Random substitutions and fractal percolation

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TR 3840-5

STELLINGEN BIJ HET PROEFSCHRIFT

Random Substitutions and Fractal Percolation

DOOR PETER VAN DER WAL

1. Beschouw een rij van onafhankelijke gelijk verdeelde stochasten $X_1, \ldots, X_n$ met $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 1) = \frac{1}{2}$ en een rij reëele getallen $\xi_1, \ldots, \xi_n$ met $\xi_1^2 + \ldots + \xi_n^2 = 1$. Dan kan men bewijzen dat

$$\mathbb{P}(|\sum_{k=1}^{n} \xi_k X_k| \leq 1) \geq \tfrac{1}{3}.$$

Deze ondergrens is een aanzienlijke verbetering van een resultaat van Ben-Tal, Nemirovski en Roos en is inmiddels opgenomen in [1]. Computer simulaties doen vermoeden dat bovenstaande kans altijd groter of gelijk is aan een half.

[1] A. Ben-Tal, A. Nemirovski, C. Roos, *Robust solutions of uncertain quadratic and conic-quadratic problems*, (2001), work in progress.

2. Een rij $x = (x_k)_k$ is gelijk verdeeld in het eenheids interval, als

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{[a,b]}(x_k)$$

bestaat en gelijk is aan $b - a$ voor alle $0 \leq a \leq b \leq 1$. Laat $H_n(x)$ de hoogte van de binaire zoekboom zijn gegenereerd door de eerste n elementen $x_1, \ldots, x_n$ van een gelijk verdeelde rij $x = (x_k)_k$ in $[0,1]$. Dan geldt dat $H_n(x) = o(n)$ als $n \to \infty$.
Vergelijk dit met het volgende resultaat van Devroye en Goudjil [2] over Weyl rijen $x_\alpha = (x_k) = (\{\alpha k\})_{k=1}^{\infty}$, waar α irrationaal is en $\{\alpha k\}$ staat voor αk modulo 1. Laat (h_n) een monotone rij van reëele getallen zijn die willekeurig langzaam naar 0 convergeren. Dan bestaat er een α met de eigenschap dat $H_n(x_\alpha) \geq nh_n$ voor oneindig veel n.

[2] L. Devroye en A. Goudjil, *A study of random Weyl trees*, Random Structures Algorithms **12(3)** (1998), 271–295.

[3] F.M. Dekking en P. van der Wal, *Uniform distribution modulo* 1 *and binary search trees*, (2000), submitted.

3. Voor alle $c \in (1,2]$ kan men een gelijk verdeelde rij $x(c)$ in het eenheids interval construeren met de eigenschap dat

$$H_n(x(c)) = \lfloor \frac{\log n}{\log c} \rfloor$$

voor alle n groot genoeg.

4. Laat $p_c(M)$ de kritische waarde zijn voor fractale percolatie in dimensie d, waarbij ieder d-dimensionaal blok wordt opgedeeld in M^d sub-blokken. Dan kan men bewijzen dat $p_c(M) \geq p_c(M^2) \geq p_c(M^3) \geq \ldots$ voor alle $M \geq 2$. Vermoedelijk geldt zelfs $p_c(2) \geq p_c(3) \geq p_c(4) \geq \ldots$

5. Beschouw een M-systeem (A, M, σ, u), waarbij A een eindig alfabet is, σ een substitutie met afhankelijkheid tussen de nakomelingen, M de substitutie lengte en u een tweezijdig oneindig startwoord. Neem aan dat σ *N-onafhankelijk* is, dat wil zeggen, als twee sub-woorden verder dan N posities uit elkaar liggen, dan worden ze onafhankelijk gesubstitueerd. Voor preciese definities, zie [4]. Dan kan een vertakkend cellulair automaat $(\bar{A}, M, N, \bar{\sigma}, \bar{u})$ en een homomorfisme $\pi : \bar{A}^* \to A^*$ worden geconstrueerd met de eigenschap dat de rijen $(\sigma^n(u))_n$ en $(\pi(\bar{\sigma}^n(\bar{u})))_n$ dezelfde verdeling hebben. Een constructie de andere kant uit is ook mogelijk.

[4] J. Peyrière, *Processus de naissance avec interaction des voisins, évolution de graphes*, Ann. Inst. Fourier (Grenoble) **31(4)** (1981), 187–218.

6. Zij μ een Bernoulli maat op $\{0,1\}^{\mathbb{N}}$, laat $x = (x_k)_k$ een element van $\{0,1\}^{\mathbb{N}}$ zijn en definieer

$$d_n(x) = \binom{n}{\sum_{k=1}^{n} x_k}.$$

In [5] merken Petersen en Schmidt op dat als λ een eigenwaarde is van de Pascal-adische transformatie, dan geldt dat $\lambda = e^{2\pi i \phi}$, waarbij

$$\phi \in E = \{0 \leq \theta < 1 : e^{2\pi i d_n(x)\theta} \to 1 \text{ voor } \mu \text{ bijna alle } x\}.$$

Het volgende kan bewezen worden: als E meetbaar is, dan $m(E) = 0$, waarbij m de Lebesgue maat is.

Schets van het bewijs: Laat $R = \{r_1, r_2, \ldots\}$ een aftelling zijn van de rationale getallen in $[0,1)$. Met elementaire eigenschappen van de Pascal driehoek kan bewezen worden dat $\{\theta + r\} \notin E$ voor alle $\theta \in E$ en $r \in R$. Definieer $E_k = \{\{\theta + r_k\} : \theta \in E\}$ voor $k = 1, 2, \ldots$ De verzamelingen E_k zijn disjunct en als we veronderstellen dat E meetbaar is, dan is ook E_k meetbaar voor alle k. Aangezien

$$1 = m([0,1)) \geq m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$$

en omdat $m(E_k) = m(E)$ voor alle k, volgt dat $m(E) = 0$.

[5] K. Petersen en K. Schmidt, *Symmetric Gibbs measures*, Trans. Amer. Math. Soc. **349** (1997), 2775–2811.

Random Substitutions and Fractal Percolation

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR
AAN DE TECHNISCHE UNIVERSITEIT DELFT,
OP GEZAG VAN DE RECTOR MAGNIFICUS PROF. DR. IR. J.T. FOKKEMA,
VOORZITTER VAN HET COLLEGE VOOR PROMOTIES,
IN HET OPENBAAR TE VERDEDIGEN OP MAANDAG 25 MAART 2002 OM 16.00 UUR

DOOR PETER VAN DER WAL
DOCTORANDUS IN DE WISKUNDE
GEBOREN TE BENNEKOM.

Dit proefschrift is goedgekeurd door de promotor:
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THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

ISBN 90-6464-064-5

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Chapter 1

Introduction

Although we shall touch on quite complicated phenomena in this thesis, like turbulence in a fluid and the propagation of gene information through generations, we start from a simple mathematical object: that of substitutions on words. A word is a sequence of symbols, for instance, 0's and 1's. An example of a substitution on words is the Fibonacci substitution that replaces 0's by 01 and 1's by 0. If we apply this substitution on the word 0101 we obtain the word 010010. We get a sequence of words by repeatedly applying the substitution on the outcome of the previous substitution. Starting with a 0, we obtain the sequence 0, 01, 010, 01001, 01001010, 0100101001001 and so on. The lengths of these words, i.e., $1, 2, 3, 5, 8, 13, \ldots$, yield the famous Fibonacci numbers.

Substitutions can also be applied to sets in the plane. In Figure 1.1, a sequence of sets $(C_0, C_1, \ldots)$ is obtained by replacing triangles by three smaller triangles. The limit set C is known as the Sierpiński gasket and is an example of a fractal set. The gasket consists of three scaled and shifted copies of itself. To be precise,

$$C = f_1(C) \cup f_2(C) \cup f_3(C),$$

where $f_1(C) = \frac{1}{2}(C + (-1, 0))$, $f_2(C) = \frac{1}{2}(C + (1, 0))$ and $f_3(C) = \frac{1}{2}(C + (0, \sqrt{3}))$. The set of functions $\{f_1, f_2, f_3\}$ is called an iterated function system (IFS) and the Sierpiński gasket is the attractor of the IFS. Figure 1.2 shows the attractor of another IFS, called the Heighway dragon. The boundary of the dragon looks very complex. A measure for the complexity of sets is Hausdorff dimension. The Hausdorff dimensions of the Sierpiński gasket and of the boundary of the Heighway dragon can be calculated and are equal to $\log(3)/\log(2) = 1.5849\ldots$, respectively $2\log\lambda/\log 2 = 1.5236\ldots$, where λ is the largest real zero of $\lambda^3 - \lambda^2 - 2$.

Often, real life phenomena are better described by random substitutions. Consider for example the two trees in Figure 1.3. The left tree is generated by a deterministic substitution and the right one by a random substitution. Although you would not expect to see either one of them ever in a forest, the random tree looks more like a real tree than the deterministic one.

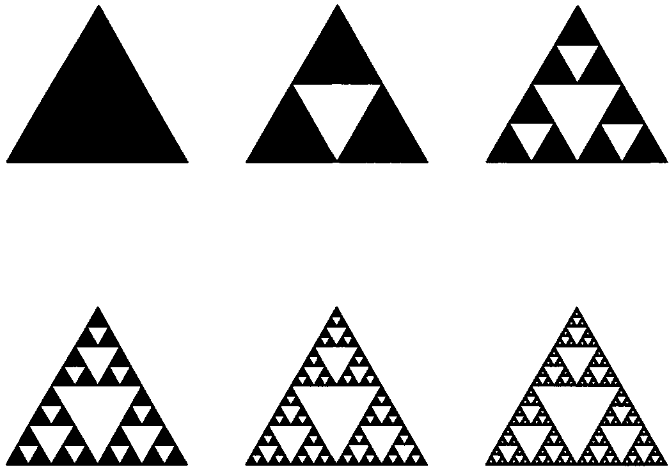


Figure 1.1: First six stages of the Sierpiński gasket.

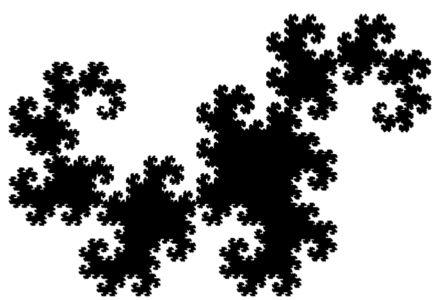


Figure 1.2: The Heighway dragon.

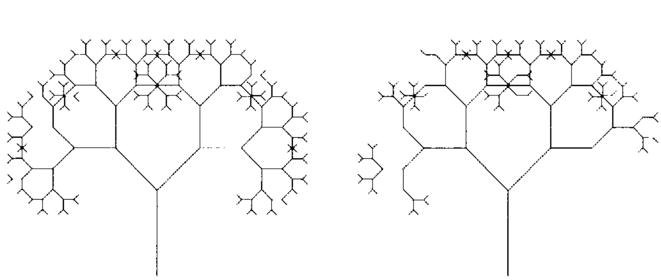


Figure 1.3: A deterministic and a random tree.

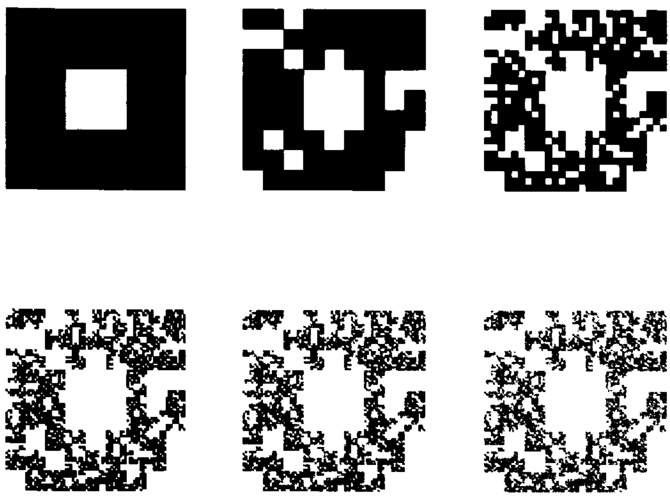


Figure 1.4: A realisation of the sets $K_1, \ldots, K_6$ of fractal percolation for $p = 0.75$.

In the early seventies, Mandelbrot proposed the following random substitution model for the phenomenon of turbulence [17]. Let p be a parameter between 0 and 1 and let K_0 be the unit square and color it black. Divide K_0 into nine sub-squares and for each of the nine sub-squares, color it black with probability p and color it white with probability $1-p$, independently of the eight other sub-squares. Let K_1 be the set consisting of all black sub-squares. Similarly, we obtain the set K_2 from K_1 by dividing all black sub-squares in K_1 into 9 sub-sub-squares, coloring them black with probability p and white with probability $1-p$. Repeating this procedure arbitrarily often, we obtain a sequence $K_0, K_1, \ldots$ of random sets. This model for obtaining random sets is commonly referred to as fractal percolation or Mandelbrot percolation. Figure 1.4 shows a realisation of the sets $K_1, \ldots, K_6$ for $p = 0.75$. Fractal percolation can be generalized in a straightforward way to dimension d where every d-dimensional block is subdivided into M^d sub-blocks. The sequence (K_n) is monotone decreasing and therefore converges to a limit set $K = \bigcap_{n=0}^{\infty} K_n$. With a branching process argument one can easily show that K is the empty set if $pM^d \leq 1$ and that K is non-empty with positive probability if $pM^d > 1$. Whereas K is a random set, its Hausdorff dimension is not so random, in fact, if K is non-empty, then its dimension is almost surely constant and equal to $\log(pM^d)/\log M$. In Chapter 2 and 3, we will explore fractal percolation in greater detail and give more examples of random substitutions and fractal sets.

Fractal percolation is not a very realistic model for turbulence. Siebesma et al. suggested that a model allowing for neighbour interaction would describe turbulence more accurately [27]. An example of such a model is majority fractal

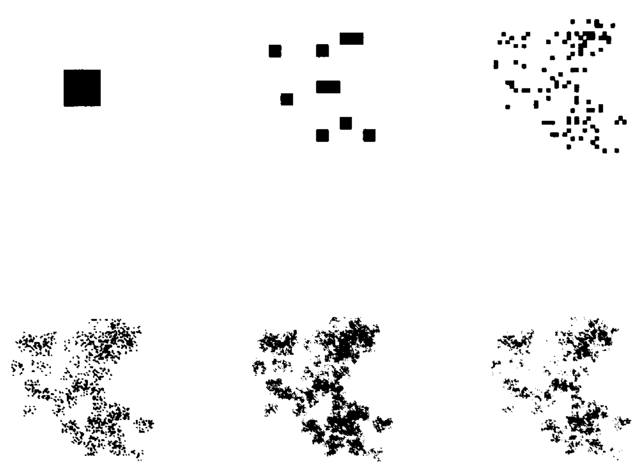


Figure 1.5: A realisation of the sets $K_0, \ldots, K_5$ of majority fractal percolation for $p = 0.15$.

percolation. Divide each level n square I into 9 level $n+1$ squares $J_1, \ldots, J_9$. But now, the probability that a sub-square J_i will be colored black will not only depend on the color of I, but also on the color of the squares surrounding I. To be precise, the probability that J_i is colored black is $1-(1-p)^N$, where N is the number of black squares among I and its 8 neighbours. In Figure 1.5, a realisation of the first six sets of majority fractal percolation is plotted for $p = 0.15$.

Fractal percolation and majority fractal percolation are examples of branching cellular automata (BCA), studied in Chapter 4. We associate sequences of sets $K_0, K_1, \ldots$ to a BCA that are generated by random substitutions with neighbour interaction. In this general framework, the sets $K_0, K_1, \ldots$ do not necessarily converge and we will present sufficient conditions for convergence. The proofs concerning extinction and dimension of the limit set of fractal percolation heavily rely on the fact that there is no neighbour interaction. We develop other techniques to prove that if a sequence $K_0, K_1, \ldots$ associated with a BCA converges to a limit set K, then K is empty if $\lambda < 1$ and is non-empty with positive probability if $\lambda > 1$, where λ is an eigenvalue of a certain offspring matrix. Moreover, if the limit set is non-empty, then the Hausdorff dimension is a constant and equal to $\log \lambda / \log M$. We will show that the (topological) boundary of the limit set of a BCA is again the limit set of a BCA.

Recently, there has been an interest in attractors of iterated function systems and their boundaries for use in image compression and wavelet theory. For a class of nicely behaving IFS's, i.e., those satisfying the strong open set condition, and for specific examples of not so nicely behaved IFS's, the attractors and their boundaries have been extensively studied. In Chapter 5, we study a subclass of the recurrent iterated function systems, a generalization of ordinary IFS's where the attractor is a vector of sets satisfying a system of self-similarity equations. Although this subclass, the M-recurrent iterated function systems (M-RIFS), does not contain all ordinary IFS's, it does contain IFS's that do not satisfy the strong open set condition. We show that M-RIFS and deterministic BCA's are equivalent in the following sense. Starting with the attractor $(C_0, \ldots, C_r)$ of an M-RIFS, we construct a BCA with limit set K such that $C_0 = K$, and, starting with a deterministic BCA with limit set K, we construct an M-RIFS with attractor $(C_0, \ldots, C_r)$ such that $K = C_0$. Using the results from Chapter 4 concerning the dimension and the boundary of the limit set of a BCA, it follows that the boundary of a component of the attractor of an M-RIFS is again a component of the attractor of an M-RIFS and we can calculate its Hausdorff dimension.

In Chapter 6 we will investigate connectivity properties of the limit set K of fractal percolation. In dimension two, we say that the limit set percolates if K contains a connected component that intersects both the left and the right side of the unit square. Let $\theta_M(p)$ denote the probability that K percolates for parameter p and subdivision into M^2 sub-squares. Then it is obvious that $\theta_M(0) = 0$ and $\theta_M(1) = 1$, but what about the values of p in between 0 and 1? Using a coupling method, it can be shown that $\theta_M(p)$ is increasing in p. Define the critical value $p_c(M)$ by

$$p_c(M) = \inf\{p : \theta_M(p) > 0\}.$$

The exact value of $p_c(M)$ is up to now unknown, but several bounds have been given. The first ones to establish non-triviality of the critical value were Chayes, Chayes and Durrett [4], who proved that $p_c(M) \geq 1/\sqrt{M}$ for $M \geq 2$ and $p_c(M) \leq p^*(M)$ for $M \geq 3$, where $p^*(M)$ is the infimum over p for which $x = (px)^{M^2} + (px)^{M^2-1}(1-px)$ has a root in the half open interval $(0, 1]$. Dekking and Meester [5] reinterpreted the Chayes, Chayes and Durrett proof for the upper bound in terms of multi-valued substitutions and improved the upper bound to $p_c(3) \leq 0.991$. Recently, White [30] established $p_c(2) \geq 0.810$ by a sophisticated lattice construction. In Chapter 6, we will generalize the methods of Chayes, Chayes and Durrett, Dekking and Meester and White to a method for finding both upper and lower bounds for the critical value. We prove that $p_c(3) \leq 0.965$ and describe techniques to obtain even sharper bounds. In addition, we explain how our method can be applied in the context of the passing of genes in a family tree.

Chapter 2

A Variety of Substitutions

2.1 Substitutions on Words

Let A be a finite set and let A^* denote the free semi-group generated by A, denoting the identity element by ε. We adopt the terminology of word combinatorics, so we call A an alphabet, we refer to its elements as letters, to A^* as the set of finite words, to the group operation as concatenation and to the identity element ε as the empty word. Let A^k denote the set of words of length k, i.e., obtained by concatenating k letters from the alphabet A. By default, we will index the letters of a word $w \in A^k$ by the set $\{0, \dots, k-1\}$, so $w = w_0 \dots w_{k-1}$.

Definition 2.1 *A substitution σ is a homomorphism on A^*. If for all $a \in A$ the words $\sigma(a)$ have length M, then the substitution is said to be of constant length and M is called the substitution length. It is a non-erasing substitution if $\sigma(a) \neq \varepsilon$ for all $a \in A$.*

Since σ is a homomorphism, it is completely determined by its image on the letters. By σ^n we denote the n-fold iterate of σ. By convention, σ^0 is the identity map.

Example 2.1 (Cantor Substitution) Consider the substitution σ on $\{0,1\}^*$ that substitutes a 0 by 000 and a 1 by 101. Then σ is of constant length 3 and is called the Cantor substitution.

Let $A^{\mathbb{Z}}$ denote the set of all bi-infinite words, i.e., sequences $\dots u_{-1}u_0u_1 \dots$ of letters in A. There is a straightforward way to extend a non-erasing substitution σ to a map on $A^{\mathbb{Z}}$. For finite words $u = u_0 \dots u_l$ and $0 \leq k \leq l$ and for bi-infinite words $u = \dots u_{-1}u_0u_1 \dots \in A^{\mathbb{Z}}$ and $k \in \mathbb{Z}$ define

$$L_u(k) = \begin{cases} \text{length of } \sigma(u_0 \dots u_{k-1}) & k \geq 1 \\ 0 & k = 0 \\ -\text{length of } \sigma(u_k \dots u_{-1}) & k \leq -1 \end{cases}$$

Define the image $\sigma(u)$ of a bi-infinite word u by

$$(\sigma(u))_{L_u(k)} \dots (\sigma(u))_{L_u(k+1)-1} = \sigma(u_k).$$

If the substitution is of constant length M, then $L_u(k) = kM$ for $k \in \mathbb{Z}$.

2.1.1 Offspring

For a substitution σ and a word u we define the letter $(\sigma(u))_l$ to be a first generation descendant, also called a child, of the letter u_k if $(\sigma(u))_l$ is a letter of the word that has been substituted for u_k, i.e., if $L_u(k) \le l \le L_u(k+1) - 1$. We recursively define $(\sigma^n(u))_l$ to be an n^{th} generation descendant of the letter u_k if $(\sigma^n(u))_l$ is a child of an $(n-1)^{\text{th}}$ generation descendant of u_k. The set of all n^{th} generation descendants of a letter is called its n^{th} generation offspring. If σ is a substitution of constant length M, then the n^{th} generation offspring of u_k is easily traced to be the set of letters

$$\{(\sigma^n(u))_{kM^n}, \dots, (\sigma^n(u))_{(k+1)M^n-1}\}.$$

2.2 Random substitutions

Let $(\sigma_k)_{k\in\mathbb{N}}$ be a sequence of independent identically distributed random maps from A to A^*. Define a random map σ on A^* by $\sigma(\varepsilon) = \varepsilon$ and for $u = u_0 \dots u_k \in A^*$ define

$$\sigma(u) = \sigma_0(u_0) \dots \sigma_k(u_k).$$

The random map σ on A^* is called a random substitution. Analogous to the deterministic case, a non-erasing random substitution can be extended to a random map on $A^{\mathbb{Z}}$. We define the n-fold iterate σ^n to be the composition of n independent copies of the substitution σ.

Example 2.2 (Fractal Percolation) Let $A = \{0, 1\}$, let p be a parameter between 0 and 1 and choose a substitution length $M \ge 2$. Consider a random substitution σ with $\sigma(0) = 0^M$ and such that $(\sigma(1))_0, \dots, (\sigma(1))_{M-1}$ are independent Bernoulli(p) random variables. This substitution is commonly referred to as fractal percolation.

2.3 Substitutions with Neighbour Dependence

Let N be a non-negative integer, referred to as the interaction length, and let σ be a map from A^{2N+1} to A^*. We will extend σ to a map on A^* by defining for $u = u_0 \dots u_k \in A^*$

$$\sigma(u) = \begin{cases} \sigma(u_0 \dots u_{2N}) \dots \sigma(u_{k-2N} \dots u_k) & \text{if } k \ge 2N \\ \varepsilon & \text{if } k \le 2N - 1 \end{cases}$$

This extended map is called a substitution with neighbour dependence. In fact, σ is the projection of a substitution on $(A^{2N+1})^*$. To see this, define a map $\zeta : A^* \to (A^{2N+1})^*$ by

$$\zeta(u_0 \dots u_k) = \begin{cases} (u_0 \dots u_{2N}) \dots (u_{k-2N} \dots u_k) & \text{if } k \geq 2N \\ \varepsilon & \text{if } k \leq 2N-1 \end{cases}$$

and define a substitution τ on $(A^{2N+1})^*$ by

$$\tau(u_0 \dots u_{2N}) = \zeta(\sigma(u_0 \dots u_{2N})).$$

Then $\sigma = \zeta^{-1}\tau\zeta$. Similarly to the case without neighbour dependence, σ can be extended to a substitution on $A^{\mathbb{Z}}$ if it is non-erasing.

2.3.1 Types

Consider a substitution σ with substitution length $M \geq 2$ and interaction length N and let v be a word in A^{2k+1}. It is easily shown by induction that the length of $\sigma^n(v)$ is at least $2k + M^n$ for all n, if and only if k is at least $\lceil \frac{MN}{M-1} \rceil$. A set $T = A^{2R+1}$ is said to be a set of types for σ if $R \geq \lceil \frac{MN}{M-1} \rceil$. For $k \geq 2R$ and $v = v_0 \dots v_k$ we define the type of a letter v_i with $R \leq i \leq k-R$ to be the word $v_{i-R} \dots v_{i+R}$ in T. Note that the type of a letter v_i completely determines the types of all descendants of v_i. If $M \geq N+1$, then $T = A^{2N+3}$ is a set of types for σ, since $\lceil \frac{MN}{M-1} \rceil = \lceil N + \frac{N}{M-1} \rceil \leq \lceil N + \frac{M-1}{M-1} \rceil = N+1$.

To emphasize that types are neighbourhoods of letters, we will always index types $t \in T = A^{2R+1}$ by the set $\{-R, \dots, R\}$ instead of $\{0, \dots, 2R\}$, so $t = t_{-R} \dots t_R$. Likewise, the indices of $\sigma^n(t)$ will be such that the leftmost descendant of the letter t_0 has index 0. Using this index convention, the n^{th} generation offspring of t_0 consists of the letters

$$(\sigma^n(t))_0, \dots, (\sigma^n(t))_{M^n-1}.$$

2.4 Random Substitutions with Neighbour Dependence

Let $(\sigma_k)_{k \in \mathbb{N}}$ be a sequence of independent and identically distributed random maps from A^{2N+1} to A^*. Define a random map on A^* by defining for $u = u_0 \dots u_k \in A^*$

$$\sigma(u) = \begin{cases} \sigma_N(u_0 \dots u_{2N}) \dots \sigma_{k-N}(u_{k-2N} \dots u_k) & \text{if } k \geq 2N \\ \varepsilon & \text{if } k \leq 2N-1 \end{cases}$$

The random map σ on A^* is called a random substitution with neighbour dependence. Analogous to the case without neighbour dependence, a non-erasing random substitution with neighbour dependence can be extended to a random map on $A^{\mathbb{Z}}$. We define the n-fold iterate σ^n to be the composition of n independent copies of the substitution σ.

Consider a random substitution with substitution length $M \geq 2$ and interaction length N and let $T = A^{2R+1}$ be a set of types for σ, i.e., $R \geq \lceil \frac{MN}{M-1} \rceil$. Then the type of a letter completely determines the (joint) distribution of the types of its offspring.

Example 2.3 (Majority Fractal Percolation) There are many ways to introduce neighbour dependence in fractal percolation (Example 2.2) and we will describe one of them. Choose a parameter $0 \leq p \leq 1$ and let σ be a random substitution on the alphabet $A = \{0, 1\}$ with substitution length M and interaction length N. We will assume that both 0's and 1's are substituted by M independent Bernoulli random variables. For words $u = u_0 \ldots u_{2N}$ and $0 \leq k \leq M - 1$ define $\mathbb{P}((\sigma(u))_k = 1) = 1 - (1-p)^{n(u)}$, where $n(u)$ is the number of 1's in u. This defines the neighbour dependent substitution σ to which we will refer as majority fractal percolation.

2.5 Multi-valued Substitutions

Let $\mathcal{A}^*$ be the set of all finite subsets of A^* and consider two binary operations on $\mathcal{A}^*$:

$$\begin{aligned} V \cup W &= \{u : u \in V \text{ or } u \in W\} \qquad \text{(union)} \\ VW &= \{vw : v \in V \text{ and } w \in W\} \qquad \text{(concatenation)}. \end{aligned}$$

A *multi-valued* substitution is a homomorphism on $\mathcal{A}^*$ respecting unions and concatenations. Since $\mathcal{A}^*$ is generated by the singletons, i.e., the sets containing one letter, a multi-valued substitution Φ is completely determined by the images $(\Phi(i))_{i \in A}$ of the singletons.

2.6 M-systems

As a general framework for a number of constructions by Mandelbrot, Peyrière introduced the notion of M-systems (see [22], [23], [24] and [25]). An M-system is a substitution that allows offspring interaction. Whereas Peyrière constructed a very general setup for M-systems acting on graphs, we will restrict ourselves to defining M-systems on words.

For all $u = u_0 \ldots u_k \in A^*$, let $\sigma_0(u), \ldots, \sigma_k(u)$ be a sequence of random words and define a random map on A^* by $\sigma(\varepsilon) = \varepsilon$ and $\sigma(u) = \sigma_0(u) \ldots \sigma_k(u)$. The random map σ is called an M-system if for all $u = u_0 \ldots u_k$ and $0 \leq i \leq j \leq k$, the words $\sigma(u_i \ldots u_j)$ and $\sigma_i(u) \ldots \sigma_j(u)$ are identically distributed.

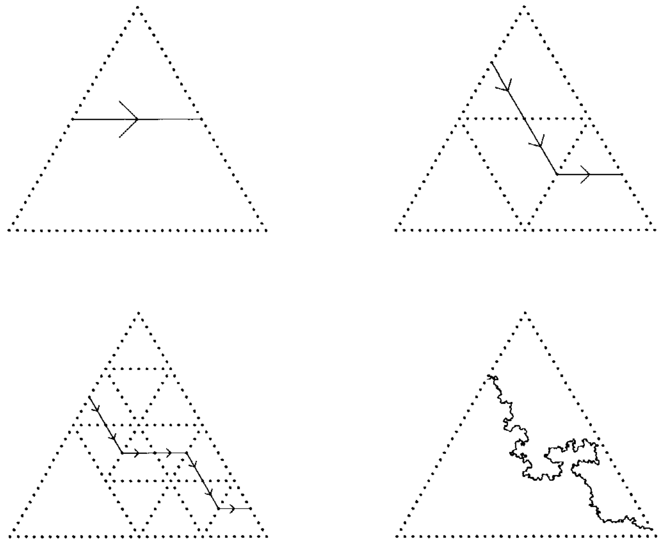


Figure 2.1: Four stages of a realisation of the stochastic river described in Example 2.4.

Example 2.4 (Stochastic River) A typical construction that can be described by an M-system is the stochastic river. Start with a level 0 triangle T_0, label one edge to be the entrance and another to be the the exit edge. After subdivision of T_0 into four similar level 1 triangles, we arbitrarily choose one of the two level 1 edges that lie on the entrance edge to be an entrance and we do the same for the exit edge. There is a natural river flowing from the level 1 entrance to the level 1 exit, crossing one or three triangles. Edges of a triangle that are crossed by the river to enter a triangle are labelled entrance and edges crossed to exit are labelled exit. The process is now repeated on the level 1 triangles that are crossed by the river. We have plotted the first three and the eighth stage of a realisation of the stochastic river in Figure 2.1.

Depending on the orientation of a triangle and the direction in which it is crossed, we will label segments of the river by a or b as is indicated in Figure 2.2. A river is represented by the word formed by the consecutive labels. For example, the paths of the first three stages of the river in Figure 2.1 are represented by the words a, bba and $bbaabba$. In the construction described above, letters are substituted by one or three letter words. Note that substitutions of neighbouring letters are dependent. The corresponding M-system σ can be described as follows. Let $A = \{a, b\}$ be the alphabet, let $\Phi(a) = \{a, bba, aaa, abb\}$ be the set of words that can be substituted for an a and $\Phi(b) = \{b, aab, bbb, baa\}$ the words that

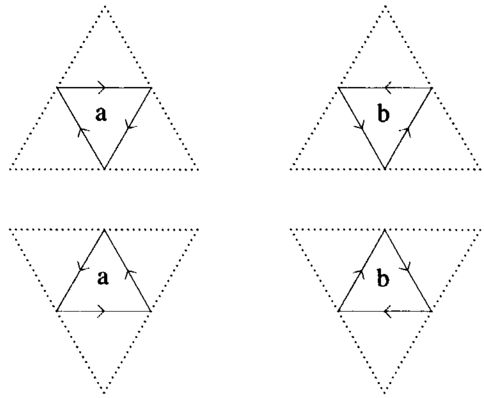


Figure 2.2: Labeling of segments of the stochastic river described in Example 2.4.

can be substituted for a b. For letters $u_0, \ldots, u_k \in A$ and words $w_0, \ldots, w_k$ with $w_i \in \Phi(u_i)$ we denote $\mathbb{P}(\sigma_0(u_0 \ldots u_k) = w_0, \ldots, \sigma_k(u_0 \ldots u_k) = w_k)$ by $p_{u_0 \ldots u_k}(w_0, \ldots, w_k)$. Define

$$\begin{aligned} p_{u_0}(w_0) &= \tfrac{1}{4} \\ p_{u_0 u_1}(w_0, w_1) &= \begin{cases} \frac{1}{8} & \text{if } w_0 \text{ is } a \text{ or } **b \text{ and } w_1 \text{ is } b \text{ or } a** \\ \frac{1}{8} & \text{if } w_0 \text{ is } b \text{ or } **a \text{ and } w_1 \text{ is } a \text{ or } b** \\ 0 & \text{else,} \end{cases} \end{aligned}$$

where a $*$ can be replaced by any letter from the alphabet and define

$$p_{u_0 \ldots u_k}(w_0, \ldots, w_k) = 4^{k-1} \prod_{i=0}^{k-1} p_{u_i u_{i+1}}(w_i, w_{i+1}).$$

One can be check that for $u = u_0 \ldots u_k$ the sequences of random words $\sigma_0(u), \ldots, \sigma_k(u)$ determine an M-system σ that describes the stochastic river.

The stochastic river can also be described by a neighbour dependent random substitution σ' with interaction length 1. Since this type of substitution is not really designed to deal with offspring interaction, its construction is rather elaborate and involves an 11 letter alphabet A'. We will therefore not give the distribution of σ' explicitly, but stick to a loose description.

By straightforward calculations and using that letters at least one position apart are substituted independently, we can also write the distribution $p_{u_0 \ldots u_k}(w_0, \ldots, w_k)$ for k even as

$$\prod_{i \text{ even}} p_{u_i}(w_i) \prod_{i \text{ odd}} \mathbb{P}(\sigma_i(u) = w_i | \sigma_{i-1}(u) = w_{i-1}, \sigma_{i+1}(u) = w_{i+1}).$$

This implies that we can break up the substitution in two stages: first we substitute the letters at the even positions independently and then we substitute each letter at an odd position, conditioned on what has been substituted for its two neighbours. On an even position $2i$ we will therefore place a pair consisting of a letter u_{2i} from $\{a, b\}$ and a word uniformly chosen from $\Phi(u_{2i})$ which will be substituted for u_{2i} in the next stage of the substitution. At the odd positions we just place the letters u_{2i+1}. To avoid substitution problems at the begin and end of a word, we place infinitely many 0's in front and behind the word and consider the substitution on bi-infinite words. The 11 letter alphabet A' will hence be $\{a, b, 0\} \cup \{a\} \times \Phi(a) \cup \{b\} \times \Phi(b)$. The substitution σ' replaces each letter by a 1 or 3 letter word. A letter at an even position is replaced by a 1 or 3 letter word from which the first and the last letter are pairs and a letter at an odd position is replaced by a word from which the first and last letter are single letters. Hence, at any stage of the substitution, the letters at even positions are pairs and at odd positions are single letters. A pair is substituted independently of its neighbours and a single letter from $\{a, b\}$ is substituted in accordance with its two neighbours. A 0 is of course always substituted by a 0. If for example the first 4 stages of the stochastic river are

$$\begin{aligned}&a\\&bba\\&b\,baa\,bba\\&bbb\,aab\,abb\,abb\,b\,baa\,a\end{aligned}$$

then the first 3 stages of the corresponding river in the BCA coding are

$$\begin{aligned}&(a, bba)\\&(b, b)\,b\,(a, bba)\\&(b, bbb)\,b\,(a, abb)\,a\,(b, b)\,b\,(a, a),\end{aligned}$$

where we omitted the 0's. If these substitutions are executed with the right probabilities, then the resulting substitution σ' is a neighbour dependent substitution with interaction length 1.

2.7 Iterated Function Systems

A lot of fractal sets can be conveniently described by means of iterated function systems (see e.g. [9]). An iterated function system (IFS) is a set $\{f_1, \ldots, f_m\}$ of maps on for example $\mathbb{R}^d$. It is often assumed that the maps $f_1, \ldots, f_m$ are contractions on $\mathbb{R}^d$ with respect to the Euclidean distance δ, i.e., all f_i are maps on $\mathbb{R}^d$ and there are constants $c_i < 1$ such that $\delta(f_i(x), f_i(y)) < c_i\delta(x, y)$ for all $x, y \in \mathbb{R}^d$. In this case it can be shown that there is a unique set C^* in H_0,

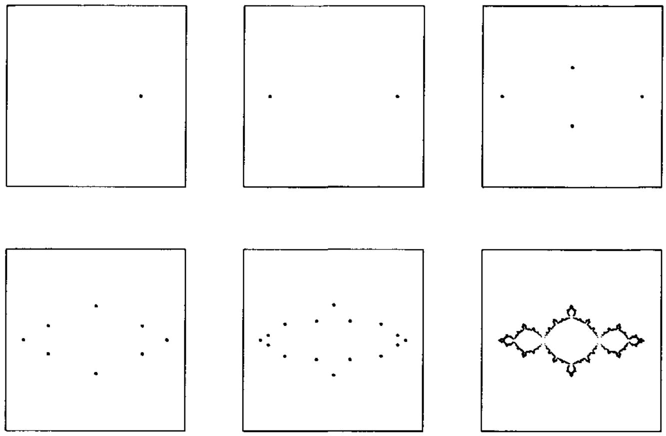


Figure 2.3: Six stages of the iterated function system described in Example 2.5.

the set of all non-empty compact sets in $\mathbb{R}^d$, such that $C^* = \bigcup_{i=1}^m f_i(C^*)$, where $f_i(C^*)$ denotes the set $\{f_i(x) : x \in C^*\}$. This set C^* is called the attractor or the invariant set of the IFS. Because of the implicit definition, the set C^* is often not easy tractable. However, for any non-empty compact set C, the attractor C^* can be approximated by sets $f^n(C)$, where $f(C) = \bigcup_{i=1}^m f_i(C)$ and f^n denotes the n-fold iterate of f. As a consequence of the contraction mapping theorem (see for example [9]), the sets $f^n(C)$ converge to C^* in Hausdorff metric for all starting sets $C \in H_0$. The function f can be viewed as a substitution, not on words, but on non-empty compact sets C in $\mathbb{R}^d$, since it replaces each point $x \in C$ by the set $\{f_1(x), \dots, f_m(x)\}$.

Example 2.5 (Julia Set) Consider an iterated function system $\{f_1, f_2\}$, where $f_1(z) = \sqrt{z+1}$ and $f_2(z) = -\sqrt{z+1}$ are maps on $\mathbb{C}$. Note that these maps are not contractions on $\mathbb{C}$. However, the sets $f^n(\{1\})$ do converge to an invariant set J which is called the Julia set for the map $z \to z^2 - 1$. In Figure 2.3 we plotted $f^n(\{1\})$ for $n = 0, 1, 2, 3, 4$ and 12.

2.8 Substitutions on Higher Dimensional Words

In dimension two and higher, the definition of a substitution gets a little bit more tricky. We will focus on the 2-dimensional case and leave the d-dimensional case to the reader. Two dimensional words are blocks of letters from an alphabet A.

Define $A^{k,l}$ to be the set of all words

$$\begin{array}{ccc} a_{0l} & \cdots & a_{kl} \\ \vdots & & \vdots \\ a_{00} & \cdots & a_{k0} \end{array}$$

with $a_{ij} \in A$ and define $A^* = \bigcup_{k,l \geq 0} A^{k,l}$, where $A^{k,l} = \{\varepsilon\}$ if k or l is 0 and ε denotes the empty word. We define horizontal concatenation $\circ_H : \bigcup_{k,l,m} A^{k,l} \times A^{m,l} \to A^*$ by

$$\begin{array}{ccc} a_{0l} & \cdots & a_{kl} \\ \vdots & & \vdots \\ a_{00} & \cdots & a_{k0} \end{array} \circ_H \begin{array}{ccc} b_{0l} & \cdots & b_{ml} \\ \vdots & & \vdots \\ b_{00} & \cdots & a_{m0} \end{array} = \begin{array}{cccccc} a_{0l} & \cdots & a_{kl} & b_{0l} & \cdots & b_{ml} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{00} & \cdots & a_{k0} & b_{00} & \cdots & b_{m0} \end{array}$$

and vertical concatenation $\circ_V : \bigcup_{k,l,m} A^{k,l} \times A^{k,m} \to A^*$ by

$$\begin{array}{ccc} a_{0l} & \cdots & a_{kl} \\ \vdots & & \vdots \\ a_{00} & \cdots & a_{k0} \end{array} \circ_V \begin{array}{ccc} b_{0m} & \cdots & b_{km} \\ \vdots & & \vdots \\ b_{00} & \cdots & b_{k0} \end{array} = \begin{array}{ccc} b_{0m} & \cdots & b_{km} \\ \vdots & & \vdots \\ b_{00} & \cdots & b_{k0} \\ a_{0l} & \cdots & a_{kl} \\ \vdots & & \vdots \\ a_{00} & \cdots & a_{k0} \end{array}$$

Instead of $v \circ_H w$ we will also write vw and instead of $v \circ_V w$ we will write $\begin{array}{c} w \\ v \end{array}$. Unfortunately, horizontal and vertical concatenation are not binary operations on A^*, so A^* does not have a semi-group structure as in the 1-dimensional case. However, every word in A^* can be obtained by horizontally and vertically concatenating letters from A. For example,

$$\begin{array}{ccc} a_{01} & a_{11} & a_{21} \\ a_{00} & a_{10} & a_{20} \end{array} = (a_{00} \circ_H a_{10} \circ_H a_{20}) \circ_V (a_{01} \circ_H a_{11} \circ_H a_{21})$$
$$= (a_{00} \circ_V a_{01}) \circ_H (a_{10} \circ_V a_{11}) \circ_H (a_{20} \circ_V a_{21}).$$

We define a 2-dimensional substitution to be a map $\sigma : A^* \to A^*$ that respects horizontal and vertical concatenations, i.e., $\sigma(vw) = \sigma(v)\sigma(w)$ and $\sigma(\begin{array}{c} w \\ v \end{array}) = \begin{array}{c} \sigma(w) \\ \sigma(v) \end{array}$. A 2-dimensional substitution is completely defined by the images $\sigma(a)$ of the letters $a \in A$, since A^* is generated by the letters $a \in A$. Note that the definition of a 2-dimensional substitution implies that the blocks $\sigma(a)$ all have the same size.

Chapter 3

From Substitutions to Fractals

3.1 Sets in $\mathbb{R}^d$ Associated with Substitutions

In this section, we associate sets in $\mathbb{R}^d$ to substitutions on d-dimensional words. For reasons of notational convenience, we restrict ourselves to 1-dimensional substitutions and leave it to the reader to generalize the definitions to higher dimensions. The sets and substitutions in the examples and figures will be 2-dimensional.

Consider a substitution σ with substitution length $M \geq 2$ and a word $u \in A^k$. For every letter $a \in A$ we define a sequence of sets $K_0(a), K_1(a), \ldots$ as follows. Let

$$J_n(a) = \{i : (\sigma^n(u))_i = a\}$$

and define

$$K_n(a) = \bigcup_{j \in J_n(a)} I_n(j),$$

where $I_n(j) = [jM^{-n}, (j+1)M^{-n}]$ is the j^{th} level n M-adic interval. Observe that the sets $K_n(a)$ are compact and contained in the interval $[0, k]$.

If we consider the extended substitution on a bi-infinite word $u \in A^{\mathbb{Z}}$, the construction above does not necessarily produce compact sets anymore. In order to obtain compact uniformly bounded sets, we assume that the alphabet contains a special symbol 0 such that $\sigma(0) = 0^M$, where 0^M denotes the M-letter word consisting of only 0's. Assume also that u is a starting word, i.e., u contains only finitely many non-zero letters. If u_l is the left most non-zero letter and u_r the right most, then all sets $K_n(a)$ with $a \neq 0$ will be contained in $[l, r]$.

Example 3.1 (Sierpiński carpet) Let σ be the 2-dimensional analogue of the

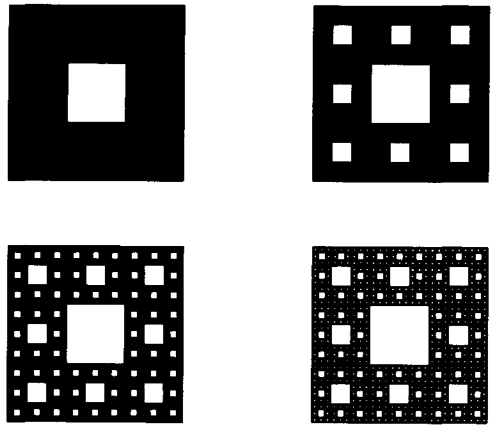


Figure 3.1: Sets $K_1, \dots, K_4$ of the 2 dimensional Sierpiński carpet (see Example 3.1).

Cantor substitution (Example 2.1) given by

$$0 \to \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad 1 \to \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}$$

For notational convenience, we will often denote the associated sets $K_n(1)$ by K_n. Starting the substitution with the letter 1, we plotted the associated sets $K_1, \dots, K_4$ in Figure 3.1. The sets K_n are decreasing and we call the limit set $K = \bigcap_{n=0}^{\infty} K_n$ the 2 dimensional Sierpiński carpet.

Example 3.2 (Fractal Percolation) Let σ be the 2 dimensional analogue of fractal percolation with parameter p and substitution length M described in Example 2.2, i.e., a 0 is substituted by an $M \times M$ block of 0's and a 1 is substituted by an $M \times M$ block of independent Bernoulli(p) random variables. Starting the substitution with the letter 1, we plotted the sets $K_1, \dots, K_5$ and K_8 of a realisation of fractal percolation with $M = 2$ and $p = 0.75$ in Figure 3.2, where K_n denotes $K_n(1)$. Since a 0 is substituted by only 0's, the sets K_n are decreasing in n.

For substitutions with neighbour dependence we will generalize the construction. Let σ be a substitution with substitution length M and interaction length N. Assume that $0 \in A$ and that $\sigma(0^{2N+1}) = 0^M$ and let $u \in A^{\mathbb{Z}}$ be a starting word. Let T be a set of types for σ, and let $\bar{0}$ denote the type consisting of only

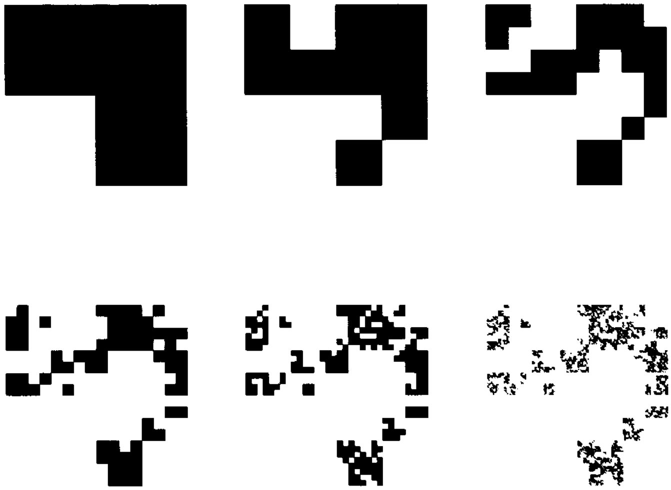


Figure 3.2: Sets $K_1, \ldots, K_5$ and K_8 of a realisation of fractal percolation with $M = 2$ and $p = 0.75$ (see Example 3.2).

zeros. Define for sets $S \subseteq T$

$$J_n(S) = \{i : \text{the type of } (\sigma^n(u))_i \text{ is an element of } S\}$$

and define

$$K_n(S) = \bigcup_{j \in J_n(S)} I_n(j),$$

where $I_n(j)$ is the j^{th} level n M-adic interval. Note that if $\bar{0} \notin S$, then the sets $K_n(S)$ are compact and uniformly bounded. For $a \in A$, we will denote by $K_n(a)$ the set $K_n(S_a)$, where $S_a = \{t \in T : t_0 = a\}$ is the set of types for which the middle letter is an a.

Example 3.3 (Ink Model) The following model is a very simplistic way to describe the spreading of a black ink drop on a white piece of paper. Let $A = \{0, 1\}$ where a 1 represents a square filled with ink and consider the following 2 dimensional neighbour dependent substitution σ on $A^{\mathbb{Z}\times\mathbb{Z}}$ with substitution length 2 and interaction length 1. Let $u \in A^{\mathbb{Z}\times\mathbb{Z}}$ be the starting word of the substitution consisting of a 1 at position $(0,0)$ and 0's elsewhere. For a letter v_{kl} in a word $v \in A^{\mathbb{Z}\times\mathbb{Z}}$ we call its north neighbour $v_{k,l+1}$ and its east neighbour $v_{k+1,l}$ the uncles of its upper right child $\sigma(v)_{2k+1,2l+1}$. In the same spirit we define two uncles for each of the other three children. Now the substitution is such that a

Figure 3.3: Sets $K_1, \ldots, K_6$ of a realisation of the ink model of Example 3.3 with $p = 0.5$.

child of a letter 1 is always 1 and a child of a 0 is 1 independently of its brothers with probability $1 - (1-p)^m$, where m is the number of its uncles that are 1. In Figure 3.3 we plotted the sets $K_1, \ldots, K_6$ of a realisation of the ink model for $p = 0.5$, where K_n denotes $K_n(1)$. In this model the sets K_n are increasing and each K_n is a connected set.

For the ink model, the set $T = A^{5\times 5}$ is a set of types. Recall that we index a type $t \in T$ symmetrically, so

$$t = \begin{matrix} t_{-2,2} & \cdots & t_{2,2} \\ \vdots & & \vdots \\ t_{-2,-2} & \cdots & t_{2,-2} \end{matrix}$$

Let $S \subset T$ be the set of types for which at least 1 and at most 8 of the middle 9 letters are 1, so

$$S = \{t \in T : 1 \leq \sum_{-1 \leq i,j \leq 1} t_{ij} \leq 8\}.$$

In Figure 3.4 we plotted the sets $K_1(S), \ldots, K_6(S)$ of the same realisation of the ink model as we used for Figure 3.3. From these two figures, one gets the impression that the sets K_n converge to a limit set K, that the sets $K_n(S)$ converge to a limit set $K(S)$ and that $K(S)$ is the boundary of K. In section 4.5 this impression is proved to be right.

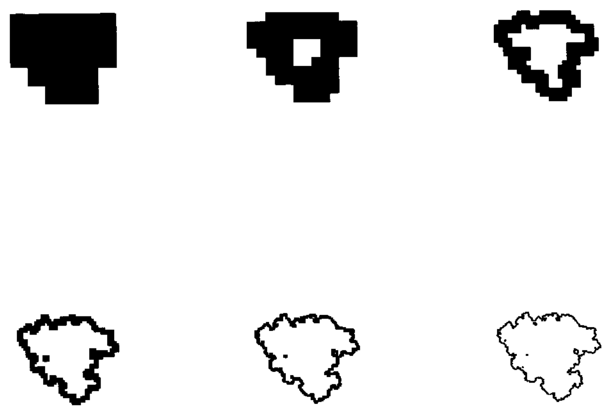


Figure 3.4: Sets $K_1(S), \ldots, K_6(S)$ of the same realisation as in Figure 3.3 of the ink model (Example 3.3), where $S \subset T = A^{5\times 5}$ is the set of all types for which at least 1 and at most 8 of the middle 9 letters are 1.

Example 3.4 (Majority Fractal Percolation) Let σ be the 2 dimensional analogue of neighbour dependent fractal percolation with parameter p, substitution length M and interaction length N described in Example 2.3. As a starting word for the substitution, we take the word in $\{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$ consisting of a 1 at position $(0,0)$ and 0's elsewhere. Again, we abbreviate the associated sets $K_n(1)$ by K_n. In Figure 3.5 we plotted the sets $K_0, \ldots, K_5$ of a realisation of majority fractal percolation with $M = 2$, $N = 1$ and $p = 0.15$. Note that the sets are not decreasing anymore, since a 0 is not necessarily substituted by only 0's.

3.2 Convergence

The Hausdorff metric m_H is a metric on the set of non-empty compact sets in $\mathbb{R}$ defined by

$$m_H(K_0, K_1) = \inf\{\varepsilon > 0 : K_0 \subset K_1^\varepsilon, K_1 \subset K_0^\varepsilon\},$$

where K_0^ε denotes the set of points that have (Euclidean) distance less than ε to a point in K_0.

On the set of compact sets that are contained in the interval $[l, r]$ with $l < r$

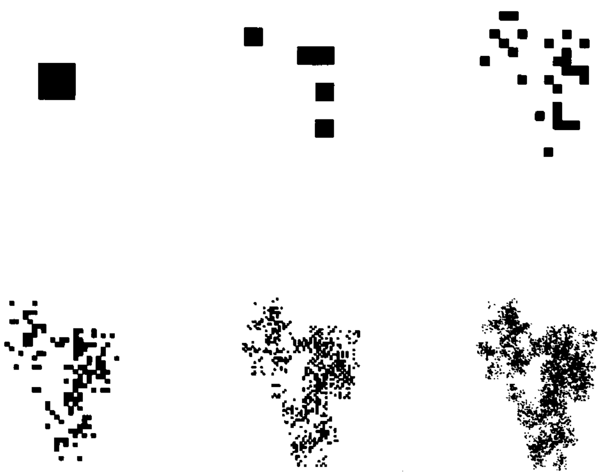


Figure 3.5: Sets $K_0, \ldots, K_4$ and K_8 of a realisation of majority fractal percolation, where $M = 2$, $N = 1$ and $p = 0.15$ (see Example 3.4).

(including the empty set) we consider a similar metric m, defined by

$$m(K_0, K_1) = \begin{cases} m_H(K_0, K_1) & \text{if } K_0, K_1 \neq \emptyset \\ 0 & \text{if } K_0 = K_1 = \emptyset \\ r - l & \text{else.} \end{cases}$$

To this metric we will also refer as the Hausdorff metric and we will also denote it by m_H.

For a fixed substitution and starting word, sequences of associated sets will always be uniformly bounded. If we make statements about their convergence, this will always be with respect to Hausdorff metric.

The following lemma implies convergence of the sets $K_0, K_1, \ldots$ in the Examples 3.1, 3.2 and 3.3. The proof is straightforward and left to the reader.

Lemma 3.1 *Let $K_0, K_1, \ldots$ be a decreasing sequence or a bounded increasing sequence of compact sets in $\mathbb{R}^d$. Then $K_0, K_1, \ldots$ converges with respect to Hausdorff metric.*

We order C_0, the set of compact sets that are contained in the interval $[l, r]$, by the standard ordering $\subset$. The supremum of a set $X \subset C_0$ is the smallest element (with respect to $\subset$) in C_0 that is larger than all elements of X and the infimum is the largest element that is smaller than all elements of X. The $\liminf_{\mathrm{H}}$ and

$\limsup_{\mathrm{H}}$ of a sequence $K_0, K_1, \ldots$ are defined by

$$\begin{aligned} \liminf{}_{\mathrm{H}} K_n &= \lim_{n\to\infty} \inf(K_n, K_{n+1}, \ldots) \\ &= \{x : B_\varepsilon(x) \cap K_n \neq \emptyset \text{ eventually for all } \varepsilon > 0\} \end{aligned}$$

and

$$\begin{aligned} \limsup{}_{\mathrm{H}} K_n &= \lim_{n\to\infty} \sup(K_n, K_{n+1}, \ldots) \\ &= \{x : B_\varepsilon(x) \cap K_n \neq \emptyset \text{ infinitely often for all } \varepsilon > 0\}, \end{aligned}$$

where $B_\varepsilon(x)$ denotes the ε-ball around x.

Lemma 3.2 *Let $(K_n)_n$ be a bounded sequence of compact sets in $\mathbb{R}^d$. Then $\liminf_{\mathrm{H}} K_n$ and $\limsup_{\mathrm{H}} K_n$ are compact. The sequence $(K_n)_n$ converges in Hausdorff metric, if and only if $\liminf_{\mathrm{H}} K_n = \limsup_{\mathrm{H}} K_n$.*

Proof The sets $\liminf_{\mathrm{H}} K_n$ and $\limsup_{\mathrm{H}} K_n$ are bounded, since the sets K_n are uniformly bounded. To prove that for example $\liminf_{\mathrm{H}} K_n$ is compact, it suffices to show that if a sequence $(x_n)_n$ with $x_n \in \liminf_{\mathrm{H}} K_n$ converges, then its limit x is also an element of $\liminf_{\mathrm{H}} K_n$. To see this, fix $\varepsilon > 0$. Then there is an n_0 such that $x_n \in B_{\varepsilon/2}(x)$ for all $n \geq n_0$. Since $x_n \in \liminf_{\mathrm{H}} K_n$, it follows that $B_{\varepsilon/2}(x) \cap K_n \neq \emptyset$ eventually and hence $B_\varepsilon(x) \cap K_n \neq \emptyset$ eventually. Hence $x \in \liminf_{\mathrm{H}} K_n$.

For the left to right implication in the second part of the lemma, assume that $\liminf_{\mathrm{H}} K_n \neq \limsup_{\mathrm{H}} K_n$. Then there is an $x \in \limsup_{\mathrm{H}} K_n$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \cap K_n = \emptyset$ infinitely often. Since $x \in \limsup_{\mathrm{H}} K_n$, we can find sequences $m_0 < m_1 < \ldots$ and $n_0 < n_1 < \ldots$ such that $B_\varepsilon(x) \cap K_{m_k} = \emptyset$ and $B_{\varepsilon/2}(x) \cap K_{n_k} \neq \emptyset$. Since $m_H(K_{m_k}, K_{n_k}) > \varepsilon/2$ for all k, the sequence (K_n) does not converge in Hausdorff metric.

For the right to left implication, assume that (K_n) does not converge. Then there are $\varepsilon > 0$ and sequences $m_0 < m_1 < \ldots$ and $n_0 < n_1 < \ldots$ such that $K_{m_k} \not\subseteq K^\varepsilon_{n_k}$. This implies that there is a sequence (x_k) such that $x_n \in K_{m_k}$ and $B_\varepsilon(x_k) \cap K_{n_k} = \emptyset$. Since the sets K_n are uniformly bounded, there is an x and a subsequence (x_{l_k}) of (x_k) such that $x_{l_k} \in B_{\varepsilon/2}(x)$ for all k. Hence $B_{\varepsilon/2}(x) \cap K_{l_k} \neq \emptyset$ and $B_{\varepsilon/2}(x) \cap K_{n_k} = \emptyset$ for all k, and therefore $x \in \limsup_{\mathrm{H}} K_n$ and $x \notin \liminf_{\mathrm{H}} K_n$. □

Let $l < r$ be integers and let $J = (J_n)_{n\geq 0}$ be a sequence of sets with $J_n \subseteq \{lM^n, \ldots, rM^n - 1\}$ and let the sequence $(K_n)_{n\geq 0}$ be defined by $K_n = \bigcup_{j\in J_n} I_n(j)$. For $m \geq 0$ and $k \in \mathbb{Z}$ define

$$Z_n(m,k) = |\{j \in J_{m+n} : kM^n \leq j \leq (k+1)M^n - 1\}|,$$

where $|\cdot|$ denotes cardinality.

Lemma 3.3 *One has*

$$\limsup_{\mathrm{H}} K_n = \bigcap_{m=0}^{\infty} \bigcup_{\{k: Z_n(m,k)>0 \ i.o.\}} I_m(k)$$

and

$$\liminf_{\mathrm{H}} K_n \supseteq \bigcap_{m=0}^{\infty} \bigcup_{\{k: Z_n(m,k)>0 \ eventually\}} I_m(k).$$

Proof Let $x \in \limsup_{\mathrm{H}} K_n$. Then $B_\varepsilon(x) \cap K_n \neq \emptyset$ infinitely often for all $\varepsilon > 0$. This implies that we can find a sequence $k_0, k_1, \ldots$ such that:

1. $I_0(k_0) \supseteq I_1(k_1) \supseteq \ldots$
2. $x \in I_m(k_m)$ for all m
3. $Z_n(m, k_m) > 0$ for infinitely many n.

Since $\bigcap_{m=0}^{\infty} I_m(k_m) = x$, we have that $x \in \bigcap_{m=0}^{\infty} \bigcup_{\{k: Z_n(m,k)>0 \text{ i.o.}\}} I_m(k)$. Let $x \in \bigcap_{m=0}^{\infty} \bigcup_{\{k: Z_n(m,k)>0 \text{ i.o.}\}} I_m(k)$, fix $\varepsilon > 0$ and let m be such that $M^{-m} < \varepsilon$. Then there is a k such that $I_m(k) \subseteq B_\varepsilon(x)$ and $Z_n(m,k) > 0$ for infinitely many m. Hence $B_\varepsilon(x) \cap K_n \neq \emptyset$ for infinitely many n and $x \in \limsup_{\mathrm{H}} K_n$. The proof that $\liminf_{\mathrm{H}} K_n \supseteq \bigcap_{m=0}^{\infty} \bigcup_{\{k: Z_n(m,k)>0 \text{ eventually}\}} I_m(k)$ is similar. □

Example 3.5 (Converging Sets) Let $0 \leq p \leq 1$ be a parameter and consider the 2 dimensional substitution σ with substitution length 2 and interaction length 1 on the alphabet $\{-1, 0, 1\}$. For $w \in A^{3\times 3}$ define $(\sigma(w))_{kl}$ to be $-w_{11}$ if $|\sum w_{ij}| < 9$ and else to be an independent random variable which is w_{11} with probability p and $-w_{11}$ with probability $1-p$. From Theorem 4.2 it follows that for $p > 0$ and for all bi-infinite starting words u, the sequences $(K_n(-1))_n$ and $(K_n(1))_n$ converge. In Figure 3.6 and 3.7 we plotted realisations of the sets $K_2(1), \ldots, K_7(1)$ for $p = 0.9$ and $p = 0.1$, started with the word consisting of a 1 at position $(0,0)$, surrounded by 8 -1's and 0's elsewhere.

Example 3.6 (Non-converging Sets) Let $A = \{-1, 0, 1\}$ and let σ be a 1-dimensional neighbour dependent substitution given by $\sigma(a_1, a, a_2) = b_1 b_2$, where

$$b_i = \begin{cases} 0 & \text{if } a_i = 0 \\ -a & \text{else.} \end{cases}$$

If the starting word u is equal to $\ldots 000\ldots$, $\ldots 0(-1)10\ldots$ or $\ldots 01(-1)0\ldots$, then the sequences $(K_n(-1))_n$ and $(K_n(1))_n$ converge. For all other starting words, the sequences do not converge.

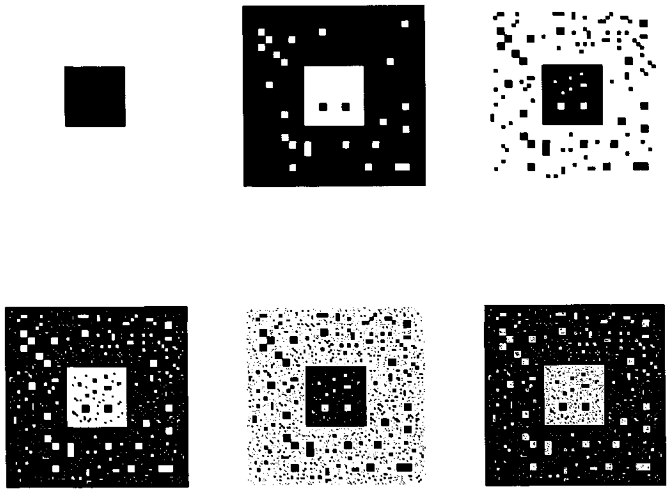


Figure 3.6: Sets $K_2, \ldots, K_7$ of a realisation of the substitution described in Example 3.5 for $p = 0.1$.

3.3 Hausdorff Dimension

Let $K \subseteq \mathbb{R}^d$, $d \geq 0$ and let $(U_i)_{i\geq 0}$ be a sequence of sets $U_i \subseteq \mathbb{R}^d$. For $\delta > 0$, we say that $(U_i)_{i\geq 0}$ is a δ-cover of K if $K \subseteq \bigcup_{i=0}^{\infty} U_i$ and $|U_i| < \delta$ for all i, where $|U_i|$ denotes the diameter of the set U_i. Let

$$\mathcal{H}_\delta^\alpha(K) = \inf(\sum_{i=1}^{\infty} |U_i|^\alpha : (U_i)_{i\geq 0} \text{ is a } \delta\text{-cover of } K)$$

and define the α-dimensional Hausdorff measure of K by

$$\mathcal{H}^\alpha(K) = \lim_{\delta\to 0} \mathcal{H}_\delta^\alpha(K).$$

The graph of $\mathcal{H}^\alpha(K)$ as a function of α does not look very interesting. There is a point α^* such that $\mathcal{H}^\alpha(K) = \infty$ if $\alpha < \alpha^*$ and $\mathcal{H}^\alpha(K) = 0$ if $\alpha > \alpha^*$. The discontinuity point α^* is called the Hausdorff dimension of K and is denoted $\dim_{\mathrm{H}}(K)$. To show that this definition of dimension corresponds to the intuitive idea of dimension, we list some properties of the Hausdorff dimension. For $K \subseteq \mathbb{R}^d$ the Hausdorff dimension satisfies:

- $0 \leq \dim_{\mathrm{H}}(K) \leq d$
- if K is countable, then $\dim_{\mathrm{H}}(K) = 0$

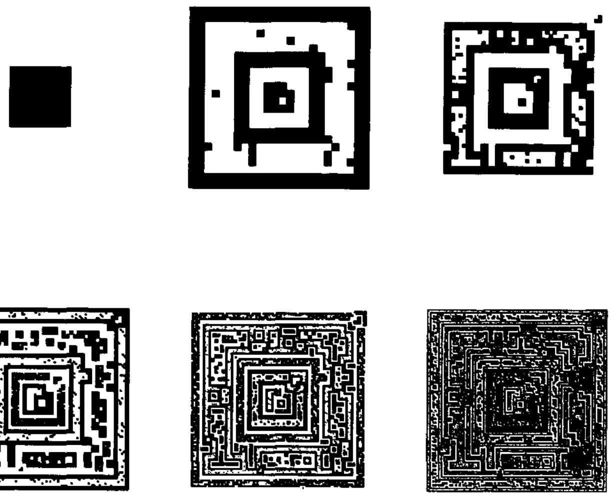


Figure 3.7: Sets $K_2, \ldots, K_7$ of a realisation of the substitution described in Example 3.5 for $p = 0.9$.

- if K is open, then $\dim_{\mathrm{H}}(K) = d$
- if $K_0, K_1, \ldots \subseteq \mathbb{R}^d$, then $\dim_{\mathrm{H}}(\bigcup_{n=0}^{\infty} K_n) = \sup_n \dim_{\mathrm{H}}(K_n)$
- if K is a smooth k-dimensional manifold, then $\dim_{\mathrm{H}}(K) = k$
- if $f : \mathbb{R}^d \to \mathbb{R}^d$ is bi-Lipschitz, then $\dim_{\mathrm{H}}(f(K)) = \dim_{\mathrm{H}}(K)$. A function f is bi-Lipschitz, if there are $c_1, c_2 > 0$ such that $c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y|$ for all $x, y \in \mathbb{R}^d$.

3.3.1 Fractal Percolation

Let σ be 1-dimensional fractal percolation (Example 2.2) with parameter p and substitution length M and let $K_0, K_1, \ldots$ denote its associated sets as in Example 3.2. From Lemma 3.1 it follows that the sets converge in Hausdorff metric to $K = \bigcap_{i=0}^{\infty} K_i$ almost surely. In this section we will prove that if the limit set K is non-empty, then its Hausdorff dimension is constant almost surely and equal to $\log(pM)/\log M$. We will subsequently proof that $\dim_{\mathrm{H}} K$ is constant on $K \neq \emptyset$, that $\dim_{\mathrm{H}} K \leq \log(pM)/\log M$ and that $\mathbb{P}_p(\dim_{\mathrm{H}}(K) \geq \log(pM)/\log M) > 0$ for $p > \frac{1}{M}$. Since $\mathbb{P}_p(K \neq \emptyset) = 0$ for $p \leq \frac{1}{M}$, these three ingredients establish the claim.

Lemma 3.4 *On* $\{K \neq \emptyset\}$, $\dim_{\mathrm{H}} K$ *is constant* $\mathbb{P}_p$*-almost surely.*

Proof Let Z_n denote the number of 1's in $\sigma^n(1)$. By self similarity of K, we have that $\mathbb{P}_p(\mathcal{H}^\alpha(K) = 0) = G_{Z_1}(\mathbb{P}_p(\mathcal{H}^\alpha(K) = 0))$, where $G_{Z_1}(s) = \mathbb{E}_p(s^{Z_1})$ is the probability generating function of Z_1. It is a classical result from branching theory that the only roots of G_{Z_1} in the interval $[0,1]$ are 1 and $\mathbb{P}_p(Z_n = 0 \text{ eventually})$. Since $\{K = \emptyset\} = \{Z_n = 0 \text{ eventually}\}$ is contained in $\{\mathcal{H}^\alpha(K) = 0\}$, we have either $\mathcal{H}^\alpha(K) = 0$ almost surely or $\{\mathcal{H}^\alpha(K) = 0\} = \{K = \emptyset\}$ almost surely. So either $\mathcal{H}^\alpha(K) = 0$ on $\{K \neq \emptyset\}$ or $\mathcal{H}^\alpha(K) > 0$ on $\{K \neq \emptyset\}$. Since $\mathcal{H}^\alpha(K)$ is decreasing in α, there is a constant α^* such that on $\{K \neq \emptyset\}$, $\mathcal{H}^\alpha(K) > 0$ if $\alpha < \alpha^*$ and $\mathcal{H}^\alpha(K) = 0$ if $\alpha > \alpha^*$. By definition, $\alpha^* = \dim_{\mathrm{H}} K$ on $\{K \neq \emptyset\}$ and so $\dim_{\mathrm{H}} K$ is a constant. □

Lemma 3.5 *We have that* $\dim_{\mathrm{H}} K \leq \log(pM)/\log M$, $\mathbb{P}_p$*-almost surely.*

Proof Since the n^{th} level M-adic squares contained in K_n cover the limit set K, it follows that $\mathcal{H}^\alpha(K) \leq M^{-n\alpha} Z_n$, where Z_n is the number of 1's in $\sigma^n(1)$. For $\varepsilon > 0$ we have

$$\begin{aligned} \mathbb{P}_p(\mathcal{H}^\alpha(K) > \varepsilon) &\leq \mathbb{P}_p(M^{-n\alpha} Z_n > \varepsilon) \\ &\leq \tfrac{1}{\varepsilon} M^{-n\alpha} \mathbb{E}_p(Z_n) \\ &= \tfrac{1}{\varepsilon} M^{-n\alpha} (pM)^n, \end{aligned}$$

where we used the Markov inequality to obtain the second inequality. Since this last expression tends to 0 as n tends to infinity for $\alpha > \log(pM)/\log M$, it follows that $\dim_{\mathrm{H}} K \leq \log(pM)/\log M$. □

For the proof of the last lemma ,we need the following lemma due to Russell Lyons ([16], p. 933), which gives a lower bound for the Hausdorff dimension of a non-random set B in $[0,1]$.

Lemma 3.6 *Let* B *be a subset of* $[0,1]$. *If* $\mathbb{P}_p(K \cap B \neq \emptyset) > 0$, *then* $\dim_{\mathrm{H}} B \geq -\log p/\log M$.

Proof Let $\mathcal{K}^\alpha(B)$ be the M-adic α dimensional Hausdorff measure of B, i.e., the analogue of $\mathcal{H}^\alpha(B)$ obtained by only considering covers of B that consist of M-adic intervals. It can be shown that $\dim_{\mathrm{H}} B = \inf\{\alpha : \mathcal{K}^\alpha(B) = 0\} = \sup\{\alpha : \mathcal{K}^\alpha(B) > 0\}$. Let $(I_i)_i$ be a cover of B consisting of M-adic squares. Then

$$\begin{aligned} \mathbb{P}_p(K \cap B \neq \emptyset) &\leq \mathbb{P}_p(K \cap \bigcup_{i=0}^{\infty} I_i \neq \emptyset) \\ &\leq \sum_{i=0}^{\infty} \mathbb{P}_p(K \cap I_i \neq \emptyset) \\ &\leq \sum_{i=0}^{\infty} |I_i|^{\frac{-\log p}{\log M}}, \end{aligned}$$

since for any n^{th} level M-adic interval I,

$$\mathbb{P}_p(K \cap I \neq \emptyset) \leq \mathbb{P}_p(I \subseteq K_n) = p^n = |I|^{\frac{-\log p}{\log M}}.$$

If we take $\alpha = -\log p/\log M$, then $\mathcal{K}^\alpha(B) \geq \mathbb{P}_p(K \cap B \neq \emptyset)$ and if $\mathbb{P}_p(K \cap B \neq \emptyset) > 0$, this implies that $\dim_{\mathrm{H}} B \geq \alpha$. □

Lemma 3.7 *Let $p > \frac{1}{M}$. Then $\mathbb{P}_p(\dim_{\mathrm{H}}(K) \geq \log(pM)/\log M) > 0$.*

Proof Denote the underlying probability space of fractal percolation by $(\Omega, \mathcal{F}, \mathbb{P}_p)$. By Lemma 3.6 we have for $\tilde{p} > 0$

$$\begin{aligned}
&\mathbb{P}_p\big(\dim_{\mathrm{H}}(K) \geq -\frac{\log \tilde{p}}{\log M}\big) \\
&= \mathbb{P}_p\big(\omega : \mathbb{P}_{\tilde{p}}(\tilde{\omega} : K(\omega) \cap K(\tilde{\omega}) \neq \emptyset) > 0\big) \\
&\geq \int_\Omega \mathbb{P}_{\tilde{p}}(\tilde{\omega} : K(\omega) \cap K(\tilde{\omega}) \neq \emptyset) \, \mathrm{d}\,\mathbb{P}_p(\omega) \\
&= \int_{\Omega \times \Omega} \mathbf{1}_{\{K(\omega) \cap K(\tilde{\omega}) \neq \emptyset\}}(\omega, \tilde{\omega}) \, \mathrm{d}\,\mathbb{P}_p \times \mathbb{P}_{\tilde{p}} \\
&= \mathbb{P}_p \times \mathbb{P}_{\tilde{p}}\big((\omega, \tilde{\omega}) : K(\omega) \cap K(\tilde{\omega}) \neq \emptyset\big),
\end{aligned}$$

where we used Fubini's theorem for the third step. Using induction it is not hard to prove that $\mathbb{P}_p \times \mathbb{P}_{\tilde{p}}(K_n(\omega) \cap K_n(\tilde{\omega}) \neq \emptyset) = \mathbb{P}_{p\tilde{p}}(K_n \neq \emptyset)$ for all n, and taking limits we obtain

$$\mathbb{P}_p\big(\dim_{\mathrm{H}}(K) \geq -\frac{\log \tilde{p}}{\log M}\big) \geq \mathbb{P}_{p\tilde{p}}(K_n \neq \emptyset).$$

Since $\mathbb{P}_{p\tilde{p}}(K_n \neq \emptyset) > 0$ whenever $p\tilde{p} > \frac{1}{M}$, the lemma follows if we let $\tilde{p} \downarrow \frac{1}{pM}$. □

3.4 Connectivity

For $x, y \in \mathbb{R}^d$ we write $x < y$ if $x_i < y_i$ for $1 \leq i \leq d$. For $x < y$ we define the d-dimensional rectangle $R(x, y)$ by $R(x, y) = \{z : x \leq z \leq y\}$. We say that a set $K \subseteq \mathbb{R}^d$ percolates through $R(x, y)$ if there is a connected component in $K \cap R(x, y)$ that intersects the plane $\{z : z_1 = x_1\}$ and the plane $\{z : z_1 = y_1\}$. If K percolates through the unit cube, we simply say that K percolates.

Let $A = \{0, 1\}$ and let $u \in A^{k \times \dots \times k}$ be a d-dimensional word. We say that u percolates if there is a sequence of d-dimensional indices $x^1, \dots, x^m \in \{0, \dots, k-1\}^d$ with $x_1^1 = 0$, $x_1^m = k - 1$ and $\sum_{j=1}^d |x_j^i - x_j^{i+1}| = 1$ for $1 \leq i \leq m - 1$ such that $u_{x^i} = 1$ for all $1 \leq i \leq m$.

Consider two dimensional fractal percolation with substitution length M and parameter p (see Example 3.2), let σ be the associated substitution and $K_0, K_1, \dots$ the associated sets. Define the percolation function $\theta_M(p)$ to be

$$\theta_M(p) = \mathbb{P}_p(K \text{ percolates}),$$

where $K = \bigcap_{n=0}^{\infty} K_n$ is the limit set of the sequence $(K_n)_n$. It can be shown that $\theta_M(p) = \mathbb{P}_p(\sigma^n(1)$ percolates for all $n)$. The critical value $p_c(M)$ is defined to be $\inf\{p : \theta_M(p) > 0\}$. From a coupling argument it follows that $\theta_M(p)$ is increasing in p and therefore $p_c(M) = \sup\{\theta_M(p) = 0\}$. There has been a lot of research on the behavior of the function θ_M and the value of $p_c(M)$ and we will discuss some of the results (see for example [10] and [5]).

The function $\theta_M(p)$ is right continuous and at the critical value it is discontinuous. The critical value is non-trivial, i.e., it is not equal to 0 or 1. The exact value of $p_c(M)$ is yet unknown for all $M \geq 2$, but many authors have given bounds for it.

A quite trivial lower bound can be obtained by comparing fractal percolation with a branching process. Let Z_n be the number of 1's in $\sigma^n(1)$. Then it can be checked that Z_n is an ordinary branching process with $\mathbb{E}(Z_1) = pM^2$. The process dies out if $p \leq \frac{1}{M^2}$ and therefore $p_c(M) \geq \frac{1}{M^2}$.

The first authors to give an upper bound were Chayes, Chayes and Durrett [4]. We will briefly review this upper bound, where we restrict ourselves to the case $M = 3$. The idea behind the CCD proof that $p_c(3) \leq 0.993$ is basically the following. Associate to each realisation of $(\sigma^n(1))_{n=0}^{\infty}$ a tree, where the letters in $\sigma^n(1)$ that are equal to 1 are the level n nodes in the tree and $\sigma^0(1)$ is the root. A letter $(\sigma^{n+1}(1))_{ij}$ is a child of $(\sigma^n(1))_{kl}$ in the tree if $(\sigma^{n+1}(1))_{ij}$ is one of the letters in the 3×3 block that has been substituted for the letter $(\sigma^n(1))_{kl}$. The key observation is that if the associated tree contains a full 8-ary tree rooted at $\sigma^0(1)$, then the words $\sigma^n(1)$ percolate for all n. Denote the probability that this event occurs by $\pi(p)$ and let $\pi_n(p)$ denote the probability that $\sigma^0(1)$ is the root of a of an 8-ary tree of depth at least n. One can easily show that the $\pi_n(p)$ converge to $\pi(p)$. Observe that $\sigma^0(1)$ is the root of an 8-ary tree of depth at least $n+1$ if and only if at least 8 of its children are the root of an 8-ary tree of depth at least n. This implies that the $\pi_n(p)$ satisfy the recursion $\pi_{n+1}(p) = (p\pi_n(p))^9 + 9(p\pi_n(p))^8(1 - p\pi_n(p))$ and hence $\pi(p)$ is the largest root of $x = (px)^9 + 9(px)^8(1-px)$ in the interval $[0,1]$. Using standard analysis one can show that for $p \geq 0.993$ this root is larger than 0 and therefore $\theta_3(0.993) \geq \pi(0.993) > 0$ and thus $p_c(3) \leq 0.993$.

The CCD proof can be easily generalized to obtain upper bounds on $p_c(M)$ for $M \geq 3$, but it breaks down for $M = 2$. However, by a coupling argument one can show that $p_c(2) \leq 1 - \left(1 - \sqrt{p_c(4)}\right)^4$ which is strictly smaller than one.

Chapter 4

Branching Cellular Automata

4.1 Introduction

In this chapter we study generalisations of a family of random sets introduced by Benoit Mandelbrot in [17]. Mandelbrot coined the name canonical curdling for these sets, but they are commonly known as fractal percolation. Let p be a number with $0 \leq p \leq 1$ and $[0,1]^d$ be the unit cube in $\mathbb{R}^d$. We furthermore choose an integer base $M \geq 2$. Random sets $K_0 = [0,1]^d, K_1, \ldots, K_n$ are generated by a recursive construction. The set K_0 is a union of M^d subcubes with side lengths M^{-1}. Generate K_1 by retaining each of these subcubes with probability p, or discarding it with probability $1-p$, independently of each other. In general K_n is a union of M-adic cubes of order n, i.e., with side lengths M^{-n}, and K_{n+1} is obtained by retaining or discarding each of the order $n+1$ M-adic subcubes of these cubes with probability p respectively $1-p$, independently of each other, and of all the previous choices. The limit set $K = \cap_{n=0}^{\infty} K_n$ is a fractal set with a.s. Hausdorff dimension $\log(pM^d)/\log M$, conditioned on being non-empty (see Section 3.3.1, but also [4], [13], and [11]).

Mandelbrot introduced fractal percolation as an alternative model for turbulence in a fluid in a critique of Kolmogorov's model. However fractal percolation is not more than a metaphor for turbulence. In the paper [27] the authors argue that physically there is dependence on the activity in neighbouring regions in turbulence, and that therefore the independent evolution of the M-adic cubes would be an important deficiency of Mandelbrot's model. They then propose neighbour interaction to obtain a model which admittedly is still phenomenological. Moreover, they merely study one specific example in the one dimensional case $d = 1$. In this chapter we develop a general theory of fractal percolation with neighbour interaction. To assess the success of such a model in the goal of modeling turbulence from a phenomenological point of view, we invite the reader to compare ordinary fractal percolation in Figure 4.1 with an example involving neighbour interaction in Figure 4.2. We mention that the interest in fractal percolation goes beyond an attempt to model turbulence. Recently, Yuval Peres has revealed a surprising relationship between fractal percolation (for specific values of p) and the path of Brownian motion [21].

The fractal percolation process can conveniently be defined on the space of M-adic trees, but allowing interaction with the neighbours destroys the tree property. We have chosen to construct these random sets by way of the iteration of random substitutions. We shall call the corresponding process a branching cellular automaton. (See [24] for another approach.) This will be done in Section 4.2, where we furthermore indicate the importance of multi-type branching processes (with dependent offspring) for the analysis of branching cellular automata (BCA's). In Section 4.3 we consider the question of extinction of these multi-type branching processes. Since the sets (K_n) will not necessarily be decreasing anymore in our general model, the question of convergence (in the Hausdorff metric) of the (K_n) arises. This problem is considered in Section 4.4, where sufficient conditions are given for (K_n) to converge. In Section 4.8 we determine the almost sure Hausdorff dimension of the limit set K, using Lyons' percolation method ([16]). In order to do this we need the notion of a product of two BCA's which is introduced in Section 4.7. For many BCA's the set K has a non-empty interior, and therefore K is *not* a fractal set (see e.g. the example analyzed in [3]). However, K will often have a fractal boundary. To determine the Hausdorff dimension of this boundary, it is therefore very useful that we show in Section 4.5 (see Theorem 4.3) that the boundary itself is again a limit set of a BCA.

4.2 Branching Cellular Automata

Let A be a finite set, acting as our alphabet and let σ be a random neighbour dependent substitution on $A^{\mathbf{Z}}$ with substitution length M and interaction length N. For a definition of a random neighbour dependent substitution we refer the reader to Section 2.4. Define a Branching Cellular Automaton (BCA) as the quintuple (A, M, N, σ, u), where $u \in A^{\mathbf{Z}}$ serves as the starting word for the random substitution σ. Assuming that $M \geq N+1$, the set $T = A^{2N+3}$ is a set of types for σ (see Section 2.3.1). As described in Section 3.1, for $S \subseteq T$ a sequence of sets $(K_n(S))_n$ is associated to a BCA. We briefly review how these sets are obtained. Define for sets $S \subseteq T$

$$J_n(S) = \{i : \text{the type of } (\sigma^n(u))_i \text{ is an element of } S\}$$

and define

$$K_n(S) = \bigcup_{j \in J_n(S)} I_n(j),$$

where $I_n(j)$ is the j^{th} level n M-adic interval.

We assume that A contains a special symbol 0 such that $\sigma(0^{2N+1}) = 0^M$. Under this assumption, the sets $K_n(S)$ are uniformly bounded for all $S \subseteq T\backslash\{\bar{0}\}$, where $\bar{0} = 0^{2N+3}$ is the type consisting of only 0's.

For ease of notation, the definitions and results presented in this chapter are for 1-dimensional BCA's, but they can be easily extended to higher dimensions (see also Section 2.8). For the figures in this chapter we use 2-dimensional BCA's.

4.2.1 Mean-offspring Matrix

Recall that types $t \in T$ and words $\sigma^n(t)$ are indexed such that the leftmost descendant in the n^{th} generation has index 0 (see Section 2.3.1). For $t \in T$ and $S \subseteq T$ define

$$\begin{aligned} J_n(t,S) &= \{0 \le k \le M^n - 1: \text{ the type of } (\sigma(t))_k \text{ is an element of } S\} \\ Z_n(t,S) &= \text{Card}(J_n(t,S)) \\ K_n(t,S) &= \bigcup_{j \in J_n(t,S)} I_n(j). \end{aligned}$$

The *mean-offspring matrix* $\mathcal{M} = (m_{s,t})_{s,t\in T}$ is defined by

$$\begin{aligned} m_{s,t} &= \text{expected number of children with type } t, \\ & \text{generated by a parent of type } s \\ &= \mathbb{E}(Z_1(s,t)), \end{aligned}$$

where $Z_1(s,t)$ is short for $Z_1(s,\{t\})$. Define $\mathcal{M}(n) = (m_{s,t}(n))_{s,t\in T}$ by $m_{st}(n) = \mathbb{E}(Z_n(s,t))$.

Lemma 4.1 *For $n = 1, 2, \ldots$*

$$\mathcal{M}(n) = \mathcal{M}^n.$$

In the proof of this lemma we use the following random variables. Fix n and define for $s, v, t \in T$ and $k = 1, \ldots, Z_n(s,v)$

$$\begin{aligned} \zeta_k(s,v,t) &= \text{number of children with type } t, \text{ generated by} \\ & \text{the } k^{\text{th}} \text{ type-}v \text{ letter in } (\sigma^n(s))_0, \ldots, (\sigma^n(u))_{M^n-1}, \end{aligned}$$

To make the $\zeta_k(s,v,t)$'s random variables on the whole probability space, we define $\zeta_k(s,v,t)$ for $k = Z_n(s,v)+1, \ldots, M^n$ as independent copies of $Z_1(v,t)$. Note that $\zeta_1(s,v,t), \ldots, \zeta_{M^n}(s,v,t)$ are identically distributed, that each $\zeta_k(s,v,t)$ is independent of $Z_n(s,v)$ and that

$$Z_{n+1}(s,t) = \sum_{v\in T} \sum_{k=1}^{Z_n(s,v)} \zeta_k(s,v,t).$$

However, the variables $\zeta_1(s,v,t), \ldots, \zeta_{M^n}(s,v,t)$ do not need to be independent.

Proof (of Lemma 4.1) The proof is by induction. We have

$$\begin{aligned}
m_{st}(n+1) &= \mathbb{E}(Z_{n+1}(s,t)) \\
&= \mathbb{E}(\sum_{v\in T}\sum_{k=1}^{Z_n(s,v)} \zeta_k(s,v,t)) \\
&= \sum_{v\in T}\mathbb{E}(\sum_{k=1}^{Z_n(s,v)} \zeta_k(s,v,t)) \\
&= \sum_{v\in T}\mathbb{E}(Z_n(s,v))\, m_{vt} \\
&= \sum_{v\in T} m_{sv}(n)\, m_{vt}.
\end{aligned}$$

□

4.3 Extinction

In this section we study the random variables $Z_n(t,C)$, where $C \subseteq T$ is a communicating class.

Definition 4.1 (Communicating class) *Let $\mathcal{M} = (m_{st})_{s,t\in T}$ be a non-negative matrix. For $s,t \in T$, we write $s \to t$ if $(\mathcal{M}^r)_{st} > 0$ for some $r \geq 0$. We say that types s and t communicate if $s \to t$ and $t \to s$. The* communicating class $C(t)$ *consists of all types in T that communicate with t.*

Denote the restriction of the matrix $\mathcal{M}$ to the communicating class C by $\mathcal{M}_C = (m_{st})_{s,t\in C}$ and denote the Perron-Frobenius eigenvalue of $\mathcal{M}_C$ by λ_C.

Note that if $t \in C$, then the events $\{Z_n(t,C) > 0 \text{ i.o.}\}$ and $\{Z_n(t,C) > 0 \text{ for all } n\}$ are the same events $\mathbb{P}$–almost surely, where *i.o.* is short for *infinitely often*.

Theorem 4.1 *Let $t \in C$, with C a communicating class of types.*

(i) If $\lambda_C < 1$, then

$$\mathbb{P}(Z_n(t,C) = 0 \textit{ eventually}) = 1.$$

(ii) If $\lambda_C > 1$, then

$$\mathbb{P}(Z_n(t,C) > 0 \textit{ infinitely often}) > 0.$$

Moreover, for all $\varepsilon > 0$ there are $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that

$$\mathbb{P}(c_1\,\lambda_C^n \leq Z_n(t,C) \leq c_2\,\lambda_C^n \textit{ for all } n \,|\, Z_n(t,C) > 0 \textit{ i.o.}) \geq 1-\varepsilon.$$

If $\lambda_C = 0$, then $\mathbb{P}(Z_n(t,C) = 0 \text{ for } n = 1,2,\ldots) = 1$, so assume in the following that $\lambda_C > 0$. Furthermore, we assume from now on that our BCA is such, that if $\lambda_C = 1$, then $\mathbb{P}(Z_n(t,C) = 0 \text{ eventually}) = 1$ for $t \in C$.

Write C as $C = \{t_1, \ldots, t_r\}$, where r is the cardinality of C. In the remaining part of this section we fix $t \in C$ and assume that $t = t_1$. Furthermore, let $m_{ij} := m_{t_i,t_j}$ and write $\mathcal{M}_C$ as $\mathcal{M}_C = (m_{ij})_{1\le i,j\le r}$. Let v_C be a row vector such that its transpose, denoted by v'_C, is a right eigenvector of $\mathcal{M}_C$ corresponding to λ_C, with all entries strictly positive. Define an r-dimensional row-vector $Z_n = (Z_n(1), \ldots, Z_n(r))$ by

$$\begin{aligned} Z_n(i) &= Z_n(t,t_i) \\ &= \text{number of type } t_i \text{ letters in } (\sigma^n(t))_0 \ldots (\sigma^n(t))_{M^n-1}. \end{aligned}$$

Note that $Z_n(t,C) = Z_n(1) + \ldots + Z_n(r)$. Let $(v_C)_1$ denote the first entry of the vector v_C and let $(v_C)_{(1)}$ denote the smallest entry.

Lemma 4.2 *We have*

$$\mathbb{E}(Z_n(t,C)) \le \frac{(v_C)_1}{(v_C)_{(1)}} \lambda_C^n.$$

In the sequel we will write v for v_C and λ for λ_C.

Proof (Lemma 4.2) Let e_1 denote the r–dimensional row vector with a 1 at the first entry and 0's elsewhere. Then for all n

$$\begin{aligned} \mathbb{E}(Z_n(t,C)) &\le \tfrac{1}{v_{(1)}}\mathbb{E}(Z_n v') \\ &= \tfrac{1}{v_{(1)}} e_1 \mathcal{M}_C^n v' \qquad \text{by Lemma 4.1} \\ &= \frac{v_1}{v_{(1)}} \lambda^n. \end{aligned}$$

□

For $n = 0, 1, \ldots$ define

$$\mathcal{F}_n = \text{ the } \sigma\text{–algebra generated by } \sigma^0, \ldots, \sigma^n.$$

Lemma 4.3 *Assume $\lambda > 1$. Then the sequence*

$$\left(\frac{Z_n v'}{\lambda^n}\right)_{n\ge 0}$$

is a uniformly integrable martingale with respect to $(\mathcal{F}_n)_{n\ge0}$.

Fix $n \geq 0$ and define for $i, j = 1, \ldots, r$ and $k = 1, \ldots, M^n$

$$\zeta_k(i,j) = \zeta_k(t, t_i, t_j).$$

Hence, for $k = 1, \ldots, Z_n(i)$

$$\zeta_k(i,j) = \text{number of children with type } t_j, \text{ generated by the } k^{\text{th}} \text{ type-}t_i \text{ letter in } (\sigma^n(t))_0, \ldots, (\sigma^n(t))_{M^n-1}.$$

Note that $\mathbb{P}$-a.s.

$$Z_{n+1}(j) = \sum_{i=1}^{r} \sum_{k=1}^{Z_n(i)} \zeta_k(i,j).$$

Proof (Lemma 4.3) The fact that the sequence $\left(\frac{Z_n v'}{\lambda^n}\right)_{n\geq 0}$ is a martingale is proved in the same way as in the case that $(Z_n)_{n\geq 0}$ is a multi-type Galton-Watson branching process. To establish uniform integrability, it suffices to show that the sequence $\left(\mathrm{Var}(\frac{Z_n v'}{\lambda^n})\right)_{n\geq 0}$ is uniformly bounded. We have

$$\begin{aligned}
\mathrm{Var}(Z_{n+1}v') &= \mathrm{Var}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r} \sum_{k=1}^{Z_n(i)} \zeta_k(i,j)\Big) \\
&= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r} \sum_{k=1}^{Z_n(i)} \zeta_k(i,j) - \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r} \sum_{k=1}^{Z_n(i)} \zeta_k(i,j)\Big)\Big)^2 \\
&= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r} \sum_{k=1}^{Z_n(i)} \big(\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))\big) \\
&\quad + \sum_{j=1}^{r} v_j \sum_{i=1}^{r} Z_n(i)\, \mathbb{E}(\zeta_1(i,j)) - \sum_{j=1}^{r} v_j \sum_{i=1}^{r} \mathbb{E}(Z_n(i))\mathbb{E}(\zeta_1(i,j))\Big)^2,
\end{aligned}$$

since the $\zeta_1(i,j), \ldots, \zeta_{M^n}(i,j)$ are identically distributed and each one is independent of $Z_n(i)$.

$$\begin{aligned}
\mathrm{Var}(Z_{n+1}v') &= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r}\sum_{k=1}^{Z_n(i)} (\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))) \\
&\quad + \sum_{i=1}^{r} (Z_n(i) - \mathbb{E}(Z_n(i))) \sum_{j=1}^{r} v_j m_{ij}\Big)^2 \\
&= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r}\sum_{k=1}^{Z_n(i)} (\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))) \\
&\quad + \lambda(Z_n v' - \mathbb{E}(Z_n v'))\Big)^2 \\
&= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r}\sum_{k=1}^{Z_n(i)} (\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j)))\Big)^2 + \lambda^2 \mathrm{Var}(Z_n v').
\end{aligned}$$

The last equality follows by writing out the square. To see that the cross-term cancels, condition on Z_n. We will derive an upper bound for the first term in the last expression.

$$\begin{aligned}
A_n &:= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r}\sum_{k=1}^{Z_n(i)} (\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j)))\Big)^2 \\
&\leq \sum_{i=1}^{r}\sum_{j=1}^{r} (r v_j)^2 \mathbb{E}\Big(\sum_{k=1}^{Z_n(i)} (\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j)))\Big)^2
\end{aligned}$$

The inequality follows by applying $(\sum_{j=1}^{r} x_j)^2 \leq r \sum_{j=1}^{r} x_j^2$ twice. Condition on $Z_n(i) = m$ to obtain

$$\begin{aligned}
A_n &\leq \sum_{i=1}^{r}\sum_{j=1}^{r} (r v_j)^2 \sum_{m=0}^{M^n}\sum_{k=1}^{m}\sum_{l=1}^{m} \mathrm{Cov}\big(\zeta_k(i,j), \zeta_l(i,j)\big)\, \mathbb{P}(Z_n(i) = m) \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r} (r v_j)^2 \sum_{m=0}^{M^n}\sum_{k=1}^{m}\sum_{|k-l|\leq 2}^{m} \mathrm{Cov}\big(\zeta_k(i,j), \zeta_l(i,j)\big)\, \mathbb{P}(Z_n(i) = m),
\end{aligned}$$

since if $|k-l| > 2$, the k^{th} and the l^{th} type t_i letter in $\sigma^n(u)$ are at least 2 places apart, which implies that the types of the children of the k^{th} type t_i letter and the types of the children of the l^{th} type t_i letter are independent. Bring two summations inside the expectation and apply the inequality $2xy \leq x^2 + y^2$ to obtain

$$\begin{aligned} A_n &\le \sum_{i=1}^{r}\sum_{j=1}^{r}(rv_j)^2\sum_{m=0}^{M^n}\mathbb{E}\Big(5\sum_{k=1}^{m}\big(\zeta_k(i,j)-\mathbb{E}(\zeta_k(i,j))\big)^2\Big)\,\mathbb{P}(Z_n(i)=m) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}5(rv_j)^2\,\mathbb{E}(Z_n(i))\,\mathrm{Var}(\zeta_1(i,j)) \\ &\le \mathbb{E}(Z_n(t,C))\Big(\sum_{j=1}^{r}5(rv_j)^2\max_{1\le i\le r}\big(\mathrm{Var}(\zeta_1(i,j))\big)\Big). \end{aligned}$$

By Lemma 4.2, we can bound $\mathbb{E}(Z_n(t,C))$ by $\frac{v_1}{v_{(1)}}\lambda^n$. Writing

$$c = 5r^2\frac{v_1}{v_{(1)}}\sum_{j=1}^{r}(v_j)^2\max_{1\le i\le r}\big(\mathrm{Var}(\zeta_1(i,j))\big),$$

we have found that $A_n \le c\lambda^n$, and therefore

$$\mathrm{Var}(Z_{n+1}v') \le c\lambda^n + \lambda^2\,\mathrm{Var}(Z_nv').$$

This recursive inequality implies that

$$\mathrm{Var}(Z_{n+1}v') \le c\lambda^n\frac{\lambda^{n+1}-1}{\lambda-1}.$$

Hence, for $\lambda > 1$

$$\mathrm{Var}(\frac{Z_{n+1}v'}{\lambda^{n+1}}) \le c\frac{1}{\lambda-1}$$

and so our martingale sequence is uniformly integrable. □

Proof (Theorem 4.1) The first part of the theorem is easy to prove. For all $n = 0, 1, \ldots$, writing $\lambda = \lambda_C$ and $v = v_C$

$$\begin{aligned} \mathbb{P}(Z_n(t,C) > 0) &\le \mathbb{E}(Z_n(t,C)) \\ &\le \frac{v_1}{v_{(1)}}\lambda^n \end{aligned}$$

by Lemma 4.2. If we assume that $\lambda < 1$, then $\mathbb{P}(Z_n(t,C) > 0)$ tends to 0 as $n \to \infty$. This implies that

$$\mathbb{P}(Z_n(t,C) > 0 \text{ for all } n) = 0.$$

For the second part where $\lambda > 1$, we use the fact that the sequence

$$\big(X_n\big)_{n\ge 0} = (\frac{Z_nv'}{\lambda^n})_{n\ge 0}$$

is a uniformly integrable martingale sequence. This implies that the X_n converge to a limit X with $0 < \mathbb{E}(X) = \mathbb{E}(X_1) = v_1 < \infty$. Hence,

$$\mathbb{P}(\text{there is a } c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \text{ for all } n) > 0$$

and

$$\mathbb{P}(\text{there is a } c_2 \text{ such that } Z_n v' \leq c_2 \lambda^n \text{ for all } n) = 1.$$

We will show that

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \text{ for all } n) > 0$$

implies

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \ \forall n \mid Z_m(t, C) > 0 \ \forall m) = 1.$$

To see this, define for each $t_i \in C$ and $n \geq 0$ an r–dimensional row-vector $Y_n(i) := (Y_n(i,1), \ldots, Y_n(i,r))$ by

$$Y_n(i,j) := Z_n(t_i, t_j).$$

Note that since $t = t_1$, we have $Z_n = Y_n(1)$. Define

$$\begin{aligned} \rho_i &:= \mathbb{P}(\exists c_1 > 0 \text{ such that } Y_n(i) v' \geq c_1 \lambda^n \ \forall n) \\ \rho &:= \min\{\rho_i : 1 \leq i \leq r\}. \end{aligned}$$

Note that $\rho > 0$. For all m,

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \ \forall n \,|\mathcal{F}_m) \geq \rho$$

almost surely on $\{Z_m(t, C) > 0\}$. So on $\{Z_m(t, C) > 0 \text{ for all } m\}$ we have

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \ \forall n \,|\mathcal{F}_m) \geq \rho$$

for all m almost surely. By Lévy's 0–1 law,

$$\mathbb{P}(\exists c_1 > 0 : \ Z_n v' \geq c_1 \lambda^n \ \forall n \,|\mathcal{F}_m) \to \mathbf{1}_{\{\exists c_1 > 0: \ Z_n v' \geq c_1 \lambda^n \ \forall n\}} \qquad \text{a.s.}$$

Hence, on $\{Z_m(t, C) > 0 \text{ for all } m\}$ we have $\mathbf{1}_{\{\exists c_1 > 0: \ Z_n v' \geq c_1 \lambda^n \ \forall n\}} = 1$ a.s., which means that

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \ \forall n \mid Z_m(t, C) > 0 \ \forall m) = 1.$$

We conclude that

$$\mathbb{P}(\exists c_1, c_2 > 0 \text{ such that } c_1 \lambda^n \leq Z_n v' \leq c_2 \lambda^n \ \forall n \mid Z_m(t, C) > 0 \ \forall m) = 1$$

and since v' has all entries positive

$$\mathbb{P}(\exists c_1, c_2 > 0 \text{ such that } c_1 \lambda^n \leq Z_n(t, C) \leq c_2 \lambda^n \ \forall n \mid Z_m(t, C) > 0 \ \forall m) = 1.$$

The second part of the theorem follows from this. □

The last part of the proof is very similar to the technique to show that a branching process (X_n) with $\mathbb{P}(X_1 = 1) < 1$ satisfies

$$\mathbb{P}(\lim_{n \to \infty} X_n = 0 \text{ or } \infty) = 1.$$

4.4 Convergence

Consider a BCA (A, M, N, σ, u) and recall that for $S \subseteq T\backslash\{\bar{0}\}$ the sets $K_n(S)$ are uniformly bounded (see Section 3.1). In this section we study convergence of the sets $K_n(S)$ with respect to Hausdorff metric (see Section 3.2). In the first part of Theorem 4.2 we will give sufficient conditions for sets $K_n(t, S)$ to converge almost surely. From this part of the theorem one can immediately conclude that the sets $K_n(t, C)$ converge to a limit set $K(t, C)$ for any $t \in T$ and communicating class $C \subseteq T$. The second part of Theorem 4.2 states that $\limsup_{\mathrm{H}} K_n(t, S)$ is equal to the union of sets $K(t, C)$ over a collection of communicating classes. From Theorem 4.2 we derive Corollary 4.1 concerning convergence of the sets $K_n(S)$.

Recall from Section 3.2 that for a sequence $(K_n)_n$ of uniformly bounded closed sets

$$\begin{aligned} \liminf{}_{\mathrm{H}} K_n &= \{x : B_\varepsilon(x) \cap K_n \neq \emptyset \text{ eventually for all } \varepsilon > 0\} \\ \liminf{}_{\mathrm{H}} K_n &= \{x : B_\varepsilon(x) \cap K_n \neq \emptyset \text{ infinitely often for all } \varepsilon > 0\}, \end{aligned}$$

where $B_\varepsilon(x)$ denotes the ε-ball around x. In Lemma 3.2 it was proved that the sequence $(K_n)_n$ converges, if and only if $\liminf_{\mathrm{H}} K_n = \limsup_{\mathrm{H}} K_n$.

Definition 4.2 (Period) *Let t be a type in a communicating class C. Define the* period $d(t)$ *of t as the greatest common divisor of integers $r \geq 1$ for which $(\mathcal{M}^r)_{tt} > 0$. It can be shown that the periods of all $t \in C$ are the same. The period of C is defined as the common value d for the periods of the types in C. If $d > 1$, then C is called* periodic *and if $d = 1$, then C is called* aperiodic.

Definition 4.3 (Cyclic classes and extended cyclic classes) *Let C be a communicating class of T with period d. If t is a type in C, then the* cyclic class $H(t)$ *consists of all types s in C that can be reached from t in a multiple of d steps, i.e. $(\mathcal{M}^{nd})_{ts} > 0$ for some $n = 0, 1, \ldots$. By $H_0, \ldots, H_{d-1}$ we denote the d cyclic classes of C. We assume that the numbering of the classes is such that if $s \in H_i$ and $\mathcal{M}_{st} > 0$, then $t \in H_{(i+1) \bmod d}$.*
If t is a type in C, then the extended cyclic class $\bar{H}(t)$ *consists of all types in T that can be reached from t in a multiple of d steps. By $\bar{H}_0, \ldots, \bar{H}_{d-1}$ we denote the d extended cyclic classes of C.*

Denoting the Perron–Frobenius eigenvalue of a communicating class C by λ_C, define for $t \in T$ and $S \subseteq T$

$$\begin{aligned} \mathcal{C} &= \{C : C \text{ is a communicating class for which } \lambda_C > 1\} \\ \mathcal{C}(t, S) &= \{C \in \mathcal{C} : \text{ there are } t' \in C \text{ and } t'' \in S \text{ such that } t \to t' \to t''\} \\ \mathcal{C}(S) &= \{C : C \in \mathcal{C}(t, S) \text{ and } t \text{ is the type} \\ &\qquad \text{of a letter in the starting word } u\}. \end{aligned}$$

Theorem 4.2 *Let $t \in T$ and $S \subseteq T$. If for each communicating class $C \in \mathcal{C}(t,S)$*

$$\bar{H}_i \cap S \neq \emptyset \qquad \text{for } 0 \le i \le d-1,$$

where d is the period of C, and $\bar{H}_0, \ldots, \bar{H}_{d-1}$ are the extended cyclic classes of C, then $(K_n(t,S))_{n\geq 0}$ converges to $K(t,S)$, $\mathbb{P}$-almost surely. Moreover,

$$\limsup\nolimits_{\mathrm{H}} K_n(t,S) = \bigcup_{C \in \mathcal{C}(t,S)} K(t,C),$$

$\mathbb{P}$-almost surely.

We first make some comments on the previous result. The first part of Theorem 4.2 implies that the sequence $(K_n(t,C))_{n\geq 0}$ converges for all $t \in T$ and for all communicating classes C. If $t \in C$ and $\lambda_C < 1$, then the sequence converges to the empty set with probability 1, by Theorem 4.1. If $t \in C$ and $\lambda_C > 1$, then the sequence converges with probability one, and it converges to a non-empty set with positive probability. If $t \in C$ and $\lambda_C = M$, i.e., the communicating class C is closed, then the sequence converges with probability one to the unit cube.

For the proof of Theorem 4.2 we need the following lemma's. Let $t \in T$, $S \subseteq T$, $m \geq 0$ and $0 \leq k \leq M^m - 1$. Define for $n \geq 0$

$$\begin{aligned} J_n(t,S,m,k) &= \{i : kM^n \le i < (k+1)M^n, \\ &\qquad \text{the type of } (\sigma^{m+n}(t))_i \text{ is an element of } S\} \end{aligned}$$

and define $Z_n(t,S,m,k) = |J_n(t,S,m,k)|$. Note that $Z_n(t,S) = Z_n(t,S,0,0)$.

Lemma 4.4 *Let $t \in T$ and $S \subseteq T$. The event*

$$\{Z_n(t,S) > 0 \ i.o\}$$

is contained in

$$\bigcup_{m,k} \bigcup_{C \in \mathcal{C}(t,S)} \{Z_n(t,C,m,k) > 0 \text{ for all } n\},$$

$\mathbb{P}$–almost surely.

Proof Fix $\omega \in \{Z_n(t,S) > 0 \text{ i.o.}\}$. We can find a sequence of integers $(k_m)_{m\geq 0}$ such that

i) $I_0(k_0) \supseteq I_1(k_1) \supseteq \ldots$

ii) for all $m \geq 0$ we have $Z_n(t,S,m,k)(\omega) > 0$ for infinitely many n.

We can find a subsequence $(k_{m_i})_i$ of $(k_m)_m$ and a type $s \in T$ such that for all $i = 0, 1, \dots$ we have that $Z_0(t, s, m_i, k_{m_i})(\omega) = 1$. Hence $Z_0(t, s, m_0, k_{m_0})(\omega) = 1$, $Z_n(t, s, m_0, k_{m_0})(\omega) > 0$ for infinitely many n and $Z_0(t, S, m_0, k_{m_0})(\omega) > 0$ for some $n \geq 0$. It follows that

$$\{Z_n(t,S) > 0 \text{ i.o.}\} \subseteq \bigcup_{m,k} \bigcup_{s \in T} \{Z_0(t,s,m,k) = 1, Z_n(t,s,m,k) > 0 \text{ i.o.}, \\ Z_0(t,S,m,k) > 0 \text{ for some } n\}.$$

This implies by Theorem 4.1 that $\mathbb{P}$–a.s.

$$\begin{aligned} &\{Z_n(t,S) > 0 \text{ i.o.}\} \\ &\subseteq \bigcup_{m,k} \bigcup_{C \in \mathcal{C}(t,S)} \bigcup_{s \in C} \{Z_0(t,s,m,k) = 1, Z_n(t,s,m,k) > 0 \text{ i.o.}\} \\ &\subseteq \bigcup_{m,k} \bigcup_{C \in \mathcal{C}(t,S)} \{Z_0(t,C,m,k) = 1, Z_n(t,C,m,k) > 0 \text{ i.o.}\} \\ &= \bigcup_{m,k} \bigcup_{C \in \mathcal{C}(t,S)} \{Z_n(t,C,m,k) > 0 \text{ for all } n\}. \end{aligned}$$

□

Let C be a communicating class with period d and let $H_0, \dots, H_{d-1}$ be the cyclic classes of C.

Lemma 4.5 *Let $t \in H_0$ and assume that $\bar{H}_0 \cap S \neq \emptyset$. The event*

$$\{Z_{nd}(t, H_0) > 0 \text{ for all } n\}$$

is contained in

$$\{Z_{nd}(t, S) > 0 \text{ eventually}\}$$

$\mathbb{P}$–almost surely.

Proof By contradiction, assume that $\mathbb{P}(Z_{nd}(t, H_0) > 0$ for all n, $Z_{nd}(t, S) = 0$ i.o.$) > 0$. By Theorem 4.1, for all $\varepsilon > 0$ there is a $c > 0$ such that

$$\begin{aligned} 0 &< \mathbb{P}(Z_{nd}(t,H_0) > 0 \text{ for all } n,\ Z_{nd}(t,S) = 0 \text{ i.o.}) \\ &\leq \mathbb{P}(Z_{nd}(t,H_0) \geq c\lambda_C^{nd} \text{ for all } n,\ Z_{nd}(t,S) = 0 \text{ i.o.}) + \varepsilon. \end{aligned}$$

This implies that we can find $v \in H_0$, $w \in S$, $l \in \mathbb{N}$ and $c' > 0$ such that

$$(\mathcal{M}^{ld})_{vw} > 0$$

and

$$\mathbb{P}(E_n \text{ i.o.}) > 0,$$

where

$$E_n = \{Z_{nd}(t,v) \geq c'\,\lambda_C^{nd},\ Z_{(n+l)d}(t,w) = 0\}.$$

Define

$$\rho = \frac{1}{M^{ld}}(\mathcal{M}^{ld})_{vw}.$$

Then

$$\begin{aligned} \mathbb{P}(Z_{ld}(v,w) > 0) &\geq \frac{1}{M^{ld}}\mathbb{E}(Z_{ld}(v,w)) \\ &= \frac{1}{M^{ld}}(\mathcal{M}^{ld})_{vw} \\ &= \rho. \end{aligned}$$

Fix n and define for all $k = 1, \ldots, Z_{nd}(t,v)$

$$\begin{aligned} \xi_k(t,v,w) &:= \text{number of type } w \text{ descendants in } \sigma^{(n+l)d}(t)\text{, generated by} \\ &\quad\ \text{the } k^{\text{th}} \text{ type } v \text{ letter in } (\sigma^{nd}(t))_0, \ldots, (\sigma^{nd}(t))_{M^{nd}-1}. \end{aligned}$$

If $\mathbb{P}(Z_{nd}(t,v) \geq c'\,\lambda_C^{nd}) > 0$, we have

$$\begin{aligned} \mathbb{P}(E_n) &= \mathbb{P}(Z_{nd}(t,v) \geq c'\,\lambda_C^{nd},\ Z_{(n+l)d}(t,w) = 0) \\ &\leq \mathbb{P}(Z_{(n+l)d}(t,w) = 0 \mid Z_{nd}(t,v) \geq c'\,\lambda_C^{nd}) \\ &\leq \mathbb{P}(\sum_{k=1}^{Z_{nd}(t,v)} \xi_k(t,v,w) = 0 \mid Z_{nd}(t,v) \geq c'\,\lambda_C^{nd}) \\ &\leq \mathbb{P}(\sum_{k=1}^{\lceil \frac{1}{3}c'\,\lambda_C^{nd} \rceil} \xi_{3k-2}(t,v,w) = 0). \end{aligned}$$

Note that if $k \neq l$, the $(3k-2)^{\text{th}}$ and the $(3l-2)^{\text{th}}$ type v letter in $\sigma^{nd}(t)$ are at least 2 places apart, which implies that the two letters generate the types of their offspring independently. Therefore,

$$\begin{aligned} \mathbb{P}(E_n) &\leq \mathbb{P}(\xi_1(t,v,w) = 0)^{\frac{1}{3}c'\lambda_C^{nd}} \\ &\leq (1-\rho)^{\frac{1}{3}c'\lambda_C^{nd}}. \end{aligned}$$

By the Borel-Cantelli lemma $\mathbb{P}(E_n \text{ i.o.}) = 0$, which is a contradiction. □

Lemma 4.6 *Let $t \in T$, $S \subseteq T$ and $C \in \mathcal{C}(t,S)$. Then*

$$\{Z_n(t,C) > 0 \ i.o.\} \subseteq \{Z_n(t,S) > 0 \ i.o.\},$$

$\mathbb{P}$-almost surely. If in addition

$$\bar{H}_i \cap S \neq \emptyset \qquad \text{for } 0 \leq i \leq d-1,$$

where d is the period of C, and $\bar{H}_0, \ldots, \bar{H}_{d-1}$ are the extended cyclic classes of C, then

$$\{Z_n(t,C) > 0 \text{ i.o.}\} \subseteq \{Z_n(t,S) > 0 \text{ eventually}\},$$

$\mathbb{P}$-almost surely.

Proof By Lemma 4.4, the event $\{Z_n(t,C) > 0 \text{ i.o.}\}$ is contained in

$$\bigcup_{m,k} \bigcup_{C' \in \mathcal{C}(t,C)} \bigcup_{s \in C'} \{Z_0(t,s,m,k) = 1, Z_n(t,C',m,k) > 0 \text{ for all } n\}.$$

Fix m, k, C' and let $H'_0, \ldots, H'_{d'-1}$ be the cyclic classes of C' and assume that $s \in H'_0$. Then $\{Z_0(t,s,m,k) = 1, Z_n(t,C',m,k) > 0 \text{ for all } n\}$ is contained in $\bigcap_{i=0}^{d'-1} \{Z_{nd'+i}(t,H'_i,m,k) > 0 \text{ for all } n\}$. Since $C' \in \mathcal{C}(t,C)$ and $C \in \mathcal{C}(t,S)$, it follows that $C' \in \mathcal{C}(t,S)$.

To prove the first inclusion of the lemma, let i be such that $\bar{H}'_i \cap S \neq \emptyset$. Then by Lemma 4.5

$$\begin{aligned} \{Z_{nd'+i}(t,H'_i,m,k) > 0 \text{ for all } n\} &\subseteq \{Z_{nd'+i}(t,S,m,k) > 0 \text{ eventually}\} \\ &\subseteq \{Z_n(t,S,m,k) > 0 \text{ i.o.}\}. \end{aligned}$$

Hence

$$\begin{aligned} \{Z_n(t,C) > 0 \text{ i.o.}\} &\subseteq \bigcup_{m,k} \{Z_n(t,S,m,k) > 0 \text{ i.o.}\} \\ &= \{Z_n(t,S) > 0 \text{ i.o.}\}. \end{aligned}$$

To prove the second inclusion, assume that $\bar{H}_i \cap S \neq \emptyset$ for all $i = 0, \ldots, d-1$. It is not difficult to see that this implies that $\bar{H}'_i \cap S \neq \emptyset$ for all $i = 0, \ldots, d'-1$. Then by Lemma 4.5

$$\begin{aligned} &\bigcap_{i=0}^{d'-1} \{Z_{nd'+i}(t,H'_i,m,k) > 0 \text{ for all } n\} \\ &\quad \subseteq \bigcap_{i=0}^{d'-1} \{Z_{nd'+i}(t,S,m,k) > 0 \text{ for all } n\} \\ &\quad = \{Z_n(t,S,m,k) > 0 \text{ eventually}\}. \end{aligned}$$

Hence

$$\begin{aligned} \{Z_n(t,C) > 0 \text{ i.o.}\} &\subseteq \bigcup_{m,k} \{Z_n(t,S,m,k) > 0 \text{ eventually}\} \\ &\subseteq \{Z_n(t,S) > 0 \text{ eventually}\}. \end{aligned}$$

□

Proof (Theorem 4.2) By Lemma 3.3 we have that

$$\limsup_{\mathrm{H}} K_n(t,S) = \bigcap_{m=0}^{\infty} \bigcup_{\{k: Z_n(t,S,m,k)>0 \text{ i.o.}\}} I_m(k).$$

Under the conditions stated in the first part of the theorem, we have $\mathbb{P}$-almost surely by Lemma 4.4 and Lemma 4.6 that

$$\{Z_n(t,S) > 0 \text{ i.o.}\} = \{Z_n(t,S) > 0 \text{ eventually}\}.$$

As a consequence, also

$$\{Z_n(t,S,m,k) > 0 \text{ i.o.}\} = \{Z_n(t,S,m,k) > 0 \text{ eventually}\}$$

for all $m \geq 0$ and $0 \leq k \leq M^m - 1$. Hence by Lemma 3.3

$$\begin{aligned} \limsup_{\mathrm{H}} K_n(t,S) &= \bigcap_{m=0}^{\infty} \bigcup_{\{k: Z_n(t,S,m,k)>0 \text{ eventually}\}} I_m(k) \\ &\subseteq \liminf_{\mathrm{H}} K_n(t,S). \end{aligned}$$

For the second part of the theorem, it follows from Lemma 4.4 and Lemma 4.6 that

$$\{Z_n(t,S,m,k) > 0 \text{ i.o.}\} = \bigcup_{C \in \mathcal{C}(t,S)} \{Z_n(t,C,m,k) > 0 \text{ i.o.}\}$$

for all $m \geq 0$ and $0 \leq k \leq M^m - 1$. Hence

$$\limsup_{\mathrm{H}} K_n(t,S) = \limsup_{\mathrm{H}} \bigcup_{C \in \mathcal{C}(t,S)} K_n(t,C).$$

Since the sets $\bigcup_{C \in \mathcal{C}(t,S)} K_n(t,C)$ converge by the first part of the theorem, one has $\limsup_{\mathrm{H}} K_n(t,S) = \bigcup_{C \in \mathcal{C}(t,S)} K(t,C)$. □

Corollary 4.1 *Let $S \subseteq T \backslash \{\bar{0}\}$. If for each communicating class $C \in \mathcal{C}(S)$*

$$\bar{H}_i \cap S \neq \emptyset \qquad \text{for } 0 \leq i \leq d-1,$$

where d is the period of C, and $\bar{H}_0, \ldots, \bar{H}_{d-1}$ are the extended cyclic classes of C, then $(K_n(S))_{n \geq 0}$ converges to $K(S)$, $\mathbb{P}$-almost surely. Moreover,

$$\limsup_{\mathrm{H}} K_n(S) = \bigcup_{C \in \mathcal{C}(S)} K(C),$$

$\mathbb{P}$-almost surely.

Proof If we define $K_n(S,k) = K_n(S) \cap [k, k+1]$, then $K_n(S) = \bigcup_k K_n(S,k)$. Let t denote the type of letter u_k in u. Since $\mathcal{C}(t,S) \subseteq \mathcal{C}(S)$, we have by Theorem 4.2 that $(K_n(t,S))$ converges to $K(t,S)$ under the conditions stated above. This implies that $(K_n(S,k))$ converges to $K(S,k)$ and hence $(K_n(S))$ converges to $K(S) = \bigcup_k K(S,k)$. From the second part of Theorem 4.2 it follows that $\limsup_{\mathrm{H}} K_n(t,S) = \bigcup_{C\in\mathcal{C}(t,S)} K(t,C)$ which is equal to $\bigcup_{C\in\mathcal{C}(S)} K(t,C)$. This implies that $\limsup_{\mathrm{H}} K_n(S,k) = \bigcup_{C\in\mathcal{C}(S)} K(C,k)$ and hence $\limsup_{\mathrm{H}} K_n(S) = \limsup_{\mathrm{H}} \bigcup_k K_n(S,k) = \bigcup_k \bigcup_{C\in\mathcal{C}(S)} K(C,k) = \bigcup_{C\in\mathcal{C}(S)} K(C)$. □

4.5 The boundary of a BCA

Consider a BCA (A, M, N, σ, u) with set of types $T = A^{2N+3}$ and let $S = T\backslash\{\bar{0}\}$. By Corollary 4.1, the sequence $(K_n(S))_n$ converges to a limitset K(S). In this section we study the boundary $\partial K(S)$ of the limit set $K(S)$. Under the assumption that our BCA is 'non-lattice' (see definition 4.4), we prove that $\partial K(S)$ is equal to the limit set $K(\partial S)$, where $\partial S = \{t \in T : t \to \bar{0}, t \neq \bar{0}\}$. As a corollary we obtain that $K(\partial S)$ is also the boundary of the limit set $K(S')$, where $S' = \{t \in T : t_0 \neq 0\}$. Although the results are stated in terms of 1-dimensional BCA's, they can be generalized to higher dimensions in a straightforward manner.

Definition 4.4 *We call a BCA 'non-lattice' (with respect to the set of types T), if for all $n \geq 0$ the probability is 0 that the types s and t of two neighbouring letters in $\sigma^n(u)$ are such that $s = \bar{0}$ and $t \in S\backslash\partial S = \{t \in T : t \not\to \bar{0}\}$.*

Example 4.1 An example of a BCA that is lattice (not non-lattice) is the following. Let $A = \{0,1\}$, $M = 2$, $N = 1$, $\sigma(v) = 11$ for all $v \neq 000$ and u is the word with a 1 at position 0 and 0's elsewhere. Let $T = A^5$ be the set of types. Then the deterministic limit set $K(S)$ is equal to the interval $[-2,3]$. Observe that $\partial S = \emptyset$ and hence the type of the letter u_2 is an element of $S\backslash\partial S$. Since the type of u_3 is $\bar{0}$, this BCA is lattice.

If we would have chosen the set of types to be A^{2N+5} instead of A^{2N+3}, any BCA is non-lattice. In fact, if s and t are the $(2N+5)$- types of two neighbouring letters in any word $v \in A^{\mathbb{Z}}$, then the probability is 0 that $s = 0^{2N+5}$ and $t \in \{t' \in A^{2N+5} : t' \not\to 0^{2N+5}\}$. To see this, let $v \in A^{Z}$ be such that the $(2N+5)$-type of v_0 is 0^{2N+5}, i.e., $v_{-N-2} \dots v_{N+2} = 0^{2N+5}$. Consider the leftmost child $(\sigma(v))_M$ of v_1. Since $(\sigma(v))_{M-(2M-1)} \cdots (\sigma(v))_{M+(2M-1)} = 0^{4M-1}$ and since $4M-1 \geq 2M + 2(N+1) - 1 \geq 4 + 2(N+1) - 1 = 2N+5$, it follows that the $(2N+5)$-type of the leftmost child of v_1 is 0^{2N+5}. Therefore the $(2N+5)$-type of v_1 cannot be an element of $\{t \in A^{2N+5} : t \not\to 0^{2N+5}\}$. Since the same holds for v_{-1}, the other neighbour of v_0, the BCA is non-lattice with respect to $T = A^{2N+5}$.

Assuming that our BCA is non-lattice, we have the following theorem.

Theorem 4.3 *Let $S = T\backslash\{\bar{0}\}$ and $\partial S = \{t \in T : t \to \bar{0}, t \neq \bar{0}\}$. Then $\partial K(S) = K(\partial S)$ almost surely.*

Let $l < r$ be integers such that letters u_k in the starting word u have type $\bar{0}$ for $k \leq l$ and $k \geq r$. For $S \subseteq T$, define sets $\bar{K}_n(S)$ by $\bar{K}_n(S) = K_n(S) \cap [l, r]$. Observe that $\bar{K}_n(S) = K_n(S)$ for $S \subseteq T\backslash\{\bar{0}\}$ and that $K_n(S) \cup \bar{K}_n(\bar{0}) = [l, r]$ if $S = T\backslash\{\bar{0}\}$. Since $\{\bar{0}\}$ is a closed communicating class, it follows from Theorem 4.2 that $(\bar{K}_n(\bar{0}))_n$ increases to a limit set $\bar{K}(\bar{0})$ and that $\bar{K}(\bar{0}) = \bigcup_{C \in \mathcal{C}(\bar{0})} \bar{K}(C)$. We will denote the interior of a set $X \subseteq \mathbb{R}$ by $\mathrm{int}(X)$ and its closure by $\mathrm{cl}(X)$.

Lemma 4.7 *Let $S = T\backslash\{\bar{0}\}$. Then almost surely*

1. $\mathrm{cl}(\mathrm{int}(\bar{K}(\bar{0}))) = \bar{K}(\bar{0})$

2. $\mathrm{int}(K(S)) \cap \bar{K}(\bar{0}) = \emptyset$.

Proof In this and the following proofs, all statements are almost sure.

1. This follows from letting n tend to infinity in

$$\bar{K}_n(\bar{0}) = \mathrm{cl}(\mathrm{int}(\bar{K}_n(\bar{0}))) \subseteq \mathrm{cl}(\mathrm{int}(\bar{K}(\bar{0}))) \subseteq \bar{K}(\bar{0}).$$

   Here we used that $(\bar{K}_n(\bar{0}))_{n\geq 0}$ increases to $\bar{K}(\bar{0})$, since $\{\bar{0}\}$ is a closed communicating class.

2. This follows since $(K_n(S))_{n\geq 0}$ is an decreasing sequence and for all n

$$\begin{aligned} \mathrm{cl}(\mathrm{int}(\bar{K}_n(\bar{0}))) &= \bar{K}_n(\bar{0}) \\ \mathrm{int}(\bar{K}_n(\bar{0})) \cap \mathrm{int}(K_n(S)) &= \emptyset. \end{aligned}$$

□

Proof (Theorem 4.3) We will first proof that $K(\partial S) \subseteq \partial K(S)$. Since $\partial S \subseteq S$, we have that $K(\partial S) \subseteq K(S)$. From Theorem 4.2 it follows that $K(\partial S) = \bigcup_{C \in \mathcal{C}(\partial S)} K(C)$. Since $\mathcal{C}(\partial S) \subseteq \mathcal{C}(\bar{0})$, we have $K(\partial S) \subseteq \bigcup_{C \in \mathcal{C}(\bar{0})} \bar{K}(C) = \bar{K}(\bar{0})$. Hence

$$\begin{aligned} K(\partial S) &\subseteq K(S) \cap \bar{K}(\bar{0}) \\ &= \left(\partial K(S) \cap \bar{K}(\bar{0})\right) \cup \left(\mathrm{int}(K(S)) \cap \bar{K}(\bar{0})\right) \\ &\subseteq \partial K(S), \end{aligned}$$

since $\mathrm{int}(K(S)) \cap \bar{K}(\bar{0}) = \emptyset$ by Lemma 4.7, part 2.

To prove that $\partial K(S) \subseteq K(\partial S)$, assume by contradiction that with positive probability there is an $x \in \partial K(S)$ such that $x \notin K(\partial S)$. Since $K(S) \cup \bar{K}(\bar{0}) =$

$[l, r]$ and $K(\partial S)$ is closed, we can find with positive probability rational $y \in [l, r]$ and rational $\varepsilon > 0$ such that

$$\begin{aligned} B_\varepsilon(y) \cap K(S) &\neq \emptyset \\ B_\varepsilon(y) \cap \bar{K}(\bar{0}) &\neq \emptyset \\ B_{2\varepsilon}(y) \cap K(\partial S) &= \emptyset. \end{aligned}$$

This last event is a union over rational y and ε and hence we can find non-random z and $\eta > 0$ such that with positive probability

$$\begin{aligned} B_\eta(z) \cap K(S) &\neq \emptyset \\ B_\eta(z) \cap \bar{K}(\bar{0}) &\neq \emptyset \\ B_{2\eta}(z) \cap K(\partial S) &= \emptyset. \end{aligned}$$

Since $(K_n(S))_n$ and $(K_n(\partial S))_n$ are decreasing sequences and $(\bar{K}_n(\bar{0}))_n$ is increasing, there is an n_0 such that with positive probability

$$\begin{aligned} B_\eta(z) \cap K_{n_0}(S) &\neq \emptyset \\ B_\eta(z) \cap \bar{K}_{n_0}(\bar{0}) &\neq \emptyset \\ B_\eta(z) \cap K_{n_0}(\partial S) &= \emptyset. \end{aligned}$$

This obviously implies that there are neighbouring n_0-adic intervals I and J such that with positive probability $I \subseteq K_{n_0}(S \backslash \partial S)$ and $J \subseteq \bar{K}_{n_0}(\bar{0})$. This however, contradicts the non-lattice assumption. □

The following lemma implies that the boundary of $K(S')$, where $S' = \{t \in T : t_0 \neq 0\}$, is also equal to $K(\partial S)$.

Lemma 4.8 *Let $S = T\backslash\{\bar{0}\}$ and $S' = \{t \in T : t_0 \neq 0\}$. Then $(K_n(S'))_n$ converges to a limit set $K(S')$ almost surely and $K(S') = K(S)$.*

Proof Since $S' \subseteq S$, we have that $K_n(S') \subseteq K_n(S)$ for all n. If we let $\varepsilon_n = M^{-n}(N + 1)$, then $K_n(S) \subseteq K_n^{\varepsilon_n}(S')$, where $K_n^{\varepsilon_n}(S')$ is the set of points within an ε_n-distance of $K_n(S')$. Since $(K_n(S))_n$ converges to $K(S)$ by Corollary 4.1, also $K_n(S')$ converges to $K(S)$. □

4.6 Examples

Example 4.2 In this example we present a BCA with interaction length N that corresponds to fractal percolation in dimension 2 with parameter p (see Example 2.2 and 3.2). Consider the BCA (A, M, N, σ, u), with $A = \{0, 1\}$ and u

Figure 4.1: A realisation of K_9 of ordinary fractal percolation (Example 4.2) with parameter $p = 0.75$ and $M = 2$.

is the 2-dimensional word with $u_{00} = 1$ and 0's elsewhere. For $v \in A^{(2N+1)\times(2N+1)}$ write

$$v = \begin{matrix} v_{-N,N} & \cdots & v_{N,N} \\ \vdots & & \vdots \\ v_{-N,-N} & \cdots & v_{N,-N} \end{matrix}$$

and

$$\sigma(v) = \begin{matrix} (\sigma(v))_{0,M-1} & \cdots & (\sigma(v))_{M-1,M-1} \\ \vdots & & \vdots \\ (\sigma(v))_{00} & \cdots & (\sigma(v))_{M-1,0} \end{matrix}$$

The probability distribution $\mathbb{P}_p$ of σ is such that all letters $(\sigma(v))_{ij}$ are 0 if $v_{00} = 0$ and they are independent Bernoulli(p) random variables if $v_{00} = 1$. Let $C \subseteq T = A^{2N+3}$ be $C = \{s \in T : s_{00} = 1\}$ and write $K_n = K_n(C)$. Since C is a communicating class, $(K_n)_n$ converges $\mathbb{P}_p$ almost surely by Theorem 4.2.

Example 4.3 Consider the 2-dimensional BCA (A, M, N, σ, u), where $A = \{0, 1\}$, u is the 2-dimensional word with $u_{00} = 1$ and 0's elsewhere and let σ be majority fractal percolation (see Example 2.3 and 3.4). Again, $T = A^{(2N+1)\times(2N+1)}$ is the set of types and let $S \subseteq T$ be $S = \{s \in T : s_{00} = 1\}$. In this example, S is not a communicating class for $N \geq 1$, but

$$C = \{s \in T : s_{ij} = 1 \text{ for some } i, j \in \{-(N+1), \ldots, N+1\}\}$$

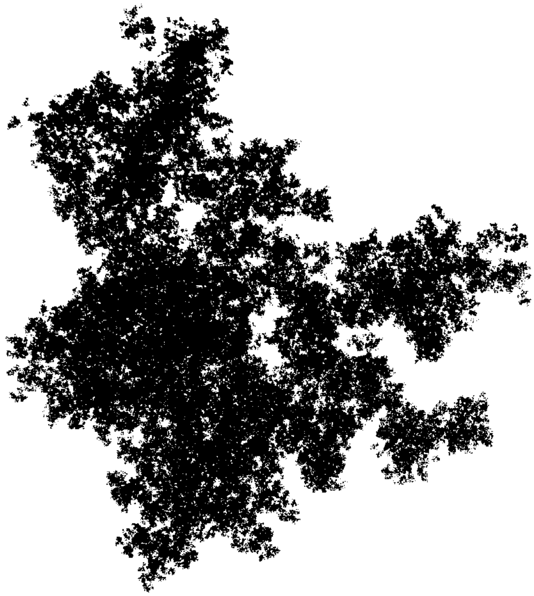


Figure 4.2: A realisation of K_9 of majority fractal percolation, where $M = 2$, $N = 1$ and $p = 0.15$ (see Example 4.3).

is. In fact, $S \subseteq C$ and $\mathcal{C}(S) = \{C\}$. Since C is aperiodic, we have by Theorem 4.2 that $K_1(S), K_2(S), \dots$ almost surely converge to $K(S) = K(C)$.

Example 4.4 Consider the 2-dimensional BCA (A, M, N, σ, u), where $A = \{0, 1\}$, $M = 2$, $N = 1$ and u is the all 0's word except for $u_{00} = 1$. Let σ be the neighbour dependent substitution from the ink model with parameter p (see Example 3.3). For this example, it suffices to take $T = A^{3\times 3}$ as set of types instead of $A^{5\times 5}$, since one is still able to determine the distribution of the types of the offspring, based on the type of the parent. Let $S = \{s \in T : s_{00} = 1\}$ and $\partial S = \{s \in T : s \to \bar{0}, s \neq \bar{0}\}$ and write $K_n = K_n(S)$. Note that ∂S is equal to the set $\{s \in T : s_{00} = 0, s \neq \bar{0}\}$. The BCA is non-lattice, since if v_k and v_l are neighbouring letters in $v \in A^{\mathbb{Z}\times\mathbb{Z}}$ and the type of v_k is $\bar{0}$, then the type of v_l is either $\bar{0}$ or an element of ∂S. By Theorem 4.3 and Lemma 4.8, we have that $\partial K(S) = K(\partial S)$ almost surely.

Figure 4.3: A realisation of $K_9(S)$ of the BCA in Example 4.4 with $p = \frac{1}{2}$.

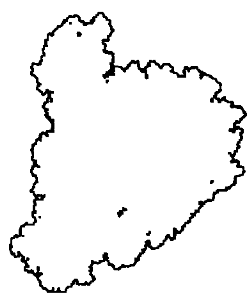


Figure 4.4: $K_9(\partial S)$ of the same realisation as in Figure 4.3.

4.7 Product BCA

Let $\mathbb{B} = (A, M, N, \sigma, u)$ and $\mathbb{B}' = (A', M, N, \sigma', u')$ be two BCA's with the same substitution length M and interaction length N. Denote the probability measure associated with $\mathbb{B}$ by $\mathbb{P}$ and the probability measure associated with $\mathbb{B}'$ by $\mathbb{P}'$. We are going to define a product BCA $\hat{\mathbb{B}} = (\hat{A}, M, N, \hat{\sigma}, \hat{u})$ with associated probability measure $\hat{\mathbb{P}}$ as follows.

Let $\hat{A} = A \times A'$. For $k \geq 0$ and $v = v_1 \dots v_k \in \hat{A}^k$ with $v_i = (v_i^{(1)}, v_i^{(2)})$, we denote the word $v_1^{(1)} \dots v_k^{(1)} \in A^k$ by $v^{(1)}$ and $v_1^{(2)} \dots v_k^{(2)} \in (A')^k$ by $v^{(2)}$. Define a neighbour dependent substitution $\hat{\sigma}$ on the alphabet $\hat{A}$ by

$$\hat{\mathbb{P}}\big(\hat{\sigma}(v) = w\big) = \mathbb{P}\big(\sigma(v^{(1)}) = w^{(1)}\big)\, \mathbb{P}'\big(\sigma'(v^{(2)}) = w^{(2)}\big)$$

for $v \in \hat{A}^{2N+1}$ and $w \in \hat{A}^M$.

Let $\hat{u}$ be the starting word defined by $\hat{u}^{(1)} = u$ and $\hat{u}^{(2)} = u'$. Then the BCA $\hat{\mathbb{B}} = (\hat{A}, M, N, \hat{\sigma}, \hat{u})$ is called the *product* BCA of $\mathbb{B}$ and $\mathbb{B}'$. Each random variable and quantity concerning $\mathbb{B}'$, respectively the product BCA $\hat{\mathbb{B}}$ will have a $'$, respectively a $\hat{}$ attached to it.

Let $t \in T$, $t' \in T'$ and define $\hat{t} \in \hat{T}$ by $\hat{t}^{(1)} = t$, $\hat{t}^{(2)} = t'$. Furthermore, let $S \subseteq T$, $S' \subseteq T'$ and let $\hat{S} \subseteq \hat{T}$ be defined by

$$\hat{S} = \{s \in \hat{T} : s^{(1)} \in S, s^{(2)} \in S'\}.$$

Then we have the following rather obvious lemma, its proof being left to the reader.

Lemma 4.9 *Let t, t', $\hat{t}$ and S, S', $\hat{S}$ be as defined above. Then*

$$\hat{\mathbb{P}}(\hat{K}_n(\hat{t}, \hat{S}) = \emptyset) = \mathbb{P} \times \mathbb{P}'(K_n(t, S) \cap K_n'(t', S') = \emptyset).$$

4.8 Dimension

Consider a 1-dimensional BCA $\mathbb{B} = (A, M, N, \sigma, u)$ with set of types $T = A^{2N+3}$. Let $C \subseteq T$ be a communicating class with Perron-Frobenius eigenvalue $\lambda > 1$ and let $t \in C$. By Theorem 4.2, the sets $K_n(t, C)$ converge to $K(t, C)$ almost surely. For ease of notation, we denote $K_n(t, C)$ by K_n and $Z_n(t, C)$ by Z_n. Define the event 'non-extinction' as $\{Z_n > 0 \text{ infinitely often}\}$. By Theorem 4.1, non-extinction has positive probability. In this section we show that conditioned on non-extinction, the Hausdorff dimension of the limit set K equals $\log \lambda / \log M$ almost surely. The setup of the proof resembles the proof of the dimension result for fractal percolation, as described in Section 3.3.1 Once more, all results in this section extend easily to the higher dimensional case (including Lemma 4.11 by Lyons).

Lemma 4.10 *Conditioned on non-extinction,* $\dim_H K$ *is a constant* $\mathbb{P}$*-a.s.*

Proof Write

$$\begin{aligned} D &= \dim_H(K(t,C)) \\ d &= \sup\{x : \mathbb{P}(D \leq x) < 1\}. \end{aligned}$$

Fix $\varepsilon > 0$ and let $\rho(\varepsilon) = \mathbb{P}(D > d - \varepsilon)$. Let $\mathcal{F}_n$ be the σ-algebra generated by $\sigma^0, \ldots, \sigma^n$. Then for all n,

$$\mathbb{P}(D > d - \varepsilon | \mathcal{F}_n) \geq \rho(\varepsilon)$$

on $\{Z_n(t,t) > 0\}$ almost surely. Hence for infinitely many n

$$\mathbb{P}(D > d - \varepsilon | \mathcal{F}_n) \geq \rho(\varepsilon)$$

on $\{Z_n(t,t) > 0 \text{ i.o.}\}$ almost surely. Since $\{Z_n(t,C) > 0 \text{ i.o.}\} \subseteq \{Z_n(t,t) > 0 \text{ i.o.}\}$ by Lemma 4.6 and since $\mathbb{P}(D > d - \varepsilon | \mathcal{F}_n)$ converges to $\mathbf{1}_{\{D \geq d - \varepsilon\}}$ almost surely by Lévy's 0-1 law, one has that $D > d - \varepsilon$ on $\{Z_n(t,C) > 0 \text{ i.o.}\}$ almost surely. If we let ε tend to 0, it follows that $\dim_H K(t,C) = d$ almost surely, conditioned on non-extinction. □

In this section we will denote by $\mathbb{B}' = (A', M, N, \sigma', u')$ ordinary fractal percolation with parameter p (see Example 4.2). So $A' = \{0,1\}$ and $u' = \ldots 010 \ldots \in (A')^{\mathbb{Z}}$. All quantities and random variables concerning fractal percolation will be written with a prime. We will write K'_n for $K'_n(t', C')$, where $t' = 0 \ldots 010 \ldots 0 \in T'$ and $C' = \{s \in T' : s_0 = 1\}$.

In the proof of Theorem 4.4, we will use Lemma 3.6. For ease of reference, we will state it here once more.

Lemma 4.11 *Let* B *be a set in* $[0,1]$*. If* $\mathbb{P}'_p(B \cap K' \neq \emptyset) > 0$*, then* $\dim_H B \geq -\frac{\log p}{\log M}$.

Consider the product BCA $(\hat{A}, M, N, \hat{\sigma}, \hat{u})$ of our initial BCA $\mathbb{B}$ with fractal percolation $\mathbb{B}'$ with parameter p. So $\hat{A} = A \times A' = A \times \{0,1\}$ and $\hat{u}^{(1)} = u$, $\hat{u}^{(2)} = u'$. All quantities and random variables concerning the product BCA will be written with a hat. We will write $\hat{K}_n = \hat{K}_n(\hat{t}, \hat{C})$, where $\hat{t}$ is such that $\hat{t}^{(1)} = t$, $\hat{t}^{(2)} = t'$ and $\hat{C}$ is the communicating class in $\hat{T}$ which contains $\hat{t}$.

Although $\hat{C}$ does not need to be equal to the set $\hat{S} = \{s \in \hat{T} : s^{(1)} \in C, s^{(2)} \in C'\}$, the sets $\hat{K}_n = \hat{K}_n(\hat{t}, \hat{C})$ and $\hat{K}_n(\hat{t}, \hat{S})$ are equal for all n. To see this, observe that $\hat{C} \subseteq \hat{S}$ and that $\hat{t} \to s$ for each $s \in \hat{S}$. This implies that if s is a type in $\hat{S} \backslash \hat{C}$, then $\hat{t} \not\twoheadrightarrow s$.

Let $\hat{\mathcal{M}}_p = (\hat{m}_{st})_{s,t \in \hat{T}}$ be the mean offspring matrix of the product BCA.

Lemma 4.12 *If we denote the Perron-Frobenius eigenvalue of $\hat{\mathcal{M}}_p$ restricted to $\hat{C}$ by $\hat{\lambda}_p$, then*

$$\hat{\lambda}_p = p\lambda,$$

where $\lambda = \lambda_C$ is the Perron–Frobenius eigenvalue of $\mathcal{M}$ restricted to the communicating class C.

For the proof we need the following lemma due to Furstenberg (cf. [20]).

Lemma 4.13 (Furstenberg) *Let $A = (a_{ij})_{1\le i,j\le n}$ be a non-negative, irreducible $n\times n$-matrix and let $B = (b_{ij})_{1\le i,j\le r}$ be a non-negative, irreducible $r\times r$-matrix, where $1 \le r \le n$. Let $I_1, \dots, I_r$ be a partition of the index set $\{1, \dots, n\}$, where all sets I_k are non-empty. Assume that for A, B and $\{I_k : k = 1, \dots, r\}$ the following relation holds. For all $1 \le i, j \le r$ we have*

$$\text{for all } l \in I_i \qquad \sum_{k\in I_j} a_{lk} = b_{ij}.$$

Then the Perron–Frobenius eigenvalues of the matrices A and B are the same.

Proof Let λ_A and λ_B be the Perron–Frobenius eigenvalues of A and B and let v_A and v_B be associated left eigenvectors with all entries strictly positive. Furthermore, let w_B^{T} be a right eigenvector associated with λ_B having all entries strictly positive. Define

$$R = (r_{ij})_{1\le i\le n,\ 1\le j\le r}$$

by

$$r_{ij} = \begin{cases} 1 & \text{if } i \in I_j \\ 0 & \text{else.} \end{cases}$$

Then $AR = RB$. Define $x \in \mathbb{R}^r$ as $x = v_A R$. Since $\{I_1, \dots, I_r\}$ is a partition, all entries of x are strictly positive. Then

$$xB = v_A RB = v_A AR = \lambda_A v_A R = \lambda_A x,$$

and therefore λ_A is an eigenvalue of B.

Furthermore, we have

$$\lambda_A x w_B^{\mathrm{T}} = xBw_B^{\mathrm{T}} = \lambda_B x w_B^{\mathrm{T}}.$$

Since both x and w_B have all entries strictly positive, we conclude that $\lambda_A = \lambda_B$.
□

Proof (of Lemma 4.12) Consider the partition $\{I_s\}_{s\in C}$ of $\hat{C}$, where

$$I_s = \{v \in \hat{C} : v^{(1)} = s\}.$$

Then we claim that for all $s, t \in C$ and for all $v \in I_s$

$$\sum_{w\in I_t} \hat{m}_{vw} = p\, m_{st}.$$

To see this define for $\hat{t} \in \hat{T}$, $S \subseteq T$ and $S' \subseteq T'$

$$\begin{aligned} \hat{J}(\hat{t}, S, S') \quad = \quad & \{0 \le k \le M-1 : \text{ the type } v \text{ of } (\hat{\sigma}(\hat{t}))_k \\ & \text{has } v^{(1)} \in S \text{ and } v^{(2)} \in S'\}. \end{aligned}$$

So we have

$$\hat{J}(\hat{t}, S, S') = \hat{J}(\hat{t}, S, T') \cap \hat{J}(\hat{t}, T, S').$$

Recall that for $t \in T$ and $S \subseteq T$

$$J_1(t, S) \quad = \quad \{0 \le k \le M-1 : \text{ the type of } (\sigma(t))_k \text{ is an element of } S\}.$$

Writing $Z(s,t)$ for $Z_1(s, \{t\})$ and $J(s,t)$ for $J_1(s, \{t\})$, we have for all $v \in I_s$

$$\begin{aligned} \sum_{w\in I_t} \hat{m}_{vw} &= \sum_{w\in I_t} \hat{\mathbb{E}}_p(\hat{Z}(v,w)) \\ &= \hat{\mathbb{E}}_p(\hat{Z}(v, I_t)) \\ &= \hat{\mathbb{E}}_p(\sum_{k=0}^{M-1} \mathbf{1}_{\hat{J}(v,t,C')}(k)) \\ &= \sum_{k=0}^{M-1} \hat{\mathbb{E}}_p(\mathbf{1}_{\hat{J}(v,t,T')}(k)\, \mathbf{1}_{\hat{J}(v,T,C')}(k)) \\ &= \sum_{k=0}^{M-1} \mathbb{E}(\mathbf{1}_{J(v^{(1)},t)}(k))\, \mathbb{E}'_p(\mathbf{1}_{J'(v^{(2)},C')}(k)) \\ &= p\,\mathbb{E}(Z(s,t)) \\ &= p\, m_{st}. \end{aligned}$$

By Lemma 4.13, the Perron–Frobenius eigenvalues of $\hat{\mathcal{M}}_p$ restricted to $\hat{C}$ and $p\mathcal{M}$ restricted to C are the same. Therefore, $\hat{\lambda}_p = p\,\lambda$. □

Lemma 4.14 *If $p > \frac{1}{\lambda}$, then*

$$\mathbb{P} \times \mathbb{P}'_p\,(K \cap K' \neq \emptyset) > 0.$$

Proof By Lemma 4.9 we have

$$\begin{aligned}
\hat{\mathbb{P}}_p(\hat{K}_n = \emptyset) &= \hat{\mathbb{P}}_p(\hat{K}_n(\hat{t}, \hat{C}) = \emptyset) \\
&= \hat{\mathbb{P}}_p(\hat{K}_n(\hat{t}, \hat{S}) = \emptyset) \\
&= \mathbb{P} \times \mathbb{P}'_p(K_n(t, C) \cap K'_n(t', C') = \emptyset) \\
&= \mathbb{P} \times \mathbb{P}'_p(K_n \cap K'_n = \emptyset)
\end{aligned}$$

for all $n = 0, 1, \ldots$. Since $\{\hat{K}_n = \emptyset\} \downarrow \{\hat{K} = \emptyset\}$ and similarly $\{K_n \cap K'_n = \emptyset\} \downarrow \{K \cap K' = \emptyset\}$, we have

$$\hat{\mathbb{P}}_p(\hat{K} = \emptyset) = \mathbb{P} \times \mathbb{P}'_p(K \cap K' = \emptyset).$$

If $p > \frac{1}{\lambda}$, then the Perron–Frobenius eigenvalue $\hat{\lambda}_p = p\lambda$ is strictly larger than 1. So by Theorem 4.1,

$$\hat{\mathbb{P}}_p(\hat{K} \neq \emptyset) = \mathbb{P} \times \mathbb{P}'_p(K \cap K' \neq \emptyset) > 0.$$

□

Our main theorem specifies the Hausdorff dimension of the limit set K of our BCA.

Theorem 4.4 *Let $K = K(t, C)$ be a set generated by a BCA with $t \in C$ and C a communicating class with $\lambda_C > 1$. Conditioned on non-extinction, we have*

$$\dim_{\mathrm{H}} K = \frac{\log \lambda}{\log M} \qquad \mathbb{P}\text{-}a.s.$$

Proof The easy part of the proof is showing that $\dim_{\mathrm{H}} K \leq \frac{\log \lambda}{\log M}$ a.s. Fix $\varepsilon > 0$. By Theorem 4.1 we can find a constant c such that

$$\mathbb{P}(Z_n \leq c\,\lambda^n \text{ for all } n) \geq 1 - \varepsilon.$$

Hence

$$\mathbb{P}(K_n \text{ can be covered with less than } c\,\lambda^n \; n^{\text{th}}\text{-level } M\text{-adic intervals, for all } n) > 1 - \varepsilon.$$

This implies that

$$\mathbb{P}(\mathcal{H}^\alpha(K) < \infty) > 1 - \varepsilon,$$

where $\alpha = \frac{\log \lambda}{\log M}$ and $\mathcal{H}^\alpha(K)$ is the α-dimensional Hausdorff measure of K. Since this holds for all ε, we conclude that $\dim_{\mathrm{H}} K \leq \frac{\log \lambda}{\log M}$ a.s.

To prove the converse, $\dim_{\mathrm{H}} K \geq \frac{\log \lambda}{\log M}$ a.s., we will use the previous lemma's. Let $\varepsilon > 0$ and $p = p(\varepsilon) = \frac{1}{\lambda} + \varepsilon$. By Lemma 4.14 we have $\mathbb{P} \times \mathbb{P}'_p\,(K \cap K' \neq \emptyset) > 0$. This implies by Fubini's theorem that the set

$$B = \{\omega : \mathbb{P}'_p(K(\omega) \cap K' \neq \emptyset) > 0\}$$

has positive $\mathbb{P}$ measure.

By Lemma 4.11, $\dim_{\mathrm{H}} K \geq -\frac{\log p}{\log M}$ with positive probability. Since conditioned on non-extinction $\dim_{\mathrm{H}} K$ is a constant a.s. (Lemma 4.10), we have in this case that $\dim_{\mathrm{H}} K \geq -\frac{\log p}{\log M}$ a.s. If we let $\varepsilon \to 0$, then $p(\varepsilon) \to \frac{1}{\lambda}$ and so we have $\dim_{\mathrm{H}} K \geq \frac{\log \lambda}{\log M}$ a.s., conditioned on non-extinction. □

Example 4.5 Consider fractal percolation with parameter p in dimension 2 (Example 4.2). Recall that $K_n := K_n(t, C)$, where

$$C = \{s \in T : s_{00} = 1\}$$

and t is the type with a 1 in the middle and 0's elsewhere. By Lemma 4.13, the largest eigenvalue of $\mathcal{M}_p$ is equal to $p\,M^2$. Hence by Theorem 4.4, we have that conditioned on non-extinction

$$\dim_{\mathrm{H}} K = 2 + \frac{\log p}{\log M} \qquad \text{a.s.}$$

Example 4.6 Consider the BCA described in Example 4.4, parametrised by p. Recall that $T = A^{3\times 3}$ and

$$\begin{aligned} S &= \{s \in T : s_{00} = 1\} \\ \partial S &= \{s \in T : s_{00} = 0, s \neq \bar{0}\}. \end{aligned}$$

Note that ∂S is a communicating class and that $K(S)$ and $K(\partial S)$ are non-empty almost surely. As a result of Theorem 4.4 we obtain that

$$\begin{aligned} \dim_{\mathrm{H}} K(S) &= 2 \\ \dim_{\mathrm{H}} \partial K(S) &= \frac{\log \lambda_p}{\log M}, \end{aligned}$$

where λ_p is the Perron–Frobenius eigenvalue associated with ∂S.

Chapter 5

Iterated Function Systems

5.1 Introduction

Recently, several authors have studied the geometry of self-similar tiles, in particular the nature of their (topological) boundaries ([6], [7], [15], [28], [29], [12]). Typically these authors consider compact sets C in $\mathbb{R}^d$ which satisfy the self-similarity equation

$$l(C) = \bigcup_{d \in D} (C + d) \tag{5.1}$$

for some expanding similarity l and some finite set D of points in $\mathbb{R}^d$. Moreover, it is assumed that l preserves a lattice $\mathcal{R}$ and that $D \subset \mathcal{R}$. Finally a tiling condition is added, namely that the interiors of $C + d$ and $C + e$ do not intersect for $d \neq e \in D$. One would not call C a tile if it has empty interior. A sufficient condition for C to have non-empty interior and to satisfy the tiling condition is that D is a *digit set*, i.e., a complete residue system for $\mathcal{R}/l(\mathcal{R})$.

Under this digit set condition, Duvall, Keesling and Vince ([7]) and Strichartz and Wang ([28]) show how to obtain the boundary ∂C of C as a union of compact sets $C_1, \ldots, C_r$ which satisfy the more general version of equation (5.1)

$$l(C_i) = \bigcup_{j=1}^{r} \bigcup_{d \in D_{ij}} (C_j + d) \qquad 1 \le i \le r \tag{5.2}$$

for some finite sets $D_{ij} \subseteq \mathcal{R}$, and where the tiling condition is generalized to requiring that the interiors of the sets $C_j + d$ and $C_k + e$ do not intersect for $d \in D_{ij}$, $e \in D_{ik}$ and $1 \le i, j, k \le r$ (except of course when $j = k$ and $d = e$).

The goal of the present work is to construct and to characterize the boundaries of self-similar sets C satisfying equation (5.1) and more generally of sets $C_1, \ldots, C_r$ satisfying equation (5.2), without the tiling condition. Our approach is to generate the boundary by iterates of a d-dimensional substitution. An advantage of this approach is that the iterates always converge to the boundary of the tile, since the d-dimensional substitutions can distinguish in some sense between boundary points and interior points of the limiting set (cf. Theorem 5.4). This shows that the 'well-behavedness' condition on the boundary which Duvall et al. ([7]) require in their analysis, is in some sense superfluous (cf. their Remark on p. 10).

Let $(C_1, \ldots, C_r)$ be a vector of non-empty compact sets in $\mathbb{R}^d$ that satisfy equations (5.2), where we only assume that $\bigcup_{j=1}^r D_{ij}$ is non-empty for each i. For reasons of simplicity, we will assume that l is just an integer scaling, i.e., $l(x) = Mx$, where $M \geq 2$ is an integer, and that the sets D_{ij} are subsets of $\mathbb{Z}^d$. However, the results in this chapter can be generalized to the case where l is an expansive similarity preserving a lattice $\mathcal{R}$ and $D_{ij} \subset \mathcal{R}$. We will call the pair $(M, (D_{ij})_{1\leq i,j\leq r})$ an M-recurrent IFS.

To analyze this M-recurrent IFS, we introduce another way to generate compact sets in $\mathbb{R}^d$, by means of (deterministic) branching cellular automata (BCA), which can also be viewed as d-dimensional substitutions. Stochastic BCA's were already explored in Chapter 4. A finite set T is associated with each BCA, a set which we will refer to as the set of types. For any subset S of T, a sequence $(K_n(S))_{n\geq 0}$ of closed and uniformly bounded subsets of $\mathbb{R}^d$ is defined. In Chapter 4 results were given concerning convergence of the sets $(K_n(S))_{n\geq 0}$ to a limit set $K(S)$ (w.r.t Hausdorff metric) and the Hausdorff dimension of $K(S)$. The boundary $\partial K(S)$ of $K(S)$ was shown to be the limit set of a sequence $(K_n(\partial S))_{n\geq 0}$, for specific subsets S and ∂S of T. In this chapter we re-formulate these results for deterministic BCA's.

We prove an equivalence result for M-recurrent IFS's and BCA's. Consider an M-recurrent IFS with attractor $(C_1, \ldots, C_r)$ and fix a set $C = C_i$ for some $1 \leq i \leq r$. Then we can find a BCA with set of types T and a set $S \subseteq T$ such that the associated sets $(K_n(S))_{n\geq 0}$ converge to C. On the other hand, consider a BCA with set of types T and let $S \subseteq T$ be such that the associated sets $(K_n(S))_{n\geq 0}$ converge to a non-empty limit set $K(S)$. Then we can find an M-recurrent IFS with attractor $(C_0, C_1, \ldots, C_r)$ such that $C_0 = K(S)$.

Our main results follow from the results about BCA's and the relation between recurrent IFS's and BCA's. Let C be a component of the attractor $(C_1, \ldots, C_r)$ of an M-recurrent IFS. Then we can find another M-recurrent IFS with attractor $(B_0, B_1, \ldots, B_l)$, such that the boundary of C is equal to B_0. In addition, we can calculate the Hausdorff dimension of C and of its boundary.

5.2 An Introductory Example

Consider a 2-dimensional tile C defined by

$$2C = C + D = \bigcup_{d\in D} (C + d),$$

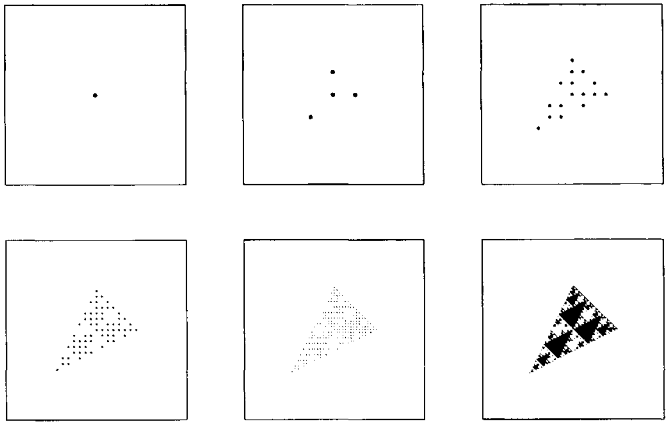


Figure 5.1: *From top left to bottom right the approximating sets* $C^{(0)}, \ldots, C^{(5)}$ *of the tile* C *generated by the* 2*-recurrent IFS in Section 5.2.*

where $2C = \{2x : x \in C\}$, and

$$D = \{(-1,-1), (0,0), (1,0), (0,1)\}.$$

Since D is a digit set for the lattice $\mathbb{Z}^2$, the dimension of the boundary of C can be calculated using the contact matrix as described in [7] and [28]. However, with this example we will demonstrate the use of branching cellular automata.

We approximate the tile C as follows. Let $C^{(0)} = \{(0,0)\}$ and recursively define sets $C^{(1)}, C^{(2)}, \ldots$ by

$$C^{(n+1)} = C^{(n)} + \frac{1}{2^{n+1}} D.$$

Then C is the Hausdorff limit of the sequence $C^{(0)}, C^{(1)}, \ldots$ (cf. Figure 5.1).

These approximating sets can also be obtained by means of a branching cellular automaton, i.e., a 2-dimensional substitution with neighbour dependence. Let $A = \{0,1\}$ be our alphabet and define a map $\sigma : A^{3\times 3} \to A^{2\times 2}$ by

$$\begin{array}{ccc} * & * & 0 \\ * & 0 & * \\ * & * & * \end{array} \to \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \qquad \begin{array}{ccc} * & * & 1 \\ * & 0 & * \\ * & * & * \end{array} \to \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}$$

$$\begin{array}{ccc} * & * & 0 \\ * & 1 & * \\ * & * & * \end{array} \to \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \qquad \begin{array}{ccc} * & * & 1 \\ * & 1 & * \\ * & * & * \end{array} \to \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}$$

Here a $*$ stands for either 0 or 1. We extend σ to a substitution $v \to \sigma(v) = w$ with $v, w \in A^{\mathbb{Z}\times\mathbb{Z}}$ in the obvious way (see Section 2.3). So for example

$$\sigma(v_{ij}) = \begin{matrix} w_{2i,2j+1} & w_{2i+1,2j+1} \\ w_{2i,2j} & w_{2i+1,2j} \end{matrix} = \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} \qquad \text{if } v_{ij} = 1 \text{ and } v_{i+1,j+1} = 0.$$

We take as starting word for the substitution the word u, which has a 1 at position $(0,0)$ and $0's$ elsewhere:

$$u = \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 1 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

Define sets $K_0, K_1, \ldots$ in the plane by

$$K_n = \frac{1}{2^n}\{(i,j) : (\sigma^n(u))_{ij} = 1\},$$

where σ^n is the n-fold iterate of σ. It is easy to check that the substitution σ is constructed in such a way, that $C^{(n)} = K_n$ for all n.

Define the *type* of a letter v_{ij} in a word $v \in A^{\mathbb{Z}\times\mathbb{Z}}$ to be the word

$$\begin{matrix} v_{i,j+1} & v_{i+1,j+1} \\ v_{ij} & v_{i+1,j} \end{matrix}$$

and let $T = A^{2\times 2}$ be the set of all possible types. Let $S = T \backslash \{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \}$. We define the 'boundary' ∂S of S by

$$\partial S = S \backslash \left\{ \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \right\},$$

and we define sets $K_0(\partial S), K_1(\partial S), \ldots$ by

$$K_n(\partial S) = \frac{1}{2^n}\{(i,j) : \text{ the type of } (\sigma^n(u))_{ij} \text{ is an element of } \partial S\}.$$

It turns out that the sets $K_n(\partial S)$ converge to a set $K(\partial S)$ (Theorem 5.1), which is the boundary of the tile C (Theorem 5.4). In Figure 5.2 we plotted $K_0(\partial S), \ldots, K_5(\partial S)$.

Note that the type of a letter v_{ij} in a word $v \in A^{\mathbb{Z}\times\mathbb{Z}}$ determines the types of the letters in $\sigma(v)$ that have been substituted for v_{ij}, i.e., of the four letters $(\sigma(v))_{2i,2j}$, $(\sigma(v))_{2i+1,2j}$, $(\sigma(v))_{2i,2j+1}$ and $(\sigma(v))_{2i+1,2j+1}$.

Let s be a type in T and let $v \in A^{\mathbb{Z}\times\mathbb{Z}}$ be such that the type of v_0 is s. Define the *offspring* matrix $\mathcal{M} = (m_{st})_{s,t\in T}$ by

$$m_{st} = \text{Card}\big(\{(i,j) : i,j \in \{0,1\}, \text{ the type of } (\sigma(v))_{ij} \text{ is } t\}\big).$$

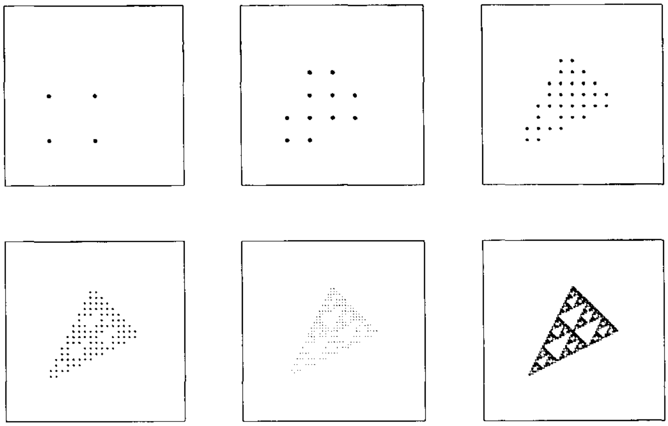


Figure 5.2: *From top left to bottom right the sets $K_0(\partial S), \ldots, K_5(\partial S)$ approximating the boundary ∂C of the tile C from Section 5.2.*

In this example, $\mathcal{M}$ is a 16×16 matrix. Let $\mathcal{M}_{\partial S}$ denote the matrix $\mathcal{M}$ restricted to ∂S (which is a 14×14 irreducible matrix), and let $\lambda_{\partial S}$ denote the largest eigenvalue of $\mathcal{M}_{\partial S}$. If we denote the boundary of C by ∂C, then by Theorem 5.3 we can express the dimension of ∂C in terms of $\lambda_{\partial S}$:

$$\dim_{\mathrm{H}} \partial C = \dim_{\mathrm{H}} K(\partial S) = \frac{\log \lambda_{\partial S}}{\log 2}.$$

A simple computation shows that $\lambda_{\partial S} = 3$, hence the dimension of the boundary of C equals $\log 3 / \log 2$.

5.3 Recurrent Iterated Function Systems

Let $m, r \geq 1$ be integers and $f_1, \ldots, f_m$ be contractions on $\mathbb{R}^d$. For $1 \leq i, j \leq r$ let Q_{ij} be a subset of $\{1, \ldots, m\}$ such that $\bigcup_{j=1}^{r} Q_{ij}$ is non-empty for each i. Then the pair

$$\left((f_k)_{1 \leq k \leq m}, (Q_{ij})_{1 \leq i,j \leq r}\right)$$

is called a *recurrent iterated function system.*

Let $\mathcal{H}$ be the set of compact subsets of $\mathbb{R}^d$ and let $\mathcal{H}_0$ be $\mathcal{H}$ without the empty set. On the space $\mathcal{H}^r$, define g by

$$g(A_1, \ldots, A_r) = \left(\bigcup_{j=1}^{r} \bigcup_{k \in Q_{1j}} f_k(A_j), \ldots\ldots, \bigcup_{j=1}^{r} \bigcup_{k \in Q_{rj}} f_k(A_j) \right).$$

In [1] it is shown that, restricted to $\mathcal{H}_0^r$, g has a unique fixed point $(C_1, \ldots, C_r)$, i.e.,

$$g(C_1, \ldots, C_r) = (C_1, \ldots, C_r),$$

and $(C_1, \ldots, C_r)$ is an attractor in $\mathcal{H}_0^r$, i.e., for all $(A_1, \ldots, A_r) \in \mathcal{H}_0^r$

$$\lim_{n\to\infty} g^n(A_1, \ldots, A_r) = (C_1, \ldots, C_r),$$

where the limit is with respect to the Hausdorff metric.

We shall always approximate the C_i by sets $C_i^{(n)}$ given by

$$\left(C_1^{(n)}, \ldots, C_r^{(n)}\right) = g^n\left(\{0^d\}, \ldots, \{0^d\}\right) \qquad \text{for } n \geq 0,$$

where 0^d denotes the origin in $\mathbb{R}^d$.

Consider the following special case of a recurrent IFS. For $1 \leq i, j \leq r$, let D_{ij} be finite subsets of $\mathbb{Z}^d$, such that for each i, $\bigcup_{j=1}^r D_{ij}$ is non-empty. Let $m = \text{Card}(\bigcup_{1\leq i,j\leq r} D_{ij})$, and let $b : \{1, \ldots, m\} \to \bigcup_{1\leq i,j\leq r} D_{ij}$ be a bijection. Define contractions $f_1, \ldots, f_m$ by

$$f_k(x) = M^{-1}(x + b(k)),$$

where $M \geq 2$ is an integer and define

$$Q_{ij} = \{k : b(k) \in D_{ij}\}$$

to obtain the recurrent IFS $\big((f_k), (Q_{ij})\big)$. The pair

$$(M, (D_{ij})_{1\leq i,j\leq r})$$

will be called an *M-recurrent IFS.*

For an M-recurrent IFS, the fixed point equations can be written as

$$C_i = \bigcup_{j=1}^r \bigcup_{k\in Q_{ij}} f_k(C_j) = \bigcup_{j=1}^r \bigcup_{k\in Q_{ij}} M^{-1}(C_j + b(k)) = \bigcup_{j=1}^r \bigcup_{e\in D_{ij}} M^{-1}(C_j + e),$$

for $1 \leq i \leq r$, which is equivalent to

$$MC_i = \bigcup_{j=1}^r (C_j + D_{ij}),$$

where $C_j + D_{ij}$ is short for $\{x + y : x \in C_j, y \in D_{ij}\}$. By convention, $C_j + D_{ij} = \emptyset$ when C_j or D_{ij} is empty.

To obtain a refinement of the sets $C_i^{(n)}$ (for an example see Figure 5.3 in Section 5.8), define sets $C_i^{(n)}(j)$ for $1 \le i, j \le r$ and $n \ge 0$ by

$$\begin{aligned}
((C_1^{(n)}(1), \ldots, C_r^{(n)}(1)) &= g^n(\{0^d\}, \emptyset, \ldots\ldots, \emptyset) \\
((C_1^{(n)}(2), \ldots, C_r^{(n)}(2)) &= g^n(\emptyset, \{0^d\}, \emptyset, \ldots, \emptyset) \\
\vdots \qquad & \quad \vdots \qquad \qquad \vdots \\
((C_1^{(n)}(r), \ldots, C_r^{(n)}(r)) &= g^n(\emptyset, \ldots\ldots, \emptyset, \{0^d\}).
\end{aligned}$$

It is easily proved by induction that for $n \ge 0$

$$C_i^{(n)} = \bigcup_{j=1}^{r} C_i^{(n)}(j), \qquad 1 \le i \le r.$$

For an M-recurrent IFS, the sets $C_i^{(n)}(j)$ satisfy a nice recursion relation.

Lemma 5.1 *For $i, j \in \{1, \ldots, r\}$ and $n = 0, 1, \ldots$ we have*

$$C_i^{(n+1)}(j) = \bigcup_{h=1}^{r} \left(C_i^{(n)}(h) + M^{-(n+1)} D_{hj} \right).$$

Proof The proof will be by induction. For $n = 0$ both sides of the equation equal $M^{-1}D_{ij}$. Assume that the lemma is proved for $n - 1$. Then, using the induction hypothesis in the third step,

$$\begin{aligned}
C_i^{(n+1)}(j) &= \bigcup_{l=1}^{r} \bigcup_{k \in Q_{il}} f_k(C_l^{(n)}(j)) \\
&= \bigcup_{l=1}^{r} M^{-1}(C_l^{(n)}(j) + D_{il}) \\
&= \bigcup_{l=1}^{r} M^{-1}\Big(\bigcup_{h=1}^{r} (C_l^{(n-1)}(h) + M^{-n} D_{hj}) + D_{il}\Big) \\
&= \bigcup_{h=1}^{r} \Big(\bigcup_{l=1}^{r} M^{-1}(C_l^{(n-1)}(h) + D_{il}) + M^{-(n+1)} D_{hj}\Big) \\
&= \bigcup_{h=1}^{r} \left(C_i^{(n)}(h) + M^{-(n+1)} D_{hj} \right).
\end{aligned}$$

□

5.4 Results for Deterministic BCA's

For notational convenience, the definitions in this section are for dimension 1, but they can be easily extended to higher dimensions. Consider a deterministic BCA (A, M, N, σ, u), i.e., A is a finite alphabet, $M \geq 2$ and $N \geq 0$ are integers, σ is a deterministic neighbour dependent substitution and $u \in A^{\mathbb{Z}}$ serves as the starting word of the substitution. For a definition of a deterministic neighbour dependent substitution, we refer the reader to Section 2.3. We assume that A contains a special element 0, that $\sigma(0^{2N+1}) = 0^M$ and that u consists of only finitely many non-zero letters. Let $T = A^{2R+1}$ be the set of types, where $R = 1 + \lceil\frac{MN}{M-1}\rceil$. This choice of R ensures that the BCA is non-lattice with respect to T (see Section 4.5). Fix a set $S \subseteq T$ which does not contain the type $\bar{0}$, the type consisting of only 0's. In this chapter we will define the associated sets $K_n(S)$ as a union of points, rather than a union of M-adic intervals. Define for $n \geq 0$

$$K_n(S) \quad = \quad M^{-n}\{k \in \mathbb{Z} : \text{ type of } (\sigma^n(u))_k \text{ is an element of } S\}.$$

Since u contains only finitely many non-zero's and $\sigma(0^{2N+1}) = 0^M$, the sets $K_n(S)$ are closed and uniformly bounded. The offspring matrix $\mathcal{M} = (m_{st})_{s,t \in T}$ is defined by

$$m_{st} = \text{Card}\big(J(s,t)\big),$$

where

$$J(s,t) \quad = \quad \{0 \leq k \leq M-1 : \text{ the type of } (\sigma(s))_k \text{ is } t\}.$$

For $S \subseteq T$ define

$$\begin{aligned} S^{\geq} &= \{t \in T : \text{ there is an } s \in S \text{ such that } t \to s\} \\ S^{\leq} &= \{t \in T : \text{ there is an } s \in S \text{ such that } s \to t\}. \end{aligned}$$

The following two results concerning the convergence of the sequence $(K_n(S))_{n \geq 0}$ follow from Theorem 4.2.

Theorem 5.1 *Let S be a subset of T. If $S = S^{\geq}$, $S = S^{\leq}$ or S is a communicating class, then $(K_n(S))_{n\geq 0}$ converges.*

A communicating class $U \subseteq T$ is called non-trivial if for all $s \in C$ there is an $n \geq 1$ such that $m^n_{ss} > 0$. Define for $t \in T$ and $S \subseteq T$

$$\begin{aligned} \mathcal{U} &= \{U \subseteq T : U \text{ is a non-trivial communicating class}\} \\ \mathcal{U}(t,S) &= \{U \in \mathcal{U} : \text{ there are } t' \in U \text{ and } t'' \in S \text{ such that } t \to t' \to t''\ \} \\ \mathcal{U}(S) &= \{U \subseteq T : U \in \mathcal{U}(t,S), \text{ where } t \text{ is the} \\ &\qquad \text{type of a letter in the starting word } u\}. \end{aligned}$$

Theorem 5.2 *Assume that $(K_n(S))_{n\geq 0}$ converges to $K(S)$. Then*

$$K(S) = \bigcup_{U \in \mathcal{U}(S)} K(U).$$

Fix a set $S \subseteq T$ such that $(K_n(S))_{n\geq 0}$ converges. For a communicating class $U \subseteq \mathcal{U}(S)$ let λ_U denote the Perron-Frobenius eigenvalue of U. If $\mathcal{U}(S) = \emptyset$, then $K(S) = \emptyset$. If $\mathcal{U}(S) \neq \emptyset$, define

$$\lambda_S = \max_{U \in \mathcal{U}(S)} \lambda_U.$$

The following theorem concerning the dimension of the limit set $K(S)$ follows from Theorem 4.4.

Theorem 5.3 *If $\mathcal{U}(S) \neq \emptyset$, then*

$$\dim_{\mathrm{H}} K(S) = \frac{\log \lambda_S}{\log M}.$$

The following result shows how to obtain the boundary of the limit set of a BCA by simply considering the appropriate subset of types. See Theorem 4.3 for a proof.

Theorem 5.4 *Let $S = \{t \in T : t \neq \bar{0}\}$ and let $\partial S = \{t \in T : t \neq \bar{0}, t \to \bar{0}\}$. Then*

$$\partial K(S) = K(\partial S),$$

where $\partial K(S)$ denotes the boundary of $K(S)$.

5.5 From M-Recurrent IFS to BCA

We consider the attractor $(C_1, \ldots, C_r)$ of a d-dimensional M-recurrent IFS $(M, (D_{ij})_{1\leq i,j\leq r})$, i.e., $(C_1, \ldots, C_r)$ is the unique non-empty solution of

$$MC_i = \bigcup_{j=1}^{r} (C_j + D_{ij}), \qquad 1 \leq i \leq r.$$

Fix an $i_0 \in \{1, \ldots, r\}$ and write C for C_{i_0}. In this section , we will construct a BCA $\mathbb{B}$ with alphabet A the set of all subsets of $\{1, \ldots, r\}$, with designated element $\emptyset$, and with set of types T such that $C = K(S)$, where $S = \{t \in T : t \neq \bar{\emptyset}\}$. Here, $\bar{\emptyset}$ denotes the type in T with all entries equal to $\emptyset$.

For $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, define

$$x \bmod M = (x_1 \bmod M, \ldots, x_d \bmod M).$$

Construction 5.1 Define a BCA as follows. Let the alphabet A be the set of all subsets of $\{1, \dots, r\}$ and let $\emptyset$ be the designated element. Choose the substitution length to be M. Define $D = \bigcup_{1 \le i,j \le r} D_{ij}$. Let the interaction length N be the smallest integer l which satisfies

$$e - e \bmod M \subseteq M\{-l, \dots, l\}^d, \qquad \text{for all } e \in D.$$

We will define a substitution σ in the following way. Let v be a word in the set $A^{(2N+1)\times\dots\times(2N+1)}$, indexed by $\{-N, \dots, N\}^d$. Define for $1 \le i, j \le r$

$$\begin{aligned} G_i &= \{x \in \{-N, \dots, N\}^d : i \in v_x\}, \\ H_j &= \bigcup_{i=1}^{r} (MG_i + D_{ij}). \end{aligned}$$

The word $\sigma(v)$ in $A^{M\times\dots\times M}$, indexed by $\{0, \dots, M-1\}^d$, is then defined by

$$(\sigma(v))_x = \{i \in \{1, \dots, r\} : x \in H_i\}, \qquad x \in \{0, \dots, M-1\}^d.$$

Let the starting word u be defined by $u_{0^d} = \{i_0\}$ and $u_x = \emptyset$ for $x \ne 0^d$. Denote the constructed BCA (A, M, N, σ, u) by $\mathbb{B}$.

Theorem 5.5 *Let $\mathbb{B}$ be the BCA constructed in Construction 5.1 and denote its set of types by T. Then $\mathbb{B}$ is a BCA with alphabet A the set of all subsets of $\{1, \dots, r\}$ and with designated element $\emptyset$ such that $C = K(S)$, where $S = \{t \in T : t \ne \bar{\emptyset}\}$.*

The theorem will be proved in two steps. First we show that $K(S) = K(\tilde{S})$, where

$$\tilde{S} = \{t \in T : t_{0^d} \ne \emptyset\},$$

t_{0^d} denoting the central letter of type t. Then we prove that $C = K(\tilde{S})$.

Lemma 5.2 *The sequences $(K_n(S))_{n\ge 0}$ and $(K_n(\tilde{S}))_{n\ge 0}$ converge to the same limit set.*

For $v \in A^{\mathbb{Z}^d}$, $x \in \mathbb{Z}^d$ and $r \in \mathbb{N}$ define $\beta_r^x(v)$ to be the $(2r+1) \times \dots \times (2r+1)$ sub-word of v centered around v_x.

Proof (cf. Lemma 4.8) By Theorem 5.1, $(K_n(S))_{n\ge 0}$ converges to $K(S)$. Since $\tilde{S} \subseteq S$, we have for all $n \ge 0$ that $K_n(\tilde{S}) \subseteq K_n(S)$. If $M^{-n}x$ is a point in $K_n(S)$, then $(\sigma^n(u))_x$ has a type s in S. Since $s \ne \bar{0}$, we can find a $y \in \beta_R^x(\sigma^n(u))$ such that $(\sigma^n(u))_y \ne 0$. So $M^{-n}y \in K_n(\tilde{S})$ and $\|x - y\| \le \sqrt{d}R$. Therefore,

$$K_n(S) \subseteq K_n(\tilde{S}) + \{x \in \mathbb{R}^d : \|x\| \le \sqrt{d}RM^{-n}\}.$$

It follows from the definition of the Hausdorff metric that $(K_n(\tilde{S}))_{n\ge 0}$ converges to $K(S)$. □

Define for $1 \leq j \leq r$

$$S_j = \{t \in T : j \in t_{0^d}\}.$$

Then $\tilde{S} = \bigcup_{j=1}^r S_j$ and $K_n(\tilde{S}) = \bigcup_{j=1}^r K_n(S_j)$ for $n = 0, 1, \ldots$.

Lemma 5.3 *Let $n \geq 0$ and $1 \leq j \leq r$. Then*

$$K_{n+1}(S_j) = \bigcup_{i=1}^{r} \left(K_n(S_i) + M^{-(n+1)} D_{ij}\right).$$

Proof To prove that $\bigcup_{i=1}^r \left(K_n(S_i) + M^{-(n+1)} D_{ij}\right)$ is contained in $K_{n+1}(S_j)$, let $M^{-(n+1)}x$ be an element of $K_n(S_i) + M^{-(n+1)} D_{ij}$ for some $1 \leq i \leq r$. Write

$$x = My + e \qquad \text{with } y \in M^n K_n(S_i) \text{ and } e \in D_{ij}.$$

Let

$$z = \tfrac{1}{M}(x - x \bmod M)$$

and note that, since $x \bmod M = My + e \bmod M = e \bmod M$,

$$\begin{aligned} y - z &= \tfrac{1}{M}(x - e) - \tfrac{1}{M}(x - x \bmod M) \\ &= \tfrac{1}{M}(x \bmod M - e) \\ &= \tfrac{1}{M}(e \bmod M - e) \end{aligned}$$

and thus $y - z \in \{-N, \ldots, N\}$ by our construction of N. Define v in the set $A^{(2N+1)\times\ldots\times(2N+1)}$ by

$$v = \beta_N^z(\sigma^n(u))$$

and index v by $\{-N, \ldots, N\}^d$. Since $y \in M^n K_n(S_i)$, we have

$$i \in (\sigma^n(u))_y = v_{y-z}.$$

Hence

$$y - z \in G_i = \{a \in \{-N, \ldots, N\}^d : i \in v_a\},$$

and so, since $x \bmod M = M(y - z) + e$

$$x \bmod M \in H_j = \bigcup_{h=1}^{r} (MG_h + D_{hj}).$$

Therefore, indexing $\sigma(v)$ by $\{0, \ldots, M-1\}^d$

$$j \in \{h : x \bmod M \in H_h\} = (\sigma(v))_{x \bmod M} = (\sigma^{n+1}(u))_x$$

and so $x \in K_{n+1}(S_j)$.

To prove that $K_{n+1}(S_j)$ is contained in $\bigcup_{i=1}^{r}\left(K_n(S_i) + M^{-(n+1)}D_{ij}\right)$, let the point $M^{-(n+1)}x$ be an element of $K_{n+1}(S_j)$. Again, let

$$z = \tfrac{1}{M}(x - x \bmod M)$$

and define $v \in A^{(2N+1)\times\ldots\times(2N+1)}$ by

$$v = \beta_N^z(\sigma^n(u)).$$

Then we have, indexing v by $\{-N,\ldots,N\}^d$ and $\sigma(v)$ by $\{0,\ldots,M-1\}^d$

$$j \in (\sigma^{n+1}(u))_x = (\sigma(v))_{x \bmod M} = \{i : x \bmod M \in H_i\},$$

hence $x \bmod M \in H_j$, where

$$H_j = \bigcup_{i=1}^{r}(MG_i + D_{ij}) \text{ and } G_i = \{a \in \{-N,\ldots,N\}^d : i \in v_a\}.$$

Hence we can find $i \in \{1,\ldots,r\}$, $a \in \{-N,\ldots,N\}^d$ with $i \in v_a$ and $e \in D_{ij}$ such that

$$x \bmod M = Ma + e.$$

So, defining $y = z + a$, we can write

$$x = Mz + x \bmod M = Mz + M(y-z) + e = My + e,$$

where $y \in M^n K_n(S_i)$, since $i \in v_a = v_{y-z} = (\sigma^n(u))_y$, and $e \in D_{ij}$. Hence

$$M^{-(n+1)}x \ \in \ \bigcup_{i=1}^{r}\left(K_n(S_i) + M^{-(n+1)}D_{ij}\right).$$

□

Proof (of Theorem 5.5) We will actually prove that $C^{(n)} = K_n(\tilde{S})$, where $C^{(n)}$ is the n^{th} approximation of C. Combined with Lemma 5.2 this yields the statement of the theorem. Recall that $C^{(n)} = \bigcup_{j=1}^{r} C^{(n)}(j)$ (by Lemma 5.1) and that $K_n(S) = \bigcup_{j=1}^{r} K_n(S_j)$. We will prove by induction that for all $1 \le j \le r$ and $n \ge 0$ $C^{(n)}(j) = K_n(S_j)$.

- Recall that the starting word u is such that $u_{0^d} = \{i_0\}$ and $u_x = \emptyset$ for all $x \neq 0^d \in \mathbb{Z}^d$. Hence

$$\begin{array}{rcccll} C^{(0)}(i_0) & = & \{0^d\} & = & K_0(S_{i_0}), & \\ C^{(0)}(j) & = & \emptyset & = & K_0(S_j) & \text{for } j \neq i_0. \end{array}$$

- Assume that for all $1 \leq j \leq r$ we have $C^{(n)}(j) = K_n(S_j)$. If we fix $1 \leq j \leq r$, then

$$\begin{aligned} C^{(n+1)}(j) &= \bigcup_{i=1}^{r} \left(C^{(n)}(i) + M^{-(n+1)} D_{ij}\right) \\ &= \bigcup_{i=1}^{r} \left(K_n(S_i) + M^{-(n+1)} D_{ij}\right) \\ &= K_{n+1}(S_j), \end{aligned}$$

where the first equality follows from Lemma 5.1, the second equality from the induction hypothesis, and the third from Lemma 5.3.

□

5.6 From BCA to M-Recurrent IFS

Consider a BCA (A, M, N, σ, u) with set of types T and offspring matrix $\mathcal{M} = (m_{st})_{s,t \in T}$. Let S be a subset of T with $\bar{0} \notin S$ such that the sequence $(K_n(S))_{n \geq 0}$ converges to a non-empty limit set $K(S)$. In this section, we will construct an M-recurrent IFS $\mathbb{I}(S)$ with attractor $(C_0, \ldots, C_r)$, such that $K(S) = C_0$. The recurrent IFS $\mathbb{I}(S)$ will not be constructed from S directly, but from the subset $\bar{S}$ of T defined by

$$\bar{S} = \{t \in T : \text{ there is an } s \in S \text{ such that } m_{ts}^k > 0 \text{ for infinitely many } k\},$$

where $m_{ts}^k = (\mathcal{M}^k)_{ts}$.

Lemma 5.4 *Let S be a subset of T. Then the following holds.*

1. *For all $s \in \bar{S}$ there is a $t \in \bar{S}$ such that $m_{st} > 0$.*
2. *If $s \in \bar{S}$ and $t \in T$ are such that $t \to s$, then $t \in \bar{S}$.*

Proof 1. If $s \in \bar{S}$, then there is an $s' \in S$ such that $m_{ss'}^k > 0$ for infinitely many k. Hence there is a $t \in T$ such that $m_{st} > 0$ and $m_{ts'}^{k-1} > 0$ for infinitely many k, which implies that $t \in \bar{S}$.

2. If $s \in \bar{S}$, then there is an $s' \in S$ such that $m_{ss'}^k > 0$ for infinitely many k. If $t \to s$, then there is an l such that $m_{ts}^l > 0$. Hence $m_{ts'}^{l+k} > 0$ for infinitely many k. Hence $t \in \bar{S}$.

□

Lemma 5.5 *Let S be a subset of T not containing type $\bar{0}$. Then $(K_n(\bar{S}))_{n \geq 0}$ converges to a limit set $K(\bar{S})$. Moreover, if $(K_n(S))_{n \geq 0}$ converges to a limit set $K(S)$, then $K(S) = K(\bar{S})$.*

Proof By Lemma 5.4 part ii) it follows that $\bar{S} = \bar{S}^{\geq}$. Hence by Theorem 5.1, the sequence $(K_n(\bar{S}))_{n\geq 0}$ converges. Assume that $(K_n(S))_{n\geq 0}$ converges to a limit set $K(S)$. By Theorem 5.2, $K(S) = \bigcup_{U\in\mathcal{U}(S)} K(U)$. We claim that $\mathcal{U}(S) = \mathcal{U}(\bar{S})$. If $U \in \mathcal{U}(\bar{S})$, then U is also an element of $\mathcal{U}(S)$, since $\bar{S}^{\geq} = \bar{S}$ is obviously contained in $S^{\geq}$.

On the other hand, let $U \in \mathcal{U}(S)$ be a non-trivial communicating class and let $s \in U$. Then $m_{ss}^k > 0$ for infinitely many k. Since $s \in S^{\geq}$, there is an $l \geq 0$ and there is a $t \in S$ such that $m_{st}^l > 0$. Therefore $m_{st}^{l+k} > 0$ for infinitely many k, and so $s \in \bar{S}$. Since $s \in \bar{S} = \bar{S}^{\geq}$ for every $s \in U$, it follows that $U \subseteq \bar{S}^{\geq}$. Hence $\mathcal{U}(S) = \mathcal{U}(\bar{S})$ and thus, applying Theorem 5.2 twice

$$K(S) = \bigcup_{U\in\mathcal{U}(S)} K(U) = \bigcup_{U\in\mathcal{U}(\bar{S})} K(U) = K(\bar{S}).$$

□

Construction 5.2 Let S be a subset of T with $\bar{0} \notin S$ such that the sequence $(K_n(S))_{n\geq 0}$ converges to a non-empty limit set $K(S)$. Write

$$\bar{S} = \{s_1, \ldots, s_r\},$$

where r is the cardinality of $\bar{S}$. Define for $s, t \in T$

$$J(s,t) = \{e \in \{0, \ldots, M-1\}^d : \text{ the type of } (\sigma(w))_e \text{ is } t\},$$

where $w \in A^{\mathbb{Z}^d}$ is such that the type of w_{0^d} is s. Define $D_{00} = \emptyset$, and let D_{i0}, D_{0j} and D_{ij} for $1 \leq i, j \leq r$ be defined as follows:

$$D_{i0} = \emptyset, \quad D_{0j} = \{x \in \mathbb{Z}^d : \text{ the type of } (\sigma(u))_x \text{ is } s_j\}, \quad D_{ij} = J(s_i, s_j).$$

Since by Lemma 5.5 and Theorem 5.2

$$\emptyset \neq K(S) = K(\bar{S}) = \bigcup_{U\in\mathcal{U}(\bar{S})} K(U),$$

we have that $\mathcal{U}(\bar{S}) \neq \emptyset$. Since by Lemma 5.4 part ii) $\bar{S}^{\geq} = \bar{S}$, it follows that the starting word u contains a letter of a type in $\bar{S}$. Hence $\bigcup_{j=0}^{r} D_{0j}$ is non-empty by Lemma 5.4 part i). Since $\bar{0} \notin S$, we have $\bar{0} \notin \bar{S}$, so $\bigcup_{j=0}^{r} D_{0j}$ is finite. The sets $\bigcup_{j=0}^{r} D_{ij}$ are also non-empty for $i = 1, \ldots, r$, again by Lemma 5.4 part i). Denote the constructed M-recurrent IFS $(M, (D_{ij})_{0\leq i,j\leq r})$ by $\mathbb{I}(S)$.

Theorem 5.6 *Let S be a subset of T with $\bar{0} \notin S$ such that the sequence $(K_n(S))_{n\geq 0}$ converges to a non-empty limit set $K(S)$. Let $\mathbb{I}(S)$ be the M-recurrent IFS constructed in Construction 5.2 and denote its attractor by $(C_0, \ldots, C_r)$. Then*

$$K(S) = C_0.$$

Let $S_j = \{s_j\}$ for $1 \leq j \leq r$ be the singleton subsets of $\bar{S} = \{s_1, \dots, s_r\}$. Then $\bar{S} = \bigcup_{j=1}^r S_j$ and $K_n(\bar{S}) = \bigcup_{j=1}^r K_n(S_j)$ for $n = 0, 1, \dots$.

Lemma 5.6 *Let $n \geq 0$ and $1 \leq j \leq r$. Then*

$$K_{n+1}(S_j) = \bigcup_{i=1}^{r} \left(K_n(S_i) + M^{-(n+1)} D_{ij}\right).$$

Proof To prove that $\bigcup_{i=1}^r \left(K_n(S_i) + M^{-(n+1)} D_{ij}\right)$ is contained in $K_{n+1}(S_j)$, let $M^{-(n+1)}x$ be an element of $K_n(S_i) + M^{-(n+1)}D_{ij}$ for some $1 \leq i \leq r$. Write

$$x = My + e \qquad \text{with } y \in M^n K_n(S_i) \text{ and } e \in D_{ij}.$$

So $e \in \{0, \dots, M-1\}^d$ and if $w \in A^{\mathbb{Z}^d}$ is such that the type of w_{0^d} is s_i, then the type of $(\sigma(w))_e$ is s_j. Since $y \in M^n K_n(S_i)$, we have that the type of $(\sigma^n(u))_y$ is s_i and therefore the type of $(\sigma^{n+1}(u))_{My+e}$ is s_j. Since $x = My + e$, it follows that $M^{-(n+1)}x \in K_{n+1}(S_j)$.

To prove that $K_{n+1}(S_j)$ is contained in $\bigcup_{i=1}^r \left(K_n(S_i) + M^{-(n+1)}D_{ij}\right)$, let the point $M^{-(n+1)}x$ be an element of $K_{n+1}(S_j)$. Let $f = x \bmod M$ and $z = \frac{1}{M}(x - f)$, i.e.

$$x = Mz + f \qquad \text{with } z \in \mathbb{Z}^d \text{ and } f \in \{0, \dots, M-1\}^d.$$

It follows from Lemma 5.4 part ii) that the type of $(\sigma^n(u))_z$ is s_i for some $1 \leq i \leq r$. This implies that $f \in D_{ij}$. Hence

$$M^{-(n+1)}x = M^{-n}z + M^{-(n+1)}f \in \bigcup_{i=1}^{r} \left(K_n(S_i) + M^{-(n+1)}D_{ij}\right).$$

□

Proof (of Theorem 5.6) We will actually prove that $K_n(\bar{S}) = C^{(n)}$ for all $n \geq 1$, where $C^{(n)} = C_0^{(n)}$ is the n^{th} approximation of C_0. Combined with Lemma 5.5 this yields the statement of the theorem. Recall that $C^{(n)} = \bigcup_{j=0}^r C^{(n)}(j)$ (by Lemma 5.1) and that $K_n(\bar{S}) = \bigcup_{j=1}^r K_n(S_j)$. We will prove by induction that for all $n \geq 1$ $C^{(n)}(0) = \emptyset$ and $C^{(n)}(j) = K_n(S_j)$ for all $1 \leq j \leq r$.

- We have

$$C^{(1)}(0) = \bigcup_{i=0}^{r} \left(C^{(0)}(i) + M^{-1}D_{i0}\right) = C^{(0)}(0) + M^{-1}D_{00} = \emptyset,$$

and for $1 \leq j \leq r$

$$\begin{aligned} C^{(1)}(j) &= \bigcup_{i=0}^{r} \left(C^{(0)}(i) + M^{-1}D_{ij}\right) \\ &= C^{(0)}(0) + M^{-1}D_{0j} \\ &= M^{-1}\{x \in \mathbb{Z}^d : \text{ the type of } (\sigma(u))_x \text{ is } s_j\} \\ &= K_1(S_j). \end{aligned}$$

- Assume that for all $1 \leq j \leq r$ we have $C^{(n)}(0) = \emptyset$ and $C^{(n)}(j) = K_n(S_j)$. Then

$$C^{(n+1)}(0) = \bigcup_{i=0}^{r} \left(C^{(n)}(i) + M^{-(n+1)}D_{i0}\right) = \emptyset,$$

and if we fix $1 \leq j \leq r$

$$\begin{aligned} C^{(n+1)}(j) &= \bigcup_{i=0}^{r} \left(C^{(n)}(i) + M^{-(n+1)}D_{ij}\right) \\ &= \bigcup_{i=1}^{r} \left(K_n(S_i) + M^{-(n+1)}D_{ij}\right) \\ &= K_{n+1}(S_j), \end{aligned}$$

where the second and the third equality follow from the induction hypothesis, and the fourth from Lemma 5.6.

□

5.7 The Boundary of an IFS is an IFS

Consider a d-dimensional M-recurrent IFS $\mathbb{I} = (M, (D_{ij})_{1\leq i,j \leq r})$ with attractor $(C_1, \ldots, C_r)$. Fix an $i_0 \in \{1, \ldots, r\}$ and write C for C_{i_0}. In this section we will construct another M-recurrent IFS $\mathbb{J}$ with attractor $(B_0, \ldots, B_l)$ such that B_0 is the boundary of C. We will also determine the Hausdorff dimension of C and its boundary.

Let $\mathbb{B}$ be the BCA we obtain from $\mathbb{I}$ by Construction 5.1. Then the alphabet A of $\mathbb{B}$ is the set of all subsets of $\{1, \ldots, r\}$ and the designated element is $\emptyset$. Denoting the set of types by T, define

$$S = \{t \in T : t \neq \bar{\emptyset}\} \quad \text{and} \quad \partial S = \{t \in T : t \to \bar{\emptyset}, t \neq \bar{\emptyset}\}.$$

Define $\mathbb{J}$ to be the M-recurrent IFS $\mathbb{I}(\partial S)$, obtained from the BCA $\mathbb{B}$ and the set ∂S by Construction 5.2.

Theorem 5.7 *Denote the attractor of* $\mathbb{J}$ *by* $(B_0, \ldots, B_l)$. *Then*

$$\partial C = B_0.$$

Proof By Theorem 5.5, $C = K(S)$ and by Theorem 5.4, $\partial K(S) = K(\partial S)$. Since $K(\partial S)$ is non-empty, $K(\partial S) = B_0$ by Theorem 5.6, and hence the theorem follows. □

Recall that λ_S is the maximum over the Perron-Frobenius eigenvalues of the communicating classes in $\mathcal{U}(S)$.

Theorem 5.8 *We have*

$$\dim_{\mathrm{H}} C = \frac{\log \lambda_S}{\log M}, \quad \textit{and} \quad \dim_{\mathrm{H}} \partial C = \frac{\log \lambda_{\partial S}}{\log M}.$$

Proof Since $C = K(S)$ by Theorem 5.5, and $\partial C = \partial K(S) = K(\partial S)$ by Theorem 5.4, the statement of the theorem follows directly from Theorem 5.3. □

We remark that we have determined the dimensions of the sets without requiring the open set condition. However, it is not hard to see that our class of M-recurrent IFS's satisfies the weak separation property from Lau and Ngai ([14], see also [31]). See [26] for other work on overlapping IFS's.

5.8 Examples

Example 5.1 In this example we present a 2-dimensional 2-recurrent IFS having small sets D_{ij}, which enables us to easily follow the evolution of the approximating sets during the first five steps. Let the 2-recurrent IFS be defined by

$$D_{11} = \{(1,0)\}, \quad D_{12} = \{(-1,0)\},$$
$$D_{22} = \{(0,1)\}, \quad D_{21} = \{(0,-1)\}.$$

Let (C_1, C_2) be the attractor, i.e. the vector satisfying

$$\begin{aligned} 2C_1 &= (C_1 + (1,0)) \cup (C_2 + (-1,0)) \\ 2C_2 &= (C_1 + (0,-1)) \cup (C_2 + (0,1)). \end{aligned}$$

We choose $i_0 = 1$, i.e., $C = C_1$. See Figure 5.3 for $C^{(n)}(1)$ and $C^{(n)}(2)$, for $n = 0, \ldots, 5$.

Example 5.2 In this example we consider a 2-dimensional 2-recurrent IFS such that the attractor sets do not tile $\mathbb{R}^2$, in particular the open set condition is not satisfied. Let the 2-recurrent IFS be defined by $r = 2$ and

$$D_{11} = \{(0,0)\}, \quad D_{12} = \{(-1,-1), (-1,0), (0,1), (1,0)\},$$
$$D_{22} = \{(0,0)\}, \quad D_{21} = \{(-1,0), (0,-1), (1,0), (1,1)\}.$$

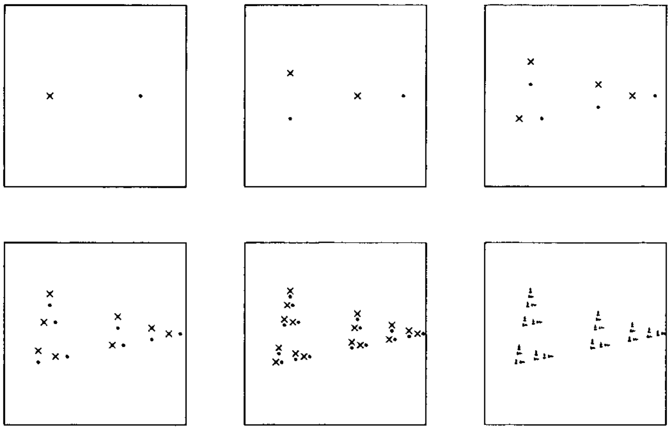


Figure 5.3: *From top left to bottom right the sets* $C_1^{(1)}, \ldots, C_1^{(5)}$ *and* $C_1^{(10)}$ *approximating the first component* C_1 *of the attractor generated by the 2-recurrent IFS in Example 5.1. The points of the sets* $C_1^{(1)}(1), \ldots, C_1^{(5)}(1)$ *are marked by* $\cdot$ *and the points of the sets* $C_1^{(1)}(2), \ldots, C_1^{(5)}(2)$ *are marked by* $\times$.

Let (C_1, C_2) denote its attractor. In Figure 5.4 the first six approximating sets for C_1 are shown. We will calculate the dimension of the boundary of C_1. By means of construction 5.1, we obtain a BCA with

$$A = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}, \qquad M = 2, \qquad N = 1,$$

and the starting word $u \in A^{\mathbb{Z}^2}$ is such that $u_{(0,0)} = \{1\}$ and $u_x = \emptyset$ for $x \neq (0,0)$. Hence $R = 1 + \lceil \frac{MN}{M-1} \rceil = 3$ and $T = A^{7\times 7}$.

We will denote $\bar{\emptyset}$, the type in T with all entries equal to $\emptyset$, by $\emptyset^{7\times7}$. If

$$S = \{t \in T : t \neq \emptyset^{7\times7}\} \quad \text{and} \quad \partial S = \{t \in T : t \to \emptyset^{7\times7}, t \neq \emptyset^{7\times7}\},$$

then by Theorem 5.8,

$$\dim_{\mathrm{H}} \partial C_1 = \dim_{\mathrm{H}} K(\partial S) = \frac{\log \lambda}{\log 2},$$

where $\lambda = \max(\lambda_U : U \in \mathcal{U}(\partial S))$.

Since $\mathrm{Card}(T) = 4^{49}$, we will first project the set of types on the smaller set $\tilde{T} = A^{2\times2}$, which consists of only 256 elements. Let π be the projection

$$\pi : T \to \tilde{T}$$

$$\begin{matrix} t_{(-3,3)} & \cdots & t_{(3,3)} \\ \vdots & & \vdots \\ t_{(-3,-3)} & \cdots & t_{(3,-3)} \end{matrix} \quad \mapsto \quad \begin{matrix} t_{(0,1)} & t_{(1,1)} \\ t_{(0,0)} & t_{(1,0)} \end{matrix}$$

Let $\mathcal{M} = (m_{st})_{s,t\in T}$ denote the offspring matrix and fix $\tilde{s}, \tilde{t} \in \tilde{T}$. Then it can be checked that

$$\sum_{t:\pi(t)=\tilde{t}} m_{st}$$

does not depend on s for all $s \in T$ with $\pi(s) = \tilde{s}$. Define a matrix $\tilde{\mathcal{M}} = (\tilde{m}_{\tilde{s}\tilde{t}})_{\tilde{s},\tilde{t}\in\tilde{T}}$ by

$$\tilde{m}_{\tilde{s}\tilde{t}} = \sum_{t:\pi(t)=\tilde{t}} m_{st},$$

where $s \in T$ is such that $\pi(s) = \tilde{s}$. Define a set $Z \subseteq \partial S$ by

$$Z = \{t \in T : t \to \emptyset^{7\times 7}, \pi(t) \neq \emptyset^{2\times 2}\},$$

where $\emptyset^{2\times 2}$ is short for

$$\begin{matrix} \emptyset & \emptyset \\ \emptyset & \emptyset \end{matrix} \in A^{2\times 2}.$$

Analogously to Lemma 5.2, one can prove that $K(Z) = K(\partial S)$. Hence $\lambda = \max(\lambda_U : U \in \mathcal{U}(Z))$. Let

$$\tilde{S} = \{\tilde{t} \in \tilde{T} : \tilde{t} \neq \emptyset^{2\times 2}\} \quad \text{and} \quad \partial\tilde{S} = \{\tilde{t} \in \tilde{T} : \tilde{t} \to \emptyset^{2\times 2}, \tilde{t} \neq \emptyset^{2\times 2}\},$$

then since $s \to t$ for some $s, t \in T$ implies $\pi(s) \to \pi(t)$, we have that $\partial\tilde{S} = \pi(Z)$.

We claim that $\tilde{\lambda} = \lambda$. To see this, let $s \in T$ be a type such that $U(s)$, the communicating class containing s, is an element of $\mathcal{U}(Z)$. Let $\lambda_{\pi(U(s))}$ be the Perron-Frobenius eigenvalue of $\tilde{\mathcal{M}}$ restricted to $\pi(U(s))$. Then by a lemma of Furstenberg 4.13, $\lambda_{U(s)} = \lambda_{\pi(U(s))}$. Since $\pi(U(s)) \subseteq U(\pi(s)) \in \mathcal{U}(\partial\tilde{S})$, we have

$$\lambda_{U(s)} = \lambda_{\pi(U(s))} \leq \lambda_{U(\pi(s))}$$

and hence $\lambda \leq \tilde{\lambda}$. If we consider a non-trivial communicating class $\tilde{U} \in \mathcal{U}(\partial\tilde{S})$, then we can find a communicating class $U \in \mathcal{U}(Z)$ such that $\pi(U) = \tilde{U}$. Again by Furstenberg's Lemma, $\lambda_{\tilde{U}} = \lambda_U$ and hence $\tilde{\lambda} \leq \lambda$. We may conclude that $\tilde{\lambda} = \lambda$.

The reduced set of types $\tilde{T}$ has only 256 states. Constructing and analyzing $\tilde{\mathcal{M}}$ with a computer package yields the following. Let $V \subseteq T$ denote the set of types appearing in the starting word u and define $\tilde{V} = \pi(V)$. The set $\tilde{V}^{\leq}$

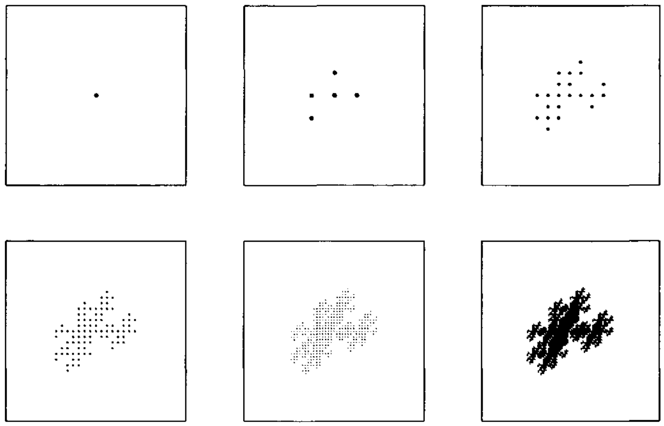


Figure 5.4: *From top left to bottom right the sets* $C_1^{(0)}, \ldots, C_1^{(5)}$ *approximating the first component* C_1 *of the attractor generated by the* 2*-recurrent IFS in Example 5.2.*

contains 110 elements and 5 non-trivial aperiodic classes $\tilde{U}_1, \ldots, \tilde{U}_5$, where

$$\begin{aligned}
\tilde{U}_1 &= \left\{ \begin{array}{ll} \emptyset & \emptyset \\ \emptyset & \emptyset \end{array} \right\}, \\
\tilde{U}_2 &= \left\{ \begin{array}{ll} \{1,2\} & \{1,2\} \\ \{1,2\} & \{1,2\} \end{array} \right\}, \\
\tilde{U}_3 &= \left\{ \begin{array}{ll} \{2\} & \{1,2\} \\ \{1,2\} & \{1,2\} \end{array} \right\}, \\
\tilde{U}_4 &= \left\{ \begin{array}{ll} \{1,2\} & \{1,2\} \\ \{1,2\} & \{1\} \end{array} \right\}, \\
\tilde{U}_5 &= \tilde{V}^{\leq} \setminus (\tilde{U}_1 \cup \tilde{U}_2 \cup \tilde{U}_3 \cup \tilde{U}_4).
\end{aligned}$$

We find that $\mathcal{U}(\partial\tilde{S}) = \{\tilde{U}_5\}$ and $\lambda = \tilde{\lambda} = \lambda_{\tilde{U}_5} = 3.0491\ldots$. Hence

$$\dim_{\mathrm{H}} \partial C_1 = \frac{\log \lambda}{\log 2} = 1.6084\ldots.$$

Chapter 6

Random and Multi-valued Substitutions

6.1 Introduction

In [17], Mandelbrot introduced fractal percolation as a model for turbulence. In dimension 2 the model can be described as follows. Choose an integer $M \geq 2$ and a parameter $0 \leq p \leq 1$. On $\{0,1\}^*$, the set of all finite two dimensional words of 0's and 1's, let σ be a random substitution which substitutes a 0 by an $M \times M$ block of 0's and a 1 by an $M \times M$ block of independent random letters being 1 with probability p and 0 with probability $1-p$. Writing σ^n for the composition of n independent copies of σ, we obtain a sequence of 2-dimensional random words $\sigma(1), \sigma^2(1), \ldots$. See Figure 6.1 for a realisation of fractal percolation with $M = 3$ and $p = 0.7$.

We say that a two dimensional word percolates if there is a path of 1's from the left to the right side of the word. For example, the word

$$\begin{matrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{matrix}$$

percolates. If we consider $\theta_M(p) = \mathbb{P}_p(\sigma^n(1)$ percolates for all $n)$ as a function of the parameter p, then it can be shown that θ_M is an increasing function with $\theta_M(0) = 0$ and $\theta_M(1) = 1$. The critical value of fractal percolation $p_c(M)$ is defined as

$$p_c(M) = \inf\{0 \leq p \leq 1 : \theta_M(p) > 0\}.$$

For all values of $M \geq 2$ the critical value is unknown, but various bounds have been given. It is easy to see that for 2-dimensional fractal percolation the

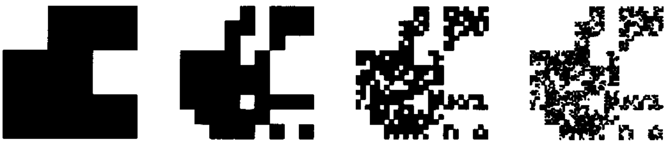


Figure 6.1: Fractal percolation with $M = 3$ and $p = 0.7$. From left to right realisations of the words $\sigma^1(1), \ldots, \sigma^4(1)$, where 1's have been replaced by black squares and 0's by white squares.

critical value is at least $\frac{1}{M^2}$ by observing that the sequence $(Z_n)_{n=0}^{\infty}$, where Z_n is the number of 1's in $\sigma^n(1)$, is an ordinary branching process. The first upper bound and non-trivial lower bound are due to Chayes, Chayes and Durrett [4]. In their proof, they construct functions $\pi_M(p)$ and $\rho_M(p)$ such that $\pi_M(p) \leq \theta_M(p)$ for $M \geq 2$ and $\rho_M(p) \geq \theta_M(p)$ for $M \geq 3$. The critical values of π_M and ρ_M can be computed and since the critical value of π_M is always larger than $p_c(M)$ and the critical value of ρ_M always smaller, this leads to upper and lower bounds for $p_c(M)$. They prove that $p_c(M) \geq \frac{1}{\sqrt{M}}$ for $M \geq 2$ and that $p_c(M) \leq p^*(M)$ for $M \geq 3$, where $p^*(M)$ is the infimum over p for which $x = (px)^{M^2} + (px)^{M^2-1}(1-px)$ has a root in the half open interval $(0, 1]$. We refer the reader to Section 3.4 for a brief outline of the method to obtain the upper bound.

Dekking and Meester [5] reinterpreted the Chayes, Chayes and Durrett proof for the upper bound in terms of multi-valued substitutions. A multi-valued substitution is a substitution on sets of words (see Section 6.2 of this chapter for a definition). They construct multi-valued substitutions Φ_M (see Example 6.2) such that the function π_M can be written as $\pi_M(p) = \mathbb{P}_p(\sigma^n(1) \in \Phi^n(1) \text{ for all } n)$, where $\Phi(1)$ denotes $\Phi(\{1\})$. Dekking and Meester observed that for any multi-valued substitution Φ on the alphabet $\{0, 1\}$, the probabilities $\pi_n(p) = \mathbb{P}_p(\sigma^n(1) \in \Phi^n(1))$ satisfy a recursion relation that can be written as $\pi_{n+1}(p) = \pi_1(p\pi_n(p))$. To analyze these probabilities it therefore suffices to study the recursion map $F_p(x) = \pi_1(px)$.

Since the paper by Chayes, Chayes and Durrett, various other bounds have been published, obtained by different techniques: $p_c(3) \leq 0.991$ [5], $p_c(3) \geq 0.634$ and recently White proved $p_c(2) \geq 0.810$ [30].

In this chapter we generalize the ideas of Dekking and Meester to obtain upper and lower bounds on the critical value of fractal percolation using multi-valued substitutions. For random substitutions σ and multi-valued substitutions Φ on a general alphabet A we analyze probabilities $\mathbb{P}(\sigma^n(i) \in \Phi^n(j))$, $i, j \in A$. These probabilities can be used to give upper and lower bounds on $p_c(M)$.

Another context that can be translated in terms of random and multi-valued substitutions is the context of broadcasting on trees [8]. Consider an alphabet A, a Markov matrix P and an M-ary tree. We uniformly pick a letter i from the alphabet and place it at the root of the tree. This letter is transmitted to each of the M children of the root. However, due to noise on the channel from the root to a child, the letter switches from an i to a j with probability P_{ij}, independently for each child. Similarly, the M letters received by the children of the root are transmitted to their children, and so on. The reconstruction problem is to find back the letter at the root with a probability bounded away from $\frac{1}{|A|}$ if only the letters at the nodes of the n^{th} generation of the tree are given, when n tends to infinity. If this is possible than the reconstruction problem is said to be solvable.

For ease of computation and robustness, biologists often prefer reconstructing the letter at the root from the n^{th} generation by first reconstructing each letter in the $(n-1)^{\text{th}}$ generation based on its M children, then each letter in the $(n-2)^{\text{th}}$ generation and so on, until the bit at the root is reconstructed. If it is possible to solve the reconstruction problem using this recursive approach, then the reconstruction problem is said to be recursively solvable.

In Section 6.3, we describe how to translate the recursive reconstruction problem in terms of random and multi-valued substitutions that can be analyzed with the techniques presented in this chapter.

6.2 Random and Multi-valued Substitutions

For ease of notation, our definitions for random and multi-valued substitutions will be for dimension 1, but they can be easily generalized to higher dimensions.

6.2.1 Random Substitutions

Let A be a finite set called the alphabet and denote by A^* the set of all finite words of letters in A. Let $(\sigma_k)_{k\in\mathbb{N}}$ be a sequence of independent identically distributed random maps from A to A^*. Define a random map σ on A^* by

$$\sigma(u) = \sigma_0(u_0)\dots\sigma_k(u_k)$$

for $u = u_0 \dots u_k \in A^*$. The random map σ on A^* is called a random substitution. We define the n-fold iterate σ^n to be the composition of n independent copies of the substitution σ.

In this chapter, we will only consider a special type of random substitutions. Let $M \geq 2$ be an integer and P be a Markov matrix indexed by $A \times A$, i.e., all entries are non-negative and the rows sum up to 1. Let σ be a random substitution such that for all $i \in A$,

$$\sigma(i) = (\sigma(i))_0 \dots (\sigma(i))_{M-1}$$

is a random word in A^M and the $(\sigma(i))_k$ are independent random letters in A with $\mathbb{P}_P((\sigma(i))_k = j) = P_{ij}$ for $j \in A$.

For $A = \{0,1\}$ and $P = \begin{pmatrix} 1 & 0 \\ 1-p & p \end{pmatrix}$ we have fractal percolation with parameter p.

6.2.2 Multi-valued substitutions

Let $\mathcal{A}^*$ be the set of all finite subsets of A^* and consider two binary operations on $\mathcal{A}^*$:

$$\begin{aligned} V \cup W &= \{u : u \in V \text{ or } u \in W\} \qquad \text{(union)} \\ VW &= \{vw : v \in V \text{ and } w \in W\} \qquad \text{(concatenation)}. \end{aligned}$$

A *multi-valued* substitution is a homomorphism on $\mathcal{A}^*$ respecting unions and concatenations. Since $\mathcal{A}^*$ is generated by the singletons, i.e., the sets containing one letter, a multi-valued substitution Φ is completely determined by the images $(\Phi(i))_{i\in A}$ of the singletons. In this chapter however, we will only consider multi-valued substitutions for which $(\Phi(i))_{i\in A}$ is a partition of A^M, where M is a given substitution length. By Φ^n we denote the n-fold iterate of Φ. It is easily shown by induction that $(\Phi^n(i))_{i\in A}$ is a partition of A^{M^n} for all $n = 1, 2, \ldots$. We will often write $\Phi(i_1, \ldots, i_k)$ instead of $\Phi(\{i_1, \ldots, i_k\})$.

Example 6.1 Let $M = 2$, $\Phi(0) = \{00, 01, 10\}$ and $\Phi(1) = \{11\}$. Then

$$\begin{aligned} \Phi(10, 101) &= \Phi(\{10\} \cup \{101\}) \\ &= \Phi(10) \cup \Phi(101) \\ &= \Phi(1)\Phi(0) \cup \Phi(1)\Phi(0)\Phi(1) \\ &= \{1100, 1101, 1110, 110011, 110111, 111011\}. \end{aligned}$$

Example 6.2 Dekking and Meester [5] reinterpreted the upper bound proof by Chayes, Chayes and Durrett (see Section 3.4) in terms of a multi-valued substitution as follows. Let σ be fractal percolation with parameter p and $M = 3$ and consider a multi-valued substitution Φ given by

$$\begin{aligned} \Phi(0) &= \{w \in A^{3\times 3} : \text{ number of 1's in } w \ \leq 7\} \\ \Phi(1) &= \{w \in A^{3\times 3} : \text{ number of 1's in } w \ \geq 8\}. \end{aligned}$$

It is not hard to see that $\sigma^0(1)$ is the root of an 8-ary tree of depth at least n if and only if $\sigma^n(1) \in \Phi^n(1)$.

6.3 Reconstruction Problem

The following problem is known as the reconstruction problem for noisy M-ary trees [8]. Consider an alphabet A, a Markov matrix P indexed by $A \times A$ and an M-ary tree. From the alphabet a letter X is uniformly chosen and placed at the root of the tree. This letter is broadcast down the tree and at each channel from parent to child the letter switches from an i to a j with probability P_{ij}. The reconstruction problem is to find back the letter at the root with a probability bounded away from $\frac{1}{|A|}$ if only the letters at the nodes of the n^{th} generation of the tree are given, when n tends to infinity. If this is possible, the reconstruction problem is said to be solvable.

Observe that the word formed by the n^{th} generation nodes of the tree can be obtained by applying n independent copies of a random substitution σ to the letter X at the root, where $\mathbb{P}((\sigma(i))_k = j) = P_{ij}$ for $k = 1, \ldots, M$ and $i, j \in A$.

It can be shown that the reconstruction problem is solvable if and only if for some $i, j \in A$ the total variation distance between $\mathbb{P}_P(\sigma^n(i) \in \cdot)$ and $\mathbb{P}_P(\sigma^n(j) \in \cdot)$ does not tend to 0, i.e., there are sets $\Psi_n \subseteq A^{M^n}$ for $n \geq 0$ and $i \in A$ such that $|\mathbb{P}_P(\sigma^n(i) \in \Psi_n) - \mathbb{P}_P(\sigma^n(j) \in \Psi_n)| \nrightarrow 0$. In this case, we say that the sets Ψ^n solve the reconstruction problem.

For reasons of low complexity and robustness, in biology and computation theory one often prefers a recursive strategy to reconstruct the bit at the root. Let $(\Phi(i))_{i \in A}$ be a partition of the M-letter words A^M. Given the word $\sigma^n(X)$, a letter in the $(n-1)^{\text{th}}$ generation is reconstructed as an i if the word formed by its M children is an element of $\Phi(i)$. In the same way, the $(n-2)^{\text{th}}$ generation is reconstructed and so on, until the letter at the root is reconstructed. If it is possible to solve the reconstruction problem using this recursive approach, then the reconstruction problem is said to be recursively solvable.

The recursive reconstruction problem can be easily translated in terms of multi-valued substitutions. The sets $\Phi(i)$ used to reconstruct the letter at the root, define a multi-valued substitution Φ. Observe that the letter at the root X is reconstructed as an i, if and only if $\sigma^n(X) \in \Phi^n(i)$. So the reconstruction problem is recursively solvable if there is a multi-valued substitution and $i, j, k \in A$ such that $|\mathbb{P}_P(\sigma^n(i) \in \Phi^n(k)) - \mathbb{P}_P(\sigma^n(j) \in \Phi^n(k))| \nrightarrow 0$ and in this case Φ is said to solve the recursive reconstruction problem.

In general, it is not known for which P the reconstruction problem is solvable or recursively solvable.

Example 6.3 Popular multi-valued substitutions for $A = \{0, 1\}$ are majority multi-valued substitutions, also called parsimony multi-valued substitutions, which satisfy

$$\begin{aligned} \Phi(0) &\subseteq \{w \in A^M : \text{ number of 1's in } w \leq M/2\} \\ \Phi(1) &\subseteq \{w \in A^M : \text{ number of 1's in } w \geq M/2\}. \end{aligned}$$

If M is odd, this substitution is unique.

Example 6.4 In [18] and [2] the case of binary symmetric channels is studied, i.e., $A = \{0, 1\}$ and

$$P(p) = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} \qquad 0 \le p \le \tfrac{1}{2}.$$

Bleher, Ruiz and Zagrebnov proved that the reconstruction problem is solvable if and only if $M\lambda_2^2(P) > 1$, where $\lambda_2(P)$ is the smallest eigenvalue of P. Mossel shows that the bit reconstruction problem is recursively solvable if and only if $p < p_c$, where p_c is defined by

$$p_c = \begin{cases} \frac{1}{2} - \frac{2^M}{4M}\binom{M-1}{\frac{M}{2}}^{-1} & \text{if } M \text{ is even} \\ \frac{1}{2} - \frac{2^M}{4M}\binom{M-1}{\frac{M-1}{2}}^{-1} & \text{if } M \text{ is odd.} \end{cases}$$

Moreover, the multi-valued substitution Φ solves the reconstruction problem for all $p < p_c$, if and only if Φ is a majority multi-valued substitution.

Example 6.5 A reconstruction problem is called *count solvable* if the letter at the root X can be reconstructed with a probability bounded away from $\frac{1}{|A|}$, when only the number of occurrences of each letter in $\sigma^n(X)$ is given, as n tends to infinity. In [19] it is proved that a reconstruction problem P is count solvable if $M\lambda_2^2(P) > 1$ and is not count solvable if $M\lambda_2^2(P) < 1$, where $\lambda_2(P)$ is the second largest eigenvalue of P.

In [19] the case of binary asymmetric channels is studied, i.e., $A = \{0, 1\}$ and

$$P(p) = \begin{pmatrix} 1-p_0 & p_0 \\ 1-p_1 & p_1 \end{pmatrix} \qquad 0 \le p \le \tfrac{1}{2}.$$

Mossel shows that the bit reconstruction problem is solvable if p_1 is sufficiently small and $M\lambda_2(P) = M(p_1 - p_0) > 1$. This result implies that there are reconstruction problems that are solvable, but not count solvable. In fact, the proof suggests that there are reconstruction problems that are recursively solvable, but not count solvable. In Example 6.9 we provide an explicit example of such a reconstruction problem.

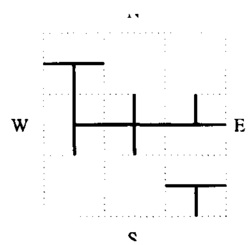
6.4 The TOX Model

In this section we will construct a multi-valued substitution on a 6 letter alphabet to estimate the percolation probability from below. We will only consider the two dimensional case with $M = 3$, but the results extend easily to higher dimensions and $M \ge 3$.

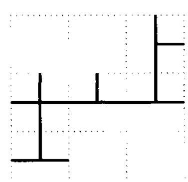
Consider the following ways to connect the 4 sides north, east, south and west of a square:

$+$	$=$	all four edges are connected
$\top$	$=$	E, S and W are connected
$\perp$	$=$	N, E and W are connected
$\vdash$	$=$	N, E and S are connected
$\dashv$	$=$	N, S and W are connected
$\circ$	$=$	no edges are connected.

Let $A = \{+, \top, \perp, \dashv, \vdash, \circ\}$ be the alphabet. We say that two sides of a word $w \in A^{m\times m}$ are connected, if they are connected in the graphical representation of w. For example, in the word



the sides E and W are connected and E and S are connected. In the word above, the dotted lines separate the letters and the letter $\circ$, referred to as the zero letter, is denoted by a blank. Two sides in a word $w \in A^{m\times m}$ are *strongly connected* if there is a connected component in w that intersects both sides twice. For example, in



E and W are strongly connected. Note that if for example N and S are strongly connected and E and S are strongly connected, then also N and E are strongly connected. Hence strongly connectedness is a transitive relation.

Define the 2-dimensional multi-valued substitution Φ with $M = 3$ by

$$
\begin{aligned}
\Phi(+) &= \{w \in A^{3\times 3} : \text{ all edges are strongly connected}\} \\
\Phi(\top) &= \{w \notin \Phi(+) : \ E,\ S \text{ and } W \text{ are strongly connected}\} \\
\Phi(\perp) &= \{w \notin \Phi(+) : \ N,\ E \text{ and } W \text{ are strongly connected}\} \\
\Phi(\vdash) &= \{w \notin \Phi(+) : \ N,\ E \text{ and } S \text{ are strongly connected}\} \\
\Phi(\dashv) &= \{w \notin \Phi(+) : \ N,\ S \text{ and } W \text{ are strongly connected}\} \\
\Phi(\circ) &= A^{3\times 3} \backslash \Phi(+, \top, \perp, \vdash, \dashv).
\end{aligned}
$$

Note that these sets indeed form a partition of $A^{3\times 3}$.

We say that a word in $w \in A^{m\times m}$ percolates if the east and the west side of w are connected. By an inductive argument it is easy to see that for all n all words in $\Phi^n(+, \top, \perp)$ percolate.

Fix a parameter $0 \leq p \leq 1$ and consider fractal percolation on the letters $\circ$ and $+$, i.e., the random substitution σ distributed according to the matrix $P = P(p) = (P_{ij})_{i,j\in A}$, where

$$
P_{ij} = \begin{cases} p & \text{if } i = j = + \\ 1 - p & \text{if } i = +, j = \circ \\ 1 & \text{if } i \neq +, j = \circ \\ 0 & \text{else.} \end{cases}
$$

Instead of $\mathbb{P}_{P(p)}$ we will write $\mathbb{P}_p$.

We can easily estimate the percolation probability by

$$
\begin{aligned}
&\mathbb{P}_p(\sigma^n(+) \text{ percolates for all } n) \\
&\qquad \geq \ \mathbb{P}_p(\sigma^n(+) \in \Phi^n(+, \top, \perp) \text{ for all } n).
\end{aligned}
$$

We refer to A, σ and Φ as the TOX-model. In Section 6.7.1 we show that the critical value of $\mathbb{P}_p(\sigma^n(+) \in \Phi^n(+, \top, \perp)$ for all $n)$ is less than 0.965.

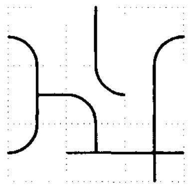
6.5 The TOXIC-model

A way to improve on the upper bound of the TOX-model is to further extend the alphabet. The extended model, called the TOXIC-model, also provides a lower bound. Again, we describe the $M = 3$ case, which can be generalized to $M \geq 3$ for the upper bound and $M \geq 2$ for the lower bound. In Section 6.7.2 we analyse the TOXIC-model and show that this leads to $p_c(3) \leq 0.965$ and $p_c(2) \geq 0.74$.

We start by describing the TOXIC upper bound model. Let

$$
A = \{+, \top, \dashv, \perp, \vdash, \text{╰}, \text{╭}, \text{╮}, \text{╯}, -, |, \text{╳}, \text{╳}, \circ\}
$$

be the alphabet. Again, we say that two sides of a word are connected if they are connected in the graphical representation of the word, and that they are strongly connected if there is a connected component that intersects both sides twice. For example, in



E and W are strongly connected. In the same spirit as for the TOX-model, we define a multi-valued substitution Φ_u by

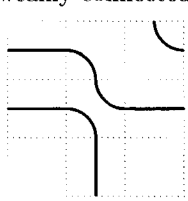
$$\begin{aligned}
\Phi_u(+) &= \{w \in A^{3\times 3} : \text{ all edges are strongly connected}\} \\
\Phi_u(\top) &= \{w \notin \Phi_u(+) : \ E,\ S \text{ and } W \text{ are strongly connected}\} \\
&\ \vdots \\
\Phi_u(\llcorner\!\urcorner) &= \{w \notin \Phi_u(+) : \ N,\ E \text{ and } S,\ W \text{ are strongly connected}\} \\
&\ \vdots \\
\Phi_u(-) &= \{w \notin \Phi_u(+,\top,\perp) : \ E \text{ and } S \text{ are strongly connected}\} \\
&\ \vdots \\
\Phi_u(\llcorner) &= \{w \notin \Phi_u(+,\perp,\vdash,\llcorner\!\urcorner) : \ N \text{ and } E \text{ are strongly connected}\} \\
&\ \vdots \\
\Phi_u(\circ) &= A^{3\times 3}\backslash\Phi_u(+,\top,\dashv,\perp,\vdash,\llcorner,\ulcorner,\urcorner,\lrcorner,-,|,\llcorner\!\urcorner,\lrcorner\!\ulcorner).
\end{aligned}$$

By induction one can show that all words in $\Phi_u^n(+,\top,\perp,-)$ percolate. If σ is fractal percolation on the symbols $\{+,\circ\}$, we can estimate the percolation probability by

$$\mathbb{P}_p(\sigma^n(+) \text{ percolates for all } n) \geq \mathbb{P}_p(\sigma^n(+) \in \Phi_u^n(+,\top,\perp,-) \text{ for all } n).$$

Since for all $i \in \{+,\top,\dashv,\perp,\vdash\}$ the sets $\Phi^n(i)$ of the TOX-model are contained in $\Phi_u^n(i)$, this estimation will lead to a sharper upper bound. We refer to A, σ and Φ_u as the TOXIC upper bound model.

On the TOXIC alphabet, one can also construct a multi-valued substitution that leads to a lower bound for $p_c(3)$. For a word $w \in A^{m\times m}$, we say that two sides $S_1, S_2 \in \{N,E,S,W\}$ are weakly connected if they are connected or if S_1 is connected to a side which is weakly connected to S_2. For example, in



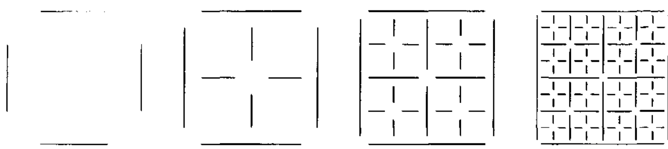


Figure 6.2: Lattices $L_1, \ldots, L_4$ from the sequence $L_0, L_1, \ldots$

N and S are weakly connected. Weakly connectedness is also a transitive relation. Define a multi-valued substitution Φ_l by replacing 'strongly connected' by 'weakly connected' in the definition of Φ_u in the TOXIC upper bound model. It is not hard to see that all words in $\Phi_l^n(\dashv, \vdash, \llcorner, \ulcorner, \urcorner, \lrcorner, |, \llcorner\lrcorner, \lrcorner\ulcorner)$ do not percolate. Hence

$$\begin{aligned} &\mathbb{P}_p(\sigma^n(+) \text{ percolates for all } n) \\ &\quad \le \ \mathbb{P}_p(\sigma^n(+) \in \Phi_l^n(+, \top, \bot, -) \text{ for all } n). \end{aligned}$$

We refer to A, σ and Φ_l as the TOXIC lower bound model.

Note that the TOXIC lower bound model can be easily extended to general $M \geq 2$. The idea behind the construction of Φ_l originates from another technique to obtain a lower bound on $p_c(M)$ using growing lattices. A variant of this technique was used by White [30] to show that $p_c(2) \geq 0.81$.

Let $M = 2$ and let $L_0, L_1, \ldots$ be a sequence of lattices, where $L_0 = \emptyset$, $L_1, \ldots, L_4$ as in Figure 6.2 and $L_5, L_6, \ldots$ the obvious continuation of the sequence. Denote by K_n the graphical representation of $\sigma^n(+)$, scaled to fit in the lattice L_n. Then of course

$$\mathbb{P}_p(\sigma^n(+) \text{ percolates for all } n) \leq \mathbb{P}_p(K_n \cup L_n \text{ percolates for all } n).$$

By an induction argument, it can be proved that $\sigma^n(+) \in \Phi_l^n(+, \top, \bot, -)$ if and only if $K_n \cup L_n$ percolates.

6.6 Some Analysis

6.6.1 A Recursion Formula

Let A be a finite alphabet and let $\mathbb{M}$ be the set of all Markov matrices indexed by $A \times A$. For $n = 1, 2, \ldots$ define a map $\Pi^n : \mathbb{M} \to \mathbb{M}$ by $\Pi^n(P) = (\Pi_{ij}^n(P))_{i,j \in A}$, where

$$\Pi_{ij}^n(P) = \mathbb{P}_P\big(\sigma^n(i) \in \Phi^n(j)\big).$$

The matrices $\Pi^n(P)$ satisfy a recursion relation.

Lemma 6.1 *For* $n = 1, 2, \ldots$

$$\Pi^{n+1}(P) = \Pi^1(P\Pi^n(P)).$$

Proof For $i, j \in A$,

$$\begin{aligned}
\Pi_{ij}^{n+1}(P) &= \mathbb{P}_P\big(\sigma^{n+1}(i) \in \Phi^{n+1}(j)\big) \\
&= \sum_{v \in \Phi(j)} \prod_{k=1}^{M} \Big(\sum_{l=1}^{M} P_{il}\, \Pi_{lv_k}^{n}(P)\Big) \\
&= \sum_{v \in \Phi(j)} \prod_{k=1}^{M} \big(P\Pi^n(P)\big)_{iv_k} \\
&= \mathbb{P}_{P\Pi^n(P)}\big(\sigma(i) \in \Phi(j)\big) \\
&= \big(\Pi^1(P\Pi^n(P))\big)_{ij}.
\end{aligned}$$

□

If we define for matrices $X \in \mathbb{M}$

$$F_P(X) = \Pi^1(PX),$$

then it follows from Lemma 6.1 that

$$\Pi^n(P) = F_P^n(I)$$

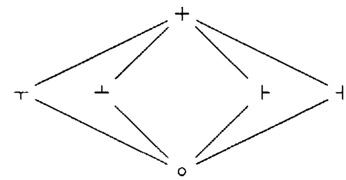
for $n = 1, 2, \ldots$, where F_P^n denotes the n-fold iterate of F_P and I denotes the identity matrix in $\mathbb{M}$.

6.6.2 Increasing Multi-valued Substitutions

Let $\prec$ be a partial ordering of the alphabet A. For words $v, w \in A^m$ we write $v \preceq w$ if $v_1 \preceq w_1, \ldots, v_m \preceq w_m$. A set $W \subseteq A^m$ is called *increasing* with respect to $\prec$ if $v \in W$ and $v \preceq w$ implies that $w \in W$. From now on we will assume that the multi-valued substitution Φ is increasing, i.e., $\Phi(J)$ is increasing for all increasing sets $J \subseteq A$. This is equivalent to the requirement that the sets $\bigcup_{j \succeq i} \Phi(j)$ are increasing for all i.

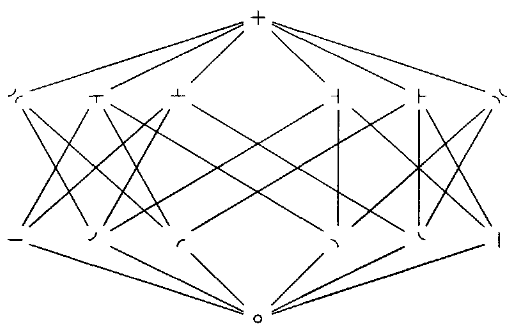
Example 6.6 If $A = \{0, 1\}$, M odd and $0 \prec 1$, then the parsimony multi-valued substitution (Example 6.3) is increasing.

Example 6.7 Consider the TOX-model presented in Section 6.4. If $\prec$ is the obvious partial ordering on A, i.e.,



then the multi-valued substitution of the TOX-model is increasing.

Example 6.8 Consider the $TOXIC$-model presented in Section 6.5. If $\prec$ is the obvious partial ordering on A, i.e.,



then the multi-valued substitutions Φ_u and Φ_l are increasing.

Define a cone S in $\mathbb{R}^{A\times A}$ by

$$S = \{X \in \mathbb{R}^{A\times A} : \sum_{j\in A} X_{ij} = 0 \text{ and } \sum_{j\in J} X_{ij} \geq 0$$
$$\text{for all } i \in A \text{ and increasing } J \subseteq A\}.$$

Lemma 6.2 *Let $X, X' \in \mathbb{M}$ be such that $X' \in X+S$. Then $F_P(X') \in F_P(X)+S$.*

Proof Let $X = (X_{ij}), X' = (X'_{ij}) \in \mathbb{M}$ be such that $X' \in X + S$. Then $\sum_{j\in J} X_{ij} \leq \sum_{j\in J} X'_{ij}$ for all $i \in A$ and increasing $J \subseteq A$. Fix $i \in A$ and define for $j \in A$

$$\mu_X(j) = X_{ij}.$$

If V is an increasing subset of A^M, then for $1 \leq k \leq M$ we have

$$\begin{aligned} \mu_X^k \times \mu_{X'}^{M-k}(V) &= \sum_{v\in A^{k-1}} \sum_{w\in A^{M-k}} \sum_{a\in A} \mu_X^{k-1}(v)\mu_{X'}^{M-k}(w) X_{ia} 1_V(vaw) \\ &= \sum_{v\in A^{k-1}} \sum_{w\in A^{M-k}} \mu_X^{k-1}(v)\mu_{X'}^{M-k}(w) \sum_{a\in V(v,w)} X_{ia}, \end{aligned}$$

where $V(v,w) = \{a : vaw \in V\}$ which is an increasing subset of A. Hence,

$$\begin{aligned}\mu_X^k \times \mu_{X'}^{M-k}(V) &\le \sum_{v\in A^{k-1}} \sum_{w\in A^{M-k}} \mu_X^{k-1}(v)\mu_{X'}^{M-k}(w) \sum_{a\in V(v,w)} X'_{ia} \\ &= \mu_X^{k-1} \times \mu_{X'}^{M-k+1}(V).\end{aligned}$$

By induction it follows that $\mu_X^M(V) \le \mu_{X'}^M(V)$.

Let J be an increasing subset of A. Since $PX, PX' \in \mathbb{M}$, $PX' \in PX + S$ and since $\Phi(J)$ is an increasing subset of A^M, we have that

$$\begin{aligned}\sum_{j\in J}(F_P(X))_{ij} &= \mu_{PX}^M(\Phi(J)) \\ &\le \mu_{PX'}^M(\Phi(J)) \\ &= \sum_{j\in J}(F_p(X'))_{ij}.\end{aligned}$$

Hence, $F_p(X') \in F_p(X) + S$. □

Lemma 6.3 *Let P and Y^* be Markov matrices such that $F_P(Y^*) \in Y^* + S$. If $F_P^{n_0}(I) \in Y^* + S$ for some n_0, then $\Pi^n(P) \in Y^* + S$ for all $n \ge n_0$.*

Notice that if $I = F_P^0(I) \in Y^* + S$, then it follows from the lemma above that $\Pi^n(P) \in Y^* + S$ for all n.

Proof The lemma is proved by induction. Assume that $\Pi^n(P) \in Y^* + S$ for $n \ge n_0$. Then by Lemma 6.2, $\Pi^{n+1}(P) \in F_P(Y^*)+S$. Since $F_P(Y^*) \in Y^*+S$ and since S is a cone, we have that $F_P(Y^*) + S \subseteq Y^* + S$ and so $\Pi^{n+1}(P) \in Y^* + S$. □

Example 6.9 Consider the reconstruction problem for asymmetric binary channels from Example 6.5, where $A = \{0,1\}$ and

$$P = \begin{pmatrix} 1-p_0 & p_0 \\ 1-p_1 & p_1 \end{pmatrix}.$$

Let $M = 16$ and let the multi-valued substitution Φ be defined by

$$\begin{aligned}\Phi(0) &= \{w : \text{ number of 1's in } w < 15\} \\ \Phi(1) &= \{w : \text{ number of 1's in } w \ge 15\}.\end{aligned}$$

We will use the results of this section to prove that if $p_0 \le 0.796$ and $p_1 \ge 0.998$, then Φ solves the reconstruction problem. This is an example of a reconstruction problem which is recursively solvable but not count solvable.

If we define a partial ordering $\prec$ by $0 \prec 1$, then Φ is an increasing multi-valued substitution. Define the cone S_1 by

$$S_1 = \left\{ \begin{pmatrix} -x_0 & x_0 \\ -x_1 & x_1 \end{pmatrix} : x_0, x_1 \geq 0 \right\}.$$

If we choose

$$Y_1^* = \begin{pmatrix} 1 & 0 \\ 0.003 & 0.997 \end{pmatrix}$$

then $\left(F_P(Y_1^*)\right)_{11} \geq 0.9971$ whenever $p_1 \geq 0.998$ and therefore $F_P(Y_1^*) \in Y_1^* + S_1$. Since the identity matrix also is an element of $Y_1^* + S_1$, it follows from Lemma 6.3 that $\Pi^n(P) \in Y_1^* + S_1$ for all n and hence $\Pi_{11}^n(P) \geq 0.997$ for all n.

If we define a partial ordering $\prec$ by $1 \prec 0$, then again Φ is an increasing multi-valued substitution. Define the cone S_2 by

$$S_2 = \left\{ \begin{pmatrix} -x_0 & x_0 \\ -x_1 & x_1 \end{pmatrix} : x_0, x_1 \leq 0 \right\}.$$

If we choose

$$Y_2^* = \begin{pmatrix} 0.58 & 0.42 \\ 0 & 1 \end{pmatrix}$$

then $\left(F_P(Y_2^*)\right)_{01} \leq 0.4197$ whenever $p_0 \leq 0.796$ and therefore $F_P(Y_2^*) \in Y_2^* + S_2$. Since the identity matrix also is an element of $Y_2^* + S_2$, it follows from Lemma 6.3 that $\Pi^n(P) \in Y_2^* + S_2$ for all n and hence $\Pi_{01}^n(P) \leq 0.796$ for all n.

We may conclude that $\Pi^n(P) \in (Y_1^* + S_1) \cap (Y_2^* + S_2)$ and hence Φ solves the reconstruction problem.

6.6.3 Decreasing Random Substitutions

Let σ be a random substitution with associated Markov matrix P and let Φ be an increasing multi-valued substitution. In this section, we will assume that σ is decreasing with respect to Φ, i.e., we assume that P is such that $\mathbb{P}_P(\sigma(i) \in \bigcup_{j \preceq i} \Phi(j)) = 1$ for all $i \in A$.

Lemma 6.4 *For all $i \in A$ and increasing sets $J \subseteq A$, the events*

$$\{\sigma^n(i) \in \Phi^n(J)\}$$

are decreasing in n, $\mathbb{P}_P$-almost surely.

Proof Suppose $\sigma^n(i) \in \Phi^n(g)$ for some $g \notin J$ and write $\sigma^n(i) = u = u_0 \dots u_{M^n-1}$ and $\sigma^{n+1}(i) = v = v_0 \dots u_{M^{n+1}-1}$. Since σ is decreasing with respect to Φ, we have that

$$v_{kM} \dots v_{(k+1)M-1} \in \bigcup_{h \preceq u_k} \Phi(h)$$

for $0 \leq k \leq M^n - 1$. Let $w = w_0 \dots w_{M^n-1}$ be such that

$$v_{kM} \dots v_{(k+1)M-1} \in \Phi(w_k).$$

Then $w \preceq u$ and since $\bigcup_{h \preceq g} \Phi^n(h)$ is decreasing, it follows that $w \in \bigcup_{h \preceq g} \Phi^n(h)$. This implies that

$$v \in \Phi(\bigcup_{h \preceq g} \Phi^n(h)) = \bigcup_{h \preceq g} \Phi^{n+1}(h).$$

Hence $\sigma^{n+1}(i) = v \notin \Phi^{n+1}(J)$. □

Lemma 6.5 *For all $i, j \in A$,*

$$\{\sigma^n(i) \in \Phi^n(j) \textit{ eventually}\} = \{\sigma^n(i) \in \Phi^n(j) \textit{ infinitely often}\}$$

$\mathbb{P}_P$-almost surely.

Proof Fixing $i, j \in A$, we have that $\{\sigma^n(i) \in \Phi^n(j) \text{ eventually}\}$ is equal to

$$\{\sigma^n(i) \in \bigcup_{h \succeq j} \Phi^n(h) \text{ eventually}\} \cap \{\sigma^n(i) \in \bigcup_{h \succ j} \Phi^n(h) \text{ infinitely often}\}^c.$$

Since $\{\sigma^n(i) \in \bigcup_{h \succ j} \Phi^n(h)\}_n$ is a decreasing sequence by Lemma 6.4, it follows that

$$\{\sigma^n(i) \in \bigcup_{h \succ j} \Phi^n(h) \text{ infinitely often}\} = \{\sigma^n(i) \in \bigcup_{h \succ j} \Phi^n(h) \text{ eventually}\}$$

and hence $\{\sigma^n(i) \in \Phi^n(j) \text{ eventually}\}$ is equal to

$$\{\sigma^n(i) \in \bigcup_{h \succeq j} \Phi^n(h) \text{ eventually}\} \cap \{\sigma^n(i) \in \bigcup_{h \succ j} \Phi^n(h) \text{ eventually}\}^c,$$

which is equal to $\{\sigma^n(i) \in \Phi^n(j) \text{ infinitely often}\}$. □

Lemma 6.6 *For all $i \in A$ and increasing sets $J \subseteq A$,*

$$\{\sigma^n(i) \in \Phi^n(J) \textit{ for all } n\} = \bigcup_{j \in J} \{\sigma^n(i) \in \Phi^n(j) \textit{ eventually}\}$$

$\mathbb{P}_P$-almost surely.

Proof Since the sequence $\{\sigma^n(i) \in \Phi^n(J)\}_n$ is decreasing by Lemma 6.4, we have that

$$\begin{aligned}\{\sigma^n(i) \in \Phi^n(J) \text{ for all } n\} &= \{\sigma^n(i) \in \Phi^n(J) \text{ infinitely often}\}\\ &= \bigcup_{j\in J}\{\sigma^n(i) \in \Phi^n(j) \text{ infinitely often}\}.\end{aligned}$$

Applying Lemma 6.5, it follows that

$$\{\sigma^n(i) \in \Phi^n(J) \text{ for all } n\} = \bigcup_{j\in J}\{\sigma^n(i) \in \Phi^n(j) \text{ eventually}\}.$$

□

The following corollary follows directly from Lemma 6.5.

Corollary 6.1 *The sequence of matrices $(\Pi^n(P))_n$ converges componentwise. Moreover, the limit matrix $\Pi^\infty(P) = (\Pi^\infty_{ij}(P))_{i,j\in A}$ satisfies*

$$\Pi^\infty_{ij}(P) = \mathbb{P}_P(\sigma^n(i) \in \Phi^n(j) \textit{ eventually}).$$

We say that the random substitution σ is decreasing with respect to the partial ordering $\prec$ if $P_{ij} > 0$ implies that $i \succeq j$ for all $i, j \in A$. Define the set $\mathbb{L} \subset \mathbb{M}$ by

$$\mathbb{L} = \{X \in \mathbb{M} : \text{ if } X_{ij} > 0 \text{ then } i \succeq j\}.$$

Lemma 6.7 *Let $P \in \mathbb{L}$ and assume that $i^M \in \bigcup_{j\preceq i}\Phi(j)$ for all $i \in A$. Then $\Pi^n(P) \in \mathbb{L}$ for all n and the sequence $(\Pi^n(P))_n$ converges to $\Pi^\infty(P) \in \mathbb{L}$. Moreover, if $Y^* \in \mathbb{L}$ is such that $F_P(Y^*) \in Y^* + S$, then $\Pi^\infty(P) \in Y^* + S$.*

Proof Since $i^M \in \bigcup_{j\preceq i}\Phi(j)$ for all $i \in A$, it follows that $\Pi^1(X) \in \mathbb{L}$. Since $\mathbb{L}$ is closed under multiplication, we have that $PX \in \mathbb{L}$ and therefore $F_P(X) = \Pi^1(PX) \in \mathbb{L}$. As a consequence, $\Pi^n(P) \in \mathbb{L}$ for all n, since $\Pi^n(P) = F^n_P(I)$ and $I \in \mathbb{L}$. Notice that $P \in \mathbb{L}$ and $i^M \in \bigcup_{j\preceq i}\Phi(j)$ for all $i \in A$ implies that σ is decreasing with respect to Φ. Hence by Corollary 6.1, the sequence $(\Pi^n(P))_n$ converges and since $\mathbb{L}$ is a closed set, $\Pi^\infty(P) \in \mathbb{L}$. The second part of the lemma follows from Lemma 6.3 and the fact that $I \in Y^* + S$ for all $Y^* \in \mathbb{L}$. □

6.7 Calculating Bounds for the Critical Value

6.7.1 The TOX model

Consider the TOX model from Section 6.4 and equip the alphabet A with the partial ordering $\prec$ from Example 6.7. Recall that

$$\begin{aligned}&\mathbb{P}_p(\sigma^n(+) \text{ percolates for all } n)\\ &\quad\geq \mathbb{P}_p(\sigma^n(+) \in \Phi^n(+) \cup \Phi^n(\top) \cup \Phi^n(\perp) \text{ for all } n).\end{aligned}$$

Since σ is decreasing with respect to Φ and since $\{+, \top, \perp\}$ is an increasing subset of A, we have by Lemma 6.6 that

$$\begin{aligned}
&\mathbb{P}_p(\sigma^n(+) \text{ percolates for all } n) \\
&\quad \geq \mathbb{P}_p(\sigma^n(+) \in \Phi^n(+) \text{ eventually}) + \mathbb{P}_p(\sigma^n(+) \in \Phi^n(\top) \text{ eventually}) \\
&\quad\quad + \mathbb{P}_p(\sigma^n(+) \in \Phi^n(\perp) \text{ eventually}) \\
&\quad = \Pi^\infty_{++}(P) + \Pi^\infty_{+\top}(P) + \Pi^\infty_{+\perp}(P),
\end{aligned}$$

where the last equality follows from Corollary 6.1 and $P = P(p)$ is the matrix associated with fractal percolation on the letters $\{\circ, +\}$ with parameter p.

Since $P \in \mathbb{L}$ and $i^{3\times 3} \in \bigcup_{j \preceq i} \Phi(j)$ for all $i \in A$, it follows from Lemma 6.7 that $\Pi^n(P) \in \mathbb{L}$ for all n. In fact, all matrices $\Pi^n(P)$ and $\Pi^\infty(P)$ are elements of the smaller set

$$\mathbb{K} = \{X \in \mathcal{P} : X_{i\,\circ} = 1 \text{ for all } i \neq + \text{ and } X_{+\top} = X_{+\perp} = X_{+\vdash} = X_{+\dashv}\}.$$

To see this, observe that the sets $\Phi(\perp)$, $\Phi(\dashv)$, $\Phi(\vdash)$ can be obtained by rotating the words from $\Phi(\top)$. Therefore we have for all $X \in \mathbb{K}$ that $F_I(X) \in \mathbb{K}$. Since $\mathbb{K}$ is closed under multiplication and since $P \in \mathbb{K}$, it follows that $F_P(X) = F_I(PX) \in \mathbb{K}$ for all $X \in \mathbb{K}$. As a consequence, $\Pi^n(P) = F^n_P(I) \in \mathbb{K}$ for all n and since $\mathbb{K}$ is closed, also $\Pi^\infty \in \mathbb{K}$.

Note that an element X of $\mathbb{K}$ is completely determined by $X_{+\top}$ and X_{++}, in other words, the map $\rho : X \mapsto (X_{+\top}, X_{++})$ is a bijection from $\mathbb{K}$ to $\bar{\mathbb{K}}$, where we define

$$\bar{\mathbb{K}} = \{(x_\top, x_+) : x_\top, x_+ \geq 0, 1 - 4x_\top - x_+ \geq 0\}$$

Define

$$\bar{F}_P(x_\top, x_+) = \rho(F_P(\rho^{-1}(x_\top, x_+)))$$

and write $\bar{F}_p$ for $\bar{F}_P$. If we let

$$\bar{S} = \{(x_\top, x_+) : x_+ \geq 0, 4x_\top + x_+ \geq 0\},$$

then we have by Lemma 6.7 that if $(y^*_\top, y^*_+) \in \bar{\mathbb{K}}$ is such that $\bar{F}_p(y^*_\top, y^*_+) \in (y^*_\top, y^*_+) + \bar{S}$, then $\rho(\Pi^\infty(P)) \in (y^*_\top, y^*_+) + \bar{S}$. Note that $\bar{F}_p(x_\top, x_+) = \bar{F}_1(p\,x_\top, p\,x_+)$. Writing $\bar{F}_1(x_\top, x_+) = (\bar{F}_\top(x_\top, x_+), \bar{F}_+(x_\top, x_+))$ and $x_\circ = 1 - 4x_\top - x_+$, we found with help of Matlab

$$\begin{aligned}
\bar{F}_\top(x_\top, x_+) = {} & 3x_+^7 x_\top^2 + 10x_+^7 x_\top^1 x_\circ^1 + 3x_+^7 x_\circ^2 + 110x_+^6 x_\top^3 + 301x_+^6 x_\top^2 x_\circ^1 \\
& +120x_+^6 x_\top^1 x_\circ^2 + 9x_+^6 x_\circ^3 + 1516x_+^5 x_\top^4 + 3454x_+^5 x_\top^3 x_\circ^1 + 1393x_+^5 x_\top^2 x_\circ^2 \\
& +122x_+^5 x_\top^1 x_\circ^3 + 2x_+^5 x_\circ^4 + 10278x_+^4 x_\top^5 + 19007x_+^4 x_\top^4 x_\circ^1 + 6796x_+^4 x_\top^3 x_\circ^2 \\
& +597x_+^4 x_\top^2 x_\circ^3 + 14x_+^4 x_\top^1 x_\circ^4 + 36223x_+^3 x_\top^6 + 52248x_+^3 x_\top^5 x_\circ^1 + 15748x_+^3 x_\top^4 x_\circ^2 \\
& +1362x_+^3 x_\top^3 x_\circ^3 + 36x_+^3 x_\top^2 x_\circ^4 + 64528x_+^2 x_\top^7 + 70224x_+^2 x_\top^6 x_\circ^1 + 17756x_+^2 x_\top^5 x_\circ^2 \\
& +1486x_+^2 x_\top^4 x_\circ^3 + 40x_+^2 x_\top^3 x_\circ^4 + 52071x_+^1 x_\top^8 + 42148x_+^1 x_\top^7 x_\circ^1 + 8966x_+^1 x_\top^6 x_\circ^2 \\
& +698x_+^1 x_\top^5 x_\circ^3 + 17x_+^1 x_\top^4 x_\circ^4 + 14160x_\top^9 + 8559x_\top^8 x_\circ^1 + 1528x_\top^7 x_\circ^2 + 106x_\top^6 x_\circ^3 \\
& +2x_\top^5 x_\circ^4
\end{aligned}$$

and

$$\begin{aligned}\bar{F}_+(x_\top, x_+) \;=\;& x_+^9 + 36x_+^8x_\top^1 + 9x_+^8x_\circ^1 + 564x_+^7x_\top^2 + 248x_+^7x_\top^1x_\circ^1 + 20x_+^7x_\circ^2 \\ &+4936x_+^6x_\top^3 + 2700x_+^6x_\top^2x_\circ^1 + 296x_+^6x_\top^1x_\circ^2 + 25834x_+^5x_\top^4 + 14688x_+^5x_\top^3x_\circ^1 \\ &+1562x_+^5x_\top^2x_\circ^2 + 80800x_+^4x_\top^5 + 42090x_+^4x_\top^4x_\circ^1 + 3812x_+^4x_\top^3x_\circ^2 + 144622x_+^3x_\top^6 \\ &+63328x_+^3x_\top^5x_\circ^1 + 4760x_+^3x_\top^4x_\circ^2 + 136372x_+^2x_\top^7 + 48152x_+^2x_\top^6x_\circ^1 + 3064x_+^2x_\top^5x_\circ^2 \\ &+58494x_+^1x_\top^8 + 16672x_+^1x_\top^7x_\circ^1 + 934x_+^1x_\top^6x_\circ^2 + 8096x_\top^9 + 1930x_\top^8x_\circ^1 + 100x_\top^7x_\circ^2.\end{aligned}$$

If we let $p = 0.965$ and $(y_\top^*, y_+^*) = (0.012, 0.934)$, then

$$\bar{F}_p(y_\top^*, y_+^*) = (0.011720\ldots, 0.935756\ldots) \in (y_\top^*, y_+^*) + \bar{S}.$$

This implies that $\rho(\Pi^\infty(P)) \in (y_\top^*, y_+^*) + \bar{S}$ and hence

$$\Pi^\infty_{++}(P) + \Pi^\infty_{+\top}(P) + \Pi^\infty_{+\perp}(P) \geq 2y_\top^* + y_+^* = 0.958.$$

We may conclude that $\mathbb{P}_p(\sigma^n(+)$ percolates for all $n) \geq 0.958$ and hence $p_c(3) \leq 0.965$.

6.7.2 The TOXIC model

In the spirit of the previous section, it is possible to obtain $p_c(3) \leq 0.958$ using the TOXIC upper bound model. In this section we will prove that $p_c(2) \geq 0.74$ using the TOXIC lower bound model. Analogous to the TOX model, $\mathbb{P}_p(\sigma^n(+)$ percolates for all $n) \leq \Pi^\infty_{++}(P) + \Pi^\infty_{+\top}(P) + \Pi^\infty_{+\perp}(P) + \Pi^\infty_{+-}(P)$. To prove that the last expression is 0 for $p = 0.74$, we use the following lemma. We will write F_p for $F_P = F_{P(p)}$.

Lemma 6.8 *If* $(F_p^n(I))_{+\circ} > 1 - \frac{1-p}{96p^2}$ *for some* n, *then* $\Pi^\infty_{+\circ}(P) = 1$.

Proof To prove this lemma, we study the iterations of F_p on $\mathbb{R}^{|A|\times|A|}$ via a weight function ρ and the iterations of a function G_p on $\mathbb{R}$. Note that the iterates $F_p^n(I)$ stay within the set

$$\mathbb{K} \;=\; \{X \in \mathbb{M} : X_{i\circ} = 1 \text{ for all } i \neq +\}.$$

For matrices $X \in \mathbb{K}$ we will denote the entry X_{+i} by X_i. Define sets $J_1, J_2 \subset A$ by $J_1 = \{$ ╲, ╭, ╮, ╯ $\}$ and $J_2 = \{-,$ ╷, ┬, ┤, ┴, ├, ╲╱, ╱╲, $+\} = A\backslash(J_1 \cup \{\circ\})$. Define

$$\begin{aligned}\rho : \mathbb{K} &\to \mathbb{R} \\ X &\mapsto \sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j\end{aligned}$$

and

$$\begin{aligned}G_p : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto px + 24p^2x^2.\end{aligned}$$

We claim that $\rho(F_p(X)) \leq G_p(\rho(X))$ for all $X \in \mathbb{K}$. To see this, note that $\rho(F_p(X)) = \rho(F_{PX}(I))$ and that $G_p(\rho(X)) = G_1(p\,\rho(X)) = G_1(\rho(PX))$. Since $PX \in \mathbb{K}$, we may replace PX by X, and it suffices to show that

$$\begin{aligned}\rho(F_X(I)) &= \sum_{j\in J_1} \mathbb{P}_X(\sigma(+) \in \Phi(j)) + 4\sum_{j\in J_2} \mathbb{P}_X(\sigma(+) \in \Phi(j)) \\ &\leq \sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j + 24(\sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j)^2 \\ &= G_1(\rho(X))\end{aligned}$$

for all $X \in \mathbb{K}$. First we will prove that

$$3\sum_{j\in J_2} \mathbb{P}_X(\sigma(+) \in \Phi(j)) \leq 18(\sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j)^2$$

and then that

$$\begin{aligned}\mathbb{P}_X(\sigma(+) \notin \Phi(\circ)) &= \sum_{j\in J_1} \mathbb{P}_X(\sigma(+) \in \Phi(j)) + \sum_{j\in J_2} \mathbb{P}_X(\sigma(+) \in \Phi(j)) \\ &\leq \sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j + 6(\sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j)^2.\end{aligned}$$

The first inequality follows from

$$\begin{aligned}\sum_{j\in J_2} \mathbb{P}_X(\sigma(+) \in \Phi(j)) &= \mathbb{P}_X(\sigma(+) \in \Phi(J_2)) \\ &\leq \mathbb{P}_X(\sigma(+) \text{ contains at least 2 non-zero letters}) \\ &\leq \binom{4}{2}(\sum_{j\neq\circ} X_j)^2 \\ &\leq 6(\sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j)^2.\end{aligned}$$

For the second inequality, observe that if $\sigma(+) \notin \Phi(\circ)$, then $\sigma(+) = \begin{matrix}0 & 0\\ * & 0\end{matrix}$ with $* \in \{\urcorner\} \cup J_2$ or, $\sigma(+) = \begin{matrix}0 & 0\\ 0 & *\end{matrix}$ with $* \in \{\ulcorner\} \cup J_2$ or, $\sigma(+) = \begin{matrix}* & 0\\ 0 & 0\end{matrix}$ with $* \in \{\lrcorner\} \cup J_2$ or, $\sigma(+) = \begin{matrix}0 & *\\ 0 & 0\end{matrix}$ with $* \in \{\llcorner\} \cup J_2$ or, $\sigma(+)$ contains at least 2 non-zero letters. From this observation it follows that

$$\begin{aligned}\mathbb{P}_X(\sigma(+) \notin \Phi(\circ)) &\leq \sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j + 6(\sum_{j\neq\circ} X_j)^2 \\ &\leq \sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j + 6(\sum_{j\in J_1} X_j + 4\sum_{j\in J_2} X_j)^2.\end{aligned}$$

We have now established our claim that $\rho(F_p(X)) \leq G_p(\rho(X))$ for all $X \in \mathbb{K}$.

With the claim we can prove that $\rho(F_p^n(X)) \leq G_p^n(\rho(X))$ for every n and since ρ is continuous,

$$\begin{aligned} \rho(\Pi^\infty(P)) &= \rho(\lim_{n\to\infty} F_p^n(I)) \\ &= \lim_{n\to\infty} \rho(F_p^n(F_p^m(I))) \\ &\leq \lim_{n\to\infty} G_p^n(\rho(F_p^m(I)). \end{aligned}$$

Suppose that $(F_p^m(I))_{+\circ} > 1 - \frac{1-p}{96p^2}$. Then

$$\begin{aligned} \rho(F_p^m(I)) &= \sum_{j\in J_1}(F_p^n(I))_{+j} + 4\sum_{j\in J_2}(F_p^n(I))_{+j} \\ &\leq 4\sum_{j\neq \circ}(F_p^n(I))_{+j} \\ &= 4(1-(F_p^n(I))_{+\circ}) < \frac{1-p}{24p^2}. \end{aligned}$$

An easy computation yields that $\lim_{n\to\infty} G_p^n(x) = 0$ for all $0 \leq x < \frac{1-p}{24p^2}$, which implies that $\rho(\Pi^\infty(P)) = 0$. Hence $(\Pi^\infty(P))_{+\circ} = 1$. □

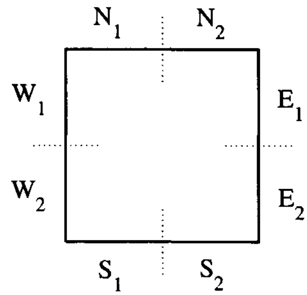
It can be checked that $(F_p^{50}(I))_{+\circ} > 0.9997$ for $p = 0.74$. Since $1 - \frac{1-0.74}{96\cdot 0.74^2} = 0.9951\ldots$, it follows from Lemma 6.8 that $\Pi^\infty_{+\circ}(P) = 1$. As a consequence, $\mathbb{P}_{0.74}(\sigma^n(+)$ percolates for all $n) = 0$ and hence $p_c(2) \geq 0.74$.

6.8 Further Improvements

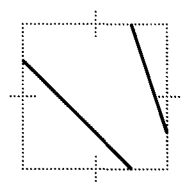
A way to improve the bounds obtained by the TOXIC model is to increase the size of the alphabet. In this section we will describe a way to do that. Unfortunately, the computation of the recursion function F_P is so much harder for large alphabets, that we cannot offer any numerical results.

The TOXIC alphabet can be seen as equivalence relations on the set $\{N, E, S, W\}$, the four sides of the unit square. If two sides $S_1, S_2 \in \{N, E, S, W\}$ are connected in the graphical representation of a letter, then the pair (S_1, S_2) is an element of the corresponding equivalence relation. For example, the letter  is represented by the equivalence relation $\{(N,N),(N,W),(E,E),(E,S),(S,E),(S,S),(W,N),(W,W)\}$. Therefore, we can use the set of all equivalence relations on $\{N, E, S, W\}$ as an alphabet for the TOXIC-model.

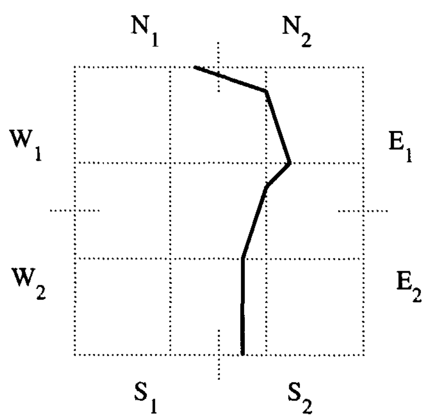
More generally, we can divide each of the four sides N, E, S and W into k subsides of equal length to obtain the set of sides $\{N_1, \ldots, N_k, E_1, \ldots, E_k, S_1, \ldots, S_k, W_1, \ldots, W_k\}$. From now on, we will assume that $k = 2$, so that the subdivision is



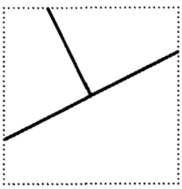
Let A be the 4140 letter alphabet consisting of all equivalence relations on the sides $\{N_1, N_2, E_1, E_2, S_1, S_2, W_1, W_2\}$ equipped with the natural partial ordering $\prec$, i.e., $a \prec b$ if $a \subset b$. Each letter $a \in A$ has a natural graphical representation. For example, the relation $a = \{(N_1, N_1), (N_2, N_2), (N_2, E_2), (E_1, E_1), (E_2, N_2), (E_2, E_2), (S_1, S_1), (S_2, S_2), (S_2, W_1), (W_1, S_2), (W_1, W_1), (W_2, W_2)\}$ can be visualized as



The maximal element of A, denoted $*$, in the partial ordering is $A \times A$, and the minimal element, denoted 0, is $\{(a, a) : a \in A\}$. Let σ be fractal percolation on the letters $\{0, *\}$ with substitution length $M = 3$ and parameter $0 \leq p \leq 1$. For a word $u \in A^{3\times3}$, we say that two sides $Z_1, Z_2 \in \{N_1, N_2, E_1, E_2, S_1, S_2, W_1, W_2\}$ are connected in u, if they are connected in the graphical representation of u. For example, in



the sides N_1 and S_2 are connected. We say that sides Z_1 and Z_2 are strongly connected, if there is a connected component in u that intersects both Z_1 and Z_2 in at least 2 places. For general M that would be in at least $\lceil \frac{M+1}{2} \rceil$ places. Similar to the TOXIC case, two sides Z_1 and Z_2 are weakly connected if they are connected or if Z_1 is connected to a side that is weakly connected to Z_2. Analogous to the TOXIC model, we use strongly and weakly connectedness to define increasing multi-valued substitutions Φ_u and Φ_l. If for example a is equal to



then

$$\Phi_u(a) = \{w \in A^{3\times 3} : N_1, E_1, W_2 \text{ are strongly connected in } w \text{ and } w \notin \Phi_u(b) \text{ for any } b \succ a\}.$$

and

$$\Phi_l(a) = \{w \in A^{3\times 3} : N_1, E_1, W_2 \text{ are weakly connected in } w \text{ and } w \notin \Phi_u(b) \text{ for any } b \succ a\}.$$

Let Ξ be the set of letters a for which $(Z_1, Z_2) \in a$, where Z_1 is E_1 or E_2 and Z_2 is W_1 or W_2. Then

$$\mathbb{P}_p(\sigma^n(*) \text{ percolates for all } n) \leq \mathbb{P}_p(\sigma^n(*) \in \Phi_l^n(\Xi) \text{ for all } n)$$

and

$$\mathbb{P}_p(\sigma^n(*) \text{ percolates for all } n) \geq \mathbb{P}_p(\sigma^n(*) \in \Phi_u^n(\Xi) \text{ for all } n).$$

Note that Ξ is an increasing set and that σ is decreasing with respect to Φ_l and Φ_u. Therefore, the probabilities $\mathbb{P}_p(\sigma^n(*) \in \Phi_l^n(\Xi) \text{ for all } n)$ and $\mathbb{P}_p(\sigma^n(*) \in \Phi_u^n(\Xi) \text{ for all } n)$ can be analyzed with the techniques presented in this chapter.

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Samenvatting

Stochastische Substituties en Fractale Percolatie

Hoewel we in dit proefschrift hele ingewikkelde verschijnselen zullen beschouwen, zoals turbulentie in vloeistoffen en het doorgeven van geninformatie tussen verschillende generaties, zullen we beginnen met een eenvoudig wiskundig model: substituties op woorden. Een woord is een rijtje symbolen, bijvoorbeeld nullen en enen. Een voorbeeld van een substitutie op woorden is de Fibonacci substitutie, die 0-len vervangt door 01 en 1-en vervangt door een 0. Dus als we deze substitutie op het woord 0101 krijgen we het woord 010010. We krijgen een rij van woorden als we de substitutie herhaaldelijk toepassen op de uitkomst van de vorige substitutie. Startend met een 0 verkrijgen we zo de rij 0, 01, 010, 01001, 01001010, 0100101001001 enzovoorts. De lengtes van deze woorden, $1, 2, 3, 5, 8, 13, \ldots$, vormen de welbekende Fibonacci getallen.

Substituties kunnen ook worden toegepast op verzamelingen in het vlak. In Figuur 1.1 wordt een rij verzamelingen $(C_0, C_1, \ldots)$ verkregen door driehoeken te vervangen door drie kleinere driehoeken. De limietverzameling C staat bekend als de Sierpiński driehoek en is een voorbeeld van een fractale verzameling. De driehoek bestaat uit drie verschoven en geschaalde kopieën van zichzelf. Om precies te zijn,

$$C = f_1(C) \cup f_2(C) \cup f_3(C),$$

waar $f_1(C) = \frac{1}{2}(C + (-1, 0))$, $f_2(C) = \frac{1}{2}(C + (1, 0))$ en $f_3(C) = \frac{1}{2}(C + (0, \sqrt{3}))$. De functieverzameling $\{f_1, f_2, f_3\}$ wordt een geïtereerd functiesysteem genoemd (IFS) en de Sierpiński driehoek is de attractor van de IFS. In Figuur 1.2 is de attractor van een andere IFS afgebeeld, de zogenaamde Heighway draak. De rand van de draak ziet er heel grillig en complex uit. Een wiskundige maat voor complexiteit is de Hausdorff dimensie. De Hausdorff dimensies van de Sierpiński driehoek en van de rand van de Heighway draak kunnen berekend worden en zijn gelijk aan $\log(3)/\log(2) = 1.5849\ldots$, respectievelijk $2\log\lambda/\log 2 = 1.5236\ldots$, waar λ de grootste reëele wortel is van $\lambda^3 - \lambda^2 - 2$.

In de praktijk kunnen bepaalde verschijnselen vaak beter beschreven worden met stochastische substituties. Vergelijk bijvoorbeeld de twee bomen in Figuur 1.3. De linker boom is gegenereerd door een deterministische substitutie en de rechter door een stochastische substitutie. Hoewel je niet verwacht om één van beide in een echt bos te zien, lijkt de stochastische boom toch meer op een echte boom dan de deterministische.

Begin jaren zeventig bedacht Mandelbrot het volgende stochastische substitutie model om turbulentie te beschrijven [17]. Laat p een parameter zijn tussen 0 en 1 en laat K_0 een zwart gekleurd eenheidsvierkant zijn. Verdeel K_0 in negen subvierkanten op de voor de hand liggende manier en kleur ieder sub-vierkant zwart met kans p en wit met kans $1-p$, onafhankelijk van de andere acht sub-vierkanten. Laat K_1 de verzameling zijn die bestaat uit de zwarte sub-vierkanten. Op vergelijkbare wijze verkrijgen we de verzameling K_2 door alle zwarte sub-vierkanten in K_1 op te delen in 9 sub-sub-vierkanten, ze zwart te kleuren met kans p en wit met kans $1-p$. Als we deze procedure willekeurig vaak herhalen, dan krijgen we een rij $K_0, K_1, \ldots$ van stochastische verzamelingen. Dit model om stochastische verzamelingen te genereren staat bekend als fractale percolatie of Mandelbrot percolatie. Figuur 1.4 toont een realisatie van de verzamelingen $K_1, \ldots, K_6$ voor $p = 0.75$. Fractale percolatie kan op een voor de hand liggende manier worden gegeneraliseerd naar dimensie d, waarbij ieder d-dimensionaal blok wordt onderverdeeld in M^d sub-blokken. De rij (K_n) is monotoon dalend en convergeert daarom naar een limietverzameling $K = \bigcap_{n=0}^{\infty} K_n$. Met een vertakkingsproces argument kan eenvoudig worden bewezen dat de limietverzameling leeg is als $pM^d \leq 1$ en dat K niet leeg is met positieve kans als $pM^d > 1$. Hoewel K een stochastische verzameling is, is de Hausdorff dimensie van K helemaal niet zo stochastisch, sterker nog, als K niet leeg is, dan is zijn dimensie constant en gelijk aan $\log(pM^d)/\log M$. In Hoofdstuk 2 en Hoofdstuk 3 gaan we dieper in op eigenschappen van fractale percolatie en geven we meer voorbeelden van stochastische substituties en fractale verzamelingen.

Fractale percolatie is niet een heel realistisch model voor turbulentie. Siebesma et al. opperden dat een model met buurafhankelijkheid turbulentie beter zou beschrijven [27]. Een voorbeeld van zo'n model is meerderheids fractale percolatie. Deel ieder niveau n vierkant I in 9 sub-vierkanten $J_1, \ldots, J_9$. De kans dat een sub-vierkant J_i zwart wordt gekleurd hangt in dit model niet alleen af van de kleur van I, maar ook van de kleur van de vierkanten die I omgeven. Om precies te zijn, de kans dat J_i zwart wordt gekleurd is $1-(1-p)^N$, waar N het aantal zwarte vierkanten is onder I en zijn 8 buren. In Figuur 1.5 is een realisatie afgebeeld van de eerste zes verzamelingen van meerderheids fractale percolatie voor $p = 0.15$.

Fractale percolatie en meerderheids fractale percolatie zijn voorbeelden van vertakkende cellulaire automaten (BCA), die we in Hoofdstuk 4 zullen bestuderen. We associëren rijen verzamelingen $K_0, K_1, \ldots$ met een BCA, die gegenereerd zijn door een stochastische substitutie met buurafhankelijkheid. In deze algemene opzet hoeven de verzamelingen $K_0, K_1, \ldots$ niet noodzakelijkerwijs te convergeren en we zullen voldoende voorwaarden voor convergentie geven. De bewijzen voor het uitsterven en de dimensie van de limietverzameling van fractale percolatie maken sterk gebruik van het ontbreken van buurafhankelijkheid. We ontwikkelen technieken om te bewijzen dat als een rij $K_0, K_1, \ldots$ geassocieerd met een BCA convergeert naar een limietverzameling K, dan is K leeg als $\lambda < 1$ en dan is K niet leeg met positieve kans als $\lambda > 1$, waar λ een eigenwaarde van een zogenaamde nakomelingenmatrix is. Verder is de Hausdorff dimensie van K constant en gelijk aan $\log \lambda / \log M$ als K niet leeg is. We zullen ook laten zien dat de rand van een limietverzameling van een BCA weer gelijk is aan de limietverzameling van een BCA.

Recentelijk is er veel belangstelling voor attractoren van geïteerde functie systemen en hun randen ontstaan vanwege hun toepasbaarheid in beeldcompressie en wavelet theorie. Er is diepgaand onderzoek verricht naar de attractoren van IFS'en die aan de sterke open verzameling voorwaarde voldoen en van specifieke voorbeelden van IFS'en die wat wilder van aard zijn. In Hoofdstuk 5 zullen we een subklasse bestuderen van de wederkerende geïtereerde functie systemen, een generalisatie van gewone IFS'en waarvoor de attractor een vector is van verzamelingen die aan een systeem van zelfgelijkvormigheids-vergelijkingen voldoen. Hoewel deze subklasse, de klasse van M-wederkerende geïtereerde functie systemen (M-RIFS), niet alle gewone IFS'en bevat, bevat zij wel IFS'en die niet aan de sterke open verzameling voorwaarde voldoen. We laten zien dat M-RIFS en deterministische BCA's equivalent zijn op de volgende wijze. Als we beginnen met een attractor $(C_0, \ldots, C_r)$ van een M-RIFS, dan kunnen we een BCA construeren met limietverzameling K zó, dat $C_0 = K$, en beginnend met een deterministische BCA met limietverzameling K, kunnen we een M-RIFS met attractor $(C_0, \ldots, C_r)$ construeren met de eigenschap dat $K = C_0$. Als we nu de resultaten uit hoofdstuk 4 toepassen aangaande de dimensie en de rand van limietverzamelingen van BCA's, dan volgt dat de rand van een component van de attractor van een M-RIFS weer gelijk is aan een component van de attractor van een M-RIFS en bovendien kunnen we de Hausdorff dimensie van de rand berekenen.

In Hoofdstuk 6 zullen we verbondenheidseigenschappen van de limietverzameling K van fractale percolatie onderzoeken. We zeggen dat de limietverzameling percoleert in dimensie twee als K een verbonden component bevat die zowel de linker als de rechter zijde van het eenheidsvierkant doorsnijdt. Zij $\theta_M(p)$ de kans dat K percoleert voor parameter p en onderverdeling in M^2 sub-vierkanten. Dan is het evident dat $\theta_M(0) = 0$ en $\theta_M(1) = 1$, maar hoe zit het met de waarden van p tussen 0 en 1? Met een koppelingstechniek kan eenvoudig worden bewezen dat $\theta_M(p)$ stijgend is in p. Definieer de kritische waarde $p_c(M)$ door

$$p_c(M) = \inf\{p : \theta_M(p) > 0\}.$$

De exacte waarde van $p_c(M)$ is tot op heden onbekend, maar er zijn verscheidene onder- en bovengrenzen gegeven. Niet-trivialiteit van de kritische waarde werd als eerste aangetoond door Chayes, Chayes en Durrett [4], die bewezen dat $p_c(M) \geq$

$1/\sqrt{M}$ voor $M \geq 2$ en $p_c(M) \leq p^*(M)$ voor $M \geq 3$, waar $p^*(M)$ het infimum is over p waarvoor $x = (px)^{M^2} + (px)^{M^2-1}(1 - px)$ een wortel heeft in het half open interval $(0, 1]$. Dekking en Meester [5] interpreteerden het bewijs van Chayes, Chayes and Durrett in termen van meerwaardige substituties en verbeterden de bovengrens tot $p_c(3) \leq 0.991$. Kort geleden bewees White [30] dat $p_c(2) \geq 0.810$ met behulp van een slim gekozen rooster. In Hoofdstuk 6 generaliseren we de methoden van Chayes, Chayes en Durrett, Dekking en Meester en White tot een methode om zowel boven- als ondergrenzen voor de kritische waarde te vinden. We bewijzen dat $p_c(3) \leq 0.965$ en we beschrijven technieken om nog scherpere grenzen te krijgen. Bovendien leggen we uit hoe onze methode toepasbaar is in de context van het doorgeven van genen in een generatieboom.

Dankwoord

Voor de totstandkoming van dit proefschrift ben ik veel mensen dank verschuldigd. In de eerste plaats wil ik mijn begeleider, Michel Dekking, bedanken voor vier en een half jaar intensief samenwerken. Je hebt me voortdurend van ideeën, inspiratie en kritiek voorzien en ik heb mogen profiteren van je ervaring en connecties in de kansrekening. Je hebt me veel laten zien van het vakgebied en me de vrijheid gegund mijn eigen weg daarin te zoeken. Ik wil de leden van mijn promotiecommissie bedanken voor het lezen van mijn proefschrift. Tijdens mijn promotie onderzoek was ik verbonden aan de afdeling CROSS van de Technische Universiteit Delft. Ik wil iedereen van de afdeling CROSS bedanken voor hun interesse en gezelligheid, met name mijn kamergenoot Marten Klok voor stimulerende discussies en spannende dartspelletjes. Van juni tot september 2002 heb ik Richard Kenyon bezocht aan de Université Paris-Sud XI in Orsay. Thank you Rick for some stimulating discussions and a wonderful summer in Orsay. Van september 2002 tot maart 2003 bezocht ik Bob Burton aan de Oregon State University in Corvallis. Thanks Bob for squeezing a couple of nice meetings with me and Larry into your tight schedule. Tijdens mijn verblijf in Corvallis heb ik veel mensen ontmoet, die mijn verblijf zeer de moeite waard hebben gemaakt. In het bijzonder wil ik Larry Pierce, Beth Timmons en T.T. Ton bedanken. Thank you Larry for inviting me to stay at your place for six months before we even met. I had a great time hanging out with you and doing math. Thank you Beth and T.T. for your company and for showing me Oregon. Ik wil mijn familie en vrienden bedanken voor hun betrokkenheid in mijn bezigheden en voor heel veel praktische hulp. Tenslotte wil ik mijn vriendin Sacha Brietenstein bedanken, die ik tijdens mijn verblijf in het buitenland ruim een half jaar niet heb gezien. Je bent het nooit helemaal eens geweest met mijn wilde reisplannen, maar je hebt me toch altijd gesteund. Ik ben je daar heel erg dankbaar voor.

Curriculum Vitae

Peter van der Wal is geboren op 1 november 1972 in Bennekom. In september 1991 begon hij zijn wiskundestudie aan de Universiteit Utrecht. Hij studeerde af op een onderwerp in percolatietheorie in mei 1997. Van juni 1997 tot maart 2002 was hij werkzaam als promovendus in de kansrekening aan de Technische Universiteit te Delft. Zijn werkzaamheden in deze periode hebben tot dit proefschrift geleid. In zijn periode als promovendus bezocht hij de Université Paris-Sud XI in Frankrijk (juni tot september 2001) en de Oregon State University in de Verenigde Staten (september 2001 tot maart 2002).