The Asymptotic Behavior of a Random Walk Among a Field of Traps

by

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Contents

Abstract

We consider the problem of random walks moving around on a lattice $\mathbb{Z}^d$ with an initial Poisson distribution of traps. We consider both static and moving traps. In the static case, we prove that the survival time has a decay of $e^{-ct^{d/d+2}}$ based on a heuristic argument. In the moving case we aim to prove the sub-exponential decay of the survival time in dimensions 1 and 2, as well as the exponential decay of survival in dimensions 3 and higher. We achieve the former by first expressing the survival probability in the range of a random walk and by showing that the asymptotic behavior of said range behaves in a sub-exponential and exponential way for dimensions 1/2 and ≥ 3 respectively. Furthermore, we also show an upper bound for the survival time of the form $\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(T \geq t) < 0$.

Following this we look at the situation where traps decay as $||x|| \to \infty$. Meaning less traps will be distributed further away from the origin. We show that in the static case, if the decay rate satisfies the condition $p_x \leq p(||x||)$ where $p(r)$ is non-increasing and $r\,p(r)$ is integrable and convergent, that the random walk will be transient, meaning that there will be a strictly positive chance of survival. Lastly, we then show that for the dynamically moving traps case, if the decay rate is "fast enough", meaning that if the Poisson parameter of the distribution of the traps $\rho(x)$ is of the form $1/||x||^{2+\alpha}$ where $\alpha > d-2$, that there will also be a strictly positive probability of survival.

Layman Summary

We will consider the problem of a random walk. A random walk is essentially a person walking, and choosing the direction they walk in based on random probabilities. If at the same time there are other people walking in the same random way, one could imagine that they can bump into each other. When this happens we say that our original person is "killed" and he stops walking. Naturally we can ask ourselves how long it will take on average to bump into someone else in this scenario. We will show that as time goes on, there is a slower than exponential rate at which this survival probability goes down when we walk in 1 or 2 dimensions, and exactly exponential as we walk in 3 or more. After this problem we consider problems where there are less and less people as we go further away from our starting point, and show that then under the right circumstances, there is always a chance we can survive.

1 Introduction

In the mathematical field of probability, the study of random walks has existed for over 100 years. While many results have already been proved and at this point are well known results, the study of random walks moving around in a field of traps is "relatively" new. Essentially what we will be studying is the movement of the random walk on a lattice, where simultaneously other random walks (traps) move around as well. When the random walker meets one of these traps, it is killed immediately and the process is stopped. We will make a distinction between two types of trapping systems: the static trapping system, which implies that from the start of the process the trap locations are known and the traps do not move, implying the random walker only has to avoid these sites, and the dynamic trapping system, where the traps move at the same time as the random walker.

The natural question that arises when viewing this problem is how the survival time of the random walker behaves as time evolves. We consider the general case, but also up to a given time t. In the case of static traps, we will show that the optimal strategy of survival is by staying in a ball that is completely devoid of traps, and staying in this ball for as long as possible. In the dynamic case, we will prove that the survival time decays sub-exponentially in dimensions 1 and 2, and exponentially in dimensions 3 or higher. We will do this by first proving lower bounds for the survival time, and then moving on to showing the upper bounds of the same order. We do this through adapting and showing the proofs of Drewitz et al from [4] and [9] for the lower bounds, and using a simplified proof of [9] for the upper bound, which uses a greatest eigenvalue technique to show the exponential decay.

After this, we will move on to a situation where there is so-called decay of traps. This means that the distribution of traps on the lattice goes to 0 as we get further away from the origin. We will first use a proof from Den Hollander, Menshikov, and Volkov [1], where they show that under certain conditions, in the non-moving case the random walk will have a positive probability of survival. From there we move on to the moving case where we will also show certain conditions for the random walk to have a positive chance of "escaping" the traps at any point, if the decay is fast enough.

2

Markov Chains and Random Walks

In this section we will discuss and prove some elementary results of random walks that are needed for any further discussion on the topic. We will start by defining Markov chains and what it means for them to be recurrent or transient, then we show that any random walk in dimension 3 or higher is transient, and recurrent otherwise. We also introduce random walks and show some basic properties. Lastly we will talk about the range of a random walk and the probability of hitting a distinct point.

2.1. Markov Chains

We start by defining a Markov chain through the Markov property:

Definition 2.1.1. *A sequence of random variables $X_1, X_2, \dots, X_{n+1}$ is called a discrete* ***Markov chain*** *if*
$\mathbb{P}(X_{n+1} = x_{n+1} | X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n)$

What we will use often, is that the Markov property inherently means that any movement of a random walk (or any Markovian process) starting at a point $x \in \mathbb{Z}^d$, is the same as movement starting from the origin.

Definition 2.1.2. *A process $\{X_t : t \geq 0\}$ is called a* ***continuous*** *Markov chain if it satisfies the continuous time Markov property*
$\mathbb{P}(X(t+s) = j | X(s) = i, X(u) = k, 0 \leq u < s) = \mathbb{P}(X(t+s) = j | X(s) = i)$.

If in combination with satisfying the Markov property we also have that $\mathbb{P}(X(t+s) = j | X(s) = i)$ is independent of s, then we call the Markov chain to be stationary.

Definition 2.1.3. *A state i is* ***recurrent*** *if $\mathbb{P}(T_i < \infty) = 1$*
where T_i is the first passage time, meaning the first time the sequence $X_1 \dots X_n$ reaches the state i.

If a state is not recurrent we call it transient. Intuitively this can be understood as the fact that a recurrent state i will always be reached at some point by the Markov process, given infinite time or steps. We say that a *process* where all states are "connected" is recurrent when all states are recurrent. Connected here means the ability to reach all states from every state, which is of course the case for a random walk in $\mathbb{Z}^d$. The definition of a transient process is analogous. The following theorem will be used extensively in the later parts of this paper.

Theorem 2.1.1. *A simple symmetric random walk on the lattice $\mathbb{Z}^d$ is recurrent when d = 1,2. But transient when $d \geq 3$.*

This also means that a random walk when $d = 1,2$, will reach every point on the lattice infinitely many times, given infinite time.

2.2. Random Walks and Basic Properties

After considering Markov chains we now move on to the main topic of this paper, which is random walks. We start by defining a discrete random walk on $\mathbb{Z}^d$.

Definition 2.2.1. *Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables, where every X_i takes values in $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_d\}$ with probabilities $\{p_1, p_2, \ldots, p_n\}$, here $\boldsymbol{e}_i$ are the base vectors of $\mathbb{Z}^d$.*

*The process $M_n := \sum_{i=0}^n X_i$ is called a **discrete random walk**.*

If $p_1 = p_2 = \ldots = p_n$, then we call the random walk **symmetric**. In this paper we will consider continuous time random walks. Note that these still take discrete steps on the lattice $\mathbb{Z}^d$, just not with discrete steps. The continuous time random walk has *jump rates* $c(x, y)$ that give the rate at which the random walk $X(t)$ moves from x to y, it is obvious that $c(x, y) = 0$ if x and y are **not** neighbors on the lattice.

Note that every random walk is a Markov chain, and so all the results that we have shown here for general Markov chains also apply to the case of the symmetric random walk that we will discuss throughout this paper.

Following this we would also like to find an expression for the probability of the random walk ever reaching a distinct point $x \in \mathbb{Z}^d$. We are able to do this through the following theorem.

Theorem 2.2.1. *The probability of a random walk X_n ever reaching the state x is given by $\mathbb{P}(T_x < \infty) = \frac{G(0,x)}{G(0,0)}$ where G is the Green's function associated with a random walk.*

Proof. Let us take the random walk X_n and define $f(0, x, m)$ as the probability to reach x at m for the first time. We define $g(0, x, n)$ to be the probability to be at state x at time n. Now let

$$G(0, x, \delta) = \sum_{n=0}^{\infty} g(0, x, n)\delta^n \text{ and}$$
$$F(0, x, \delta) = \sum_{n=0}^{\infty} f(0, x, n)\delta^n$$

For $\delta < 1$. Now note that if we let $\delta \to 1, F(0, x, \delta)$ becomes the probability to ever reach x, starting from 0. Let

$$G(0, x, \delta) = \sum_{n=0}^{\infty} g(0, x, n)\delta^n = \sum_n \sum_{j=1}^{n} f(0, x, j) g(x, x, n - j)\delta^j \delta^{n-j} =$$
$$\sum_n \sum_{j=1}^{n} f(0, x, j)\delta^j g(x, x, n - j)\delta^{n-j} = F(0, x, \delta)G(x, x, \delta)$$

Now it is easy to see that $F(0, x, \delta) = \frac{G(0,x,\delta)}{G(x,x,\delta)}$. By translation invariance of the random walk, we have that $G(x, x, \delta) = G(0, 0, \delta)$. Now letting $\delta \to 1$ we arrive at the desired formula. □

Definition 2.2.2. *For a continuous random walk $X(t)$ on $\mathbb{Z}^d$, we define the **range** of $X(t)$ to be*

$$R(X, t) = \sum_x \mathbb{1}_{\{X(s)=x, \exists s \in [0,t)\}}$$

Many results about how the range behaves are well known. What we need is the asymptotic behavior of the expectation of the range which Erdös and Dvoretzky [5] showed behaves in the following way.

Theorem 2.2.2. *For a Random walk $X(t)$ on the lattice $\mathbb{Z}^d$, we have that*

$$\mathbb{E}[R(X,t)] = \begin{cases} \sqrt{t}, & d=1 \\ \frac{\pi t}{\log t} + O\left(\frac{t \log\log t}{\log^2 t}\right), & d=2 \\ t\gamma_d + O(t^{2-d/2}), & d \geq 3 \end{cases}$$

Here γ_d is the probability of never returning to the origin. We omit the proof and refer you to [5] for a more detailed overview.

3

Trapping Systems and Lower Bounds on the Tail Survival Probabilities

In this chapter we will start by introducing traps and different trapping systems, namely static and moving traps. We will continue by discussing the best strategy to survive in the case where the traps are static, which will turn out to give us an expression of the form that decays sub-exponentially, in the form $e^{-ct^{\frac{d}{d+2}}}$. Then we move on to the case of moving traps, where we will show lower bounds which imply that the survival probability decays sub-exponentially in dimensions 1 and 2, but exponentially when $d \geq 3$.

3.1. Trapping Systems

We start by introducing the system as a whole. From now on we let $X = X(t)_{t\geq 0}$ be a continuous simple symmetric random walk on the lattice $\mathbb{Z}^d$, starting at the origin. We might refer to this random walk as the random *walker*. Now let $Y(t)^y$ be another random walk on the same lattice, starting from y. The collection $(Y_i^y)_{1\leq j\leq \eta(y), y\in\mathbb{Z}^d}$ is the collection of these random walks, where $\eta(y)$ is the amount of random walks starting at $y \in \mathbb{Z}^d$. We will mainly discuss the problem where $\eta(y)$ is Poisson distributed with mean ρ and the random walks Y_i^y have jump rate ν. We will call the collection **traps** from this point forward. The amount of traps at time t at point $x \in \mathbb{Z}^d$ will be denoted by $\xi(t,x)$. Which is defined as follows:

$$\xi(t,x) = \sum_{y\in\mathbb{Z}^d, 1\leq i\leq \eta(y)} \delta_x(Y_i^y(t)).$$

Where $\delta_x(Y)$ is equal to 1 if $Y(t) = x$ and 0 otherwise. The main problem that we are interested in, is that as the traps and the random walker move on the same lattice, there is always a probability that our random walker will meet one of the traps at the same lattice point. When this happens we say the random walker is killed with a rate $\gamma \in (0,\infty]$ and stops moving. We are interested in how long our walker can survive, and we will call this the survival probability. The probability of survival up to time t will be denoted by $\mathbb{P}(T \geq t)$. We will also leave γ arbitrary in most calculations, however in the case of any intuitive arguments we will always assume that $\gamma = \infty$, which means that with probability 1 the walker is killed immediately when it meets a trap.

Something important to note is that if we assume that the Poisson distribution of the traps is homogeneous, that under the process of independent random walkers the distribution is invariant. Meaning that even after the traps move, they are still Poisson distributed. When started from a possible in-homogeneous Poisson distribution, one has that at every $t > 0$, we have that under the evolution of independent random walks we have again a product of Poisson measures.

We will discuss two main trapping systems with one major distinction. We will talk about dynamic (moving) traps, and static (non-moving) traps.

3.2. Static Non-Moving Traps

We start by discussing the problem of a static trapping system. Traps appear at a point y on the lattice $\mathbb{Z}^d$ with a certain probability p (this system can still be Poisson distributed). The traps are still considered continuous random walks just with jump rate $v = 0$. We will discuss how to find the optimal survival strategy for the random walker such that the survival time is maximized.

Originally in two papers by Donsker and Varadhan [2] & [3], they proved the survival rate for the (spacially) continuous analog of a random walk, a Brownian motion. They showed that

Theorem 3.2.1. *There exists a $k(v,d) > 0$, such that for any $x \in \mathbb{R}^d$ and $\epsilon > 0$,*

$$\lim_{t\to\infty} \frac{1}{t^{\frac{d}{d+2}}} \log \mathbb{E}_x \left[e^{-v|C_t^{\epsilon}(\beta(\cdot))|} \right] = -k(v,d). \tag{3.1}$$

We do not provide further details of $k(v,d)$ here as it is beyond the scope of this paper. We will however discuss the more relevant parts. In (3.1) $|C_t^{\epsilon}(\beta(\cdot))|$ denotes the d-dimensional volume of a Wiener Sausage, which is a body that takes all points within a neighborhood of the path that a Brownian motion has taken. $\mathbb{E}_x$ here denotes the expectation on the Brownian motion paths starting at x. Through analysis of the survival time in the static trap case, we will be able to see where the decay term $t^{d/d+2}$ comes from.

So how do we find the survival probability of our random walker? As there is no randomness involved within the movement of the traps, only where they come into existence, it would make sense for us to find a way for our random walker to stay away from any existing traps. The basic strategy here is to find a d-dimensional ball that contains no traps, and have the walker stay in this ball for as long as possible. If we do this, it means that there are two components of randomness at play when trying to find this trap-free ball. One is the probability of the walker not leaving the ball, and the other is the probability of the ball being devoid of traps. Logically the size of the ball will influence both probabilities, with a bigger ball causing the probability of the walker not leaving the safe area to be larger, but the probability of the ball being trap-free will naturally be smaller. This causes us to consider some sort of optimal radius, dependent on t, for the ball, where these probabilities have the same rate of decay.

Now to maximize this survival probability, we first look at the two probabilistic events at work. The probability to have a ball of radius R, in which there are **no** traps present, is given by $e^{-c_1 R^d}$, where c_1 is a constant. The probability for the random walker to not leave the ball with radius R until time t, is given by $e^{-c_2 \frac{t}{R^2}}$. Now we want to find an optimal R as a function of t, where the rate of decay for both probabilities is the same. If we decide to choose $R = t^{\alpha}$, for some α, then by plugging it into both of the probabilities and equating the rates of decay yields

$$e^{-c_1 t^{\alpha d}} \approx e^{-c_2 t(1/t^{2\alpha})}. \tag{3.2}$$

$$t^{\alpha d} = \frac{t}{t^{2\alpha}} \tag{3.3}$$

This then of course implies that $\alpha = \frac{1}{d+2}$, meaning that the optimal radius of the ball is given by $t^{1/d+2}$. Now note that the survival probability Z_t of the random walker can be expressed using the range. We want to confine the range to be only within this ball of radius $t^{1/d+2}$, at least until time t. Meaning that the survival probability can be expressed as:

$$Z_t = \mathbb{E}_0^X \left[e^{R(X,t)\ln p} \right] \approx e^{-Ct^{\frac{d}{d+2}}}. \tag{3.4}$$

The connection to Theorem 3.2.1 can now be seen through observing the appearance of the term $t^{\frac{d}{d+2}}$ in both cases. Note that also both incorporate the volume of some sort of body in which we would like to stay. There are more generalized results for the survival probability when the field of traps is not for example i.i.d Bernoulli distributed, but we will not discuss them in this paper.

3.3. First Lower Bound of the Survival Probability with Moving Traps

We now move on to start discussing the problem of moving traps. We will first prove an initial lower bound for the survival probability based on the paper of Redig [9], where we will use the Poisson character of the traps to express the survival probability in terms of the range of a random walk, and then use Jensen's inequality to prove a lower bound. We will assume that the traps follow a Poisson distribution on the lattice $\mathbb{Z}^d$ with mean ρ.

Theorem 3.3.1. *Let T be the survival time of a simple random walker in a system of dynamic traps on $\mathbb{Z}^d$ that start from a Poisson distribution with mean ρ, denoted μ_ρ. Then:*

$$\mathbb{P}(T \geq t) = \mathbb{E}_X[e^{-\rho \mathbb{E}_Y[R(X-Y,t)]}]. \tag{3.5}$$

Here $\mathbb{E}_X$ denotes the expectation with respect to X, and $R(X-Y,t)$ is the range of the random walk $X-Y$ up to time t.

Proof.

$$\begin{aligned}
\mathbb{P}(T \geq t) &= \int \mathbb{P}\left(Y_i^y(s) - X(s) \neq 0, \forall i = 1, \ldots, \eta(y), \forall y, \forall s \in [0,t)\right) \mu_\rho(d\eta) \\
&= \int \left(\int \prod_y \prod_{i=1}^{\eta(y)} \left(\int \mathbb{1}_{\{Y_i(s) \neq X(s)-y, \forall s \in [0,t)\}} d\mathbb{P}_{Y_i} \right) \mu_\rho(d\eta) \right) d\mathbb{P}_X \\
&= \int \left(\int \prod_y \left(\int \mathbb{1}_{\{Y(s) \neq X(s)-y, \forall s \in [0,t)\}} d\mathbb{P}_Y \right)^{\eta(y)} \mu_\rho(d\eta) \right) d\mathbb{P}_X
\end{aligned} \tag{3.6}$$

Here we used the Tonelli-Fubini theorem for the 1st equality. In the 3rd equality we use that the traps Y_i move as independent random walks. Moving on from (3.6) we find that

$$\begin{aligned}
\mathbb{P}(T \geq t) &= \int \left(\int \prod_y \mathbb{P}\left(Y(s) \neq X(s) - y, \forall s \in [0,t)\right)^{\eta(y)} \mu_\rho(d\eta) \right) d\mathbb{P}_X \\
&= \int \left(\prod_y \sum_{\eta=0}^{\infty} e^{-\rho} \frac{\rho^\eta}{\eta!} \mathbb{P}\left(Y(s) \neq X(s) - y, \forall s \in [0,t)\right)^{\eta(y)} \right) d\mathbb{P}_X \\
&= \int \left(\prod_y e^{-\rho} \sum_{\eta=0}^{\infty} \frac{\left(\rho \mathbb{P}\left(Y(s) \neq X(s) - y, \forall s \in [0,t)\right)\right)^{\eta(y)}}{\eta!} \right) d\mathbb{P}_X \\
&= \int \left(\prod_y e^{-\rho} e^{\rho \mathbb{P}(Y(s) \neq X(s) - y, \forall s \in [0,t))} \right) d\mathbb{P}_X
\end{aligned}$$

$$
\begin{aligned}
&= \int \left(\prod_y e^{-\rho(1-\mathbb{P}(Y(s)\neq X(s)-y, \forall s\in[0,t))} \right) d\mathbb{P}_X \\
&= \int \left(\prod_y e^{-\rho\mathbb{P}(Y(s)=X(s)-y, \exists s\in[0,t))} \right) d\mathbb{P}_X \\
&= \int \left(e^{-\rho \int \sum_x \mathbb{1}_{\{Y(s)-X(s)=-y, \exists s\in[0,t)\}} d\mathbb{P}_Y} \right) d\mathbb{P}_X
\end{aligned}
$$

Now recall Definition 2.2.2 where the range of a random walk was defined, from there we see

$$
\begin{aligned}
\mathbb{P}(T \geq t) &= \int e^{-\rho \int R(Y-X,t) d\mathbb{P}_Y} d\mathbb{P}_X \\
&= \mathbb{E}_X[e^{-\rho \mathbb{E}_Y[R(X-Y,t)]}]. \qquad (3.7)
\end{aligned}
$$

□

Where we used the commonly known expectation rule for the last equality. Now we can move on to using Jensen's inequality to find a lower bound for the survival probability.

Theorem 3.3.2. *For a simple random walk Z on the lattice $\mathbb{Z}^d$ we have that*

$$
\mathbb{P}(T \geq t) \geq e^{-\rho \mathbb{E}[R(Z,2t)]}. \qquad (3.8)
$$

Proof. By Theorem 3.3.1 we have that

$$
\mathbb{P}(T \geq t) = \mathbb{E}_X[e^{-\rho \mathbb{E}_Y[R(X-Y,t)]}],
$$

now by Jensen's inequality we get

$$
\mathbb{E}_X[e^{-\rho \mathbb{E}_Y[R(X-Y,t)]}] \geq e^{-\rho \mathbb{E}_X \mathbb{E}_Y[R(X-Y,t)]}.
$$

And by the known fact that the random walk $X - Y$ behaves the same as a random walk Z at twice the speed, we have that (3.8) follows. □

Now that we have a lower bound on the survival probability, Theorem 2.2.2 combined with Theorem 3.8 and letting $t \to \infty$ yields a sub-exponential lower bound in dimensions 1 & 2, and an exponential lower bound for $d \geq 3$.

This is a first result in showing the behavior of the random walk moving through a field of dynamic traps. It turns out we can actually find a sharp lower bound when d = 1,2. Furthermore, we can find upper bounds of the same order through methods that use spectral methods, or through means of something called the Pascal principle. Theorem 1.1 from [4], which is stated below, is an expression of the expectation of the survival probability, here denoted by $\mathbb{E}^\xi[Z^\gamma_{t,\xi}]$, that we will prove some parts of. As some of the ingredients needed for the lengthy proof are beyond the scope of this paper. Seen immediately is that the expression in (3.9) is of the same order as the one in (3.5), meaning that even the first lower bound we found is already of the right magnitude.

Theorem 3.3.3. [Drewitz et al, (2011)] *Let $\gamma \in (0,\infty], \kappa \geq 0, \rho > 0, \nu > 0$, then:*

$$
\mathbb{E}^\xi[Z^\gamma_{t,\xi}] = \begin{cases} e^{-\rho\sqrt{\frac{8\nu t}{\pi}}(1+o(1))}, & d=1 \\ e^{-\rho\pi\nu\frac{t}{\log t}(1+o(1))}, & d=2 \\ e^{-\lambda_{d,\gamma,\kappa,\nu,\rho}t(1+o(1))}, & d\geq 3 \end{cases} \qquad (3.9)
$$

where $\lambda_{d,\gamma,\kappa,\nu,\rho}$ depends on $d, \gamma, \kappa, \nu, \rho$, and is called the annealed Lyapunov exponent. Furthermore $\lambda_{d,\gamma,\kappa,\nu,\rho} \geq \lambda_{d,\gamma,0,\nu,\rho} = \rho\gamma/(1+\frac{\gamma G_d(0)}{\nu})$, where $G_d(0) := \int_0^\infty p_t(0)dt$ is the Green function of a simple symmetric random walk on $\mathbb{Z}^d$ with jump rate 1 and transition kernel $p_t(\cdot)$.

The proof of Theorem 3.3.3 consists of a lot of different steps. In this paper we will prove the existence of the Lyapunov exponent $\lambda_{d,\gamma,\kappa,\nu,\rho}$, and sketch the proof for the sharp lower bounds. In Section 4 we will discuss the Pascal principle and show upper bounds for $\mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}]$.

3.4. Lyapunov Exponent

We will start by proving that $\lambda = \lambda_{d,\gamma,\kappa,\nu,\rho}$ exists and is finite for $d \geq 3$. For the purpose of showing its existence we define

$$\lambda_{d,\gamma,\kappa,\nu,\rho} := -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}]. \tag{3.10}$$

We will require a few tools to show that the limit exists. First we require the subadditivity lemma (sometimes known as Fekete's lemma, named after the mathematician who proved it in [6]).

Lemma 3.4.1. [Subadditivity Lemma] *Let $\{a_n\}_{n\geq 1}$ be a* ***subadditive*** *sequence with $a_i \geq 0$. That is for any a_i, a_j we have*

$$a_{i+j} \leq a_i + a_j$$

then the following limit exists and is equal to

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf \frac{a_n}{n}$$

The analogous also holds for measurable functions. The proof of this is pretty short and concise so we will state it here.

Proof. Let $\epsilon > 0$ and set $L = \inf \frac{a_n}{n}$, as well as $M = \max_{1\leq i\leq n} a_i$. Then choose n such that $a_n < n(L+\epsilon)$. Now choose a $m \geq n$ with $m = kn + r$, provided $r < n$. Then through the subadditive property:

$$a_m = a_{kn+r} \leq ka_n + a_r \leq ka_n + M$$

Now

$$\frac{a_m}{m} \leq \frac{ka_n + M}{m} = \frac{ka_n}{m} + \frac{M}{m} < \frac{kn(L+\epsilon)}{m} + \frac{M}{m}$$

Now as we note that $\frac{kn}{m} \to 1$ as $m \to \infty$. We instantly see that $\frac{kn(L+\epsilon)}{m} + \frac{M}{m} \to (L+\epsilon)$, as $m \to \infty$ which completes the proof. □

The next tool we need is the following form of the Feynman-Kac formula.

Theorem 3.4.1. [Feynman-Kac] *Let X_t be a Markov process starting from $X_0 = x$, with generator L. The differential equation*

$$\begin{aligned}&\frac{\partial}{\partial t} v(t,x) = Lv(t,\cdot)(x) + V(t,x)v(t,x),\\ &v(0,\cdot) = 1,\end{aligned}$$

is solved by

$$v(t,x) = \mathbb{E}^X_x[e^{\int_0^t V(t-s,X_s)ds}1]$$

Note that the generator of a random walk $Lf(x)$, is given by $\sum_{y\sim x} c(x,y)(f(y)-f(x))$ where $c(x,y)$ is the jump rate going from x to y of the random walk. If the jump rate is constant and we call it κ, which is the case in the upcoming proof, it can be taken out and we can simply write $\kappa Lf(x) = \kappa \sum_{y\sim x}(f(y)-f(x))$.

We now prove the existence of the Lyapunov exponent (3.10). We start by finding an expression for $\mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}]$ through the Feynman-Kac formula that will allow us to prove the assumption needed for the subadditivity lemma.

$$\mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}] = \mathbb{E}^{X}_{0}\mathbb{E}^{\xi}\left[e^{-\gamma\int_0^t \xi(t-s,X(s))ds}\right] = \mathbb{E}^{X}_{0}\left[e^{\kappa\sum_{y\in\mathbb{Z}^d}(v_X(t,y)-1)}\right] \tag{3.11}$$

where we condition on X and $v_X(t,y)$ has the following expression.

$$v_X(t,y) = \mathbb{E}^{Y}\left[e^{-\gamma\int_0^t \delta_0(Y(s)-X(t-s))ds}\right] \tag{3.12}$$

Now we can recognize the form of Theorem 3.4.1 and see that $v_X(t,y)$ solves the differential equation

$$\frac{\partial}{\partial t}v_X(t,x) = \kappa L v_X(t,\cdot)(x) - \gamma\delta_X(t)(y)v_X(t,x), \tag{3.13}$$

with inital condition $v(0,\cdot)=1$. If we then set $\Sigma_X(t) := \sum_{y\in\mathbb{Z}^d}(v_X(t,y)-1)$ we see that $\Sigma_X(t)$ solves the equation

$$\begin{aligned}&\frac{\partial}{\partial t}\Sigma_X(t) = -\gamma v_X(t,X(t))\\&\Sigma_X(0)=0\end{aligned}$$

Integrating from 0 to t then yields

$$\Sigma_X(t) = -\gamma\int_0^t v_X(s,X(s))ds. \tag{3.14}$$

This we can then plug back into our original expression (3.11) and we get back

$$\mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}] = \mathbb{E}^{X}_{0}\left[e^{-\kappa\gamma\int_0^t v_X(s,X(s))ds}\right]. \tag{3.15}$$

Now we will use the subadditivity lemma 3.4.1 to show that the limit exists. For this we require that

$$-\log\mathbb{E}^{\xi}[Z^{\gamma}_{t_1+t_2,\xi}] \le -\left(\log\mathbb{E}^{\xi}\left[Z^{\gamma}_{t_1,\xi}\right] + \log\mathbb{E}^{\xi}\left[Z^{\gamma}_{t_2,\xi}\right]\right)$$

We define the *temporal shift* $(\theta_{t_1}X)(s) = X(t_1+s)-X(t_1)$. And use the fact that for any $s>t_1$ we have

$$\begin{aligned}v_X(s,X(s)) &= \mathbb{E}^{Y}_{X(s)}\left[e^{-\gamma\int_0^s \delta_0(Y(r)-X(s-r))dr}\right]\\&\le \mathbb{E}^{Y}_{X(s)}\left[e^{-\gamma\int_0^{s-t_1} \delta_0(Y(r)-X(s-r))dr}\right]\\&= \mathbb{E}^{Y}_{X(s)}\left[e^{-\gamma\int_0^{s-t_1} \delta_0(Y(r)-X(s-r-t_1)+X(t_1)dr}\right]\\&= \mathbb{E}^{Y}_{X(s)}\left[e^{-\gamma\int_0^{s-t_1} \delta_0(Y(r)-(X(s-r-t_1)-X(t_1))dr}\right]\\&= \mathbb{E}^{Y}_{X(s)}\left[e^{-\gamma\int_0^{s-t_1} \delta_0(Y(r)-((\theta_{t_1}X)(s-t_1)dr}\right]\\&= v_{\theta_{t_1}X}(s-t_1,(\theta_{t_1}X)(s-t_1))\end{aligned}$$

Then for $t_1,t_2>0$ we get

$$\begin{aligned}\mathbb{E}^{\xi}[Z^{\gamma}_{t_1+t_2,\xi}] &= \mathbb{E}^{X}_{0}\left[e^{-\kappa\gamma\int_0^{t_1+t_2} v_X(s,X(s))ds}\right]\\&= \mathbb{E}^{X}_{0}\left[e^{-\kappa\gamma\left(\int_0^{t_1} v_X(s,X(s))ds+\int_{t_1}^{t_1+t_2} v_X(s,X(s))ds\right)}\right]\\&= \mathbb{E}^{X}_{0}\left[e^{-\kappa\gamma\int_0^{t_1} v_X(s,X(s))ds}\right]\mathbb{E}^{X}_{0}\left[e^{-\kappa\gamma\int_{t_1}^{t_1+t_2} v_X(s,X(s))ds}\right]\\&\ge \mathbb{E}^{X}_{0}\left[e^{-\kappa\gamma\int_0^{t_1} v_X(s,X(s))ds}\right]\mathbb{E}^{X}_{0}\left[e^{-\kappa\gamma\int_0^{t_2} v_{\theta_{t_1}X}(s,\theta_{t_1}X(s))ds}\right]\\&= \mathbb{E}^{\xi}\left[Z^{\gamma}_{t_1,\xi}\right]\mathbb{E}^{\xi}\left[Z^{\gamma}_{t_2,\xi}\right]\end{aligned}$$

Now from this inequality we see that $-\log \mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}]$ satisfies the subadditive property, and so by Lemma 3.4.1 we see that (3.10) exists and is equal to $-\sup_{t>0} \frac{1}{t} \log \mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}]$. Which completes the proof.

3.5. Improved lower bounds

We can improve on the bound acquired in (3.8) through Lemma 3.5.1, the bounds in $d = 1,2$ are sharp by making it sharp in the cases $d = 1,2$. Instead of proving it completely we will instead sketch the proof and omit a few details. For the full detailed proof we refer you to [4].

Lemma 3.5.1.

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}] \geq -\rho\sqrt{\frac{8\nu}{\pi}}, \qquad d = 1,$$
$$\liminf_{t\to\infty} \frac{\ln t}{t} \log \mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}] \geq -\rho\pi\nu, \qquad d = 2.$$

Sketch of proof of Lemma 3.5.1 The way to prove these inequalities is similar to how we prove the survival probability for static traps. There are multiple forces at work so to speak, that all have a cost associated with their respective probabilities. What we want is to make the Random walk stay in a ball B_r that contains no traps up to time t. We can express this with three events. Namely E_t, F_t, and G_t. E_t is the event that $\{\eta(y) = 0, \forall y \in B_r\}$. Recall that $\eta(y)$ is the amount of traps starting at a point $y \in \mathbb{Z}^d$, meaning that E_t is the event that no traps start within the ball. F_t is the event that $\{Y^y_i \notin B_r, \forall y \notin B_r\}$, or the even that no traps that start outside of the ball initially, move inside the ball. Lastly G_t is the event that the random walk $X(t)$ starting from the origin, does not leave the ball before time t. With these three events defined we can now find a lower bound on $\mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}]$ expressed in their respective probabilities. Namely:

$$\mathbb{E}^{\xi}[Z^{\gamma}_{t,\xi}] \geq \mathbb{P}(E_t \cap F_t \cap G_t) = \mathbb{P}(E_t)\mathbb{P}(F_t)\mathbb{P}(G_t). \tag{3.16}$$

Now all that's left is to estimate the three possibilities. Considering a scale function R_t which moves slower than $\sqrt{t}$, (but faster than 1), we can find expressions for all the probabilities dependent on R_t. We leave out the derivations of these probabilities, but note that after obtaining them all thats left is to choose an appropriate form of R_t to achieve the desired lower bound. It turns out that choosing $R_t = \sqrt{\frac{t}{\ln t}}$ for $d = 1$, and $R_t = \ln t$ for $d = 2$ get the job done. Lemma 3.5.1 then follows.

4

Upper bounds

In this section we are going to use two ways to derive an upper bound for the survival probability. We will first use a method utilizing spectral techniques adapted from [9], where we use a greatest eigenvalue technique to derive an upper bound of the form

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(T \geq t) < 0 \tag{4.1}$$

The second technique is the one mentioned earlier used in the paper of Drewitz et al. [4], which uses the Pascal principle to find an upper bound.

4.1. Greatest Eigenvalue Method

We will start by introducing the setting for this problem. The traps move according to a process $\{\eta(t) : t \geq 0\}$. Where $\eta(t)_x$ is the number of traps at x, at time t. We call this process the *configuration process,* and the generator of this process we will denote by L_0. We will assume that this process has a reversible (and hence stationary) measure μ which is also translation invariant. Reversible here means that

$$\int g(L_0(f))\, d\mu = \int L_0(g) f\, d\mu, \tag{4.2}$$

meaning that the generator is symmetric in $L^2(\mu)$. From Chapter 4 section 4 of [7] we then find that this also implies the self-adjointness of L_0. Simultaneously with the configuration process, we of course also have our random walker moving across the lattice. From the "perspective" of the random walker the traps are at a different position than from a general observer, namely they are at some distance x away from the random walker. The traps seen from the position x of the random walker is denoted by another process $\{\tau_{x(t)}\eta(t) : t \geq 0\}$. This process we will call the *environment process.* We define the shift τ_x by

$$\tau_x \eta(y) = \eta(x+y). \tag{4.3}$$

Now the generator of this process is the sum of the two process embedded within, meaning that the generator of the environment process $L = L_0 + L_1$, where L_1 is the shift part due to the motion of the random walker. The precise definition is then as follows

$$\begin{aligned}
&Lf(\eta) = L_0 f(\eta) + L_1 f(\eta), \text{ where} \\
&L_0 f(\eta) = \lim_{t\downarrow 0} \frac{\mathbb{E}^T_\eta f(\eta_t) - f(\eta)}{t}, \\
&L_1 f(\eta) = \sum_y p(0,y)(f(\tau_y \eta) - f(\eta)).
\end{aligned}$$

$\mathbb{E}_\eta^T$ here denotes the expectation with regards to the traps, also note $\eta_0 = \eta$. We know that by assumption L_0 is reversible and self-adjoint. Because L_1 is the generator of a symmetric random walk we can easily see that it is a self-adjoint bounded operator, adding a bounded self-adjoint operator to an unbounded one causes no problems, so we see that L is also a self-adjoint operator, and thus the measure μ is also reversible for the process $\{\tau_x\eta(t)\}$.

We will need to introduce some notation and acquire some new tools before we can prove an upper bound. Recall the definition of γ, which we will from now on denote by $\gamma(\eta)$ and call a killing function. Then we say that $x \in \mathbb{Z}^d$ is a trap, iff $\gamma(\tau_x\eta) > 0$. Or in words, if γ at a position x, as seen from the view of our walker, is greater than 0, it is a trap. We will now first prove a few lemma's after which the upper bound will follow.

Lemma 4.1.1. *Let $\{\eta_t : t \geq 0\}$ as before, then*

$$\mathbb{P}(T \geq t) \leq \mathbb{E}\left[e^{-\int_0^t \gamma(\tau_{X_s}\eta_s)ds}\right] \tag{4.4}$$

Proof.

$$\begin{aligned}\mathbb{P}(T \geq t) &= \mathbb{P}(\gamma(\tau_{X_s}\eta_s) = 0, \forall s \in [0,t)) \\ &= \mathbb{P}\left(\int_0^t \gamma(\tau_{X_s}\eta_s)ds = 0\right) \\ &= \mathbb{P}\left(e^{-\int_0^t \gamma(\tau_{X_s}\eta_s)ds} \geq 1\right) \\ &\leq \mathbb{E}\left[e^{-\int_0^t \gamma(\tau_{X_s}\eta_s)ds}\right] \end{aligned} \tag{4.5}$$

□

Where the last step follows from the Markov inequality. The following lemma will provide us an even better expression for the upper bound.

Lemma 4.1.2. *Let Λ be the greatest eigenvalue of the operator $L - \gamma$. Then we have that*

$$\frac{1}{t}\log\mathbb{E}\left[e^{-\int_0^t \gamma(\tau_{X_s}\eta_s)ds}\right] \leq \Lambda. \tag{4.6}$$

Proof. Note that by the Feynman-Kac formula we have that

$$\psi(\eta, t) = e^{t(L-\gamma)}f(\eta) = \mathbb{E}_\eta\left[f(\eta)e^{-\int_0^t \gamma(\tau_{X_s}\eta_s)ds}\right].$$

$f(\eta_t, t)$ is the unique solution of the differential equation

$$\begin{aligned}&\frac{\partial}{\partial t}\psi(\eta, t) = L\psi(\eta, t) - \gamma(\eta)\psi(\eta, t) \\ &\psi(\eta, 0) = f(\eta)\end{aligned}$$

now take $f(\eta) = 1$ and see that

$$\mathbb{E}_\eta\left[e^{-\int_0^t \gamma(\tau_{X_s}\eta_s))ds}\right] = e^{t(L-\gamma)}1(\eta), \tag{4.7}$$

integrating over the probability measure μ then yields

$$\int \mathbb{E}_\eta\left[e^{-\int_0^t \gamma(\tau_{X_s}\eta_s))ds}\right] d\mu = (1, e^{t(L-\gamma)1}) \tag{4.8}$$

where $(\cdot,\cdot)$ denotes the inner product in L^2. Now we can see that

$$(1, e^{t(L-\gamma)}) \leq e^{t\Lambda} \tag{4.9}$$

where Λ is the greatest eigenvalue of the operator $L-\gamma$. Thus we have

$$\begin{aligned} \mathbb{E}\left[e^{-\int_0^t \gamma(\tau_{X_s}\eta_s)ds}\right] &\leq e^{t\Lambda}, \text{and hence} \\ \frac{1}{t}\log\mathbb{E}\left[e^{-\int_0^t \gamma(\tau_{X_s}\eta_s)ds}\right] &\leq \Lambda \end{aligned} \tag{4.10}$$

□

The next lemma will show the negativity of Λ after which the upper bound quickly follows.

Lemma 4.1.3. *Assume that the largest eigenvalue of the operator $L_0-\gamma$ is <0, then*

$$\Lambda < 0. \tag{4.11}$$

Proof.

$$\begin{aligned} \Lambda &= \sup_{g:\int g^2 d\mu=1}\big(((L-\gamma)g,g)\big) \\ &= \sup_{g:\int g^2 d\mu=1}\big(((L_0+L_1-\gamma)g,g)\big) \\ &= \sup_{g:\int g^2 d\mu=1}\big((-\gamma g,g)+(g,L_0 g)+(g,L_1 g)\big) \\ &= \sup_{g:\int g^2 d\mu=1}\left(\int -\gamma(\eta)g^2(\eta)d\mu+(g,L_0 g)+(g,L_1 g)\right) \\ &\leq \sup_{g:\int g^2 d\mu=1}\left(\int -\gamma(\eta)g^2(\eta)d\mu+(g,L_0 g)\right) \\ &\text{Now by assumption} \\ &< 0. \end{aligned} \tag{4.12}$$

The inequality here follows from the fact that $(g,L_1 g)<0$. □

Now by combining Lemma's 4.1.1, 4.1.2, and 4.1.3. We find that

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(T\geq t)<0. \tag{4.13}$$

The assumption in Lemma 4.1.3 is not strictly necessary, and can be made more rigorous through the means of imposing large deviation conditions on the configuration process, technically this is all thats needed to prove the upper bound. This is elaborated upon in [9], however it is beyond the scope of this paper. So for more general, less restrictive conditions for this upper bound we would like to refer you to his paper.

4.2. Pascal Principle

We can find a different upper bound as well. One with less restrictions that still provides us with an upper bound that decays exponentially. We do this by means of the Pascal principle. First proved in the paper of Moreau et al [8] in the discrete time case, Drewitz et al [4] then extended it to the continuous time case to be able to apply it in the setting that we have been working on.

Theorem 4.2.1. [Pascal Principle]

Let ξ be the random field generated by a collection of symmetric random walks $\{Y_i^y\}$ on $\mathbb{Z}^d$ with jump rate $\kappa > 0$. Then for all piecewise constant $X : [0,t] \to \mathbb{Z}^d$ with a finite number of discontinuities, we have

$$\mathbb{E}^{\xi}\left[e^{-\gamma\int_0^t \xi(s,X(s))ds}\right] \leq \mathbb{E}^{\xi}\left[e^{-\gamma\int_0^t \xi(s,0)ds}\right] \tag{4.14}$$

Effectively Theorem 4.2.1 tells us that the probability of survival is maximized when $X(t)$ is contained to the origin and does not move. This of course provides an upper bound at all times.

5

Recurrence and Transience in a Decaying Field of Traps

We have gathered results about the behavior of a random walk in the case of moving and non-moving traps and shown that it decays sub-exponentially in $d = 1,2$ and exponentially in $d = 3$. We will now move to a new situation where the traps decay as they get further away from the origin. Meaning that the probability of site x being a trap at $t = 0$ goes to 0 as $||x|| \to \infty$. We will show a few results about the recurrence and transience of random walks in this situation and show different conditions for when which state holds.

We know the results about recurrence and transience in the usual sense, however in this section we will slightly re-define what these terms mean. Now, when we call a random walk recurrent, we will understand it to mean that the probability of survival when $t \to \infty$ is 0. And analogous, a random walk being transient will imply that the probability of survival is *strictly* positive when $t \to \infty$. What we will aim to show are the different conditions for recurrence and transience for the random walk $X(t)$ on $\mathbb{Z}^d$. By p_x we will denote the probability of a trap existing at lattice site x, and as we will consider the decay of said traps we will assume that $p_x \to 0$, as $||x|| \to \infty$, where $|\cdot|$ is the Euclidean norm.

The first thing that is important to note is that when $d < 3$, a symmetric random walk is always recurrent. No matter the decay or existence of traps, the random walk $X(t)$ will reach every $y \in \mathbb{Z}^d$ infinitely many times, given infinite time. Meaning that even if there is only one trap, the random walker will eventually reach it. This means that for non-trivial results we will constrict ourselves to look at the case where $d \geq 3$.

What we are interested in looking at are the different conditions for the random walk to be either recurrent or transient. When there is no decay of traps the random walks are of course always recurrent, as Theorem 3.3.3 shows. But we will find that when certain conditions hold there will be a probability strictly greater than 0 of survival.

5.1. Conditions for the Non-Moving Case

We will start by considering the situation where the traps do not move. We will adapt a proof here from the paper of Den Hollander, Menshikov, and Volkov [1], that uses potential theory to arrive at the following theorem.

Theorem 5.1.1. *A simple random walk with non-moving traps is:*
(a) recurrent, if $p_x \geq \alpha/||x||^2$ for large $||x||$ and some $\alpha > 0$;
(b) transient, if $p_x \leq p(||x||)$, with $r \to p(r)$ non-increasing and

$$\int_0^\infty r p(r) dr < \infty.$$

We will work to prove this theorem, but to do so we need a few other results first. We will omit the proofs of the following lemmas and theorems, and refer you to [1] if one wishes to read them.

Lemma 5.1.1. *For a simple random walk on $\mathbb{Z}^d$, $d \geq 3$, there exists a $c > 0$ such that the Green's function $G(x, y)$ behaves like*

$$G(x, y) \simeq \frac{c}{||x-y||^{d-2}} \tag{5.1}$$

as $||x-y|| \to \infty$.

Now we want to define what it means for a subset of $\mathbb{Z}^d$ to be *massive*. Intuitively this means that the subset is "big enough" for the random walk $X(t)$ to *always* enter it at some point.

Definition 5.1.1. *Let $A \subset \mathbb{Z}^d$, define*

$$\pi_A(x) = \mathbb{P}\left(X(t) \in A, \exists t \in [0,\infty) | X(0) = x\right).$$

We call the set A ***massive*** *if*

$$\pi_A(x) = 1, \text{ for all } x.$$

And call A non-massive otherwise.

We now define the following sets for any arbitrary set $A \in \mathbb{Z}^d$

$$\begin{aligned}
I_n &= [0, \alpha 2^n)^{d-2} \bigcap \mathbb{Z}^{d-2} \\
D_n &= \{x \in \mathbb{Z}^d : 2^n \leq ||x|| \leq 2^{n+1}\} \\
D_n^{'}(z) &= \left\{(z^{'}, z) \in \mathbb{Z}^d : z^{'} \in [2^n, \beta 2^n)^2 \right\} \\
A_n &= D_n \cap A
\end{aligned}$$

Where $\alpha, \beta > 0$, their exact choice is not very important for the proof we want to complete. We now have the following lemma which we will use to prove part (a) of Theorem 5.1.1

Lemma 5.1.2. *If for $A \subset \mathbb{Z}^d$, there exist $\gamma > 0$ and n_0 such that for all $n > n_0$*

$$\left| \{z \in I_n : D_n^{'}(z) \cap A \neq \emptyset \right| \geq \gamma |I_n| \tag{5.2}$$

then A is massive

Now to prove part (b) of Theorem 5.1.1, we need the following lemma

Lemma 5.1.3. *Choose $A \subset \mathbb{Z}^d$. If*

$$\sum_{n=0}^{\infty} \frac{|A_n|}{2^{n(d-2)}} < \infty \tag{5.3}$$

then A is non-massive

We now start the proof of Theorem 5.1.1 for the trap set T.

Proof. **Part (a)**

First, we note that $|D_n'(z)| \geq K2^{2n}$ for some constant $K > 0$, and all $z \in I_n$. Now note that by assumption we have that $p_x \geq \alpha/||x||^2$. Which causes us to find the following inequality

$$\begin{aligned}\mathbb{P}(D_n'(z) \cap T = \emptyset) = 1 - \mathbb{P}(D_n'(z) \cap T \neq \emptyset) &= 1 - \mathbb{P}(x \in T, x \in D_n'(z))\\ &\leq (1 - \frac{\alpha}{2^{2n}})^{K2^{2n}}\end{aligned}$$

for large n. This now implies that a n_1 and β exist such that

$$\mathbb{P}(D_n' \cap T \neq \emptyset) \geq \beta, \text{for } n > n_1. \tag{5.4}$$

Now we fix $n > n_1$, for $z \in I_n$, let

$$\phi(z) = \begin{cases} 1, & \text{if } D_n'(z) \cap T \neq \emptyset \\ 0, & \text{else} \end{cases} \tag{5.5}$$

Of course $\phi(z)$ are independent Bernoulli random variables with parameter $p = \mathbb{P}(D_n'(z) \cap T \neq \emptyset) \geq \beta$. Which means $\mathbb{E}[\phi(z)] \geq \beta$, and by the fact that it is a Bernoulli random variable also $\text{Var}(\phi(z)) \leq 1/4$ holds. Now we define

$$\sigma_n = \sum_{z \in I_n} \phi(z)$$

which is exactly the LHS of lemma 5.1.2 applied to the trap set T. Now we use Chebyshev's inequality to see

$$\begin{aligned}\mathbb{P}(\sigma_n \leq \frac{1}{2}\mathbb{E}[\sigma_n]) &\leq \mathbb{P}\left(|\sigma_n - \mathbb{E}[\sigma_n] \geq \frac{1}{2}\mathbb{E}[\sigma_n]\right)\\ &\leq \frac{\text{Var}(\sigma_n)}{\left(\frac{1}{2}\mathbb{E}[\sigma_n]\right)^2}\\ &= \frac{\sum_{z \in I_n} \text{Var}(\phi(z))}{\left(\frac{1}{2}\sum_{z \in I_n} \mathbb{E}[\phi(z)]\right)^2}\\ &\leq \frac{\sum_{z \in I_n} 1/4}{1/4\left(\sum_{z \in I_n} \mathbb{E}[\phi(z)]\right)^2}\\ &= \frac{\sum_{z \in I_n} 1}{\left(\sum_{z \in I_n} \mathbb{E}[\phi(z)]\right)^2}\\ &\leq \frac{|I_n|}{|I_n|^2\beta^2}\\ &\leq \frac{1}{\beta^2 2^n}\end{aligned}$$

Now we define events $\sigma_n \leq \frac{1}{2}\mathbb{E}[\sigma_n]$, and by the Borell-Cantelli lemma we find that

$$\sum_{n=0}^{\infty} \mathbb{P}(\sigma_n \leq \frac{1}{2}\mathbb{E}[\sigma_n]) \leq \sum_{n=0}^{\infty} \frac{1}{2^n \beta^n} < \infty$$

which implies that $\mathbb{P}(\sigma_n \leq \frac{1}{2}\mathbb{E}[\sigma_n], i.o) = 0$. Which in turn means that there a.s. exists some $n_0 > n_1$ such that $\sigma_n > \frac{1}{2}\mathbb{E}[\sigma_n]$ a.s. for all $n > n_0$. Now recall that σ_n is exactly the LHS of Lemma 5.1.2, and we see that

$$\left|\{z \in I_n : D_n'(z) \cap T \neq \emptyset\right| \geq \frac{1}{2}\beta|I_n|. \tag{5.6}$$

Then by Lemma 5.1.2, we see that the trap set T is massive, and this implies that a simple random walk with non-moving traps is recurrent.

Part (b)

For $x \in \mathbb{Z}^d$ we define

$$\zeta(x) = \begin{cases} 1, & \text{if } x \in T \\ 0, & \text{else.} \end{cases} \tag{5.7}$$

Let $T_n = D_n \cap T$, then clearly $|T_n| = \sum_{x\in D_n} \zeta(x)$. The $\zeta(x)$ are independent with $\mathbb{P}(\zeta(x) = 1) = p_x$. Then recall the definition of the set D_n and see that by assumption that $p_x \le p(||x||) \le p(2^n)$ for $x \in D_n$. Clearly $\zeta(x)$ is a Bernoulli random variable and so

$$\begin{aligned} \mathbb{E}[\zeta(x)] &= p_x \le p(2^n) \\ \mathrm{Var}(\zeta(x)) &= p_x(1-p_x) \le p(2^n). \end{aligned}$$

Then because there exists some constant K such that $|D_n| \le K2^{nd}$ we find

$$\begin{aligned} \mathbb{E}[|T_n|] &= \sum_{x\in D_n} \mathbb{E}[\zeta(x)] \le \sum_{x\in D_n} p(2^n) \le K2^{nd} p(2^n) \\ \mathrm{Var}(|T_n|) &\le K2^{nd} p(2^n) \end{aligned}$$

Now we define the independent random variables

$$\xi_n = \frac{|T_n|}{2^{n(d-2)}}.$$

Now note that if we show that

$$\sum_{n=0}^{\infty} \frac{|T_n|}{2^{n(d-2)}} < \infty,$$

then by Lemma 5.1.3 T a.s. satisfies the condition, and is non-massive, and thus the random walk is transient. So this is what we will prove. Note that

$$\begin{aligned} \mathbb{E}[\xi_n] &\le K2^{nd} p(2^n) 2^{-n(d-2)} \\ \mathrm{Var}(\xi_n) &\le K2^{nd} p(2^n) 2^{-2n(d-2)}. \end{aligned}$$

Now we use Chebyshev's inequality to see that

$$\begin{aligned} \mathbb{P}\left(|\xi_n - \mathbb{E}[\xi_n]| \ge 2^{-n/2}\right) &\le \frac{\mathrm{Var}(\xi_n)}{2^{-n}} \\ &\le K2^{nd} p(2^n) 2^{-2n(d-2)} 2^n \\ &= K2^{nd+n-2n(d-2)} p(2^n) \\ &= K2^{-nd+5n} p(2^n) \\ &= K2^{n(3-d)} 2^{2n} p(2^n) \\ &\le K2^{2n} p(2^n) \qquad \text{(Note: } d \ge 3\text{).} \end{aligned} \tag{5.8}$$

Because $r \to p(r)$ non-increasing and $\int_0^\infty r p(r) dr < \infty$ we have that

$$K \sum_{n=0}^{\infty} 2^{2n} p(2^n) < \infty. \tag{5.9}$$

By the Borel-Cantelli lemma we can now find that

$$\mathbb{P}\left(\xi_n - \mathbb{E}[\xi_n] \geq 2^{-n/2}, i.o\right) = 0. \tag{5.10}$$

This implies that we can find some n_0 such that for all $n > n_0$ we have

$$\xi_n < \mathbb{E}[\xi_n] + 2^{-n/2} \leq K2^{2n} p(2^n) + 2^{-n/2} \tag{5.11}$$

Then clearly

$$\sum_{n=0}^{\infty} \xi_n < \sum_{n=0}^{\infty} K2^{2n} p(2^n) + 2^{-n/2} < \infty. \tag{5.12}$$

As that sum is convergent, we see immediately that by Lemma 5.1.3 the trap set T is non-massive, and so the random walk is transient. Thus the proof is complete. □

5.2. The Case of Moving Traps with Decaying Density

We now move on to the moving traps, where we will assume that just as before they are Poisson distributed with mean ρ. However, as we now discuss the instance where there is decay, we will instead consider $\rho(x)$. Where of course $\rho(x) \to 0$ as $||x|| \to \infty$. We will start by expressing the survival probability in terms of the range again just like before in Theorem 3.3.1, then we will cover multiple choices of $\rho(x)$. The situation is the same as in Chapter 2. The only difference is now the distribution of the traps has mean $\rho(x)$. We also follow the same steps as in Theorem 3.3.1 up to the 6th equality. We continue from there

$$\begin{aligned}
\mathbb{P}(T \geq t) &= \int \left(\prod_y e^{-\rho(y)} e^{\rho(y)(1-\mathbb{P}_Y(Y(s)-X(s)=-y, \exists s \in [0,t)))} \right) d\mathbb{P}_X \\
&= \int \left(\prod_y e^{-\rho(y)\mathbb{P}_Y(Y(s)-X(s)=-y, \exists s \in [0,t))} \right) d\mathbb{P}_X \\
&= \int \left(\prod_y e^{-\rho(y)\mathbb{P}(-y \in R(X-Y,t)} \right) d\mathbb{P}_X \\
&= \mathbb{E}_X \left[\prod_y e^{-\rho(y)\mathbb{P}(-y \in R(X-Y,t)} \right] \\
&= \mathbb{E}_X \left[e^{\sum_y -\rho(y)\mathbb{P}(-y \in R(X-Y,t)} \right] \\
&= \mathbb{E}_X \left[e^{\mathbb{E}_Y \left[\sum_y -\rho(y) \mathbb{1}_{\{-y \in R(X-Y,t)\}} \right]} \right] \\
&= \mathbb{E}_X \left[e^{\mathbb{E}_Y \left[\sum_{y \in R(X-Y,t)} -\rho(-y) \right]} \right].
\end{aligned} \tag{5.13}$$

Now we can find a very natural inequality that arises from this expression. Take note that obviously $R(X-Y, t_1) \geq R(X-Y, t_2) for t_1 > t_2$. As we can of course never "lose" points that we have visited, we can only find more with a non-zero probability. Knowing that we can find $\mathbb{E}_X \left[e^{\mathbb{E}_Y \left[\sum_{y \in R(X-Y,t)} -\rho(-y) \right]} \right] \geq \mathbb{E}_X \left[e^{\mathbb{E}_Y \left[\sum_{y \in R(X-Y,\infty)} -\rho(-y) \right]} \right]$.

Example 4.1. Let us work out an example where $\rho(x) = \mathbb{1}_{x=0}$. Of course we could pick any point for the trap to start, but because of the translation-invariance of the random walks and the difference of a random walk, it effectively does not matter. In this case the expression (5.13) would turn into

$$\begin{aligned}&\mathbb{E}_X\left[e^{\mathbb{E}_Y\left[\sum_{y\in R(X-Y,t)}-\mathbb{1}_{x=0}\right]}\right]\\&=\mathbb{E}_X\left[e^{-\mathbb{P}_Y(0\in R(X-Y,t)}\right]-\frac{1}{e}\end{aligned}$$

Where we have to subtract $1/e$ as it is already counted in the first term of the Poisson distributed traps. Then after this we can use the afore mentioned inequality to see that

$$\mathbb{E}_X\left[e^{-\mathbb{P}_Y(0\in R(X-Y,t)}\right]-\frac{1}{e}\geq\mathbb{E}_X\left[e^{-\mathbb{P}_Y(0\in R(X-Y,\infty)}\right]-\frac{1}{e} \tag{5.14}$$

Within equation (5.14) it is clear that if $d \leq 2$ and we have a recurrent random walk, that the probability of ever reaching zero is 1. And the inequality turns to ≥ 0, and in fact it is exactly 0. If $d \geq 3$ and we have a transient random walk, this probability is < 1, meaning that the actual survival probability is strictly greater than 0. There will always be a positive probability that we can escape the trap.

Of course this example only works for 1 moving trap, so we will now move on to finding a condition for $\rho(x)$ to decay "fast" enough that there will be a positive chance of survival in the transient case. This leads us to find the following result

Theorem 5.2.1. *If there exists an $\alpha > 0$, such that $\rho(x) = \frac{1}{||x||^{2+\alpha}}$, where $\rho(x)$ is the Poisson parameter of the distribution of traps on $\mathbb{Z}^d$, $d \geq 3$, then for a $\delta > 0$ independent of t, we have*

$$\mathbb{P}(T \geq t) > \delta > 0. \tag{5.15}$$

Proof. We start by writing $P(T \geq t)$ in the form of $\mathbb{E}_X\left[e^{\sum_y -\rho(y)\mathbb{P}(-y\in R(X-Y,t))}\right]$. Now we note that by Jensen's inequality it holds that

$$\mathbb{E}_X\left[e^{\sum_y -\rho(y)\mathbb{P}(-y\in R(X-Y,t))}\right]\geq e^{\mathbb{E}_Z\sum_y -\rho(y)\mathbb{P}(-y\in R(Z,\infty))}. \tag{5.16}$$

Then by combining Theorem 2.2.1 and Lemma 5.1.1, we see that the expression (5.16) becomes

$$e^{-\sum_y \rho(y)\frac{c}{||y||^{d-2}}}. \tag{5.17}$$

Now we plug in the assumed form of $\rho(x)$ to find

$$e^{-\sum_y \frac{1}{||y||^{2+\alpha}}\frac{c}{||y||^{d-2}}} \tag{5.18}$$

Note that for this proof we want the expression (5.18) to satisfy > 0. This means that we want the sum in the exponent to be convergent and find the smallest α needed for this. Note now that for this sum to be convergent we can use the integral test for convergence to find if the sum is divergent or not. This means we require

$$\int \frac{r^{d-1}}{r^{d+\alpha}}dr < \infty \tag{5.19}$$

Here the term r^{d-1} comes from the fact that we are effectively integrating over a d-dimensional ball, and transforming into polar coordinates gives the extra term that has to be taken into account. We can now gather the exponents in the denominator and see

$$\int \frac{1}{r^{1+\alpha}}dr, \tag{5.20}$$

which is convergent if $\alpha > 0$. □

Theorem 5.2.1 of course always holds for $\alpha > 0$, but we would like to impose some sort of "triviality" condition as well. One could imagine that if alpha is very large there are effectively no traps or very little to even avoid, meaning a strictly positive survival probability is easily achieved. To find such a condition we note that the probability to survive until time t *and* and for the trap set to not be empty is

$$e^{-\sum_y \rho(y)\frac{G(0,y)}{G(0,0)}} - e^{-\sum_y \rho(y)}. \tag{5.21}$$

Now we would like $\sum_y \rho(y) = \infty$. We find through similar means as the proof of Theorem 5.2.1, that $\int \frac{r^{d-1}}{r^{2+\alpha}}$ leads to the requirement of $1 - d + 2 + \alpha < 1$, which then implies that for a non trivial solution we require $\alpha < d - 2$.

5.3. Conclusions and Future Considerations

In this paper we discussed the different facets of random walks moving among a field of traps. We have shown the different behaviors of the survival time depending on whether we consider a static or moving field of traps, and when there is spacial decay of the traps. There are of course more things we would have liked to consider in this paper that due to time constraints we were unable to investigate. We will name a few interesting problems that in the future may yield interesting results.

Problems for future considerations:

- Some kind of interpolation between the Donsker-Varadhan decay of $t^{d/d+2}$ and the decay of Theorem 3.3.3. An even distribution of moving and non-moving traps would yield trivial results as the former would dominate the decay. However if one was able to find some sort of joint distribution dependent on perhaps a parameter λ that could interpolate between the two, non-trivial results may exist.
- In Section 5 we considered decay in a Poisson distribution of traps. This might be extended to a general distribution where $p_x \to 0$ as $||x|| \to \infty$.
- In general there are many more trapping systems that can be considered in combination with the decay of traps or in combination with any of the other considered elements of the entire process. There are freezing traps (traps that start out moving but after some time stay in place), traps where $\gamma < \infty$, meaning the probability of killing the random walk is < 1, or Sisyphus random walks where after meeting a trap the random walk starts back at the origin again, and many others to consider as well.

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