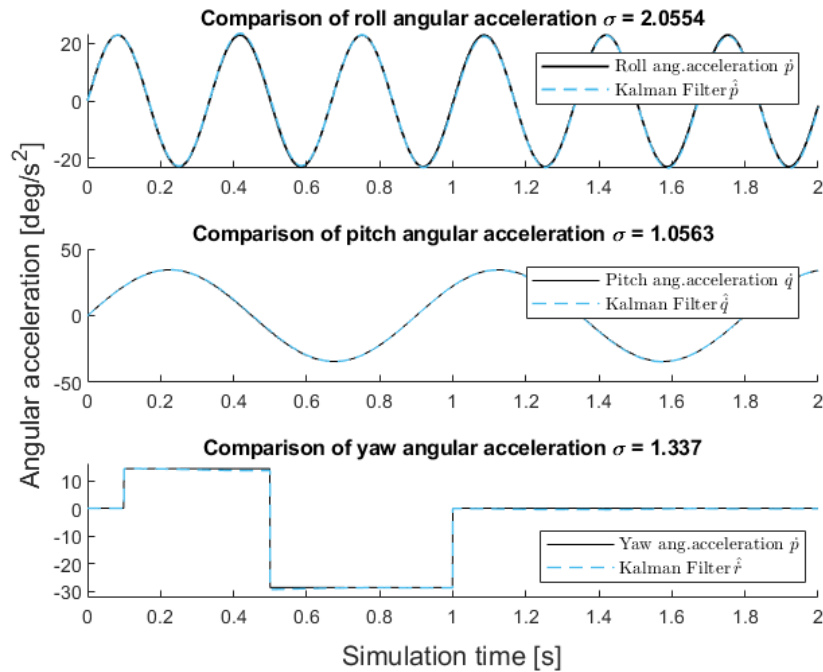


Figure 5-3: Comparison of simulated angular acceleration and Kalman Filter angular acceleration estimations

The accuracy of the KF is remarkably poorer than the EKF, but the estimations show an acceptable performance, and the changes in angular acceleration are correctly estimated.

For this reason, the constant angular acceleration assumption and the observer model are considered positively for this scenario.

Stationary Kalman Filter with Nonlinear Output

The standard deviation matrices considered for the NOSKF are the same than those used in the previous observers. As explained in Section 4-2-2, the noise matrices $Q = EE^T$, $R = FF^T$ are used to solve the Discrete-Algebraic Riccati Equation (DARE) to obtain the observer gain, which has the following components determining the angular acceleration estimations:

$$L_{\hat{\alpha}, e_{\omega}} = 10^{-9} \begin{bmatrix} 16.49 & -8.18 & 1.76 \\ -0.14 & 8.46 & 1.19 \\ 0.13 & -0.09 & 0.04 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} 2.78 & 2.78 & -5.56 \\ -8.33 & -8.33 & -16.68 \\ 13.89 & 13.89 & 5.56 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} 5.56 & 13.89 & 13.89 \\ -16.67 & -8.34 & -8.33 \\ -5.56 & 2.78 & 2.78 \end{bmatrix}$$

The angular acceleration estimations obtained with the NOSKF are shown in the Figure 5-4.

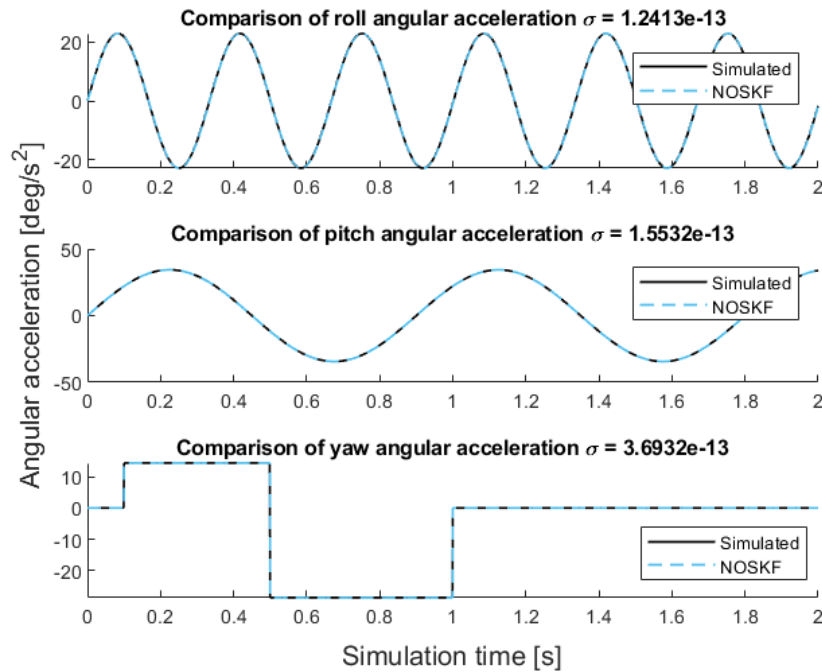


Figure 5-4: Comparison of simulated angular acceleration and Stationary Kalman Filter with Nonlinear Output angular acceleration estimations

Although the NOSKF is a constant-gain observer, its estimations show a similar accuracy to those of the EKF and higher than the KF, what highlights the advantage of using the

nonlinear expression of the measurement model of the linear accelerometers in the observer output equation.

Stationary Kalman Filter with Modified Input

The standard deviation matrices considered for the MISKF are the same than those used in the previous observers and are omitted here for brevity. The observer gain of the MISKF is the same than that of the NOSKF shown in the previous subsection, since both are designed by solving the same DARE with the same noise matrices. The angular acceleration estimations obtained with the MISKF are shown in Figure 5-5.

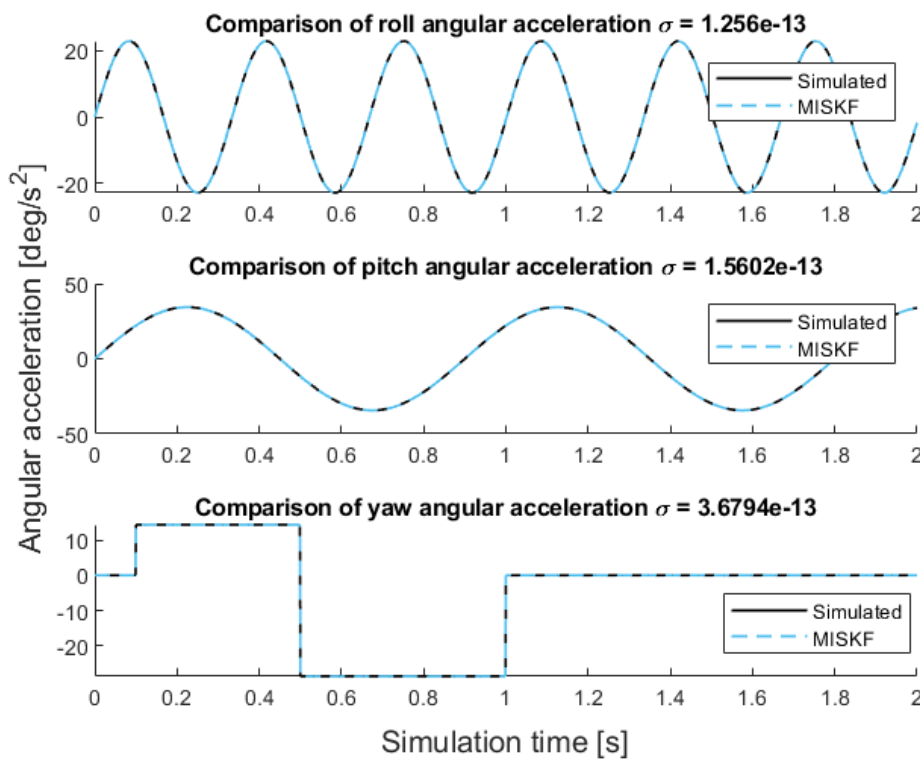


Figure 5-5: Comparison of simulated angular acceleration and Stationary Kalman Filter with Modified Input angular acceleration estimations

The previous results allow to conclude that the MISKF provides estimations with similar accuracy to those of the EKF in the considered scenario although it is a Linear-Time-Invariant (LTI) system, what additionally implies that it is computationally lightweight.

The analysis of the observers has shown that they can correctly estimate the angular acceleration of the body under input torques of different shapes, such as steps or sinusoids, despite that the angular acceleration is assumed to be constant in the system model. The satisfactory performance under this assumption allows to use compact observer models such as the NOSKF or the MISKF, which have proven to provide accurate estimations when using noiseless measurements and working near the linearization point. The next section covers the

applicability of the analyzed observers in flight states distant from the origin, while relying on the system model successfully tested in this scenario.

5-2-2 Ideal sensors and flight conditions different from the linearization point

In this section, constant torques are applied in every axis of the rigid body until it reaches rotational speeds in the range from 10 to 15 deg/s , what is considered far from the linearization point used for the KF, the NOSKF, and the MISKF. Since it is possible to reach these angular speeds during flight, it is necessary to ensure a correct performance of the observers in this condition.

Extended Kalman Filter

The EKF linearizes the system dynamics every sampling instant, and its gain is recalculated online based on the linearized system matrices and the noise covariances. Since the measurements from the gyroscopes and the linear accelerometers are still noiseless, their associated standard deviations are the same than in the previous scenario:

$$F_{\omega} = \begin{bmatrix} 10^{-3} & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix} \quad F_{\Delta a_{2,1}} = F_{\Delta a_{3,2}} = \begin{bmatrix} 10^{-3} & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix} \quad (5-12)$$

The assumption of a constant angular acceleration is known to be incorrect, since the simulation is designed to generate angular accelerations in the range from 10 to 20 deg/s^2 . For that reason, the process standard deviation matrices are designed to indicate a low confidence on the process equation of the angular acceleration α , as well as in the previous scenario:

$$E_{\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{\alpha} = \begin{bmatrix} 10^3 & 0 & 0 \\ 0 & 10^3 & 0 \\ 0 & 0 & 10^3 \end{bmatrix} \quad (5-13)$$

In contrast to the previous scenario, in which the flight state changed continuously and the observer gain did not converge to a specific set of values, in this case it converges after the angular accelerations drop to zero and the angular speeds reach their maximum value. The observer gain after convergence shows the following components which determine the angular acceleration estimations:

$$L_{\hat{\alpha}, e_{\omega}} = 10^{-6} \begin{bmatrix} -22.5 & 0.36 & -8.2 \\ 15.88 & 8.35 & -0.79 \\ 21.44 & 5.14 & -1.58 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} 2.78 & 2.77 & -5.55 \\ -8.34 & -8.33 & -16.67 \\ 13.88 & 13.91 & 5.54 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} 5.55 & 13.88 & 13.91 \\ -16.67 & -8.32 & -8.34 \\ -5.54 & 2.77 & 2.77 \end{bmatrix}$$

The relative weight that the angular speed and the linear accelerations difference have in the angular acceleration estimations is very similar to that of the previous scenario, being the gain on the latter four orders of magnitude higher than on the former.

The angular speed and angular acceleration estimations obtained with the EKF are shown in the Figures 5-6 and 5-7, respectively.

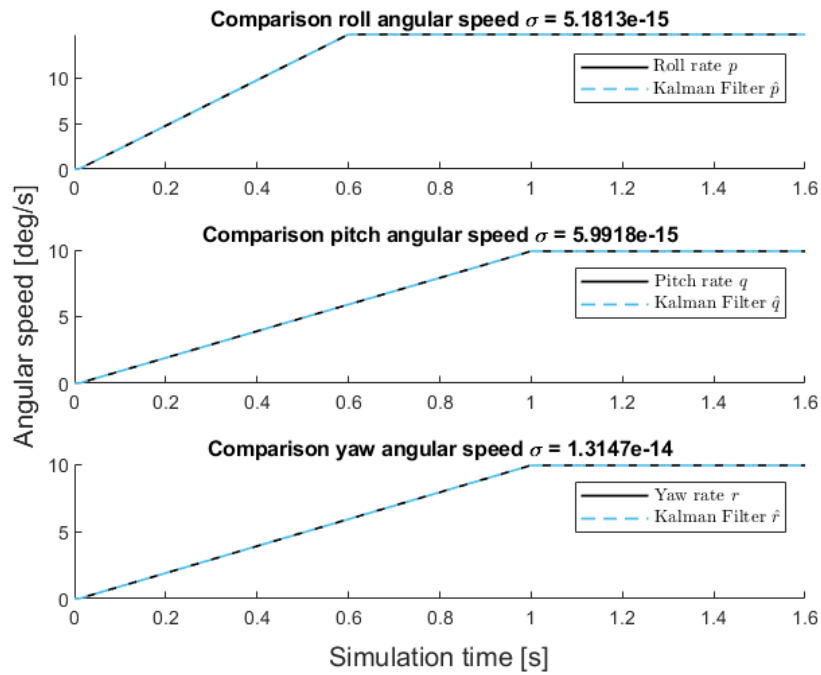


Figure 5-6: Comparison of realistic angular speed and Kalman Filter angular speed estimations

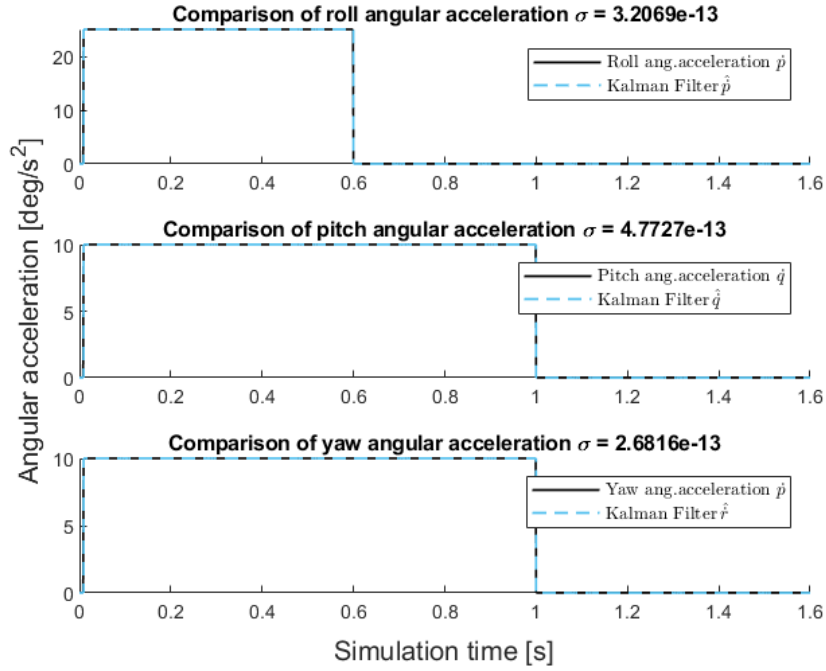


Figure 5-7: Comparison of realistic angular acceleration and Kalman Filter angular acceleration estimations

The angular speeds are correctly estimated and, since they are also measured and the estimations do not show any inconsistencies with the measurements for any of the observers presented hereafter, the angular speed estimations obtained in the second scenario are not shown again for the sake of conciseness.

The accuracy of the angular acceleration estimations is remarkably high, as well as in the previous scenario. It can be noticed that the performance of the EKF is not affected by the change in the operating point of the observer, what highlights the ability of this estimator to deal with the smooth nonlinearities present in the model of the linear accelerometers, i.e., quadratic functions due to the term Ω^2 .

Kalman Filter

The measurement standard deviation of the angular speed is set very low, as in the previous section, due to the fact that the gyroscopes are considered noiseless. However, this is not the case for Δa , since, although the linear accelerometers are also considered noiseless, the measurement model is used at a point different from rest (zero angular speed and zero angular acceleration), where it was linearized.

Therefore, in order to take into account the possible impact of the error due to the nonlinearity in the measurement model of the linear accelerometers, the measurement standard deviation

matrices are set as follows:

$$F_\omega = \begin{bmatrix} 10^{-3} & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix} \quad F_{\Delta a_{2,1}} = F_{\Delta a_{3,2}} = \begin{bmatrix} 10^2 & 0 & 0 \\ 0 & 10^2 & 0 \\ 0 & 0 & 10^2 \end{bmatrix} \quad (5-14)$$

On the other hand, the considerations regarding the process equation are still considered valid, and thus the process standard deviations E_ω and E_α are set in the same way as for the EKF:

$$E_\omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_\alpha = \begin{bmatrix} 10^3 & 0 & 0 \\ 0 & 10^3 & 0 \\ 0 & 0 & 10^3 \end{bmatrix} \quad (5-15)$$

This set of state and measurement standard deviations leads to an observer gain with the following components:

$$L_{\hat{\alpha}, e_\omega} = 10^{-1} \begin{bmatrix} 4.61 & -2.77 & 0.92 \\ -1.54 & 8.31 & -2.77 \\ 0.92 & -2.77 & 4.61 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} 2.77 & 2.77 & -5.56 \\ -8.32 & -8.32 & -16.65 \\ 13.88 & 13.88 & 5.55 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} 5.55 & 13.88 & 13.88 \\ -16.65 & -8.32 & -8.32 \\ -5.56 & 2.77 & 2.77 \end{bmatrix}$$

The estimations obtained with the KF are shown in Figure 5-8:

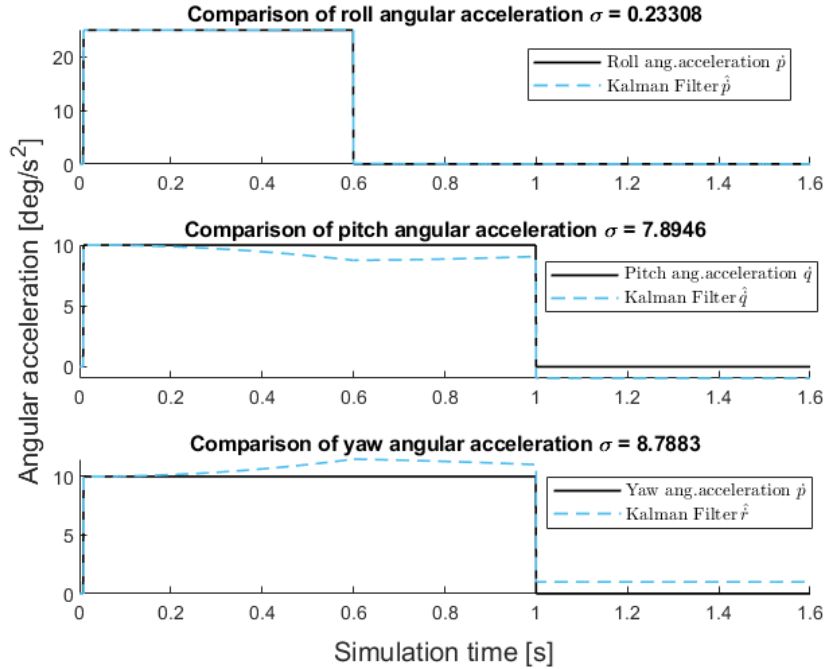


Figure 5-8: Comparison of realistic angular acceleration and Kalman Filter angular acceleration estimations

One can observe that, given the previous state and measurement noise covariances, the angular acceleration estimations do not converge from the moment that the body has been accelerating for more than 0.5 s, i.e., when its angular speed is in the range of 5 to 10 *deg/s* or higher.

At this point, it is hypothesized that the cause of the error is that the linearization approximation applied to the nonlinear equations of Δa (Eq. 4-14) is not valid at the angular velocities considered in this scenario. Based on this idea, the measurement standard deviations are modified to assign a lower relevance on the estimation to the measurement Δa obtained from the linear accelerometers. In this way, the observer is forced to work as a pseudo-differentiator in order to reduce the influence of the inaccuracies due to the nonlinearities of the linear accelerometers expressions:

$$F_{\Delta a_{2,1}} = F_{\Delta a_{3,2}} = \begin{bmatrix} 10^5 & 0 & 0 \\ 0 & 10^5 & 0 \\ 0 & 0 & 10^5 \end{bmatrix} \quad (5-16)$$

$$L_{\hat{\alpha}, e_{\omega}} = \begin{bmatrix} 997.30 & -0.87 & 10^{-8} \\ -10^{-5} & 972.01 & -10^{-5} \\ 10^{-8} & -0.87 & 997.30 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-6} \begin{bmatrix} 10^{-8} & 10^{-8} & -299.19 \\ -10^{-5} & -10^{-5} & -0.03 \\ 299.19 & 299.19 & 10^{-5} \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-6} \begin{bmatrix} 10^{-5} & 299.19 & 299.19 \\ -0.03 & -10^{-5} & -10^{-5} \\ -299.19 & 10^{-8} & 10^{-8} \end{bmatrix}$$

The angular acceleration estimations obtained with the new observer gain are shown in Figure 5-9:

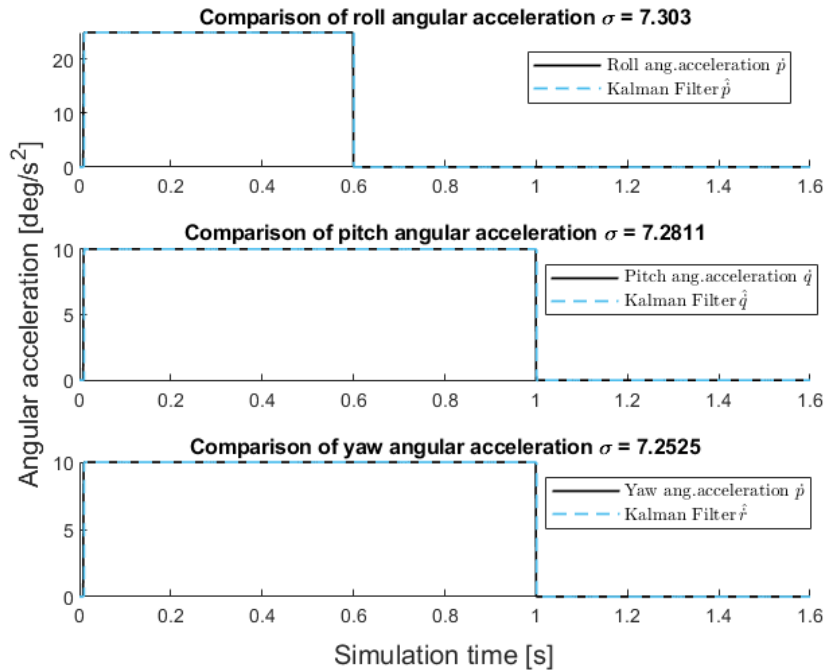


Figure 5-9: Comparison of realistic angular acceleration and Kalman Filter angular acceleration estimations

The modified standard deviation matrices lead to accurate angular acceleration estimations, and thus confirms the hypothesis that relates the steady-state error of the estimations to the inaccuracies of the linear accelerometers model far from its linearization point. However, the previous approach is based on noiseless measurements, what is not always applicable for realistic sensor models, as will be shown in the next scenario.

Stationary Kalman Filter with Nonlinear Output

The gain of the NOSKF is designed based on the system dynamics linearized at rest (zero angular speed and zero angular acceleration), and thus the observer is forced to work in states far from its design point. The process and measurement standard deviations are chosen to be the same than in the previous scenario and are not repeated here for the sake of conciseness, as well as the components of the observer gain.

The angular acceleration estimations obtained with the NOSKF in this scenario are shown in the Figure 5-10.

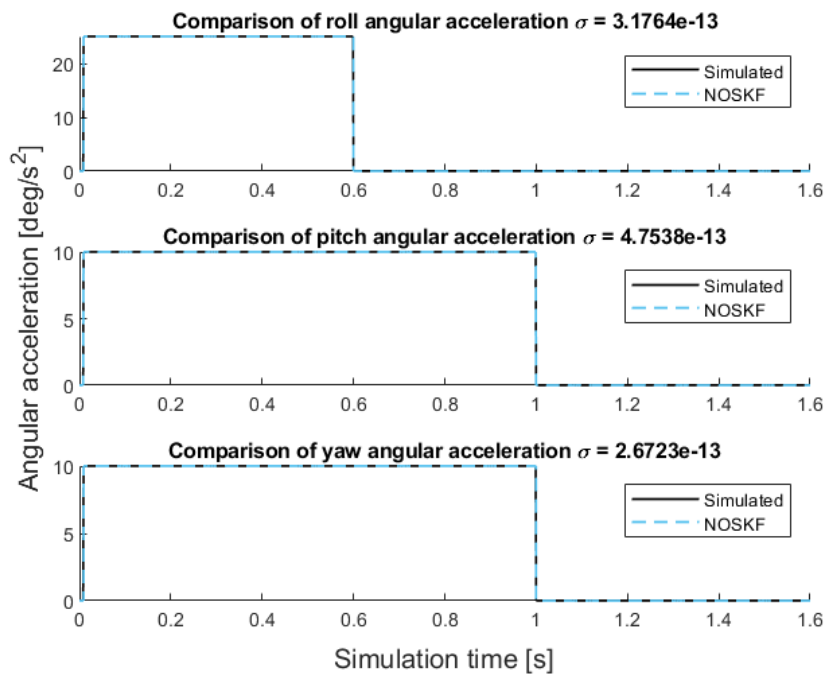


Figure 5-10: Comparison of simulated angular acceleration and Nonlinear-Output Kalman Filter angular acceleration estimations

It is remarkable that the accuracy of the NOSKF is comparable to that of the EKF. Moreover, the estimations obtained with the NOSKF in the current and the previous scenarios have been performed using the same observer gain. This highlights the broad applicability of the NOSKF in many flight states, while keeping the simplicity and low computational weight of a constant gain observer.

Stationary Kalman Filter with Modified Input

As previously stated, the observer gain of the MISKF coincides with that of the NOSKF when the used noise matrices are the same. That is the case for this scenario, and thus the noise matrices and the observer gain are not shown again here. The angular acceleration estimations provided by the MISKF are shown in Figure 5-11.

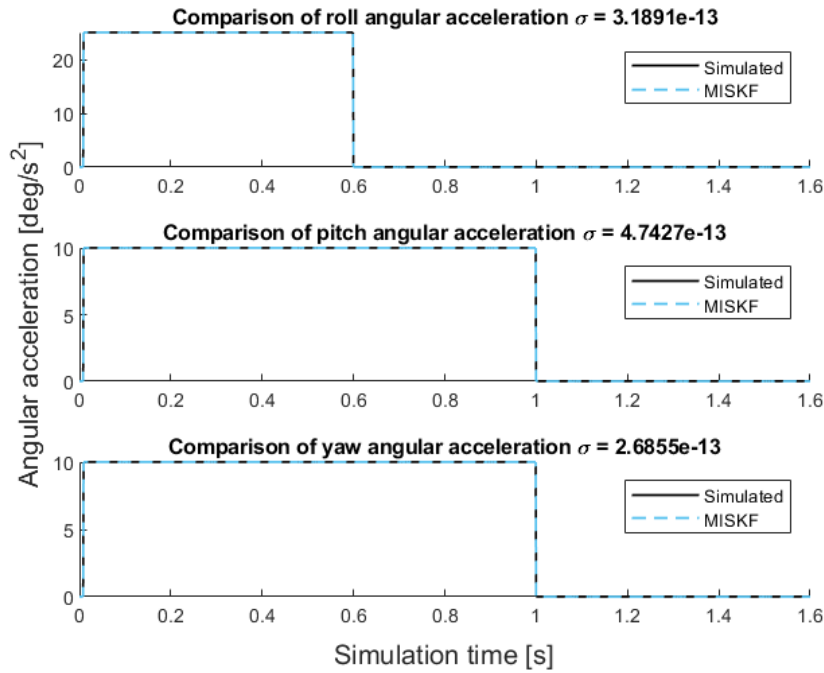


Figure 5-11: Comparison of simulated angular acceleration and Stationary Kalman Filter with Modified Input angular acceleration estimations

It can be concluded that the MISKF has a performance comparable to that of the EKF, while being remarkably more compact. In addition to this, it allows to deal with the nonlinearities of the linear accelerometers model in an alternative way to the NOSKF, offering the same broad applicability but keeping the observer linear.

In this scenario, it has been shown that all the observers provide accurate angular acceleration estimations even far from their linearization point when the measurements are noiseless. For the case of the EKF, the NOSKF, and the MISKF, it has not been necessary to modify their associated noise matrices. In contrast, the KF requires to heavily rely on the angular speed measurements in order to provide angular acceleration estimations without steady-state error. The next section covers the scenario in which the sensor measurements are corrupted by noise and its effect on the observers performance.

5-2-3 Noisy sensors and flight conditions different from the linearization point

In this scenario, the same torques than in the previous one are applied to the rigid body helicopter model. In addition to that, the angular speed and the linear acceleration measure-

ments are corrupted with zero-mean white Gaussian noise, with a variance of $2 \times 10^{-5} \text{ deg/s}$ and $5 \times 10^{-3} \text{ m/s}^2$, respectively.

Extended Kalman Filter

The measurement noise matrices used to design the observer are set slightly different to the exact noise used in the simulation in order to simulate that it is not perfectly known:

$$F_\omega = \begin{bmatrix} \sqrt{8 \times 10^{-5}} & 0 & 0 \\ 0 & \sqrt{8 \times 10^{-5}} & 0 \\ 0 & 0 & \sqrt{8 \times 10^{-5}} \end{bmatrix} \quad (5-17)$$

$$F_{\Delta a_{2,1}} = F_{\Delta a_{3,2}} = \begin{bmatrix} \sqrt{12 \times 10^{-3}} & 0 & 0 \\ 0 & \sqrt{12 \times 10^{-3}} & 0 \\ 0 & 0 & \sqrt{12 \times 10^{-3}} \end{bmatrix} \quad (5-18)$$

On the other hand, the standard deviation of the angular acceleration process equation E_α is tuned as follows:

$$E_\omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_\alpha = 10^{-3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (5-19)$$

Given the current set of state and measurement standard deviations, a sample of the observer gain after convergence is shown hereafter:

$$L_{\hat{\alpha}, e_\omega} = 10^{-4} \begin{bmatrix} 4.55 & -5.04 & 2.51 \\ -1.44 & 8.51 & -4.15 \\ -1.201 & -0.65 & 3.91 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{1,3}}} = 10^{-2} \begin{bmatrix} 2.47 & 3.82 & -1.18 \\ 0.28 & -0.307 & 3.12 \\ -3.12 & 1.12 & -0.80 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} 0.304 & 0.303 & -2.42 \\ -1.10 & -1.09 & -3.82 \\ 3.52 & 0.80 & -2.43 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} 0.80 & 3.52 & 3.53 \\ -3.82 & 1.10 & -1.09 \\ -2.43 & 0.29 & 0.29 \end{bmatrix}$$

It can be noticed that the ratio between the weights allocated to the angular speed and to the linear accelerations difference is lower than in the previous cases. While in the previous scenarios there was an approximate difference of four orders of magnitude, in this one the difference is only of two orders of magnitude. This can be explained by the noise associated to Δa via the matrices $F_{\Delta a_{2,1}}$, $F_{\Delta a_{3,2}}$, which is higher than the noise of the angular speed measurement F_ω .

The angular acceleration estimation obtained with the tuned observer is shown in Figure 5-12. It can be highlighted that the NOSKF is able to reject the measurement noise without introducing a significant time delay, while showing a maximum error of 11 % in the pitch axis.

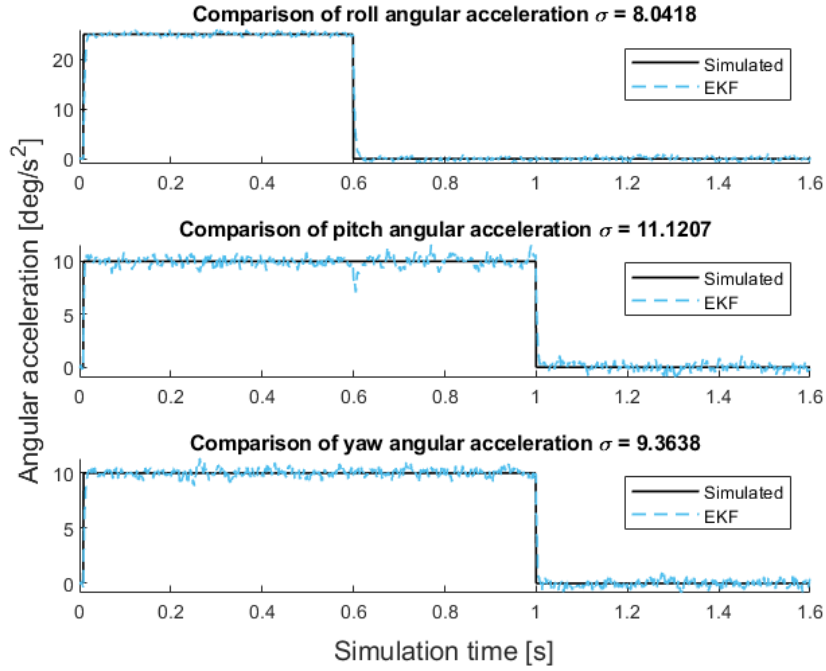


Figure 5-12: Comparison of simulated angular acceleration and Extended Kalman Filter angular acceleration estimations

Kalman Filter

The measurement noise matrices for the KF are set the same than for the EKF. Meanwhile, the standard deviation of the process noise is initially set as follows:

$$E_{\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{\alpha} = 10^{-2} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad (5-20)$$

The component of the observer gain which determine the angular acceleration estimates after convergence are the following:

$$L_{\hat{\alpha}, e_{\omega}} = 10^{-4} \begin{bmatrix} 5.1 & -5.7 & 2.1 \\ -5.7 & 12.9 & -5.2 \\ 2.1 & -5.2 & 3.9 \end{bmatrix} \quad (5-21)$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} 1.96 & 1.96 & -5.79 \\ -6.35 & -6.35 & -13.93 \\ 12.4 & 12.4 & 4.39 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} 4.22 & 11.97 & 11.97 \\ -13.77 & -6.18 & -6.18 \\ -6.05 & 1.96 & 1.96 \end{bmatrix} \quad (5-22)$$

The estimations obtained with the observer are shown in Figure 5-13:

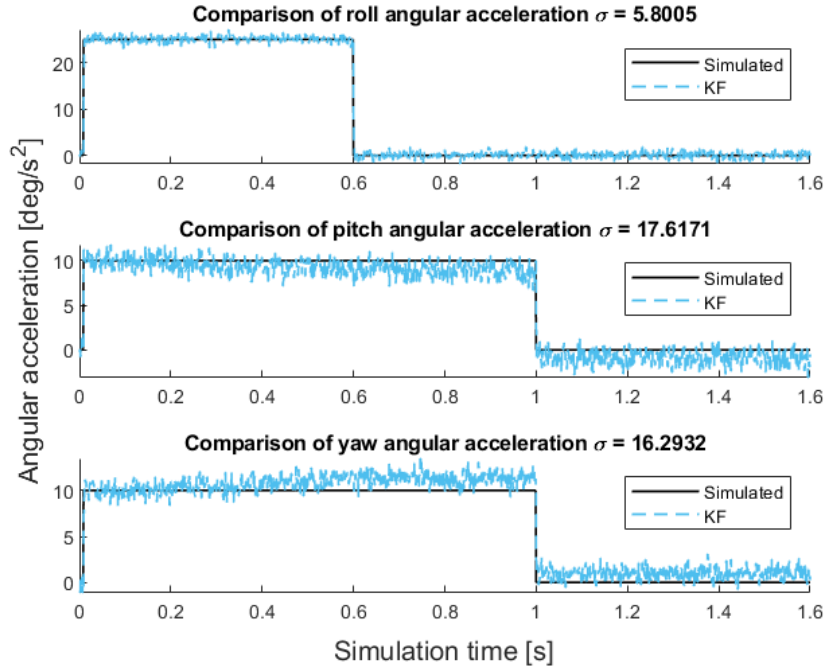


Figure 5-13: Comparison of realistic angular acceleration and Kalman Filter angular acceleration estimations

The figure shows that the estimations obtained with the previous set of noise matrices do not converge to the simulated value, and the angular accelerations (mostly in pitch and yaw) show a steady-state error. In the same fashion than in the previous scenario, it is decided to reduce the weight of the linear acceleration difference Δa in the angular acceleration estimations. This is performed by increasing the standard deviation assigned to the linear accelerometer measurements $F_{\Delta a_{2,1}}$, $F_{\Delta a_{3,2}}$:

$$F_{\Delta a_{2,1}} = F_{\Delta a_{3,2}} = 10^3 \begin{bmatrix} \sqrt{12 \times 10^{-3}} & 0 & 0 \\ 0 & \sqrt{12 \times 10^{-3}} & 0 \\ 0 & 0 & \sqrt{12 \times 10^{-3}} \end{bmatrix} \quad (5-23)$$

This set of state and measurement standard deviations leads to an observer gain with the following components:

$$L_{\hat{\alpha}, e_{\omega}} = \begin{bmatrix} 4.12 & -1.56 & 0.39 \\ -1.56 & 6.09 & -1.61 \\ 0.41 & -1.67 & 4.2 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-7} \begin{bmatrix} 0.11 & 0.11 & -1.54 \\ -0.51 & -0.51 & -2.1 \\ 2.46 & 2.46 & 0.4 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-7} \begin{bmatrix} 0.34 & 1.99 & 1.99 \\ -2.05 & -0.45 & -0.45 \\ -1.95 & 0.11 & 0.11 \end{bmatrix}$$

The estimations obtained using the observer with the modified noise matrices are shown in the Figure 5-14.

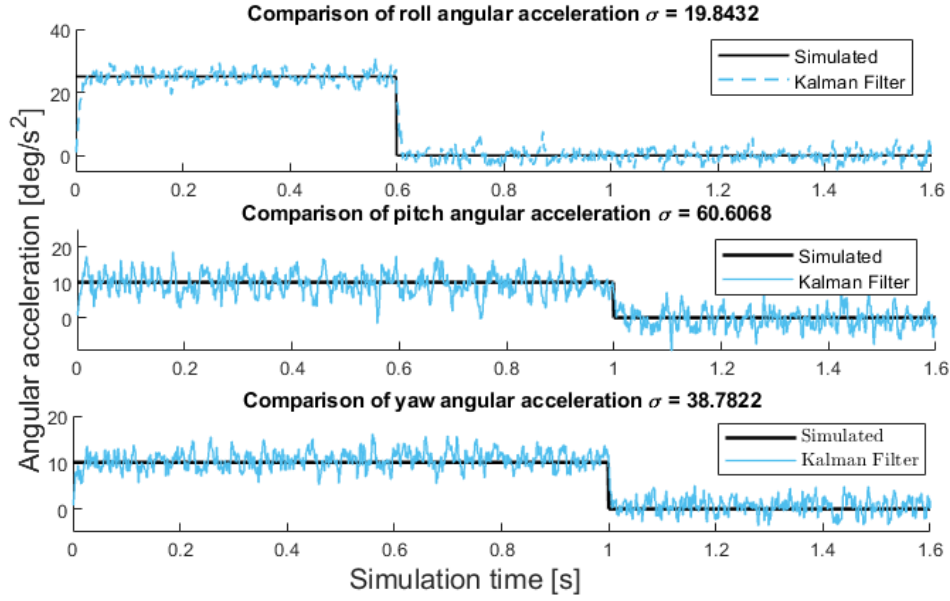


Figure 5-14: Comparison of realistic angular acceleration and Kalman Filter angular acceleration estimations

At this point, the estimations converge to the true value, but it can be observed that they are remarkably noisier than in the previous case. The reason for this is that now the observer relies mostly on the gyroscope measurements, and thus ω is used to estimate α and the observer behaves as a pseudo-differentiator, leading to noise amplification. The standard deviation of the process equation can be further tuned to obtained a less noisy estimation, at the cost of introducing a higher time delay:

$$E_{\alpha} = \begin{bmatrix} 10^{-2} & 0 & 0 \\ 0 & 10^{-2} & 0 \\ 0 & 0 & 10^{-2} \end{bmatrix} \quad (5-24)$$

This set of state and measurement standard deviations leads to an observer gain with the following components:

$$L_{\hat{\alpha}, e_{\omega}} = \begin{bmatrix} 1.09 & -3.13 \times 10^{-5} & 2.98 \times 10^{-8} \\ -3.13 \times 10^{-5} & 1.10 & -3.13 \times 10^{-5} \\ 2.98 \times 10^{-8} & -3.13 \times 10^{-5} & 1.09 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-8} \begin{bmatrix} 2.76 & 2.77 & -5.56 \\ -8.32 & -8.32 & -16.65 \\ 13.88 & 13.88 & 5.55 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-8} \begin{bmatrix} 5.55 & 13.88 & 13.88 \\ -16.65 & -8.32 & -8.32 \\ -5.56 & 2.77 & 2.77 \end{bmatrix}$$

The estimations obtained with the observer are shown in the Figure 5-15

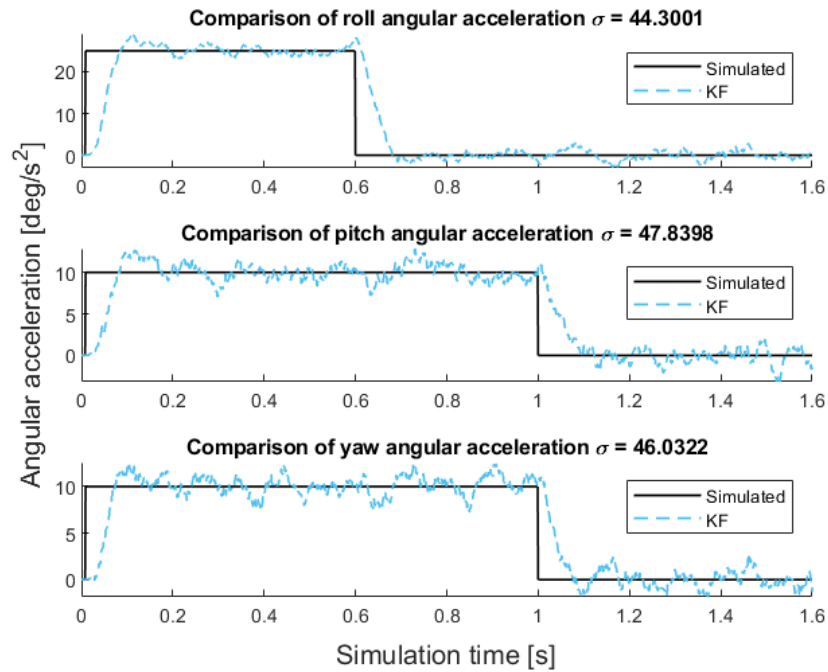


Figure 5-15: Comparison of realistic angular acceleration and Kalman Filter angular acceleration estimations

In order to give an overview of the tuning process of the KF, the angular acceleration estimations in the pitch axis obtained with the three previous observers are shown in Figure 5-16.

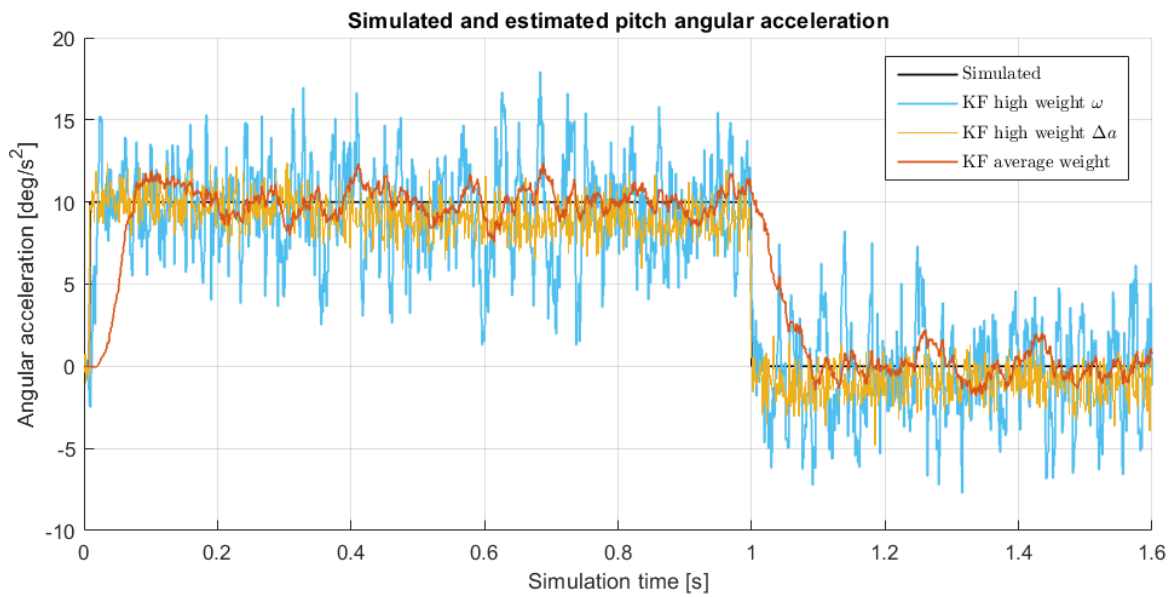


Figure 5-16: Kalman Filter tuning for realistic flight data

It can be concluded that using the considered KF in realistic flight conditions, i.e., with noisy measurements and far from its linearization point, it is only possible to obtain angular acceleration estimations without steady-state error if a low weight is assigned to the linear accelerometers measurements. In addition to this, if it is required to limit the noise amplification in the estimations, it is only possible to observe the angular acceleration with a high time delay compared to the other observers, which is a significant drawback for the goal of this master thesis.

Stationary Kalman Filter with Nonlinear Output

The process and measurement standard deviations used to obtain the NOSKF observer gain are the same than those of the EKF:

$$F_\omega = \begin{bmatrix} \sqrt{8 \times 10^{-5}} & 0 & 0 \\ 0 & \sqrt{8 \times 10^{-5}} & 0 \\ 0 & 0 & \sqrt{8 \times 10^{-5}} \end{bmatrix}$$

$$F_{\Delta a_{2,1}} = F_{\Delta a_{3,2}} = \begin{bmatrix} \sqrt{12 \times 10^{-3}} & 0 & 0 \\ 0 & \sqrt{12 \times 10^{-3}} & 0 \\ 0 & 0 & \sqrt{12 \times 10^{-3}} \end{bmatrix}$$

$$E_\omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_\alpha = 10^{-3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This set of state and measurement standard deviations leads to an observer gain with the following components:

$$L_{\hat{\alpha}, e_\omega} = 10^{-4} \begin{bmatrix} 3.25 & -2.29 & 0.82 \\ -2.29 & 6.06 & -2.39 \\ 0.82 & -2.39 & 3.69 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} 0.32 & 0.32 & -2.69 \\ -1.15 & -1.15 & -5.34 \\ 2.82 & 2.82 & 0.83 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} 1.13 & 4.14 & 4.14 \\ -5.63 & -8.34 & -1.45 \\ -1.66 & 0.32 & 0.32 \end{bmatrix}$$

The estimations obtained with the NOSKF are shown in Figure 5-17. The accuracy of the observed angular acceleration is similar to that of the EKF, while the measurement noise injected in the sensors is also correctly rejected.

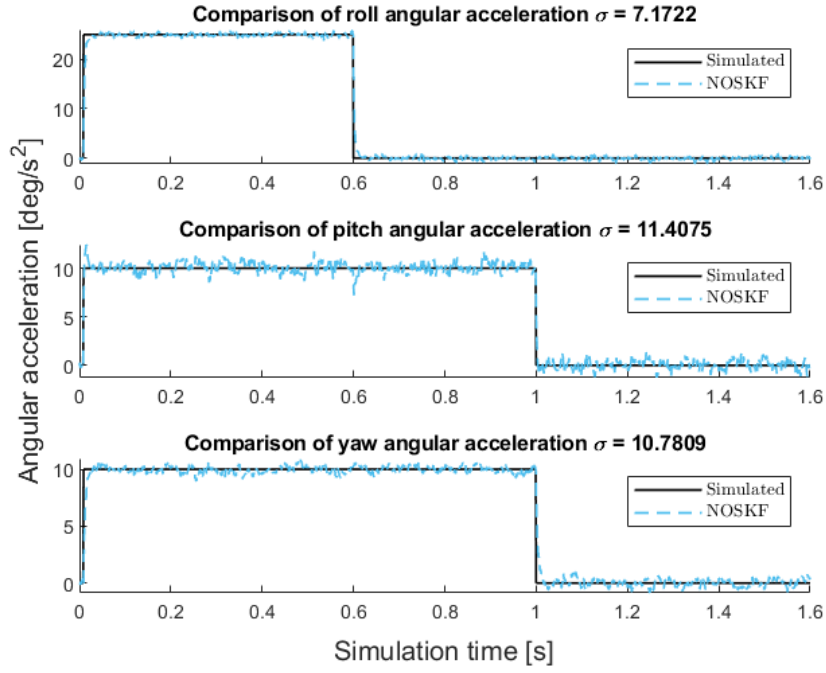


Figure 5-17: Comparison of simulated angular acceleration and Nonlinear-Output Kalman Filter angular acceleration estimations

Stationary Kalman Filter with Modified Input

Since the measurement equation of the MISKF is different to that of the previous observers, its measurement noise matrices have to be tuned in a different way. The measurement equation, here repeated for convenience

$$\begin{aligned}\Delta\tilde{a}_{2,1} &= \Delta D_{2,1}\alpha \\ \Delta\tilde{a}_{3,2} &= \Delta D_{3,2}\alpha\end{aligned}$$

where

$$\begin{aligned}\Delta\tilde{a}_{2,1} &= \Delta a_{2,1} - \Omega^2 \Delta d_{2,1} = \Delta a_{2,1} - (\omega\omega^T - \omega^T\omega I)\Delta d_{2,1} \\ \Delta\tilde{a}_{3,2} &= \Delta a_{3,2} - \Omega^2 \Delta d_{3,2} = \Delta a_{3,2} - (\omega\omega^T - \omega^T\omega I)\Delta d_{3,2}\end{aligned}$$

is the modified version of the measurements, to which the nonlinear term has been subtracted.

Therefore, the measurement noise matrices can be designed using the standard deviation of the term $\Delta\tilde{a}$ with respect to $\Delta D\alpha$ for the sensors S_1 , S_2 , S_3 :

$$\begin{aligned}F_{\Delta\tilde{a}_{2,1}} &= \sqrt{\frac{1}{N-1} \sum_{i=1}^N (\Delta D_{2,1}\alpha_i - \Delta\tilde{a}_{2,1i})^2} \\ F_{\Delta\tilde{a}_{3,2}} &= \sqrt{\frac{1}{N-1} \sum_{i=1}^N (\Delta D_{3,2}\alpha_i - \Delta\tilde{a}_{3,2i})^2}\end{aligned}\tag{5-25}$$

where α_i is the ideal angular acceleration estimate and N is the length of the measurement vector.

It is important to highlight that, while the noise matrices of the previous observers can be tuned based on knowledge about the measurement noise, that is not the case for the MISKF, since the ideal angular acceleration reached during the flight is also required to calculate $F_{\Delta\tilde{a}}$, as shown in Eqs. 5-26. This implies that the observer tuning process is susceptible to overfitting, since the estimation performance can be optimized for the flight which data is used to tune the noise matrices (via the angular acceleration of that flight, used to calculate $F_{\Delta\tilde{a}}$), but might lead to an increased estimation errors in other flights.

In order to avoid overfitting, two measures are taken. On the one hand, the measurement standard deviations are slightly modified in order to simulate that the noise characteristics are not perfectly known. The matrices $F_{\Delta\tilde{a}_{2,1}}$, $F_{\Delta\tilde{a}_{3,2}}$ are multiplied by a factor of 2.5, while the same F_ω used for tuning the previous observers is used here:

$$F_\omega = \begin{bmatrix} \sqrt{8 \times 10^{-5}} & 0 & 0 \\ 0 & \sqrt{8 \times 10^{-5}} & 0 \\ 0 & 0 & \sqrt{8 \times 10^{-5}} \end{bmatrix}$$

$$F_{\Delta\tilde{a}_{2,1}} = 2.5 \times 10^{-2} \begin{bmatrix} 6.93 & 0 & 0 \\ 0 & 7.13 & 0 \\ 0 & 0 & 7.14 \end{bmatrix} \quad F_{\Delta\tilde{a}_{3,2}} = 2.5 \times 10^{-2} \begin{bmatrix} 7.09 & 0 & 0 \\ 0 & 7.18 & 0 \\ 0 & 0 & 6.92 \end{bmatrix} \quad (5-26)$$

In addition to this, the standard deviation matrices of the process equation are tuned using a common, round up value for all axes, instead of specific gains for every axis:

$$E_\omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_\alpha = 10^{-2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (5-27)$$

This set of state and measurement standard deviations leads to an observer gain with the following components:

$$L_{\hat{\alpha}, e_\omega} = 10^{-3} \begin{bmatrix} 4.67 & -4.91 & 1.76 \\ -5.36 & 14.38 & -5.32 \\ 1.92 & -5.29 & 7.14 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta\tilde{a}_{2,1}}} = 10^{-2} \begin{bmatrix} 0.55 & 0.52 & -6.32 \\ -1.76 & -1.67 & -5.06 \\ 5.21 & 4.92 & 1.15 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta\tilde{a}_{3,2}}} = 10^{-2} \begin{bmatrix} 1.60 & 8.32 & 8.94 \\ -5.58 & -2.07 & -2.23 \\ -3.30 & 0.51 & 0.55 \end{bmatrix}$$

The angular acceleration estimations obtained with the MISKF are shown in the Figure 5-18. It can be observed that its accuracy is similar or even higher (mainly in the roll axis) than that of the EKF and the NOSKF. However, this high accuracy might be reached because the measures taken to avoid overfitting during the tuning process of $F_{\Delta\tilde{a}_{2,1}}$ and $F_{\Delta\tilde{a}_{3,2}}$ were not strict enough. Nevertheless, since the goal of this chapter is to give an estimated assessment of the observers performance in realistic flight conditions, as well as the feasibility of the model assumptions, the small differences in accuracy are not further investigated at this point.

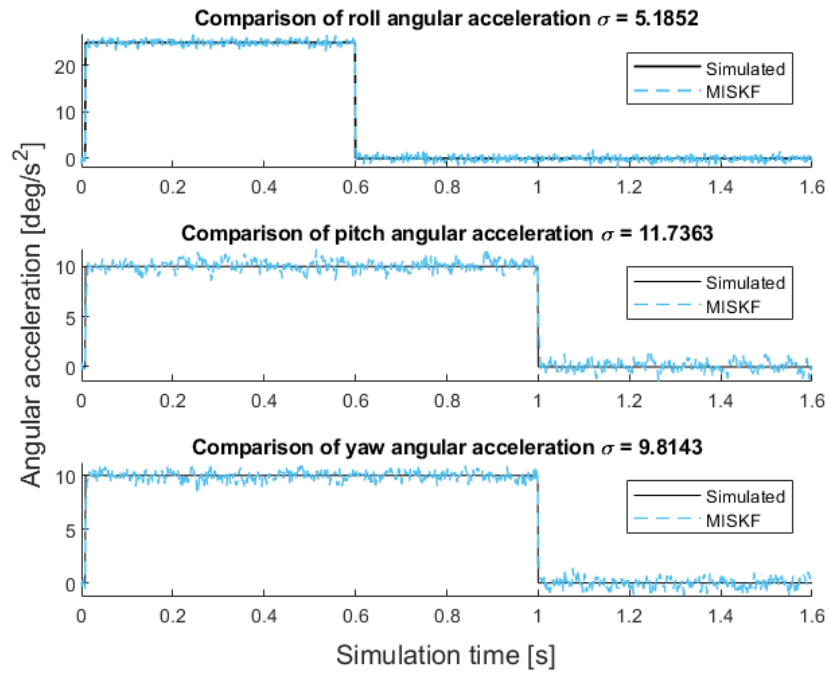


Figure 5-18: Comparison of simulated angular acceleration and Stationary Kalman Filter with Modified Input angular acceleration estimations

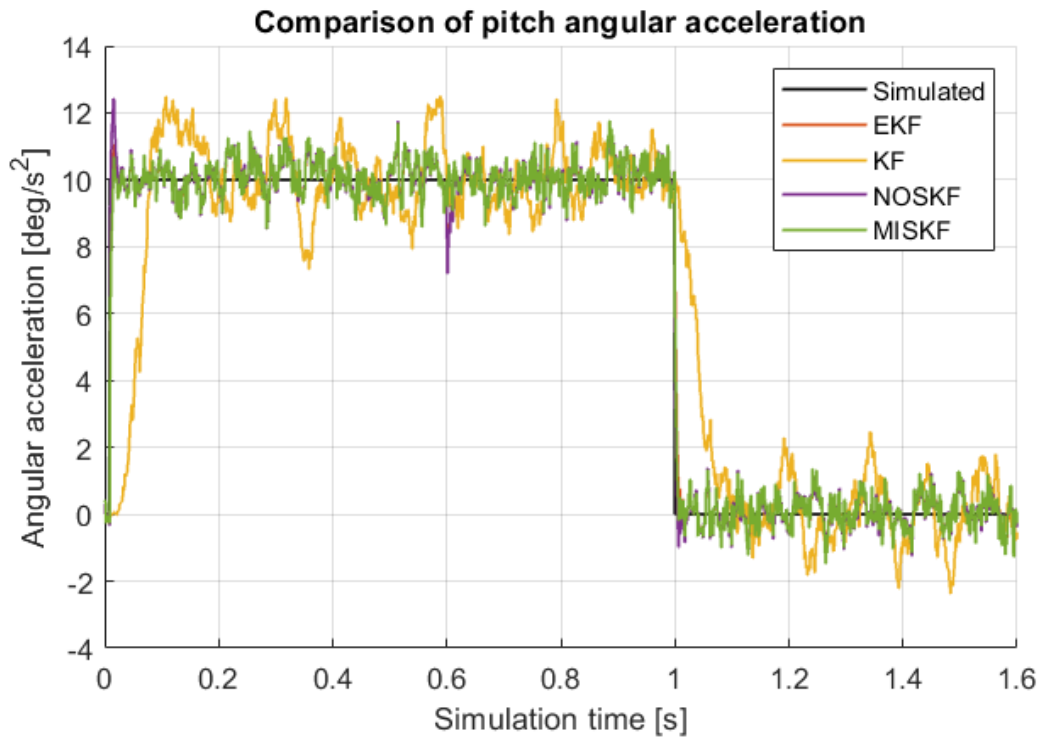


Figure 5-19: Comparison of realistic angular acceleration and the angular acceleration estimations calculated with the proposed observers

Figure 5-19 shows the comparison of the pitch angular acceleration estimations of all the observers under realistic flight conditions and noisy measurements. It can be observed that the KF is not able to provide a good performance, mainly due to the impact of the nonlinearities of the system. On the other hand, the EKF, the NOSKF and the MISKF show a similar performance, with a low time delay and noise amplification.

5-3 Conclusions on the analyzed observers

The analysis of the estimations obtained in the rigid body simulation has shown that the constant angular acceleration assumption taken in the observer model is valid, since all the proposed observers are able to accurately follow the simulated angular acceleration in the first scenario. In addition to this, the analysis performed in the second scenario has allowed to identify a limitation of the KF to estimate the angular acceleration when it operates far from its linearization point. Although that difficulty could be counteracted by relying more on the angular speed measurements, the third scenario has led to the conclusion that it is not feasible in a realistic scenario.

Nevertheless, it has been proven that the EKF, the NOSKF, and the MISKF are able to provide adequate angular acceleration estimations in realistic flight conditions and under white and Gaussian measurement noise. For this reason, the three observers are tested with real flight data in the next chapter.

Observer design for real flight data

In the previous chapter it has been shown that the proposed estimators can be used to obtain accurate angular acceleration estimations of a system if it behaves as a rigid body and if the measurement noise is white and Gaussian. In this chapter, the observers are tested using real flight data in order to address whether they can work correctly in the helicopter. In this scenario, several assumptions which were respected in the simulation are not fulfilled. For instance, the measurements recorded in flight include coloured noise, instead of white Gaussian noise for which Kalman Filters are especially suited. In addition to this, the helicopter does not behave as a rigid body, what violates a relevant assumption on which the dynamic model of the observers is based.

As an initial step, it is desired to analyze the performance of the observers when the flight data is smoothed out to remove its noise. The angular acceleration estimations obtained with the Extended Kalman Filter (EKF) using noiseless data are shown in Figure 6-1, as well as the estimations obtained using Least Squares (LS) and the Lanczos differentiator.

The observer fuses the angular speed and the linear acceleration measurements to find a compromise between the delayed and almost noiseless estimate obtained with the Lanczos differentiator, and the estimation provided by LS regression, which is very reactive but has a larger estimation error. In other words, the proposed EKF can be tuned to cover the full range of solutions between the one offered by the Lanczos and that of the LS, with the constrain of respecting a trade-off between time delay and noise amplification in the estimations. This comparison shows the potential of the proposed observer approach, which encompasses the two other estimation methods in one single system, as well as its capability to adapt in order to provide estimations which fulfill the requirements of the designer.

Since this estimation has been performed with noiseless data, it can be concluded that the observer performance shown here is -approximately- the best one it can reach with the given measurements. Hence, the ability of each observer to deal with the noise present in the recorded flight data will determine the change in the estimation performance that they will show in the next sections.

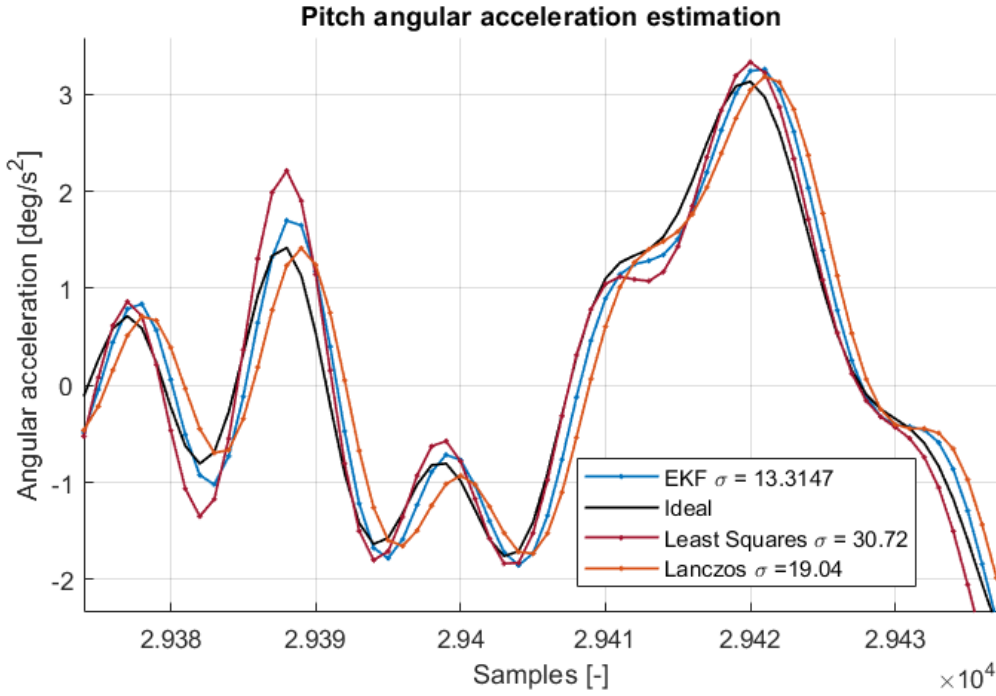


Figure 6-1: Comparison of the estimations of the pitch angular acceleration obtained with the Lanczos differentiator and the EKF using smoothed data.

6-1 Estimation with the Extended Kalman Filter

Data from three different flights has been used to tune the noise matrices of the EKF. In the same fashion than in the previous chapter, the process noise of the angular speed E_ω has been set equal to zero to express that the angular acceleration is the exact derivative of the angular speed. Since the observer model is based on the assumption that the angular acceleration is constant, the standard deviation of the jerk is used as its process noise E_α . The jerk has been approximated by differentiating the ideal angular acceleration estimate.

The measurement noise matrices of the linear accelerations differences $F_{\Delta a_{i,j}}$ have been tuned using the Equation 4-2, hereafter repeated for convenience:

$$\Delta a_{i,j} = (\Omega_{LPF}^2 + \dot{\Omega}_{LPF}) \Delta d_{i,j}$$

where $i, j = \{1, 2, 3\}$ are the indexes of the linear accelerometers.

In the previous section, the helicopter was modeled as a rigid body, and thus the left and the right parts of the previous equation were exactly equal except for the noise injected the linear accelerometers measurements during the simulation. However, the analysis performed in Section 4-1-2 showed this cannot be assumed for the real helicopter, since the elasticity effects introduce a relevant error in the linear acceleration measurements. For that reason, the noise matrices $F_{\Delta a_{i,j}}$ have been designed using the standard deviation of the difference between the measurements $\Delta a_{i,j}$ and the expression at the right part of Equation 4-2, which is calculated using a filtered version of the angular speeds Ω_{LPF} , and the ideal angular acceleration estimate $\dot{\Omega}_{LPF}$.

The measurement noise matrix of the angular speed has been calculated using the standard deviation of the difference between the measured angular speed and its filtered version. Although the exact values of the noise matrices are not shown in this document, representative values are presented hereafter:

$$F_{\omega} = 10^{-6} \begin{bmatrix} 8.5 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad (6-1)$$

$$F_{\Delta a_{2,1}} = 10^{-4} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad F_{\Delta a_{3,2}} = 10^{-4} \begin{bmatrix} 3.5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \quad (6-2)$$

$$E_{\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{\alpha} = 10^{-2} \begin{bmatrix} 2.23 & 0 & 0 \\ 0 & 0.83 & 0 \\ 0 & 0 & 1.32 \end{bmatrix} \quad (6-3)$$

As explained in Section 5-2, the weight that the EKF assigns to each measurement variable (ω , $\Delta a_{2,1}$, $\Delta a_{3,2}$) is analyzed via the components of the observer gain L :

$$L = \begin{bmatrix} L_{\hat{\omega},e_{\omega}} & L_{\hat{\omega},e_{\Delta a_{2,1}}} & L_{\hat{\omega},e_{\Delta a_{3,2}}} \\ L_{\hat{\alpha},e_{\omega}} & L_{\hat{\alpha},e_{\Delta a_{2,1}}} & L_{\hat{\alpha},e_{\Delta a_{3,2}}} \end{bmatrix} \quad (6-4)$$

The analysis is focused on the components of L which affect the angular acceleration estimate $\hat{\alpha}$, which are placed in the second row of the matrix, and are henceforth referred to as $L_{\hat{\alpha}}$.

The EKF adapts its gain every sampling period by linearizing the system model and updating its covariance matrix. Since the flight state is constantly changing, it can be concluded that the gain of the EKF does not converge to a specific value. However, inspecting the observer gain throughout the flight it can be noticed that it reaches a point around which it oscillates depending on the flight state. The values of $L_{\hat{\alpha}}$ at this point are shown hereafter:

$$L_{\hat{\alpha},e_{\omega}} = \begin{bmatrix} 6.75 & -2.43 & -0.36 \\ -0.38 & 10.36 & 0.28 \\ -0.30 & 0.99 & 3.49 \end{bmatrix}$$

$$L_{\hat{\alpha},e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} -1.11 & -5.50 & 0.27 \\ 5.39 & 0.76 & -2.21 \\ 1.23 & 11.51 & -0.19 \end{bmatrix} \quad L_{\hat{\alpha},e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} -0.89 & -31.24 & 22.91 \\ 7.50 & 0.98 & 5.87 \\ -7.51 & -8.35 & -0.23 \end{bmatrix}$$

The angular acceleration estimations obtained with the EKF in the pitch axis are shown in Figure 6-2, as well as the benchmark estimation provided by the Lanczos differentiator, and the ideal estimate.

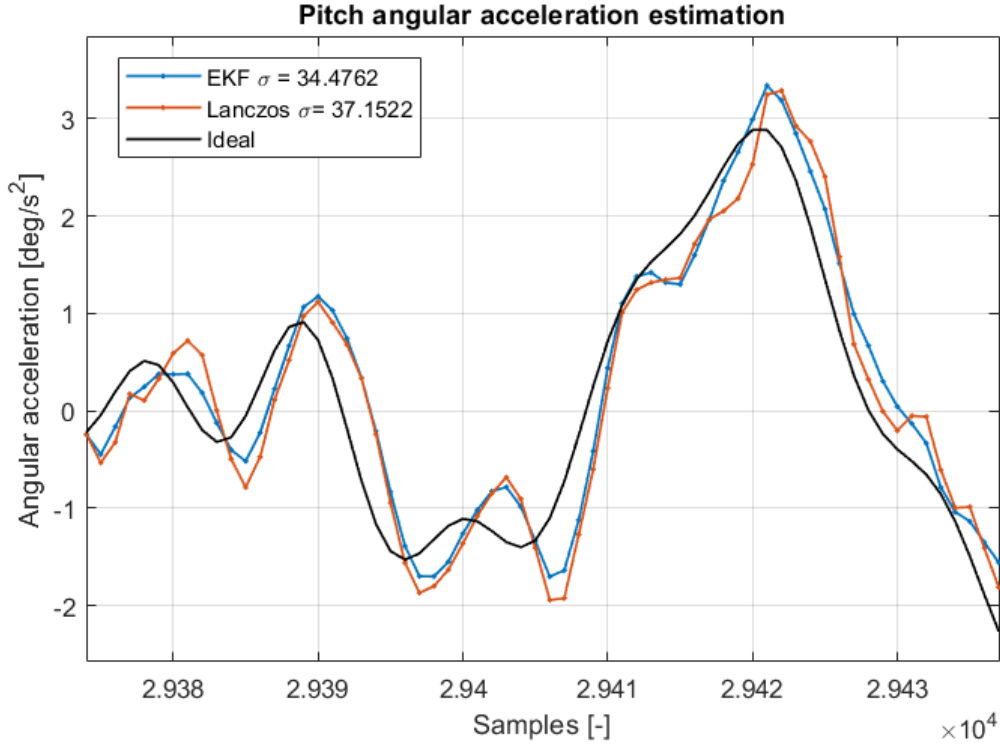


Figure 6-2: Comparison of the estimations of the pitch angular acceleration obtained with the Lanczos differentiator and the EKF.

The accuracy of the EKF improves that of the benchmark. However, the estimations introduce an average time delay of 0.87 samples, which is not substantially less than that associated to the Lanczos differentiator. This behavior differs remarkably from that observed in Figure 6-1 when filtered data was used. The reason for this difference is that in that case, the noise level of both the linear accelerometers and the gyroscopes is equally low, since their measurements are filtered. Hence, the observer balances the weight that each sensor has on the estimations based just on a trade-off between time delay and noise level.

In contrast, when using unfiltered data, the noise of each measurement source plays an important role to define the observer gain. Since the angular velocity measurements has a noise level approximately one order of magnitude lower than the linear accelerations, the observer assigns them a stronger weight, and thus the EKF behaves similarly to a differentiator, leading to time delay in the estimations.

However, since the main performance criteria is the reactivity of the estimator, it is desired to reduce its time delay, even at the cost of some decrease in precision. To do so, the value of the measurement noise matrices $F_{\Delta a}$ is reduced to force the observer to increase the weight assigned to the linear acceleration measurements. A decrease of 30 % leads to the following components of $L_{\hat{a}}$:

$$L_{\hat{\alpha}, e_{\omega}} = \begin{bmatrix} 4.09 & -2.69 & -0.29 \\ -0.43 & 7.81 & 0.21 \\ -0.18 & 0.94 & 1.74 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta a_{2,1}}} = 10^{-2} \begin{bmatrix} -2.69 & -8.62 & 0.82 \\ 9.13 & 1.72 & -3.75 \\ 2.25 & 17.02 & -0.35 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta a_{3,2}}} = 10^{-2} \begin{bmatrix} -2.38 & -45.08 & 31.96 \\ 12.64 & 2.34 & 9.03 \\ -9.58 & -10.67 & -0.22 \end{bmatrix}$$

The observer obtained with the new noise matrices provides remarkably faster estimations, with an average time delay of 0.34 samples, while its error is slightly lower than that of the Lanczos differentiator. In Figure 6-3 one can notice the higher reactivity of the new angular acceleration obtained with the EKF.

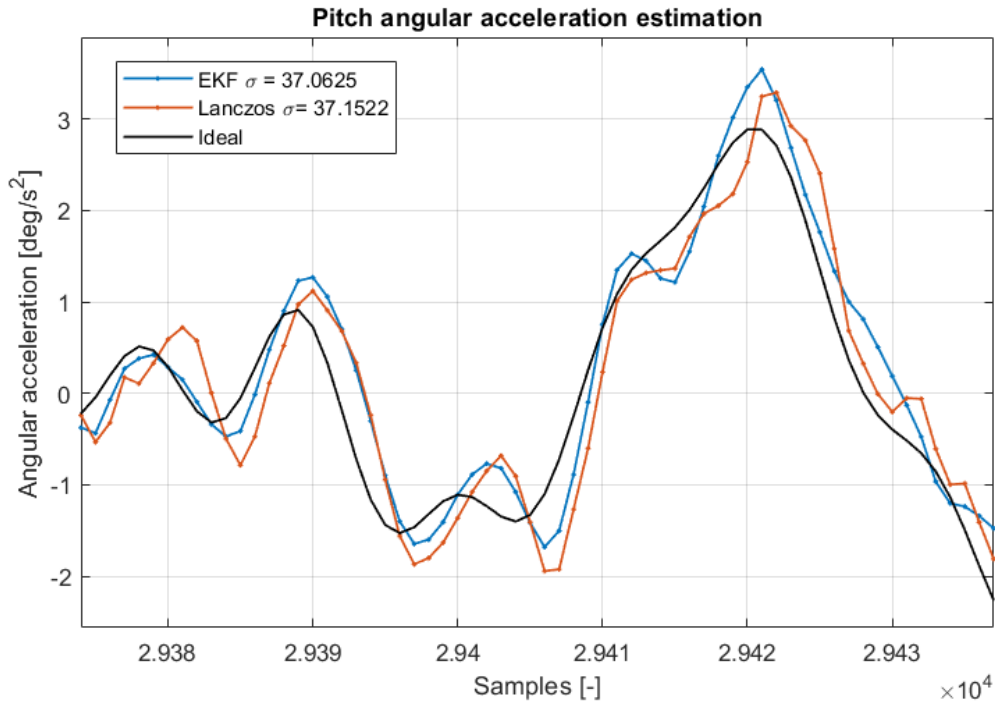


Figure 6-3: Comparison of the estimations of the pitch angular acceleration obtained with the Lanczos differentiator and the EKF.

Table 6-1 shows the estimation error in all axes, from which can be concluded that the proposed EKF provides more accurate angular acceleration estimations than the Lanczos differentiator while introducing approximately one third of its time delay.

Table 6-1: Estimators performance

	Lanczos n=1	Least Squares	EKF
Time delay [samples]	1	0	0.34
Error [%]	$\sigma_x=43.11$	$\sigma_x=65.72$	$\sigma_x=28.16$
	$\sigma_y=37.15$	$\sigma_y=90.55$	$\sigma_y=37.06$
	$\sigma_z=47.20$	$\sigma_z=59.78$	$\sigma_z=26.84$

6-2 Estimation with the Nonlinear-Output Stationary Kalman Filter

The observer gain of the Stationary Kalman Filter with Nonlinear Output (NOSKF) has been obtained by solving the Discrete-Algebraic Riccati Equation (DARE) with the process and measurement noise matrices $Q = EE^T$, $R = FF^T$, where E and F take the same value than for the EKF of the previous section to force a higher relevance of the linear accelerometers measurements. The estimation error is similar to that obtained with the EKF, as shown in Figure 6-4, as well as the introduced time delay, which is 0.35 samples.

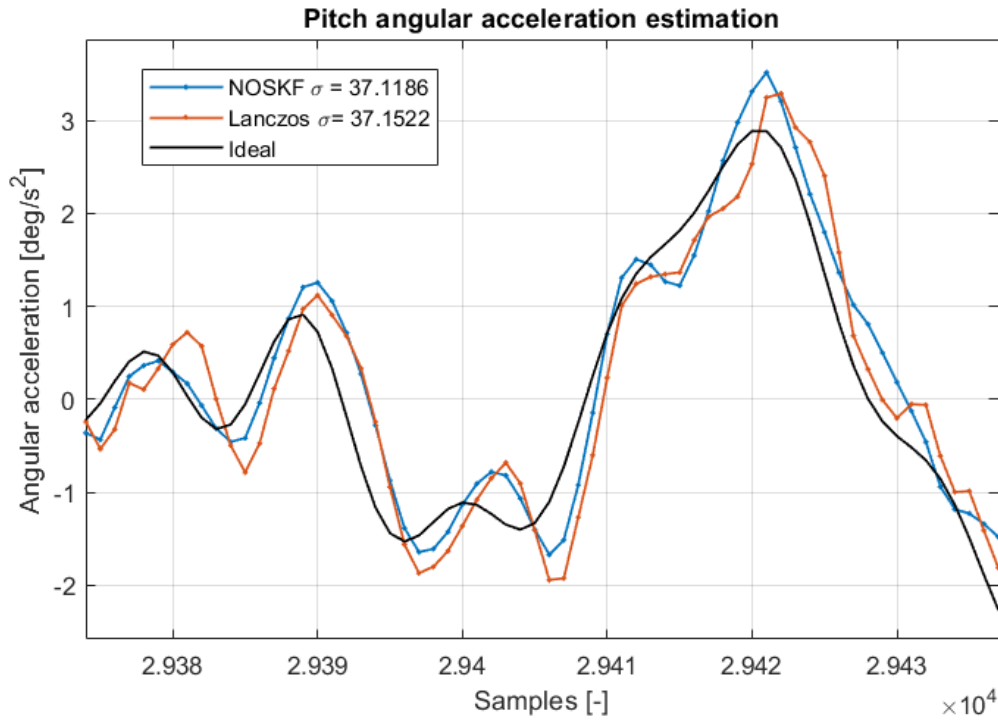


Figure 6-4: Comparison of the estimations of the pitch angular acceleration obtained with the Lanczos differentiator and the NOSKF.

Table 6-2: Estimators performance

	Lanczos n=1	Least Squares	EKF	NOSKF
Time delay [samples]	1	0	0.34	0.35
Error [%]	$\sigma_x=43.11$	$\sigma_x=65.72$	$\sigma_x=28.16$	$\sigma_x=27.92$
	$\sigma_y=37.15$	$\sigma_y=90.55$	$\sigma_y=37.06$	$\sigma_y=37.12$
	$\sigma_z=47.20$	$\sigma_z=59.78$	$\sigma_z=26.84$	$\sigma_z=26.91$

6-3 Estimation with the Modified-Input Stationary Kalman Filter

The process equation of the Stationary Kalman Filter with Modified Input (MISKF) is the same than that of the EKF and the NOSKF, and thus the same standard deviation matrices E_ω , E_α are used to obtain its observer gain. However, the measurement equation is different for the MISKF, and its noise matrices are designed using the Equation 4-25:

$$\Delta \tilde{a}_{i,j} = \Delta D_{i,j} \alpha_{LPF} \quad (6-5)$$

where $i, j = \{1, 2, 3\}$ are the indices of the linear accelerometers, and

$$\Delta \tilde{a}_{i,j} = \Delta a_{i,j} - \Omega^2 \Delta d_{i,j} = \Delta a_{i,j} - (\omega \omega^T - \omega^T \omega I) \Delta d_{i,j} \quad (6-6)$$

The noise matrices $F_{\Delta \tilde{a}_{i,j}}$ are designed using the standard deviation of the difference between the left and the right side of the Equation 4-25. The terms at the left are calculated using measured values, while the expression at the right side is calculated with the ideal angular acceleration estimate. The DARE is solved with this set of process and measurement noise matrices, which leads to the following components of the observer gain $L_{\hat{\alpha}}$:

$$L_{\hat{\alpha}, e_\omega} = \begin{bmatrix} 4.42 & -3.18 & -0.84 \\ -0.52 & 8.04 & 0.29 \\ -0.54 & 1.27 & 2.88 \end{bmatrix}$$

$$L_{\hat{\alpha}, e_{\Delta \tilde{a}_{2,1}}} = 10^{-2} \begin{bmatrix} -2.61 & -8.59 & 0.81 \\ 9.34 & 1.79 & -3.78 \\ 2.11 & 15.70 & -0.39 \end{bmatrix} \quad L_{\hat{\alpha}, e_{\Delta \tilde{a}_{3,2}}} = 10^{-2} \begin{bmatrix} -2.35 & -45.56 & 32.41 \\ 12.70 & 2.32 & 9.22 \\ -9.64 & -10.83 & -0.22 \end{bmatrix}$$

In a similar fashion to the previous sections, the noise matrices of the modified inputs $\Delta \tilde{a}_{2,1}$, $\Delta \tilde{a}_{3,2}$ are multiplied by a factor of 0.7 to simulate that their noise is lower and thus force the observer to assign them a higher weight in the estimations.

The angular acceleration estimations in the pitch axis are shown in Figure 6-5, as well as the benchmark solution and the ideal estimate. The angular acceleration estimations obtained with the MISKF show a slightly lower time delay than the previous observers (0.33 samples), while their estimation error is similar to that of the EKF and lower than that of the NOSKF. Table 6-5 summarizes the estimation errors and the associated time delays of all the analyzed estimators.

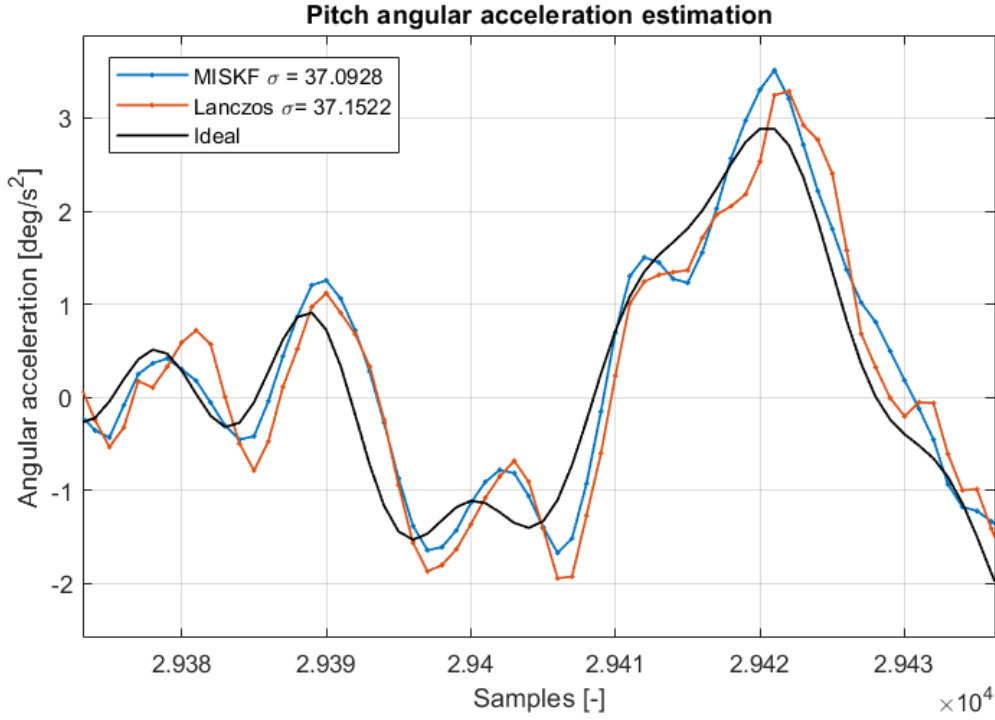


Figure 6-5: Comparison of the estimations of the pitch angular acceleration obtained with the Lanczos differentiator and the MISKF.

Table 6-3: Estimators performance

	Lanczos n=1	Least Squares	EKF	NOSKF	MISKF
Time delay [samples]	1	0	0.34	0.35	0.33
Error [%]	$\sigma_x=43.11$	$\sigma_x=65.72$	$\sigma_x=28.16$	$\sigma_x=27.92$	$\sigma_x=27.87$
	$\sigma_y=37.15$	$\sigma_y=90.55$	$\sigma_y=37.06$	$\sigma_y=37.12$	$\sigma_y=37.09$
	$\sigma_z=47.20$	$\sigma_z=59.78$	$\sigma_z=26.84$	$\sigma_z=26.91$	$\sigma_z=27.58$

Although the Lanczos differentiator and the LS have some advantages such as a low complexity, the capability of observers to balance estimation error and time delay clearly supports their suitability for the problem analyzed in this master thesis. Therefore, the Lanczos and the LS are disregarded as estimation methods, and the comparison is focused on the three observers, which is covered in the next section.

6-4 Comparison of the observers performance

The estimation performance of the proposed observers is compared in Figures 6-6, 6-7, and 6-8, respectively. The benchmark estimations are included for completeness.

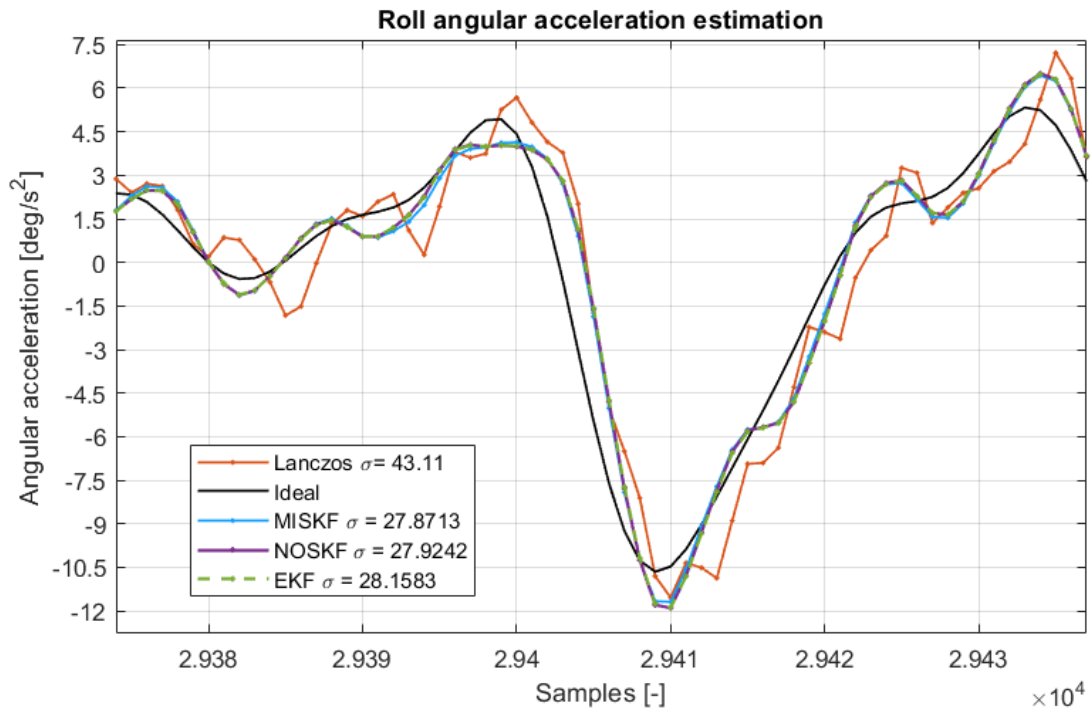


Figure 6-6: Comparison of the estimations of the roll angular acceleration obtained with the Lanczos differentiator, the EKF, the NOSKF, and the MISKF.

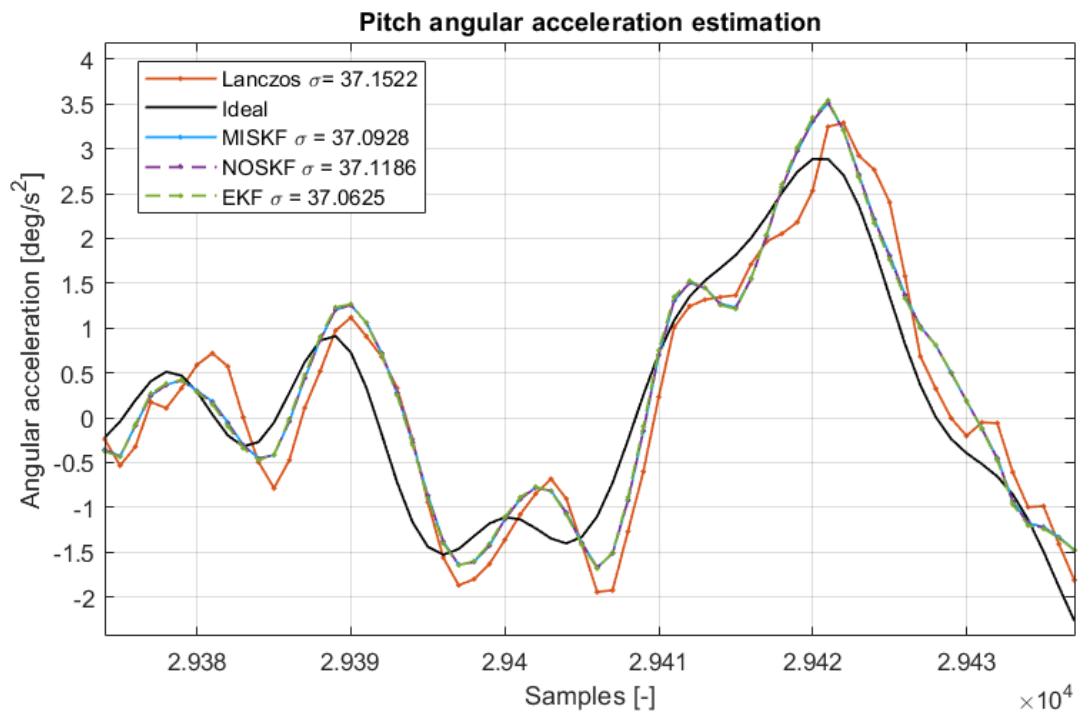


Figure 6-7: Comparison of the estimations of the pitch angular acceleration obtained with the Lanczos differentiator, the EKF, the NOSKF, and the MISKF.

The MISKF shows the best performance in terms of reactivity, since it provides estimations with an average time delay of 0.33 samples, in comparison to 0.35 in the case of the NOSKF and 0.34 for the EKF. In addition to this, it outperforms the NOSKF in the estimations of the roll and the pitch angular accelerations. On the other hand, the estimation error of the MISKF is slightly larger than that of the EKF in the pitch and the yaw axes. However, the fact that the MISKF is the most lightweight observer is considered a decisive factor, since it allows to use less computational resources in flight and facilitates implementation, which are crucial aspects in the aerospace industry.

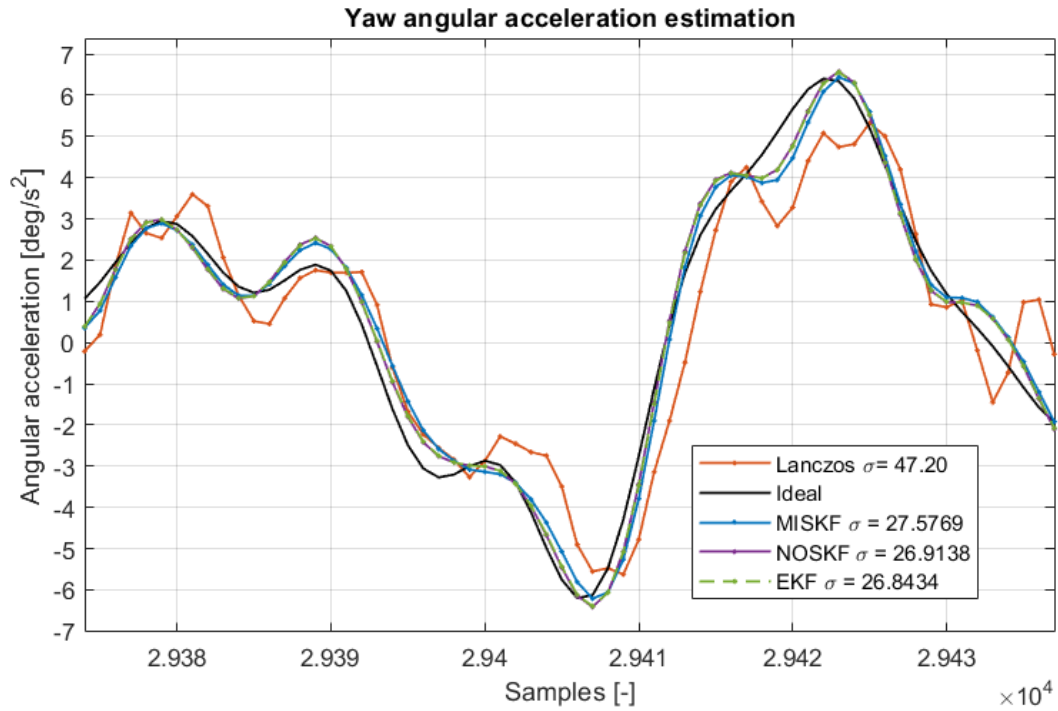


Figure 6-8: Comparison of the estimations of the yaw angular acceleration obtained with the Lanczos differentiator, the EKF, the NOSKF, and the MISKF.

In conclusion, the MISKF provides more accurate estimations than the benchmark solution, while introducing a time delay approximately one third lower. In addition to this, since it is a Linear-Time-Invariant (LTI) observer, its stability and performance analyses are simpler than for the other considered observers, what remarkably simplifies the testing and certification processes for airborne systems.

Conclusions, open points, and further analysis

In this master thesis, a novel method for angular acceleration estimation has been introduced and compared to the benchmark solution using real flight data. The benchmark estimator is a Lanczos differentiator with three sampling points. The proposed approach is based on the Stationary Kalman Filter with Modified Input (MISKF), a state observer which fuses angular speed and linear acceleration measurements provided by gyroscopes and linear accelerometers. Since these sensors are usually already present inside an Inertial Measurement Unit (IMU) in the majority of the helicopters, the method does not imply adding extra weight to the vehicle, which is a major benefit in the aerospace industry. In addition to this, the observer does not require the dynamic model of the system whose angular acceleration is to be estimated, neither it needs the location of its Center of Gravity (C.G.), and thus the range of vehicles to which this method can be applied is remarkably broad.

The MISKF is a Linear-Time-Invariant (LTI) and lightweight observer, and provides estimations with a higher accuracy than the benchmark solution, while introducing approximately one third of its time delay. Moreover, the Signal-to-noise Ratio (SNR) of the linear acceleration difference between two linear accelerometers increases quadratically with the distance between the sensors. Hence, the noise content in the estimations can be reduced without introducing any time delay by spreading out the linear acceleration sensors, as far as the weight and space constraints allow it. However, since the IMUs of the helicopter which has been used for this thesis could not be displaced from their original location, it was not possible to test this aspect, and thus it remains as an open point to be studied.

Mathematical proofs and derivations

A-1 Derivation of the translational and rotational equations of motion

The position of a point Q which is defined in a reference frame A ($p_{Q(A)}$) can be expressed in a reference frame B using a coordinate transformation composed of three successive rotations $\beta_1, \beta_2, \beta_3$ such that

$$\begin{aligned} p_{Q(B)} &= R_{BA} p_{Q(A)} \\ R_{BA} &= R_{\beta_3} R_{\beta_2} R_{\beta_1} \\ R_{\beta_1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta_1 & -\sin \beta_1 \\ 0 & \sin \beta_1 & \cos \beta_1 \end{bmatrix} \\ R_{\beta_2} &= \begin{bmatrix} \cos \beta_2 & 0 & \sin \beta_2 \\ 0 & 1 & 0 \\ -\sin \beta_2 & 0 & \cos \beta_2 \end{bmatrix} \\ R_{\beta_3} &= \begin{bmatrix} \cos \beta_3 & -\sin \beta_3 & 0 \\ \sin \beta_3 & \cos \beta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{A-1}$$

If the frame A is fixed to the vector p so that it rotates and translates with it, the change of p over time can be expressed in B as follows:

$$\dot{p}_{Q(B)} = \dot{R}_{BA} p_{Q(A)} + R_{BA} \dot{p}_{Q(A)} \tag{A-2}$$

Introducing the expressions for the coordinate transformations A-1 and using the chain rule, one obtains

$$\begin{aligned}
\dot{p}_{Q(B)} &= \dot{R}_{BA} p_A + R_{BA} \dot{p}_{Q(A)} = (\dot{R}_{\beta_3} R_{\beta_2} R_{\beta_1} + R_{\beta_3} \dot{R}_{\beta_2} R_{\beta_1} + R_{\beta_3} R_{\beta_2} \dot{R}_{\beta_1}) p_{Q(A)} + R_{BA} \dot{p}_{Q(A)} = \\
&= \left(\begin{bmatrix} -\sin \beta_3 & -\cos \beta_3 & 0 \\ \cos \beta_3 & -\sin \beta_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\beta}_3 \begin{bmatrix} \cos \beta_2 & 0 & \sin \beta_2 \\ 0 & 1 & 0 \\ -\sin \beta_2 & 0 & \cos \beta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta_1 & -\sin \beta_1 \\ 0 & \sin \beta_1 & \cos \beta_1 \end{bmatrix} + \right. \\
&+ \begin{bmatrix} \cos \beta_3 & -\sin \beta_3 & 0 \\ \sin \beta_3 & \cos \beta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin \beta_2 & 0 & \cos \beta_2 \\ 0 & 1 & 0 \\ -\cos \beta_2 & 0 & -\sin \beta_2 \end{bmatrix} \dot{\beta}_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta_1 & -\sin \beta_1 \\ 0 & \sin \beta_1 & \cos \beta_1 \end{bmatrix} + \\
&+ \left. \begin{bmatrix} \cos \beta_3 & -\sin \beta_3 & 0 \\ \sin \beta_3 & \cos \beta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \beta_2 & 0 & \sin \beta_2 \\ 0 & 1 & 0 \\ -\sin \beta_2 & 0 & \cos \beta_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin \beta_1 & -\cos \beta_1 \\ 0 & \cos \beta_1 & -\sin \beta_1 \end{bmatrix} \dot{\beta}_1 \right) p_{Q(A)} + R_{BA} \dot{p}_{Q(A)} \quad (A-3)
\end{aligned}$$

Rearranging the previous equation, it can be expressed as follows

$$\begin{aligned}
\dot{p}_{Q(B)} &= \left(\begin{bmatrix} \cos \beta_3 & -\sin \beta_3 & 0 \\ \sin \beta_3 & \cos \beta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \beta_2 & 0 & \sin \beta_2 \\ 0 & 1 & 0 \\ -\sin \beta_2 & 0 & \cos \beta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta_1 & -\sin \beta_1 \\ 0 & \sin \beta_1 & \cos \beta_1 \end{bmatrix} \right. \\
&\left. \begin{bmatrix} 0 & \sin \beta_1 \dot{\beta}_2 - \cos \beta_1 \cos \beta_2 \dot{\beta}_3 & \cos \beta_1 \dot{\beta}_2 + \sin \beta_1 \cos \beta_2 \dot{\beta}_3 \\ -\sin \beta_1 \dot{\beta}_2 + \cos \beta_1 \cos \beta_2 \dot{\beta}_3 & 0 & -\dot{\beta}_1 + \sin \beta_2 \dot{\beta}_3 \\ -\cos \beta_1 \dot{\beta}_2 - \sin \beta_1 \cos \beta_2 \dot{\beta}_3 & \dot{\beta}_1 - \sin \beta_2 \dot{\beta}_3 & 0 \end{bmatrix} \right) p_{Q(A)} + \\
&+ R_{BA} \dot{p}_{Q(A)} = (R_{\beta_3} R_{\beta_2} R_{\beta_1} \Gamma) p_{Q(A)} + R_{BA} \dot{p}_{Q(A)} = R_{BA} \Gamma p_{Q(A)} + R_{BA} \dot{p}_{Q(A)} \quad (A-4)
\end{aligned}$$

where

$$\Gamma = \begin{bmatrix} 0 & \sin \beta_1 \dot{\beta}_2 - \cos \beta_1 \cos \beta_2 \dot{\beta}_3 & \cos \beta_1 \dot{\beta}_2 + \sin \beta_1 \cos \beta_2 \dot{\beta}_3 \\ -\sin \beta_1 \dot{\beta}_2 + \cos \beta_1 \cos \beta_2 \dot{\beta}_3 & 0 & -\dot{\beta}_1 + \sin \beta_2 \dot{\beta}_3 \\ -\cos \beta_1 \dot{\beta}_2 - \sin \beta_1 \cos \beta_2 \dot{\beta}_3 & \dot{\beta}_1 - \sin \beta_2 \dot{\beta}_3 & 0 \end{bmatrix} \quad (A-5)$$

Combining Eqs. A-4 and A-2 leads to

$$\begin{aligned}
\dot{p}_{Q(B)} &= \dot{R}_{BA} p_{Q(A)} + R_{BA} \dot{p}_{Q(A)} = R_{BA} \Gamma p_{Q(A)} + R_{BA} \dot{p}_{Q(A)} \\
\dot{R}_{BA} p_{Q(A)} + R_{BA} \dot{p}_{Q(A)} &= R_{BA} \Gamma p_{Q(A)} + R_{BA} \dot{p}_{Q(A)} \quad (A-6) \\
\dot{R}_{BA} &= R_{BA} \Gamma
\end{aligned}$$

Moreover, realizing that the skew-symmetric matrix Γ corresponds to a cross-product, Equation A-4 can be rewritten as

$$\dot{p}_Q(B) = R_{BA} \dot{p}_Q(A) + R_{BA} (\gamma \times p_Q(A)) \quad (\text{A-7})$$

where

$$\gamma = \begin{bmatrix} \dot{\beta}_1 - \sin \beta_2 \dot{\beta}_3 \\ \cos \beta_1 \dot{\beta}_2 + \sin \beta_1 \cos \beta_2 \dot{\beta}_3 \\ -\sin \beta_1 \dot{\beta}_2 + \cos \beta_1 \cos \beta_2 \dot{\beta}_3 \end{bmatrix} \quad (\text{A-8})$$

A-1-1 Kinematic equations of a 6 Degrees of Freedom rigid-body

The kinematic equations are a set of differential equations that relate the position, velocity, and acceleration of the body in different reference frames. Since a rigid body can move in six Degrees of Freedom (DoF), three rotations and three translations, the kinematic equations can be separated into two sets of three rotational and three translational kinematic equations.

Rotational kinematic equations

The kinematic equations relate the angular velocity of a body to its change in attitude, i.e., the angular velocities in Body-fixed reference frame (F_b) $\omega = [p \ q \ r]^T$ to the rate of change of the Euler Angles $\dot{\Theta} = [\dot{\phi} \ \dot{\theta} \ \dot{\psi}]^T$, referred to as Euler Rates ($\dot{\Theta}$) from here onward.

In the previous section, the Euler Angles were introduced to define the attitude of the body as an orientation of F_b with respect to Vehicle-carried normal Earth reference frame (F_E). This transformation is composed of three sequential rotations and makes use of two intermediate reference frames $F_{E'}$ and $F_{E''}$: a rotation in yaw from F_E to $F_{E'}$, performed by a multiplication of the rotation matrix $R(\psi)$, a rotation in pitch from $F_{E'}$ to $F_{E''}$, performed by a multiplication by $R(\theta)$, and a rotation in roll from $F_{E''}$ to F_b by $R(\phi)$, such that:

$$\mathbf{F}_b = R_{bE''}(\phi) R_{E''E'}(\theta) R_{E'E}(\psi) \mathbf{X}_E = R_{bE} \mathbf{F}_E \quad (\text{A-9})$$

The relative angular velocity between F_b and F_E is ω , and taking into account that each Euler Rate component ($\dot{\phi}, \dot{\theta}, \dot{\psi}$) is expressed in a different coordinate system ($F_E, F_{E'}$ and $F_{E''}$, respectively), one can relate ω and $\dot{\Theta}$ by rotating each Euler Rate component from their individual reference frame to F_b and then adding them:

$$\omega = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + R_{bE''}(\phi) \left(\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + R_{E''E'}(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \right) \quad (\text{A-10})$$

Introducing the expressions for $R_{bE''}(\phi)$ and $R_{E''E'}(\theta)$ shown in Eqs. 2-5 to 2-7, one obtains the rotational kinematic equation:

$$\omega = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (\text{A-11})$$

which can be expressed in vectorial form as

$$\omega = \mathbf{E} \dot{\Theta}$$

The expression of the Euler Rates $\dot{\Theta}$ can be obtained by inverting the previous equation:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \frac{\sin \phi}{\cos \theta} & \frac{\cos \phi}{\cos \theta} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (\text{A-12})$$

or in vectorial form

$$\dot{\Theta} = \mathbf{B} \omega$$

At this point, it can be noticed that the expression of the body angular velocity ω in Eq. A-11 is the same than the expression of γ in Eq. A-8 if the generic angles $\beta_1, \beta_2, \beta_3$ are replaced by the Euler Angles ϕ, θ, ψ :

$$\omega = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\gamma = \begin{bmatrix} \dot{\beta}_1 - \sin \beta_2 \dot{\beta}_3 \\ \cos \beta_1 \dot{\beta}_2 + \sin \beta_1 \cos \beta_2 \dot{\beta}_3 \\ -\sin \beta_1 \dot{\beta}_2 + \cos \beta_1 \cos \beta_2 \dot{\beta}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \beta_2 \\ 0 & \cos \beta_1 & \sin \beta_1 \cos \beta_2 \\ 0 & -\sin \beta_1 & \cos \beta_1 \cos \beta_2 \end{bmatrix} \begin{bmatrix} \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{bmatrix} \quad (\text{A-13})$$

This fact implies the remarkable advantage that γ , which expression was found to be a matrix of products of the rotation angles and their derivatives, can be substituted by the angular velocity between the reference frames A, B ¹. Therefore, the Equation A-7 can be rewritten as follows:

$$\dot{p}_{Q(B)} = R_{BA} \dot{p}_{Q(A)} + R_{BA} (\omega \times p_{Q(A)}) \quad (\text{A-14})$$

and the Equation A-6 can be expressed as

$$\dot{R}_{BA} = R_{BA} \Omega \quad (\text{A-15})$$

where

$$\Omega = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

is the skew-symmetric matrix corresponding to ω , in an analogous way to γ and Γ , but with a clear physical meaning (the angular velocity of the body) and a heavily simplified expression. Equation A-15 will be frequently used in the next subsections to derive the equations of motion.

¹Since the rotation from A to B is performed in the reversed order than that from F_E to F_b , the substitution of the rotation angles might not be straightforward and is derived in the appendix A-3.

Translational kinematic equations

This section covers the relative position, velocity and acceleration of a rigid body from different reference frames which are in relative motion. Figure A-1 shows the reference frame a generic reference frame C with origin at point Q, which is in relative motion with respect to an inertial reference frame I, with origin at A. Point P is moving with respect to both frames and has a position $p_{P(I)}$ with respect to I, which can be related to the position in C as follows:

$$p_{P(I)} = p_{Q(I)} + R_{IC} p_{P(C)} \quad (\text{A-16})$$

where R_{IC} is the rotation matrix from C to I.

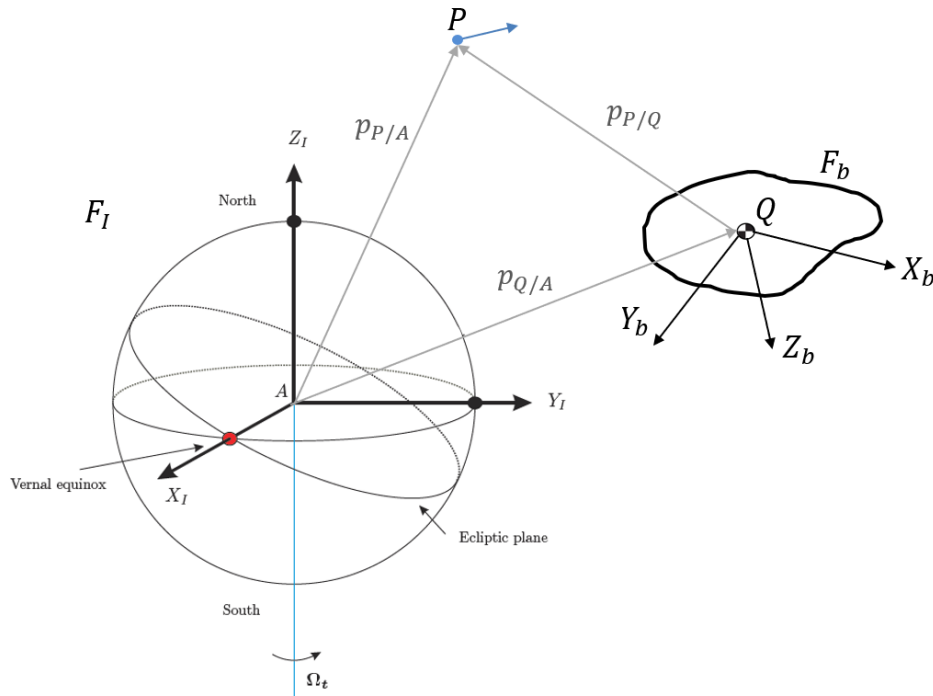


Figure A-1: A body-fixed reference frame C in relative motion to an inertial reference frame I

Taking the time derivative of Equation A-16 one obtains

$$\dot{p}_{P(I)} = \dot{p}_{Q(I)} + (\dot{R}_{IC} p_{P(C)}) \quad (\text{A-17})$$

Since $p_{P(I)}$ and $p_{Q(I)}$ are defined in an inertial reference frame I, their derivatives are directly velocities in I $\dot{p}_{P(I)} = v_{P(I)}$ and $\dot{p}_{Q(I)} = v_{Q(I)}$. However, this is not the case for the derivative of $R_{IC} p_{P(C)}$, since it is defined in C, which is not an inertial reference frame. The chain rule can be used to expand the previous equation:

$$\dot{p}_{P(I)} = v_{P(I)} = \dot{p}_{Q(I)} + (\dot{R}_{IC} p_{P(C)}) = v_{Q(I)} + \dot{R}_{IC} p_{P(C)} + R_{IC} \dot{p}_{P(C)} \quad (\text{A-18})$$

Moreover, using the result derived in Equation A-15 ($\dot{R}_{BA} = R_{BA} \Omega$) the previous equation can be simplified as follows

$$v_{P(I)} = v_{Q(I)} + R_{IC} (\Omega_{(C)} p_{P(C)} + \dot{p}_{P(C)}) \quad (\text{A-19})$$

where $\Omega_{(C)}$ is the angular velocity of the reference frame C expressed in C.

Taking the time derivative of the previous expression, and taking into account that the derivatives of $v_{P(I)}$ and $v_{Q(I)}$ are accelerations $\dot{v}_{P(I)} = a_{P(I)}$ and $\dot{v}_{Q(I)} = a_{Q(I)}$ because they are defined in an inertial reference system, one obtains

$$\begin{aligned} a_{P(I)} &= a_{Q(I)} + \dot{R}_{IC} (\Omega_{(C)} p_{P(C)} + \dot{p}_{P(C)}) + R_{IC} (\dot{\Omega}_{(C)} p_{P(C)} + \Omega_{(C)} \dot{p}_{P(C)} + \ddot{p}_{P(C)}) \\ &= a_{Q(I)} + R_{IC} (\Omega_{(C)}^2 p_{P(C)} + \Omega_{(C)} \dot{p}_{P(C)}) + R_{IC} (\dot{\Omega}_{(C)} p_{P(C)} + \Omega_{(C)} \dot{p}_{P(C)} + \ddot{p}_{P(C)}) \\ &= a_{Q(I)} + R_{IC} (\ddot{p}_{P(C)} + \Omega_{(C)}^2 p_{P(C)} + 2\Omega_{(C)} \dot{p}_{P(C)} + \dot{\Omega}_{(C)} p_{P(C)}) \end{aligned} \quad (\text{A-20})$$

The skew-symmetric matrix of angular velocities can be substituted for the cross product formulation and the following components of the acceleration can be distinguished:

$$a_{P(I)} = a_{Q(I)} + R_{IC} \left(\underbrace{\ddot{p}_{P(C)}}_{\text{Relative Acc.}} + \underbrace{\dot{\omega}_{(C)} \times p_{P(C)}}_{\text{Tangential Acc.}} + \underbrace{\omega_{(C)} \times \omega_{(C)} \times p_{P(C)}}_{\text{Centripetal Acc.}} + \underbrace{2\omega_{(C)} \times \dot{p}_{P(C)}}_{\text{Coriolis Acc.}} \right) \quad (\text{A-21})$$

A-1-2 Dynamic equations of motion of a 6-DOF rigid-body

The Equations of Motion (EoM) of a body relate the acceleration of its Center of Gravity (C.G.) to the torques and forces which act upon it, allowing to calculate the time-evolution of the body. The main tools used to obtain this relationship are Newton's laws and the concepts of kinematics derived in the previous section.

Rotational equations of motion

The rotational EoM describe the evolution of a body subjected to external torques. From classical mechanics, one can calculate the change in angular momentum L from the net torque applied around the C.G. of the body:

$$\sum T_{CG(I)} = \dot{L}_{CG(I)} \quad (\text{A-22})$$

The previous equation can be expressed in F_b as follows

$$R_{bI} \left(\sum T_{CG(I)} \right) = \sum T_{CG(b)} = R_{bI} \dot{L}_{CG(I)} \quad (\text{A-23})$$

The right hand side term can be obtained using the chain rule and Equation A-15 ($\dot{R} = R\Omega$):

$$\begin{aligned} L_{(I)} &= R_{Ib} L_{(b)} \\ \dot{L}_{(I)} &= \dot{R}_{Ib} L_{(b)} + R_{Ib} \dot{L}_{(b)} = R_{Ib} \left(\Omega_{(b)} L_{(b)} + \dot{L}_{(b)} \right) \\ R_{bI} \dot{L}_{(I)} &= R_{bI} R_{Ib} (\Omega_{(b)} L_{(b)} + \dot{L}_{(b)}) = \Omega_{(b)} L_{(b)} + \dot{L}_{(b)} \end{aligned} \quad (\text{A-24})$$

which can be substituted in Equation A-23 to obtain

$$\sum T_{CG(b)} = \Omega_{(b)} L_{(b)} + \dot{L}_{(b)} \quad (\text{A-25})$$

where $\Omega_{(b)}$ is the angular velocity of the body expressed in F_b .

On the other hand, the angular momentum of a differential mass element δm around the C.G. of the rigid body is defined as follows

$$\delta L_{CG(b)} \equiv r_{\delta m(b)} \times (v_{\delta m(b)} \delta m) \quad (\text{A-26})$$

where $r_{\delta m(b)}$ is the vector from the C.G. to δm .

In order to relate $r_{\delta m(b)}$ and $v_{\delta m(b)}$ it is not correct to write $\dot{r}_{\delta m(b)} = v_{\delta m(b)}$ since the rigid body frame is not an inertial reference frame. Using the chain rule and Equation A-15 ($\dot{R} = R\Omega$) one can write

$$\begin{aligned} r_{\delta m(I)} &= R_{Ib} r_{\delta m(b)} \\ \dot{r}_{\delta m(I)} &= \dot{R}_{Ib} r_{\delta m(b)} + R_{Ib} \dot{r}_{\delta m(b)} = R_{Ib} (\Omega_{(b)} r_{\delta m(b)} + \dot{r}_{\delta m(b)}) \end{aligned} \quad (\text{A-27})$$

Since $\dot{r}_{\delta m(I)} = v_{\delta m(I)} = R_{Ib} v_{\delta m(b)}$, the velocity of δm in inertial and body frames can be related as follows:

$$\dot{r}_{\delta m(I)} = R_{Ib} v_{\delta m(b)} = R_{Ib} (\Omega_{(b)} r_{\delta m(b)} + \dot{r}_{\delta m(b)}) \quad (\text{A-28})$$

and thus

$$v_{\delta m(b)} = \Omega_{(b)} r_{\delta m(b)} + \dot{r}_{\delta m(b)} \quad (\text{A-29})$$

Due to the rigid body assumption, the distance from the C.G. to δm is constant and thus $\dot{r}_{\delta m(b)} = 0$. Therefore, Equation A-26 can be rewritten as

$$\delta L_{CG(b)} = r_{\delta m(b)} \times (\Omega_{(b)} r_{\delta m(b)}) \delta m \quad (\text{A-30})$$

$L_{CG(b)}$ can be obtained by integrating $\delta L_{CG(b)}$ over the mass of the rigid body. Omitting the subscripts inside the integral for clarity, one can write

$$\begin{aligned} L_{CG(b)} &= \int_m r \times (\Omega r) dm \\ &= \int_m r \times (\omega \times r) dm \\ &= \int_m (\omega r^T r - r r^T \omega) dm \\ &= \underbrace{\int_m [r^T r I_{3 \times 3} - r r^T] dm}_{J_{(b)}} \cdot \omega = J_{(b)} \cdot \omega_{(b)} \end{aligned} \quad (\text{A-31})$$

The term $J_{(b)}$ is the inertia matrix, which is constant in the rigid body frame, and is composed of the moments of inertia J_{xx} , J_{yy} , J_{zz} in its diagonal and the products of inertia $J_{i,j}$ for $i, j = \{x, y, z\}$ in the off-diagonal entries:

$$J = \begin{bmatrix} J_{xx} & -J_{xy} & -J_{xz} \\ -J_{xy} & J_{yy} & -J_{yz} \\ -J_{xz} & -J_{yz} & J_{zz} \end{bmatrix} = \begin{bmatrix} \int (y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int (x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int (x^2 + y^2) dm \end{bmatrix} \quad (\text{A-32})$$

The expression for $L_{CG(b)}$ can be substituted in Equation A-25:

$$\sum T_{CG(b)} = J_{(b)} \cdot \dot{\omega}_{(b)} + \omega_b \times J_{(b)} \cdot \omega_{(b)} \quad (\text{A-33})$$

and rearranged to solve for $\dot{\omega}_b$ as follows

$$\dot{\omega}_{(b)} = J_{(b)}^{-1} \left[\sum T_{CG(b)} - \omega_{(b)} \times J_{(b)} \omega_{(b)} \right] \quad (\text{A-34})$$

The helicopter model is generally considered symmetric in the X_b, Z_b -plane, and thus the inertia matrix can be simplified as shown in Appendix A-4 to obtain:

$$J = \begin{bmatrix} J_{xx} & 0 & -J_{xz} \\ 0 & J_{yy} & 0 \\ -J_{xz} & 0 & J_{zz} \end{bmatrix} \quad (\text{A-35})$$

The simplified rotational dynamic equations can also be expressed in scalar form as follows:

$$\begin{aligned} \dot{p} &= \frac{J_{zz}}{J^*} M_x + \frac{J_{xz}}{J^*} M_z + \frac{(J_{xx} - J_{yy} + J_{zz})J_{xz}}{J^*} pq + \frac{(J_{yy} - J_{zz})J_{zz} - J_{xz}^2}{J^*} qr \\ \dot{q} &= \frac{M_y}{J_{yy}} + \frac{J_{xz}}{J_{yy}} (r^2 - p^2) + \frac{J_{zz} - J_{xx}}{J_{yy}} pr \\ \dot{r} &= \frac{J_{xz}}{J^*} M_x + \frac{J_{xx}}{J^*} M_z + \frac{(J_{xx} - J_{yy})J_{xx} + J_{xz}^2}{J^*} pq + \frac{(-J_{xx} + J_{yy} - J_{zz})J_{xz}}{J^*} qr \end{aligned} \quad (\text{A-36})$$

where $J^* = J_{xx}J_{zz} - J_{xz}^2$ and $\sum T_{CG} = [M_x \ M_y \ M_z]^T$ are the net torques in the X_b , Y_b , and Z_b -axis, respectively.

Translational equations of motion

The translational EoM relate the acceleration of the C.G. of the body to the external forces acting on the body via Newton's second law. Assuming that the body has a constant mass M one can write:

$$\sum F_{(I)} = M a_{CG(I)} \quad (\text{A-37})$$

where $a_{CG(I)}$ is the acceleration of the body's C.G. in Earth-centered Inertial reference frame (F_I).

At this point one can use the expression obtained in the previous section for the acceleration of a point P in inertial frame (Eq. A-21), here repeated for convenience:

$$a_{P(I)} = a_{Q(I)} + R_{IC} \left(\underbrace{\ddot{p}_{P(C)}}_{\substack{\text{Relative} \\ \text{Acc. wrt. } C}} + \underbrace{\dot{\Omega}_{(C)} p_{P(C)}}_{\text{Tangential Acc.}} + \underbrace{\Omega_{(C)}^2 p_{P(C)}}_{\text{Centripetal Acc.}} + \underbrace{2\Omega_{(C)} \dot{p}_{P(C)}}_{\text{Coriolis Acc.}} \right) \quad (\text{A-38})$$

The previous generic I, C frames are now defined to be the F_I and Earth-centered, Earth-fixed reference frame (F_C), respectively, and point P to be the C.G. of the body. Since F_I and F_C have the origin at the same point ($A \equiv Q$) the term $a_{Q(I)}$ vanishes from the acceleration of P, now the acceleration of CG.

The angular velocity of the Earth can be considered constant and thus $\dot{\Omega}_{(C)} \approx 0$, what makes the tangential acceleration negligible. In addition to this, since the master thesis is not focused on guidance or navigation systems for long distance flights but rather on short displacements, the influence of the Coriolis acceleration due to the spinning velocity of the Earth is also neglected. Moreover, a local reference frame F_E is defined at the longitude and latitude of the Earth at which the body motion is analyzed. Its $X_E - Y_E$ plane is parallel to the local Earth surface, and its Z_E -axis points to the center of the Earth, as introduced in Subsection 2-1-1 and shown in Figure A-2. Since the analysis of the body motion is restricted to the surroundings of F_E , for short displacements the $X_E - Y_E$ plane does not deviate significantly from the surface of the Earth, and thus the centripetal acceleration due to the curvature of the Earth's surface is considered negligible.

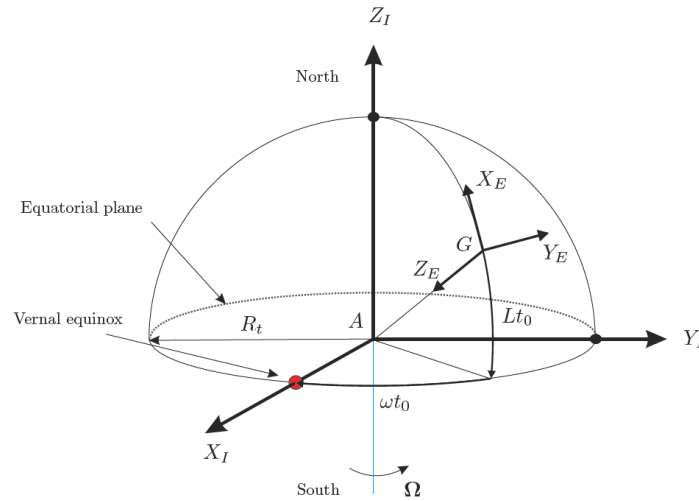


Figure A-2: Definition of the Vehicle-carried normal Earth reference frame F_E [2]

Based on these assumption, F_C can be now considered an inertial reference frame, with respect to which the displacement, velocity, and acceleration of the body (F_b) are defined. Including these approximations, Eq. A-38 is simplified as follows:

$$a_{CG(I)} \approx R_{IC} \underbrace{\ddot{p}_{CG(C)}}_{\substack{\text{Relative} \\ \text{Acc. wrt. } C}} \approx R_{IC} \underbrace{a_{CG(C)}}_{\substack{\text{Absolute} \\ \text{Acc. in } C}} \quad (\text{A-39})$$

Introducing Equation A-39 into Equation A-37 yields

$$\sum F_{(I)} = M R_{IC} a_{CG(C)} \quad (\text{A-40})$$

The forces that act on the body can be separated into gravitational (F_g) and non-gravitational (aerodynamic and thrust forces) as follows:

$$\sum F_{(I)} + F_{g(I)} = M R_{IC} a_{CG(C)} \quad (\text{A-41})$$

Since the aerodynamic and thrust forces are usually defined in body frame, it is useful to express Eq. A-41 in Body reference frame.

$$\begin{aligned} R_{bI} \left(\sum F_{(I)} + F_{g(I)} \right) &= R_{bI} \left(M R_{IC} a_{CG(C)} \right) = M R_{bI} R_{IC} a_{CG(C)} = M R_{bC} a_{CG(C)} \\ \sum F_{(b)} + F_{g(b)} &= M a_{CG(b)} \end{aligned} \quad (\text{A-42})$$

The gravitational force in Body reference frame depends on the orientation of the body with respect to the gravity vector. Since the Z -axis of F_E is perpendicular to Earth surface, it can be assumed that it coincides with the gravity vector, and thus the gravitational force in F_b can be calculated as a function of the orientation of F_b with respect to F_E through the rotation matrix R_{bE} , as explained in Section 2-1-2:

$$F_{g(b)} = R_{bE} g_{(E)} M \quad (\text{A-43})$$

where the gravity acceleration g_E corresponds to the gravity at the location of the Earth in which the local frame F_E is defined (usually approximated to $g_E = [0 \ 0 \ 9.81]^T$).

Finally, it is useful to relate the forces to the derivative of the velocities in Body frame F_b , but it is not correct to write $M a_{CG(b)} = M \dot{v}_{CG(b)}$ since F_b is not an inertial reference frame. In contrast, one can use the equivalence $v_{CG(C)} = R_{Cb} v_{CG(b)}$ and the assumption that F_C is an inertial reference frame to use the chain rule as follows:

$$\dot{v}_{CG(C)} = a_{CG(C)} = \dot{R}_{Cb} v_{CG(b)} + R_{Cb} \dot{v}_{CG(b)} = R_{Cb} (\Omega_{(b)} v_{CG(b)} + \dot{v}_{CG(b)}) \quad (\text{A-44})$$

Since $a_{CG(C)} = R_{Cb} a_{CG(b)}$, one can retrieve the expression for $a_{CG(b)}$:

$$a_{CG(b)} = \Omega_{(b)} v_{CG(b)} + \dot{v}_{CG(b)} \quad (\text{A-45})$$

and introduce it in Equation A-42 to obtain

$$\sum F_{(b)} + R_{bE} g_{(E)} M = M (\Omega_{(b)} v_{CG(b)} + \dot{v}_{CG(b)}) \quad (\text{A-46})$$

or

$$\frac{1}{M} \sum F_{(b)} + R_{bE} g_{(E)} = \dot{v}_{CG(b)} + \Omega_{(b)} v_{CG(b)} \quad (\text{A-47})$$

Introducing the notation $V \equiv v_{CG(b)}$, $\dot{V} \equiv \dot{v}_{CG(b)}$ for the linear velocity and linear acceleration with respect to F_C expressed in F_b , the evolution of the velocity of the C.G. of the body in F_b can be expressed as follows:

$$\dot{V} = \frac{1}{M} \sum F_{(b)} + R_{bE} R_{IC} g_{(E)} - \Omega_{(b)} V \quad (A-48)$$

or in scalar form:

$$\begin{aligned} \dot{u} &= rv - qw + \frac{X}{M} - g \sin \theta \\ \dot{v} &= pw - ru + \frac{Y}{M} + g \cos \theta \sin \phi \\ \dot{w} &= qu - pv + \frac{Z}{M} + g \cos \theta \cos \phi \end{aligned} \quad (A-49)$$

where $V = [u \ v \ w]^T$, $\dot{V} = [\dot{u} \ \dot{v} \ \dot{w}]^T$, and X, Y, Z are the net forces excluding gravity in the X_b, Y_b , and Z_b -axis, respectively.

A-2 Derivation of the linear accelerometers measurement expression

The measurements provided by linear accelerometers are highly dependent on their location. Figure A-3 shows a simplified schematic of one linear accelerometer at a distance R from the C.G. of a rigid body which represents the helicopter. One can observe that this device only allows the displacement of the proof mass m in one dimension, along the coordinate d . In reality, every Inertial Measurement Unit (IMU) usually includes a triaxial set of linear accelerometers, and thus Figure A-3 would include two additional sensor cases with their own proof masses, placed orthogonally to the one currently shown. Hereafter, the expression of the linear acceleration provided by a one-axis linear accelerometers is derived, and the result is later generalized to three sensors.

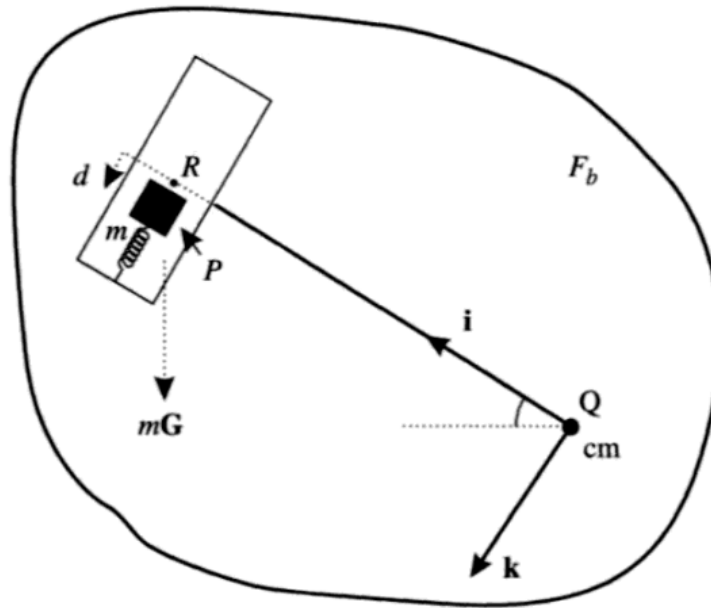


Figure A-3: One-axis linear accelerometer (*Stevens, 2003 [28]*).

The forces which act upon the proof mass m are gravity, the spring force, and the damping force of the device. Applying Newton's second law one can obtain the acceleration of the centre of gravity P of the mass m along the coordinate d (which is aligned with the axis $\mathbf{k}$ of F_b):

$$\frac{\sum F_{(b)}}{m} = a_{P(b)} = R_{bE} g_{(E)} - \frac{k_s}{m} d_b - \frac{b_s}{m} \dot{d}_b \quad (\text{A-50})$$

where $g = [0 \ 0 \ 9.81]^T$ in the F_E , R_{bE} is the rotation matrix from F_E to F_b , and k_s, b are the spring and damping coefficients of the sensor.

On the other hand, $a_{P(b)}$ can be calculated using Eq.A-21, obtained in the derivation of the translational kinematics equations:

$$\begin{aligned} a_{P(b)} &= R_{bI} a_{P(I)} \\ a_{P(I)} &= a_{Q(I)} + R_{Ib} \left(\underbrace{\ddot{p}_{P(b)}}_{\substack{\text{Relative} \\ \text{Acc. wrt. } F_b}} + \underbrace{\dot{\Omega}_{(b)} p_{P(b)}}_{\text{Tangential Acc.}} + \underbrace{\Omega_{(b)}^2 p_{P(b)}}_{\text{Centripetal Acc.}} + \underbrace{2 \Omega_{(b)} \dot{p}_{P(b)}}_{\text{Coriolis Acc.}} \right) \end{aligned} \quad (\text{A-51})$$

where $a_{P(I)}$ and $a_{Q(I)}$ are the acceleration of the proof mass and the helicopter C.G. expressed in inertial frame, and $p_{P(b)}$, $\dot{p}_{P(b)}$, and $\ddot{p}_{P(b)}$ are the relative position, velocity, and acceleration of the proof mass m with respect to Q expressed in F_b , respectively. Point Q is the C.G. of the helicopter, and $\Omega_{(b)}$ is the helicopter angular velocity expressed in F_b .

Combining the two Equations in A-51 one obtains

$$a_{P(b)} = \underbrace{R_{bI} a_{Q(I)}}_{a_{Q(b)}} + \ddot{p}_{P(b)} + \dot{\Omega}_{(b)} p_{P(b)} + \Omega_{(b)}^2 p_{P(b)} + 2 \Omega_{(b)} \dot{p}_{P(b)} \quad (\text{A-52})$$

The acceleration of the C.G. of a body expressed in F_b was obtained in Eq. A-45 from the translational dynamic EoM:

$$a_{CG(b)} = \dot{V} + \Omega_{(b)} V \quad (\text{A-53})$$

and can be substituted for $a_{Q(b)}$ in Eq.A-52:

$$a_{P(b)} = \dot{V} + \Omega_{(b)} V + \ddot{p}_{P(b)} + \dot{\Omega}_{(b)} p_{P(b)} + \Omega_{(b)}^2 p_{P(b)} + 2 \Omega_{(b)} \dot{p}_{P(b)} \quad (\text{A-54})$$

From Figure A-3 one can notice that the position vector $p_{P(b)}$ can be decomposed as $p_{P(b)} = p_{R(b)} + d_b$, where $p_{R(b)}$ is the relative distance between the location R of the sensor and the body's C.G., and the vector d is the displacement of the proof mass, and thus $\dot{p}_{P(b)} = \dot{d}$, $\ddot{p}_{P(b)} = \ddot{d}$, which allows to write Eq. A-54 as

$$a_P = \dot{V} + \Omega V + \ddot{d} + \dot{\Omega} (p_R + d) + \Omega^2 (p_R + d) + 2 \Omega \dot{d} \quad (\text{A-55})$$

where the subscript (b) which indicates that the variables are expressed in F_b has been omitted for clarity.

Each linear accelerometer physically restricts its proof mass to displace only in one direction, called the sensitive axis, for example, along the axis $\mathbf{k}$ in the Figure A-3. Since the acceleration

measurement is based on the displacement of its proof mass, each linear accelerometer only measures the acceleration in the direction in which its mass is allowed to move, and thus they only provide the acceleration component in the sensitive axis out of the three components of the vectorial equation (A-55).

For instance, the sensitive axis $d = 0\vec{i} + 0\vec{j} + d\vec{k}$ of the linear accelerometer shown in Figure A-3 coincides with Z_b of the F_b , and the acceleration of the proof mass can be obtained as follows:

$$\begin{aligned}\vec{k}^T \cdot a_P &= \vec{k}^T \cdot (\dot{V} + \Omega V + \ddot{d} + \dot{\Omega} p_R + \dot{\Omega} d + \Omega^2 p_R + \Omega^2 d + 2\Omega \dot{d}) \\ &= [\dot{w} - qu + pv + \ddot{d} - d(p^2 + q^2) + p_R(pr - \dot{q})] \vec{k}\end{aligned}\quad (\text{A-56})$$

where the acceleration terms which are a cross-product of d or its derivatives ($\dot{\Omega} d$ and $2\Omega \dot{d}$) are zero along the sensitive axis of the linear accelerometer.

Therefore, if three orthogonal linear accelerometers (triaxial accelerometer) aligned with the axes of F_b are located at point R, and assuming each of them has a proof mass m , then each linear accelerometer will measure the acceleration of each axis of F_b , which can be appended and expressed in vectorial form as

$$a_P = \dot{V} + \Omega V + \ddot{d}_s + \dot{\Omega} p_R + \Omega^2 p_R + \Omega^2 d_s \quad (\text{A-57})$$

where $d_s = [d_x \ d_y \ d_z]^T$ is the vector of orthogonal displacements of the proof masses of the triaxial accelerometer.

The Equations A-50 and A-57 can be combined to obtain

$$\ddot{d}_s + \frac{b}{m} \dot{d}_s + \frac{k_s}{m} d_s + \Omega^2 d_s = -(\dot{V} + \Omega V + \dot{\Omega} p_R + \Omega^2 p_R) + R_{bE} g \quad (\text{A-58})$$

The linear accelerometers measure the displacements d_s of the proof masses, and thus knowing the masses m , and the value of the damping and spring coefficients, the value of the expression at the right hand side of Eq.A-58 can be obtained with an error of $\Omega^2 d_s$. However, in high-sensitivity accelerometers, this error is eliminated using a force rebalancing technique, which measures the force required to maintain d_b and its derivatives very close to zero [28]. This required force, divided by the known mass m , and changed of sign, is the acceleration signal provided by the linear accelerometers, referred to as a , which can be expressed as follows:

$$a = \dot{V} + \Omega V + \dot{\Omega} p_R + \Omega^2 p_R - R_{bE} g \quad (\text{A-59})$$

or in scalar form

$$\begin{aligned}a_x &= \dot{u} - rv + qw - x(q^2 + r^2) + y(pq - \dot{r}) + z(pr + \dot{q}) + g \sin \theta \\ a_y &= \dot{v} - pw + ru - y(p^2 + r^2) + z(qr - \dot{p}) + x(pq + \dot{r}) - g \cos \theta \sin \phi \\ a_z &= \dot{w} - qu + pv - z(p^2 + q^2) + x(pr - \dot{q}) + y(qr + \dot{p}) - g \cos \theta \cos \phi\end{aligned}\quad (\text{A-60})$$

where $p_R = [x \ y \ z]^T$.

A-3 Substitution of the rotation angles

It can be noticed that the order of rotation from the generic frame A to B is the inverse of that from the F_E to the F_b , here repeated for convenience.

$$p_Q(B) = R_{BA} p_Q(A)$$

$$R_{BA} = R_{\beta_3} R_{\beta_2} R_{\beta_1}$$

$$R_{bE} = R_{bE''}(\phi) R_{E''E'}(\theta) R_{E'E}(\psi)$$

$$R_{\beta_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta_1 & -\sin \beta_1 \\ 0 & \sin \beta_1 & \cos \beta_1 \end{bmatrix}$$

$$R_{E'E}(\psi) = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{\beta_2} = \begin{bmatrix} \cos \beta_2 & 0 & \sin \beta_2 \\ 0 & 1 & 0 \\ -\sin \beta_2 & 0 & \cos \beta_2 \end{bmatrix}$$

$$R_{E''E'}(\theta) = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$R_{\beta_3} = \begin{bmatrix} \cos \beta_3 & -\sin \beta_3 & 0 \\ \sin \beta_3 & \cos \beta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{bE''}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix}$$

For instance, the structure of R_{β_1} reveals that it is a rotation around the X -axis, while the first rotation from F_E to F_b is around the Z -axis ($R(\psi)$). However, if the motion described by R_{BA} is the same than R_{bE} but in reversed order, the rotation angles $\beta_3, \beta_2, \beta_1$ must be the same than the ones from F_E to F_b but in reversed order, and thus $\beta_1 = \phi, \beta_2 = \theta$, and $\beta_3 = \psi$. This substitution can be confirmed taking into account that

$$R_{BA}^T = (R_{\beta_3} R_{\beta_2} R_{\beta_1})^T = R_{\beta_1}^T R_{\beta_2}^T R_{\beta_3}^T$$

and realizing that $R_{\beta_1}^T = R_{bE''}(\phi)$, $R_{\beta_2}^T = R_{E''E'}(\theta)$, $R_{\beta_3}^T = R_{E'E}(\psi)$ leads to $R_{BA}^T = R_{bE}$.

A-4 Simplification of the inertia matrix

If the rigid body is symmetric in the X, Z -plane, then $Y = -Y$ and the products of inertia $\int xy$ and $\int xz$ are zero as hereafter proved:

$$\begin{aligned} \int xy \, dm &= \rho \int_{-a}^a \int_{-b}^b \int_{-c}^c xy \, dx \, dy \, dz \\ &= \rho \int_{-a}^a \int_{-b}^b xy \underbrace{[z]_{-c}^c}_{2c} \, dx \, dy \\ &= 2c\rho \left(\int_{-a}^a \int_{-b}^0 xy \, dx \, dy + \int_{-a}^a \int_0^b xy \, dx \, dy \right) \\ &= 2c\rho \int_{-a}^a \left(\int_{-b}^0 xy \, dy + \int_0^b xy \, dy \right) dx \\ &= 2c\rho \int_{-a}^a \left(-\int_0^b xy \, dy + \int_0^b xy \, dy \right) dx = 0 \end{aligned} \tag{A-61}$$

The same procedure can be followed to prove that the product of inertia $\int xz$ is zero.

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Glossary

List of Acronyms

C.G.	Center of Gravity
DARE	Discrete-Algebraic Riccati Equation
DoF	Degrees of Freedom
D-LG-EKF	Discrete Lie-Group Extended Kalman Filter
EKF	Extended Kalman Filter
EoM	Equations of Motion
F_b	Body-fixed reference frame
F_C	Earth-centered, Earth-fixed reference frame
F_E	Vehicle-carried normal Earth reference frame
F_I	Earth-centered Inertial reference frame
IMU	Inertial Measurement Unit
INS	Inertial Navigation Systems
KF	Kalman Filter
LS	Least Squares
LPF	Low-Pass-Filter
LTI	Linear-Time-Invariant
NOSKF	Stationary Kalman Filter with Nonlinear Output
NED	North-East-Down
MEMS	Micro-electromechanical System
MISKF	Stationary Kalman Filter with Modified Input
SNR	Signal-to-noise Ratio

