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The Riesz transform on a complete Riemannian manifold with Ricci curvature bounded from below

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"The Riesz transform on a complete Riemannian manifold with Ricci curvature bounded from below"

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Summary

In this thesis we study the Riesz transform and Hodge-Dirac operator on a complete Riemannian manifold with Ricci curvature bounded from below. The motivation for this is the paper '*Étude des transformation de Riesz dans les variétés riemanniennes à courbure de Ricci minorée*' by D. Bakry ([6]). In this paper, Bakry proves the boundedness of the Riesz transform acting on k-forms under the assumption that the Ricci-curvature, as well as related quadratic forms, are bounded from below. The analysis of this paper is one of two main goals in this thesis. In the second part we extend the operators defined by Bakry beyond L^2 to L^p-spaces for arbitrary $1 \leq p < \infty$ and analyse the Hodge-Dirac operator $\Pi = \mathrm{d} + \delta$ on L^p. For this, we will follow the lines of the paper '*Quadratic estimates and functional calculi of perturbed Dirac operators*' by A. Axelsson, S. Keith and A. McIntosh ([4]) and also of the paper '*Boundedness of Riesz transforms for elliptic operators on abstract Wiener spaces*' by J.Maas and J.M.A.M van Neerven ([25]).

Before we turn to the analysis of the paper however, we first need to introduce some basic theory of differential geometry and strongly continuous semigroups. We collect the necessary definitions and results in these areas to create a basic understanding of these subjects. For a more detailed discussion of these subjects one should look in the references made in chapters 2 and 3.

After the basic theory is discussed, we thoroughly discuss the paper of Bakry up to and including the section on the Riesz transform on k-forms. We first start out by introducing the Witten-Laplacian on smooth functions and 1-forms. These turn out to be self-adjoint on L^2, and via the spectral theory one can define the strongly continuous semigroups they generate. The lower bound for the Ricci-curvature is used to get useful estimates for these semigroups. Next, we define subordinated semigroups, the generators of which turn out to be useful in the proof of the boundedness of the Riesz transform. The final tools needed are two estimates, one of which is proved in a probabilistic manner, while the other is purely analytic. These tools are then combined to prove the boundedness of the Riesz transform on functions. In the final section we show that under minor adjustments, one can follow a similar approach in proving the boundedness of the Riesz transform on k-forms. It is this result that is most important for the remainder of the thesis.

We then present a general discussion of the theory of sectorial and bisectorial operators. We give the definitions of such operators, and also introduce the concept of R-(bi)sectoriality. We furthermore construct the H^∞-functional calculus for sectorial and bisectorial operators, which is based on the Dunford functional calculus. We will introduce the concept of a bounded H^∞-functional calculus and collect some results that we wish to use.

Finally, we extend the operators defined by Bakry only for smooth functions and k-forms to L^p for $1 \leq p < \infty$ and introduce the Hodge-Dirac operator $\Pi = \mathrm{d} + \delta$ on L^p. We then show that the Riesz transform on k-forms is also bounded on L^p for $1 < p < \infty$. From this, we deduce gradient bounds, which in turn imply the R-bisectoriality of the Hodge-Dirac operator. From the R-bisectoriality we deduce that Π has a bounded H^∞-functional calculus. We finish our results by showing that this again implies the boundedness of the Riesz-transform.

Preface

This thesis is the result of my work for my graduation for the master Applied Mathematics at TU Delft.

The subject of this thesis is based on the seminar 'Stochastic analysis on manifolds' organized by the Analysis department of DIAM. I attended all meetings so far, and also spoke at two of them. It really helped me to grasp the theory I was studying, and I also gained some experience in presenting mathematical material.

Additionally, the work done in this thesis is a preparation for the future PhD research that I intend to carry out at the TU Delft. This opportunity is offered to me by the Peter Paul Peterich fund, which presented me with a scholarship for PhD research, for which I am extremely thankful. Although the final results found in this thesis are not directly related to the projected PhD research, the theory studied to obtain the results form a firm basis for the future research. In this way, I was able to lay a good foundation for the coming years, while also broadening my knowledge even further.

I want to thank my supervisor Prof. Dr. J.M.A.M. van Neerven for guiding me through the project and teaching me about the many things one should think about when working in the field of (unbounded) operators. I furthermore want to thank Prof. Dr. F.H.J. Redig for quite some hours helping me out filling in details in proofs that I studied. Last, but not least, I want to thank family and friends for the mental support, allowing me to finish this thesis.

Now it only remains to say that I hope it is an enjoyable read!

Kind regards,

Rik Versendaal

Contents

Chapter 1

Introduction

In this thesis we study the Riesz transform and Hodge-Dirac operator on a complete Riemannian manifold M with Ricci curvature bounded from below. This gives us a natural way to divide the thesis into two parts. In the first part we discuss the boundedness of the Riesz transform acting on so called differential forms. For this, we analyse the paper '*Étude des transformation de Riesz dans les variétés riemanniennes à courbure de Ricci minorée*' by D. Bakry ([6]) in which this is ultimately proved. In the second part we turn to the analysis of the Hodge-Dirac operator on $L^p(\Lambda TM)$ for $1 < p < \infty$. We prove various properties such as the R-sectoriality and the fact that it has a bounded H^∞-functional calculus. These ideas are in light of the paper '*Quadratic estimates and functional calculi of perturbed Dirac operators*' by A. Axelsson, S. Keith and A. McIntosh ([4]) and can also be found in the paper '*Boundedness of Riesz transforms for elliptic operators on abstract Wiener spaces*' by J.Maas and J.M.A.M van Neerven ([25]).

1.1 The Riesz transform

On $\mathbb{R}^n$ we can consider the Laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ a priori defined on $C_0^\infty(\mathbb{R}^n)$. It can be shown that Δ is essentially self-adjoint on $L^2(\mathbb{R}^n)$, from which it follows that its closure, again denoted by Δ, is a self-adjoint operator on $L^2(\mathbb{R}^n)$. One can also show that Δ is a negative operator, and consequently, we can define the operator $(-\Delta)^{1/2}$ via the spectral theorem. The Riesz transforms can now be defined as the operators $R_i f = \frac{\partial}{\partial x_i}(-\Delta)^{-1/2} f$ which map $(-\Delta)^{1/2} f$ to $\frac{\partial f}{\partial x_i}$. The boundedness of the Riesz transforms on $L^p(\mathbb{R}^n)$ then entails that these operators are well-defined and bounded on $L^p(\mathbb{R}^n)$. This is equivalent to stating that there exist constants $c, C > 0$ independent of f such that

$$c||\nabla f||_p \leq ||(-\Delta)^{1/2} f||_p \leq C||\nabla f||_p.$$

In particular, this estimate implies that for $1 < p < \infty$ the Riesz transforms are bounded on $L^p(\mathbb{R}^n)$.

However, on an arbitrary complete Riemannian manifold M, this need not hold. As shown in [6] it turns out that under the assumption that the Ricci curvature is bounded from below the claim remains true. The Riesz transform on a complete Riemannian manifold has been the subject of various other studies. In [3] Hardy spaces of differential forms are constructed to study the Riesz transform on such forms. This is done in such a way that the Riesz transform is automatically bounded on them. These constructions are made under the additional assumption that the manifold satisfies the doubling property: If $V(x, r)$ denotes the measure of the geodesic ball $B(x, r)$ with centre x and radius r, then there must exist a $C > 0$ such that for all $x \in M$

and all $r > 0$
$$V(x, 2r) \leq CV(x, r).$$
This assumption simply means that if we double the radius of a ball, its volume increases at most by some uniformly fixed factor.

In the process, the authors define the Hodge-Dirac operator $\mathrm{d} + \mathrm{d}^*$ on $L^2(\Lambda TM)$. Here, ΛTM is the space of differential forms of any order and $L^2(\Lambda TM)$ denotes all square integrable sections. In that case the Hodge-de Rham Laplacian is given as $\Delta = (\mathrm{d} + \mathrm{d}^*)^2 = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$. The following result confirms the relation to the Riesz transform $(\mathrm{d} + \mathrm{d}^*)\Delta^{-1/2}$ as mentioned above. This is corollary 1.3 in [3].

Corollary 1.1.1. *Assume that M has the doubling property. Then for all $1 \leq p \leq \infty$, $(\mathrm{d} + \mathrm{d}^*)\Delta^{-1/2}$ is $H^p(\Lambda TM)$ bounded. Consequently, it is $H^1(\Lambda TM) - L^1(\Lambda TM)$ bounded.*

However, in the same paper it is left open whether $H^p(\Lambda TM)$ can be described in terms of $L^p(\Lambda TM)$ for some or all $p \in (1, \infty) \setminus \{2\}$. The relation between $H^p(\Lambda TM)$ and $L^p(\Lambda TM)$ is only given in the form of two inclusions, which can be found in corollary 1.2 in [3].

The topic of Hardy spaces is further studied in [2], which also fills a gap in [3]. Although in this thesis we are not directly interested in Hardy spaces, it is closely related to our study and it shows in which direction current research is heading.

1.2 Hodge-Dirac operator

We are also interested in the Hodge-Dirac operator $\mathrm{d} + \mathrm{d}^*$, which was already briefly mentioned in the previous section. In [4] a study is caried out of Dirac type operators on a Hilbert space $\mathcal{H}$, of which the Hodge-Dirac operator on $L^2(\Lambda TM)$ is a special case. It is shown that one has the orthogonal decomposition
$$L^2(\Lambda TM) = N(\mathrm{d} + \mathrm{d}^*) \oplus \overline{R(d)} \oplus \overline{R(d^*)}.$$
In this paper it is furthermore shown that $\mathrm{d} + \mathrm{d}^*$ is bisectorial in the sense that its spectrum is contained in a double sector Σ_ω and that it satisfies the resolvent bounds
$$||(I + \tau(\mathrm{d} + \mathrm{d}^*))^{-1}|| \lesssim \frac{|\tau|}{\mathrm{dist}(\tau, \Sigma_\omega)}.$$
for all $\tau \in \mathbb{C} \setminus \Sigma_\omega$. Here $\Sigma_\omega = \Sigma_\omega^+ \cup (-\Sigma_\omega^+)$, where $\Sigma_\omega^+ = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \omega\}$. Additionally, it is shown in theorem 2.10 that $\mathrm{d} + \mathrm{d}^*$ has a bounded holomorphic functional calculus in $L^2(\Lambda TM)$. We aim to extend this study to the Hodge-Dirac operator on $L^p(\Lambda TM)$ for $1 < p < \infty$.

The idea to achieve this is inspired by [25] in which the boundedness of the Riesz transform is proved in an arbitrary UMD space, where one considers the Malliavin derivative D rather than the exterior derivative d. In this paper the Hodge-Dirac operator is considered in matrix form on $L^p \oplus \overline{R_p(D)}$, where $1 < p < \infty$, defined as
$$\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}.$$
It is shown that the boundedness of the Riesz transform implies R-gradient bounds, which are essentially off-diagonal estimates. These are then used to show that the Hodge-Dirac operator is R-bisectorial. Under assumptions on the operator $\underline{L} = DD^*$ it is shown that the Hodge-Dirac operator has a bounded H^∞-functional calculus, which in turn implies the boundedness of the Riesz transform. This will be the path that we follow in chapter 6.

1.3 Relating our work

In the papers that are discussed in this introduction, the Hodge-Dirac operator $\mathrm{d} + \mathrm{d}^*$ is only defined for the Hilbert space $L^2(\Lambda TM)$. In [3] the Riesz transform is then extended to the space $H^p(\Lambda TM)$ for all $1 \leq p \leq \infty$ as stated in theorem 5.11 of the paper. Instead of considering the spaces $H^p(\Lambda TM)$, we consider the spaces $L^p(\Lambda TM)$ for $1 < p < \infty$. Furthermore, we not only consider the Riesz transform on $L^p(\Lambda TM)$ for $p \in (1, \infty)$, but we also define the Hodge-Dirac operator as a closed operator on $L^p(\Lambda TM)$. This can be found in section 6.1.

Additionally, theorem 6.3.12 is a special case of the work done in [4] in the case where $p = 2$. Although we restrict ourselves to a specific operator, we show that the results found for this operator in [4] may be extended beyond the Hilbert space $L^2(\Lambda TM)$ to the spaces $L^p(\Lambda TM)$ for $1 < p < \infty$.

Finally, we consider the Witten Laplacian instead of the usual Hodge Laplacian, the difference being that we also consider a potential. We thus obtain results for the Hodge Laplacian as a special case of the theorems presented in chapter 6.

This thesis is structured as follows: In chapter 2, we give an overview of the most important objects concerning differential geometry. We introduce smooth manifolds and tangent vectors. We then turn to differential forms and the exterior derivative d. We also define the Riemannian metric and consequently also Riemannian manifolds. Following this, we discuss the covariant derivative and curvature of a Riemannian manifold. We finish the chapter by introducing the volume measure on a Riemannian manifold, from which we can define the divergence and Laplace-Beltrami operator. Chapter 3 is devoted to strongly continuous semigroups of linear operators. We give the standard definitions, and collect various results that we need in later chapters.

In chapter 4 we analyse the paper [6] of Bakry. We go through the first five sections, working out the various details which are typically left to reader. We finally end up with the boundedness of the Riesz transforms on differential k-forms.

With this at hand, we can move on to the second part. Before we present our results, we first introduce the concepts of sectorial and bisectorial operators in chapter 5. We go over R-sectoriality and the H^∞-functional calculus based on the Dunford functional calculus. In chapter 6 we then turn to the analysis of the Hodge-Dirac operator on $L^p(\Lambda TM)$ for $1 < p < \infty$. We first extend the operators defined on L^2 in chapter 4 to L^p and show that these turn out to be consistent. We also show that the boundedness of the Riesz transform remains true. From hereon we follow the lines of [25] as discussed above.

Chapter 2

Differential geometry

In this chapter we will introduce the most important aspects of differential geometry. Our aim is to give an intuitive understanding of the objects used wherever possible. We will also collect various results which are either important for the general overview of the theory, or are of use to us in later chapters. The first three sections follow the lines of [32]. The material in the fourth section can be found in [21]. For the last section we used [14] and [30]. For a more intuitive introduction to manifolds, we refer to [5].

We will start out by defining smooth manifolds and tangent vectors. Afterwards we will concern ourselves with differential forms. Once we have these basic concepts, we will move on to Riemannian metrics. Following this, we will have a look at the covariant derivative and curvature of a manifold. We finish the discussion by introducing a volume measure, together with the Laplace-Beltrami operator.

2.1 Manifolds and tangent vectors

Before we can make any progress, we will first need to define what we mean by a manifold. The most simple definition is as follows:

Definition 2.1.1 (Manifold)**.** Let (M, d) be a metric space. We call M a manifold if for all $x \in M$ there exists an open neighbourhood U of x and some integer $n \geq 0$ such that U is homeomorphic to $\mathbb{R}^n$. Here we consider $\mathbb{R}^n$ with its usual Eucledian metric.

However, we will restrict our attention to those manifolds for which there exists a countable base for the topology introduced by the metric. This definition corresponds to the one in [14]. This technical detail will be of importance if we are going to consider a volume measure on a Riemannian manifold.

It turns out that the integer $n \geq 0$ required in the definition is uniquely determined by the point $x \in M$. This follows from the fact that if $n \neq m$, then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic. If we can use the same n for the entire manifold, we will say that the manifold is n-dimensional.

Intuitively speaking, a manifold is nothing more than a metric space which looks locally like $\mathbb{R}^n$. A homeomorphism between a neighbourhood U of $x \in M$ and $\mathbb{R}^n$ can be used to define coordinates on the patch $U \subset M$. We will now give some examples of manifolds.

Example 2.1.2. 1. The easiest example of a manifold is $\mathbb{R}^n$ itself with a metric which is equivalent to the Eucledian metric. The equivalence of the metrics implies that the identity is a homeomorphism on the whole space, thus for any $x \in \mathbb{R}^n$ we may take $U = \mathbb{R}^n$ together with the identity as homeomorphism.

2. Any open subset O of $\mathbb{R}^n$ with the induced metric is also a manifold. Indeed, for $x \in O$ we can find an open ball $x \in B \subset O$. As any open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ the claim now follows.

3. The unit circle S^1 is a 1-dimensional manifold. For simplicity, we will consider S^1 as a subset in $\mathbb{R}^2$. First take $x \in S^1$, $x \neq (0,1)$. Take $U = S^1 \setminus \{(0,1)\}$. One can now construct a homeomorphism from U to $\mathbb{R} \times \{-1\}$ by mapping a point $y \in U$ to the point on $\mathbb{R} \times \{-1\}$ where the line through y and $(0,1)$ intersects the line $\mathbb{R} \times \{-1\}$. This idea is sketched in figure 2.1. From this construction we deduce the homeomorphism $f : U \to \mathbb{R}$ given by $f(x,y) = \frac{2x}{1-y}$. If $x = (0,1)$, one can make a similar homeomorphism, now defined on $U = S^1 \setminus \{(0,-1)\}$. Note that this idea may also be applied to the sphere S^n for general $n \in \mathbb{N}$, where one should replace the line by a hyperplane.

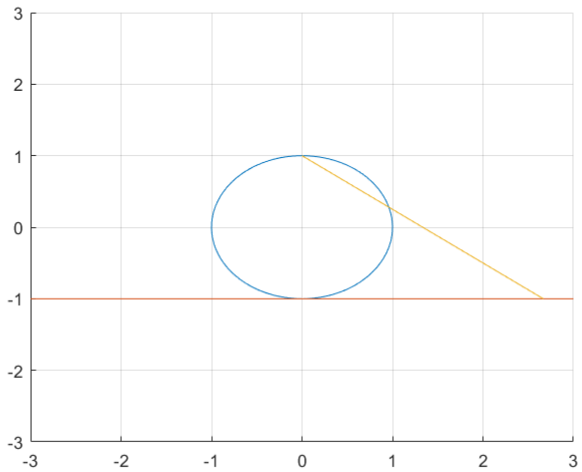

Figure 2.1: Construction of a homeomorphism to show that S^1 is a manifold.

2.1.1 Smooth manifolds

As a manifold M is a metric space, it is perfectly sensible to talk about continuous functions $f : M \to \mathbb{R}$. However, unlike for functions on $\mathbb{R}^n$, we cannot yet speak about the differentiability of a function. A naive attempt to define what we mean by a function f on a manifold to be differentiable, is to take a homeomorphism $\phi : U \subset M \to \mathbb{R}^n$ and say that f is differentiable if $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable. However, we could also pick another homeomorphism $\psi : V \subset M \to \mathbb{R}^n$ with $U \cap V \neq \emptyset$. It is then not necessarily true that also $f \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable. The point is that on $U \cap V$ we may write

$$f \circ \psi^{-1} = f \circ \phi^{-1} \circ (\phi \circ \psi^{-1}).$$

The differentiability thus carries over if we know that $\phi \circ \psi^{-1}$ is differentiable. This observation lies at the heart of the definition of a smooth manifold, where smooth is understood to be C^∞. In the remainder of the discussion, unless otherwise stated, by differentiable we mean infinitely differentiable.

Definition 2.1.3. Let $U, V \subset M$ and suppose that $x : U \to x(U) \subset \mathbb{R}^n$ and $y : V \to y(V) \subset \mathbb{R}^n$ are two homeomorphisms. We say that these maps are C^∞-related if the maps

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$$

$$x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$$

are smooth maps.

A collection of mutually C^∞-related homeomorphisms whose domains cover all of M is called an *atlas* for M. An element (x, U) of an atlas is referred to as a chart or coordinate system. The reason for this is that the map x, or better x^{-1}, induces coordinate lines on the set U. We call an atlas *maximal* if no further charts can be added which are C^∞-related to all other charts in the atlas. It is possible to show that any atlas for M is contained in a unique maximal atlas for M. We are now able to define what we mean by a smooth manifold.

Definition 2.1.4 (Smooth manifold)**.** A pair $(M, \mathcal{A})$ is called a smooth manifold if $\mathcal{A}$ is a maximal atlas for the manifold M.

Example 2.1.5. The sphere S^n with the maximal atlas generated by the homeomorphisms as discussed in example 2.1.2 is smooth. Note that it suffices to check only for the charts discussed that they are C^∞-related. We show this for S^1. Let $U = S^1 \setminus \{(0,1)\}$ and $V = S^1 \setminus \{(0,-1)\}$ with homeomorphism $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ given by $f(x,y) = \frac{2x}{1-y}$ and $g(x,y) = \frac{2x}{1+y}$. Then $g^{-1}(t) = (\frac{8t}{2t^2+8}, \frac{-2t^2+8}{2t^2+8})$. It follows that $f \circ g^{-1}(t) = \frac{16t^3}{(t^2+4)^2}$ which is smooth on $g(U \cap V) = \mathbb{R} \setminus \{0\}$. The case $g \circ f^{-1}$ can be treated similarly.

We are now able to define differentiability of a function between two manifolds M and N. We call a function $f : M \to N$ differentiable if for any pair of charts (x, U) for M and (y, V) for N the map $y \circ f \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable. As all charts used are C^∞-related, this definition does not depend on the choice of chart. If $f : M \to \mathbb{R}$ we can also identify the derivative in a chart. Indeed, we may define

$$\frac{\partial f}{\partial x^i}(p) = D_i(f \circ x^{-1})(x(p)).$$

It turns out that in a chart one can (almost) forget that we are working on a manifold instead of on $\mathbb{R}^n$. Indeed, rules like the chain rule and product rule are still valid in a chart.

2.1.2 Tangent vectors

In this section we will introduce the concept of tangent vectors to a smooth manifold M. We will follow an intuitive explanantion, but we warn the reader that this is not an easy concept when following the precise treatment (see for example [32]). When we think of a manifold as living in some higher dimensional Eucledian space, tangent vectors are indeed precisely what they should be: tangent vectors to the surface formed by our manifold. Generally, these tangent vectors point in the various directions that one could travel when 'walking' on the manifold.

To be just a little more precise, there are two ways to define tangent vectors which are intuitively understandable. First of all, we wish to attach to every point $p \in M$ a so called tangent space T_pM, which contains all possible tangent vectors at the point p, and is a space of the same dimension as the manifold at the point p. This means that tangent vectors at different points come from different spaces, hence we cannot in general try to compare them, and do arithmetic with them!

A first way now to define tangent vectors at a point $p \in M$, is to consider smooth curves $c : (-\epsilon, \epsilon) \to M$ with $c(0) = p$. For a chart (x, U) we then call two curves c_1 and c_2 equivalent if the maps $x \circ c_1$ and $x \circ c_2$ have the same derivative in 0. This defines an equivalence relation, independent of the chart. We then define the tangent space at p to consist of all equivalence classes. The interpretation of this approach is that a tangent vector defined via a chart (x, U) around p is the derivative of $x \circ c$ in the point 0. Tangent vectors thus become the 'speed' of curves in the manifold through the point p.

A second way, and also the most commonly used method for computations, is to think of tangent vectors as directions in which we want to differentiate functions. For this, we say that a linear operator l acting on the smooth functions is a derivation at p if $l(fg) = f(p)l(g)+g(p)l(f)$. We now define the tangent space at p to be the set of all linear derivations at the point p. It turns out that this space has dimension equal to the manifold M at the point p, and is spanned in a coordinate chart (x, U) by the operators $\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^n}|_p$.

In an abstract way we can put all tangent spaces together to form the tangent bundle $TM = \bigsqcup_{p \in M} T_pM$, which can itself be made in a smooth manifold again. This means that we can talk about smooth functions on TM, which we will refer to as sections. Such a smooth section is called a *vector field*. We thus get a vector field by attaching to each point in M a tangent vector from the corresponding tangent space, so that the tangent vectors vary in a smooth way.

2.2 Differential k-forms

In this section we introduce differential k-forms, and the exterior derivative operator d acting on such forms.

2.2.1 Differential of a function

The construction of the tangent bundle by pasting tangent spaces together can also be done with different spaces. Instead of the tangent spaces, we could consider their duals $(T_pM)^*$. These can also be put together to form a bundle T^*M. For a smooth function $f : M \to \mathbb{R}$, we define the smooth section $\mathrm{d}f$ of T^*M by

$$\mathrm{d}f(p)(X) = X(f)$$

for all $X \in T_pM$. We call $\mathrm{d}f$ the differential of f. It is easy to see that in a coordinate chart (x, U), the differentials $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ form a basis for $(T_pM)^*$. Consequently every section ω can be uniquely represented in a chart as

$$\omega(p) = \sum_{i=1}^{n} \omega_i(p)\mathrm{d}x^i(p)$$

which we will write as

$$\omega = \sum_{i=1}^{n} \omega_i \mathrm{d}x^i.$$

It holds that a section ω is smooth precisely when its coordinate functions ω_i are smooth. For the differential of a smooth function f we get the classical formula.

Theorem 2.2.1. *If (x, U) is a chart for M and $f : M \to \mathbb{R}$ is smooth, then on U*

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\mathrm{d}x^i.$$

The proof of such theorems often simply amount to applying both sides to an arbitrary tangent vector and concluding that the result is the same. The above theorem is no exception!

2.2.2 Alternating tensors and the wedge product

The differentials of functions defined above are examples of what we call 1-forms. Before we define exactly what these, and k-forms in general, are, we first make a construction based on an arbitrary vector space V of dimension n. We will then later apply this to the case where V is a tangent space. Denote by $\mathcal{T}^k(V)$ the set of all multilinear maps from V^k to $\mathbb{R}$. Here multilinear means that it is linear in each of its entries. Let $\Lambda^k(V) \subset \mathcal{T}^k(V)$ be the subspace consisting of all alternating elements of $\mathcal{T}^k(V)$. Note that we call such a map T alternating if $T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = 0$ whenever $v_i = v_j$ for some $1 \leq i, j \leq k$. It holds that an alternating map is also skew-symmetric, i.e.,

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

For elements $T \in \mathcal{T}^k(V)$ and $S \in \mathcal{T}^l(V)$, we can define the tensor product $T \otimes S \in \mathcal{T}^{k+l}(V)$ by

$$T \otimes S(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = T(v_1, \ldots, v_k)S(v_{k+1}, \ldots, v_{k+l}).$$

However, for $\Lambda^k(V)$, this tensor product is not suitable, as the product is not necessarily alternating again. We therefore need to find an alternative, which we will call the 'wedge product'.

In order to do this, denote by S_k the set of all permutations of the set $\{1, \ldots, k\}$. For a k-tuple $(v_1, \ldots, v_k)$ and $\sigma \in S_k$ we denote

$$\sigma \cdot (v_1, \ldots, v_k) = (v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

For an element $T \in \mathcal{T}^k(V)$ we can now define its alternation by

$$\text{Alt } T = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}\sigma \cdot T \circ \sigma$$

i.e., applied to a k-tuple $(v_1, \ldots, v_k)$ we get that

$$\text{Alt } T(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}\sigma \cdot T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

The alternation satisfies the following expected properties:

Proposition 2.2.2. *1. If $T \in T^k(V)$, then Alt(T) $\in \Lambda^k(V)$.*

2. If $\omega \in \Lambda^k(V)$, then Alt $\omega = \omega$.

3. If $T \in T^k(V)$, then Alt(Alt(T)) = Alt(T).

For an element $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$ we now define the wedge product $\omega \wedge \eta \in \Lambda^{k+l}(V)$ by

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta).$$

Note that the wedge product satisfies the following properties:

Proposition 2.2.3. *1. $\wedge$ is bilinear.*

2. $\wedge$ is anti-commutative, i.e., for $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$ we have that $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.

3. If $\omega \in \Lambda^k(V), \eta \in \Lambda^l(V)$ and $\theta \in \Lambda^m(V)$ then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta).$$

We will now try to find a basis for $\Lambda^k(V)$. Let $v_1, \ldots, v_n$ be a basis for V and denote by $\phi_1, \ldots, \phi_n$ the corresponding dual basis. It turns out that the following theorem holds:

Theorem 2.2.4. *The set*

$$\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis for $\Lambda^k(V)$, and consequently $\Lambda^k(V)$ has dimension $\begin{pmatrix} n \\ k \end{pmatrix}$. Note that in particular $\Lambda^k(V) = \{0\}$ for $k > n$.

Based on the above theorem, we often write I for a multi-index $(i_1, \ldots, i_k)$ where $1 \leq i_1 < \cdots < i_k \leq n$ and write ϕ_I for $\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}$. In that case any element $\omega \in \Lambda^k(V)$ may be uniquely expressed as

$$\sum_I \omega_I \phi_I.$$

Finally, we call $\Lambda V := \bigoplus_{k=0}^n \Lambda^k V$ the *exterior algebra* over V.

2.2.3 Differential k-forms

We will now apply the construction in the previous section to $V = T_pM$. To each point of the manifold we attach the space $\Lambda^k(T_pM)$. These can again be made in a bundle, similar to the tangent bundle. We will denote this bundle by $\Lambda^k(TM)$. A section of $\Lambda^k(TM)$ is called a k-form.

Note that in a coordinate chart (x, U) the 1-forms $\mathrm{d}x^1 \ldots, \mathrm{d}x^n$ form the dual basis to $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$. A k-form ω can thus be uniquely written as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

which, using the convection that $\mathrm{d}x^I$ stands for $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, can be written as

$$\omega = \sum_I \omega_I \mathrm{d}x^I.$$

For convenience we will set $\Lambda^0(T_pM) = \mathbb{R}$, in which case the 0-forms are precisely the C^∞ functions on M. The wedge product with a function is then understood to simply mean multiplication.

Remember that we defined the differential of a function f by $\mathrm{d}f(X) = X(f)$. This is a 1-form, given in a chart (x, U) by

$$\mathrm{d}f = \sum_{j=1}^n \frac{\partial f}{\partial x^j} \mathrm{d}x^j.$$

If ω is a k-form, say

$$\omega = \sum_I \omega_I \mathrm{d}x^I$$

we can define a $(k+1)$-form $\mathrm{d}\omega$, the differential of ω, as

$$\mathrm{d}\omega = \sum_I \mathrm{d}\omega_I \wedge \mathrm{d}x^I = \sum_I \sum_{j=1}^n \frac{\partial \omega_I}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^I.$$

It turns out that this definition does not depend on the chosen chart, and that this operator d, called the *exterior derivative*, is the unique extension of the differential on functions with the following properties:

Proposition 2.2.5. *1.* $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$.

2. If ω_1 *is a* k*-form, then*

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2.$$

3. $\mathrm{d}(\mathrm{d}\omega) = 0$, *i.e.,* $\mathrm{d}^2 = 0$.

We conclude the introduction of k-forms by defining the class C_0^∞ of smooth, compactly supported k-forms. The k-forms as defined above are smooth by definition, as we assume them to be smooth sections of $\Lambda^k(TM)$. Note that this is equivalent to stating that its coefficients in any given chart are smooth.

By the support of a k-form we mean the closure of the set of points on the manifold M where ω is not identically 0, i.e.,

$$\mathrm{supp}(\omega) = \overline{\{p \in M : \omega(p) \not\equiv 0\}}.$$

We call ω compactly supported if its support is a compact set in M.

2.3 Riemannian manifolds

We will now turn to Riemannian manifolds, which essentially are smooth manifolds on which we can speak about length. In order to do this, we need to have an inner product. In our case, by an inner product we mean a symmetric, positive definite bilinear form (sesquilinear if we work over $\mathbb{C}$ instead of $\mathbb{R}$.) Note that an innerproduct on a vector space V is an element of $\mathcal{T}^2(V)$. For now, we will denote the inner product by $\langle \ , \ \rangle$.

Suppose that $v_1, \ldots, v_n$ is a basis for V, and denote by $\phi_1, \ldots, \phi_n$ its dual basis. If we define $g_{ij} = \langle v_i, v_j \rangle$ for all $1 \leq i, j \leq n$, we can write

$$\langle \ , \ \rangle = \sum_{i,j=1}^{n} g_{ij} \phi_i \otimes \phi_j.$$

Given an inner product, we define a norm in the usual way by setting $||v|| = \sqrt{\langle v, v \rangle}$.

Apart from speaking about length, an inner product also allows us to identify V with V^*. Indeed, by the Riesz-representation theorem, for any $\omega \in V^*$ there exists a unique $w \in V$ such that for all $v \in V$ it holds that $\omega(v) = \langle w, v \rangle$. Consequently, we can also define the inner product of two elements of V^*. In order to find an expression for this inner product, we first observe that as $G := (g_{ij})$ is positive definite, it is in particular invertible, as all eigenvalues are greater than 0. Hence we may write g^{ij} for the components of G^{-1}, in which case we can write the corresponding inner product on V^* as

$$\langle \ , \ \rangle^* = \sum_{i,j=1}^{n} g^{ij} v_i \otimes v_j$$

where $v_1, \ldots, v_n$ is again the basis for V as before.

Observe that we can apply the above construction to $V = T_pM$ for any point $p \in M$. A smooth selection of inner products at each point $p \in M$ is called a *Riemannian metric* on M.

A smooth manifold together with a Riemannian metric is called a *Riemannian manifold.* In a coordinate chart (x, U) on M we can represent the Riemannian metric as

$$\langle \ , \ \rangle = \sum_{i,j=1}^{n} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

where the g_{ij} are smooth functions, satisfy $g_{ij} = g_{ji}$ by the symmetry of the inner product, and $\det(g_{ij}) > 0$ as the inner product is positive definite. As in the case before we can define the inner product of two 1-forms by

$$\langle \ , \ \rangle^* = \sum_{i,j=1}^{n} g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

We can also define the inner product between arbitrary k-forms. By linearity of the inner product, it is sufficient to define the inner product between k-forms which are the wedge product of 1-forms. Say $\omega = \omega_1 \wedge \cdots \wedge \omega_k$ and $\eta = \eta_1 \wedge \cdots \wedge \eta_k$ are such k-forms. We then define their inner product as

$$\langle \omega, \eta \rangle = \det[(\langle \omega_i, \eta_j \rangle^*)]$$

Using this innerproduct, we define the length of a k-form ω as $|\omega| = \sqrt{\langle \omega, \omega \rangle}$.

We finish the section by defining completeness for a Riemannian manifold. Although there are various ways to characterise this, we will only give the one that we use in the future, and that we can give without the need to introduce other objects. Proposition C.4.1 in [8] assures that we can do this.

Definition 2.3.1 (Completeness)**.** A smooth Riemannian manifold M is called complete if there exists a sequence $(h_n)_n$ of smooth compactly supported functions on M such that $|\mathrm{d}h_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} h_n = 1$ pointwise.

2.4 Covariant derivative and curvature

In this section we will introduce the concept of covariant derivative, which is a way to differentiate a vector field in the direction of some other vector field. Along with this, we will also define the total covariant derivative. These objects can be defined for arbitrary smooth manifolds. In the specific case of a Riemannian manifold, more can be said. Finally, we will give a concise description of curvature of a Riemannian manifold.

2.4.1 Covariant derivative

The question of how one should define the derivative of a vector field is far from trivial. Indeed, as we stressed earlier, comparing tangent vectors attached to different points is a priori impossible, as they come from different spaces. It turns out that in general, there is no unique way to define the derivative of a vector field in the direction of some other vector field such that it satisfies the 'usual' properties. Therefore, we have the following definition, where $\Gamma(TM)$ stands for the space of smooth sections in TM, i.e., the space of vector fields.

Definition 2.4.1 (Linear connection)**.** A linear connection is a map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

written as $(X, Y) \mapsto \nabla_X Y$, which satisfies the following properties:

1. $\nabla_X Y$ is linear over C^∞ in X, i.e., for all $f, g \in C^\infty$

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y.$$

2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y, i.e., for all $a, b \in \mathbb{R}$

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2.$$

3. For all $f \in C^\infty$, ∇ satisfies the following Leibniz rule

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y.$$

$\nabla_X Y$ is called the covariant derivative of Y in the direction of X.

The following property of the covariant derivative is good to keep in mind:

Proposition 2.4.2. *Suppose that ∇ is a linear connection on a manifold M, and that $X, Y \in \Gamma(TM)$. Then $\nabla_X Y|_p$ depends only on values of Y in a neighbourhood of p and the value of X at p.*

When working with manifolds, we often restrict ourselves to a chart (x, U). Remember that $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ forms a basis for the tangent space. Consequently, $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$ has a unique expression in terms of this basis, which we will write as

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

This defines functions Γ_{ij}^k which are called the Christoffel symbols of ∇ in the given coordinate chart.

In order to assure the existence of connections, let us give an example.

Example 2.4.3 (Euclidean connection)**.** Consider $\mathbb{R}^n$ with its usual Eucledian metric, which is obviously a manifold. Define a connection by

$$\nabla_X \left(\sum_{j=1}^n Y^j \frac{\partial}{\partial x^j} \right) = \sum_{j=1}^n (XY^j) \frac{\partial}{\partial x^j}.$$

What this connection does, is simply differentiating the coefficient functions in the direction of X. For completeness, we will verify that this indeed defines a connection. First suppose that $f, g \in C^\infty$. Then

$$\nabla_{fX_1+gX_2} Y = \sum_{j=1}^n fX_1(Y^j) \frac{\partial}{\partial x^j} + gX_2(Y^j) \frac{\partial}{\partial x^j} = f\nabla_{X_1} Y + g\nabla_{X_2} Y.$$

In the same way we find for $a, b \in \mathbb{R}$ that

$$\nabla_X(aY_1 + bY_2) = \sum_{j=1}^n aX(Y_1^j) \frac{\partial}{\partial x^j} + bX(Y_2^j) \frac{\partial}{\partial x^j} = a\nabla_X Y_1 + b\nabla_X Y_2$$

where we used the linearity of X. Finally, as X is a derivation we find for a function $f \in C^\infty$ that

$$\nabla_X(fY) = \sum_{j=1}^n X(fY^j) \frac{\partial}{\partial x^j} = \sum_{j=1}^n fX(Y^j) \frac{\partial}{\partial x^j} + Y^j X(f) \frac{\partial}{\partial x^j} = f\nabla_X Y + YX(f).$$

Finally, let us compute the Christoffel symbols for a coordinate chart (x, U). In that case we find

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \frac{\partial \delta_{jk}}{\partial x^i} \frac{\partial}{\partial x^k} = 0$$

where δ_{jk} denotes the Dirac-delta interpreted as a constant function on the chart (x, U). We conclude that all Christoffel symbols are 0.

It turns out that having a linear connection is not a propery of special manifolds, as the following proposition holds:

Proposition 2.4.4. *Every smooth manifold admits a linear connection.*

One can prove this by first showing that in a chart one can always find a linear connection, and then paste those together in a suitable way using a partition of unity.

A linear connection on vector fields automatically induces a linear connection on all tensor bundles under some assumptions as stated in lemma 4.6 in [21]. In particular, for the covariant derivative of a function f in the direction of a vector field X we require that

$$\nabla_X f = Xf.$$

Our main interest for this extended connection is that we can compute the covariant derivative of k-forms. If ω is a k-form and X is a vector field, we have for vector fields $Y_1, \ldots, Y_k$ that

$$(\nabla_X \omega)(Y_1, \ldots, Y_k) = X(\omega(Y_1, \ldots, Y_k)) - \sum_{j=1}^{k} \omega(Y_1, \ldots, \nabla_X Y_j, \ldots, Y_k).$$

This is formula (4.7) on page 54 of [21].

We will finish this section by defining the total covariant derivative. Although one can do this for arbitrary tensor fields, we will restrict ourselves to k-forms, as these form the only instance in which we will use it. Suppose that ω is a k-form. We define its total covariant derivative as the map $\nabla \omega : \Gamma(TM) \times (\Gamma(TM))^k \to C^\infty$ by

$$\nabla \omega(X, Y_1, \ldots, Y_k) = \nabla_X(Y_1, \ldots, Y_k).$$

Note that here we follow the convention as in [6] by using the first entry as the direction in which we differentiate, while in [21] the last entry is used for this.

An important example of the use of the total covariant derivative is the Hessian of a function $f \in C_0^\infty(M)$, which is defined as $\nabla\nabla f$. Note that this definition makes sense as we can consider a function as a 0-form. In proposition A.1.3 in the appendix it is shown that for vector fields X, Y it holds that

$$\nabla\nabla f(X, Y) = X(Yf) - (\nabla_X Y)f.$$

2.4.2 Levi-Civita connection and normal coordinates

It may be clear from the definition of a connection as above that it is not uniquely defined for a manifold. In the case of a Riemannian manifold, we can choose a unique connection with certain properties, which we will call the Levi-Civita connection.

First of all, we want our connection to be compatible with the Riemannian metric g. By this we mean that for any vector fields X, Y, Z it must hold that

$$\nabla_X \langle X, Y\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

The other requirement is that the connection is symmetric, by which we mean that for vector fields X and Y it must hold that

$$\nabla_X Y - \nabla_Y X \equiv [X, Y].$$

Here $[X, Y] = XY - YX$ is the commutator of X and Y.

With these notions at hand, we get the following theorem.

Theorem 2.4.5 (Fundamental Lemma of Riemannian Geometry)**.** *Let (M, g) be a Riemannian manifold. There exists a unique linear connection ∇ on M that is compatible with g and symmetric.*

This connection is referred to as the Levi-Civita connection. Unless otherwise stated, we will always assume that a Riemannian manifold is equipped with this connection.

As a by-product of the proof of the above theorem, we also get a nice expression for the Christoffel symbols of the Levi-Civita connection in a chart. It holds that

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{n} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

where (g_{ij}) is the matrix of the metric in the chart, and (g^{ij}) denotes its inverse.

We will finish this section by briefly mentioning so called normal coordinates, as we will use them to derive certain identities. Normal coordinates are a special type of chart which can be defined around any point p, which is called its centre. We will not go into any detail on these, but simply state the most important properties.

Proposition 2.4.6 (Normal coordinates)**.** *For all $p \in M$ there exists a chart (x, U) around p with the following properties:*

1. *The coordinates of p are $(0, \dots, 0)$.*
2. *The components of the metric at p are $g_{ij} = \delta_{ij}$.*
3. *The first partial derivatives of g_{ij} and the Christoffel symbols vanish at p.*

Such a chart is referred to as normal coordinates around p.

2.4.3 Curvature

We can now finally consider the idea of curvature of a Riemannian manifold. Before we define any quantities, let us first given an idea of how one can visualize curvature. For this, let $M = S^2$, which is a manifold as discussed in example 2.1.2(3). We can visualize M as the unit sphere in $\mathbb{R}^3$. In that case we can see a tangent vector in TM as a vector in $\mathbb{R}^3$ which is attached to a point in S^2 'along tangent lines'.

Using the above visualization we will explain how one can see that the sphere is curved. Suppose we start with the tangent vector $(0, 0, 1)$ at the point $(0, 1, 0)$. Suppose we 'carry' this tangent vector to the north pole (the point $(0, 0, 1)$) along the circle in the (y, z)-plane, so that it keeps pointing in the same direction on the manifold. We can do this by keeping the angle between the tangent vector and the direction in which we travel constant. At the north pole, this tangent vector has then become $(0, -1, 0)$. If we now carry it in the same way to the point $(1, 0, 0)$ along the circle in the (x, z)-plane, we end up with the tangent vector $(0, -1, 0)$ but now at $(1, 0, 0)$. Carrying it back to the point $(0, 1, 0)$ along the circle in the (x, y) plane, we end up with the tangent vector $(1, 0, 0)$, which is different than the one we started with. This process is sketched in figure 2.2 on page 16. We thus see that, although we assured that the tangent vector kept pointing in the same direction with respect to the manifold, it still turned around overall. The reason for this, is that the sphere is curved.

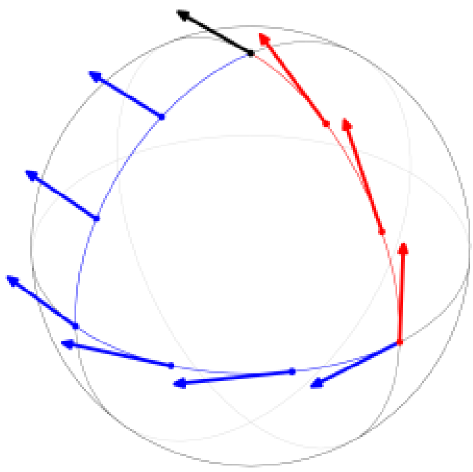

Figure 2.2: Visualization of curvature. We start on the right-hand side with a red tangent vector point up and move it along the specified route, while keeping its angle with the direction in which we travel constant. The fact that we end up with a different tangent vector when we are back at the starting point is the result of the curvature of the sphere. (Source: `https://en.wikipedia.org/wiki/Connection_(mathematics)`)

Let us now define some objects which help us identify curvature. It turns out that the situation sketched above occurs because the second order covariant derivatives do not commute in a satisfying way. On $\mathbb{R}^n$ the following commutativity relation holds

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z.$$

The above relation is referred to as the *flatness criterion*. This motivates the following definition.

Definition 2.4.7 (Riemann curvature endomorphism)**.** We call the map $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

the Riemann curvature endomorphism.

By repeatedly applying properties (1) and (3) of the linear connection as in definition 2.4.1 one can show that R is linear over $C^\infty(M)$ in any of its variables.

Based on this curvature endomorphism, we can also define the Riemann curvature tensor $Rm : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(TM)$ by

$$Rm(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle$$

for vector fields X, Y, Z, W. As R is multilinear over $C^\infty(M)$, it is easy to see that this also holds for Rm. By theorem 2 in chapter 4 of [32] we then find that Rm is indeed a 4-tensor. By duality we can deduce from this that R is in fact a $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$-tensor field. In a coordinate system (x, U) we will write it as

$$R = R_{ijk}{}^l \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \frac{\partial}{\partial x^l}$$

where the coefficients are defined by

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \sum_{l=1}^n R_{ijk}{}^l \frac{\partial}{\partial x^l}.$$

In the same way we can also express Rm in coordinates giving that

$$Rm = R_{ijkl} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l$$

where $R_{ijkl} = \sum_{m=1}^n g_{lm} R_{ijk}{}^m$ by the duality via the metric g.

The whole point in defining this Riemann curvature tensor is the following theorem. We call a Riemannian manifold *flat* if it is locally isometric to Eucledian space.

Theorem 2.4.8. *A Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

We finish this section by defining one more object, the Ricci-curvature, which can be derived from the Riemann curvature tensor. We will denote the Ricci curvature by Ric and it is a 2-tensor. If the components of the Riemann curvature tensor are given by R_{ijkl}, then the components of the Ricci curvature are given by

$$Ric_{ij} = \sum_{k=1}^n \sum_{m=1}^n g^{km} R_{kijm}.$$

A geometrical interpration of the Ricci-curvature can be found in [24]. In this paper it is explained that one can interpret the Ricci curvature as an indication of how the a small volume changes if we move it along a geodesic. What this actually gives us is that the Ricci curvature somehow governs how a geodesic ball[1] differs from a usual Eucledian ball of the same radius.

2.5 Volume measure and the Laplace-Beltrami operator

We finish the chapter by introducing some analysis on Riemannian manifolds. We will introduce the volume measure, which then allows us to define the divergence of k-forms. This puts us into the position to define the Laplace-Beltrami operator, which is the manifold version of the standard Laplacian in $\mathbb{R}^n$. On k-forms it is known as the Laplace-de Rham operator. Our discussion follows the lines of chapter 3 in [14].

[1] A geodesic ball of radius $r > 0$ around $p \in M$ is obtained by travelling along geodesics starting in p for a distance r. Doing this in every direction gives us a geodesic ball. For a more detailed explanation, see p.76 in [21].

2.5.1 Volume measure

Let M be a smooth manifold and denote by $\mathcal{B}(M)$ the smallest σ-algebra generated by the open sets in M. In a chart, we can use the identification with $\mathbb{R}^n$ and the Lebesgue measure there to define a measure and speak about measurability. Indeed, let $E \in \mathcal{B}(M)$ be a Borel set and (x, U) a chart. As U is open in M we find that $E \cap U$ is a Borel set in U. As x is a homeomorphism, we find that $x(U)$ is a Borel set in $\mathbb{R}^n$ and is thus Lebesgue measurable. This shows that we can indeed use the Lebesgue measure on $\mathbb{R}^n$ to construct a measure on $\mathcal{B}(M)$, which is stated in the following theorem, which is theorem 3.11 in [14].

Theorem 2.5.1 (Volume measure)**.** *Let (M, g) be a Riemannian manifold. There exists a unique measure ν on $\mathcal{M}$ such that in any chart U it holds that*

$$\mathrm{d}\nu = \sqrt{\det g}\mathrm{d}\lambda$$

where g is the matrix belonging to the Riemannian metric and λ is the Lebesgue measure in U.

Furthermore, the measure ν is complete, $\nu(K) < \infty$ for any compact set $K \subset M$, $\nu(\Omega) > 0$ for any non-empty open set $\Omega \subset M$ and ν is regular in the following sense: for any set $A \in \mathcal{M}$,

$$\nu(A) = \sup\{\nu(K) : K \subset A, K \textit{ compact}\}$$

and

$$\nu(A) = \sup\{\nu(\omega) : A \subset \Omega, \Omega \textit{ open}\}.$$

The measure ν is called the (Riemannian) volume measure. In the future, we will often write $\mathrm{d}x$ instead of ν, copying the convention in the case of the Lebesgue measure.

The above theorem is one of the reasons why we assumed a manifold to have a countable base for the topology induced by the (topological) metric. Indeed, to prove the above theorem, one uses the following lemma, which makes use of this countable base.

Lemma 2.5.2. *For any manifold M there is a countable family of relatively compact charts $\{U_i\}_{i\in\mathbb{N}}$ such that $M \subset \bigcup_{i=1}^{\infty} U_i$ and every closure $\overline{U}_i$ is contained in some other chart.*

Remark 2.5.3. Note that this lemma also implies that the volume measure ν is σ-finite. Indeed, each compact set has finite measure, and by the lemma we know that M is contained in a countable union of compact sets.

2.5.2 Divergence and the Laplace-Beltrami operator

We will now consider the adjoint d^* of the exterior derivative d with respect to the $L^2(\mathrm{d}x)$ inner product, which is given as

$$\int_M \langle \omega, \eta \rangle \, \mathrm{d}x$$

for smooth, compactly supported k-forms ω and η. Here $\mathrm{d}x$ is as defined in the previous section. Note that the divergence is given as $\mathrm{div} = -\mathrm{d}^*$. Before we can prove an existence result, we will first need to define the Hodge-star operator. We will follow te lines of [30]. Remember that in a chart (x, U), our volume form is written as $\sqrt{\det g}\mathrm{d}\lambda$, where λ denotes Lebesgue measure. With the idea in mind that the Lebesgue measure should correspond to the top-order form $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, we define the coefficients

$$e_{1\ldots n} = \sqrt{\det g}.$$

From this, we define the other coefficients as

$$e_{i_1\dots i_n} = \mathrm{sgn}(\sigma)e_{1\dots n}$$

where σ is the permutation given by $\sigma(j) = i_j$ for $1 \leq j \leq n$, and $\mathrm{sgn}(\sigma)$ denotes its sign. We will now define the Hodge dual, denoted by $*$.

Definition 2.5.4. Let ω be a k-form given in a chart (x, U) by

$$\omega = \sum_{1\leq i_1<\cdots<i_k\leq n} \omega_{i_1\dots i_k}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

We define its Hodge dual $*\omega$ as the $(n-k)$-form given by

$$*\omega = \sum_{1\leq j_1<\cdots<j_{n-k}\leq n} (*\omega)_{j_1\dots j_{n-k}}\mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_{n-k}}$$

where

$$(*\omega)_{j_1\dots j_{n-k}} = \sum_{1\leq i_1<\cdots<i_k\leq n} e_{i_1\dots i_k j_1\dots j_{n-k}}\omega^{i_1\dots i_k}.$$

Here,

$$\omega^{i_1\dots i_k} = \sum_{1\leq r_1,\dots,r_k\leq n} g^{i_1 r_1}\cdots g^{i_k r_k}\omega_{r_1\dots r_k}.$$

It turns out that the Hodge dual allows us to write a useful expression for d^*, which we will state as a lemma.

Lemma 2.5.5. *Suppose that ω is a k-form such that $*\omega \in D(\mathrm{d})$. Then $\omega \in D(\mathrm{d}^*)$ and*

$$\mathrm{d}^*\omega = (-1)^{n(k+1)+1} * \mathrm{d} * \omega.$$

Note that in [30], d^* is called the metric transpose, and is denoted by δ.

The above lemma is used in the existence part of the following theorem.

Theorem 2.5.6 (Divergence theorem)**.** *Let (M, g) be a Riemannian manifold and denote by $\mathrm{d}x$ the Riemannian volume measure. For a smooth, compactly supported $(k+1)$-form ω, there exists a unique compactly supported k-form η such that for all smooth, compactly supported k-forms ϕ it holds that*

$$\int_M \langle\eta, \phi\rangle\mathrm{d}x = \int_M \langle\omega, \mathrm{d}\phi\rangle\mathrm{d}x.$$

We write $\eta = \mathrm{d}^*\omega$ and call η the divergence of ω.

Proof. The uniqueness follows by the fact that if η' is another candidate, then

$$\int_M \langle\eta, \phi\rangle\mathrm{d}x = \int_M \langle\eta', \phi\rangle\mathrm{d}x$$

for all $\phi \in C_0^\infty(\Lambda^k TM)$, from which it follows that $\eta = \eta'$.[2]

We will now prove existence. For this, let ω be a smooth compactly supported $(k+1)$-form. Note that in a chart, $*$ only adds signs to the coefficients, and multiplies with coefficients of the metric g, which are smooth. Hence, the coefficients of $*\omega$ are again smooth. Furthermore, it is

[2] One can think of this in the L^2 sense, in which case the density of C_0^∞ gives us the result.

obvious that the support of $*\omega$ is contained in the support of ω. Consequently, $*\omega$ is a smooth, compactly supported $(n-k-1)$-form, from which it follows that $*\omega \in D(\mathrm{d})$. Consequently, by the previous lemma $\mathrm{d}^*\omega$ exists and is given by $\mathrm{d}^*\omega = *\mathrm{d}*\omega$. As d simply differentiates the coefficients, we see that $\mathrm{d}*\omega$ also has smooth coefficients, and has support contained in that of ω. Finally, applying $*$ again, we see that $\mathrm{d}^*\omega = *\mathrm{d}*\omega$ has compact support, and has smooth coefficients in any chart. This gives us that $\mathrm{d}^*\omega \in C_0^\infty(\Lambda^k TM)$ as desired. □

We will conclude this section by defining the Laplace-Beltrami operator as well as the Laplace-de Rham operator. With the divergence at hand, we are ready to define the Laplace-Beltrami operator. For a function $f \in C_0^\infty(M)$ we define

$$\Delta f = \mathrm{div} \circ \mathrm{d}f = -\mathrm{d}^*\mathrm{d}f$$

We wish to extend this definition to a C_0^∞ k-form ω. However, we cannot simply copy the definition for functions. The reason for this is that $\mathrm{d}^* f = 0$ for any $f \in C_0^\infty$. However, this is not the case for our k-form ω. Keeping this in mind, we define

$$\Delta\omega = -(\mathrm{d}^*\mathrm{d}\omega + \mathrm{d}\mathrm{d}^*\omega) = -(\mathrm{d} + \mathrm{d}^*)^2\omega$$

This operator Δ acting on k-forms is referred to as the Laplace-de Rham operator. Note that in the above the second equality holds true because $\mathrm{d}^2 = (\mathrm{d}^*)^2 = 0$.

Chapter 3

Semigroups of linear operators

In this chapter we will discuss the theory of strongly continuous semigroups of linear operators. Along with various definitions, we will also include some basic results which we will be using later. Our discussion will follow the lines of chapter 7 in [26]. The results we collect will be without proofs. Unless otherwise stated, the proofs (or references therefor) may be found in [26].

3.1 Strongly continuous semigroups

Strongly continuous semigroups arise when we try to find a solution for the abstract Cauchy problem given by

$$\begin{cases} u'(t) = Au(t) & t \in [0,T] \\ u(0) = u_0 & u_0 \in X \end{cases} \tag{3.1}$$

where A is some linear operator defined on a domain $D(A)$ contained in a Banach space X. Here A need not be a bounded operator.

In order to give an idea of how we arrive at such problems, we consider the heat equation on some open domain $D \subset \mathbb{R}^d$ given by

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) & t \in [0,T], x \in D \\ u(t,x) = 0 & t \in [0,T], x \in \partial D \\ u(0,x) = u_0(x) & x \in D \end{cases}$$

One way of trying to solve such a problem is using sepration of variables, in which we assume that there exists a solution of the form $u(t,x) = g(t)f(x)$. However, instead of trying to find a function in t and x that satisfies the partial differential equation, we could also try to reduce the problem to finding a certain function of t. However, such a function cannot simply take real values, but should take values in for example $L^p(D)$ in order to take into account the space variable, where $1 \leq p < \infty$. In this case we need to assume that $u_0 \in L^p(D)$.

In order to follow the above approach, we first need to define the (unbounded) operator Δ on $X = L^p(D)$ with domain $D(\Delta) = W^{2,p}(D) \cap W_0^{1,p}(D)$. Here $W^{k,p}(D)$ denotes the Sobolev space of functions in $L^p(D)$ of which the weak partial derivatives of order up to and including k exist and are again in $L^p(D)$. $W_0^{k,p}(D)$ is then the closure of $C_0^\infty(D)$ in $W^{k,p}(D)$. As the boundary conditions are included in the defintion of the domain of Δ, the problem reduces to

$$\begin{cases} u'(t) = \Delta u(t) & t \in [0,T] \\ u(0) = u_0 & u_0 \in L^p(D) \end{cases}$$

When looking at ordinary differential equations of such a form, one would naively suggest that the solution is given by $u(t) = e^{t\Delta}u_0$. However, the question is whether we can define $e^{t\Delta}$ in a sensible way so that this indeed works. For operators on a finite dimensional linear space we know that we can represent the operator as a matrix, for which we can define the exponential via the Taylor expansion. In more general situation we need what we call strongly continuous semigroups.

Definition 3.1.1. A family $S = (S(t))_{t\geq 0}$ of bounded linear operators acting on a Banach space X is called a strongly continuous semigroup if the following properties are satisfied:

1. $S(0) = I$.
2. $S(t)S(s) = S(t+s)$ for all $t, s \geq 0$.
3. $\lim_{t\downarrow 0} ||S(t)x - x|| = 0$ for all $x \in X$.

The infinitesimal generator, or simply the generator, of S is the linear operator A with domain $D(A)$ defined by

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x) \text{ exists in } X\}$$

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x), \qquad x \in D(A)$$

where the limit is considered to be in the norm of X.

If we consider the domain $D(A)$, we always assume that it carries the graph norm. Observe that it is again a Banach space. We furthermore note that if A generates the strongly continuous semigroup $(S(t))_{t\geq 0}$, then $A - \mu$ generates the strongly continuous semigroup $(e^{-\mu t}S(t))_{t\geq 0}$. Indeed, properties (1) and (2) of the definition are clear. Part (3) follows from the estimate

$$||e^{-\mu t}S(t)x - x|| \leq e^{-\mu t}||S(t)x - x|| + (e^{-\mu t} - 1)||x||.$$

The fact that $A - \mu$ is the generator, with $D(A - \mu) = D(A)$ follows from

$$\begin{aligned}
&\left|\left|\frac{1}{t}(e^{-\mu t}S(t)x - x) - Ax + \mu x\right|\right| \\
&\leq e^{-\mu t}\left|\left|\frac{1}{t}(S(t)x - x) - Ax\right|\right| + (e^{-\mu t} - 1)||Ax|| + \left(\frac{1}{t}(e^{-\mu t} - 1) + \mu\right)||x||.
\end{aligned}$$

Here the last term indeed also goes to zero as $-\mu$ is the derivative of $e^{-\mu t}$ in $t = 0$.

Before we collect some basic properties, we first discuss some examples of strongly continuous semigroups.

Example 3.1.2 (Translation semigroup)**.** Let $1 \leq p < \infty$ and consider the translation group on $L^p(\mathbb{R})$ defined by

$$S(t)f(x) := f(x+t), \qquad x \in \mathbb{R}, t \geq 0.$$

It is clear that $S(t)$ is linear and that $||S(t)f||_p = ||f||_p$. It is also obvious that $S(0)f = f$. Furthermore, for $t, s \geq 0$ we have that $S(t)S(s)f(x) = S(t)f(x+s) = f(x+s+t) = S(t+s)f(x)$. Finally, we will prove strong continuity. For this, first suppose that $f \in C_0(\mathbb{R})$. As f has compact support, it is obvious that we can find a compact set E such that the support of $S(t)f$ is contained in E for all $0 \leq t \leq 1$. As f is bounded and $S(t)f \to f$ almost everywhere, we find by the dominated convergence theorem (E has finite measure) that $\lim_{t\downarrow 0} S(t)f = f$ in $L^p(\mathbb{R})$.

Now pick $g \in L^p(\mathbb{R})$ arbitrary and let $\epsilon > 0$. Choose $f \in C_0(\mathbb{R})$ such that $||f - g||_p \leq \epsilon$. Finally, pick $\delta > 0$ such that $||S(t)f - f||_p \leq \epsilon$ for $t \in [0, \delta)$. For such t we have that

$$\begin{aligned} ||S(t)g - g||_p &\leq ||S(t)g - S(t)f||_p + ||S(t)f - f||_p + ||f - g||_p \\ &\leq 2||f - g||_p + ||S(t)f - f||_p \\ &\leq 3\epsilon. \end{aligned}$$

As $\epsilon > 0$ is arbitrary, this shows that $\lim_{t\downarrow 0} S(t)g = g$ in $L^p(\mathbb{R})$, finishing the proof of the strong continuity.

Now let A denote the generator of the translation semigroup. We will show that $C_0^1(\mathbb{R}) \subset D(A)$ and that $Af = f'$ for $f \in C_0^1(\mathbb{R})$. First note that as $f \in C_0^1(\mathbb{R})$, $f' \in C_0(\mathbb{R})$, where the support of f' is contained in that of f. Consequently, f' is bounded, say by some constant K. Now take again the compact set E as above. By using the mean value theorem we get that

$$\left| \frac{1}{t}(f(x + t) - f(x)) \right| = |f'(\xi)| \leq K$$

where $\xi \in (x, x + t)$. Consequently, we may use the dominated convergence theorem to conclude that

$$\lim_{t\downarrow 0} \frac{1}{t}(f(x + t) - f(x)) = f'(x)$$

in $L^p(\mathbb{R})$ as the convergence clearly is pointwise almost everywhere. This now proves the claim.

The following example can be found in [8].

Example 3.1.3 (Gaussian kernels)**.** Define for $t > 0$ and $x \in \mathbb{R}$ the Gaussian kernel

$$p_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}$$

and set $p_0 = \delta_0$, the dirac mass at 0. Let $1 < p < \infty$. Define on $L^p(\mathbb{R}^n)$ the operators $P_0 = I$ and for $t > 0$

$$P_t f(x) = f * p_t(x) = \int_{\mathbb{R}} f(y) p_t(x - y) \text{ d}y.$$

Note that these operators are clearly linear. By Young's inequality we find that $||P_t f||_p \leq ||f||_p ||p_t||_1 = ||f||_p$, as p_t is the density of a normal distribution with mean 0 and variance $2t$. Hence the operators are bounded. Furthermore, we have by definition that $P_0 = I$. Now note that p_t is a Schwarz function and that its Fourier transform is given by $\hat{p}_t(\xi) = e^{-4t\pi^2\xi^2}$. As the Fourier transform is invertible on Schwarz functions we find for all $t, s \geq 0$ that

$$p_t * p_s = \mathcal{F}^{-1}(e^{-4t\pi^2\xi^2} e^{-4s\pi^2\xi^2}) = \mathcal{F}^{-1}(e^{-4(t+s)\pi^2\xi^2}) = p_{t+s}.$$

From this we deduce that

$$P_t P_s f = p_t * p_s * f = p_{t+s} * f = P_{t+s} f.$$

Finally, we prove strong continuity. For this, first suppose that f is a Schwarz function. Observe that for any $t \geq 0$

$$S(t)f - f = \mathcal{F}^{-1}[(e^{-4t\pi^2\xi^2} - 1)\mathcal{F}(f)].$$

Consequently, by the dominated convergence theorem (if f is Schwarz, then so is $\mathcal{F}(f)$)

$$\lim_{t\downarrow 0} S(t)f - f = \mathcal{F}^{-1}[\lim_{t\downarrow 0}(e^{-4t\pi^2\xi^2} - 1)\mathcal{F}(f)] = 0$$

where the convergence is in $L^p(\mathbb{R})$. A similar denisty argument as in the previous example gives the strong continuity for all $f \in L^p(\mathbb{R})$.

Now denote the generator of the semigroup on $L^2(\mathbb{R})$ by A. We will show that $C_0^2(\mathbb{R})$ is contained in the domain of the generator, and that for such f we have $Af = \Delta f$. By the boundedness of the Fourier transform on $L^2(\mathbb{R})$ we find that

$$\begin{aligned} \lim_{t\downarrow 0} \frac{1}{t}(S(t)f - f) &= \mathcal{F}^{-1}\left(\left(\lim_{t\downarrow 0} \frac{e^{-4t\pi^2\xi^2} - 1}{t}\right)\mathcal{F}(f)\right) \\ &= \mathcal{F}^{-1}\left(-\xi^2\mathcal{F}(f)\right) \\ &= \Delta f. \end{aligned}$$

Here the right-hand side limit in the first line can be seen to holds in $L^2(\mathbb{R})$ by using the mean-value theorem and dominated convergence.

It is now routine to show that $C_0^2(\mathbb{R})$ is also contained in the domain of the generator if we consider the semigroup on $L^p(\mathbb{R})$ for general $1 < p < \infty$. We go over this in detail for a more difficult setting in section 6.2.1.

We now collect some basic properties of strongly continuous semigroups. The proofs can be found in [26].

Proposition 3.1.4. *Let S be a strongly continuous semigroup on a Banach space X. There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $||S(t)|| \leq Me^{\omega t}$ for all $t \geq 0$.*

Proposition 3.1.5. *Let S be a strongly continuous semigroup on a Banach space X with generator A. The following properties hold:*

1. *For all $x \in X$ the orbit $t \mapsto S(t)x$ is continuous for $t \geq 0$.*
2. *For all $x \in D(A)$ and $t \geq 0$ we have $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.*
3. *For all $x \in X$ we have $\int_0^t S(s)x\mathrm{d}s \in D(A)$ and*
$$A\int_0^t S(s)x\mathrm{d}s = S(t)x - x$$
If $x \in D(A)$, then both sides are equal to $\int_0^t S(s)Ax\mathrm{d}s$.
4. *The generator A is a closed and densely defined operator.*
5. *For all $x \in D(A)$ the orbit $t \mapsto S(t)x$ is continuously differentiable for $t \geq 0$ and*
$$\frac{\mathrm{d}}{\mathrm{d}t}S(t)x = AS(t)x = S(t)Ax, \qquad t \geq 0$$

The above properties suggest that the strongly continuous semigroup generated by the operator A is indeed the object to construct a solution of problem (3.1). The following will make this precise.

Definition 3.1.6. A classical solution of problem (3.1) is a continuous function $u : [0,T] \to X$ which belongs to $C^1((0,T];X) \cap C((0,T];D(A))$ and satisfies $u(0) = x$ and $u'(t) = Au(t)$ for all $t \in [0,T]$.

Corollary 3.1.7. *For initial values $x \in D(A)$ the problem* (3.1) *has a unique classical solution which is given by $u(t) = S(t)x$.*

This corollary tells us that we can think of a semigroup as evolving the initial value over time guided by the differential equation.

3.1.1 Strong equals weak

The continuity of a semigroup is considered in the strong sense. It turns out that this is in fact equivalent to weak continuity. The following is proposition 1.23 from [9].

Proposition 3.1.8. *If S is a semigroup on a Banach space X, i.e., S satisfies (1) and (2) in definition 3.1.1, then it also satisfies (3) if and only if*

$$w - \lim_{t \downarrow 0} S(t)x = x$$

for all $x \in X$. Here $w - \lim$ stands for the weak limit.

The same also holds for the generator A of a semigroup S. The following result is theorem 1.3 in [27]

Theorem 3.1.9. *Let S be a strongly continuous semigroup on a Banach space X with generator A. Define the operator $\bar{A}$ by*

$$D(\bar{A}) = \{x \in X : w - \lim_{t \downarrow 0} \frac{1}{t}(S(t)x - x) \text{ exists in } X\}$$

$$\bar{A}x = w - \lim_{t \downarrow 0} \frac{1}{t}(S(t)x - x)$$

for $x \in D(\bar{A})$. Then $A = \bar{A}$.

3.1.2 Analytic semigroups

We will finish this section by introducing analytic semigroups on a Banach space X. Denote by Σ_ω a sector of angle ω in the complex plane, i.e.,

$$\Sigma_\omega = \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \omega\}.$$

We call a strongly continuous semigroup $S(t)$ on X analytic if there exists a $\omega \in (0, \frac{\pi}{2})$ such that the following properties hold:

1. The mapping $t \mapsto S(t)$ can be extended to Σ_ω such that for all $z, w \in \Sigma_\omega$ it holds that $S(z)S(w) = S(z + w)$ and the mapping $z \mapsto S(z)x$ is continuous for all $x \in X$.
2. For all $z \in \Sigma_\omega \setminus \{0\}$ the mapping $z \mapsto S(z)$ is analytic in the operator norm.

Later, in chapter 6 we will see sufficient conditions for an operator to generate a strongly continuous analytic semigroup.

3.2 Resolvents and the Hille-Yosida theorem

In this section we will introduce the resolvent set of an operator, and collect a useful expression for resolvent operators in terms of a Laplace transform. We will finish the section by stating the Hille-Yosida theorem, which we will need in the future.

Definition 3.2.1. Let T be a linear operator with domain $D(T)$ on a complex Banach space X. The resolvent set of T is the set $\rho(T)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists a (necessarily unique) bounded linear operator $R(\lambda, T)$ on X such that

1. $R(\lambda, T)(\lambda - T)x = x$ for all $x \in D(T)$.
2. $R(\lambda, T)x \in D(T)$ and $(\lambda - T)R(\lambda, T)x = x$ for all $x \in X$.

The spectrum of T is the complement $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

We call $R(\lambda, T)$ the resolvent of T at λ, which we may also write as $(\lambda - T)^{-1}$. The following proposition allows us to express the resolvent of the generator of a semigroup in terms of the semigroup. The proof can again be found in [26].

Proposition 3.2.2. *Let A be the generator of a strongly continuous semigroup on a Banach space X. Let $M \geq 1$ and $\omega \in \mathbb{R}$ be such that $||S(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Then $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$ and on this set the resolvent of A is given by*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x \; \mathrm{d}t, \qquad x \in X.$$

Consequently, for $\operatorname{Re} \lambda > \omega$ we have that

$$||R(\lambda, A)|| \leq \frac{M}{\operatorname{Re} \lambda - \omega}.$$

We will now state the Hille-Yosida theorem for contraction semigroups. An operator T on a Banach X space is called a contraction if $||Tx|| \leq ||x||$ for all $x \in X$.

Theorem 3.2.3 (Hille-Yosida)**.** *Let A be a closed and densely defined linear operator on a Banach space X. The following assertions are equivalent:*

1. *A generates a strongly continuous semigroup of contractions on X.*
2. *$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\} \subset \rho(A)$ and*

$$||R(\lambda, A)|| \leq \frac{1}{\operatorname{Re} \lambda}, \qquad \operatorname{Re} \lambda > 0.$$

3.3 Markovian semigroups

We finish this chapter with a short discussion of Markovian semigroups. The definitions and results are taken from [8].

Markovian semigroups cannot be defined on arbitrary Banach spaces. Instead, we assume that we have a measurable space $(\Omega, \mathcal{F})$, which can be thought of as the state space. Before we can define Markovian semigroups, we first need the notion of invariant measure.

Definition 3.3.1 (Invariant measure)**.** Let $P = (P_t)_{t \geq 0}$ be a family of operators acting on the measurable functions of some measure space $(\Omega, \mathcal{F})$. A σ-finite measure μ is said to be invariant for P if for every bounded positive measurable function f and for all $t \geq 0$ it holds that

$$\int_\Omega P_t f \; \mathrm{d}\mu = \int_\Omega f \; \mathrm{d}\mu.$$

where we allow both sides to be infinite.

We are now able to define a Markovian semigroup as follows:

Definition 3.3.2 (Markovian semigroup)**.** A family $P = (P_t)_{t \geq 0}$ defined on the measurable functions on a state space $(\Omega, \mathcal{F})$ with invariant σ-finite measure μ is called a Markovian semigroup if it satisfies the following properties:

1. P is a strongly continuous semigroup on $L^2(\mathrm{d}\mu)$.
2. For all $t \geq 0$, P_t maps bounded measurable functions on $(\Omega, \mathcal{F})$ to bounded measurable functions.
3. $P_t 1 = 1$, where 1 is understood to be the constant function.
4. If $f \geq 0$, then $P_t f \geq 0$.

Property 5 is referred to as mass conservation, whereas propery 6 says that the semigroup is positivity preserving. It are these two properties that make the semigroup into a Markovian semigroup. It turns out that a Markovian semigroup defines a semigroup of contractions on $L^2(\mu)$.

One can associate to a Markovian semigroup $(P_t)_{t\geq 0}$ a Markov process X_t. For $x \in \Omega$, we will denote by $\mathbb{P}^x$ the law of X_t^x, which is the process that starts almost surely in x. If we write $\mathbb{E}^x$ for the expectation under $\mathbb{P}^x$, we have for any $f \in L^2(\mathrm{d}\mu)$ that

$$P_t f(x) = \mathbb{E}^x(f(X_t)).$$

Notice that this defines a Markovian semigroup. Indeed, $P_0 f(x) = \mathbb{E}^x(f(X_0)) = f(x)$ as $X_0 = x$ almost surely. Furthermore, for $t, s \geq 0$

$$\begin{aligned} P_{t+s}f(x) = \mathbb{E}^x(f(X_{t+s})) &= \mathbb{E}^x(\mathbb{E}^x(f(X_{t+s})|\mathcal{F}_t)) \\ &= \mathbb{E}^x(\mathbb{E}^x(f(X_{t+s})|X_t)) \\ &= \mathbb{E}^x(P_s f(X_t)) \\ &= P_t P_s f(x) \end{aligned}$$

where we used the Markov property in the second line. The third line follows from the fact that by the Markov property $\mathbb{E}^x(f(X_{t+s})|X_t) = \mathbb{E}(f(X_{t+s}|X_t)$ and the fact that $X_{t+s}|X_t = y$ and $X_s|X_0 = y$ have the same distribution for any $y \in M$ as $(X_t)_t$ is a Markov process. Strong continuity now follows from the fact that $f(X_t^x)$ goes to $f(x)$ almost surely and the dominated convergence theorem. This last theorem may be applied as $||P_t f||_2 \leq ||f||_2$, as conditional expectation is contractive by Jensen's inequality. Now suppose that f is bounded, say $|f| \leq M$ almost surely, then $\mathbb{E}^x(f(X_t)) \leq \mathbb{E}^x(M) = M$. Similarly, it is easy to see that $\mathbb{E}^x(1) = 1$, and $\mathbb{E}^x(f(X_t)) \geq 0$ when $f \geq 0$.

To conclude this section, we finish example 3.1.3 in the sense that we will show that the semigroup is mass conserving and positivity preserving.

Example 3.3.3 (Addition to the Gaussian kernels)**.** Let $(P_t)_t$ be the semigroup as in example 3.1.3. We will first show that it conserves mass. Indeed,

$$P_t 1(x) = \int_{\mathbb{R}} p_t(x-y)\ \mathrm{d}y = 1$$

as p_t is a probability distribution on $\mathbb{R}$.

Furthermore, if $f \geq 0$, then $f(\cdot)p_t(y - \cdot) \geq 0$ for any $y \in \mathbb{R}$. Consequently, we find that $P_t f = p_t * f \geq 0$, which shows that the semigroup is positivity preserving. All in all, we find that $(P_t)_t$ is Markovian.

This should not be much of a surprise. We already observed that p_t is the probability density of a normal distribution with mean 0 and variance $2t$. This indicates that the underlying Markov process is nothing other than a multiple of standard Brownian Motion, which is well known to indeed be generated by the Laplace operator.

Chapter 4

Study of the paper of Bakry

In this chapter we analyse the paper '*Étude des transformations de Riesz dans les variétés riemanniennes à de Ricci minorée*' by D. Bakry." We will study sections 1 through 5, each of which we discuss in a seperate section. We will go through the proofs in detail, verifying all claims made. We are mainly interested in the results concerning the boundedness of the Riesz transform, which are theorems 4.1 and 5.1 in [6]. It is nevertheless worthwile to have a detailed look into how these theorems are proved as plenty of useful techniques and theories are used.

Before we get to the theory, let us first set the scene and introduce the notation which we will use. For consistency, we will follow the notation in [6] as much as possible.

First of all, let M be a smooth complete Riemannian manifold of dimension n with a metric g. We will represent the Riemannian metric on either forms or tangent vectors by $\cdot$, i.e., if ϵ, ω are two 1-forms for example, we write their inner product as $\epsilon \cdot \omega$. The length of an element ω is denoted by $|\omega|$ and is defined in the usual way by $|\omega| = (\omega \cdot \omega)^{1/2}$. We furthermore assume that M is equipped with the Levi-Civita connection.

Let us now pick a function $\rho \in C^\infty(M), \rho > 0$ and let $\mathrm{d}x$ denote the Riemannian volume measure on M. We can then define the measure $\mathrm{d}m(x) = \rho(x)\mathrm{d}x$. If $\rho \in L^1(\mathrm{d}x)$, we will always assume that $m(M) = 1$. Furthermore, we will denote $L^p(m)$ by L^p and consequently also write $||\cdot||_p$ for $||\cdot||_{L^p(m)}$. Also, we define $\overrightarrow{L}^p$ to be the closure of the space of all 1-forms in $C_0^\infty(M)$ with respect to the norm $||\omega||_p := |||\omega|||_p$.

For simplicity we will define $\langle f \rangle := \int_M f \mathrm{d}m$, in which case the inner product on L^2 is given by $\langle f, g \rangle = \langle fg \rangle$, and on $\overrightarrow{L}^2$ it is given by $\langle \omega, \epsilon \rangle = \langle \omega \cdot \epsilon \rangle$.

We now define an operator L acting on C^∞, given by

$$Lf = \Delta f + \mathrm{d}f \cdot \mathrm{d} \log \rho.$$

Here Δ is the Laplace-Beltrami operator as discussed in section 2.5. Finally, we let R be the tensor $\mathrm{Ric} - \nabla\nabla \log \rho$, where Ric is the Ricci-curvature tensor and $\nabla\nabla \log \rho$ is the Hessian of $\log \rho$ as discussed on page 14. The main assumption in the paper is that this tensor is bounded from below in the sense that there exists a constant r_0 such that $R(X, X) \geq r_0 |X|^2$ for all tangent vectors X.

Throughout this chapter, the operators are all considered on C_0^∞ functions or forms respectively. In chapter 6 we will discuss how the operators can be extended to L^p for general $p \in (1, \infty)$. We will then also show that these operators are all consistent at least on C_0^∞. This also means that whenever we talk about forms in this chapter, we mean them to be C_0^∞, unless otherwise stated.

4.1 Generalities

In this section we take a look at the first section of [6]. This section is mainly concerned with the analysis of the operato L on L^2 as well as the related operator $\overrightarrow{L}$ on $\overrightarrow{L}^2$ which we will defined shortly. We will prove that these operators are essentially self-adjoint when defined on C_0^∞, allowing us to define the strongly continuous semigroups generated by their closures in L^2 and $\overrightarrow{L}^2$ respectively via the spectral theorem.

Before we can prove essential self-adjointness of either of the operators, let us first recall our definition of completeness of a Riemannian manifold.

Definition 4.1.1. A smooth Riemannian manifold is complete if and only if there exists an increasing sequence $(h_n)_n$ of smooth compactly supported functions on M such that $|\nabla h_n| \leq \frac{1}{n}$ for all n and $\lim_{n\to\infty} h_n = 1$ pointwise.

Intermezzo: Second order elliptic operators on a manifold

Additionally, we need to know a little bit more about second order elliptic operators. Background for the upcoming discussion can be found in [37].

Definition 4.1.2 (Second order differential operator)**.** Let T be an operator acting on the measurable functions on M. We say that T is a second order differential operator if for any chart (x, U) there exist measurable functions a_{ij}, b_i and c such that

$$Tf = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} + cf$$

Note that the definition really comes down to saying that $(x^{-1})^* T x^*$ is a second order differential operator on measurable functions on $x(U) \subset \mathbb{R}^n$.[1] Note that it in fact suffices to check the above condition for a set of charts which cover M. Indeed, a coordinate transformation is a diffeomorphism $\phi : U \to V$ between open sets $U, V \subset \mathbb{R}^n$. One can show by using the chain rule that if S is a second order differential operator on measurable functions on U, then $(\phi^{-1})^* S \phi^*$ is a second order differential operator on measurable functions on V.

We can now define what we mean by an elliptic second order differential operator.

Definition 4.1.3 (Elliptic operator)**.** Let T be a second order differential operator. We call T elliptic if for any chart (x, U) the following holds: Writing

$$Tf = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^{2} \frac{\partial f}{\partial x^i} + cf$$

it must hold for all $p \in M$ and all $\xi \in \mathbb{R}^n$ that

$$\sum_{i,j=1}^{n} a_{ij}(p) \xi^i \xi^j \geq 0.$$

We want to relate this property to a property what we will call hypo-ellipticity, which we will define next.

[1] For a function $\xi : M \to N$ we denote by ξ^* the pullback via ξ in that if $g : N \to \mathbb{R}$ is a function on N, then $\xi^* g = g \circ \xi$.

Definition 4.1.4 (Hypo-ellipticity)**.** A linear differential operator T acting on distributions on some open subset $U \subset \mathbb{R}^n$ is called hypo-elliptic if for all distributions u with $Tu \in C^\infty(U)$ it holds that $u \in C^\infty(U)$.

If T acts on distributions on a manifold M, we call T hypo-elliptic if for any chart (x, U) it holds that $(x^{-1})^* T x^*$ is hypo-elliptic on $x(U)$.

Note in the case of a manifold, if T is hypo-elliptic and Tu is smooth, then u restricted to any chart is smooth, and consequently, $u \in C^\infty(M)$.

We wish to use the following theorem, which is theorem 2.1 on page 77 of [31].

Theorem 4.1.5 (Regularity)**.** *Let T be an elliptic second order differential operator with smooth coefficients. Then T is hypo-elliptic.*

The fact that the above also holds on a manifold follows from the fact that one can show using the result in $\mathbb{R}^n$

We will finish this intermezzo by showing that L is hypo-elliptic by using this theorem.

Lemma 4.1.6 (Hypo-ellipticity of L)**.** *The operator L is hypo-elliptic.*

Proof. Let (x, U) be a chart. By the previous theorem it suffices to show that L is elliptic and has smooth coefficients in U. In (x, U) L is given by

$$L = \sum_{i,j=1}^n g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i,j=1}^n \left(\frac{1}{\rho} g^{ij} \frac{\partial \rho}{\partial x^i} + \frac{\partial g^{ij}}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

As both ρ and the coefficients of the metric are smooth, we see that L has smooth coefficients. Furthermore, as $G = (g_{ij})$ is positive definite, so is G^{-1}, which concludes the proof that L hypo-elliptic. □

We are now ready to prove the following.

Proposition 4.1.7. *The operator L, defined on C_0^∞ is essentially self-adjoint on L^2. Furthermore, for all $f, g \in C_0^\infty$ we have that*

$$\langle f, Lg \rangle = \langle g, Lf \rangle = -\langle \mathrm{d}f, \mathrm{d}g \rangle. \tag{4.1}$$

Proof. We will start by showing that the given identity holds true. By the definition of L we have that

$$\begin{aligned} \langle Lf, g \rangle &= \int_M \rho g \Delta f \ \mathrm{d}x + \int_M \rho g \mathrm{d}(\log \rho) \cdot \mathrm{d}f \ \mathrm{d}x \\ &= -\int_M \mathrm{d}(g\rho) \cdot \mathrm{d}f \ \mathrm{d}x + \int_M g \mathrm{d}\rho \cdot \mathrm{d}f \ \mathrm{d}x. \end{aligned}$$

Here we used that $\rho \mathrm{d}(\log \rho) = \mathrm{d}\rho$ and the fact that $\Delta = -\mathrm{d}^* \mathrm{d}$, where d^* denotes the adjoint of d. For this we also used that $f, g \in C_0^\infty$. As g, ρ are functions, we have that $\mathrm{d}(g\rho) = g\mathrm{d}\rho + \rho \mathrm{d}g$. Plugging this into the above shows that $\langle Lf, g \rangle = -\langle \mathrm{d}f, \mathrm{d}g \rangle$. Interchanging the roles of f and g shows that also $\langle f, Lg \rangle = -\langle \mathrm{d}f, \mathrm{d}g \rangle$.

We will now show that L is essentially self-adjoint on L^2. Note first that we can indeed consider the adjoint L^* of L, as C_0^∞ (the domain on which we consider L) is dense in L^2. Equation (4.1) shows us that $\langle Lf, f \rangle \leq 0$ for all $f \in C_0^\infty$. Under these conditions it holds that L is essentially self-adjoint precisely when there exists a positive $a > 0$ which is not an eigenvalue of L^* (see for example [29], p.136-137). We will in fact show that every $a > 0$ is not an eigenvalue of L^*. So suppose that $a > 0$. Observe that (4.1) shows us that the adjoint of L coincides with L on C_0^∞ functions. Hence, L^* is an extension of L and as L is hypo-elliptic by lemma 4.1.6, we find that if f satisties $L^* f = af$, it holds that $f \in C^\infty$. If $g \in C_0^\infty$ we get that

$$0 \leq a\langle f^2, g^2\rangle = \langle af, g^2 f\rangle = \langle L^* f, g^2 f\rangle = \langle f, L(g^2 f)\rangle = -\langle \mathrm{d}f, \mathrm{d}(g^2 f)\rangle.$$

Here we used that f satisies $L^* f = af$, the fact that L^* is the adjoint of L and (4.1). Now we compute $\mathrm{d}(g^2 f) = g^2 \mathrm{d}f + f\mathrm{d}g^2 = g^2\mathrm{d}f + 2fg\mathrm{d}g$. Inserting this in the above expression gives us that

$$0 \leq -\langle \mathrm{d}f, g^2\mathrm{d}f\rangle - 2\langle \mathrm{d}f, fg\mathrm{d}g\rangle = -\langle g^2, |\mathrm{d}f|^2\rangle - 2\langle fg, \mathrm{d}f \cdot \mathrm{d}g\rangle.$$

This gives us that

$$\begin{aligned}
\langle g^2, |\mathrm{d}f|^2\rangle &\leq -2\langle fg, \mathrm{d}f\cdot \mathrm{d}g\rangle \\
&\leq 2\int_M |fg\mathrm{d}f \cdot \mathrm{d}g|\ m(\mathrm{d}x) \\
&\leq 2\int_M |fg\mathrm{d}f||\mathrm{d}g|\ m(\mathrm{d}x) \\
&\leq 2||\mathrm{d}g||_\infty \int_M |fg\mathrm{d}f|\ m(\mathrm{d}x) \\
&\leq 2||\mathrm{d}g||_\infty ||f||_2 ||g\mathrm{d}f||_2 \\
&= 2||\mathrm{d}g||_\infty ||f||_2 \langle g^2, |\mathrm{d}f|^2\rangle^{1/2}.
\end{aligned}$$

Here the third and last inequality are simply Cauch-Schwarz (although for different inner products). We conclude that $\langle g^2, |\mathrm{d}f|^2\rangle^{1/2} \leq 2||\mathrm{d}g||_\infty ||f||_2$.

Now, let $(h_n)_n$ be a sequence as in definition 4.1.1. If we take $g = h_n$ in the above estimate, we find that $\langle h_n^2, |\mathrm{d}f|^2\rangle^{1/2} \leq \frac{2}{n}||f||_2$. Taking the limit $n \to \infty$, the right hand side goes to 0, while the left hand side becomes $|\mathrm{d}f|^2$. Hence we find that $\mathrm{d}f = 0$ and consequently, f is constant on each component of M. However, it holds that $0 = Lf = L^* f = af$ and as $a > 0$, it must be that $f \equiv 0$. (Although we only have that $f \in C^\infty$, we see that the proof of (4.1) still holds when only one of the two function f and g has compact support.) This shows that a is not an eigenvalue, hence L^* has no positive eigenvalues, from which it follows that L defined on C_0^∞ is essentialy self-adjoint in L^2. □

This proposition implies that L is closable in L^2 and that its closure is given by L^*. We will denote the closure of L by the same symbol, so that L is self-adjoint. Hence, we may write $D(L)$ for the domain of either L or the adjoint L^*. The proposition also implies that C_0^∞ is dense in $D(L^*) = D(L)$ in the graph norm given by $||f||_{D(L)} = ||f||_2 + ||Lf||_2 = ||f||_2 + ||L^* f||_2$.

We will now show that formula (4.1) also holds when $f, g \in C^\infty \cap D(L)$. In order to do this, we first prove a lemma.

Lemma 4.1.8. *Let $f \in C^\infty \cap D(L)$ and suppose that $(f_n)_n$ is a sequence of C_0^∞ functions converging to f in $D(L)$.*[2] *We have that*

1. *For all $g \in C_0^\infty$ it holds that $gf \in D(L)$ and $(gf_n)_n$ converges to gf in $D(L)$.*

2. *$(\mathrm{d}f_n)$ converges to $\mathrm{d}f$ in $\overrightarrow{L}^2$.*

[2] Such a sequence exists as C_0^∞ is dense in $D(L)$.

Proof. We first prove (1). As $f_n \to f$ in $D(L)$, we have that $f_n \to f$ and $Lf_n \to Lf$ in L^2. As g is bounded, it follows that $gf_n \to gf$ in L^2. It remains to prove that $L(gf_n) \to L(gf)$ in L^2. We show that $(L(gf_n))_n$ is Cauchy in L^2. We can compute that $\Delta(gf_n) = g\Delta f_n + 2\mathrm{d}g \cdot \mathrm{d}f_n + f_n \Delta g$ and $\mathrm{d}(gf_n) = g\mathrm{d}f_n + f_n \mathrm{d}g$. Putting these together we get that $L(gf_n) = gLf_n + 2\mathrm{d}g \cdot \mathrm{d}f_n + f_n Lg$. Again, as both g and Lg are bounded, we find that $gLf_n \to gLf$ and $f_n Lg \to fLg$ both in L^2, and are thus in particular Cauchy. For the middle part, observe that

$$||\mathrm{d}f_n \cdot \mathrm{d}g - \mathrm{d}f_m \cdot \mathrm{d}g||_2 \leq ||\mathrm{d}g||_\infty ||d(f_n - f_m)||_2$$

and also

$$||d(f_n - f_m)||_2^2 = -\langle f_n - f_m, L(f_n - f_m)\rangle \leq ||f_n - f_m||_2 ||L(f_n - f_m)||_2.$$

Here, the equality follows from formula (4.1), the estimate is simply Cauchy-Schwarz. As $(f_n)_n$ and $(Lf_n)_n$ converge in L^2, they are Cauchy. Putting the estimates together shows that $(\mathrm{d}f_n \cdot \mathrm{d}g)_n$ is Cauchy. We conclude that $L(gf_n)_n$ is Cauchy in L^2. As L is closed, and $(gf_n, L(gf_n))$ is a Cauchy sequence in the graph, we find that $gf \in D(L)$ and $gf_n \to gf$ in $D(L)$.

We will now prove (2). The above estimate shows us that for $f \in C_0^\infty$ we have that

$$||\mathrm{d}f_n - \mathrm{d}f||_2^2 \leq ||f_n - f||_2 \cdot ||Lf_n - Lf||_2 \to 0$$

as $f_n \to f$ in $D(L)$. Now if f is only in C^∞, we saw above that $(\mathrm{d}f_n)_n$ is Cauchy in $\overrightarrow{L}^2$ and by completeness it converges, say to some 1-form ω. Now pick $g \in C_0^\infty$. Then $gf_n \in C_0^\infty$, and thus $\mathrm{d}(gf_n) \to \mathrm{d}(gf) = g\mathrm{d}f + f\mathrm{d}g$ in $\overrightarrow{L}^2$ as $gf_n \to gf$ in $D(L)$ by (1). On the other hand, we find that $d(gf_n) = g\mathrm{d}f_n + f_n\mathrm{d}g \to g\omega + f\mathrm{d}g$. As limits are unique, it must thus be that $g\omega = g\mathrm{d}f$. As this holds for all $g \in C_0^\infty$, we conclude that $\omega = \mathrm{d}f$ as desired. ☐

With this lemma at hand, the following proposition is almost immediate.

Proposition 4.1.9. *Let $f, g \in C^\infty \cap D(L)$. Then $Lf = L^*f \in L^2$, $\mathrm{d}f, \mathrm{d}g \in \overrightarrow{L}^2$ and* (4.1) *holds.*

Proof. As L can also be defined on C^∞ functions, the fact that L is essentially self-adjoint implies that $Lf = L^*f$. As $f \in D(L)$, we have by definition of the adjoint that $L^*f \in L^2$, as it is retrieved via the Riesz representation theorem. This proves the first claim.

Now let $(f_n)_n$ be a sequence in C_0^∞ converging to f in $D(L)$, and let $(g_n)_n$ be a similar sequence for g. By (2) of the previous lemma, we have that $\mathrm{d}f_n \to \mathrm{d}f$ and $\mathrm{d}g_n \to \mathrm{d}g$ in $\overrightarrow{L}^2$, from which it follows that $\mathrm{d}f, \mathrm{d}g \in \overrightarrow{L}^2$ by completeness.

Finally, observe that $\lim_{n\to\infty}\langle f_n, Lg_n\rangle = \langle f, Lg\rangle$. Indeed

$$\begin{aligned} |\langle f, Lg\rangle - \langle f_n, Lg_n\rangle| &\leq |\langle f - f_n, Lg\rangle| + |\langle f_n, Lg - Lg_n\rangle| \\ &\leq ||f - f_n||_2 ||Lg||_2 + ||f_n||_2 ||Lg - Lg_n||_2 \end{aligned}$$

As $Lg, f_n \in L^2$ and $f_n \to f, g_n \to g$ both in $D(L)$, we see that the upper bound goes to 0. Using this we find that

$$\langle f, Lg\rangle = \lim_{n\to\infty}\langle f_n, Lg_n\rangle = -\lim_{n\to\infty}\langle \mathrm{d}f_n, \mathrm{d}g_n\rangle = \langle \mathrm{d}f, \mathrm{d}g\rangle$$

where we used (4.1) in the second equality. ☐

4.1.1 Heat semigroup P_t corresponding to L

As discussed above, we can consider L as a self-adjoint operator with domain $D(L)$. By the spectral theorem we may write $L = -\int_0^\infty \lambda \mathrm{d}E_\lambda$, where $(E_\lambda)_\lambda$ is the spectral family belonging to L. The integral starts at 0 as we know that L has no positive spectrum as L satisfies $\langle Lf, f\rangle \leq 0$ for all $f \in D(L)$.

We now define the heat semi-group by $P_t = \int_0^\infty e^{-t\lambda}\mathrm{d}E_\lambda$ with generator L. The corresponding bilinear form given by $\Gamma : (f,g) \mapsto -\langle f, Lg\rangle$ satisfies $\Gamma(f,f) \geq 0$ by (4.1). The extensive discussion of generators of a semigroup in connection with such a bilinear form as done in [8] gives us that P_t is a sub-Markovian semigroup in the sense that $0 \leq P_t f \leq 1$ whenever $0 \leq f \leq 1$. However, as the curvature is bounded from below, we in fact get that the semigroup is Markovian, meaning that in this case it actually holds that $P_t 1 = 1$.

Furthermore, L is an elliptic operator, and for all $f \in L^2$, the function $\bar{f}(x,t) = P_t f(x)$ solves the parabolic equation $(\frac{\mathrm{d}}{\mathrm{d}t} - L)P_t f = 0$ in the sense of distributions. By the general theory of elliptic equations we then find that $P_t f \in C^\infty(M \times (0,\infty))$.

Probalistic interpretation of P_t

There is also a probabilistic interpretation of the semigroup P_t. As the semigroup is Markovian, we can associate a Markov process X_t. For $x \in M$ we denote by $\mathbb{P}^x$ the law of X_t^x, i.e., the process starting almost surely in x. In that case we can write $P_t f(x) = \mathbb{E}^x(f(X_t))$, where $\mathbb{E}^x$ denotes the expectation under $\mathbb{P}^x$.

Using this process, we can also form the Dynkin martingale

$$M_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\mathrm{d}s$$

for $f \in C_0^\infty$. We will show that this is indeed a martingale. Let us denote by $(\mathbb{F}_s)_s$ the natural filtration of the process $(X_t)_t$. We have that

$$\begin{aligned}
\mathbb{E}(M_t - M_s|\mathbb{F}_s) &= \mathbb{E}\left(f(X_t) - f(X_s) - \int_s^t Lf(X_r)\mathrm{d}r\Big|\mathbb{F}_s\right) \\
&= P_{t-s}f(X_s) - f(X_s) - \int_s^t P_{r-s}Lf(X_r)\mathrm{d}r \\
&= P_{t-s}f(X_s) - f(X_s) - \int_s^t \frac{\mathrm{d}}{\mathrm{d}r}P_{r-s}f(X_r)\mathrm{d}r \\
&= P_{t-s}f(X_s) - f(X_s) - [P_{r-s}f(X_r)]_{r=s}^t \\
&= 0.
\end{aligned}$$

Here we used in the first step that $(X_t)_t$ is Markov, hence the conditioning may be done only on X_s and the process can be considered starting at time s. By definition of the semigroup P_t, we then have that $\mathbb{E}(f(X_t)|X_s) = P_{t-s}f(X_s)$. The second part follows similarly after applying Fubini's theorem. Furthermore, $f(X_s)$ is $\mathbb{F}_s$ measurable, so we can take it out of the conditional expectation. The next step simply follows from the fact that L generates the semigroup $(P_t)_t$.

Finally, we will show that m is a stationary distribution of the Markov process, as this will be used later on. In fact, from proposition 4.1.7 we find for $f \in C^\infty$ and $g \in C_0^\infty$ that

$$\int_M gLf \,\mathrm{d}m(x) = \int_M fLg \,\mathrm{d}m(x).$$

If we take $f = 1$ in this expression, we find for all $g \in C_0^\infty$ that

$$\int_M Lg \, \mathrm{d}m(x) = \int_M gL(1) \, \mathrm{d}m(x) = 0$$

which shows that m is stationary for L. Now note that as P_t is generated by L, we also have the symmetry property for P_t. If we then again take $f = 1$ and remember that $P_t 1 = 1$ we see that the stationarity of m translates to

$$\int_M P_t g \, \mathrm{d}m(x) = \int_M g P_t 1 \, \mathrm{d}m(x) = \int_M g \, \mathrm{d}m(x)$$

which is the form in which we will use it.

4.1.2 The case for 1-forms

We will now define and analyze the analogue of the operator L acting on 1-forms. We will write d^* for the the adjoint of the exterior derivative with respect to the inner product given by $\langle \omega, \eta \rangle = \int_M \omega \cdot \eta \, \mathrm{d}x$. We define δ in the same way, only now with respect to the measure $m(\mathrm{d}x)$. We will relate these two notions in the following proposition. Before we can state it however, we first need to introduce what we mean by contraction on the first entry.

Definition 4.1.10 (Contraction on the first entry)**.** Let ω be a k-form and suppose that X is a tangent vector. We define $\iota(X)\omega$ as the $(k-1)$-form given by

$$\iota(X)\omega(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1})$$

for tangent vectors $Y_1, \ldots, Y_{k-1}$. We refer to ι as contraction on the first entry.

Proposition 4.1.11. *If ω is a k-form, then*

$$\delta\omega = \mathrm{d}^*\omega - \iota((\mathrm{d}\log\rho)^*)\omega$$

where $\mathrm{d}(\log\rho)^*$ *is the tangent vector dual to the 1-form* $\mathrm{d}(\log\rho)$ *and ι denotes contraction on the first entry as defined above.*

Proof. Suppose that ω is a k-form. We have for any $(k-1)$-form ϵ that

$$\begin{aligned}
\int_M (\mathrm{d}^*\omega - \iota((\mathrm{d}\log\rho)^*)\omega) \cdot \epsilon\rho \, \mathrm{d}x &= \int_M (\rho\epsilon) \cdot \mathrm{d}^*\omega - \iota((\mathrm{d}\log\rho)^*)\omega \cdot (\rho\epsilon) \, \mathrm{d}x \\
&= \int_M \mathrm{d}(\rho\epsilon) \cdot \omega - \iota((\mathrm{d}\log\rho)^*)\omega \cdot (\rho\epsilon) \, \mathrm{d}x \\
&= \int_M \mathrm{d}(\rho\epsilon) \cdot \omega - \iota(\rho(\mathrm{d}\log\rho)^*)\omega \cdot \epsilon \, \mathrm{d}x \\
&= \int_M \omega \cdot (\rho \mathrm{d}\epsilon + \mathrm{d}\rho \wedge \epsilon) - \iota((\mathrm{d}\rho)^*)\omega \cdot \epsilon \, \mathrm{d}x \\
&= \int_M (\omega \cdot \mathrm{d}\epsilon)\rho \, \mathrm{d}x
\end{aligned}$$

where we used that k-forms are linear over C^∞ functions to arrive at the third line. The last inequality follows from proposition A.1.10 in the appendix. From the uniqueness of δ the claim now follows.

□

Remark 4.1.12. In the case where ω is a 1-form, the identity derived in the previous proposition reduces to

$$\delta\omega = \mathrm{d}^*\omega - \omega \cdot \mathrm{d}(\log\rho).$$

Indeed, by duality we find that

$$\iota((\mathrm{d}\log\rho)^*)\omega = \omega((\mathrm{d}\log\rho)^*) = \omega \cdot (\mathrm{d}\log\rho).$$

Note that from proposition 2.5.6 it follows that $\mathrm{d}^*\omega$ is a C_0^∞ $(k-1)$-form whenever ω is a C_0^∞ k-form. Consequently, by the above proposition this also holds for δ. Indeed, as ρ is smooth, it follows that $(\mathrm{d}\rho)^*$ is a smooth vector field. The smoothness of ω now implies that $\iota((\mathrm{d}\rho)*)\omega$ is a smooth $(k-1)$-form.

Now that we have the divergence relative to the measure $m(\mathrm{d}x)$, we can define the operator $\overrightarrow{L}$ acting on C_0^∞ 1-forms.

Definition 4.1.13. We define the operator $\overrightarrow{L}$ acting on C_0^∞ 1-forms by

$$\overrightarrow{L}\omega = -(\mathrm{d}\delta + \delta\mathrm{d})\omega.$$

Remark 4.1.14. The operators L and $\overrightarrow{L}$ are related by $\mathrm{d}Lf = \overrightarrow{L}\mathrm{d}f$ for $f \in C_0^\infty$. Indeed, a simple calculation shows us that

$$\begin{aligned}\overrightarrow{L}\mathrm{d}f = -(\mathrm{d}\delta + \delta\mathrm{d})\mathrm{d}f &= -\mathrm{d}\delta\mathrm{d}f \\ &= -\mathrm{d}(\mathrm{d}^*\mathrm{d}f - \mathrm{d}f \cdot \mathrm{d}\log\rho) \\ &= \mathrm{d}(-\mathrm{d}^*\mathrm{d}f + \mathrm{d}f \cdot \mathrm{d}\log\rho) \\ &= \mathrm{d}Lf.\end{aligned}$$

In the next proposition, we will relate the operator $\overrightarrow{L}$ to the operators $\Delta = -\mathrm{d}^*\mathrm{d}$ and $\overrightarrow{\Delta} = -(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})$. Before stating the proposition, we introduce the following notation: for ω, ϵ 1-forms, we write

$$\overrightarrow{\omega}(\epsilon) := d(\omega \cdot \epsilon) + \mathrm{d}\epsilon(\omega^*, \cdot) \qquad \text{and} \qquad \omega^H(\epsilon) = \nabla\epsilon(\omega^*, \cdot).^3$$

For bilinear forms T on TM, we write $\overrightarrow{T}(\omega) = T(\omega^*, \cdot)$. Finally, for any $p \in M$ we can find an orthonormal frame $e_1, \ldots, e_d$ on some neighbourhood of p in which case we write that $|\nabla\omega|^2 = \sum_{i=1}^d \langle \nabla_{e_i}\omega, \nabla_{e_i}\omega \rangle$.

Proposition 4.1.15. *The following relations hold:*

1. $\overrightarrow{L} = \overrightarrow{\Delta} + \overrightarrow{\mathrm{d}}(\log\rho) = \Delta + \mathrm{d}(\log\rho)^H - \overrightarrow{R}$.
2. *For all 1-forms ω,* $L|\omega|^2 = 2\omega \cdot \overrightarrow{L}\omega + 2|\nabla\omega|^2 + 2R(\omega^*, \omega^*)$.

Proof. We will first prove (1). For this, let ω be a 1-form. We have that

$$\begin{aligned}\overrightarrow{L}\omega &= -(\mathrm{d}\delta + \delta\mathrm{d})\omega \\ &= -(\mathrm{d}(\mathrm{d}^*\omega - \omega\mathrm{d}(\log\rho))) - (\mathrm{d}^*\mathrm{d}\omega - \mathrm{d}\omega(\mathrm{d}(\log\rho)^*, \cdot)) \\ &= -(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\omega + \mathrm{d}(\omega \cdot \mathrm{d}(\log\rho)) + \mathrm{d}\omega(\mathrm{d}(\log\rho)^*, \cdot) \\ &= \overrightarrow{\Delta}\omega + \overrightarrow{\mathrm{d}}(\log\rho)(\omega).\end{aligned}$$

[3] The H in this notation has to do with the horizontal lift as discussed briefly on page 39.

For the second equality in (1), we use the Bochner-Lichnérowicz-Weitzenböck formula (see for example [28]), which states that $\overrightarrow{\Delta} = \Delta - \overrightarrow{\mathrm{Ric}}$. By definition of R, we have that $\overrightarrow{\mathrm{Ric}} = \overrightarrow{R} + \overrightarrow{\nabla\nabla}(\log\rho)$. Using the previous result, this gives us that

$$\overrightarrow{L} = \Delta - \overrightarrow{R} - \overrightarrow{\nabla\nabla}(\log\rho) + \overrightarrow{\mathrm{d}}(\log\rho).$$

Now for any function $h \in C^\infty$ we have that

$$\mathrm{d}(\mathrm{d}h \cdot \omega) = \nabla\omega(\cdot, \mathrm{d}h^*) + \nabla\nabla h(\cdot, \omega^*) = \overrightarrow{\nabla\nabla}h(\omega) + \nabla\omega(\mathrm{d}h^*, \cdot) - \mathrm{d}\omega(\mathrm{d}h^*, \cdot).$$

Here we used corollary A.1.2 and propositions A.1.4 and A.1.5 from the appendix. Using this expression we get that

$$\overrightarrow{\mathrm{d}h}(\omega) = \mathrm{d}(\mathrm{d}h \cdot \omega) + \mathrm{d}\omega(\mathrm{d}h^*, \cdot) = \overrightarrow{\nabla\nabla}h(\omega) + \nabla\omega(\mathrm{d}h^*, \cdot) = \mathrm{d}h^H(\omega) + \overrightarrow{\nabla\nabla}h(\omega).$$

If we now take $h = \log\rho$, we get that

$$\overrightarrow{L} = \Delta - \overrightarrow{R} + \mathrm{d}(\log\rho)^H$$

as desired.

For (2), observe that for a 1-form ω it holds by lemma A.1.9 that

$$\Delta|\omega|^2 = 2\omega \cdot \Delta\omega + 2|\nabla\omega|^2.$$

This gives us that

$$L|\omega|^2 = \Delta|\omega|^2 + \mathrm{d}|\omega|^2 \cdot \mathrm{d}(\log\rho) = 2\omega \cdot \Delta\omega + 2|\nabla\omega|^2 + \mathrm{d}|\omega|^2(\mathrm{d}(\log\rho)^*).$$

Now, for any 1-form ϵ,

$$\mathrm{d}|\omega|^2(\epsilon^*) = \epsilon^*(|\omega|^2) = 2\nabla\omega(\epsilon^*, \omega^*) = 2\omega \cdot \nabla\omega(\epsilon^*, \cdot) = 2\omega \cdot \epsilon^H(\omega).$$

Here, the first equality is lemma A.1.1 from the appendix, and the second is the duality via the metric. Applying this with $\epsilon = \mathrm{d}(\log\rho)$ and plugging this into the expression for $L|\omega|^2$ gives us that

$$L|\omega|^2 = 2\omega \cdot \Delta\omega + 2|\nabla\omega|^2 + 2\omega \cdot \mathrm{d}(\log\rho)^H(\omega).$$

Applying the second identity from (1) now gives us that

$$L|\omega|^2 = 2\omega \cdot \overrightarrow{L}\omega + 2|\nabla\omega|^2 + 2\omega \cdot \overrightarrow{R}(\omega) = 2\omega \cdot \overrightarrow{L}\omega + 2|\nabla\omega|^2 + 2R(\omega^*, \omega^*)$$

where the last equality holds again by the duality via the metrc. □

We will now show that $\overrightarrow{L}$ is essentially self-adjoint on $\overrightarrow{L}^2$. For this, we will first argue that $\overrightarrow{L}$ is symmetric on C_0^∞ 1-forms. Let ω, ϵ be smooth 1-forms with compact support. By the definition of $\overrightarrow{L}$ and δ we find that

$$\langle \omega, \overrightarrow{L}\epsilon \rangle = -\langle \omega, d\delta\epsilon \rangle - \langle \omega, \delta d\epsilon \rangle = -\langle \delta\omega, \delta\epsilon \rangle - \langle \mathrm{d}\omega, \mathrm{d}\epsilon \rangle. \tag{4.2}$$

Interchanging the roles of ω and ϵ gives us that $\langle \omega, \overrightarrow{L}\epsilon \rangle = \langle \overrightarrow{L}\omega, \epsilon \rangle$. We are now ready to prove that $\overrightarrow{L}$ defined on C_0^∞ is essentially self-adjoint on $\overrightarrow{L}^2$. The prove is very similar to that of proposition 4.1.7.

Proposition 4.1.16. *$\overrightarrow{L}$ is essentially self-adjoint on $\overrightarrow{L}^2$.*

Proof. For this proof, we will follow [34], p. 51-52. By equation (4.2) we find that $\langle \overrightarrow{L}\omega, \omega\rangle \leq 0$ for any 1-form ω. Hence it again suffices to show that the adjoint $\overrightarrow{L}^*$ has no positive eigenvalues. So let $\lambda > 0$ and suppose that $\omega \in C_0^\infty$ satisfies $\overrightarrow{L}^*\omega = \lambda\omega$. We will show that $\omega = 0$. Pick the sequence of functions $(h_n)_n$ as in the proof of proposition 4.1.7. As $\overrightarrow{L}\omega = \overrightarrow{L}^*\omega$, by formula (4.2) we find

$$0 \leq \lambda\langle h_n^2, \omega^2\rangle = \langle h_n^2\omega, \lambda\omega\rangle = \langle h_n^2\omega, \overrightarrow{L}\omega\rangle = -\langle \mathrm{d}(h_n^2\omega), \mathrm{d}\omega\rangle - \langle \delta(h_n^2\omega), \delta\omega\rangle.$$

Now we compute that $\mathrm{d}(h_n^2\omega) = h_n^2\mathrm{d}\omega + 2h_n\mathrm{d}h_n \wedge \omega$. For the other term, we find by lemma A.1.6 from the appendix that $\langle \delta(h_n^2\omega), \delta\omega\rangle = \langle h_n^2\delta\omega, \delta\omega\rangle - 2\langle h_n\mathrm{d}h_n \wedge \delta\omega, \omega\rangle$. Plugging these into the above expression gives us that

$$\begin{aligned} 0 &\leq -\langle h_n^2\mathrm{d}\omega, \mathrm{d}\omega\rangle - \langle h_n^2\delta\omega, \delta\omega\rangle - 2\langle h_n\mathrm{d}h_n \wedge \omega, \mathrm{d}\omega\rangle + 2\langle h_n\mathrm{d}h_n \wedge \delta\omega, \omega\rangle \\ &= -\langle h_n^2\mathrm{d}\omega, \mathrm{d}\omega\rangle - \langle h_n^2\delta\omega, \delta\omega\rangle - 2\langle h_n\mathrm{d}h_n \wedge \omega, \mathrm{d}\omega\rangle + 2\langle \omega, h_n\delta\omega\mathrm{d}h_n\rangle \end{aligned}$$

where in the second line we used that $\delta\omega$ is a function.

But then we find that

$$\begin{aligned} \langle h_n^2\mathrm{d}\omega, \mathrm{d}\omega\rangle + \langle h_n^2\delta\omega, \delta\omega\rangle &\leq -2\langle h_n\mathrm{d}h_n \wedge \omega, \mathrm{d}\omega\rangle + 2\langle \omega, h_n\delta\omega\mathrm{d}h_n\rangle \\ &\leq 2|\langle h_n\mathrm{d}h_n \wedge \omega, \mathrm{d}\omega\rangle| + 2|\langle \omega, h_n\delta\omega\mathrm{d}h_n\rangle| \\ &\leq 2\langle |h_n||\mathrm{d}h_n \wedge \omega|, |\mathrm{d}\omega|\rangle + 2\langle |\omega|, |h_n||\delta\omega||\mathrm{d}h_n|\rangle \end{aligned}$$

where the last line is again simply the Cauchy-Schwarz inequality. Now observe that for general 1-forms ϵ and η it holds that $|\epsilon \wedge \eta| \leq |\epsilon||\eta|$. Indeed, by definition of the inner product on 2-forms, $|\epsilon \wedge \eta|^2 = |\epsilon|^2|\eta|^2 - (\epsilon \cdot \eta)^2 \leq |\epsilon|^2|\eta|^2$, from which the claim follows.

Applying this to the above estimate, we get that

$$\begin{aligned} \langle h_n^2\mathrm{d}\omega, \mathrm{d}\omega\rangle + \langle h_n^2\delta\omega, \delta\omega\rangle &\leq 2\langle |h_n||\mathrm{d}h_n||\omega|, |\mathrm{d}\omega|\rangle + 2\langle |\omega|, |h_n||\delta\omega||\mathrm{d}h_n|\rangle \\ &\leq 2||\mathrm{d}h_n||_\infty \int_M |h_n||\omega|(|\mathrm{d}\omega| + |\delta\omega|)\ \mathrm{d}x \\ &\leq 2||\mathrm{d}h_n||_\infty||\omega||_2(||h_n\mathrm{d}\omega||_2 + ||h_n\delta\omega||_2) \end{aligned}$$

where the last inequality again follows by Cauchy-Schwarz. We conclude that

$$||h_n\mathrm{d}\omega||_2^2 + ||h_n\delta\omega||_2^2 \leq 2||\mathrm{d}h_n||_\infty||\omega||_2(||h_n\mathrm{d}\omega||_2 + ||h_n\delta\omega||_2)$$

from which it follows that[4]

$$||h_n\mathrm{d}\omega||_2 + ||h_n\delta\omega||_2 \leq 4||\mathrm{d}h_n||_\infty||\omega||_2 = \frac{4}{n}||\omega||_2$$

as $|\mathrm{d}h_n| \leq \frac{1}{n}$. Taking the limit $n \to \infty$ and recalling that $h_n \uparrow 1$, we can take the limit inside the norm by the montone convergence theorem to find that $||\mathrm{d}\omega||_2 + ||\delta\omega||_2 = 0$. But then $\mathrm{d}\omega = \delta\omega = 0$, from which it follows that $\lambda\omega = \overrightarrow{L}\omega = -(\delta d + d\delta)\omega = 0$, from which it follows that $\omega = 0$. We conclude that $\lambda > 0$ is not an eigenvalue of $\overrightarrow{L}^*$, from which it follows by an earlier remark that $\overrightarrow{L}$ is essentially self-adjoint. □

As $\overrightarrow{L}$ is essentially self-adjoint, we can find a spectral family $\overrightarrow{E}_\lambda$ such that $\overrightarrow{L} = -\int_0^\infty \lambda\mathrm{d}\overrightarrow{E}_\lambda$, where the spectrum of $\overrightarrow{L}$ is again contained in $(-\infty, 0]$ by formula (4.2). We can thus again define the heat semigroup $\overrightarrow{P}_t = \int_0^\infty e^{-t\lambda}\mathrm{d}\overrightarrow{E}_\lambda$.

[4]Using the general fact that whenever $x^2 + y^2 \leq c(|x| + |y|)$, then $|x| + |y| \leq 2c$.

Probabilistic interpretation of heat semigroup on 1-forms

In his paper, Bakry discusses the semigroup $\overrightarrow{P}_t$ in a purely probabilistic manner. We will include this discussion here, and fill in details where possible, although we were not able to verify all details due to lack of time.

As in the case for the heat semigroup P_t, we can also find a probabilistic interpretation for $\overrightarrow{P}_t$. This interpretation will be used to prove some useful estimates. As 1-forms act on tangent vectors, we need to make a more advanced construction than in the case of functions. For explanations about the (orthonormal) frame bundle and horizontal lift, we refer to [16].

Let $\pi : O(M) \to M$ denote the orthonormal frame bundle, and let $H_u : T_{\pi(u)}M \to T_uO(M)$ be the horizontal lift.[5] A frame $u = (u_1, \dots, u_n)$ can be thought of as n tangent vectors forming a basis for $T_{\pi(u)}M$. If $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$, we will take $u_j = u(e_j)$. From this, we can define vector fields on $O(M)$ by setting $X_i(u) = H_u(u_i)$. If we define $U(u) = H_u((\mathrm{d}\log\rho)^*(\pi(u)))$ as the horizontal lift of the vector field $(\mathrm{d}\log\rho)^*$, we can consider the 'horizontal' operator $L^H = \sum_{i=1}^n X_i^2 + U$. Observe that L^H maps $O(M)$ into $O(M)$. Let U_t be the Markov process generated by L^H. Denote the law of U_t starting from $u \in O(M)$ by $\mathbb{P}^u$. From the construction it follows that the process $(\pi(U_t))$ has law $\mathbb{P}^{\pi(u)}$. Finally, we remark that we can in general use the horizontal lift to define parallel transport. This allows us to see U_t as an isometry from $T_{\pi(U_0)}M$ to $T_{\pi(U_t)}$.

Let us now define $\overrightarrow{R}_*$ acting on the tangent bundle by $\overrightarrow{R}_*(X) = R(X, \cdot)^*$. We define the process V_t taking values in the tangent bundle by

$$\frac{\mathrm{d}}{\mathrm{d}t}U_t^{-1}(V_t) = -U_t^{-1}(\overrightarrow{R}_*(V_t))$$

with intial value $V_0 = V \in T_{\pi(U_0)}M$. Using Îto's formula, proposition 4.1.15(1) and integrate, we get that

$$|V_t|^2 = |V_0|^2 - 2\int_0^t R(V_s, V_s)\mathrm{d}s$$

Using that R is bounded from below, we obtain the estimate

$$|V_t|^2 \le |V_0|^2 - 2r_0\int_0^t |V_s|^2\mathrm{d}s.$$

By a special instance of Grönwell's theorem we get that

$$|V_t| \le e^{-r_0t}|V_0|.$$

If we write $\mathbb{P}^v$ for the law of (V_t) under the condition that $V_0 = V$ almost surely, we can define $\bar{P}_t(\omega)(v) = E^v(\omega(V_t))$ for any 1-form $\omega \in C_0^\infty$. We can write

$$|\bar{P}_t(\omega)(v)| \le E^v(|\omega(V_t)|) \le E^v(|\omega|(\pi(V_t))|V_t|)$$

$$\le e^{-r_0t}|V|E^{\pi(v)}(|\omega|(\pi(U_t)) = e^{-r_0t}|V|P_t|\omega|.$$

Here, the first estimate is getting the absolute value inside the expectation. The second is Cauchy-Schwarz (after identification between ω and ω^*). Afterwards we applied the estimate

[5] In the book of Hsu, this map is defined in the other direction, but we will follow the notation as in Bakry.

derived above. It follows that $|\bar{P}_t\omega| \leq e^{-r_0 t}P_t|\omega|$. Finally, note that we constructed $\bar{P}_t$ so that for all 1-forms ω with compact support it holds that

$$\bar{P}_t\omega = \omega + \int_0^t \bar{P}_s \overrightarrow{L}\omega \mathrm{d}s$$

which shows that $\bar{P}_t$ is precisely $\overrightarrow{P}_t$.

We finish this section with a proposition collecting the most important properties of $\overrightarrow{P}_t$

Proposition 4.1.17. $\overrightarrow{P}_t$ *satisfies the following properties:*

1. $|\overrightarrow{P}_t\omega| \leq e^{-r_0 t}P_t|\omega|$.
2. $||\overrightarrow{P}_t\omega||_p \leq e^{-r_0|1-2/p|t}||\omega||_p$, $1 \leq p \leq \infty$.
3. $\overrightarrow{P}_t\mathrm{d}f = \mathrm{d}P_t f$ *for* $f \in C_0^\infty$.
4. *If* $\omega \in C_0^\infty$, *we have that* $\overrightarrow{P}_t\omega \in C^\infty(M \times [0,\infty))$ *and it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{P}_t\omega = \overrightarrow{L}\,\overrightarrow{P}_t\omega = \overrightarrow{P}_t\overrightarrow{L}\omega.$$

Proof. The first part follows in the same way as above for $\bar{P}_t$. For the second part, observe that part (1) gives us that $||\overrightarrow{P}_t\omega||_1 \leq e^{-r_0 t}||\omega||_1$ and $||\overrightarrow{P}_t\omega||_\infty \leq e^{-r_0 t}||\omega||_\infty$. Furthermore, for $p = 2$ we already know that $||\overrightarrow{P}_t\omega||_2 \leq ||\omega||_2$ (this follows from the probabilistic interpretation). Using interpolation between $p = 1$ and $p = 2$, and also between $p = 2$ and $p = \infty$, we get the desired estimate for the other p.

For part (3), first remember that $\overrightarrow{L}\mathrm{d}f = \mathrm{d}Lf$. Now for all $\omega \in C_0^\infty$ we have

$$\langle \mathrm{d}P_t f, \omega\rangle = \langle P_t f, \delta\omega\rangle = \int_0^\infty e^{-\lambda t}\mathrm{d}\langle E_\lambda f, \delta\omega\rangle.$$

On the other hand, we have that

$$\langle \overrightarrow{P}_t\mathrm{d}f, \omega\rangle = \langle \mathrm{d}f, \overrightarrow{P}_t\omega\rangle = \int_0^\infty e^{-\lambda t}\mathrm{d}\langle \overrightarrow{E}_\lambda\omega, \mathrm{d}f\rangle.$$

Finally, we observe that

$$\langle E_\lambda f, \delta\omega\rangle = \langle \mathrm{d}E_\lambda f, \omega\rangle = \langle \overrightarrow{E}_\lambda\mathrm{d}f, \omega\rangle = \langle \mathrm{d}f, \overrightarrow{E}_\lambda\omega\rangle.$$

Here, we used that $\mathrm{d}E_\lambda f = \overrightarrow{E}_\lambda\mathrm{d}f$ as it also holds for L and $\overrightarrow{L}$.

The last part follows from general theory of (Markov) semigroups and the fact that $\overrightarrow{L}$ is elliptic. ☐

Final remark. As the smooth functions and 1-forms with compact support are dense in L^p respectively $\overrightarrow{L}^p$, estimates for P_t and $\overrightarrow{P}_t$ may be extended to holds on L^p and $\overrightarrow{L}^p$ instead of only holding for C_0^∞. It can even be shown that P_t and $\overrightarrow{P}_t$ extend to strongly continuous semigroups on L^p and $\overrightarrow{L}^p$ respectively. These ideas are discussed in more detail in section 6.2, especially in proposition 6.2.1.

4.2 Subordinated semigroups and harmonic extensions

We will now focus our attention on harmonic extensions of functions defined on M. If f is a function on M, we will look at its extension to $M \times \mathbb{R}_+$, defined by equations of the form

$$\left[\left(\frac{\partial}{\partial t} - (s-d)I\right)\left(\frac{\partial}{\partial t} - (s+d)I\right) + L\right] f(x,t) = 0.$$

where $s \in \mathbb{R}$ and $d \geq 0$. The reason why we do it slightly differently than in the case of $M = \mathbb{R}^n$ is that we wish to take into account the curvature of the manifold. We will show that given an initial distribution at time $t = 0$, say $f_0(x)$, the solutions of the above equations are given by $f(x,t) = Q_t^{s,d} f_0(x)$ for some semigroup $Q_t^{s,d}$ depending on P_t. Its analogue for 1-forms is denoted conform other notation by $\overrightarrow{Q}_t^{s,d}$. The semigroups that arise in this way turn out to be particularly useful since their generators are operators that we wish to study for the Riesz transforms.

Before we continue, we introduce some notation. On $\mathbb{R}_+$ we denote $\frac{\partial}{\partial t}$ by D_0 and define $L_0^{s,d}$ by

$$L_0^{s,d} = (D_0 - (s-d)I)(D_0 - (s+d)I) = D_0^2 - 2sD_0 + (s^2 - d^2)I$$

where we consider $s \in \mathbb{R}$ and $d \geq 0$. Furthermore, we denote by M_b the multiplication operator by e^{bt}. One can show that

$$M_b L_0^{s,d} M_{-b} = L_0^{s+b,d}$$

Indeed, using the product rule and collecting terms, we find that

$$L_0^{s,d} M_{-b} f = e^{-bt} D_0^2 f - 2e^{-bt}(s+b) D_0 f + e^{-bt}(b^2 + 2sb + s^2 - d^2) f$$

and hence

$$M_b L_0^{s,d} M_{-b} f = D_0^2 f - 2(s+b) D_0 f + (b^2 + 2sb + s^2 - d^2) f = L_0^{s+b,d} f$$

as $b^2 + 2sb + s^2 = (s+b)^2$.

Furthermore, on $M \times \mathbb{R}_+$ we write $L^{s,d}$ for the operator $L_0^{s,d} + L$ and $\overrightarrow{L}^{s,d} = L_0^{s,d} + \overrightarrow{L}$. If $s = d$ we simply write L_0^d, L^d and $\overrightarrow{L}^d$.

Finally, we define $|\overline{\mathrm{d}} f|^2 := |\mathrm{d}f|^2 + (D_0 f)^2$ and for a family of 1-forms $\omega(x,t)$, indexed by t, we write $|\overline{\mathrm{d}}\omega|^2 := |\mathrm{d}\omega|^2 + (D_0\omega)^2$ and $|\overline{\nabla}\omega|^2 := |\nabla\omega|^2 + (D_0\omega)^2$.

4.2.1 Subordinated semigroups

We now introduce measures $m_t(\mathrm{d}u)$ for all $t \in \mathbb{R}_+$ given by

$$m_t(\mathrm{d}u) = \frac{1}{2\sqrt{\pi}} t u^{-3/2} e^{-t^2/(4u)}\, \mathrm{d}u$$

Remember that the probability density of the hitting time of Brownian motion of a point t is given on $\mathbb{R}_+$ by $\frac{1}{\sqrt{2\pi}} t u^{-3/2} e^{-t^2/(2u)}$. This thus integrates to 1, giving us that

$$\int_0^\infty m_t(\mathrm{d}u) = 1$$

after making the substitution $u = \frac{1}{2}x$. Furthermore, we have for any $c \in \mathbb{R}$ that[6]

$$\int_0^\infty e^{-c^2 u} m_t(\mathrm{d}u) = e^{-|c|t} \tag{4.3}$$

[6] To prove this identity one could make use of the so called Modified Bessel functions. It turns out that the integral we wish to compute is the integral representation of such a function for a given parameter, and for this parameter, there also exists an explicit formula.

which is an expression that we will use multiple times in the upcoming proofs.

Armed with these measures we can now define the subordinated semigroups $Q_t^{s,d}$ and $\overrightarrow{Q}_t^{s,d}$ which we talked about earlier.

Definition 4.2.1. For all $s \in \mathbb{R}$ and $d \geq 0$ we define for $f \in L^2$ and $\omega \in \overrightarrow{L}^2$ the operators $Q_t^{s,d}$ and $\overrightarrow{Q}_t^{s,d}$ by

$$Q_t^{s,d} f = \int_0^\infty (P_u f) e^{st-d^2u} m_t(\mathrm{d}u)$$

$$\overrightarrow{Q}_t^{s,d}\omega = \int_0^\infty (\overrightarrow{P}_u \omega) e^{st-d^2u} m_t(\mathrm{d}u)$$

where P_u, $\overrightarrow{P}_u$ are as in the previous section.

The following two propositions sum up various properties of these operators.

Proposition 4.2.2. *The operators $Q_t^{s,d}$ satisfy the following properties:*

1. *$Q_t^{s,d}$ is a positive, symmetric semigroup on L^2 with generator $C^{s,d} = sI - (d^2 I - L)^{1/2}$.*
2. *$M_b Q_t^{s,d} = Q_t^{s+b,d}$.*
3. *For all $1 \leq p \leq \infty$ we have the estimate $||Q_t^{s,d} f||_p \leq e^{t(s-d)}||f||_p$.*
4. *$Q_t^{s,d} 1 = e^{t(s-d)}$ and in particular, $Q_t^{d,d} = Q_t^d$ is Markovian.*
5. *There exist constants $c(s, d)$ depending only on s, d such that for all $1 \leq p \leq \infty$ and all $f \in C_0^\infty$ we have $f \in D(C^{s,d})$ and*

$$||C^{s,d} f||_p \leq c(s, d)(||f||_p + ||Lf||_p).$$

6. *For all $f \in C_0^\infty$, the function $\bar{f}(x, t) = Q_t^{s,d} f(x)$ is in $C^\infty(M \times (0, \infty)) \cap C(M \times [0, \infty))$ and satisfies $L^{s,d}\bar{f} = 0$.*

Proposition 4.2.3. *The operators $\overrightarrow{Q}_t^{s,d}$ satisfy the following properties:*

1. *$\overrightarrow{Q}_t^{s,d}$ is a symmetric semigroup on L^2 with generator given by $\overrightarrow{C}^{s,d} = sI - (d^2 I - \overrightarrow{L})^{1/2}$.*
2. *$M_b \overrightarrow{Q}_t^{s,d} = \overrightarrow{Q}_t^{s+b,d}$.*
3. *If $d^2 \geq -r_0$, then for all C_0^∞ 1-forms ω we have that*

$$|\overrightarrow{Q}_t^{s,d}\omega| \leq Q_t^{s,(d^2+r_0)^{1/2}}|\omega|$$

4. *If $d^2 \geq -r_0|1 - 2/p|$, then $(\overrightarrow{Q}_t^{s,d})$ is a semigroups of bounded operators on $\overrightarrow{L}^p$, where the norms are bounded by $e^{t(s-(d^2+r_0|1-2/p|)^{1/2}}$.*
5. *If $d^2 \geq -r_0|1-2/p|$, then there exist constants $c(s, d)$ depending only on s, d such that for all $1 \leq p \leq \infty$ and all $\omega \in C_0^\infty$*

$$||\overrightarrow{C}^{s,d}\omega||_p \leq c(s, d)(||\omega||_p + ||\overrightarrow{L}\omega f||_p)$$

6. *For all* $\omega \in C_0^\infty$, *the function* $\bar{\omega}(x,t) = \overrightarrow{Q}_t^{s,d}\omega(x)$ *is in* $C^\infty(M \times (0,\infty)) \cap C(M \times [0,\infty))$ *and satisfies* $\overrightarrow{L}^{s,d}\bar{\omega} = 0$.

Before we turn to the proofs of these propositions, we first need a lemma, which is used in the proof of part (5) of the propositions.

Lemma 4.2.4. *The following functions, defined on* $[0,\infty)$ *are Laplace transforms of bounded measures:*

$$f_1(x) = \frac{(1+x)^{1/2}}{1+x^{1/2}}, \quad f_2(x) = \frac{1+x^{1/2}}{(1+x)^{1/2}}, \quad f_3(x) = \frac{1}{(1+x)^{1/2}}, \quad f_4(x) = \frac{x^{1/2}}{1+x}.$$

Proof. By formula (4.3) the Laplace transform of $m_t(\mathrm{d}u)$ is given by $e^{-tx^{1/2}}$. But then we find for $n_t(\mathrm{d}u)e^{t-u}m_t(\mathrm{d}u)$ that

$$\mathcal{L}(n_t(\mathrm{d}u)) = e^t\mathcal{L}(e^{-u}m_t(\mathrm{d}u)) = e^te^{-(x+1)^{1/2}t}.$$

If we now define $n(\mathrm{d}u) = \int_0^\infty e^{-t}n_t(\mathrm{d}u)$ then

$$\begin{aligned}
\mathcal{L}(n(\mathrm{d}u)) &= \int_0^\infty\int_0^\infty e^{-xu}e^{-t}n_t(\mathrm{d}u)\mathrm{d}t \\
&= \int_0^\infty e^{-t}e^te^{-(x+1)^{1/2}t}\mathrm{d}t \\
&= \int_0^\infty e^{-(x+1)^{1/2}t}\mathrm{d}t \\
&= \left[-\frac{1}{(x+1)^{1/2}}e^{-(x+1)^{1/2}t}\right]_{t=0}^\infty \\
&= \frac{1}{(x+1)^{1/2}}.
\end{aligned}$$

Now $n(\mathrm{d}u)$ is a bounded measure as $m_t(\mathrm{d}u)$ and so also $n_t(\mathrm{d}u)$ is. Hence we found a bounded measure which admits f_3 as Laplace transform.

Now observe that

$$f_2(x) - f_3(x) = \frac{x^{1/2}}{(1+x)^{1/2}} = \left(\frac{x}{1+x}\right)^{1/2} = \left(1 - \frac{1}{1+x}\right)^{1/2} = (1 - f_3(x)^2)^{1/2}.$$

But then we find that

$$f_2(x) - f_3(x) = 1 - \sum_{k=1}^\infty c_kf_3^{2k}$$

with $c_k \geq 0$, $\sum_{k=1}^\infty c_k = 1$.[7] Now observe that multiplication in the Laplace domain correspond to convolution in the usual domain, which gives us that $\mathcal{L}(n^{*2k}) = f_3^{2k}$, where the $*$ in the exponent indicates powers of convolution. Noting that $1 = \mathcal{L}(\delta_0)$, the linearity of the Laplace transform gives us that $n'(\mathrm{d}u) = \delta_0 - \sum_{k=1}^\infty c_kn^{*2k}$ has Laplace transform equal to $f_2 - f_3$. But then $n'(\mathrm{d}u) + n(\mathrm{d}u)$ has Laplace transform f_2.

Noting that $f_4 = f_3(f_2 - f_3)$, we see that f_4 is the Laplace transform of $n * n'$.

It remains to show that f_1 is the Laplace transform of some bounded measure. We first note that[8] $\int_0^\infty t^{-3/2}(1-e^{-tx})\mathrm{d}t = 2\sqrt{\pi x}$, which shows that $\frac{1}{\sqrt{2\pi}}t^{-3/2}(1-e^{-t})\mathrm{d}t$ is a bounded measure. Calculating the Laplace transform, we find that

$$\frac{1}{\sqrt{2\pi}}\int e^{-xt}t^{-3/2}(1-e^{-t})\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int t^{-3/2}e^{-xt}\mathrm{d}t - \frac{1}{\sqrt{2\pi}}\int t^{-3/2}e^{-(x+1)t}\mathrm{d}t$$

[7] This comes from the Taylor expansion of the function $g(x) = \sqrt{1+x}$.

[8] First make the substitution $u = \frac{1}{\sqrt{t}}$, then integrate by parts and then make the substitution $y = \frac{1}{u}$.

$$= \quad -x^{1/2} + (x+1)^{1/2}.$$

On the other hand, we find that $\int_0^\infty m_t(\mathrm{d}u)e^{-t}\mathrm{d}t$ has Laplace transform $\frac{1}{1+x^{1/2}}$, which can be proved similar to the Laplace transform of n. Finally, we can write

$$f_1(x) = ((1+x)^{1/2} - x^{1/2})\frac{1}{1+x^{1/2}} + \frac{x^{1/2}}{1+x^{1/2}}$$

from which we conclude that f_1 is nothing more than the multiplication and addition of Laplace transforms of bounded measures, which is again the Laplace transform of a bounded measure, as we have already used before. □

We will now turn to the proof of proposition 4.2.2.

Proof of proposition 4.2.2. We will first prove (1). Remember that $P_u = \int_0^\infty e^{-\lambda u}\mathrm{d}E_\lambda$. Plugging this into the definition of $Q_t^{s,d}$ gives us for some functions $f, g \in C_0^\infty$ that

$$\begin{aligned}
\langle Q_t^{s,d} f, g\rangle &= \left\langle \int_0^\infty (P_u f) e^{st-d^2u} m_t(\mathrm{d}u), g \right\rangle \\
&= \int_0^\infty \langle P_u f, g\rangle e^{st-d^2u} m_t(\mathrm{d}u) \\
&= \int_0^\infty \left(\int_0^\infty e^{-\lambda t}\mathrm{d}\langle E_\lambda f, g\rangle \right) e^{st-d^2u} m_t(\mathrm{d}u) \\
&= \int_0^\infty \int_0^\infty e^{-\lambda t} e^{st-d^2u} m_t(\mathrm{d}u)\mathrm{d}\langle E_\lambda f, g\rangle \\
&= \int_0^\infty e^{st} \int_0^\infty e^{-(d^2+\lambda)t} m_t(\mathrm{d}u)\mathrm{d}\langle E_\lambda f, g\rangle \\
&= \int_0^\infty e^{st} e^{-(d^2+\lambda)^{1/2}t}\mathrm{d}\langle E_\lambda f, g\rangle \\
&= \int_0^\infty e^{(s-(d^2+\lambda)^{1/2})t}\mathrm{d}\langle E_\lambda f, g\rangle \\
&= \langle e^{tC^{s,d}}, g\rangle.
\end{aligned}$$

Here we used Fubini in the second line, which we may do as all function are smooth and g has compact support, so that the integral of the absolute value is finite. In the fourth line we again use Fubini, which is justified as the functions are nonnegative. In the fifth line we used formula (4.3). As the above holds for all $g \in C_0^\infty$, by density we may conclude that $Q_t^{s,d} = e^{tC^{s,d}}$ which shows that $Q_t^{s,d}$ is a semigroup with generator $C^{s,d} = sI - (d^2 - L)^{1/2}$. The fact that this semigroup is symmetric and positive follows immediately from the fact that the semigroup (P_t) is.

For (2), pick a function f and compute that

$$\begin{aligned}
M_b Q_t^{s,d} f &= e^{bt} \int_0^\infty (P_u f) e^{st-d^2u} m_t(\mathrm{d}u) \\
&= \int_0^\infty (P_u f) e^{(s+b)t-d^2u} m_t(\mathrm{d}u) \\
&= Q_t^{s+b,d} f
\end{aligned}$$

showing that $M_b Q_t^{s,d} = Q_t^{s+b,d}$.

For (3), using that P_u is a contraction on L^p, we find for a function f that

$$\begin{aligned} ||Q_t^{s,d} f||_p &\leq \int_0^\infty e^{st-d^2u} ||P_u f||_p m_t(\mathrm{d}u) \\ &\leq ||f||_p e^{st} \int_0^\infty e^{-d^2u} m_t(\mathrm{d}u) \\ &= ||f||_p e^{st} e^{-dt} \\ &= e^{(s-d)t} ||f||_p \end{aligned}$$

where we again used formula (4.3).

For (4), note that $P_u 1 = 1$ for all u. Hence

$$Q_t^{s,d} 1 = \int_0^\infty e^{st-d^2u} m_t(\mathrm{d}u) = e^{t(s-d)}$$

as above. Especially, when $s = d$, this gives us that $Q_t^d 1 = 1$. Futhermore, as $P_u f \geq 0$ whenever $f \geq 0$, we find that if $f \geq 0$, then $Q_t^{s,d} f \geq 0$, as the integrand is nonnegative. These two remarks show that Q_t^d is in fact Markovian.

For (5) we observe that by lemma 4.2.4 there exists a bounded measure α with Laplace transform given by $\frac{x^{1/2}}{1+x}$. By the rules for the Laplace transform, we then find that $\alpha_d(\mathrm{d}u) = e^{-d^2u}\alpha(\mathrm{d}u)$ has Laplace transform $\frac{(d^2+x)^{1/2}}{1+d^2+x}$. Similar to the computation in the proof of part (1), we get that

$$(d^2 - L)^{1/2}(1 + d^2 - L)^{-1} = \int_0^\infty P_u \ \alpha_d(\mathrm{d}u).$$

This is bounded operator, of which the norm is bounded by $\int_0^\infty \alpha_d(\mathrm{d}u) \leq \int_0^\infty \alpha_d(\mathrm{d}u) =: |\alpha|$ as P_u is a contraction. But then we find for a function $f \in C_0^\infty$ that

$$||(d^2 - L)^{1/2} f||_p = \left|\left|\int_0^\infty P_u((1 + d^2)f - Lf)\alpha_d(\mathrm{d}u)\right|\right|_p \leq |\alpha|||(1 + d^2)f - Lf)||_p$$

where we used that $\int_0^\infty P_u \ \alpha_d(\mathrm{d}u)$ is bounded with norm bounded by $|\alpha|$. But then we find that

$$||(d^2 - L)^{1/2} f||_p \leq |\alpha|[(1 + d^2)||f||_p + ||Lf||_p].$$

Observing that the operator on the left hand side is simply $C^{0,d}$, we conclude that there exists a constant c only depending on d such that

$$||C^{0,d} f||_p \leq c(||f||_p + ||Lf||_p).$$

But then we find that

$$||C^{s,d} f||_p \leq s||f||_p + ||C^{0,d}||_p \leq s||f||_p + c(||f||_p + ||Lf||_p)$$

which proves the claim.

Finally, for (6), if $f \in C_0^\infty$, then the function $\bar{f}(x,t) = Q_t^{s,d} f(x)$ satisfies $D_0 \bar{f} = C^{s,d} \bar{f}$ in the L^2 sense, as $C^{s,d}$ is the generator of $Q_t^{s,d}$. Also note that on the domain of $C^{s,d}$ the operators $C^{s,d}$ and $Q_t^{s,d}$ commute. Let us now argue that for $f \in C_0^\infty$ the expression $(C^{s,d})^k f$ is well-defined.

For this, first note that as $C^{s,d} = sI - (d^2 - L)^{1/2}$ we have that $f \in D(L)$ precisely when $f, C^{s,d} f \in D(C^{s,d})$. As $C_0^\infty \subset D(L)$, we have for $f \in C_0^\infty$ that $f, C^{s,d} f \in D(C^{s,d})$. We also have that $(C^{s,d})^2 f = Lf \in C_0^\infty$ as both d and δ map C_0^∞ to C_0^∞. But then we can repeat the argument to show that arbitrary powers of $C^{s,d}$ are well-defined at least on C_0^∞. By the commutativity of $Q_t^{s,d}$ and $C^{s,d}$ on the domain of $C^{s,d}$ we conclude that $D_0^k \bar{f} = (C^{s,d})^k \bar{f}$ for any $k \geq 1$. Now using the expression for $C^{s,d}$ from part (a) we find that

$$(C^{s,d})^2 - 2sC^{s,d} + (s^2 - d^2)I = -L.$$

But then

$$L^{s,d}\bar{f} = L_0^{s,d}\bar{f} + L\bar{f} = -L\bar{f} + L\bar{f} = 0$$

which holds in the L^2 sense. It then also holds in the sense of distributions on $M \times (0, \infty)$. However, as $L^{s,d}$ is an elliptic operator, we in fact find that it must be that $\bar{f} \in C^\infty$, and consequently, it is a solution of $L^{s,d}\bar{f} = 0$ in the ordinary sense.

It remains to show the continuity at 0. For this, observe that for semigroups we have by proposition 3.1.5 that

$$\begin{aligned} ||Q_t^{s,d} f - f||_\infty &= \left|\left| \int_0^t Q_u^{s,d} C^{s,d} f \mathrm{d}u \right|\right| \\ &\leq \int_0^t ||Q_u^{s,d} C^{s,d} f||_\infty \mathrm{d}u \\ &\leq \int_0^t e^{(s-d)u} ||C^{s,d} f||_\infty \mathrm{d}u \end{aligned}$$

where we used part (c). Letting $t \to 0$, we see that the upper bound goes to 0 (as $||C^{s,d}f||_\infty < \infty$ by (e)), proving the continuity of $Q_t^{s,d} f(x) = \bar{f}(x,t)$ at $t = 0$. □

Proof of proposition 4.2.3. Parts (1), (2) and (6) are proved in the same way as done in the proof of proposition 4.2.2.

For part (3) we use proposition 4.1.17, giving us that $|\overrightarrow{P}_t \omega| \leq e^{-r_0 t} P_t |\omega|$ for a 1-form ω. Applying this estimate, we find for a 1-form ω that

$$\begin{aligned} |\overrightarrow{Q}_t^{s,d} \omega| &= \left| \int_0^\infty (\overrightarrow{P}_u \omega) e^{st - d^2 u} m_t(\mathrm{d}u) \right| \\ &\leq \int_0^\infty |\overrightarrow{P}_u \omega| e^{st - d^2 u} m_t(\mathrm{d}u) \\ &\leq \int_0^\infty e^{-r_0 u} P_u |\omega| |e^{st - d^2 u} m_t(\mathrm{d}u) \\ &= \int_0^\infty P_u |\omega| |e^{st - (d^2 + r_0) u} m_t(\mathrm{d}u) \\ &= Q_t^{s,(d^2 + r_0)^{1/2}} |\omega| \end{aligned}$$

as desired. Here we assumed that $d^2 \geq -r_0$, because we need that $d^2 + r_0 \geq 0$, as we only define the semigroups $Q_t^{s,d}$ for $d \geq 0$.[9]

For part (4), we observe that proposition 4.1.17 gives us that for any 1-form $\omega \in C_0^\infty$ we have the estimate $||\overrightarrow{P}_t \omega||_p \leq e^{-r_0 |1 - p/2| t} ||\omega||_p$. We find that

$$\begin{aligned} ||\overrightarrow{Q}_t^{s,d} \omega||_p &\leq \int_0^\infty ||P_u \omega||_p e^{st - d^2 u} m_t(\mathrm{d}u) \\ &\leq \int_0^\infty e^{-r_0 |1 - p/2| u} ||\omega||_p e^{st - d^2 u} m_t(\mathrm{d}u) \end{aligned}$$

[9]This is done for the convergence of the integral.

$$
\begin{aligned}
&= ||\omega||_p e^{st} \int_0^\infty e^{-(r_0|1-2/p|+d^2)u} m_t(\mathrm{d}u) \\
&= e^{(s-(r_0|1-2/p|+d^2)^{1/2})t} ||\omega||_p
\end{aligned}
$$

where in the last line we again used formula (4.3). This shows that $\overrightarrow{Q}_t^{s,d}$ is bounded on $\overrightarrow{L}^p$ with norm bounded by $e^{(s-(r_0|1-2/p|+d^2)^{1/2})t}$. That this again gives a semigroup on all of $\overrightarrow{L}^p$ can be proven similarly as done in proposition 6.2.1 in section 6.2.

For part (5) we only remark that this goes similarly as in the proof of part (5) of proposition 4.2.2. However, instead of using that $||P_u f||_p \leq ||f||_p$, we now use the estimate $||\overrightarrow{P}_u \omega||_p \leq e^{-r_0|1-p/2|u}||\omega||_p$. The assumption $d^2 \geq -r_0|1-2/p|$ is then necessary to assure that bounds are finite. $\square$

4.2.2 Harmonic extensions

We finish the chapter by proving some relevant equalities and inequalities concerning harmonic extensions. The proofs in the section are not particularly difficult, but are rather some extensive computations and rewritings. The results are nevertheless important when proving the boundedness of the Riesz transform later on.

Proposition 4.2.5. *Let $f \in C^\infty(M \times (0,\infty))$ be such that $L^{s_1,d_1} f = 0$. Then*

1. *For $s_2 \in \mathbb{R}, d_2 \geq 0$ we have that*

$$L^{s_2,d_2} f^2 = 2|\mathrm{d}f|^2 + 2(D_0 f + (s_1 - s_2)f)^2 + (2d_1^2 - d_2^2 - (s_2 - 2s_1)^2)f^2.$$

2. *If $d_1^2 \geq d_2^2 \geq s_1^2$, we have for all $\epsilon > 0$ that $L^{s_1,d_2}((f^2+\epsilon^2)^{1/2} - \epsilon) \geq 0$.*

Furthermore, let $\omega(x,t)$ be a family of 1-forms on M, smooth in (x,t) and suppose that $\overrightarrow{L}^{s_1,d_1}\omega = 0$. Then

1. *For $s_2 \in \mathbb{R}, d_2 \geq 0$ we have*

$$L^{s_2,d_2}|\omega|^2 \geq 2|\nabla\omega|^2 + 2|D_0\omega + (s_1 - s_2)\omega|^2 + (2d_1^2 - d_2^2 - (s_2 - 2s_1)^2 + 2r_0)|\omega|^2.$$

2. *If $r_0 + d_1^2 \geq d_2^2 \geq s_1^2$, then for all $\epsilon > 0$ it holds that $L^{s_1,d_2}((|\omega|^2+\epsilon^2)^{1/2} - \epsilon) \geq 0$.*

Proof. We will start by proving the statemensts concerning a function $f \in C^\infty(M \times (0,\infty))$. First observe that

$$L^{s_2,d_2} - L^{s_1,d_1} = L_0^{s_2,d_2} - L_0^{s_1,d_2} = 2(s_1 - s_2)D_0 + (s_2^2 - s_1^2 + d_1^2 - d_2^2)I. \tag{4.4}$$

By the chain rule we find for a function $\phi : \mathbb{R} \to \mathbb{R}$, that $D_0\phi(f) = \phi'(f)D_0 f$ and

$$D_0^2\phi(f) = \phi'(f)D_0^2 f + \phi''(f)D_0 f D_0 f = \phi'(f)D_0^2 f + \phi''(f)(D_0 f)^2.$$

Similarly, $\mathrm{d}\phi(f) = \phi'(f)\mathrm{d}f$ and $\Delta\phi(f) = \phi'(f)\Delta f + \phi''(f)|\mathrm{d}f|^2$, which gives us that

$$
\begin{aligned}
L\phi(f) &= \Delta\phi(f) + \mathrm{d}\phi(f)\cdot \mathrm{d}(\log\rho) \\
&= \phi'(f)\Delta f + \phi'(f)\mathrm{d}f \cdot \mathrm{d}(\log\rho) + \phi''(f)|\mathrm{d}f|^2 \\
&= \phi'(f)Lf + \phi''(f)|\mathrm{d}f|^2.
\end{aligned}
$$

Putting everything together, we find that (already collecting terms)

$$\begin{aligned} L^{s,d}\phi(f) &= \phi'(f)(D_0^2 f - 2sD_0 f + Lf) + \phi''(f)((D_0 f)^2 + |\mathrm{d}f|^2) + (s^2 - d^2)\phi(f) \\ &= \phi'(f)L^{s,d}f - \phi'(f)(s^2 - d^2)f + \phi''(f)|\bar{\mathrm{d}}f|^2 + (s^2 - d^2)\phi(f) \\ &= \phi'(f)L^{s,d}f + \phi''(f)|\bar{\mathrm{d}}f|^2 + (s^2 - d^2)(\phi(f) - f\phi'(f)). \end{aligned}$$

Applying this identity with $(s, d) = (s_1, d_1)$ and $\phi(x) = x^2$, we get

$$L^{s_1,d_1} f^2 = 2|\bar{\mathrm{d}}f|^2 - f^2(s_1^2 - d_1^2) = 2|\bar{\mathrm{d}}f|^2 + f^2(d_1^2 - s_1^2)$$

where we used that $L^{s_1,d_1} f = 0$ and $\phi(x) - x\phi'(x) = x^2 - 2x^2 = -x^2$. Plugging this into (4.4), we find that

$$\begin{aligned} L^{s_2,d_2} f^2 &= L^{s_1,d_1} f^2 + 2(s_1 - s_2)D_0 f^2 + (s_2^2 - s_1^2 + d_1^2 - d_2^2)f^2 \\ &= 2|\bar{\mathrm{d}}f|^2 + 4(s_1 - s_2)fD_0 f + (s_2^2 - 2s_1^2 + 2d_1^2 - d_2^2)f^2 \end{aligned}$$

where we also used that $D_0 f^2 = 2fD_0 f$. Remembering that $|\bar{\mathrm{d}}f|^2 = |\mathrm{d}f|^2 + (D_0 f)^2$, we may write

$$2|\bar{\mathrm{d}}f|^2 + 4(s_1 - s_2)fD_0 f = 2|\mathrm{d}f|^2 + 2(D_0 f + (s_1 - s_2)f)^2 - 2(s_1 - s_2)^2 f^2.$$

Observing that $s_2^2 - 2s_1^2 - 2(s_1 - s_2)^2 = -(s_2 - 2s_1)^2$, we find that

$$L^{s_2,d_2} f^2 = 2|\mathrm{d}f|^2 + 2(D_0 f + (s_1 - s_2)f)^2 + (2d_1^2 - d_2^2 - (s_2 - 2s_1)^2)f^2$$

which is the first identity.

For the second part, let $\epsilon > 0$ and suppose that $d_1^2 \geq d_2^2 \geq s_1^2$. We will use the same formulas as above, only now with $s_1 = s_2$ and $\phi(x) = (x^2 + \epsilon^2)^{1/2} - \epsilon$. Then $\phi(x) \geq 0$, as $(x^2 + \epsilon^2)^{1/2} \geq \epsilon$. Furthermore, $\phi'(x) = \frac{x}{(x^2+\epsilon^2)^{1/2}}$ and $\phi''(x) = \frac{x^2}{(x^2+\epsilon^2)^{3/2}} + \frac{1}{(x^2+\epsilon^2)^{1/2}} = \frac{\epsilon^2}{(x^2+\epsilon^2)^{3/2}} \geq 0$. We also have that

$$\phi(x) - x\phi'(x) = (x^2 + \epsilon^2)^{1/2} - \epsilon - \frac{x^2}{(x^2 + \epsilon^2)^{1/2}} = \frac{\epsilon^2}{(x^2 + \epsilon^2)^{1/2}} - \epsilon =$$

$$= \epsilon\left(\frac{\epsilon}{(x^2 + \epsilon^2)^{1/2}} - 1\right) \leq 0$$

as the fraction is less than 1.

Using the formulas above, we then find that (with $s_2 = s_1$)

$$\begin{aligned} L^{s_1,d_2}\phi(f) &= L^{s_1,d_1}\phi(f) + (d_1^2 - d_2^2)\phi(f) \\ &= \phi''(f)|\bar{\mathrm{d}}f|^2 + (s_1^2 - d_1^2)(\phi(f) - f\phi'(f)) + (d_1^2 - d_2^2)\phi(f). \end{aligned}$$

Now noticing that the assumption $d_1^2 \geq d_2^2 \geq s_1^2$ implies that $s_1^2 - d_1^2 \leq 0$ and $d_1^2 - d_2^2 \geq 0$, we find from $\phi(x), \phi''(x) \geq 0$ and $\phi(x) - x\phi'(x) \leq 0$ that

$$L^{s_1,d_2}((f^2 + \epsilon^2)^{1/2} - \epsilon) = L^{s_1,d_2}\phi(f) \geq 0$$

as desired.

Now suppose that $\omega(x, t)$ is a family of 1-forms, C^∞ in (x, t) and satisfying $\overrightarrow{L}^{s_1,d_1}\omega = 0$. From proposition 4.1.15(2) we find that

$$L|\omega|^2 = 2\omega \cdot \overrightarrow{L}\omega + 2|\nabla\omega|^2 + 2R(\omega^*, \omega^*).$$

We also have that

$$\overrightarrow{L}^{s,d}\omega = L_0^{s,d}\omega + \overrightarrow{L}\omega = D_0^2\omega - 2sD_0\omega + (s^2 - d^2)\omega + \overrightarrow{L}\omega.$$

Combining the two, we find that

$$\begin{aligned} L|\omega|^2 &= 2\omega \cdot (\overrightarrow{L}^{s,d}\omega - D_0^2\omega + 2sD_0\omega - (s^2 - d^2)\omega) + 2|\nabla\omega|^2 + 2R(\omega^*, \omega^*) \\ &= 2\omega \cdot \overrightarrow{L}^{s,d}\omega + 2\omega \cdot (2sD_0\omega - D_0^2\omega) + 2(d^2 - s^2)|\omega|^2 + 2|\nabla\omega|^2 + 2R(\omega^*, \omega^*). \end{aligned}$$

Now observe that $D_0|\omega|^2 = D_0(\omega \cdot \omega) = 2\omega \cdot D_0\omega$ and

$$D_0^2|\omega|^2 = D_0(2\omega \cdot D_0\omega) = 2D_0\omega \cdot D_0\omega + 2\omega \cdot D_0^2\omega = 2|D_0\omega|^2 + 2\omega \cdot D_0^2\omega.$$

But then

$$\begin{aligned} L^{s,d}|\omega|^2 &= L_0^{s,d}|\omega|^2 + L|\omega|^2 \\ &= 2\omega \cdot \overrightarrow{L}^{s,d}\omega + 2|\bar{\nabla}\omega|^2 + 2R(\omega^*, \omega^*) + (d^2 - s^2)|\omega|^2. \end{aligned}$$

Using that $R(\omega^*, \omega^*) \geq r_0|\omega|^2$ now gives us that

$$L^{s,d}|\omega|^2 \geq 2\omega \cdot \overrightarrow{L}^{s,d}\omega + 2|\bar{\nabla}\omega|^2 + (d^2 - s^2 + 2r_0)|\omega|^2.$$

Using the identity

$$\overrightarrow{L}^{s_2,d_2} - \overrightarrow{L}^{s_1,d_1} = L_0^{s_2,d_2} - L_0^{s_1,d_2} = 2(s_1 - s_2)D_0 + (s_2^2 - s_1^2 + d_1^2 - d_2^2)I$$

and the fact that $L^{s_1,d_1}\omega = 0$, a similar rewriting as done in part (1) above (where we remember that $D_0|\omega|^2 = 2\omega \cdot D_0\omega$) gives

$$L^{s_2,d_2}|\omega|^2 \geq 2|\nabla\omega|^2 + 2|D_0\omega + (s_1 - s_2)\omega|^2 + (2d_1^d - d_2^2 - (s_2 - 2s_1)^2 + 2r_0)|\omega|^2$$

as desired.

For the second part, pick $\epsilon > 0$ and suppose that $r_0 + d_1^2 \geq d_2^2 \geq s_1^2$. Write $|\omega|_\epsilon = (|\omega|^2 + \epsilon^2)^{1/2}$ and take $\phi(x) = (x + \epsilon^2)^{1/2} - \epsilon$ defined for $x \geq 0$. Then $\phi'(x) = \frac{1}{2}(x + \epsilon^2)^{-1/2}$, giving us that

$$\psi(x) := \phi(x) - x\phi'(x) = (x + \epsilon^2)^{1/2} - \epsilon - \frac{x}{2(x + \epsilon^2)^{1/2}} = \frac{x/2 + \epsilon^2}{(x + \epsilon^2)^{1/2}} - \epsilon.$$

Now

$$\begin{aligned} \psi(x) &= \frac{1}{2}\frac{x + \epsilon^2}{(x + \epsilon^2)^{1/2}} + \frac{1}{2}\frac{\epsilon^2}{(x + \epsilon^2)^{1/2}} - \epsilon \\ &\leq \frac{1}{2}(x + \epsilon^2)^{1/2} + \frac{1}{2}\epsilon - \epsilon \\ &= \frac{1}{2}\phi(x). \end{aligned}$$

Furthermore, $\phi''(x) = -\frac{1}{4}(x + \epsilon^2)^{-3/2}$. We now apply the formula for $L^{s,d}\phi(f)$ we found in the very beginning with the given ϕ and $f = |\omega|^2$. By the definition of $|\omega|_\epsilon$ we easily see that $\phi'(|\omega|^2) = \frac{1}{2|\omega|_\epsilon}$ and $\phi''(|\omega|^2) = -\frac{1}{4|\omega|_\epsilon^3}$. This gives us that

$$L^{s_1,d_2}\phi(|\omega|^2) = \frac{1}{2|\omega|_\epsilon}L^{s_1,d_2}|\omega|^2 - \frac{1}{4|\omega|_\epsilon^3}|\bar{\nabla}|\omega|^2|^2 + (s_1^2 - d_2^2)\psi(|\omega|^2).$$

Using that $d_2^2 \geq s_1^2$, and $\psi \leq \frac{1}{2}\phi$, and thus $(s_1^2 - d_2^2)\psi \geq \frac{s_1^2-d_2^2}{2}\phi$, we get that

$$\begin{aligned} L^{s_1,d_2}\phi(|\omega|^2) &\geq \frac{1}{|\omega|_\epsilon^3}\left(\frac{|\omega|_\epsilon^2}{2}L^{s_1,d_2}|\omega|^2 - \frac{1}{4}|\bar{\nabla}|\omega|^2|^2\right) + \frac{(s_1^2-d_2^2)}{2}\phi(|\omega|^2) \\ &= \frac{1}{|\omega|_\epsilon^3}\left(\frac{|\omega|_\epsilon^2}{2}L^{s_1,d_2}|\omega|^2 - \frac{1}{4}|\bar{\nabla}|\omega|^2|^2\right) + \frac{(s_1^2-d_2^2)}{2}(|\omega|_\epsilon - \epsilon). \end{aligned}$$

If we now apply the first part with $s_2 = s_1$, we get that

$$L^{s_1,d_2}|\omega|^2 \geq 2|\bar{\nabla}\omega|^2 + |\omega|^2(2d_1^2 - d_2^2 - s_1^2 + 2r_0).$$

Furthermore, by the fact that $D_0|\omega|^2 = 2\omega \cdot D_0\omega$ and $\nabla|\omega|^2 = 2\omega \cdot \nabla\omega$ we find

$$\begin{aligned} |\bar{\nabla}|\omega|^2|^2 &= |\nabla|\omega|^2|^2 + |D_0|\omega|^2|^2 \\ &= 4|\nabla\omega(\omega^*,\cdot)|^2 + 4|\omega\cdot D_0\omega|^2 \\ &\leq 4|\omega|^2|\nabla\omega|^2 + 4|\omega|^2|D_0\omega|^2 \\ &= 4|\omega|^2|\bar{\nabla}\omega|^2 \\ &\leq 4|\omega|_\epsilon^2|\bar{\nabla}\omega|^2 \end{aligned}$$

where the first inequality is simply Cauchy-Schwarts and the second is obvious from the defition of $|\omega|_\epsilon$. Putting everything together we deduce

$$\begin{aligned} L^{s_1,d_2}\phi(|\omega|^2) &\geq \frac{1}{|\omega|_\epsilon^3}\left(\frac{|\omega|_\epsilon^2}{2}L^{s_1,d_2}|\omega|^2 - \frac{1}{4}|\bar{\nabla}|\omega|^2|^2\right) + \frac{(s_1^2-d_2^2)}{2}(|\omega|_\epsilon - \epsilon) \\ &\geq \frac{|\omega|^2}{2|\omega|_\epsilon}(2d_1^2 - d_2^2 - s_1^2 + 2r_0) + \frac{\epsilon}{2}(d_2^2 - s_1^2) + \frac{1}{2}(s_1^2 - d_2^2)|\omega|_\epsilon \\ &= \frac{|\omega|_\epsilon}{2}(2d_1^2 - 2d_2^2 + 2r_0) - \frac{\epsilon^2}{2|\omega|_\epsilon}(2d_1^2 - d_2^2 - s_1^2 + 2r_0) + \frac{\epsilon}{2}(d_2^2 - s_1^2) \\ &\geq |\omega|_\epsilon(d_1^2 - d_2^2 + r_0) - \epsilon(d_1^2 + r_0 - d_2^2) \\ &= (|\omega|_\epsilon - \epsilon)(d_1^2 - d_2^2 + r_0). \end{aligned}$$

Here we used that $|\omega|^2 = |\omega|_\epsilon^2 - \epsilon^2$ and that $|\omega|_\epsilon \geq \epsilon$ and $d_2^2 \geq s_1^2$. We thus find that

$$L^{s_1,d_2}((|\omega|^2 + \epsilon^2)^{1/2} - \epsilon) = L^{s_1,d_2}\phi(|\omega|^2) \geq (|\omega|_\epsilon - \epsilon)(d_1^2 - d_2^2 + r_0) \geq 0$$

as $d_1^2 + r_0 \geq d_2^2 \geq d_1^2$ and $|\omega|_\epsilon \geq \epsilon$. □

4.3 Inequalities of the type of Littlewood-Paley-Stein

In this section we will be concerned with sub-harmonic functions f on $M \times \mathbb{R}_+$, i.e., functions which satisfy $L^d f \geq 0$. Here L^d is the operator $L_0^d + L$ as defined in the previous section. Our goal is to prove that under certain assumption the estimate

$$\left|\left|\int_0^\infty Q_u^d(L^d f)V_d(u)\mathrm{d}u\right|\right|_p \leq C(p)||f(\cdot,0)||_p$$

holds, where Q_u^d is the subordinated semigroup as in the previous section, and V_d is some potential which will be given explicitly. The second estimate we wish to prove holds only for $1 < p \leq 2$, and is given by

$$\left|\left|\left(\int_0^\infty L^d(f^2)V_d(u)\mathrm{d}u\right)^{1/2}\right|\right|_p \leq C(p)||f(\cdot,0)||_p.$$

Before we continue, we fix some notation. Let $x \in M$ be a point on the manifold. We denote by $\mathbb{P}^x$ the distribution with respect to the canonical path space of the process with generator L, starting almost surely in x. We will denote this process by X_t^x. Furthermore, let B_t^a be an arbitrary one-dimensional Brownian motion starting from $a > 0$. We assume that X_t^x and B_t^a are independent and denote the distribution of the couple (X_t^x, B_t^a) by $\mathbb{P}^{x,a}$ and write the expectation under this law as $\mathbb{E}^{x,a}$.

We also construct the measure $\mathbb{P}_a = \int_M \mathbb{P}^{x,a}\mathrm{d}m(x)$, which is a probability measure whenever m is. We will write (suggestively if m is not a probability measure) $\mathbb{E}_a(Z) = \int_{M\times\mathbb{R}_+} Z\mathrm{d}\mathbb{P}_a$. Even if m is not a probability measure, this will not cause any problems in the way we use it. Indeed, by remark 2.5.3 we have that $(M, \mathrm{d}x)$ is σ-finite. As $\rho \in C^\infty$, $\rho > 0$ we find that $m(\mathrm{d}x) = \rho(x)\mathrm{d}x$ is also a σ-finite measure. Our only probabilistic use of the measure m and expectation $\mathbb{E}_a$ is in the sense of conditional expectations, which do not pose any problems when working with a σ-finite measure.

4.3.1 Stopping time and martingales

We now introduce a stopping time

$$T^{a,d} = \inf\{s | B_s^a - 2ds = 0\}.$$

Observe that we indeed only need to know the values of B_u^a up to time s to conclude whether $T^{a,d} \leq s$, showing that it is in fact a stopping time. Remembering that one-dimensional Brownian motion is recurrent, we even get that $T^{a,d}$ is almost surely finite.

Let us now consider the process

$$Z_t^{x,a,d} = (X^x_{t\wedge T^{a,d}}, B^a_{t\wedge T^{a,d}} - 2d(t \wedge T^{a,d}))$$

defined on $M \times \mathbb{R}_+$ and denote by $(\mathcal{F}_t)_t$ its natural filtration. We claim that this process has generator $L^d = (D_0^2 - 2dD_0 + L)$.

Proposition 4.3.1. *Let $f \in C^\infty(M \times (0, \infty))$. Then the process*

$$f(Z_t^{x,a,d}) - f(x, a) - \int_0^t (L^d f)(Z_s^{x,a,d})\mathrm{d}s$$

is a local martingale on $[0, T^{a,d})$.

Proof. By stopping the process, we assure that $Z_t^{x,a,d}$ remains in a bounded subset of the manifold. Hence it suffices to consider the case when f has compact support, as it suffices to show that we get a local martingale up to stopping times. Now observe that we can approximate f uniformly by functions f_n which are linear combinations of functions of the form $g_n(x)h_n(t)$. In that case, $L^d f$ is the uniform limit of $L^d f_n$. As the convergence is uniform, we may interchange limit and expectation, hence it suffices to consider functions of the form $g(x)h(t)$.

As L is the generator of X_t^x, we have that $g(X_t^x) = g(x) + tLg + \mathcal{O}(t^2)$. We also have that B_t^a has generator D_0^2. Furthermore, as the process $Y_t^x = t + x$ is deterministic, we can easily see that D_0 is the generator of the process Y_t. But then we find that $D_0^2 - 2dD_0$ is the generator of the process $B_t^a - 2dt$. This gives us that $h(B_t^a - 2dt) = h(a) + t(D_0^2 h - 2dD_0 h) + \mathcal{O}(t^2)$. Combining expressions, we find that

$$g(X_t^x)h(B_t^a - 2dt) \quad = \quad g(x)h(a) + t(h(a)Lg + g(x)(D_0^2 h - 2dD_0 h)) + \mathcal{O}(t^2)$$

$$= \quad g(x)h(a) + t(L^d(hg)(x,a)) + \mathcal{O}(t^2).$$

This shows that L^d is indeed the generator of $Z_t^{x,a,d}$ at least for functions of the form $f(x,t) = g(x)h(t)$ as the processes X_t^x and $B_t^a - 2dt$ are independent. But then

$$f(Z_t^{x,a,d}) - f(x,a) - \int_0^t (L^d f)(Z_s^{x,a,d})\mathrm{d}s$$

is a local martingale as required. By the previous remark, this proves the claim. $\square$

We now take $f \in C_0^\infty(M)$ and define $\bar{f}^d(x,t) = Q_t^d f(x)$, which satisfies $L^d(\bar{f}^d) = 0$ by proposition 4.2.2(f). The proposition also tells us that $\bar{f}^d$ is C^∞. Hence, by proposition 4.3.1, we find that $\bar{f}^d(Z_t^{x,a,d})$ is a local martingale on $[0, T^{a,d})$, as the integral vanishes, and constants are not important. In fact, as $\bar{f}^d$ is bounded and continuous at $t = 0$, it turns out to be a martingale.[10] Observing that $Z_{T^{a,d}}^{x,a,d} = (X_{T^{a,d}}^x, 0)$, we find that

$$\bar{f}^d(Z_{T^{a,d}}^{x,a,d}) = Q_0^d f(X_{T^{a,d}}^x) = f(X_{T^{a,d}}^x).$$

But then

$$\begin{aligned}
\mathbb{E}^{x,a}(f(X_{T^{a,d}}^x)|\mathcal{F}_s) &= \mathbb{E}^{x,a}(\bar{f}^d(Z_{T^{a,d}}^{x,a,d})|\mathcal{F}_s) \\
&= \lim_{t\to\infty} \mathbb{E}^{x,a}(\bar{f}^d(Z_{t\wedge T^{a,d}}^{x,a,d})|\mathcal{F}_s) \\
&= \lim_{t\to\infty} \bar{f}^d(Z_{s\wedge T^{a,d}}^{x,a,d}) \\
&= \bar{f}^d(Z_s^{x,a,d}).
\end{aligned}$$

Here we made use of the martingale stopping theorem, and the fact that $s \in [0, T^{a,d})$. Also note that the interchanging of limit and integral is allowed as $\bar{f}^d$ is bounded.

Now suppose that $f \in C^\infty(M \times (0,\infty))$ is bounded, continuous in $t = 0$ and satisfies $L^d f \geq 0$. By Fatou's lemma (which we may apply by the boundedness of f), we find that

$$\begin{aligned}
&\mathbb{E}^{x,a}(f(X_{T^{a,d}}^x, 0)|\mathcal{F}_s) \\
&\geq \limsup_{t\to\infty} \mathbb{E}^{x,a}(f(Z_{t\wedge T^{a,d}}^{x,a,d})|\mathcal{F}_s) \\
&\geq \limsup_{t\to\infty} \mathbb{E}^{x,a}\left(f(Z_{t\wedge T^{a,d}}^{x,a,d}) - \int_0^{t\wedge T^{a,d}} (L^d f)(Z_r^{x,a,d})\mathrm{d}r + \int_0^{s\wedge T^{a,d}} (L^d f)(Z_r^{x,a,d})\mathrm{d}r |\mathcal{F}_s \right) \\
&= \limsup_{t\to\infty} f(Z_{s\wedge T^{a,d}}^{x,a,d}) - \int_0^{s\wedge T^{a,d}} (L^d f)(Z_r^{x,a,d})\mathrm{d}r + \int_0^{s\wedge T^{a,d}} (L^d f)(Z_r^{x,a,d})\mathrm{d}r \\
&= f(Z_s^{x,a,d}).
\end{aligned}$$

Here we assume $t > s$ in the third line, and used the martingale stopping theorem in the fourth line. We also used that the second integral is $\mathcal{F}_s$-measurable, as $Z_r^{x,a,d}$ is $\mathcal{F}_s$-measurable for $r \leq s$ and $1_{s\wedge T^{a,d}}$ is also $\mathcal{F}_s$-measurable, as $\{T^{a,d} \leq r\} \in \mathcal{F}_s$ for all $r \leq s$.

From these two results, we can deduce the following proposition.

Proposition 4.3.2. *Let $f \in C^\infty(M \times \mathbb{R}_+)$ and suppose that f is bounded, continuous in $t = 0$ and satisfies $L^d f \geq 0$. Then for all $t, u \in \mathbb{R}_+$ is holds that*

$$Q_t^d f(\cdot, u) \geq f(\cdot, t+u).$$

[10] As in general a bounded local martingale is a martingale.

Proof. By changing t to $t+u$ in f, it suffices to consider $u=0$. If we now use the two previous identities with $s=0$ and $a=t$, we obtain

$$\begin{aligned} Q_t^d f(x,0) &= \mathbb{E}^{x,t}(f(X_{T^{t,d}},0)|\mathcal{F}_0) \\ &\geq f(Z_0^{x,t,d}) \\ &= f(x,t) \end{aligned}$$

as desired. □

We will now introduce a function V_d on $\mathbb{R}_+^2$ given by

$$\begin{aligned} V_d(t,s) &= \frac{1}{d}e^{-d(s\vee(2s-t))}\sinh(d(t\wedge s)) \\ &= \frac{1-e^{-2ds}}{2d}1_{s<t} + e^{-2ds}\frac{e^{2dt}-1}{2d}1_{s\geq t} \end{aligned}$$

for $d>0$, where the equality can be found by writing out the hyperbolic sine and observing that if $s<t$ then $2s-t<2s-s=s$ and if $s\geq t$ then $2s-t\geq 2s-s=s$. If $d=0$, we define $V_0(t,s)=t\wedge s$.

The following proposition illustrates the usefulness of this function.

Proposition 4.3.3. *Let f be a positive Borel function on $M\times\mathbb{R}_+$. Then*

$$\mathbb{E}_a\left(\int_0^{T^{a,d}} f(Z_s^{a,d})\ \mathrm{d}s\right) = \int_M\int_0^\infty f(x,u)V_d(a,u)\ \mathrm{d}u\ \mathrm{d}m(x).$$

Here $Z_s^{a,d}$ is the process $(X_{s\wedge T^{a,d}}, B^a_{s\wedge T^{a,d}} - 2d(s\wedge T^{a,d}))$ with distribution $\mathbb{P}_a$.[11]

Proof. First consider a positive C^∞ function $h(t)$ with compact support in $(0,\infty)$. We will show that

$$\mathbb{E}_a\left(\int_0^{T^{a,d}} h(B_s^a - 2ds)\ \mathrm{d}s\right) = \int_0^\infty h(u)V_d(a,u)\ \mathrm{d}u.$$

To this extend, define the function $G(a)=\int_0^\infty h(u)V_d(a,u)\mathrm{d}u$. As

$$V_d(0,u) = e^{-2du}\frac{1-1}{2d} = 0$$

we find that $G(0)=0$. Also G is bounded. Indeed, suppose that the support of h is contained in $[c,d]$ with $0<c<d<\infty$. For $a>d$ we find that

$$G(a) = \int_c^d h(u)\frac{1-e^{-2du}}{2d}\mathrm{d}u \leq \int_c^d h(u)\frac{1}{2d}\mathrm{d}u =: M.$$

As h is continuous on a compact support, it is bounded, hence M is finite. In the case that $a\leq d$, we find that

$$G(a) \;=\; \int_c^a h(u)\frac{1-e^{-2du}}{2d}\mathrm{d}u + \int_a^d h(u)e^{-2du}\frac{e^{2da}-1}{2d}\mathrm{d}u$$

[11] We see it like this, because the starting point $x\in M$ is now considered to be random, and picked from M according to the law $m(\mathrm{d}x)$.

$$\begin{aligned}
&\leq M + \int_a^d h(u)\frac{e^{2d(a-u)} - e^{-2du}}{2d}\mathrm{d}u \\
&\leq M + \int_a^d h(u)\frac{1 - e^{-2du}}{2d}\mathrm{d}u \\
&\leq 2M
\end{aligned}$$

where we made use of the fact that h is positive.

Finally, G satisfies $L_0^d G = -h$. Indeed, thinking in terms of distributions, we find by the product rule that

$$D_0 V_d(t,s) = \frac{1 - e^{-2ds}}{2d}\delta_t + e^{-2ds}e^{2dt}1_{s\geq t} - \delta_t e^{-2ds}\frac{e^{2dt}-1}{2d}.$$

Hence

$$\begin{aligned}
D_0 G(a) &= \int_0^\infty h(u) D_0 V_d(a,u)\mathrm{d}u \\
&= \frac{1-e^{-2da}}{2d}h(a) - e^{-2da}\frac{e^{2da}-1}{2d}h(a) + \int_0^\infty h(u)e^{2d(a-u)}1_{u\geq a}\mathrm{d}u \\
&= \int_0^\infty h(u)e^{2d(a-u)}1_{u\geq a}\mathrm{d}u.
\end{aligned}$$

Similar as above, we find that

$$D_0(e^{2d(a-u)}1_{u\geq a}) = 2de^{2d(a-u)} - e^{2d(a-u)}\delta_a$$

giving us

$$D_0^2 G(a) = -h(a)e^{2d(a-a)} + 2d\int_0^\infty h(u)e^{2d(a-u)}1_{u\geq a}\mathrm{d}u = -h(a) + 2dD_0G(a).$$

We conclude that

$$L_0^d G = D_0^2 G - 2dD_0G = -h + 2dD_0G - 2dD_0G = -h$$

as desired.

From the above we may now conclude that

$$G(B_s^a - 2ds) - \int_0^s L_0^d G(B_u^a - 2du)\mathrm{d}u = G(B_s^a - 2ds) + \int_0^s h(B_u^a - 2du)\mathrm{d}u$$

is a martingale, which is bounded on $[0, s\wedge T^{a,d}]$ as both G and h are. As it is a martingale, taking expectations gives us that

$$\begin{aligned}
&\mathbb{E}_a\left(G(B^a_{s\wedge T^{a,d}} - 2d(s\wedge T^{a,d})) + \int_0^{s\wedge T^{a,d}} h(B_u^a - 2du)\mathrm{d}u\right) \\
&= \mathbb{E}_a\left(G(B_0^a - 2d\cdot 0) + \int_0^0 h(B_u^a - 2du)\mathrm{d}u\right) \\
&= G(a) \\
&= \int_0^\infty h(u)V_d(a,u)\mathrm{d}u.
\end{aligned}$$

As the martingale is bounded, we can take the limit $s \to \infty$ to find that

$$\begin{aligned}
&\mathbb{E}_a\left(\int_0^{T^{a,d}} h(B_u^a - 2du)\mathrm{d}u\right) \\
&= \lim_{s\to\infty} \mathbb{E}_a\left(G(B^a_{s\wedge T^{a,d}} - 2d(s\wedge T^{a,d})) + \int_0^{s\wedge T^{a,d}} h(B_u^a - 2du)\mathrm{d}u\right) \\
&= \int_0^\infty h(u)V_d(a,u)\mathrm{d}u
\end{aligned}$$

where we used that $G(0) = 0$.

By the monotone class theorem, we can now extend this result to the case where h is a Borel function.

To finish the proof, as in the proof of proposition 4.3.1, we only need to focus on $f(x,t) = g(x)h(t)$. By using Fubini and applying the independence of X_t and B_t, we get that

$$\mathbb{E}_a\left(\int_0^{T^{a,d}} f(Z_s^{a,d})\mathrm{d}s\right) = \int_0^\infty \mathbb{E}_a(h(B_s^a - 2ds)1_{s<T^{a,d}})\mathbb{E}_a(g(X_s))\mathrm{d}s.$$

By definition $\mathbb{E}^x(g(X_s)) = (P_s g)(x)$, which gives that

$$\mathbb{E}_a(g(X_s)) = \int_M g(X_s)\mathrm{d}\mathbb{P}^{x,a}\mathrm{d}m(x) = \int_M \mathbb{E}^x(g(X_s))\mathrm{d}m(x) = \int_M P_s g(x)\mathrm{d}m(x).$$

Now as $P_s 1 = 1$, we find from the self-adjointness that

$$\int_M P_s g(x)\mathrm{d}m(x) = \int_M g(x)P_s 1\mathrm{d}m(x) = \int_M g(x)\mathrm{d}m(x) \ (= \langle g\rangle).$$

We conclude that $\mathbb{E}_a(g(X_s)) = \int_M g(x)\mathrm{d}m(x)$. Using this, and applying Fubini again, we find that

$$\begin{aligned}
\mathbb{E}_a\left(\int_0^{T^{a,d}} f(Z_s^{a,d})\mathrm{d}s\right) &= \langle g\rangle \mathbb{E}_a\left(\int_0^{T^{a,d}} h(B_s^a - 2ds)\mathrm{d}s\right) \\
&= \langle g\rangle \int_0^\infty h(u)V_d(a,u)\mathrm{d}u \\
&= \int_M \int_0^\infty g(x)h(u)V_d(a,u)\ \mathrm{d}u\ \mathrm{d}m(x)
\end{aligned}$$

which is the desired equality. □

This proposition admits the following corollary.

Corollary 4.3.4. *Under the same assumptions as in the previous proposition, we have that*

$$\mathbb{E}_a\left(\int_0^{T^{a,d}} f(Z_s^{a,d})\mathrm{d}s \big| X_{T^{a,d}} = x\right) = \int_0^\infty Q_s^d f(x,s)V_d(a,s)\ \mathrm{d}s$$

Proof. By definition of conditional expectation, it suffices to prove for all $g \in C_0^\infty(M)$ that

$$\mathbb{E}_a\left(\int_0^{T^{a,d}} f(Z_s^{a,d})\mathrm{d}s g(X_{T^{a,d}})\right) = \int_0^\infty \langle Q_s^d f(x,s)g(x)\rangle V_d(a,s)\ \mathrm{d}s$$

because the distribution of $X_{T^{a,d}}$ is $m(\mathrm{d}x)$. Indeed, for any Borel function f

$$\begin{aligned}
\mathbb{E}(f(X_{T^{a,d}})) &= \int_0^\infty \mathbb{E}(f(X_{T^{a,d}})|T^{a,d}=t)\ \mathbb{P}_{T^{a,d}}(\mathrm{d}t) \\
&= \int_0^\infty \int_M \mathbb{E}^x(f(X_t))\ m(\mathrm{d}x)\ \mathbb{P}_{T^{a,d}}(\mathrm{d}t) \\
&= \int_M f(x)\ m(\mathrm{d}x) \int_0^\infty\ \mathbb{P}_{T^{a,d}}(\mathrm{d}t) \\
&= \int_M f(x)\ m(\mathrm{d}x)
\end{aligned}$$

where we used the invariance of m in the third line, along with the fact that X_t and $T^{a,d}$ are independent as $T^{a,d}$ solely depends on B_t^a.

By the symmetry of Q_t^d, rembering that $\bar{g}^d(x,t) = Q_t^d g(x)$, the right hand side above is equal to

$$\int_M \int_0^\infty f(x,s)\bar{g}^d(x,s)V_d(a,s)\ \mathrm{d}s\mathrm{d}m(x).$$

For the left hand side we can write

$$\begin{aligned}
\mathbb{E}_a\left(\int_0^{T^{a,d}} f(Z_s^{a,d})\mathrm{d}s g(X_{T^{a,d}})\right) &= \int_0^\infty \mathbb{E}_a(f(Z_s^{a,d})1_{\{s\le T^{a,d}\}}g(X_{T^{a,d}}))\mathrm{d}s \\
&= \int_0^\infty \mathbb{E}_a(\mathbb{E}_a[f(Z_s^{a,d})1_{\{s\le T^{a,d}\}}g(X_{T^{a,d}})|\mathcal{F}_s])\mathrm{d}s \\
&= \int_0^\infty \mathbb{E}_a(f(Z_s^{a,d})1_{\{s\le T^{a,d}\}}\mathbb{E}_a[g(X_{T^{a,d}})|\mathcal{F}_s])\mathrm{d}s \\
&= \mathbb{E}_a\left(\int_0^{T^{a,d}} f(Z_s^{a,d})\mathbb{E}_a[g(X_{T^{a,d}})|\mathcal{F}_s]\mathrm{d}s\right).
\end{aligned}$$

Here we used Fubini in the first and last line, the tower property for conditional expectation and the fact that $Z_s^{a,d}$ and hence also $f(Z_s^{a,d})$ are $\mathcal{F}_s$-measurable. We also used the $\mathcal{F}_s$-measurability of $1_{\{s\le T^{a,d}\}}$, which follows from the fact that $T^{a,d}$ is a stopping time.

But

$$\mathbb{E}_a(g(X_{T^{a,d}})|F_s) = \mathbb{E}^{x,a}(g(X_{T^{a,d}})|\mathcal{F}_s) = \bar{g}^d(Z_s^{a,d}).$$

Indeed, the second equality follows from the discussion after proposition 4.3.1, while the first equality follows from the observation that for any $F \in \mathcal{F}_s$ we have

$$\int_F \mathbb{E}^{x,a}(g(X_{T^{a,d}})|\mathcal{F}_s)\mathrm{d}\mathbb{P}_a = \int_F \mathbb{E}_a(g(X_{T^{a,d}})|\mathcal{F}_s))\mathrm{d}m(x) = \int_F g(X_{T^{a,d}})\mathrm{d}m(x).$$

Here the first equality follows from the definition of $\mathbb{P}_a$ and the second follows from the fact that the distribution of $X_{T^{a,d}}$ is $\mathrm{d}m(x)$.

If we now apply the previous proposition, we find that

$$E_a\left(\int_0^{T^{a,d}} f(Z_s^{a,d})\mathrm{d}s g(X_{T^{a,d}})\right) = \int_M \int_0^\infty f(x,s)\bar{g}^d(x,s)V_d(a,s)\ \mathrm{d}s\ \mathrm{d}m(x)$$

as desired. □

4.3.2 Main estimates

We now arrive at the main results of this section. We define

$$V_d(u) = \lim_{t\to\infty} V_d(t,u) = \frac{1-e^{-2du}}{2d}$$

for $d > 0$ and similarly $V_0(u) = u$. The constants $C(p)$ in the remainder of this chapter solely depend on p, where $1 < p < \infty$. In particular, they do not depend on f, d and the manifold M.

Theorem 4.3.5. *Let $f \in C^\infty(M \times (0,\infty)) \cup C(M \times [0,\infty])$ be positive and bounded, such that $L^d f \geq 0$. Then for all $1 < p < \infty$ we have*

$$\left|\left|\int_0^\infty Q_u^d(L^d f)V_d(u)\mathrm{d}u\right|\right|_p \leq C(p)||f(\cdot,0)||_p$$

for some constant $C(p)$ only depending on p.

Proof. From proposition 4.3.1 we know that

$$f(Z_t^{x,a,d}) - f(x,a) - \int_0^t (L^d f)(Z_s^{x,a,d})\mathrm{d}s$$

is a local martingale. As in the remarks after that proposition, we find that

$$Y_t = f(Z_t^{x,a,d}) - f(x,a)$$

is a sub-martingale. Because f is bounded, it is in fact a bounded sub-martingale, with

$$Y_0 = f(Z_0^{x,a,d}) - f(x,a) = f(x,a) - f(x,a) = 0.$$

The associated increasing process[12] is given by

$$A_t = \int_0^{t\wedge T^{a,d}} (L^d f)(Z_s^{x,a,d})\mathrm{d}s.$$

By theorem 3.2 from [22] we have for all $1 < p < \infty$ that

$$\mathbb{E}^{x,a}((A_{T^{a,d}})^p) \leq C(p)\mathbb{E}^{x,a}(|Y_{T^{a,d}}|^p).$$

Integrating over M with respect to $\mathrm{d}m(x)$, we get that

$$\mathbb{E}_a((A_{T^{a,d}})^p) \leq C(p)\mathbb{E}_a(|Y_{T^{a,d}}|^p).$$

By Jensen's inequality and the tower property for conditional expectation, we find that

$$\begin{aligned}
\mathbb{E}_a(\mathbb{E}_a(A_{T^{a,d}}|X_{T^{a,d}} = \cdot)^p) &\leq \mathbb{E}_a(\mathbb{E}_a(A^p_{T^{a,d}}|X_{T^{a,d}} = \cdot)) \\
&= \mathbb{E}_a(A^p_{T^{a,d}}) \\
&\leq C(p)\mathbb{E}_a(|Y_{T^{a,d}}|^p).
\end{aligned}$$

But $Y_{T^{a,d}} = f(X_{T^{a,d}},0) - f(x,a)$, hence we find that its L^p norm is bounded by $||f(\cdot,0)||_p + ||f(\cdot,a)||_p$. However, as $L^d f \geq 0$, we know from proposition 4.3.2 that $f(\cdot,a) \leq Q_a^d f(\cdot,0)$. By propisition 4.2.2, Q_a^d is a contraction. But then we find that $||f(\cdot,a)||_p \leq ||Q_a^d f(\cdot,0)||_p \leq$

[12] The deficit to a proper martingale.

$||f(\cdot,0)||_p$. Hence we can bound the L^p norm of $Y_{T^{a,d}}$ by $||f(\cdot,0)||_p + ||f(\cdot,a)||_p \le 2||f(\cdot,0)||_p$. Putting things together, we find that

$$\mathbb{E}_a((A_{T^{a,d}})^p) \le C(p)\mathbb{E}_a(|Y_{T^{a,d}}|^p) \le C(p)||f(\cdot,0)||_p^p.$$

By corollary 4.3.4 we have that

$$\begin{aligned}
\mathbb{E}_a(\mathbb{E}_a(A_{T^{a,d}}|X_{T^{a,d}}=\cdot)^p) &= \mathbb{E}_a\left(\mathbb{E}_a\left(\int_0^{t\wedge T^{a,d}}(L^d f)(Z_s^{x,a,d})\mathrm{d}s|X_{T^{a,d}}=\cdot\right)^p\right)\\
&= \mathbb{E}_a\left(\left(\left[\int_0^\infty Q_u^d(L^d f(\cdot,u))V_d(a,u)\mathrm{d}u\right](X_{T^{a,d}})\right)^p\right)\\
&= \left|\left|\int_0^\infty Q_u^d(L^d f(\cdot,u))V_d(a,u)\mathrm{d}u\right|\right|_p^p
\end{aligned}$$

where the last equality holds because $m(\mathrm{d}x)$ is the distribution of $X_{T^{a,d}}$. We thus find that

$$\left|\left|\int_0^\infty Q_u^d(L^d f(\cdot,u))V_d(a,u)\mathrm{d}u\right|\right|_p \le C(p)||f(\cdot,0)||_p.$$

Taking the limit $a\to\infty$, by Fatou's lemma we obtain

$$\left|\left|\int_0^\infty Q_u^d(L^d f(\cdot,u))V_d(u)\mathrm{d}u\right|\right|_p \le C(p)||f(\cdot,0)||_p$$

as desired. □

Theorem 4.3.6. *Let $f\in C^\infty(M\times(0,\infty))$ be a bounded and positive function. Assume that $f, D_0 f$ are continuous at $t=0$ and satisfy $L^d f\ge 0$. Suppose furthermore that*

$$\int_M\int_0^\infty (|L_0^d f^2| + |Lf^2| + |D_0 f|^2 + |\nabla f|^2)V_d(u)\mathrm{d}u\ \mathrm{d}m(x) < \infty.$$

Then for all $1<p\le 2$

$$\left|\left|\left(\int_0^\infty L^d(f^2)(\cdot,u))V_d(u)\mathrm{d}u\right)^{1/2}\right|\right|_p \le C(p)||f(\cdot,0)||_p.$$

Before proving the theorem, we first prove a lemma.

Lemma 4.3.7. *Let $f\in C^\infty(M\times(0,\infty))$, denoted as $f(x,s)=f_s(x)$, be a positive function such that $f, D_0 f$ converge for $t\to 0$. Suppose furthermore that the following are satisfied*

1. *$\int_M\int_0^\infty(|L_0 f_s| + |Lf_s|)V_d(s)\mathrm{d}s\ \mathrm{d}m(x) < \infty$.*
2. *For almost all $x\in M$, $\int_0^\infty |D_0 f|V_d(s)\mathrm{d}s < \infty$.*
3. *For almost all $s\in\mathbb{R}_+$, $\int_M |\nabla f_s|\mathrm{d}m(x) < \infty$.*

Then

$$\int_M f(x,0)\mathrm{d}m(x) \ge \int_M\int_0^\infty L^d f_s V_d(s)\mathrm{d}s\ \mathrm{d}m(x).$$

Proof. Fix s and take the sequence h_n of function as in definition 4.1.1, so that $h_n f_s$ approximates f_s pointwise. We claim that $\langle L(h_n f_s)\rangle = 0$. Indeed, $L(h_n f_s) = h_n L f_s + f_s L h_n + 2\mathrm{d}h_n \cdot \mathrm{d}f_s$. By proposition 4.1.7, we then have that

$$\langle L(h_n f_s)\rangle = \langle h_n, L f_s\rangle + \langle f_s, L h_n\rangle + 2\langle \mathrm{d}h_n, \mathrm{d}f_s\rangle = 2\langle h_n, L f_s\rangle + 2\langle \mathrm{d}h_n, \mathrm{d}f_s\rangle = 0$$

as $\langle h_n, L f_s\rangle = -\langle \mathrm{d}h_n, \mathrm{d}f_s\rangle$. By property (3) we can take $n \to \infty$ to obtain $\langle L f_s\rangle = 0$. Using Fubini's theorem, we get that $\langle \int_0^\infty L f_s V_d(s)\mathrm{d}s\rangle = 0$.

As $L^d = L_0^d + L$, it now remains to show that $\langle f_0\rangle \geq \langle \int_0^\infty L_0^d f_s V_d(s)\mathrm{d}s\rangle$. For this, we will find a pointwise estimate. First note that $D_0 V_d(t) = e^{-2dt} = -2dV_d(t) + 1$. Using this, we find by the product rule that

$$\begin{aligned} D_0(-f_t + V_d(t) D_0 f_t) &= -D_0 f_t + D_0 V_d(t) D_0 f_t + V_d(t) D_0^2 f_t \\ &= -2dV_d(t) D_0 f_t + V_d(t) D_0^2 f_t \\ &= L_0^d(f_t) V_d(t). \end{aligned}$$

This gives us that

$$\int_0^t L_0^d(f_s) V_d(s)\mathrm{d}s = f_0 - f_t + V_d(t) D_0 f_t$$

as $V_d(0) = 0$. By condition (2) we find that $\lim_{t\to\infty} V_d(t) D_0 f_t = 0$ for almost all $x \in M$. As $f_t \geq 0$, we get in the limit $t \to \infty$ that

$$\int_0^\infty L_0^d(f_s) V_d(s)\mathrm{d}s \leq f_0$$

as desired. □

Proof of theorem 4.3.6. Let $q = \frac{p}{2}$ and define for $\epsilon > 0$ the function $f_{\epsilon,q} = (f^2+\epsilon)^q - \epsilon^q$. Using the product rule for the Laplace-Beltrami operator (see corollary A.1.7 in the appendix) we find that[13]

$$L(f_{\epsilon,q}) = q(f^2+\epsilon)^{q-1} L(f^2) + q(q-1)(f^2+\epsilon)^{q-2} |\nabla f^2|^2$$

or

$$\frac{1}{q} L(f_{\epsilon,q}) = (f^2+\epsilon)^{q-1} L(f^2) + (q-1)(f^2+\epsilon)^{q-2} |\nabla f^2|^2.$$

Similarly, we get that

$$\frac{1}{q} L_0^d(f_{\epsilon,q}) = (f^2+\epsilon)^{q-1} L_0^d(f^2) + (q-1)(f^2+\epsilon)^{q-2} (D_0 f^2)^2.$$

By the assumptions we make, we may apply the previous lemma to $f_{\epsilon,q}$ to obtain

$$\frac{1}{q}\langle f_{\epsilon,q}(\cdot,0)\rangle \geq \left\langle \int_0^\infty (f^2+\epsilon)^{q-1}(L^d f^2 + (q-1)(f^2+\epsilon)^{-1} |\bar{\nabla} f^2|^2) V_d(s)\mathrm{d}s \right\rangle.$$

Again by the product rule, we find that

$$L^d f^2 = 2fL^d f + 2|\bar{\nabla} f|^2 \geq 2|\bar{\nabla} f|^2 = \frac{1}{2} f^{-2} |\bar{\nabla} f^2|^2$$

[13]Remember that for functions we have that $\mathrm{d}f = \nabla f$, where ∇ denotes the total covariant derivative as usual.

where we used that $f, L^d f \geq 0$. Here, the last equality simply follows by writing out the right hand side. As $f^2 \leq f^2+\epsilon$, we even get that $L^d f^2 \geq \frac{1}{2}(f^2+\epsilon)^{-1}|\bar{\nabla} f^2|^2$. As $V_d(s) \leq 0$, combining the estimates gives us that

$$\begin{aligned}
\frac{1}{q}\langle f_{\epsilon,q}(\cdot,0)\rangle &\geq \left\langle \int_0^\infty (f^2+\epsilon)^{q-1}(L^d f^2 + 2(q-1)L^d f^2)V_d(s)\mathrm{d}s \right\rangle \\
&= (2q-1)\left\langle \int_0^\infty (f^2+\epsilon)^{q-1} L^d f^2 V_d(s)\mathrm{d}s \right\rangle.
\end{aligned}$$

By the monotone convergence theorem we can take $\epsilon \to 0$ to obtain (after dividing by $(2q-1) > 0$, as $p > 1$)

$$\frac{1}{q(2q-1)}\langle f(\cdot,0)^p\rangle \geq \left\langle \int_0^\infty f^{p-2} L^d f^2 V_d(s)\mathrm{d}s \right\rangle$$

where the exponent of f arises as $(f^2)^q = f^{2q} = f^p$ and $(f^2)^{q-1} = f^{2q-2} = f^{p-2}$.

If we now define $f^* = \sup_s |f_s|$, and noting that $f^{p-2}f^{2-p} = f^0 = 1$, we get that

$$\begin{aligned}
\left(\int_0^\infty L^d f^2 V_d(s)\mathrm{d}s\right)^{p/2} &\leq \left(\int_0^\infty (f^*)^{2-p} f^{p-2} L^d f^2 V_d(s)\mathrm{d}s\right)^{p/2} \\
&= (f^*)^{\frac{p}{2}(2-p)}\left(\int_0^\infty f^{p-2} L^d f^2 V_d(s)\mathrm{d}s\right)^{p/2}.
\end{aligned}$$

If we now apply Hölder's inequality with exponents $\frac{2}{2-p}$ and $\frac{2}{p}$ we find that

$$\left\langle \left(\int_0^\infty L^d f^2 V_d(s)\mathrm{d}s\right)^{p/2} \right\rangle \leq ||f^*||_p^{\frac{p}{2}(2-p)} \left\langle \int_0^\infty f^{p-2} L^d f^2 V_d(s)\mathrm{d}s \right\rangle^{p/2}.$$

By proposition 4.3.2 $f(x,s) \leq Q_s^d f(x,0)$ and hence $f^*(x) \leq \sup_s Q_s^d f(x,0)$. But as Q_s^d is a symmetric Markovian semigroup, $||\sup_s |Q_s^d f(\cdot,0)|||_p \leq C(p)||f(\cdot,0)||_p$ for $1 < p < \infty$. On the other hand, we already deduced an upper bound for the second factor on the right hand side. Putting everything together, we get that

$$\left(\int_0^\infty L^d f^2 V_d(s)\mathrm{d}s\right)^{p/2} \leq C(p)||f(\cdot,0)||_p^{\frac{p}{2}(2-p)}||f(\cdot,0)||_p^{p^2/2}$$

which rewrites to (after raising to the power $\frac{1}{p}$)

$$\left|\left|\left(\int_0^\infty L^d(f^2)(\cdot,u))V_d(u)\mathrm{d}u\right)^{1/2}\right|\right|_p \leq C(p)||f(\cdot,0)||_p$$

as desired. □

Remark 4.3.8. From the proof of theorem 4.3.6 we see that it actually suffices to assume that $f^2 \in C^\infty(M \times \mathbb{R}_+)$ and that for all $\epsilon > 0$, $L^d(f^2+\epsilon)^{1/2} \geq 0$. Indeed, in the part where we use that $L^d f \geq 0$, we may also consider the function $(f^2+\epsilon)^{1/2}$ instead of f. As furthermore $L^d(f^2+\epsilon) = L^d(f^2)$ the proof still holds. Note that this observation will important for the proof of the boundedness of the Riesz transform, as we are going to apply this to $f = |\omega|$, where $\omega(x,t)$ is a family of 1-forms.

4.4 Riesz transform on functions

We are now ready to have a look at Riesz transform on functions. Let us remember that r_0 denoted the lower bound of the curvature tensor R in the sense that $R(X,X) \geq r_0|X|^2$ for any vector field X. We can assume that $r_0 \leq 0$, so that we may write $r_0 = -a^2$ for some $a \geq 0$.

The square roots of operators occurring in this section are constructed via de spectral theory. Later, in section 6.3, we will show how to define these square roots on L^p for arbitrary $1 \leq p < \infty$. It is also shown that for $p = 2$ this construction coincides with the one given by the spectral theory, and that these square roots acting on the different L^p are consistent on C_0^∞. This means that the results in this section, which are only considered for functions in C_0^∞, make sense in any L^p.

Before we get to the main result of this section, we first prove a lemma.

Lemma 4.4.1. *There exist constants c_1, c_2 such that for all $1 \leq p \leq \infty$ and all generators L of a Markovian, symmetric semigroup on L^2 and for all $a \geq 0$ we have that*

$$c_1(a||f||_p + ||(-L)^{1/2}f||_p) \leq ||(a^2I - L)^{1/2}f||_p \leq c_2(a||f||_p + ||(-L)^{1/2}f||_p)$$

for all $f \in C_0^\infty$.

Proof. First suppose that $a = 1$. As L generates a symmetric Markovian semigroup on L^2, we know from [8] that L is self-adjoint, and the semigroup is in fact a semigroup of contractions. The self-adjointness implies that the spectrum of L is contained in the real axis. By the Hille-Yosida theorem, we then find that the spectrum is in fact contained in $[0,\infty)$. Writing E_λ for the spectral family of L, we have that $-L = \int_0^\infty \lambda \mathrm{d}E_\lambda$. Consequently, the semigroup generated by L on L^2 is given by $P_t = \int_0^\infty e^{-\lambda t}\mathrm{d}E_\lambda$.

By lemma 4.2.4 the function $(1+x)^{1/2}(1+x^{1/2})^{-1}$ is the Laplace transform of some bounded measure n_1. We compute

$$\begin{aligned}\int_0^\infty P_s n_1(\mathrm{d}s) &= \int_0^\infty \int_0^\infty e^{-\lambda s} n_1(\mathrm{d}s)\mathrm{d}E_\lambda \\ &= \int_0^\infty (1+\lambda)^{1/2}(1+\lambda^{1/2})^{-1}\mathrm{d}E_\lambda \\ &= (I-L)^{1/2}(I+(-L)^{1/2})^{-1}.\end{aligned}$$

We furthermore have that

$$\left|\left|\int_0^\infty P_s f n_1(\mathrm{d}s)\right|\right|_p \leq \int_0^\infty ||P_s f||_p n_1(\mathrm{d}s) \leq ||f||_p \int_0^\infty n_1(\mathrm{d}s) =: |n_1|||f||_p$$

where $|n_1| < \infty$ as n_1 is a bounded measure. We thus see that $\int_0^\infty P_s n_1(\mathrm{d}s)$ is a bounded operator on L^p [14], giving us that

$$||(I-L)^{1/2}f||_p = \left|\left|\int_0^\infty P_s(f + (-L)^{1/2}f)n_1(\mathrm{d}s)\right|\right|_p \leq |n_1|||f + (-L)^{1/2}f||_p$$

from which we deduce that

$$||(I-L)^{1/2}f||_p \leq c_2(||f||_p + ||(-L)^{1/2}f||_p)$$

[14] Technically, it is only defined on C_0^∞, but by density, and the boundedness we can uniquely extend it to a bounded operator on L^p.

which is the second inequality for $a = 1$.

From lemma 4.2.4 we also find that the functions $(1 + x^{1/2})(1 + x)^{-1/2}$ and $(1 + x)^{-1/2}$ are the Laplace transforms of some bounded measures n_2 and n_3. Following the same argument as above gives us that

$$||f + (-L)^{1/2} f||_p \leq |n_2|||(I - L)^{1/2} f||_p$$

and

$$||f||_p \leq |n_3|||(I - L)^{1/2}||_p.$$

Putting these together, we find that

$$||f||_p + ||(-L)^{1/2}||_p \leq 2||f||_p + ||f + (-L)^{1/2} f||_p \leq c||(I - L)^{1/2}||_p$$

giving us that there is some constant c_1 such that

$$c_1(||f||_p + ||(-L)^{1/2}||_p) \leq ||(I - L)^{1/2}||_p$$

which is the first inequality in the case that $a = 1$.

Now suppose that $a \geq 0$ is arbitrary. Observe that the case $a = 0$ is trivial, hence we may suppose that $a > 0$. Using the above with the operator $\frac{1}{a^2}L$ we get that

$$c_1(||f||_p + \frac{1}{a}||(-L)^{1/2} f||_p) \leq \frac{1}{a}||(a^2 - L)^{1/2}||_p \leq c_2(||f||_p + \frac{1}{a}||(-L)^{1/2} f||_p)$$

or, after multiplying by $a > 0$

$$c_1(a||f||_p + ||(-L)^{1/2} f||_p) \leq ||(a^2 I - L)^{1/2}||_p \leq c_2(a||f||_p + ||(-L)^{1/2} f||_p)$$

as desired. □

We are now ready to prove the main result of this section, namely the boundedness of the Riesz transform on functions.

Theorem 4.4.2. *For all $p \in (1, \infty)$ there exists a constant $C(p)$ solely depending on p such that for all $f \in C_0^\infty$*

$$||\mathrm{d}f||_p \leq C(p)||(-L)^{1/2} f||_p + a||f||_p$$

where $a \geq 0$ is such that $r_0 = -a^2$.

Proof. Let $f \in C_0^\infty$ be arbitrary. It suffices to show that for all $d > a$ it holds that $||\mathrm{d}f||_p \leq C(p)||C^{0,d} f||_p$ for some constant $C(p)$, where $C^{0,d} f = (d^2 I - L)^{1/2}$. Indeed, by the previous lemma we then find for all $d > a$ that[15]

$$||\mathrm{d}f||_p \leq C(p)||C^{0,d} f||_p \leq C(p)(d||f||_p + ||(-L)^{1/2} f||_p)$$

By taking the limit $d \downarrow a$, we obtain

$$||\mathrm{d}f||_p \leq C(p)(a||f||_p + ||(-L)^{1/2} f||_p)$$

as desired.

Now denote by q the conjugate exponent of p. We start by showing that for all 1-forms $\omega \in C_0^\infty$

$$\langle \mathrm{d}f, \omega \rangle \leq C(p)||C^{0,d} f||_p ||\omega||_q.$$

[15] We use the convention to use the same symbol for a constant, even if it slightly changes.

In order to simplify notation, we will write $Q_t^{0,d} = Q_t, \overrightarrow{Q}_t^{0,d} = \overrightarrow{Q}_t, C^{0,d} = C$, and so on. We furthermore write $b^2 = d^2 - a^2 > 0$, as $d > a \geq 0$. Finally, we will write $\bar{f}(x,t) = \bar{f}_t(x) = Q_t f(x)$ and $\bar{\omega}(x,t) = \bar{\omega}_t(x) = \overrightarrow{Q}_t\omega(x)$.

From propositions 4.2.2(3) and 4.2.3(3) we find for all $1 \leq r \leq \infty$ that

$$||\bar{f}_t||_r \leq e^{-dt}||f||_r$$

and

$$||\bar{\omega}_t||_r \leq e^{-(d^2+r_0)^{1/2}t}||\omega||_r = e^{-(d^2-a^2)^{1/2}t}||\omega||_r = e^{-bt}||\omega||_r.$$

Note furthermore that

$$\begin{aligned} \mathrm{d}\bar{f}_t &= \mathrm{d}Q_t f \\ &= \int_0^\infty \mathrm{d}(P_u f)e^{st-d^2u}m_t(\mathrm{d}u) \\ &= \int_0^\infty \overrightarrow{P}_u(\mathrm{d}f)e^{st-d^2u}m_t(\mathrm{d}u) \\ &= \overrightarrow{Q}_t\mathrm{d}f \end{aligned}$$

where we used proposition 4.1.17(3). We deduce the estimate

$$||\mathrm{d}\bar{f}_t||_r \leq e^{-bt}||\mathrm{d}f||_r.$$

Now by the same considerations as in the proof of proposition 4.2.2(6) we have that

$$C^k\bar{f}_t = C^kQ_tf = Q_tC^kf = (\overline{C^kf})_t$$

and consequently

$$||D_0^k\bar{f}_t||_r = ||C^k\bar{f}_t||_r = ||(\overline{C^kf})_t||_r \leq e^{-dt}||C^kf||_r.$$

By proposition 4.2.2(e) and the commutativity of C and L (as $C = \sqrt{d^2 - L}$) we find that

$$||C^kf||_r \leq K\sum_{i=0}^k ||L^if||_r.$$

As proposition 4.2.3 is the analogue of proposition 4.2.2 for 1-forms, we get similar estimates for $D_0^k\mathrm{d}\bar{f}_t$ and $D_0^k\bar{\omega}_t$, where we need to replace d by b.

As $b, d > 0$, we thus find exponentially decreasing bounds in t, which allow the following computation

$$f = Q_0f = \bar{f}_0 = \int_0^\infty D_0^2(\bar{f}_s)s\mathrm{d}s = 4\int_0^\infty (D_0^2\bar{f})_{2s}s\mathrm{d}s.$$

Here we used that $Q_0 = I$, lemma A.2.1 from the appendix and a transformation $s \mapsto 2s$. From the above it now follows that

$$\mathrm{d}f = 4\int_0^\infty (D_0^2\mathrm{d}\bar{f})_{2s}s\mathrm{d}s$$

from which we find by Fubini (ω has compact support)

$$\langle \mathrm{d}f, \omega\rangle = 4\int_0^\infty \langle (D_0^2\mathrm{d}\bar{f})_{2s}, \omega\rangle s\mathrm{d}s.$$

Now note that

$$(D_0^2\mathrm{d}\bar{f})_{2s} = \mathrm{d}(D_0^2\bar{f})_{2s} = \mathrm{d}(C^2Q_{2s}f) = \mathrm{d}(CQ_sCQ_sf) = \overrightarrow{C}\overrightarrow{Q}_s\mathrm{d}(CQ_sf)$$

Plugging this in, we obtain

$$\langle (D_0^2 \mathrm{d}\bar{f})_{2s}, \omega\rangle = \langle \overrightarrow{C}\overrightarrow{Q}_s \mathrm{d}(CQ_s f), \omega\rangle = \langle \mathrm{d}(CQ_s f), \overrightarrow{C}\overrightarrow{Q}_s\omega\rangle = \langle \mathrm{d}C\bar{f}_s, D_0\bar{\omega}_s\rangle$$

where we used the symmetry of $\overrightarrow{C}$ and $\overrightarrow{Q}_s$ and the fact that $\overrightarrow{C}$ generates $\overrightarrow{Q}_s$.

Let us now define $\bar{g}_s = CQ_s f = Q_s Cf$ and $\bar{g}_0 = g = Cf$. Applying first Cauchy-Schwarz and then Hölder's inequality, we find that

$$\begin{aligned}
\langle \mathrm{d}f, \omega\rangle &= 4\left\langle \int_0^\infty (\mathrm{d}\bar{g}_s \cdot D_0\bar{\omega}_s) s\mathrm{d}s\right\rangle \\
&\leq 4\left\langle \left(\int_0^\infty |\mathrm{d}\bar{g}_s|^2 s\mathrm{d}s\right)^{1/2}, \left(\int_0^\infty |D_0\bar{\omega}_s|^2 s\mathrm{d}s\right)^{1/2}\right\rangle \\
&\leq 4\left|\left|\left(\int_0^\infty |\mathrm{d}\bar{g}_s|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_p \left|\left|\left(\int_0^\infty |D_0\bar{\omega}_s|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_q .
\end{aligned}$$

From propositions 4.2.2(6) and 4.2.3(6) we know that $L^{0,d}\bar{g} = 0$ and $\overrightarrow{L}^{0,d}\bar{\omega} = 0$, where in the first case we apply the proposition to Cf. From proposition 4.2.5 with $s_1 = s_2 = 0, d_1 = d$ and $d_2 = 0$, we find that

$$L^0(\bar{g}^2) = 2|\mathrm{d}\bar{g}|^2 + 2(D_0\bar{g})^2 = 2d^2\bar{g}^2 \geq 2|\mathrm{d}\bar{g}|^2 \tag{4.5}$$

and similarly

$$L^0|\bar{\omega}|^2 \geq 2|D_0\bar{\omega}|^2. \tag{4.6}$$

Furthermore, we find for all $\epsilon > 0$ (we apply the proposition to $\sqrt{\epsilon}$) that

$$L^0((|\bar{g}|^2 + \epsilon)^{1/2} - \sqrt{\epsilon}) \geq 0$$

and as $d > a$, also

$$L^0((|\bar{\omega}|^2 + \epsilon)^{1/2} - \sqrt{\epsilon}) \geq 0.$$

However, as $L^0(\sqrt{\epsilon}) = 0$, we in fact get that

$$L^0((|\bar{g}|^2 + \epsilon)^{1/2}) \geq 0 \tag{4.7}$$

and

$$L^0((|\bar{\omega}|^2 + \epsilon)^{1/2}) \geq 0. \tag{4.8}$$

We are now in a position to make the final estimates. We need to distinguish between three cases: $p > 2$, $p = 2$ and $1 < p < 2$.

First suppose that $p > 2$. By (4.6) we have that $2\int_0^\infty |D_0\bar{\omega}|^2 s\mathrm{d}s \leq \int_0^\infty L^0|\bar{\omega}|^2 s\mathrm{d}s$. We now wish to apply theorem 4.3.6 to the function $|\omega|$. Following remark 4.3.8, we already satisfy $L^0(|\bar{\omega}|^2 + \epsilon) \geq 0$. Furthermore, as ω is smooth, so is $|\omega|^2$. By the exponential bounds we found above and the fact that ω_t has compact support, we have that

$$\int_M \int_0^\infty (|L_0^d|\bar{\omega}_u|^2| + |L|\bar{\omega}_u|^2| + |D_0|\bar{\omega}_u||^2 + |\nabla|\bar{\omega}_u||^2) u \;\mathrm{d}u \;\mathrm{d}m(x) < \infty$$

where we filled in $V_0(u) = u$. Consequently, we may apply theorem 4.3.6 for $q \in (1, 2]$ (as $p > 2$) to find that

$$\left|\left|\left(\int_0^\infty |D_0\bar{\omega}_s|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_q \leq \left|\left|\left(\int_0^\infty L^0|\bar{\omega}_s|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_q \leq C(q)||\bar{\omega}_0||_q = C(q)||\omega||_q.$$

On the other hand, observe that

$$0 = \mathrm{d}L^{0,d}\bar{g}_s = \overrightarrow{L}^{0,d}\mathrm{d}\bar{g}_s.$$

Again using proposition 4.2.5 we find

$$L^0|\mathrm{d}\bar{g}_s|^2 \geq 2|\nabla\mathrm{d}\bar{g}_s|^2 + 2|D_0\mathrm{d}\bar{g}_s|^2 + (2d^2 + 2r_0)|\mathrm{d}\bar{g}_s|^2 \geq 0$$

as $d^2 > a^2 = -r_0$. But then by proposition 4.3.2

$$Q_s^0|\mathrm{d}\bar{g}_s|^2 \geq |\mathrm{d}\bar{g}_{2s}|^2.$$

By a change of variables, the above estimate and estimate (4.5) now give us that

$$\begin{aligned}\int_0^\infty |\mathrm{d}\bar{g}_s|^2 s\mathrm{d}s &= 4\int_0^\infty |\mathrm{d}\bar{g}_{2s}|^2 s\mathrm{d}s \\ &\leq 4\int_0^\infty Q_s^0|\mathrm{d}\bar{g}_s|^2\mathrm{d}s \\ &\leq 2\int_0^\infty Q_s^0(L^0\bar{g}_s^2)s\mathrm{d}s.\end{aligned}$$

Here in the last line we also made use of the fact that $Q_s^0 h \geq 0$ whenever $h \geq 0$.

If we now apply theorem 4.3.5 to the function $\bar{g}^2$ with exponent $\frac{p}{2} > 1$, we obtain

$$\begin{aligned}\left|\left|\left(\int_0^\infty |\mathrm{d}\bar{g}_s|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_p &= \left|\left|\int_0^\infty |\mathrm{d}\bar{g}_s|^2 s\mathrm{d}s\right|\right|_{p/2}^{\frac{1}{2}} \\ &\leq \sqrt{2}\left|\left|\int_0^\infty Q_s^0(L^0\bar{g}_s^2)s\mathrm{d}s\right|\right|_{p/2}^{\frac{1}{2}} \\ &\leq \sqrt{2}C(p)||\bar{g}_0^2||_{p/2}^{\frac{1}{2}} \\ &= C(p)||\bar{g}_0||_p \\ &= C(p)||g||_p.\end{aligned}$$

If we now combine all inequalities, we find that

$$\begin{aligned}\langle\mathrm{d}f,\omega\rangle &\leq 4\left|\left|\left(\int_0^\infty |\mathrm{d}\bar{g}_s|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_p \left|\left|\left(\int_0^\infty |D_0\bar{\omega}_s|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_q \\ &\leq 4C(p)||g||_p C(q)||\omega||_q \\ &:= C(q)||g||_p||\omega||_q \\ &= C(q)||C^{0,d}f||_p||\omega||_q\end{aligned}$$

which is the desired estimate. In this case, theorem 4.3.5 also holds for $p = 1$, hence the above goes through for $p = 2$ as well.

We now consider $1 < p < 2$. The argument is analogous to the one given for the case $p > 2$. Indeed, we only need to interchange the roles of $\mathrm{d}\bar{g}$ and $\bar{\omega}$, as in fact the values of p and q are interchanged. First observe that from (4.6) it follows that $L^0|\bar{\omega}|^2 \geq 0$. From proposition 4.3.2 and the fact that $Q_u^0 h \geq 0$ whenever $h \geq 0$, we then find that

$$|D_0\bar{\omega}_{2u}|^2 \leq Q_u^0|D_0\bar{\omega}_u|^2 \leq Q_u^0(L^0|\bar{\omega}_u|^2).$$

In the same way as above, we obtain that

$$\int_0^\infty |D_0\bar{\omega}_s|^2 s\mathrm{d}s \leq 2\int_0^\infty Q_s^0(L^0|\bar{\omega}|^2)s\mathrm{d}s$$

and again applying theorem 4.3.5, but now to the function $|\bar{\omega}|^2$ (which is allowed for the same reasons as above), we find with exponent $\frac{q}{2} > 1$ that

$$\left|\left|\left(\int_0^\infty |D_0\bar{\omega}_s|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_q \leq C(p)||g||_q.$$

On the other hand, (4.5) and (4.7) allow us to apply theorem 4.3.6 in the same way as discussed in the case $p > 2$ but now to the function $|\mathrm{d}\bar{g}_s|$ and with exponent p. We obtain

$$\left|\left|\left(\int_0^\infty |\mathrm{d}\bar{g}_s|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_p \leq C(p)||g||_p.$$

Putting these together, we again find that

$$\langle \mathrm{d}f, \omega\rangle \leq C(q)||C^{0,d}f||_p||\omega||_q.$$

We are now ready to finish the proof. We found that there exists a constant $C(p)$ such that for all $\omega \in C_0^\infty$ it holds that $\langle \mathrm{d}f, \omega\rangle \leq C(p)||C^{0,d}f||_p||\omega||_q$. This shows that the functional $\omega \mapsto \langle \mathrm{d}f, \omega\rangle$ is bounded on $\overrightarrow{L}^q$, with norm bounded by $C(p)||C^{0,d}f||_p$. However, from the $\overrightarrow{L}^p/\overrightarrow{L}^q$-duality we know that the norm is given by $||\mathrm{d}f||_p$. But then we find that

$$||\mathrm{d}f||_p \leq C(p)||C^{0,d}f||_p$$

for any $d > a$, which is sufficient as shown at the beginning of the proof. □

We will now prove a corollary, which gives the reverse inequality of the one in the theorem above.

Corollary 4.4.3. *Under the same assumption as in theorem 4.4.2 there exist a constant $C(p)$ only depending on p, such that for all $f \in C_0^\infty$*

$$||(a^2 - L)^{1/2}f||_p \leq C(p)(a||f||_p + ||\mathrm{d}f||_p).$$

Proof. First observe that by lemma 4.4.1 it suffices to consider $a > 0$. Indeed, by the lemma we have for any $a > 0$ that

$$||(-L)^{1/2}f||_p \leq C(p)||(a^2I - L)^{1/2}f||_p$$

Now, if we have the result for all $a > 0$, we thus find that

$$||(-L)^{1/2}f||_p \leq C(p)(a||f||_p + ||\mathrm{d}f||_p)$$

But then it is obviously also true for $a = 0$ by taking the limit $a \downarrow 0$.

So assume that $a > 0$. Remember that $(a^2 - L)^{1/2} = -C^{0,a}$. We claim that $C^{0,a}(C_0^\infty)$ is dense in L^p for any $1 \leq p < \infty$. For this, let q be the conjugate exponent of p and suppose that $f \in L^q$ is orthogonal to $C^{0,a}(C_0^\infty)$. By proposition 3.1.5 we have for all $g \in C_0^\infty$ that $\int_0^t Q_s^{0,a} g \mathrm{d}s \in D(C^{0,a})$. As $C^{0,a}$ is the generator of $Q_s^{0,a}$, by the same proposition

$$C^{0,a}\left(\int_0^t Q_s^{0,a} g \mathrm{d}s\right) = Q_t^{0,a} g - g.$$

Now observe that the operators P_t acting on different L^p spaces are consistent (see section 6.2 for details) and that the operators $Q_t^{0,a}$ are defined as integrals in L^p of the P_t. However, as the integrands are consistent, the integrals must be as well. This can be seen by switching to a subsequence which converges almost everywhere. Hence, for $g \in C_0^\infty$ we may interpret $Q_t^{0,a} g$ both in L^2 and L^p. By a similar argument this also holds for $\int_0^t Q_s^{0,a} g \ \mathrm{d}s$. Furthermore, as L is essentially selfadjoint as operator on C_0^∞ in L^2, we find that C_0^∞ is a core for L in L^2. Consequently, by proposition 3.8.2 in [1] it is also a core for $C^{0,a}$ in L^2. We can thus find a sequence $(g_n) \subset C_0^\infty$ which converges to $\int_0^t Q_s^{0,a} g \mathrm{d}s$ in $D_2(C^{0,a})$. But then $\langle f, C^{0,a} g_n\rangle = 0$ for all n and by taking the limit $n \to \infty$ we find that

$$0 = \langle f, Q_t^{0,a} g - g\rangle = -\langle f, g\rangle + \langle f, Q_t^{0,a} g\rangle.$$

Now let $(f_n)_n$ be a sequence of C_0^∞ functions such that $f_n \to f$ in L^q. By the consistency of $Q_t^{0,a}$ on C_0^∞ we may use the symmetry of $Q_t^{0,a}$ (proposition 4.2.2) to obtain that

$$\langle f, Q_t^{0,a} g\rangle = \lim_{n\to\infty} \langle f_n, Q_t^{0,a} g\rangle = \lim_{n\to\infty} \langle Q_t^{0,a} f_n, g\rangle$$

Again using consistency, we may interpret $Q_t^{0,a} f_n$ in L^q and the boundedness of $Q_t^{0,a}$ on L^q (proposition 4.2.2(3)) then implies that

$$\lim_{n\to\infty} \langle Q_t^{0,a} f_n, g\rangle = \langle Q_t^{0,a} f, g\rangle.$$

Putting everything together, we find that

$$\langle Q_t^{0,a} f, g\rangle - \langle f, g\rangle = 0.$$

As this holds for all $g \in C_0^\infty$, by density it must be that $Q_t^{a,0} f = f$. From proposition 4.2.2 we have that $||Q_t^{a,0} f||_q \leq e^{-at}||f||_q$. As $f \in L^q$ and $a > 0$, we see that $\lim_{t\to 0} ||Q_t^{a,0} f||_q = 0$, from which it follows that $||f||_q = 0$, i.e., $f = 0$. We conclude that 0 is the only element orthogonal to $C^{0,a}(C_0^\infty)$ as subset of L^p, from which it follows that it is dense in L^p.

The above, together with duality now gives us that

$$||C^{0,a} f||_p = \sup_{\{g \in C_0^\infty | ||C^{0,a} g||_q \leq 1\}} \langle C^{0,a} f, C^{0,a} g\rangle.$$

Using the symmetry of $C^{0,a}$ and proposition 4.1.7 we get that

$$\langle C^{0,a} f, C^{0,a} g\rangle = \langle f, a^2 g - Lg\rangle = a^2\langle f, g\rangle + \langle \mathrm{d}f, \mathrm{d}g\rangle.$$

By Hölder's inequality we now find that

$$\begin{aligned}\langle C^{0,a} f, C^{0,a} g\rangle &\leq a^2||f||_p||g||_q + ||\mathrm{d}f||_p||\mathrm{d}g||_q \\ &\leq (a^2||f||_p^2 + ||\mathrm{d}f||_p^2)^{1/2}(a^2||g||_q^2 + ||\mathrm{d}g||_q^2)^{1/2}.\end{aligned}$$

The last inequality holds in general, see lemma A.2.2 in the appendix. By lemma 4.4.1 and theorem 4.4.2 we obtain

$$(a^2||g||_q^2 + ||\mathrm{d}g||_q^2)^{1/2} \leq a||g||_q + ||\mathrm{d}g||_q$$

$$\begin{aligned} &\leq C(q)(a||g||_q + ||(-L)^{1/2}g||_q \\ &\leq C(q)||C^{0,a}g||_q. \end{aligned}$$

If we now combine everything, we find that

$$\begin{aligned} ||C^{0,a}f||_p &\leq \sup_{\{g\in C_0^\infty |||C^{0,a}g||_q\leq 1\}} (a^2||f||_p^2 + ||\mathrm{d}f||_p^2)^{1/2}(a^2||g||_q^2 + ||\mathrm{d}g||_q^2)^{1/2} \\ &\leq \sup_{\{g\in C_0^\infty |||C^{0,a}g||_q\leq 1\}} C(q)||C^{0,a}g||_q(a^2||f||_p^2 + ||\mathrm{d}f||_p^2)^{1/2} \\ &\leq C(q)(a^2||f||_p^2 + ||\mathrm{d}f||_p^2)^{1/2} \end{aligned}$$

which is the desired estimate, as we may well write $C(p)$ instead of $C(q)$. □

4.5 Riesz transform for k-forms

In the previous section we have seen that theorem 4.4.2 relies on the formula for $L|\omega|^2$ as presented in proposition 4.1.15, where ω is a 1-form. If we have such a formula for forms of any order, we can extend the results from the previous section to k-forms.

We will write $L^{p,k}$ for the closure of the space of C_0^∞ k-forms with respect to $||\omega||_p = |||\omega|||_p$. In this notation $\overrightarrow{L}^p = L^{p,1}$. As in section 4.1, we denote by δ the divergence, acting on forms of any order. From the definition of divergence we see that it takes k-forms to $(k-1)$-forms. The Laplace operator acting on k-forms is given by $L_k = -(d\delta + \delta d)$, which is commonly known as the Witten-Laplacian. As d sends k-forms to $(k+1)$-forms, we see that L_k sends k-forms to k-forms. In the case where $\rho \equiv 1$ we simply write Δ_k for L_k.

Proposition 4.5.1. *L_k defined on C_0^∞ is essentially self-adjoint on $L^{2,k}$ and for all $\omega \in C_0^\infty$ we have*

$$\langle L_k\omega, \omega\rangle = -\langle \mathrm{d}\omega, \mathrm{d}\omega\rangle - \langle \delta\omega, \delta\omega\rangle.$$

Proof. The proof is identical to that of proposition 4.1.16, as everything used there also applies to arbitrary k-forms. It can again also be found in [34]. □

We will now write down some usefull identities. First of all, remembering that $d^2 = \delta^2 = 0$ we immediately see that $L_k\mathrm{d} = \mathrm{d}L_{k-1}$ and $L_k\delta = \delta L_{k+1}$. This follows by simply writing out both sides of the equality. Observe that in each of the equalities, both sides do in fact act on the same type of forms, and return the same type as well.

It is furthermore known that L_k satisfies a Bochner-Lichnérowicz-Weitzenbock formula (see for example [23]). In the case where $\rho \equiv 1$, this formula is given by

$$\frac{1}{2}\Delta|\omega|^2 = \omega \cdot \Delta_k\omega + \frac{1}{k!}|\nabla\omega|^2 + \tilde{Q}_k(\omega, \omega) \tag{4.9}$$

where $\tilde{Q}_k$ is a quadratic form acting on k-forms which involves the curvature tensors, which we can represent in local coordinates. We write ${r_{ij}}^k{}_l$ for the coefficients of the curvature tensor, and $R_{ab} = \sum_{i=1}^n {r_{ia}}^i{}_b$ for the coefficients of the Ricci curvature tensor. Writing $\omega_{i_1 i_2 \ldots i_k}$ for the coefficients of the k-form ω, and writing upper indices if we have used the duality, we get that

$$\tilde{Q}_k(\omega, \omega) = \frac{1}{(k-1)!}R_{ab}\omega^{ai_2\ldots i_k}{\omega^b}_{i_2\ldots i_k} + \frac{1}{2(k-2)!}r_{pqrs}\omega^{pqi_3\ldots i_k}{\omega^{rs}}_{i_3\ldots i_k}$$

where we use Einstein's summation convention.

For arbitrary $\rho > 0$ we find using proposition 4.1.11 that

$$L_k\omega = \Delta_k\omega + \mathrm{d}(\iota(\mathrm{d}\log\rho^*)\omega) + \iota(\mathrm{d}\log\rho^*)\mathrm{d}\omega.$$

Remembering that $L|\omega|^2 = \Delta|\omega|^2 + \mathrm{d}|\omega|^2 \cdot \mathrm{d}\log\rho$ we deduce that

$$\frac{1}{2}L|\omega|^2 = \omega \cdot L_k\omega + \frac{1}{k!}|\nabla\omega|^2 + Q_k(\omega,\omega) \tag{4.10}$$

where Q_k is given by

$$Q_k(\omega,\omega) = \tilde{Q}_k(\omega,\omega) + \frac{1}{2}\mathrm{d}(\omega\cdot\omega)\cdot\mathrm{d}\log\rho - \omega\cdot\mathrm{d}(\iota(\mathrm{d}\log\rho^*)\omega) - \omega\cdot\iota(\mathrm{d}\log\rho^*)\mathrm{d}\omega$$

This formula for $L|\omega|^2$ is what we are looking for as discussed in the beginning of the section. In analogy to the assumption that the Ricci curvature is bounded from below, we will now assume that the quadratic forms Q_k are bounded from below, i.e., we assume that there exist constants $a_k \geq 0$ such that $Q_k(\omega,\omega) \geq -a_k^2|\omega|^2$ for all k-forms ω

We can now start repeating what we have done in the previous sections. By proposition 4.5.1 we have that the closure L_k in $L^{2,k}$ (denoted again by L_k) is self-adjoint and negative. Hence its spectrum is contained in $(-\infty, 0]$. By the spectral theorem we may write $-L_k = \int_0^\infty \lambda \mathrm{d}E_\lambda^k$, where $(E_\lambda^k)_\lambda$ is the spectral family belonging to L_k. We can then also define the heat semigroup $P_t^k := e^{tL_k} = \int_0^\infty e^{-\lambda t}\mathrm{d}E_\lambda^k$.

We collect some properties of P_t^k in the following theorem. These properties are analogues of those in proposition 4.1.17 and may be proved similarly.

Proposition 4.5.2. *The operators P_t^k satisfy the following properties:*

1. $|P_t^k\omega| \leq e^{a_k^2 t}P_t|\omega|$.
2. $||P_t^k\omega||_p \leq e^{a_k^2|1-2/p|t}||\omega||_p$, $1 \leq p \leq \infty$.
3. *For all k-forms $\omega \in C_0^\infty$*
$$P_t^{k+1}\mathrm{d}\omega = \mathrm{d}P_t^k\omega, \qquad P_t^{k-1}(\delta\omega) = \delta P_t^k\omega.$$
4. *If $\omega \in C_0^\infty$ is a k-form, we have that $P_t^k\omega \in C^\infty(M\times[0,\infty))$ and*
$$\frac{\mathrm{d}}{\mathrm{d}t}P_t^k\omega = L_kP_t^k\omega = P_t^kL_k\omega.$$

As done before, we will again define for $s \in \mathbb{R}$ and $d \geq 0$ the subordinated semigroups

$$Q_t^{k,s,d}\omega = \int_0^\infty P_u^k\omega e^{st-d^2u}m_t(\mathrm{d}u).$$

We get the following analogue of proposition 4.2.3.

Proposition 4.5.3. *The operators $Q_t^{k,s,d}$ satisfy the following properties:*

1. *$Q_t^{k,s,d}$ is a symmetric semigroup on $L^{2,k}$ with generator given by $C^{k,s,d} = sI - (d^2I - L_k)^{1/2}$.*
2. $M_bQ_t^{k,s,d} = Q_t^{k,s+b,d}$.

3. *If $d \geq a_k$, then for all k-forms ω we have that*

$$|Q_t^{k,s,d}\omega| \leq Q_t^{s,(d^2-a_k^2)^{1/2}}|\omega|.$$

4. *If $d^2 \geq a_k^2|1-2/p|$, then $(Q_t^{k,s,d})_t$ is a semigroup of bounded operators on $L^{p,k}$, where the norms are bounded by $e^{t(s-(d^2-a_k^2|1-2/p|)^{1/2}}$.*

5. *If $d^2 \geq a_k^2|1-2/p|$, then there exist constants $c(k,s,d)$ depending only on k,s,d such that for all $1 \leq p \leq \infty$ and all $\omega \in C_0^\infty$*

$$||\overrightarrow{C}^{k,s,d}\omega||_p \leq c(k,s,d)(||\omega||_p + ||L_k\omega||_p).$$

6. *For all k-forms $\omega \in C_0^\infty$, the function $\bar{\omega}(x,t) = Q_t^{k,s,d}\omega(x)$ is in $C^\infty(M \times (0,\infty)) \cap C(M \times [0,\infty))$ and satisfies $(L_0^{s,d} + L_k)\bar{\omega} = 0$.*

Proof. Part (1) and (2) are proved similarly as for function and 1-forms. For part (3), we observe that the proof of proposition 4.2.3(3) relies on the fact that $|P_t^1\omega| \leq e^{-r_0 t}P_t|\omega|$. In this case we have for any $k = 0,1,\ldots,n$ that $|P_t^k\omega| \leq e^{a_k^2 t}P_t|\omega|$ and the proof is the same. Part (4) and (5) also rely on the analogue of proposition 4.1.17. Finally, the proof for part (6) can also be copied from the proof of proposition 4.2.2. □

4.5.1 Boundedness of the Riesz transform on k-forms

As our semigroups $Q_t^{k,s,d}$ satisfy the same properties as in the case for functions and 1-forms, it might not be surprising that we find an analogue for theorem 4.4.2 for k-forms. Before we state the theorem we define $b_0 = a_0$ and $b_k = \max\{a_k, a_{k-1}\}$ for $k = 1,\ldots n$ and write $C^{k,d}$ for the generator of $Q_t^{k,0,d}$.

Theorem 4.5.4. *Let $1 < p < \infty$. There exists a constant $c(p,k)$ only depending on p and k such that for all k-forms $\omega \in C_0^\infty$ it holds that*

$$||\mathrm{d}\omega||_p \leq c(p,k)||C^{k,b_k}\omega||_p$$

and

$$||\delta\omega||_p \leq c(p,k)||C^{k,b_{k-1}}\omega||_p.$$

The proof of this theorem uses the same strategy as the proof of theorem 4.4.2. We therefore first prove a lemma.

Lemma 4.5.5. *For all p, $1 \leq p \leq \infty$, define $a_{k,p}^2 = a_k^2|1-2/p|$ and for all $d \geq a_{k,p}$ write $d^2 = e^2 + a_{k,p}^2$ where $e \geq 0$. There exist two constants C_1, C_2 such that for all k,p and for all k-forms ω and for all $d \geq a_{k,p}$ it holds that*

$$C_1(e||\omega||_p + ||C^{k,a_{k,p}}\omega||_p) \leq ||C^{k,d}\omega||_p \leq C_2(e||\omega||_p + ||C^{k,a_{k,p}}\omega||_p).$$

Proof. The proof is similar to that of lemma 4.4.1. First observe that the inequalities are trivial if $d = a_{k,p}$, i.e., if $e = 0$. We may thus assume that $d > a_{k,p}$.

By proposition 4.2.4 the function $f_1(x) = (1+x)^{1/2}(1+x^{1/2})^{-1}$ is the Laplace transform of some bounded measure n_1. Let $c > 0$ be a constant to be determined later. We define the operator

$$\int_0^\infty P_{cs}^k e^{-a_{k,p}^2 cs} n_1(\mathrm{d}s).$$

We have that

$$\begin{aligned}
\int_0^\infty P^k_{cs} e^{-a^2_{k,p}cs} n_1(\mathrm{d}s) &= \int_0^\infty \int_0^\infty e^{-\lambda cs} e^{-a^2_{k,p}cs} n_1(\mathrm{d}s)\mathrm{d}E^k_\lambda \\
&= \int_0^\infty (1+c(\lambda+a^2_{k,p}))^{1/2}(1+(c(\lambda+a^2_{k,p}))^{1/2})^{-1}\mathrm{d}E^k_\lambda \\
&= (I+c(a^2_{k,p}-L_k))^{1/2}(I+c^{1/2}(a^2_{k,p}-L_k)^{1/2})^{-1}.
\end{aligned}$$

Now for any C_0^∞ k-form ω we obtain

$$\begin{aligned}
\left|\left|\int_0^\infty P^k_{cs}\omega e^{-a^2_{k,p}cs} n_1(\mathrm{d}s)\right|\right|_p &\leq \int_0^\infty ||P^k_{cs}\omega||_p e^{-a^2_{k,p}cs} n_1(\mathrm{d}s) \\
&\leq \int_0^\infty e^{a^2_{k,p}cs}||\omega||_p e^{-a^2_{k,p}cs} n_1(\mathrm{d}s) \\
&= |n_1|||\omega||_p
\end{aligned}$$

which is finite as n_1 is a bounded measure. But then we find that

$$\begin{aligned}
||(I+c(a^2_{k,p}-L_k))^{1/2}\omega||_p &\leq |n_1|||(I+c^{1/2}(a^2_{k,p}-L_k)^{1/2})\omega||_p \\
&\leq |n_1|(||\omega||_p + c^{1/2}||C^{k,a_{k,p}}\omega||_p)
\end{aligned}$$

or after dividing by $c^{1/2}$

$$||(1/c+a^2_{k,p}-L_k)^{1/2}\omega||_p \leq |n_1|(c^{-1/2}||\omega||_p + ||C^{k,a_{k,p}}\omega||_p.$$

We want that $\frac{1}{c}+a^2_{k,p}=d$, or $\frac{1}{c}=d^2-a^2_{k,p}=e^2$. Plugging this into the above estimate we find that

$$||(d-L_k)^{1/2}\omega||_p \leq |n_1|(e||\omega||_p + ||C^{k,a_{k,p}}\omega||_p$$

which is the second inequality, as $C^{k,d}\omega=(d-L_k)^{1/2}\omega$.

Furthermore, by proposition 4.2.4 also the functions $f_2(x)=(1+x^{1/2})(1+x)^{-1/2}$ and $f_3(x)=(1+x)^{-1/2}$ are Laplace transforms of bounded measures n_2 and n_3. In the same way as above, we may write operators

$$(I+c^{1/2}(a^2_{k,p}-L_k)^{1/2})(I+c(a^2_{k,p}-L_k))^{-1/2} = \int_0^\infty P^k_{cs}e^{-a^2_{k,p}cs}n_2(\mathrm{d}s)$$

and

$$(I+c(a^2_{k,p}-L_k))^{-1/2} = \int_0^\infty P^k_{cs}e^{-a^2_{k,p}cs}n_3(\mathrm{d}s).$$

But then we find that

$$||\omega+c^{1/2}C^{k,a_{k,p}}\omega||_p \leq |n_2|||(I+c(a^2_{k,p}-L_k))^{1/2}\omega||_p$$

and

$$||\omega||_p \leq |n_3|||(I+c(a^2_{k,p}-L_k))^{1/2}\omega||_p$$

or again after dividing by $c^{1/2}$ and again taking $c=\frac{1}{e^2}$

$$||e\omega+C^{k,a_{k,p}}\omega||_p \leq |n_2|||C^{k,d}\omega||_p$$

and

$$e||\omega||_p \leq |n_3|||C^{k,d}\omega||_p$$

Combining the two inequalites now gives the desired estimate. $\square$

We also need the following proposition. Here we define $L^{k,s,d} = L_0^{s,d} + L_k$, which acts on families of k-forms. $L_0^{s,d}$ is as in section 4.2. We get the following computational result, which is the analogue of proposition 4.2.5 for k-forms.

Proposition 4.5.6. *Suppose that $\omega(x,t)$ is a family of k-forms which is smooth in (x,t) and satisfies $L^{k,s_1,d_1}\omega = 0$. Then*

1. *For all $s_2, d_2 \geq 0$*

$$L^{s_2,d_2}|\omega|^2 \geq \frac{2}{k!}|\nabla\omega|^2 + 2|D_0\omega + (s_1 - s_2)\omega|^2 + (2d_1^2 - d_2^2 - (s_2 - 2s_1)^2 - 2a_k^2)|\omega|^2.$$

2. *If furthermore $d_1^2 - a_k^2 \geq d_2^2 \geq s_1^2$, the for all $\epsilon > 0$*

$$L^{s_1,d_2}((|\omega|^2 + \epsilon^2)^{1/2} - \epsilon) \geq 0.$$

Proof. The proof is similar to that of proposition 4.2.5. Assume that $L^{k,s_1,d_1}\omega = 0$. Writing out $L^{k,s_2,d_2} - L^{k,s_1,d_1}$ gives us that

$$\begin{aligned} L^{k,s_2,d_2}\omega &= L^{k,s_1,d_1}\omega + 2(s_1 - s_2)D_0\omega + (s_2^2 - s_1^2 + d_1^2 - d_2^2)\omega \\ &= 2(s_1 - s_2)D_0\omega + (s_2^2 - s_1^2 + d_1^2 - d_2^2)\omega \end{aligned}$$

By equation (4.10), we have that (where we used that the part of $L_0^{s_2,d_2}$ can be treated as in the proof of proposition 4.2.5)

$$\begin{aligned} L^{s_2,d_2}|\omega|^2 &= 2\omega \cdot L^{k,s_2,d_2}\omega + 2|D_0\omega|^2 + \frac{2}{k!}|\nabla\omega|^2 + 2Q_k(\omega,\omega) - (s_2^2 - d_2^2)|\omega|^2 \\ &\geq 4(s_1 - s_2)\omega \cdot D_0\omega + (s_2^2 - 2s_1^2 + 2d_1^2 - d_2^2)|\omega|^2 + 2|D_0\omega|^2 + \frac{2}{k!}|\nabla\omega|^2 - 2a_k^2|\omega|^2 \\ &= \frac{2}{k!}|\nabla\omega|^2 + 2|D_0\omega + (s_1 - s_2)\omega|^2 + (2d_1^2 - d_2^2 - (s_2 - 2s_1)^2 - 2a_k^2)|\omega|^2. \end{aligned}$$

Here we used that $Q_k(\omega,\omega) \geq -a_k^2|\omega|^2$ in the second line, and a rewriting identical to the one in the proof of proposition 4.2.5 in the last line.

For the second part of the proposition, we can copy the proof of proposition 4.2.5, only now we use the function $\phi(x) = (\frac{x}{k!} + \epsilon^2)^{1/2} - \epsilon$. In that case we find that

$$L^{s_1,d_2}((|\omega|^2/k! + \epsilon^2)^{1/2} - \epsilon) \geq 0$$

for all $\epsilon > 0$, so in particular for $(k!)^{-1/2}\epsilon$, which gives us that

$$L^{s_1,d_2}((|\omega|^2 + \epsilon^2)^{1/2} - \epsilon) = (k!)^{1/2}L^{s_1,d_2}((|\omega|^2/k! + (\epsilon(k!)^{-1/2})^2)^{1/2} - \epsilon(k!)^{-1/2}) \geq 0$$

as desired. □

We are now ready to prove the boundedness of the Riesz transform on k-forms.

Proof of theorem 4.5.4. We will first focus on the estimate for $\mathrm{d}\omega$. It suffices to show for all $d > b_k$ that

$$||\mathrm{d}\omega||_p \leq c(p,k)||C^{k,d}\omega||_p$$

where $c(p,k)$ is some constant only depending on k and p. Indeed, if we have this, we find for all $d > b_k$ that

$$||\mathrm{d}\omega||_p \leq c(p,k)||C^{k,d}\omega||_p \leq c(p,k)((d^2-b_k^2)^{1/2}||\omega||_p + ||C^{k,b_k}\omega||_p)$$

where the last inequality follows from lemma 4.5.5. (We actually need to look into the proof of the lemma, in which case we see that it still holds with b_k instead of $a_{k,p}$ as $b_k \geq a_{k,p}$). Taking the limit $d \downarrow b_k$, we find that

$$||\mathrm{d}\omega||_p \leq c(p,k)||C^{k,b_k}\omega||_p.$$

It in fact suffices to show for all $(k+1)$-forms η in C_0^∞ that

$$\langle \mathrm{d}\omega, \eta\rangle \leq c(p,k)||C^{k,d}\omega||_p||\eta||_q$$

where q is the conjugate exponent of p. The above is indeed sufficient, as it shows that the norm of the map on $L^{q,k+1}$ given by $\eta \mapsto \langle \mathrm{d}\omega, \eta\rangle$ is bounded by $c(p,k)||C^{k,d}\omega||_p$. As this norm is in fact equal to $||\mathrm{d}\omega||_p$ we get our desired estimate.

Now fix $d > b_k$, a k-form ω and a $(k+1)$-form η which are smooth and compactly supported. We write

$$\bar{\omega}(x,t) = Q_t^{k,0,d}; \qquad \hat{\omega}(x,t) = Q_t^{k,0,d}(C^{k,d}\omega)(x); \qquad \bar{\eta}(x,t) = Q_t^{k+1,0,d}\eta(x).$$

By proposition 4.5.3 we have that $\bar{\omega}$ satisfies $L^{k,0,d}\bar{\omega} = 0$. We can thus use proposition 4.5.6 with $s_1 = s_2 = 0$, $d_1 = d$ and $d_2 = 0$, which gives us that

$$L^0|\bar{\omega}|^2 \geq \frac{2}{k!}|\nabla\bar{\omega}|^2 + 2|D_0\bar{\omega}|^2 + (2d^2 - 2a_k^2)|\omega|^2 \geq \frac{2}{k!}|\nabla\bar{\omega}|^2 \geq 2|\mathrm{d}\bar{\omega}|^2.$$

Here we used that $d > b_k \geq a_k$. The last inequality follows by using normal coordinates and showing that the estimate holds pointwise. Observe that the same holds for $\hat{\omega}$. As $d > a_{k+1}$, we also find it for $\bar{\eta}$. If we do the exact same, but leave out other terms, we furthermore have that

$$L^0|\bar{\eta}|^2 \geq 2|D_0\bar{\eta}|^2.$$

Now observe that $L^0(\epsilon) = 0$, so that from proposition 4.5.6 we get for all $\epsilon > 0$ that

$$L^0((|\bar{\omega}|^2+\epsilon)^{1/2}) \geq 0; \qquad L^0((|\hat{\omega}|^2+\epsilon)^{1/2}) \geq 0; \qquad L^0((|\bar{\eta}|^2+\epsilon)^{1/2}) \geq 0.$$

Now by combining proposition 4.5.3 with 4.2.3 we find that

$$\begin{aligned}
&||\bar{\omega}(\cdot,t)||_p \leq e^{-(d^2-a_k^2)^{1/2}t}||\omega||_p \\
&||\hat{\omega}(\cdot,t)||_p \leq e^{-(d^2-a_k^2)^{1/2}t}||C^{k,d}\omega||_p \leq ce^{-(d^2-a_k^2)^{1/2}t}(||\omega||_p + ||L_k\omega||_p) \\
&||\bar{\eta}(\cdot,t)||_p \leq e^{-(d^2-a_k^2)^{1/2}t}||\eta||_p.
\end{aligned}$$

As the proof of proposition 4.5.3(6) is the same as that of 4.2.2(6), we also get exponential bounds on the D_0 derivatives of any order. All these estimates allow us to copy the proof of theorem 4.4.2, which summarizes to

$$\langle \mathrm{d}\omega, \eta\rangle = \int_0^\infty D_0^2\langle \mathrm{d}\bar{\omega}, \eta\rangle s\mathrm{d}s \quad = \quad 4\int_0^\infty \langle \mathrm{d}\hat{\omega}, D_0\bar{\eta}\rangle s\mathrm{d}s$$

$$\leq \left|\left|\left(\int_0^\infty |\mathrm{d}\hat{\omega}|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_p \left|\left|\left(\int_0^\infty |D_0\bar{\eta}|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_q$$
.

Finally, as we have identical inequalities as in the proof of theorem 4.4.2, we find from theorems 4.3.5 and 4.3.6 that

$$\left|\left|\left(\int_0^\infty |\mathrm{d}\hat{\omega}|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_p \leq C(p)||\hat{\omega}_0||_p = C(p)||C^{k,d}\omega||_p$$

and

$$\left|\left|\left(\int_0^\infty |D_0\bar{\eta}|^2 s\mathrm{d}s\right)^{1/2}\right|\right|_q \leq C(q)||\bar{\eta}_0||_q = C(q)||\eta||_q$$

Putting everything together gives us that

$$\langle \mathrm{d}\omega, \eta\rangle \leq C(p)||C^{k,d}\omega||_p||\eta||_q$$

which is sufficient, as argued above. This completes the proof of the first inequality.

The proof for $\delta\omega$ is similar. It again suffices to show for all $d > b_{k-1}$ that

$$||\delta\omega||_p \leq c(p,k)||C^{k,d}\omega||_p$$

which again follows from

$$\langle \delta\omega, \eta\rangle \leq c(p,k)||C^{k,d}\omega||_p||\eta||_q$$

for all $(k-1)$-forms η in C_0^∞. (Note that here η is a $(k-1)$-form as $\delta\omega$ is.) The argument for proving this is identical as above, where we now have to define $\bar{\eta}(x,t) = Q_t^{k-1,0,d}\eta(x)$. This also makes clear why we need to consider b_{k-1} rather than b_k. Finally, as we also have that $\frac{2}{k!}|\nabla\bar{\omega}|^2 \geq 2|\delta\bar{\omega}|^2$, which can also be proven pointwise using normal coordinates, the remainder of the proof is identical, with which the theorem is proved. □

As before, we get the reverse inequalities as a corollary.

Corollary 4.5.7. *Let $d_{k,p} = (b_k^2 - a_{k,p}^2)^{1/2} + (b_{k-1}^2 - a_{k,p}^2)^{1/2} + 2a_{k,p}$. For all $p \in (1,\infty)$ and all $k = 0, 1\ldots, n$, there exists a constant $C(k,p)$ such that for all k-forms ω and all $e \geq d_{k,p}$ it holds that*

$$||C^{k,e}\omega||_p \leq C(k,p)(e||\omega||_p + ||\mathrm{d}\omega||_p + ||\delta\omega||_p).$$

Proof. We follow the idea of the proof of corollary 4.4.3. Observe that $e \geq d_{k,p} > a_{k,p}$. We will show that $C^{k,e}(C_0^\infty)$ is dense in $L^{p,k}$ for any $1 \leq p \leq \infty$. For this, let $\eta \in L^{q,k}$ be an arbitrary element orthogonal to $C^{k,e}(C_0^\infty)$. By proposition 3.1.5 for $\omega \in C_0^\infty$ we have that $\int_0^t Q_s^{k,0,e}\omega\mathrm{d}s \in D(C^{k,e})$ and

$$C^{k,e}\int_0^t Q_s^{k,0,e}\omega\mathrm{d}s = Q_t^{k,0,e}\omega - \omega.$$

By similar reasoning as in the proof of corollary 4.4.3 (but now with the semigroup P_t^k) we find that this is orthogonal to η. Using the symmetry of $Q_t^{k,0,e}$, we find that

$$0 = \langle Q_t^{k,0,e}\omega, \eta\rangle - \langle\omega,\eta\rangle = \langle\omega, Q_t^{k,0,e}\eta - \eta\rangle.$$

As this holds for all k-forms ω in C_0^∞, we conclude that $Q_t^{k,0,e}\eta = \eta$, or $||\eta||_q = ||Q_t^{k,0,e}\eta||_q$. But as $e > a_{k,p} \geq a_k$, we find that

$$||Q_t^{k,0,e}\eta||_q \leq e^{-(e^2-a_k^2)^{1/2}t}||\eta||_q \to 0 \quad (t \to \infty).$$

This shows that $||\eta||_q = 0$, hence $\eta = 0$. We find that the only element of $L^{q,k}$ orthogonal to $C^{k,e}(C_0^\infty)$ is 0, hence $C^{k,e}(C_0^\infty)$ is dense in $L^{p,k}$.
By duality we may now write that

$$||C^{k,e}\omega||_p = \sup_{\{\eta\in C_0^\infty | ||C^{k,e}\eta||_q \leq 1\}} \langle C^{k,e}\omega, C^{k,e}\eta\rangle.$$

Now we have that

$$\begin{aligned}
\langle C^{k,e}\omega, C^{k,e}\eta\rangle &= \langle (C^{k,e})^2\omega, \eta\rangle \\
&= \langle (e^2 - L_k)\omega, \eta\rangle \\
&= e^2\langle \omega, \eta\rangle + \langle \mathrm{d}\omega, \mathrm{d}\eta\rangle + \langle \delta\omega, \delta\eta\rangle \\
&\leq e^2||\omega||_p||\eta||_q + ||\mathrm{d}\omega||_p||\mathrm{d}\eta||_q + ||\delta\omega||_p||\delta\eta||_q \\
&\leq (e||\omega||_p + ||\mathrm{d}\omega||_p + ||\delta\omega||_p)(e||\eta||_q + ||\mathrm{d}\eta||_q + ||\delta\eta||_q).
\end{aligned}$$

In the line before last we used Hölder's inequality and in the last line we simply added some positive terms.

From theorem 4.5.4 and lemma 4.5.5 it now follows that

$$\begin{aligned}
||\mathrm{d}\eta||_q + ||\delta\eta||_q &\leq C(q,k)(||C^{k,b_k}\eta||_q + ||C^{k,b_{k-1}}\eta||_q) \\
&\leq C(q,k)([(b_k^2 - a_{k,p}^2)^{1/2} + (b_{k-1}^2 - a_{k,p}^2)^{1/2}]||\eta||_q + ||C^{k,a_{k,p}}\eta||_q).
\end{aligned}$$

But then we find that

$$\begin{aligned}
e||\eta||_q + ||\mathrm{d}\eta||_q + ||\delta\eta||_q &\leq C(q,k)[e + (b_k^2 - a_{k,p}^2)^{1/2} + (b_{k-1}^2 - a_{k,p}^2)^{1/2}]||\eta||_q + ||C^{k,a_{k,p}}\eta||_q) \\
&= C(q,k)[e + d_{k,p} - 2a_{k,p}]||\eta||_q + ||C^{k,a_{k,p}}\eta||_q).
\end{aligned}$$

Now as $e \geq d_{k,p}$, we have that

$$e + d_{k,p} \leq 2e \leq 2((e^2 - a_{k,p}^2)^{1/2} + a_{k,p})$$

which rewrites to

$$e + d_{k,p} - 2a_{k,p} \leq 2(e^2 - a_{k,p}^2)^{1/2}.$$

Furthermore, by lemma 4.5.5 we obtain

$$(e^2 - a_{k,p}^2)^{1/2}||\eta||_q + ||C^{k,a_{k,p}}\eta||_q) \leq C||C^{k,e}\eta||_q.$$

Using this, we find that

$$e||\eta||_q + ||\mathrm{d}\eta||_q + ||\delta\eta||_q \leq C(q,k)||C^{k,e}\eta||_q.$$

Finally, as

$$\sup_{\{\eta\in C_0^\infty | ||C^{k,e}\eta||_q \leq 1\}} ||C^{k,e}\eta||_q \leq 1$$

we can put everything together to obtain that

$$||C^{k,e}\omega||_p \leq C(p,k)e||\omega||_p + ||\mathrm{d}\omega||_p + ||\delta\omega||_p$$

as desired. □

4.6 Concluding remarks

We will finish this chapter with some concluding remarks. The theorems that are proven in [6] are all under the assumption that the operators act on C_0^∞ functions or forms. In chapter 6 we will carefully extend (or define where necessary) all these operators to $L^{p,k}$ for $1 < p < \infty$. We will also show that these extension agree at least on C_0^∞. This allows us to apply the results of this chapter so that we obtain estimates for these operators on $L^{p,k}$ which will ultimately lead to the results that we discuss in sections 6.3 and 6.4. Before we can turn to our results however, we first need to introduce the concepts of R-sectoriality and H^∞-functional calculi for sectorial and bisectorial operators, which are the topics of the next chapter.

Chapter 5

R-sectoriality and H^∞-calculus

In this chapter we will introduce the necessary notions for our analysis of the Hodge-Dirac operator in the next chapter. We will introduce the concept of R-boundedness of a collection of operators. Thereafter we define what we mean by a sectorial and bisectorial operator, as well as an R-sectorial and R-bisectorial operator. Finally we will say what we mean by the fact that a sectorial or bisectorial operator has a bounded H^∞-functional calculus. Along with the various definitions, we also collect some results which we want to use in the next chapter.

5.1 R-boundedness

Suppose that X, Y are two Banach spaces and denote by $\mathcal{L}(X, Y)$ the set of bounded linear operators from X to Y. Suppose that $\mathcal{T} = \{T_i | i \in I\} \subset \mathcal{L}(X, Y)$. We say that $\mathcal{T}$ is uniformly bounded if the operator norms of the T_i are uniformly bounded, i.e., if

$$\sup_{i \in I} ||T_i|| < \infty$$

When working in a Hilbert space, this notion is usually fine to work with. However, in other cases we need often to generalize this concept.

Definition 5.1.1 (R-boundedness)**.** Suppose that X, Y are Banach spaces and let $\mathcal{T} \subset \mathcal{L}(X, Y)$. Let $(r_n)_n$ be a Rademacher sequence, i.e., a sequence of independent, identically distributed random variables satisfying $\mathbb{P}(r_1 = \pm 1) = \frac{1}{2}$. We say that $\mathcal{T}$ is R-bounded if there exists a constant C such that for all $n \in \mathbb{N}$ and all $T_1, \dots, T_n \in \mathcal{T}$ and $x_1, \dots, x_n \in X$ we have that

$$\mathbb{E}\left(\left|\left|\sum_{j=1}^n r_j T_j x_j\right|\right|^2\right) \leq C^2 \mathbb{E}\left(\left|\left|\sum_{j=1}^n r_j x_j\right|\right|^2\right).$$

Notice that we may replace the 2 in the above statement by any $p \in [1, \infty)$. This will only result in a different constant. See for example [18].

We now state the relationship between R-boundedness and uniform boundedness.

Proposition 5.1.2. *Suppose that X, Y are Banach spaces and let $\mathcal{T} \subset \mathcal{L}(X, Y)$. If $\mathcal{T}$ is R-bounded, then it is also uniformly bounded. If we furthermore assume that X, Y are Hilbert spaces, the converse also holds.*

Proof. First assume that $\mathcal{T}$ is R-bounded with constant C. Pick $T \in \mathcal{T}$ arbitrary. By the R-boundedness we find for all $x \in X$ that

$$||Tx||^2 = \mathbb{E}(||r_1 Tx||^2) \leq C^2 \mathbb{E}(||r_1 x||^2) = C^2 ||x||^2$$

hence $||T|| \leq C$. As this holds for arbitrary T, we find that $\sup\{||T|| : T \in \mathcal{T}\} \leq C < \infty$. We conclude that $\mathcal{T}$ is uniformly bounded.

Now suppose that X, Y are Hilbert spaces and assume that $\mathcal{T}$ is uniformly bounded, say by a constant C. Let $k \in \mathbb{N}$ and choose $T_1, \dots, T_k \in \mathcal{T}$ and $x_1, \dots, x_k \in X$. We can write that

$$\begin{aligned}
\mathbb{E}\left(\left|\left|\sum_{n=1}^k r_n T_n x_n\right|\right|^2\right) &= \mathbb{E}\left(\left\langle \sum_{n=1}^k r_n T_n x_n, \sum_{m=1}^k r_m T_m x_m \right\rangle\right) \\
&= \sum_{n,m=1}^k \mathbb{E}(r_n r_m)\langle T_n x_n, T_m x_m\rangle \\
&= \sum_{n=1}^k \langle T_n x_n, T_n x_n\rangle \\
&= \sum_{n=1}^k ||T_n x_n||^2 \\
&\leq C^2 \sum_{n=1}^k ||x_n||^2.
\end{aligned}$$

Here the last line follows by the fact that $\mathcal{T}$ is uniformly bounded. We can use a similar rewriting to find that

$$\mathbb{E}\left(\left|\left|\sum_{n=1}^k r_n T_n x_n\right|\right|^2\right) \leq C^2 \sum_{n=1}^k ||x_n||^2 = C^2 \mathbb{E}\left(\left|\left|\sum_{n=1}^k r_n x_n\right|\right|^2\right)$$

proving that $\mathcal{T}$ is R-bounded. □

Many results concerning uniform boundedness of sets of operators on Hilbert spaces turn out to generalize to Banach spaces by replacing uniform boundedness by R-boundedness.

We conclude this section by proving an elementary property about R-boundedness of certain sets of operators.

Proposition 5.1.3. *Let X, Y, Z be Banach spaces and suppose that $\mathcal{T} \subset \mathcal{L}(X, Y)$ is R-bounded and let $B \in \mathcal{L}(Y, Z)$. Then the set $\{BT : T \in \mathcal{T}\}$ is R-bounded in $\mathcal{L}(X, Z)$.*

Proof. Let C be the constant from the R-boundedness of $\mathcal{T}$. Let $N \in \mathbb{N}$ and pick $T_1, \dots, T_N \in \mathcal{T}$ and $x_1, \dots, x_N \in X$. By the linearity and boundedness of B we get that

$$\begin{aligned}
\mathbb{E}\left(\left|\left|\sum_{n=1} r_n B T_n x_n\right|\right|^2\right) &= \mathbb{E}\left(\left|\left|B \sum_{n=1} r_n T_n x_n\right|\right|^2\right) \\
&\leq ||B||^2 \mathbb{E}\left(\left|\left|\sum_{n=1} r_n T_n x_n\right|\right|^2\right) \\
&\leq C^2 ||B||^2 \mathbb{E}\left(\left|\left|\sum_{n=1} r_n x_n\right|\right|^2\right).
\end{aligned}$$

This proves that $\{BT : T \in \mathcal{T}\}$ is R-bounded. □

5.2 R-sectorial operators

Before defining R-sectorial operators, we will first introduce the notion of a sectorial operator. For $\omega \in (0,\pi)$ we define Σ^+_ω to be the open sector of the complex plane given by

$$\Sigma^+_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\}.$$

Definition 5.2.1 (Sectorial operator)**.** A linear operator L with domain $D(L)$ acting on a Banach space X is called sectorial if there exists an $\omega \in [0,\pi)$ such that $\sigma(L) \subset \overline{\Sigma^+_\omega}$ and for all $\theta \in (\omega,\pi)$ it holds that $||(\lambda - L)^{-1}|| \lesssim_\theta \frac{1}{|\lambda|}$ for all $\lambda \notin \overline{\Sigma^+_\theta}$. The infimum over all possible ω for which the above holds is denoted by $\omega(L)$, and is called the angle of sectoriality.

We begin with a simple observation about positive self-adjoint operators on some Hilbert space $\mathcal{H}$. This can be found in chapter 11 of [20].

Proposition 5.2.2. *Suppose that L is a densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume furthermore that for all $h \in D(L)$ it holds that $\langle Lh, h\rangle \geq 0$. Then L is sectorial with $\omega(L) = 0$.*

Proof. The self-adjointness of L implies that its spectrum is real. The positivity then implies that the spectrum is in fact contained in the positive real axis.

Now suppose that $\lambda \notin [0,\infty)$. For $h \in D(L)$ with $||h|| = 1$ we find by the Cauchy-Schwarz inequality that

$$||(\lambda - L)h||||h|| \geq |\langle(\lambda - L)h, h\rangle| = |\lambda - \langle Lh, h\rangle| \geq d(\lambda, [0,\infty)).$$

Now pick $\mu > 0$ and suppose that $\lambda \notin \overline{\Sigma^+_\mu}$. If $\mathrm{Re}(\lambda) \leq 0$, then $d(\lambda,[0,\infty)) = |\lambda|$. If $\mathrm{Re}(\lambda) > 0$, then $d(\lambda,[0,\infty)) = |\mathrm{Im}(\lambda)|$. Now observe that $|\mathrm{Im}(\lambda)| \geq \sin(\mu)|\lambda|$. We thus see that $d(\lambda,[0,\infty)) \geq \min\{1,\sin(\mu)\}|\lambda|$. But then we find that $||(\lambda - L)h|| \gtrsim |\lambda|||h||^{-1}$. Consequently, $||(\lambda-L)^{-1}h|| \lesssim \frac{1}{|\lambda|}||h||$. As this holds for all $h \in D(L)$ the continuity of $(\lambda-L)^{-1}$ and density of $D(L)$ give us that $||(\lambda-L)^{-1}|| \lesssim \frac{1}{|\lambda|}$. As the above holds for all $\mu > 0$, we find that $\omega(L) = 0$. □

Example. As an example of an application of the above proposition we can consider the operators $-L_k$ on $L^{2,k}$ as defined in section 4.5 in chapter 4. By proposition 4.5.1 the operator L_k defined on C_0^∞ is essentially self-adjoint on $L^{2,k}$. Hence it is closeable in $L^{2,k}$, and its closure is self-adjoint. We denote this closure again by L_k. Proposition 4.5.1 also gives us that $\langle -L_k\omega, \omega\rangle \geq 0$ for all $\omega \in C_0^\infty$ and that C_0^∞ is dense in $D(L_k)$. The positivity for arbtirary $\omega \in D(L_k)$ now follows by approximation and the continuity of the inner product. The above proposition thus applies, and we get that $-L_k$ is sectorial on $L^{2,k}$ with $\omega(-L_k) = 0$.

The key to extending the concept of sectoriality to R-sectoriality is to observe that the second part of the definition of sectoriality can also be characterized by stating that the set $\{\lambda(\lambda - L)^{-1} | \lambda \notin \overline{\Sigma^+_\theta}\}$ is bounded for all $\theta \in (\omega,\pi)$. Replacing bounded by R-bounded, we get the following definition.

Definition 5.2.3 (R-sectorial operator)**.** A linear operator L with domain $D(L)$ acting on a Banach space X is called R-sectorial if there exists an $\omega \in [0,\pi)$ such that $\sigma(L) \subset \overline{\Sigma^+_\omega}$ and for all $\theta \in (\omega,\pi)$ it holds that the set $\{\lambda(\lambda - L)^{-1} | \lambda \notin \overline{\Sigma^+_\theta}\}$ is R-bounded. The infimum of all ω for which the above holds is denoted by $\omega_R(L)$.

5.2.1 Bisectorial operators

Instead of assuming that the spectrum is contained in a single sector, we could also consider operators of which the spectrum is contained in a double sector. If we write $\Sigma_\omega^- = -\Sigma_\omega^+$, we may define for $\omega \in (0, \frac{\pi}{2})$ a double sector of angle ω by $\Sigma_\omega = \Sigma_\omega^+ \cup \Sigma_\omega^-$. We have the following definition.

Definition 5.2.4 (Bisectorial operator)**.** A linear operator L with domain $D(L)$ acting on a Banach space X is called bisectorial if there exists an $\omega \in (0, \frac{\pi}{2})$ such that $\sigma(L) \subset \overline{\Sigma_\omega}$ and for all $\theta \in (\omega, \frac{\pi}{2})$ the set $\{\lambda(\lambda - L)^{-1} : \lambda \notin \overline{\Sigma_\theta}\}$ is bounded. The infimum over all ω for which the above holds is called the angle of bisectoriality.

As extension to the above definition, we call an operator R-bisectorial if it satisfies the above definition with bounded replaced by R-bounded.

Typically, second order differential operators are sectorial, while first order differential operators are bisectorial.

5.3 H^∞-functional calculus

In this section we will give a concise introduction to the holomorphic functional calculus for sectorial and bisectorial operators. Along with the basic definitions we will collect some results that we will need for our proofs in the next chapter. For sectorial operators we refer to [20] and [17], whereas the treatment of bisectorial operators can be found in [11].

First of all, we need to introduce the spaces $H^\infty(\Omega)$ and $H_0^\infty(\Omega)$, where Ω is either Σ_ω^+ or Σ_ω for some $\omega \in (0, \pi)$, respectively $\omega \in (0, \frac{\pi}{2})$.

Definition 5.3.1. Let Ω be as above. The space $H^\infty(\Omega)$ is defined as the Banach space of all bounded analytic functions on Ω endowed with the supremum norm. The set $H_0^\infty(\Omega)$ is defined as the linear subspace of $H^\infty(\Omega)$ consisting of those $f \in H^\infty(\Omega)$ for which there exist constants $\epsilon > 0$ and $C \geq 0$ such that

$$|f(z)| \leq C \frac{|z|^\epsilon}{(1 + |z|)^{2\epsilon}}$$

for all $z \in \Omega$.

The property for a function in $H_0^\infty(\Omega)$ simply tells us that the function must have some decay near 0 and when $|z|$ tends to infinity.

We will first focus on the case where A is a sectorial operator of angle ω on a Banach space X. We wish to define a bounded operator $f(A)$ for $f \in H_0^\infty(\Sigma_\sigma^+)$, where $\sigma > \omega$. This can be done by the Dunford integral, which is inspired by the following theorem from complex analysis.

Theorem 5.3.2 (Cauchy Integral Formula)**.** *Let U be a domain in $\mathbb{C}$ and suppose that $f : U \to \mathbb{C}$ is a holomorphic function. Define $D = \{z \in \mathbb{C} : |z - z_0| \leq r\}$, where $r > 0$ and such that $D \subset U$. Then for all a in the interior of D it holds that*

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - a} \mathrm{d}z$$

where we traverse the boundary counter-clockwise.

We will thus define

$$f(A) := \frac{1}{2\pi i}\int_{\partial\Sigma_\theta^+} f(z)(z-A)^{-1}\mathrm{d}z$$

where $\theta \in (\omega, \sigma)$ may be chosen arbitrarily as a consequence of the Cauchy integral theorem. The contour-integral is oriented counter-clockwise, in the sense that the spectrum of A is on the left-hand side when traversing the path. We will show that the integral is well defined as Bochner integral in $\mathcal{L}(X)$. For this, first note that as $\theta > \omega$, we have that $\partial\Sigma_\theta \setminus \{0\} \subset \rho(A)$. This means that for almost all $z \in \Sigma_\theta^+$ the resolvent $(z-A)^{-1}$ is well-defined. Furthermore, as A is sectorial, there exists a constant M_θ such that for all $z \in \partial\Sigma_\theta^+ \setminus \{0\}$ it holds that $||z(z-A)^{-1}|| \leq M_\theta$. This, together with the decay assumption on f gives us that

$$||f(A)|| \leq \frac{M_\theta}{2\pi}\int_{\partial\Sigma_\theta^+} |f(z)|\frac{|\mathrm{d}z|}{|z|} \leq \frac{CM}{2\pi}\int_{\partial\Sigma_\theta^+} \frac{|z|^{\epsilon-1}}{(1+|z|)^{2\epsilon}}|\mathrm{d}z| < \infty.$$

For completeness we also state the following result, which is theorem 10.16 in [17].

Theorem 5.3.3. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. The mapping $H_0^\infty(\Sigma_\sigma^+) \to \mathcal{L}(X)$ defined by $f \mapsto f(A)$ is linear and multiplicative. Furthermore, it holds that*

$$\rho(A) = A(I+A)^{-2} \text{ for } \rho(z) = \frac{z}{(1+z)^2}$$

and we have the following convergence property: if $f_n, f \in H^\infty(\Sigma_\sigma^+)$ are uniformly bounded and if $f_n(z) \to f(z)$ for all $z \in \Sigma_\sigma^+$, then for all $g \in H_0^\infty(\Sigma_\sigma^+)$ we have

$$\lim_{n\to\infty}(f_n g)(A) = (fg)(A).$$

Armed with the above construction, we define what we mean by a bounded $H^\infty(\Sigma_\theta^+)$-functional calculus.

Definition 5.3.4 (Bounded H^∞-functional calculus)**.** Let A be a sectorial operator on a Banach space X. We say that A admits a bounded $H^\infty(\Sigma_\theta^+)$-functional calculus if there exists a constant $C_\theta \geq 0$ such that for all $f \in H_0^\infty(\Sigma_\theta^+)$ and all $x \in X$ it holds that

$$||f(A)x|| \leq C_\theta ||f||_\infty ||x||.$$

The infimum over all angles θ for which the above holds is denoted by $\omega_{H^\infty}^+(A)$.

A sectorial operator A now admits a bounded H^∞-functional calculus if there exists a $\theta \in (0,\pi)$ such that A admits a $H^\infty(\Sigma_\theta^+)$-functional calculus.

For a bisectorial operator A of angle ω we can make a similar construction. Indeed, the main difference will be in the contour along which we must integrate. Instead of integrating over $\partial\Sigma_\theta^+$, we will now integrate over $\partial\Sigma_\theta^+ \cup \partial\Sigma_\theta^-$, where each piece of the contour is oriented so that the spectrum is at your left-hand side when traversing the contour. We thus get a $H_0^\infty(\Sigma_\theta)$-functional calculus for A, which satisfies similar properties as discussed above for a sectorial operator, as the proofs can be easily adjusted. However, the function ρ in theorem 5.3.3 should be replaced by the function $\tilde{\rho}(z) = \frac{iz}{(i+z)^2}$. By theorem 1.4.8 in [11] we then have that $\tilde{\rho}(A) = iA(i+A)^{-2}$. Finally, the notion of a bounded H^∞-functional calculus can be defined similar to definition 5.3.4, with the obvious adjustments.

5.3.1 Extending the functional calculus to $f \in H^\infty$

The above definitions raise the question whether it is possible to extend the H^∞-functional calculus to abitrary $f \in H^\infty$, rather then restricting to holomorphic functions with sufficient decay. There are multiple ways to make this extension, which all agree. We will give an overview of how one can do this via regularization. For the rigorous details we refer to [17] and [11].

Let A be a sectorial or bisectorial operator on a Banach space X. In the remainder of the discussion we will suppress the domains Σ_ω^+, respectively Σ_ω on which we work.

It turns out that for $f \in H^\infty$ we can in general only define $f(A)$ as a closed operator on $\overline{R(A)}$. For this reason we will restrict ourselves momentarily to the case where A has dense range, in which case $\overline{R(A)} = X$. By lemma 1.2.6 in [11] we then find that A is also injective. Furthermore, by proposition 10.10[1] in [17] we find that $D(A) \cap R(A)$ is dense in $\overline{R(A)}$. We can thus consider the part of A in $\overline{R(A)}$[2] as densely defined operator. By proposition 15.2[3] in [20] we find that $A|_{\overline{R(A)}}$ is again bisectorial with the same angle as A and is furthermore injective and has dense range.

We now want define $f(A)$ for arbitrary $f \in H^\infty$ or even for the larger class of holomorphic functions with polynomial growth at 0 and ∞, which we denote by H_P^∞. To do this, we follow the ideas in [15] and [11] by regularizing the function $f \in H_P^\infty$. By theorem 1.4.10 in [11] the following definition makes sense.

Definition 5.3.5. Let $f \in H_P^\infty$ and let $\rho(z) = \frac{z}{(1+z)^2}$ and $\tilde{\rho}(z) = \frac{iz}{(i+z)^2}$ be as in the previous section. Depending on whether A is sectorial or bisectorial, we define $f(A)$ in the following way:

1. If A is sectorial, we define $f(A) = \rho(A)^{-1}(\rho f)(A)$.
2. If A is bisectorial, we define $f(A) = \tilde{\rho}(A)^{-1}(\tilde{\rho} f)(A)$.

That we can define $(\rho f)(A)$ and $(\tilde{\rho} f)(A)$ in the above definition follows from the fact that both ρf and $\tilde{\rho} f$ are H_0^∞ functions if $f \in H_P^\infty$.

Finally, if $f \in H_0^\infty$, the definition of $f(A)$ via regularization coincides with the definition via the Dunford functional calculus. This follows readily from the multiplicativity as in theorem 5.3.3.

The following theorem sheds light on the question when $f(A)$ as defined above is a bounded operator. In the theorem we do not assume that A has dense range, in which case we can only consider $f(A)$ as a closed operator on $\overline{R(A)}$ as discussed earlier. The theorem as stated is theorem 10.25 in [17]. The statement is also true for bisectorial operators with the appropriate change of notation, see for example 1.4.20 in [11].

Theorem 5.3.6. *Let A be a sectorial operator of angle $\omega(A)$ on X and fix $\sigma \in (\omega(A), \pi)$. Let $C \geq 0$ be a constant. The following statements are equivalent:*

1. *For all $f \in H_0^\infty(\Sigma_\sigma^+)$ and $x \in \overline{R(A)}$ it holds that*

$$||f(A)x|| \leq C||f||_\infty ||x||.$$

[1]This also holds for bisectorial operators, as we can apply the proposition to iA which is sectorial if A is bisectorial.

[2]$\overline{R(A)}$ is indeed again a Banach space as it is a closed subspace of X.

[3]This also holds for bisectorial operators. The injectivity and dense range can be retrieved by applying the proposition to iA, which is sectorial. The argument that resolvent bounds carry over remains true for bisectorial operators.

2. *For all $f \in H^\infty(\Sigma_\sigma^+)$ the operator $f(A)$ extends to a bounded operator on $\overline{R(A)}$ with norm bounded by*
$$||f(A)|| \leq C||f||_\infty.$$

Under these equivalent conditions we have for all $f \in H^\infty(\Sigma_\sigma^+)$ and $x \in \overline{R(A)}$ that $f(A)x \in \overline{R(A)}$ and
$$||f(A)||_{\mathcal{L}(\overline{R(A)})} \leq C||f||_\infty.$$

Let us relate the above theorem to the definition of having a bounded $H^\infty(\Sigma_\sigma^+)$-functional calculus. If A admits a bounded $H^\infty(\Sigma_\sigma^+)$-functional calculus, there exists a constant $C \geq 0$ such that for all $f \in H_0^\infty(\Sigma_\sigma^+)$ and all $x \in X$ it holds that $||f(A)x|| \leq C||f||_\infty||x||$. This means that part (1) of the above theorem is satisfied. By the above theorem, it thus implies that for any $f \in H^\infty(\Sigma_\sigma^+)$, the operator $f(A)$ can be defined and extended to a bounded operator on $\overline{R(A)}$ with norm bounded by $||f(A)|| \leq C||f||_\infty$. This is something that we will use in the future.

Removing the dense range condition

Now suppose that A is a sectorial or bisectorial operator on a Banach space X, without the additional assumption that it has dense range and suppose that A has a bounded H^∞-functional calculus. Let us furthermore assume that X is reflexive (with the idea that we are going to apply this for L^p-spaces with $1 < p < \infty$). By the above construction, for $f \in H^\infty$, we can define a closed and bounded operator $f(A)$ on $\overline{R(A)}$ by restricting A to its part in $\overline{R(A)}$. By proposition 10.9 in [17] we have that $X = N(A) \oplus \overline{R(A)}$ if A is sectorial. Observing that iA is sectorial if A is bisectorial, one sees that the direct sum decomposition also holds if A is bisectorial. We can now extend $f(A)$ to a bounded operator on X. Indeed, we simply achieve this by setting $f(A)x = 0$ for $x \in N(A)$.

5.3.2 Some additional results

We finish this section by collecting some results that we need in the next chapter. We start out by a relation between an R-bisectorial operator and its square concerning the aspect of having bounded H^∞-functional calculi. This can be found in [25].

Proposition 5.3.7. *Let $1 < p < \infty$ and let A be an R-bisectorial operator on some closed subspace of L^p. Then A^2 is R-sectorial and for each $\omega \in (0, \frac{\pi}{2})$ the following are equivalent:*

1. *A admits a bounded $H^\infty(\Sigma_\omega)$-functional calculus.*
2. *A^2 admits a bounded $H^\infty(\Sigma_{2\omega})$-functional calculus.*

We will also need the R-boundedness of certain sets of operators constructed via the H^∞-functional calculus. The following proposition is proposition 10.31 from [17].

Proposition 5.3.8. *Let A be a sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-functional calculus on a Banach space X and let $\sigma < \nu < \pi$. For all $\phi \in H_0^\infty(\Sigma_\nu)$ the set*
$$\{\phi(zA) : |\arg(z)| < \nu - \sigma\}$$
is R-bounded in $\mathcal{L}(\overline{R(A)})$.

Remark 5.3.9. If X is reflexive, the operators $\phi(zA)$ can be extended to bounded operators on X, the norm of which is bounded by the norm of $\phi(zA)$ as operator on $\overline{R(A)}$. Consequently, if X is reflexive, we may replace $\mathcal{L}(\overline{R(A)})$ by $\mathcal{L}(X)$ in the above proposition.

Chapter 6

The Hodge-Dirac operator

In this chapter we want to extend the results of sections 4 and 5 of the paper of Bakry which are discussed in sections 4.4 and 4.5. For this, we will follow the ideas in [25]. Before coming to the results, we first need to extend the operator d beyond C_0^∞ forms. We then define the Hodge-Dirac operator $\Pi = \mathrm{d} + \delta$.

Before we continue, let us give a short recapitulation of the setting in chapter 4, and also make some slight changes in notation. We consider a complete Riemannian manifold M of dimension n, together with its volume measure $\mathrm{d}x$. We furthermore pick a positive function $\rho \in C^\infty(M)$ and define the measure $\mathrm{d}m(x) = \rho(x)\mathrm{d}x$. From now on, all L^p-spaces that occur are considered with respect to this measure $m(\mathrm{d}x)$, unless otherwise stated. Remember that the exterior algebra over the tangent bundle is denoted by $\Lambda TM = \bigoplus_{k=0}^n \Lambda^k TM$. A section of $\Lambda^k TM$ is referred to as a k-form. We will write $C_0^\infty(\Lambda^k TM)$ for the set of smooth, compactly supported k-forms. We will often simply write C_0^∞ when the order of the form is understood.

For two elements $\omega, \eta \in \Lambda TM$ we write $\omega = \sum_{k=0}^n \omega^k$ and $\eta = \sum_{k=0}^n \eta^k$, where ω^k, η^k are k-forms. Their inner product (as forms) is then defined by $\omega \cdot \eta = \sum_{k=0}^n \omega^k \cdot \eta^k$. As usual, we define the length of ω as $|\omega| = \sqrt{\omega \cdot \omega}$. For $1 \leq p < \infty$ we can now define $L^p(\Lambda^k TM)$ as the closure of $C_0^\infty(\Lambda^k TM)$ with respect to the norm $||\omega||_p^p = \int_M |\omega|^p \,\mathrm{d}m(x)$. Note that these are precisely the spaces $L^{p,k}$ as defined in chapter 4. We only used that notation there to be consistent with the paper [6] which we discussed in that chapter. However, we will still write $\langle \omega, \eta \rangle = \int_M \omega \cdot \eta \,\mathrm{d}m(x)$ for the L^2 inner product.

Additionally, we define

$$C_0^\infty(\Lambda TM) = \bigoplus_{k=0}^n C_0^\infty(\Lambda^k TM), \qquad L^p(\Lambda TM) = \bigoplus_{k=0}^n L^p(\Lambda^k TM).$$

Here, the last is understood to carry the norm $||\omega||_p^p = \int_M |\omega|^p \mathrm{d}m(x) = \sum_{k=0}^n ||\omega^k||_p^p$, where we again write $\omega = \sum_{k=0}^n \omega^k$ with ω^k a k-form. By standard estimates[1] it is easy to see that $||\omega||_p$ is equivalent to $\sum_{k=0}^n ||\omega^k||_{L^p(\Lambda^k TM)}$.

Observe furthermore that $C_0^\infty(\Lambda TM) = \bigoplus_{k=0}^n C_0^\infty(\Lambda^k TM)$ is dense in $L^p(\Lambda TM)$ for $1 \leq p < \infty$. This follows easily from the fact that for each $0 \leq k \leq n$ it holds that $C_0^\infty(\Lambda^k TM)$ is dense in $L^p(\Lambda^k TM)$.

For $k = 0, 1, \ldots, n$ we define the operators $L_k = -(\mathrm{d}_{k-1}\delta_{k-1} + \delta_k \mathrm{d}_k)$ acting on C_0^∞ k-forms. These are known in the literature as Witten-Laplacians. Here δ is the adjoint of d in the L^2 sense with respect to the measure $m(\mathrm{d}x)$. By proposition 4.5.1 each L_k is essentially self-adjoint on $L^2(\Lambda^k TM)$. Consequently, the closure in $L^2(\Lambda^k TM)$ is a self-adjoint operator, which we again

[1] For $a, b > 0$ it holds that $(a^p + b^p)^{1/p} \simeq a + b$.

denote by L_k. We define P_t^k as the strongly continuous semigroup on $L^2(\Lambda^k TM)$ generated by L_k. Finally, we recall formula (4.10) on page 69 for $L_0|\omega|^2$ where ω is a C_0^∞ k-form:

$$\frac{1}{2}L_0|\omega|^2 = \omega \cdot L_k\omega + \frac{1}{k!}|\nabla\omega|^2 + Q_k(\omega,\omega).$$

We assume that for eacht $k = 1, \ldots, n$ there exists a constant $a_k \geq 0$ such that $Q_k(\omega,\omega) \geq -a_k^2|\omega|^2$ for all $\omega \in C_0^\infty(\Lambda^k TM)$. For $k = 1$ we have that $Q_1 = R = \mathrm{Ric} - \nabla\nabla(\log\rho)$. For other k, Q_k can also be expressed in terms of the curvature tensor, but in a more complex way.

Throughout the entire chapter we assume that $R(X,X) \geq 0$ for any vector field X and that $Q_k(\omega,\omega) \geq 0$ for any k-form ω. Only in the final section we will relax this assumption by considering negative lower bounds.

6.1 Extension of d and the Hodge-Dirac operator

A priori, the exterior derivative d is only defined on smooth forms. As we wish to study the action of the operator on k-forms seperately, we will denote by d_k the exterior derivative acting on smooth k-forms. By working in coordinate charts, it is easy to see that d_k maps a C_0^∞ k-form to a C_0^∞ $(k+1)$-form. This means that we can consider d_k as a densely defined operator on $L^p(\Lambda^k TM)$ for any $1 \leq p < \infty$.

Consequently, we can define the adjoint δ_k as an unbounded, closed operator on $L^q(\Lambda^{k+1}TM)$ where q is the conjugate exponent to p. We denote the domain of δ_k as operator on $L^q(\Lambda^{k+1}TM)$ by $D_q(\delta_k)$. By the closedness of δ_k, it follows that $\omega \in D_q(\delta_k)$ precisely when there exists a $\eta \in L^q(\Lambda^k TM)$ such that for all k-forms $\phi \in C_0^\infty$ it holds that

$$\langle \mathrm{d}_k\phi, \omega\rangle = \langle \phi, \eta\rangle. \tag{6.1}$$

In that case we define $\delta_k\omega = \eta$.

Proposition 2.5.6 combined with proposition 4.1.11 gives us that if $\omega \in C_0^\infty(\Lambda^{k+1}TM)$, then $\delta_k\omega$ exists, and is in fact in $C_0^\infty(\Lambda^k TM)$. In particular, we find that $\delta_k\omega \in L^q(\Lambda^k TM)$ for all $1 < q \leq \infty$. This shows that $C_0^\infty(\Lambda^{k+1}TM) \subset D_q(\delta_k)$, and consequently, δ_k is densely defined on $L^q(\Lambda^{k+1}TM)$ for $1 < q < \infty$. Furthermore, the adjoint operators acting on different $L^q(\Lambda^{k+1}TM)$ are consistent on $C_0^\infty(\Lambda^{k+1}TM)$ in the sense that they agree on C_0^∞ $(k+1)$-forms. Indeed, let $1 < q, r \leq \infty$ and pick a $(k+1)$-form $\omega \in C_0^\infty$. The above shows that $\omega \in D_q(\delta_k) \cap D_r(\delta_k)$. As $\delta_k\omega \in C_0^\infty(\Lambda^k TM)$, we can interpret condition (6.1) in both the L^q and the L^r sense. By uniqueness of $\delta_k\omega$ it follows that δ_k as operator on $L^q(\Lambda^{k+1}TM)$ and $L^r(\Lambda^{k+1}TM)$ is consistent on $C_0^\infty(\Lambda^{k+1}TM)$ as claimed.

With the above in mind, we have the following lemma.

Lemma 6.1.1. *The exterior derivative d_k acting on C_0^∞ k-forms is closable in $L^p(\Lambda^k TM)$ for $1 \leq p < \infty$.*

Proof. Let $(\omega_n)_n$ be a sequence of C_0^∞ k-forms converging to 0 in $L^p(\Lambda^k TM)$, such that $\mathrm{d}_k\omega_n$ converges to some $(k+1)$-form η in $L^p(\Lambda^{k+1}TM)$. We will prove that $\eta = 0$. For this, pick an arbitrary $(k+1)$-form $\phi \in C_0^\infty$. Denoting q the conjugate exponent to p, we have that $\phi \in D_q(\delta_k)$. But then $\langle \omega_n, \delta_k\phi\rangle = \langle \mathrm{d}_k\omega_n, \phi\rangle$ for every $n \in \mathbb{N}$. Hölder's inequality now implies that

$$\langle \eta, \phi\rangle = \lim_{n\to\infty}\langle \mathrm{d}_k\omega_n, \phi\rangle = \lim_{n\to\infty}\langle \omega_n, \delta_k\phi\rangle = \langle 0, \delta_k\phi\rangle = 0.$$

As $C_0^\infty(\Lambda^{k+1}TM)$ is dense in $L^p(\Lambda^{k+1}TM)$, we conclude that $\eta = 0$. This proves that d_k is closable in $L^p(\Lambda^k TM)$. □

This lemma allows us to define the closure of d_k in $L^p(\Lambda^k TM)$, which we will again denote by d_k. Its domain $D_p(\mathrm{d}_k)$ consists of those $\omega \in L^p(\Lambda^k TM)$ for which there exists a sequence $(\omega_n)_n$ of C_0^∞ k-forms such that $\omega_n \to \omega$ in $L^p(\Lambda^k TM)$ and $\mathrm{d}_k\omega_n$ converges in $L^p(\Lambda^{k+1}TM)$. If $\omega \in D_p(\mathrm{d}_k)$ is not in $C_0^\infty(\Lambda^k TM)$ we define $\mathrm{d}_k\omega = \lim_{n\to\infty} \mathrm{d}_k\omega_n$ in the $L^p(\Lambda^{k+1}TM)$ sense. The closedness of the graph implies that this uniquely defines $\mathrm{d}_k\omega$.

Remark 6.1.2. If $\omega \in D_p(\mathrm{d}_k)$ then $\mathrm{d}_k\omega$ is also the exterior derivative of ω in the weak sense. Indeed, let $(\omega_n)_n$ be a sequence of C_0^∞ k-forms which converges to ω in $D_p(\mathrm{d}_k)$. By Hölder's inequality, the following holds for all $\phi \in C_0^\infty(\Lambda^{k+1}TM) \subset D_q(\delta_k)$

$$\langle \mathrm{d}_k\omega, \phi\rangle = \lim_{n\to\infty} \langle \mathrm{d}_k\omega_n, \phi\rangle = \lim_{n\to\infty} \langle \omega_n, \delta_k\phi\rangle = \langle \omega, \delta_k\phi\rangle$$

which is the desired equality.

6.1.1 The Hodge-Dirac operator

As the exterior derivative d is defined on $C_0^\infty(\Lambda TM)$, it is a densely defined operator on $L^p(\Lambda TM)$ for $1 \le p < \infty$. Consequently, we can define its adjoint δ as an unbounded, closed operator on $L^q(\Lambda TM)$ where q is the conjugate exponent to p. By linearity we find that δ restricted to $L^q(\Lambda TM)$ coincided with δ_k as defined in the previous section.

We now define the Hodge-Dirac operator $\Pi = \mathrm{d} + \delta$ a priori only on $C_0^\infty(\Lambda TM)$ by setting $\Pi\omega = \mathrm{d}\omega + \delta\omega$ for $\omega \in C_0^\infty(\Lambda TM)$. On $\bigoplus_{k=0}^n C_0^\infty(\Lambda^k TM)$ it is represented by the $(n+1)\times(n+1)$-matrix

$$\Pi = \begin{pmatrix} 0 & \delta_0 & & & \\ \mathrm{d}_0 & 0 & \delta_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mathrm{d}_{n-2} & 0 & \delta_{n-1} \\ & & & \mathrm{d}_{n-1} & 0 \end{pmatrix}$$

so that formally

$$\Pi^2 = \begin{pmatrix} -L_0 & & \\ & \ddots & \\ & & -L_n \end{pmatrix}$$

is a diagonal matrix which we denote by $-L$. This follows from the fact that $\mathrm{d}^2 = \delta^2 = 0$.

We will show that the Hodge-Dirac operator is closable on $L^p(\Lambda TM)$ for $1 \le p < \infty$.

Lemma 6.1.3. Π *as defined above on* $C_0^\infty(\Lambda TM)$ *is closable in* $L^p(\Lambda TM) = \bigoplus_{k=0}^n L^p(\Lambda^k TM)$ *for* $1 \le p < \infty$.

Proof. Let $(\omega_n)_n$ be a sequence in $C_0^\infty(\Lambda TM)$ and suppose that $\omega_n \to 0$ and $\Pi\omega_n \to \eta$ both in $L^p(\Lambda TM)$. Decomposing along the direct sum we find that $\omega_n^k \to \omega^k$ in $L^p(\Lambda^k TM)$ for $0 \le k \le n$ and $\mathrm{d}_{k-1}\omega_n^{k-1} + \delta_k\omega_n^{k+1} \to \eta^k$ in $L^p(\Lambda^k TM)$ for $1 \le k \le n-1$. For $k = 0$ we find that $\delta_0\omega_n^1 \to \eta^0$ in $L^p(\Lambda^0 TM)$ and for $k = n$ we have that $\mathrm{d}_{n-1}\omega_n^{n-1} \to \eta^n$ in $L^p(\Lambda^n TM)$.

First consider $1 \le k \le n-1$, and pick $\phi \in C_0^\infty(\Lambda^k TM)$. By Hölder's inequality we find that

$$\langle \eta^k, \phi\rangle = \lim_{n\to\infty} \langle \mathrm{d}_{k-1}\omega_n^{k-1} + \delta_k\omega_n^{k+1}, \phi\rangle = \lim_{n\to\infty} \langle \omega_n^{k-1}, \delta_{k-1}\phi\rangle + \langle \omega_n^{k+1}, \mathrm{d}_k\phi\rangle = \langle 0, \delta_k\phi\rangle + \langle 0, \mathrm{d}_k\phi\rangle = 0.$$

This is justified as both ω_n^{k+1} and ϕ are C_0^∞ forms, and thus are also in $D_q(\delta_k)$, respectively $D_q(\delta_{k-1})$, with q the conjugate exponent to p. It follows that $\eta^k = 0$ by density. The cases $k = 0, n$ are treated similarly. We conclude that $\eta^k = 0$ for all k, hence $\eta = 0$. It follows that Π is closable in $L^p(\Lambda TM)$. □

The lemma allows us to consider the Hodge-Dirac operator on $L^p(\Lambda TM)$ as the closure in $L^p(\Lambda TM)$ of Π on $C_0^\infty(\Lambda TM)$. For simplicity we will denote this closure again by Π when there is no confusion. Its domain in $L^p(\Lambda TM)$ is denoted by $D_p(\Pi)$.

6.2 Extending the semigroup P_t^k to $L^p(\Lambda^k TM)$

Now that we succesfully extended the exterior derivative on k-forms, as well as the divergence to closed and densely defined operators, it remains to extend the operator $L_k = -(\delta_k \mathrm{d}_k + \mathrm{d}_{k-1}\delta_{k-1})$, at first defined only for smooth k-forms.

In proposition 4.5.1 we showed that the operator L_k is essentially self-adjoint on $L^2(\Lambda^k TM)$. Its closure is thus a self-adjoint operator on $L^2(\Lambda^k TM)$ which we will also denote by L_k. The semigroup P_t^k is now defined as the semigroup generated by L_k on $L^2(\Lambda^k TM)$. We have the following proposition, of which the proof is inspired by the proof of theorem 1.4.1 from [10].

Proposition 6.2.1. *The strongly continuous contraction semigroup P_t^k on $L^2(\Lambda^k TM)$, more precisely, its restriction to $L^p(\Lambda^k TM) \cap L^2(\Lambda^k TM)$, can be extended to a strongly continuous contraction semigroup on $L^p(\Lambda^k TM)$ for any $p \in [1, \infty)$. Furthermore, these extensions are consistent, i.e., the semigroups agree on the intersection $L^p(\Lambda^k TM) \cap L^2(\Lambda^k TM)$.*

Proof. By proposition 4.5.2 we have for all k-forms $\omega \in C_0^\infty$ and all $1 \leq p \leq \infty$ that

$$||P_t^k \omega||_p \leq ||\omega||_p.$$

As $C_0^\infty(\Lambda^k TM)$ is dense in $L^p(\Lambda^k TM)$ we find that each P_t^k extends to a contraction on $L^p(\Lambda^k TM)$ for any $1 \leq p \leq \infty$. This also implies the consistency on $L^p(\Lambda^k TM) \cap L^2(\Lambda^k TM)$. The semigroup property follows easily by approximation and the fact that each P_t^k is bounded, hence continuous on $L^p(\Lambda^k TM)$. It remains to show strong continuity for $1 \leq p < \infty$.

First consider the case $p = 1$. Pick a k-form $\omega \in C_0^\infty$, and let the support of ω be contained in $E \subset M$, where $|E| < \infty$. By definition we have that $||P_t^k \omega||_1 = \int_M |P_t^k \omega| \, \mathrm{d}m(x)$. By the Cauchy-Schwarz inequality and strong continuity on $L^2(\Lambda^k TM)$ we find that

$$\lim_{t\to 0} \int_M |P_t^k \omega| 1_E \, \mathrm{d}m(x) = \lim_{t\to\infty} \langle |P_t^k \omega|, 1_E \rangle = \langle |\omega|, 1_E \rangle = ||\omega||_1.$$

On the other hand we have that $||P_t^k \omega||_1 \leq ||\omega||_1$. Combining the two, we find that

$$\lim_{t\to 0} \int_M 1_{M\setminus E} |P_t^k \omega| \, \mathrm{d}m(x) = 0.$$

But then

$$\begin{aligned}
\lim_{t\to 0} ||P_t^k \omega - \omega||_1 &= \lim_{t\to 0} \int_M |P_t^k \omega - \omega| 1_E \, \mathrm{d}m(x) + \lim_{t\to 0} \int_M |P_t^k \omega - \omega| 1_{M\setminus E} \, \mathrm{d}m(x) \\
&\leq \lim_{t\to 0} \int_M |P_t^k \omega - \omega| 1_E \, \mathrm{d}m(x) + \lim_{t\to 0} \int_M |P_t^k \omega| 1_{M\setminus E} \, \mathrm{d}m(x) + \lim_{t\to 0} \int_M |\omega| 1_{M\setminus E} \, \mathrm{d}m(x) \\
&= \lim_{t\to 0} \int_M |P_t^k \omega - \omega| 1_E \, \mathrm{d}m(x) \\
&\leq \lim_{t\to 0} ||P_t^k \omega - \omega||_2 |E|^{1/2} \\
&= 0
\end{aligned}$$

as $|E| < \infty$ and the fact that P_t^k is strongly continuous on $L^2(\Lambda^k TM)$. The second equality follows from the result above and the fact that ω has support inside E. Now pick $\omega \in L^1(\Lambda^k TM)$ arbitrary, and approximate it with C_0^∞ k-forms ω_n. The estimate

$$\begin{aligned} ||P_t^k\omega - \omega||_1 &\leq ||P_t^k\omega - P_t^k\omega_n||_1 + ||P_t^k\omega_n - \omega_n||_1 + ||\omega_n - \omega||_1 \\ &\leq ||\omega - \omega_n||_1 + ||P_t^k\omega_n - \omega_n||_1 + ||\omega_n - \omega||_1 \end{aligned}$$

shows that strong continuity also holds for ω.

Now let $1 < p < 2$. By interpolation we find for a k-form $\omega \in C_0^\infty$ that

$$||P_t^k\omega - \omega||_p \leq ||P_t^k\omega - \omega||_1^{2/p-1}||P_t^k\omega - \omega||_2^{2-2/p}$$

where the upper bound goes to 0 by the strong continuity for $p = 1, 2$. By a similar approximation as for the case $p = 1$ we find the strong continuity for all $\omega \in L^p(\Lambda^k TM)$.

Now pick $p \in (2, \infty)$. Denote by $q \in (1, 2)$ the conjugate exponent and let $\omega \in C_0^\infty(\Lambda^k TM)$. Then for all $\eta \in C_0^\infty(\Lambda^k TM)$ it holds by the symmetry of P_t^k and Hölder's inequality that

$$|\langle P_t^k\omega - \omega, \eta\rangle| = |\langle \omega, P_t^k\eta - \eta\rangle| \leq ||\omega||_p||P_t^k\eta - \eta||_q$$

which goes to 0 by the strong continuity for $q \in (1, 2)$. The use of the symmetry of P_t^k in the above argument is justified as we can first interpret the equality in the $L^2(\Lambda^k TM)$ sense as $\omega, \eta \in C_0^\infty(\Lambda^k TM)$. The consistency of the P_t^k then also allows us to see the equality in the sense of the L^p/L^q-duality. By approximation the above holds even for all $\eta \in L^q(\Lambda^k TM)$. This shows that $\lim_{t\to 0} P_t^k\omega = \omega$ in the weak-* topology. As $L^q(\Lambda^k TM)$ is reflexive, we also get weak continuity. By proposition 3.1.8 this implies strong continuity. □

Remark 6.2.2. When there is no confusion, we will use the same notation for the semigroups and generators on the different $L^p(\Lambda^k TM)$. For the generator L_k on $L^p(\Lambda^k TM)$, we denote its domain by $D_p(L_k)$. Observe that if $\omega \in D_p(L_k) \cap D_r(L_k)$ then $L_k\omega$ in the sense of $L^p(\Lambda^k TM)$ coincides with the one in $L^r(\Lambda^k TM)$. Indeed, the sequence $\frac{1}{t}(P_t^k\omega - \omega)$ converges in both $L^p(\Lambda^k TM)$ and $L^r(\Lambda^k TM)$ and the consistency of P_t^k implies that in both spaces this is the same sequence. By switching to a subsequence along which almost everywhere convergence holds, we find that the limits in $L^p(\Lambda^k TM)$ and $L^r(\Lambda^k TM)$ must coincide.

Remark 6.2.3. One can show that $C_0^\infty(\Lambda^k TM)$ is contained in $D_p(L_k)$ for any $1 < p < \infty$. To do this, we follow the same idea as in the proof of lemma 4.8 in [25]. Pick a k-form $\omega \in C_0^\infty$. Then $\omega \in D_2(L_k) \cap L^p(\Lambda^k TM)$. As $L^p(\Lambda^k TM)$ is reflexive, in order to show that $\omega \in D_p(L_k)$ it suffices to show that $\limsup_{t\downarrow 0} \frac{1}{t}||P_t^k\omega - \omega||_p < \infty$. By proposition 3.1.5 we have that

$$\frac{1}{t}(P_t^k\omega - \omega) = \frac{1}{t}\int_0^t P_s^k L_k\omega \,\mathrm{d}s$$

in $L^2(\Lambda^k TM)$. However, as $L_k\omega \in C_0^\infty(\Lambda^k TM)$ as both d and δ map $C_0^\infty(\Lambda TM)$ to $C_0^\infty(\Lambda TM)$, we actually have that $L_k\omega \in L^2(\Lambda^k TM) \cap L^p(\Lambda^k TM)$. This means that we can interpret the right hand side as a Bochner integral in $L^p(\Lambda^k TM)$ and consequently, we find the following estimate

$$\begin{aligned} \frac{1}{t}||P_t^k\omega - \omega||_p &\leq \frac{1}{t}\int_0^t ||P_s^k L_k\omega||_p \,\mathrm{d}s \\ &\leq \frac{1}{t}\int_0^t ||L_k\omega||_p \,\mathrm{d}s \\ &= ||L_k\omega||_p. \end{aligned}$$

But then $\limsup_{t\downarrow 0} \frac{1}{t}||P_t^k\omega - \omega||_p \leq ||L_k\omega||_p < \infty$, which proves the claim.

6.2.1 R-sectoriality of $-L_k$

Now that we have defined $-L_k$ also on $L^p(\Lambda^k TM)$, we will prove that it is R-sectorial. It turns out that this follows by applying some general theorems consecutively. We will start out by showing that the semigroup $(P_t^k)_t$ is analytic on $L^p(\Lambda^k TM)$ for $1 < p < \infty$. For this we can use the following theorem by Stein, which is theorem 1 on p.67 in [33].

Theorem 6.2.4. *Let (Ω, μ) be a σ-finite measure space and suppose that that the semigroup $(T_t)_t$ satisfies*

1. $||T_t f||_p \leq ||f||_p$ *for all* $1 \leq p \leq \infty$.
2. *For all $t \geq 0$, T_t is self-adjoint on $L^2(\Omega)$.*

Let $1 < p < \infty$. Then the map $t \mapsto T_t$ has an "analytic continuation", i.e., it extends to an analytic L_p-operator-valued function $z \mapsto T_z$, defined in the sector Σ_ω^+ where $\omega = \frac{\pi}{2}(1 - |2/p - 1|)$. On this sector it also holds that $||T_z f||_p \leq ||f||_p$.

We can now deduce that L_k generates a strongly continuous analytic contraction semigroup on $L^p(\Lambda^k TM)$ for $1 < p < \infty$.

Corollary 6.2.5. *Let $1 < p < \infty$. For all $k = 0, 1, \ldots, n$, the operator L_k generates a strongly continuous analytic contraction semigroup on $L^p(\Lambda^k TM)$ of angle $\frac{\pi}{2}(1 - |2/p - 1|)$.*

Proof. Fix $1 < p < \infty$. By proposition 6.2.1 the semigroup P_t^k generated by L_k on $L^q(\Lambda^k TM)$ is a contraction for all $1 \leq q \leq \infty$. Furthermore, proposition 4.5.1 gives us that L_k is self-adjoint on $L^2(\Lambda^k TM)$, which implies that the semigroup P_t^k on $L^2(\Lambda^k TM)$ is self-adjoint. By the previous theorem we conclude that the map $t \mapsto P_t^k$ extends to an analytic L^p-operator-valued function $z \mapsto P_z^k$ defined in the sector Σ_ω with $\omega = \frac{\pi}{2}(1 - |2/p - 1|)$. Furthermore we get that on this sector, P_z^k is a contraction.

It remains to argue that it defines a strongly continuous semigroup. Investigating the proof of theorem 6.2.4, we see that on $L^2(\Lambda^k TM)$ P_z^k is defined via the spectral theorem as e^{zL_k}, thus defining a strongly continuous semigroup. By a similar approach as in the proof of proposition 6.2.1 one can show that P_z^k also defines a strongly continuous semigroup on $L^p(\Lambda^k TM)$ which completes the proof. □

In order to retrieve R-sectoriality of $-L_k$ from the above corollary, we need to combine two more results. The first one can be found on p.217 in [35], the second one is theorem 4.2 in [36].[2]

Proposition 6.2.6. *If A generates an analytic contraction semigroup on L^q for some $1 < q < \infty$, then A has maximal L^p-regularity.*

Theorem 6.2.7. *Let A be the generator of a bounded analytic semigroup in a UMD-space X. Then A has maximal L^p-regularity for one (all) $p \in (1, \infty)$ on $\mathbb{R}_+$ if and only if one of the following equivalent conditions is fulfilled.*

1. *$\{\lambda R(\lambda, A) : \lambda \in \Sigma_\sigma\}$ is R-bounded for some $\sigma > \frac{\pi}{2}$.*
2. *$\{T_z : z \in \Sigma_\delta\}$ is R-bounded for some $\delta > 0$.*

[2]The concepts of maximal L^p-regularity and UMD spaces are of no real importance for us. The only thing we use is the well-known fact that L^q for $1 < q < \infty$ has UMD.

We will now derive the R-sectoriality of $-L_k$ from these two results.

Proposition 6.2.8 (R-sectoriality of $-L_k$)**.** *Let $1 < p < \infty$. The operator $-L_k$ is R-sectorial on $L^p(\Lambda^k TM)$ with angle $\omega < \frac{\pi}{2}$.*

Proof. Fix $1 < p < \infty$. By corollary 6.2.5 L_k generates a strongly continuous analytic contraction semigroup on $L^p(\Lambda^k TM)$. Combining the above proposition and theorem, and noting that L^p has UMD, we find that there exists a $\sigma > \frac{\pi}{2}$ such that the set $\{\lambda R(\lambda, L_k) : \lambda \in \Sigma_\sigma\}$ is R-bounded. Note that this claim also entails that the spectrum of L_k is contained in $\mathbb{C} \setminus \Sigma_\sigma$. But then we find that the spectrum of $-L_k$ is contained in $\overline{\Sigma_{\pi-\sigma}}$ and that the set $\{\lambda R(\lambda, -L_k) : \lambda \notin \overline{\Sigma_{\pi-\sigma}}\}$ is R-bounded. This shows that $-L_k$ is R-sectorial on $L^p(\Lambda^k TM)$ with angle $\omega = \pi - \sigma < \frac{\pi}{2}$. □

6.3 Boundedness of the Riesz transform

In this section we will extend theorem 4.5.4 and prove properties of the Hodge-Dirac operator, where we follow the ideas in [25]. We assume that M is a complete Riemannian manifold of dimension n. We define the quadratic forms Q_k acting on k-forms by the following formula

$$\frac{1}{2} L_0 |\omega|^2 = \omega \cdot L_k \omega + \frac{1}{k!} |\nabla \omega|^2 + Q_k(\omega, \omega) \tag{6.2}$$

which is formula (4.10) on 69. These Q_k involve the curvature of M and we assume that $Q_k \geq 0$ for $k = 1, \dots n$.

Before we are able to do this, we will first define $(-L_k)^{1/2}$ as an operator on $L^p(\Lambda^k TM)$ for $1 \leq p < \infty$. As $-L_k$ is sectorial of angle less than $\frac{\pi}{2}$, the following definition is justified by proposition 3.8.2 in [1], and coincides with the work done in [15].

Definition 6.3.1. For $1 \leq p < \infty$ we define the operator $(-L_k)^{1/2}$ on $L^p(\Lambda^k TM)$ as the closed operator with $D_p(-L_k)$ as a core, on which it is defined as

$$(-L_k)^{1/2} = \frac{1}{\pi} \int_0^\infty -s^{-1/2} (s - L_k)^{-1} L_k \, \mathrm{d}s.$$

The following formal computation shows that on $L^2(\Lambda^k TM)$ the above coincides with the square root as defined via the spectral theorem. Remember that $-L_k = \int_0^\infty \lambda \, \mathrm{d}E^k_\lambda$.

$$\begin{aligned}
\frac{1}{\pi} \int_0^\infty -s^{-1/2} (s - L_k)^{-1} L_k \, \mathrm{d}s &= \frac{1}{\pi} \int_0^\infty s^{-1/2} \int_0^\infty (s + \lambda)^{-1} \lambda \, \mathrm{d}E^k_\lambda \mathrm{d}s \\
&= \frac{1}{\pi} \int_0^\infty \int_0^\infty s^{-1/2} (s + \lambda)^{-1} \lambda \, \mathrm{d}s \mathrm{d}E^k_\lambda \\
&= \frac{1}{\pi} \int_0^\infty \left[\frac{2}{\sqrt{\lambda}} \tan^{-1}(\sqrt{s}/\sqrt{\lambda}) \right]_{s=0}^\infty \mathrm{d}E^k_\lambda \\
&= \int_0^\infty \lambda^{1/2} \, \mathrm{d}E^k_\lambda.
\end{aligned}$$

We will now prove that $C_0^\infty(\Lambda^k TM)$ is a core for $D_p((-L_k)^{1/2})$ if $1 < p < \infty$.

Lemma 6.3.2. *Let $1 < p < \infty$. For any $k = 0, 1, \dots, n$, $C_0^\infty(\Lambda^k TM)$ is dense in $D_p((-L_k)^{1/2})$.*

Proof. Pick $\omega \in D_p((-L_k)^{1/2})$ arbitrary. By proposition 3.8.2 in [1] we have that $\omega \in D_p((I-L_k)^{1/2})$. But then we can consider $(I-L)^{1/2}\omega \in L^p(\Lambda^k TM)$. From the proof of corollaries 4.4.3 and 4.5.7 we get that there exists a sequence $(\omega_n)_n$ of C_0^∞ k-forms such that $(I-L_k)^{1/2}\omega_n \to (I-L_k)^{1/2}\omega$ in $L^p(\Lambda^k TM)$. By lemma's 4.4.1 and 4.5.5 we then find that

$$||\omega_n - \omega||_{D_p((-L_k)^{1/2})} = ||\omega_n - \omega||_p + ||(-L_k)^{1/2}(\omega_n - \omega)||_p \lesssim ||(I-L_k)^{1/2}(\omega_n - \omega)||_p$$

By the choice of the sequence ω_n we find that the last term goes to 0, hence we find that ω_n converges to ω in $D_p((-L_k)^{1/2})$, proving the lemma. □

We conclude our discussion of $(-L_k)^{1/2}$ by showing that the different operators defined on the different $L^p(\Lambda^k TM)$ agree at least on $D_p(L_k) \cap D_r(L_k)$, where $1 \le p, r < \infty$. For this, first observe that P_t^k is consistent on $L^p(\Lambda^k TM) \cap L^r(\Lambda^k TM)$ by proposition 6.2.1. Now observe that $(s-L_k)^{-1} = \int_0^\infty e^{-st} P_t^k \mathrm{d}t$. As the integrand is consistent on $L^p(\Lambda^k TM) \cap L^r(\Lambda^k TM)$, we find that $(s-L_k)^{-1}$ is as well. Indeed, although the convergence of the integral must be considered in $L^p(\Lambda^k TM)$ and $L^r(\Lambda^k TM)$ respectively, switching to a subsequence so that we get almost everywhere convergence shows that the limits must agree. Now by remark 6.2.2 L_k is consistent on $D_p(L_k) \cap D_r(L_k)$ and for $\omega \in D_p(L_k) \cap D_r(L_k)$ it holds that $L_k\omega \in L^p(\Lambda^k TM) \cap L^r(\Lambda^k TM)$. We find that the integrand in the definition of $(-L_k)^{1/2}$ is consistent on $D_p(L_k) \cap D_r(L_k)$. In the same way as above, the integral must yield the same result in both $L^p(\Lambda^k TM)$ and $L^r(\Lambda^k TM)$, which shows that $(-L_k)^{1/2}$ is consistent on $D_p(L_k) \cap D_r(\Lambda^k TM)$. If $p, r > 1$, then $C_0^\infty(\Lambda^k TM) \subset D_p(L_k) \cap D_r(L_k)$ and we also find consistency on $C_0^\infty(\Lambda^k TM)$.

The main reason why we want this consistency on C_0^∞ is that now the results in the paper of Bakry as discussed in chapter 4 may be applied to the operator $(-L_k)^{1/2}$ considered as an operator on $L^p(\Lambda^k TM)$ for $1 < p < \infty$. Whenever we apply results from Bakry, this is what we keep in mind.

We are now in a position to prove the following extension of theorem 4.5.4, together with corollary 4.5.7 where we have homogeneous estimates as we assume that quadratic forms Q_k are nonnegative. Here by $\Pi_k := \mathrm{d}_k + \delta_{k-1}$ we mean the restriction of Π to $L^p(\Lambda^k TM)$ by noting that we can consider an element $\omega \in L^p(\Lambda^k TM)$ as $(0, \dots, 0, \omega, 0, \dots, 0) \in L^p(\Lambda TM)$.

Theorem 6.3.3 (Boundedness of the Riesz transform)**.** *Suppose that M is a complete Riemannian manifold of dimension n and assume that Q_l as in formula* (6.2) *is nonnegative for all $l \in \{1, \dots, n\}$. Let $1 < p < \infty$ and suppose that $0 \le k \le n$ $(= \dim M)$. Then $D_p((-L_k)^{1/2}) = D_p(\mathrm{d}_k + \delta_{k-1})$ and for all ω in the common domain it holds that*

$$||(\mathrm{d}_k + \delta_{k-1})\omega||_p \simeq_{p,k} ||(-L_k)^{1/2}\omega||_p.$$

Proof. We will first show that $D_p((-L_k)^{1/2}) \subset D_p(\mathrm{d}_k + \delta_{k-1})$ together with the estimate

$$||(\mathrm{d}_k + \delta_{k-1})\omega||_p \lesssim_{p,k} ||(-L_k)^{1/2}\omega||_p.$$

We first prove that $D_p((-L_k)^{1/2}) \subset D_p(\mathrm{d}_k + \delta_{k-1})$. For this, pick $\omega \in D_p((-L_k)^{1/2})$ arbitrary. As $C_0^\infty(\Lambda^k TM)$ is dense in $D_p((-L_k)^{1/2})$ by lemma 6.3.2, we can find a sequence ω_n of smooth k-forms converging to ω in $D_p((-L_k)^{1/2})$. By theorem 4.5.4 we then find for n, m that

$$\begin{aligned}
&||\omega_n - \omega_m||_p + ||(\mathrm{d}_k + \delta_{k-1})\omega_n - (\mathrm{d}_k + \delta_{k-1})\omega_m||_p \\
&\lesssim ||\omega_n - \omega_m||_p + ||\mathrm{d}_k\omega_n - \mathrm{d}_k\omega_m||_p + ||\delta_{k-1}\omega_n + \delta_{k-1}\omega_m||_p \\
&\lesssim ||\omega_n - \omega_m||_p + ||(-L_k)^{1/2}\omega_n - (-L_k)^{1/2}\omega_m||_p
\end{aligned}$$

which shows that $(\omega_n)_n$ is Cauchy in $D_p(\mathrm{d}_k + \delta_{k-1})$. Here, the first inequality is justified by the fact that $\omega_n, \omega_m \in C_0^\infty(\Lambda^k TM)$, in which case $(\mathrm{d}_k+\delta_{k-1})\omega_n = \mathrm{d}_k\omega_n+\delta_{k-1}\omega_n$. By the closedness of $\mathrm{d}_k + \delta_{k-1}$ we then find that this sequence converges to some $\eta \in D_p(\mathrm{d}_k + \delta_{k-1})$.

We now show that $\omega = \eta$. For this, observe that both $D_p((-L_k)^{1/2})$ and $D_p(\mathrm{d}_k + \delta_{k-1})$ are continuously embedded into $L^p(\Lambda^k TM)$. But then we find in $L^p(\Lambda^k TM)$ that $\omega_n \to \omega$ and $\omega_n \to \eta$. Consequently, it must hold that $\omega = \eta$. This proves that $\omega \in D_p(\mathrm{d}_k + \delta_{k-1})$, from which we conclude that $D_p((-L_k)^{1/2}) \subset D_p(\mathrm{d}_k + \delta_{k-1})$.

To prove the given estimate, by theorem 4.5.4 it holds for all n that

$$||(\mathrm{d}_k + \delta_{k-1})\omega_n||_p \leq ||\mathrm{d}_k\omega_n||_p + ||\delta_{k-1}\omega_n||_p \leq C(p,k)||(-L_k)^{1/2}\omega_n||_p.$$

By the continuity of the norm, and the fact that $\omega_n \to \omega$ both in $D_p((-L_k)^{1/2})$ and $D_p(\mathrm{d}_k+\delta_{k-1})$ we find that

$$||(\mathrm{d}_k + \delta_{k-1})\omega||_p \leq C(p,k)||(-L_k)^{1/2}\omega||_p$$

which concludes the first part of the proof.

The reverse inclusion and estimate may be proven in a similar manner. For this, one uses that $C_0^\infty(\Lambda^k TM)$ is dense in $D_p(\mathrm{d}_k+\delta_{k-1})$ as it is the closure of the operator defined on $C_0^\infty(\Lambda^k TM)$ by lemma 6.1.3. One furthermore uses the estimate in corollary 4.5.7 which holds with $e = 0$ by our assumption on the lower bounds a_k. Finally, the discussion of the norm on $L^p(\Lambda TM)$ at the beginning of the chapter shows that $||\mathrm{d}_k\omega||_p + ||\delta_{k-1}\omega||_p \lesssim_p ||(\mathrm{d}_k + \delta_{k-1})\omega||_p$ for all $\omega \in C_0^\infty(\Lambda^k TM)$. □

6.3.1 R-gradient bounds

We now wish to define for $1 < p < \infty$ the operator $\mathrm{d}_k/(-L_k)^{1/2} : R_p((-L_k)^{1/2}) \to R_p(\mathrm{d}_k)$ which maps $(-L_k)^{1/2}\omega$ to $\mathrm{d}_k\omega$ for any $\omega \in D_p((-L_k)^{1/2})$. Here, by R_p we mean the range of the operator considered on $L^p(\Lambda^k TM)$. The following lemma shows that this operator is well-defined and bounded.

Lemma 6.3.4. *The operator* $\mathrm{d}_k/(-L_k)^{1/2}$ *as defined above is well-defined and bounded.*

Proof. We start by showing that the operator is well-defined. For this, suppose that $(-L_k)^{1/2}\omega = (-L_k)^{1/2}\eta$ in $L^p(\Lambda^k TM)$. By the estimate in theorem 6.3.3 we find that

$$||\mathrm{d}_k\omega - \mathrm{d}_k\eta||_p \leq C||(-L_k)^{1/2}\omega - (-L_k)^{1/2}\eta||_p = 0$$

Hence $\mathrm{d}_k\omega = \mathrm{d}_k\eta$ in $L^p(\Lambda^k TM)$, which shows that the operator is well-defined. The boundedness follows by the same estimate. □

In the same way we can define the bounded operator $\delta_{k-1}/(-L_k)^{1/2} : R_p((-L_k)^{1/2}) \to R_p(\delta_{k-1})$.

Before we can continue, we first need the following lemma.

Lemma 6.3.5. *Let* $1 < p < \infty$*. Then* $D_p((-L_k)^{1/2}) \subset D_p(\mathrm{d}_k) \cap D_p(\delta_{k-1})$.

Proof. We only proof that $D_p((-L_k)^{1/2}) \subset D_p(\mathrm{d}_k)$. The case for δ_{k-1} is proved similarly.

In order to do this, pick $\omega \in D_p((-L_k)^{1/2})$ arbitrary. As $C_0^\infty(\Lambda^k TM)$ is dense in $D_p(\sqrt{-L_k})$ by lemma 6.3.2, we can find a sequence ω_n of smooth k-forms converging to ω in $D_p((-L_k)^{1/2})$. By theorem 4.5.4 we then find for n, m that

$$||\omega_n - \omega_m||_p + ||\mathrm{d}_k\omega_n - \mathrm{d}_k\omega_m||_p \lesssim ||\omega_n - \omega_m||_p + ||(-L_k)^{1/2}\omega_n - (-L_k)^{1/2}\omega_m||_p$$

which shows that $(\omega_n)_n$ is Cauchy in $D_p(\mathrm{d}_k)$. By the closedness of d_k we then find that this sequence converges to $\eta \in D_p(\mathrm{d}_k)$.

We now show that $\omega = \eta$. For this, observe that both $D_p((-L_k)^{1/2})$ and $D_p(\mathrm{d}_k)$ are continuously embedded into $L^p(\Lambda^k TM)$. But then we find that in $L^p(\Lambda^k TM)$ we have that $\omega_n \to \omega$ and $\omega_n \to \eta$. Consequently, it must hold that $\omega = \eta$. This proves that $\omega \in D_p(\mathrm{d}_k)$, from which we conclude that $D_p((-L_k)^{1/2}) \subset D_p(\mathrm{d}_k)$. □

A consequence of this lemma is that the operators $\mathrm{d}_k(I - t^2 L_k)^{-1}$ and $\delta_{k-1}(I - t^2 L_k)^{-1}$ are well-defined. Indeed, as the operator L_k defines a contraction semigroup on $L^p(\Lambda^k TM)$, we have that $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ is contained in the resolvent set of L_k by proposition 3.2.2. Hence, for all $t \neq 0$, the operator $(I - t^2 L_k)^{-1}$ is well-defined and bounded and maps $L^p(\Lambda^k TM)$ into $D_p(L_k)$. As $D_p(L_k) = D_p(-L_k) \subset D_p((-L_k)^{1/2})$, we find that $D_p(L_k) \subset D_p(\mathrm{d}_k) \cap D_p(\delta_{k-1})$. This shows that the operators $\mathrm{d}_k(I - t^2 L_k)^{-1}$ and $\delta_{k-1}(I - t^2 L_k)^{-1}$ are well-defined. We will use this in the following proposition.

Proposition 6.3.6 (R-gradient bounds). *For any $0 \leq k \leq n (= \dim(M))$ the families of operators*

$$\{t\mathrm{d}_k(I - t^2 L_k)^{-1} : t > 0\}$$

and

$$\{t\delta_{k-1}(I - t^2 L_k)^{-1} : t > 0\}$$

are R-bounded in $\mathcal{L}(L^p(\Lambda^k TM), L^p(\Lambda^{k+1} TM))$, respectively $\mathcal{L}(L^p(\Lambda^k TM), L^p(\Lambda^{k-1} TM)$ for $1 < p < \infty$.

Proof. We will only prove that the first set is R-bounded, the other one following in exactly the same way.

Note that for $t > 0$ we can write that

$$t\mathrm{d}_k(I - t^2 L_k)^{-1} = (\mathrm{d}_k/(-L_k)^{1/2})((-t^2 L_k)^{1/2}(I - t^2 L_k)^{-1}) = (\mathrm{d}_k/(-L_k)^{1/2})(\psi(-t^2 L_k))$$

where $\psi(z) = \frac{\sqrt{z}}{1+z}$.[3] Observe that $\psi \in H_0^\infty(\Sigma_\theta^+)$ for any $\theta \in (0, \frac{\pi}{2})$. By proposition 5.3.8, or more precisely the remark thereafter, we find that the set $\{\psi(-t^2 L_k) : t > 0\}$ is R-bounded in $\mathcal{L}(L^p(\Lambda^k TM), L^p(\Lambda^k TM))$. As the operator $\mathrm{d}_k/(-L_k)^{1/2}$ is bounded, we conclude by proposition 5.1.3 that the set $\{(\mathrm{d}_k/(-L_k)^{1/2})(\psi(-t^2 L_k)) : t > 0\}$ is R-bounded in $\mathcal{L}(L^p(\Lambda^k TM), L^p(\Lambda^{k+1} TM))$, which concludes the proof. □

6.3.2 The Hodge-Dirac operator

We are now able to commence our analysis of the Hodge-Dirac operator $\Pi = \mathrm{d} + \delta$. We will start by showing that Π is R-bisectorial. For this, we first need a lemma.

Lemma 6.3.7. *Let $1 \leq p < \infty$. For any $k = 0, 1, \ldots, n$ and $t \in \mathbb{R}$ the following identities hold on $D_p(\mathrm{d}_k)$ and $D_p(\delta_k)$ respectivily*

$$(I - t^2 L_{k+1})^{-1}\mathrm{d}_k = \mathrm{d}_k(I - t^2 L_k)^{-1}$$

and

$$(I - t^2 L_k)^{-1}\delta_k = \delta_k(I - t^2 L_{k+1})^{-1}.$$

[3] To see this, one could turn to the multiplicativity of the functional calculus for holomorphic functions with polynomial limits at 0 and infinity.

Proof. We will only prove the first identity, the second following in a similar manner.

If $t = 0$ the statement is trivial, so we will assume that $t \neq 0$. For a k-form $\omega \in C_0^\infty$ we have that $P_t^{k+1}\mathrm{d}_k\omega = \mathrm{d}_k P_t^k\omega$ by proposition 4.5.2. Here, the right-hand side is well-defined as $P_t^k\omega \in D_p(L_k) \subset D_p(\mathrm{d}_k)$. Now pick $\omega \in D_p(\mathrm{d}_k)$ and let $\omega_n \in C_0^\infty(\Lambda^k TM)$ be a sequence converging to $\omega \in D_p(\mathrm{d}_k)$. Such a sequence exists by the definition of d_k as a closed operator. In that case we have that $\omega_n \to \omega$ and $\mathrm{d}_k\omega_n \to \mathrm{d}_k\omega$ in $L^p(\Lambda^k TM)$ respectively $L^p(\Lambda^{k+1}TM)$. The boundedness of P_t^k and P_t^{k+1} then implies that $P_t^k\omega_n \to P_t^k\omega$ and $P_t^{k+1}\mathrm{d}_k\omega_n \to P_t^{k+1}\mathrm{d}_k\omega$ in $L^p(\Lambda^k TM)$ respectively $L^p(\Lambda^{k+1}TM)$. As $P_t^{k+1}\mathrm{d}_k\omega_n = \mathrm{d}_k P_t^k\omega_n$ for every n, and as the left hand side converges, we obtain that $\mathrm{d}_k P_t^k\omega_n$ converges in $L^p(\Lambda^{k+1}TM)$. The closedness of d_k shows that $P_t^k\omega \in D_p(\mathrm{d}_k)$ and that $P_t^{k+1}\mathrm{d}_k\omega = \mathrm{d}_k P_t^k\omega$.

If we now take Laplace transforms on both sides and plug in the point $\lambda = t^{-2} \in \rho(L_k) \cap \rho(L_{k+1})$ (as $t^{-2} > 0$) we get by proposition 3.2.2 that

$$(t^{-2} - L_{k+1})^{-1}\mathrm{d}_k\omega = \mathrm{d}_k(t^{-2} - L_k)^{-1}\omega$$

which gives us that

$$t^2(I - t^2L_{k+1})^{-1}\mathrm{d}_k\omega = t^2\mathrm{d}_k(I - t^2L_k)^{-1}\omega$$

from which one deduces the desired identity. □

Theorem 6.3.8 (R-bisectoriality of Π)**.** *Suppose that M is a complete Riemannian manifold of dimension n and assume that Q_k as in formula* (6.2) *is nonnegative for all $k \in \{1, \dots, n\}$. Let $1 < p < \infty$. Then the Hodge-Dirac operator Π is R-bisectorial on $L^p(\Lambda TM)$.*

Proof. We will start by showing that the set $\{it : t \neq 0\}$ is contained in the resolvent set of Π. We will do this by showing that $(I - it\Pi)$ has a two-sided bounded inverse.[4] We claim that

$$(I - it\Pi)^{-1}_{kl} = \begin{cases} (I - t^2L_k)^{-1} & k = l \\ it\mathrm{d}_{k-2}(I - t^2L_{k-2})^{-1} & k = l + 1 \\ it\delta_{k-1}(I - t^2L_k)^{-1} & k = l - 1 \\ 0 & \text{otherwise} \end{cases}$$

This gives a matrix with three diagonals, looking like

$$\begin{pmatrix} (I - t^2L_0)^{-1} & it\delta_0(I - t^2L_1)^{-1} & & & \\ it\mathrm{d}_0(I - t^2L_0)^{-1} & (I - t^2L_1)^{-1} & it\delta_1(I - t^2L_2)^{-1} & & \\ & \ddots & \ddots & \ddots & \\ & & it\mathrm{d}_{n-2}(I - t^2L_{n-2})^{-1} & (I - t^2L_{n-1})^{-1} & it\delta_{n-1}(I - t^2L_n)^{-1} \\ & & & it\mathrm{d}_{n-1}(I - t^2L_{n-1})^{-1} & (I - t^2L_n)^{-1} \end{pmatrix}.$$

By the R-sectoriality of $-L_k$ (proposition 6.2.8) and the R-gradient bounds (proposition 6.3.6) we have that all entries are bounded. It only remains to check that it is in fact a two-sided inverse. Let us first multiply by $(I - it\Pi)$ from the left. It suffices to compute the three diagonals, as the other elements of the product are clearly 0. It is easy to see that for any k the k-th diagonal element becomes

$$t^2\mathrm{d}_{k-2}\delta_{k-2}(I - t^2L_{k-1})^{-1} + (I - t^2L_{k-1})^{-1} + t^2\delta_{k-1}\mathrm{d}_{k-1}(I - t^2L_{k-1})^{-1} =$$

$$= (I - t^2L_{k-1})(I - t^2L_{k-1})^{-1} = I$$

[4] As we show this for all $t \neq 0$, rewriting then also gives that $(it - \Pi)$ has two-sided bounded inverse for any $t \neq 0$.

as $L_{k-1} = -(\mathrm{d}_{k-2}\delta_{k-2} + \delta_{k-1}\mathrm{d}_{k-1})$. For $k = 1, n$ the obvious adjustment should be made to the above expression. For the two other diagonals it is easy to see that one gets two terms which will cancel.

Let us now multiply by $(I - it\Pi)$ on the right side. Again computing only the three diagonals, and using lemma 6.3.7 one easily sees that the product is again the identity.

It remains to show that the set $\{it(it - \Pi)^{-1} : t \neq 0\} = \{(it - \Pi)^{-1} : t \neq 0\}$ is R-bounded. For this, observe that the diagonal entries are R-bounded by the R-sectoriality of $-L_k$. The R-boundedness of the other entries follows from the R-gradient bounds. Now observe that a set of operator matrices is R-bounded precisely when each entry is R-bounded. We conclude that Π is R-bisectorial. $\square$

We now wish to give sufficient conditions under which Π has a bounded H^∞-functional calculus on a bisector. Before we can do this, we first need the following proposition. Although this result seems trivial, one also needs to show that the domains are equal, which needs a little more effort.

Proposition 6.3.9. *Let $1 < p < \infty$. Then $\Pi^2 = -L$ as operators on $L^p(\Lambda TM)$.*

Proof. It suffices to show for any $k \in \{0, 1, \dots, n\}$ that $\Pi_k^2 := \mathrm{d}_{k-1}\delta_{k-1} + \delta_k \mathrm{d}_k = -L_k$. Here Π_k^2 is simply the restriction of Π^2 to $L^p(\Lambda^k TM)$ by considering $\omega \in L^p(\Lambda^k TM)$ as $(0, \dots, 0, \omega, 0, \dots, 0) \in L^p(\Lambda TM)$.

By the previous theorem Π is bisectorial and consequently, Π^2 is sectorial. But then Π_k^2 is also sectorial. By proposition 6.2.8 we also have that $-L_k$ is sectorial. We thus find that the negative real axis is contained in the resolvent set of both Π_k^2 and $-L_k$, and hence the positive real axis is contained in the resolvent set of both $-\Pi_k^2$ and L_k. Now pick $\lambda > 0$. As the operators acting on different $L^p(\Lambda^k TM)$ are consistent on $C_0^\infty(\Lambda^k TM)$, we may consider the operators in $L^2(\Lambda^k TM)$ where $L_k\omega = -\Pi_k^2\omega$ for $\omega \in C_0^\infty(\Lambda TM)$ by definition. But then we also have that $(\lambda + \Pi_k^2)\omega = (\lambda - L_k)\omega$ for all $\omega \in C_0^\infty(\Lambda^k TM)$.

We now prove that $(\lambda - L_k)C_0^\infty$ is dense in $L^p(\Lambda^k TM)$. For this, let q be the conjugate exponent to p and let $\eta \in L^q(\Lambda^k TM)$ be orthogonal to $(\lambda - L_k)C_0^\infty$ and pick a k-form $\omega \in C_0^\infty$ arbitrary. Remember that L_k generates the strongly continuous semigroup of contraction P_t^k on $L^r(\Lambda^k TM)$ for all $1 < r < \infty$ which is consistent in the sense that the operators agree $L^{r_1}(\Lambda^k TM) \cap L^{r_2}(\Lambda^k TM)$ for any $1 < r_1, r_2 < \infty$. Now observe that $L_k - \lambda$ generates the semigroup $e^{-\lambda t}P_t^k$. By proposition 3.1.5 we have that

$$(\lambda - L_k)\left(\int_0^t e^{-\lambda s} P_s^k \omega \, \mathrm{d}s\right) = e^{-\lambda t} P_t^k \omega - \omega.$$

As $e^{-\lambda s}P_s^k$ is consistent on the different $L^p(\Lambda^k TM)$ then so is $\int_0^t e^{-\lambda s} P_s^k \omega \, \mathrm{d}s$. Indeed, although we have to consider the integral as limit in different L^p-spaces, by switching to a subsequence along which we have almost everywhere convergence shows that the limits must coincide. Furthermore, by proposition 4.5.1 L_k defined on $C_0^\infty(\Lambda^k TM)$ is essentially self-adjoint on $L^2(\Lambda^k TM)$, hence $C_0^\infty(\Lambda^k TM)$ is dense in $D_2(L_k)$. But then we can find a sequence $(\zeta_n)_n$ of C_0^∞ k-forms which converges to $\int_0^t e^{-\lambda s} P_s^k \omega \, \mathrm{d}s$ in $D_2(L_k)$. By the choice of η we find for every n that $\langle \eta, \zeta_n \rangle = 0$. By taking the limit $n \to \infty$ we find by Cauchy-Schwarz that

$$0 = \langle \eta, e^{-\lambda t} P_t^k \omega - \omega \rangle = -\langle \eta, \omega \rangle + \langle \eta, e^{-\lambda t} P_t^k \omega \rangle.$$

Now let $(\eta_n)_n$ be a sequence of k-forms converging to η in $L^q(\Lambda^k TM)$. By the consistency of P_t^k we may use the symmetry of P_t^k on $L^2(\Lambda^k TM)$ to obtain that

$$\langle \eta, e^{-\lambda t} P_t^k \omega \rangle = \lim_{n \to \infty} \langle \eta_n, e^{-\lambda t} P_t^k \omega \rangle = \lim_{n \to \infty} \langle e^{-\lambda t} P_t^k \eta_n, \omega \rangle.$$

Again using consistency, we can interpret $e^{-\lambda t}P_t^k\eta_n$ in $L^q(\Lambda^kTM)$. By the boundedness of $e^{-\lambda t}P_t^k$ we then obtain that

$$\langle\eta, e^{-\lambda t}P_t^k\omega\rangle = \langle e^{-\lambda t}P_t^k\eta, \omega\rangle.$$

Combining the above equalities, we find that

$$\langle e^{-\lambda t}P_t^k\eta - \eta, \omega\rangle = 0.$$

As this holds for all $\omega \in C_0^\infty(\Lambda^kTM)$ we find by density that $e^{-\lambda t}P_t^k\eta = \eta$ and consequently, $||\eta||_p = e^{-\lambda t}||P_t^k\eta||_p$. Using that P_t^k is a contraction in $L^p(\Lambda^kTM)$, we obtain that $||\eta||_p \leq e^{-\lambda t}||\eta||_p$. Taking the limit $t \to \infty$ and recalling that $\lambda > 0$, we find that $||\eta||_p = 0$ from which it follows that $\eta = 0$. We thus find that 0 is the only element in $L^q(\Lambda^kTM)$ which is orthogonal to $(\lambda - L_k)C_0^\infty$, from which it follows that $(\lambda - L_k)C_0^\infty$ is dense in $L^p(\Lambda^kTM)$.

We can now finish the proof. Observe that $\omega = (\lambda + \Pi_k^2)^{-1}(\lambda - L_k)\omega$ for all $\omega \in C_0^\infty$. But then $(\lambda + \Pi_k^2)^{-1} = (\lambda - L_k)^{-1}$ at least on $(\lambda - L_k)C_0^\infty$. As both operators are bounded on $L^p(\Lambda^kTM)$, and $(\lambda - L_k)C_0^\infty$ is dense in $L^p(\Lambda^kTM)$, we find that $(\lambda + \Pi_k^2)^{-1} = (\lambda - L_k)^{-1}$ holds on all of $L^p(\Lambda^kTM)$. From this we obtain that

$$D_p(\Pi_k^2) = D_p(\lambda + \Pi_k^2) = \overline{R((\lambda + \Pi_k^2)^{-1})} = \overline{R((\lambda - L_k)^{-1})} = D_p(\lambda - L_k) = D_p(-L_k).$$

Finally, pick $\omega \in D_p(\Pi_k^2) = D_p(-L_k)$. Let $\eta \in L^p(\Lambda^kTM)$ be such that $\omega = (\lambda - L_k)^{-1}\eta$. Then

$$(\lambda + \Pi_k^2)\omega = (\lambda + \Pi_k^2)(\lambda - L_k)^{-1}\eta = \eta = (\lambda - L_k)(\lambda - L_k)^{-1}\eta = (\lambda - L_k)\omega$$

from which it follows that $\Pi_k^2\omega = -L_k\omega$. We conclude that $\Pi_k^2 = -L_k$, finishing the proof. □

We will now give a sufficient condition under which the Hodge-Dirac operator Π has a bounded H^∞-functional calculus. We will discuss this condition in more detail in section 6.5, where we will also relate to the results in the next section.

Theorem 6.3.10 (Bounded H^∞-functional calculus Π). *Suppose that M is a complete Riemannian manifold of dimension n and assume that Q_l as in formula* (6.2) *is nonnegative for all $l \in \{1, \dots, n\}$. Let $1 < p < \infty$. Then the Hodge-Dirac operator Π on $L^p(\Lambda^kTM)$ has a bounded H^∞-functional calculus on a bisector if and only if for all $k \in \{0, 1, \dots, n\}$ the operator $-L_k$ has a bounded H^∞-functional calculus on $L^p(\Lambda^kTM)$.*

Proof. As Π is R-bisectorial and $\Pi^2 = -L$, we find by proposition 5.3.7 that Π has a bounded H^∞-functional calculus in $L^p(\Lambda TM)$ on a bisector if and only if $-L$ has a bounded H^∞-functional calculus in $L^p(\Lambda TM)$ on a sector. As $-L$ is a diagonal matrix, this last statement is equivalent to stating that each $-L_k$ has a bounded H^∞-functional calculus in $L^p(\Lambda^kTM)$. □

Finally, we will show that from this bounded H^∞-functional calculus we can again retrieve the Riesz bounds.

Theorem 6.3.11 (Riesz bounds from bounded H^∞-functional calculus). *Suppose that M is a complete Riemannian manifold of dimension n and assume that Q_k as in formula* (6.2) *is nonnegative for all $k \in \{1, \dots, n\}$. Let $1 < p < \infty$ and suppose that Π has a bounded H^∞-functional calculus on a bisector. Then $D_p(\Pi) = D_p((-L)^{1/2})$ and for all ω in the common domain it holds that*

$$||\Pi\omega||_p \simeq_p ||(-L)^{1/2}\omega||_p.$$

Proof. Pick $\theta > \omega_{H^\infty}(\Pi)$ and let us consider the function sgn $\in H^\infty(\Sigma_\theta)$ given by $\mathrm{sgn}(z) = 1_{\Sigma_\theta^+} - 1_{\Sigma_\theta^-} = \frac{z}{\sqrt{z^2}} = \frac{\sqrt{z^2}}{z}$. By the bounded H^∞-functional calculus for Π we can define the bounded operator $\mathrm{sgn}(\Pi)$ on all of $L^p(\Lambda^k TM)$ by the discussion at the end of section 5.3.1.

Now observe that $\sqrt{\Pi^2} = (-L)^{1/2}$. By theorem 15.18 in [20] we see that this definition comes from the extended functional calculus for $-L$, and consequently also from the functional calculus for Π with the function $f(z) = \sqrt{z^2} \in H_P(\Sigma_\theta)$. It then follows from theorem 1.4.12 in [11] that

$$\mathrm{sgn}(\Pi) \circ \sqrt{\Pi^2} \subset \Pi \qquad \text{and} \qquad \mathrm{sgn}(\Pi) \circ \Pi \subset \sqrt{\Pi^2}$$

from which we conclude that $D_p(\mathrm{sgn}(\Pi) \circ \sqrt{\Pi^2}) \subset D_p(\Pi)$ and $D_p(\mathrm{sgn}(\Pi) \circ \Pi) \subset D_p(\sqrt{\Pi^2})$. However, as $\mathrm{sgn}(\Pi)$ is bounded, this reduces to $D_p(\sqrt{\Pi^2}) \subset D_p(\Pi)$ and $D_p(\Pi) \subset D_p(\sqrt{\Pi^2})$, from which we conclude that $D_p(\sqrt{\Pi^2}) = D_p(\Pi)$. Consequently, we even have that

$$\mathrm{sgn}(\Pi) \circ \sqrt{\Pi^2} = \Pi \qquad \text{and} \qquad \mathrm{sgn}(\Pi) \circ \Pi = \sqrt{\Pi^2}.$$

But then we find for $\omega \in D_p(\Pi) = D_p(\sqrt{\Pi^2})$ that

$$||\Pi\omega||_p = ||\mathrm{sgn}(\Pi) \circ \sqrt{\Pi^2}\omega||_p \lesssim_p ||\sqrt{\Pi^2}\omega||_p$$

and

$$||\sqrt{\Pi^2}\omega||_p = ||\mathrm{sgn}(\Pi) \circ \Pi\omega||_p \lesssim_p ||\Pi\omega||_p.$$

Remembering that $\sqrt{\Pi^2} = (-L)^{1/2}$ gives the result.

□

We will conclude this section by summarizing the results in one theorem.

Theorem 6.3.12. *Suppose that M is a complete Riemannian manifold of dimension n and let $1 < p < \infty$. Assume that for all $k \in \{0, 1, \ldots, n\}$ the operator $-L_k$ has a bounded H^∞-functional calculus on $L^p(\Lambda^k TM)$. Then the following assertions are equivalent:*

1. *For all $k \in \{0, 1, \ldots, n\}$, $D_p((-L_k)^{1/2}) = D_p(\mathrm{d}_k + \delta_{k-1})$ and for ω in this common domain it holds that*
$$||(\mathrm{d}_k + \delta_{k-1})\omega||_p \sim_{p,k} ||(-L_k)^{1/2}\omega||_p.$$

2. *The Hodge-Dirac operator Π is R-sectorial on $L^p(\Lambda TM)$.*

3. *The Hodge-Dirac operator Π has a bounded H^∞-functional calculus on a bisector.*

6.4 Nonzero lower bounds

In this section M is a complete Riemannian manifold of dimension n. We again define the quadratic forms Q_k acting on k-forms by the following formula

$$\frac{1}{2}L_0|\omega|^2 = \omega \cdot L_k\omega + \frac{1}{k!}|\nabla\omega|^2 + Q_k(\omega, \omega) \tag{6.3}$$

which is formula (4.10) on 69. These Q_k involve the curvature of M and we assume that $Q_k \geq 0$ for $k = 1, \ldots n$. We assume that for all $k \in \{1, \ldots, n\}$ there exists a constant $a_k \geq 0$ such that $Q_k(\omega, \omega) \geq -a_k^2|\omega|$ for all $\omega \in C_0^\infty(\Lambda^k TM)$.

Our aim is to discuss the validity of the analogues of the results in the previous section. If we investigate the proof of theorem 4.5.4 we see that we in fact have the following theorem.

Theorem 6.4.1. *Suppose that M is a complete Riemannian manifold of dimension n and assume that Q_l as in formula (6.3) is bounded from below for all $l \in \{1, \dots, n\}$. Then for $1 < p < \infty$ and $k = 0, 1, \dots, n$ there exists a constant $C(p, k)$ such that for all k-forms $\omega \in C_0^\infty$ the following estimates hold:*

1. *For all $d \geq b_k$, $||\mathrm{d}_k\omega||_p \leq C(p, k)||(d^2 - L_k)^{-1}||_p$.*
2. *For all $d \geq b_{k-1}$, $||\delta_{k-1}\omega||_p \leq C(p, k)||(d^2 - L_k)^{-1}||_p$.*

Here $b_k = \max\{a_k, a_{k+1}\}$ as was defined in section 4.5. Hence, denote $r = \max\{a_k : k = 0, 1, \dots, n\} = \max\{b_k : k = 0, 1, \dots, n\}$. In that case we have the above estimates for all k with the operator $(r^2 - L_k)^{-1}$.

As L_k generates the semigroup P_t^k, the operator $L_k - r^2$ generates the semigroup $e^{-r^2t}P_t^k$. Now remember that for all $1 \leq p \leq \infty$ the semigroup P_t^k satisfies the estimate $||P_t^k\omega||_p \leq e^{a_k^2 t}||\omega||_p$ (see proposition 4.5.2). As $r \geq a_k \geq 0$ for any k, we then find that the semigroup $e^{-r^2t}P_t^k$ satisfies $||e^{-r^2t}P_t^k||_p \leq e^{-r^2t}e^{a_k^2t}||\omega||_p \leq ||\omega||_p$.

Furthermore, as P_t^k is self-adjoint on $L^2(\Lambda^kTM)$, it is obvious that $e^{-r^2t}P_t^k$ is also self-adjoint on $L^2(\Lambda^kTM)$. These observations show that we can follow the proof of proposition 6.2.8, but now applied to the operator $L_k - r^2$ to obtain the following result.

Proposition 6.4.2 (R-sectoriality of $r^2 - L_k$)**.** *For any $1 < p < \infty$, the operator $r^2 - L_k$ is R-sectorial on $L^p(\Lambda^kTM)$ with angle less than $\frac{\pi}{2}$.*

We now wish to define the operator $(r^2 - L_k)^{1/2}$. For this, we can follow the same construction as was done in the previous section when we defined $(-L_k)^{1/2}$. The operator $(r^2 - L_k)^{1/2}$ satisfies similar properties, such as that $C_0^\infty(\Lambda^kTM)$ is a core for $D_p((r^2 - L_k)^{1/2})$ if $1 < p < \infty$ and that for $1 \leq p, p' < \infty$ the operators are consistent on $D_p(r^2 - L_k) \cap D_{p'}(r^2 - L_k)$, so in particular also on $C_0^\infty(\Lambda^kTM)$ if $p, p' > 1$.

Similar to theorem 6.3.3, we get the following extension of theorem 6.4.1.

Theorem 6.4.3 (Boundedness of the Riesz transform, nonzero lower bounds)**.** *Suppose that M is a complete Riemannian manifold of dimension n and assume that Q_l as in formula (6.3) is bounded from below for all $l \in \{1, \dots, n\}$. Let $1 < p < \infty$ and suppose that $k \in \{0, 1, \dots, n\}$. Then $D_p((r^2 - L_k)^{1/2}) \subset D_p(\mathrm{d}_k + \delta_{k-1})$ and there exists a constant $C(p, k)$ only depending on p, k, such that for all $\omega \in D_p((r^2 - L_k)^{1/2})$ the following estimate holds:*

$$||(\mathrm{d}_k + \delta_{k-1})\omega||_p \leq C(p, k)||(r^2 - L_k)^{1/2}\omega||_p.$$

We will now state some analogue results to those of section 6.3.2. By applying the results from section 6.3.1 to the operator $L_k - r^2$ instead of L_k we obtain the following.

Proposition 6.4.4 (R-gradient bounds)**.** *Let $1 < p < \infty$. For any $0 \leq k \leq n (= \dim(M))$ the families of operators*

$$\{t\mathrm{d}_k(I - t^2(L_k - r^2))^{-1} : t > 0\}$$

and

$$\{t\delta_{k-1}(I - t^2(L_k - r^2))^{-1} : t > 0\}$$

are R-bounded in $\mathcal{L}(L^p(\Lambda^kTM), L^p(\Lambda^{k+1}TM))$, respectively $\mathcal{L}(L^p(\Lambda^kTM), L^p(\Lambda^{k-1}TM))$.

We will now focus ourselves on the spectrum of the Hodge-Dirac operator Π. Note that we cannot hope that Π remains bisectorial. Indeed, if it would be, then its square, $-L_k$, would be sectorial, which is not necessarily the case. Before we can state the result, we first have to go over two lemmas.

Lemma 6.4.5. *Let $1 < p < \infty$. For any $k \in \{0, 1, \ldots, n-1\}$ and for any $t \in \mathbb{R} \setminus \{0\}$ such that $|t| < \frac{1}{r}$ the operators*

$$\mathrm{d}_k (I - t^2 L_k)^{-1} \qquad \text{and} \qquad \delta_{k-1}(I - t^2 L_k)^{-1}$$

are well-defined on $L^p(\Lambda^k TM)$.

Proof. We will only prove that the first operator is well-defined, the second following in a similar manner. As $||P_t^k|| \leq e^{r^2 t}$ we find by proposition 3.2.2 that $\{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > r^2\} \subset \rho(L_k)$. Consequently, for $t \neq 0$ such that $|t| < \frac{1}{r}$ we have that $t^{-2} > r^2$ and thus $(t^{-2} - L_k)^{-1}$ is well defined and bounded. Consequently, we also find that $(I - t^2 L_k)^{-1}$ is well-defined and bounded and maps $L^p(\Lambda^k TM)$ into $D_p(L_k)$. Now observe that

$$D_p(L_k) = D_p(r^2 - L_k) \subset D_p((r^2 - L_k)^{1/2}) \subset D_p(\mathrm{d}_k)$$

where the last inclusion follows by applying the results from section 6.3.1 to the operator $L_k - r^2$. From this we conclude that $\mathrm{d}_k(I - t^2 L_k)^{-1}$ is well defined. □

Lemma 6.4.6. *Let $1 \leq p < \infty$. For any $k = 0, 1, \ldots, n$ and $t \in \mathbb{R}$ with $|t| < \frac{1}{r}$ the following identities hold on $D_p(\mathrm{d}_k)$ and $D_p(\delta_k)$ respectively:*

$$(I - t^2 L_{k+1})^{-1} \mathrm{d}_k = \mathrm{d}_k (I - t^2 L_k)^{-1}$$

and

$$(I - t^2 L_k)^{-1} \delta_k = \delta_k (I - t^2 L_{k+1})^{-1}.$$

Proof. We will again only proof the first identity, the second following in a similar manner.

Pick t such that $|t| < \frac{1}{r}$. Observe that by the previous lemma all operators are well-defined. For $t = 0$ the statement is trivial, so we may assume that $t \neq 0$. From the proof of lemma 6.3.7 we get for $\omega \in D_p(\mathrm{d}_k)$ that $P_t^{k+1} \mathrm{d}_k \omega = \mathrm{d}_k P_t^k \omega$. Observe that $t^{-2} > r^2$ which implies that $t^{-2} \in \rho(L_k) \cap \rho(L_{k+1})$. By taking Laplace transforms we then find that

$$(t^{-2} - L_{k+1})^{-1} \mathrm{d}_k \omega = \mathrm{d}_k (t^{-2} - L_k)^{-1} \omega$$

which rewrites to

$$t^2 (I - t^2 L_{k+1})^{-1} \mathrm{d}_k \omega = t^2 \mathrm{d}_k (I - t^2 L_k)^{-1} \omega$$

from which the desired identity follows.

□

With these lemmas at hand we can prove the following.

Theorem 6.4.7 (Spectrum of Π)**.** *Suppose that M is a complete Riemannian manifold of dimension n and assume that Q_l as in formula* (6.3) *is bounded from below for all $l \in \{1, \ldots, n\}$. Let $1 < p < \infty$. Then $\{it : |t| > r\} \subset \rho(\Pi)$ when considering Π as an operator on $L^p(\Lambda TM)$.*

Proof. First observe that $(it - \Pi)^{-1} = \frac{1}{it}(I - \frac{1}{it}\Pi)^{-1}$. Hence is suffices to show that $(I - it\Pi)$ is boundedly invertible for $|t| < \frac{1}{r}$. To do this, we can simply use the representation deduced in the proof of theorem 6.3.8 and observe that the entries are indeed bounded in the case when $|t| < \frac{1}{r}$. The above lemma guarantees that the necessary manipulations are in fact allowed.

□

We will leave out the results concerning the bounded H^∞-functional calculus for Π for reasons which we will explain in the next section.

6.5 Remarks and conjecture

We will finish this chapter by discussing the results from the previous two sections concercing the boundedness of the H^∞-functional calculus for the Hodge-Dirac operator Π. In theorem 6.3.10 we stated that Π has a bounded H^∞-functional calculus precisely when $-L_k$ has one for all $k \in \{0, 1, \ldots, n\}$. Our belief is that this is indeed satisfied in the situation of section 6.3, the reason for which is that in theorem 6.3.3 we have two-sided Riesz estimates. Our conjecture is that these two-sided Riesz estimates for $-L_k$ are equivalent to the fact that $-L_k$ has a bounded H^∞-functional calculus. The reason for this is that we think it likely that we can prove two-sided square-function estimates for $-L_k$ from the two-sided Riesz estimates, which is the approach as taken in [25]. These two-sided square-function estimates are equivalent to having a bounded H^∞-functional calculus.

This immediately leads us to explaining why we left out the results concercing the bounded H^∞-functional calculus in the previous section. In this case, we namely only have one-sided Riesz estimates in theorem 6.4.3 and corollary 4.5.7 suggests that we do not obtain the reverse inequalities, as we can no longer take $e = 0$. But then we can no longer expect to get two-sided square-function estimates for $-L_k$, and consequently we should not expect that $-L_k$ has a bounded H^∞-functional calculus.

Chapter 7

Conclusion

We will conclude this thesis by recollecting in broad lines the work we have done. We have studied the Riesz transform on a complete Riemannian manifold with Ricci curvature bounded from below. We were particularly interested in the connection between the boundedness of the Riesz transform and the boundedness of the H^∞-functional calculus of the Hodge-Dirac operator $\mathrm{d} + \delta$. In chapters 2, 3 and 5 we discussed some general theory concerning differential geometry, strongly continuous semigroups and the H^∞-functional calculus for sectorial and bisectorial operators. Chapters 4 and 6 form the most important chapters and deserve some more attention.

7.1 Boundedness of the Riesz transform on a complete Riemannian manifold

The basis for our study of the Riesz transform on complete Riemannian manifolds is formed by the paper '*Étude des transformation de Riesz dans les variétés riemanniennes à courbure de Ricci minorée*' by D. Bakry ([6]). We thoroughly discussed this paper in chapter 4. The paper first aimed at proving the boundedness of the Riesz transform on functions. To this extend, the (Witten-)Laplacian was defined as $\mathrm{d}\delta + \delta\mathrm{d}$ both for functions and 1-forms, denoted by L and $\overrightarrow{L}$ respectively. Here, δ denotes the adjoint of d in L^2 with respect to the measure $m(\mathrm{d}x) = \rho(x)\mathrm{d}x$ for some smooth function $\rho > 0$. It is shown that L and $\overrightarrow{L}$ defined on C_0^∞ are essentially self-adjoint on L^2 and via the spectral theory the semigroups generated by the closure of L and $\overrightarrow{L}$ were defined, which are denoted by P_t and $\overrightarrow{P}_t$ respectively. In order to show that the Riesz transform on functions is bounded, it was furthermore assumed that the Ricci-curvature was bounded from below. This allowed for useful bounds on these semigroups generated by L and $\overrightarrow{L}$ respectively.

The approach in proving the boundedness of the Riesz transform was to make use of subordinated semigroups formed from P_t and $\overrightarrow{P}_t$. It turned out that the generators of these subordinated semigroups were exactly the operators that came up in proving bounds for the Riesz transform. However, in order to make full use of them, it was necessary to derive two important estimates, which is done in section 3 of the paper. One of these estimates was proven with purely analytic methods, while the other made clever use of the concept of martingales and conditional expectation from probability theory.

In the final section from the paper of Bakry that we studied (section 5), it was shown that all ideas used to prove the boundedness of the Riesz transform on functions can also be used with minor adjustments to prove the boundedness of the Riesz transform on k-forms. In this section, the Witten-Laplacian on k-forms is denoted as L_k and can be defined in the same way as for 1-forms, as d acts on forms of any order and we can again define its adjoint. With this result at hand, we turned to an analysis of th Hodge-Dirac operator $\mathrm{d} + \delta$.

7.2 Hodge-Dirac operator

In chapter 6 we analysed the Hodge-Dirac operator $\Pi = \mathrm{d} + \delta$ on a complete Riemannian manifold with Ricci-curvature bounded from below. The inspiration for the results that we acquired is formed by the paper '*Quadratic estimates and functional calculi of perturbed Dirac operators*' by A. Axelsson, S. Keith and A. McIntosh ([4]) as well as the paper '*Boundedness of Riesz transforms for elliptic operators on abstract Wiener spaces*' by J.Maas and J.M.A.M van Neerven ([25]).

We started out by proving that the operators as defined in [6], which are primarily considered only on L^2 can be extended to L^p. We also showed that the boundedness of the Riesz transform still holds in this interpretation of the operators, rather than only considering them to be acting on C_0^∞. This allowed us to consider the Hodge-Dirac operator acting on L^p for arbitrary $1 \leq p < \infty$. As the boundedness of the Riesz transform only holds for $1 < p < \infty$, the results are obviously restricted to that case.

Armed with the right extensions, we first showed that the boundedness of the Riesz transform implies R-gradient bounds (propositions 6.3.6 and 6.4.4). These gradient bounds then in turn imply the R-bisectoriality of Π. From this we deduced that Π has a bounded H^∞-functional calculus on a bisector. Finally, we proved that one can again retrieve the boundedness of the Riesz-transform from this bounded H^∞-functional calculus, which concluded our study.

7.3 Future considerations

Finally, we will point out a direction in which the results may be extended. In the light of [4] (and also [25]), we could ask ourselves the question what happens if we consider perturbed Dirac-type operators. As an example we could consider bounded operators B_1 and B_2 and form the operators $\mathrm{d}_B = B_2^* \mathrm{d} B_1^*$ and $\delta_B = B_1 \delta B_2$. The Hodge-Dirac operator we would then study is $\Pi_B = \mathrm{d} + \delta_B$. This is a special case of what is done in [4] for L^2, where they also replace the operator d by a more general operator Γ which must satisfy $\Gamma^2 = 0$, just like d. Additionally, some assumptions are made for the operators B_1 and B_2. We could then ask ourselves if these results also hold on L^p for arbitrary $1 < p < \infty$. Our results in sections 6.3.2 and 6.4 would then be special cases of this.

Appendix A

A.1 Identities from differential geometry

In all results that follow we assume that we work on a Riemannian manifold M with the Levi-Civita connection ∇. The identities and results are mainly used in chapter 4. We therefore use the notation from that chapter. That means that the inner product between tangent vectors and forms is denoted by $\cdot$, while the L^2 innerproduct is written as $\langle\ ,\ \rangle$.

Lemma A.1.1. *Let η be a 1-form and denote by η^* the corresponding tangent vector from the duality via the metric. Let X, Y be tangent vectors. Then*

$$\nabla\eta(X,Y) = \nabla_X\eta^* \cdot Y.$$

Proof. By the compatibility with the metric, we have that

$$X(\eta^* \cdot Y) = \nabla_X\eta^* \cdot Y + \eta^* \cdot \nabla_X Y.$$

Using this, we find

$$\begin{aligned}
\nabla\eta(X,Y) &= \nabla_X\eta(Y) \\
&= X(\eta(Y)) - \eta(\nabla_X Y) \\
&= X(\eta^* \cdot Y) - \eta^* \cdot \nabla_X Y \\
&= \nabla_X\eta^* \cdot Y.
\end{aligned}$$

□

Corollary A.1.2. *For any smooth function h, and 1-form ω we have*

$$\mathrm{d}(\mathrm{d}h \cdot \omega) = \nabla\omega(\cdot, \mathrm{d}h^*) + \nabla\nabla h(\cdot, \omega^*).$$

Proof. Let X be an arbitrary vector field. Then $\mathrm{d}(\mathrm{d}h \cdot \omega)(X) = X(\mathrm{d}h \cdot \omega) = X(\mathrm{d}h^* \cdot \omega^*)$. By the compatibility of ∇ with the metric and lemma A.1.1, we find that

$$X(\mathrm{d}h^* \cdot \omega^*) = \nabla_X \mathrm{d}h^* \cdot \omega^* + \mathrm{d}h^* \cdot \nabla_X\omega^* = \nabla\mathrm{d}h(X, \omega^*) + \nabla\omega(X, \mathrm{d}h^*).$$

Noticing that $\mathrm{d}h = \nabla h$ for function gives the desired equality. □

Proposition A.1.3 (Hessian)**.** *For any function h, we have that $\nabla h = \mathrm{d}h$ and $\nabla\nabla h(X,Y) = X(Yh) - (\nabla_X Y)h$.*

Proof. Let X, Y be arbitrary tangent vectors. Then

$$\nabla h(X) = \nabla_X h = Xh = \mathrm{d}h(X)$$

which shows that $\nabla h = \mathrm{d}h$.

Furthermore, we have that

$$\nabla\nabla h(X, Y) = \nabla_X \mathrm{d}h(Y) = X(\mathrm{d}h(Y)) - \mathrm{d}h(\nabla_X Y) = X(Yh) - (\nabla_X Y)h$$

which proves the second identity. □

Proposition A.1.4 (Symmetry Hessian). *The Hessian $\nabla\nabla h$ of a function h is symmetric.*

Proof. Let X, Y be tangent vectors. Then

$$\begin{aligned} \nabla\nabla h(X, Y) &= X(Yh) - (\nabla_X Y)h \\ &= [X, Y]h + Y(Xh) - (\nabla_X Y)h \\ &= Y(Xh) - (\nabla_Y X)h \\ &= \nabla\nabla h(Y, X) \end{aligned}$$

where we used proposition A.1.3 in the first line and the symmetry of the connection in the third line. □

Proposition A.1.5 (Commutativity rule for covariant derivative of 1-forms). *Let ω be a 1-form. Then for tangent vectors X, Y it holds that*

$$\nabla\omega(X, Y) = \nabla\omega(Y, X) - \mathrm{d}\omega(Y, X).$$

Proof. Using the symmetry of the connection we find that

$$\begin{aligned} \nabla\omega(Y, X) - \mathrm{d}\omega(Y, X) &= \nabla_X\omega(Y) - (X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])) \\ &= X(\omega(Y)) - \omega(\nabla_X Y) - X(\omega(Y)) + Y(\omega(X)) + \omega([X, Y]) \\ &= Y(\omega(X)) - \omega(\nabla_Y X) \\ &= \nabla\omega(X, Y) \end{aligned}$$

which is the desired equality. The expression for $\mathrm{d}\omega$ used in the first line follows from theorem 13 in chapter 7 of [32]. □

Lemma A.1.6 ('Product rule' for divergence on k-forms). *Denote by δ the divergence of k-forms. For any function f, k-form ω and $(k-1)$-form ϵ we have*

$$\langle \delta(f\omega), \epsilon\rangle = \langle f\delta\omega, \epsilon\rangle - \langle \omega, \mathrm{d}f \wedge \epsilon\rangle$$

Proof. We have that

$$\begin{aligned} \langle f\delta\omega, \epsilon\rangle &= \langle \delta\omega, f\epsilon\rangle \\ &= \langle \omega, \mathrm{d}(f\epsilon)\rangle \\ &= \langle \omega, \mathrm{d}f \wedge \epsilon + f\mathrm{d}\epsilon\rangle \\ &= \langle \omega, \mathrm{d}f \wedge \epsilon\rangle + \langle f\omega, \mathrm{d}\epsilon\rangle \\ &= \langle \omega, \mathrm{d}f \wedge \epsilon\rangle + \langle \delta(f\omega), \epsilon\rangle \end{aligned}$$

which rewrites to the desired equality. □

Corollary A.1.7 (Product rule for the Laplace-Beltrami operator)**.** *Let $f, g \in C_0^\infty$, and denote by $\Delta = \delta \mathrm{d}$ the Laplace-Beltrami operator. Then*

$$\Delta(fg) = f\Delta g + g\Delta f - 2\mathrm{d}f \cdot \mathrm{d}g.$$

Proof. Applying lemma A.1.6 to a 1-form ω and arbitrary $f, h \in C_0^\infty$, we find that

$$\langle \delta(f\omega), h\rangle = \langle f\delta\omega - \omega \cdot \mathrm{d}f, h\rangle.$$

As this holds for all $h \in C_0^\infty$, by density we find that $\delta(f\omega) = f\delta\omega - \mathrm{d}f \cdot \omega$.

This identity now gives us that

$$\Delta(fg) = \delta(f\mathrm{d}g + g\mathrm{d}f) = f\delta\mathrm{d}g - \mathrm{d}f \cdot \mathrm{d}g + g\delta\mathrm{d}f - \mathrm{d}f \cdot \mathrm{d}g = f\Delta g + g\Delta f - 2\mathrm{d}f \cdot \mathrm{d}g.$$

□

Lemma A.1.8 (Norm of wedge product)**.** *Suppose that ϵ is k-form and η a l-form. Then*

$$|\epsilon \wedge \eta| \leq |\epsilon||\eta|$$

Proof. Notice that the estimate is pointwise. Pick $p \in M$ and let $x_1, \dots, x_n$ be normal coordinates. We can write $\epsilon = \sum_I \epsilon_I \mathrm{d}x^I$ and $\eta = \sum_J \eta_J \mathrm{d}x^J$, where I and J are multi-indices of length k respectively l. As the metric at the point p is given by $G = I$, the following holds at p

$$\begin{aligned} |\epsilon \wedge \eta|^2 &= \sum_{I,J} |\epsilon_I \eta_J \mathrm{d}x^I \wedge \mathrm{d}x^J|^2 \\ &\leq \sum_{I,J} |\epsilon_I|^2 |\eta_J|^2 \\ &= \left(\sum_I |\epsilon_I|^2\right)\left(\sum_J |\eta_J|^2\right) \\ &= |\epsilon|^2|\eta|^2 \end{aligned}$$

Here the first line follows by the Pythagorean theorem and the second by using that $|\mathrm{d}x^I \wedge \mathrm{d}x^J|^2$ is either 0 or 1, depending if I and J contain common indices. □

Lemma A.1.9. *For a 1-form ω it holds that*

$$\Delta|\omega|^2 = 2\omega \cdot \Delta\omega + 2|\nabla\omega|^2$$

Proof. We prove the equality locally. Let $\{x^1, \dots, x^d\}$ be normal coordinates, and write $\partial_i = \frac{\partial}{\partial x^i}$. In that case $\nabla_{\partial_i}\partial_i = 0$ and hence

$$\nabla^2_{\partial_i\partial_i} := \nabla_{\partial_i}\nabla_{\partial_i} - \nabla_{\nabla_{\partial_i}\partial_i} = \nabla_{\partial_i}\nabla_{\partial_i}$$

As $\Delta = \mathrm{Tr}(\nabla\nabla) = \sum_{i=1}^d \nabla^2_{\partial_i\partial_i}$, we find that

$$\begin{aligned} \Delta|\omega|^2 &= \sum_{i=1}^d \nabla_{\partial_i}\nabla_{\partial_i}(\omega \cdot \omega) \\ &= \sum_{i=1}^d \nabla_{\partial_i}(\nabla_{\partial_i}\omega \cdot \omega + \omega \cdot \nabla_{\partial_i}\omega) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{d} 2(\nabla_{\partial_i}\nabla_{\partial_i}\omega)\cdot\omega + 2\nabla_{\partial_i}\omega\cdot\nabla_{\partial_i}\omega \\
&= 2\Delta\omega\cdot\omega + 2|\nabla\omega|^2
\end{aligned}
$$

where we used compatibility with the metric twice and the fact that $|\omega|^2 = \omega\cdot\omega$. □

Proposition A.1.10. *Let ρ be a smooth function, and suppose that ω is a k-form and ϵ a $(k-1)$-form. Then*

$$\omega\cdot(\mathrm{d}\rho\wedge\epsilon) = \iota(\mathrm{d}\rho^*)\omega\cdot\epsilon$$

where ι denotes contraction on the first entry.

Proof. Working in a chart, by linearity it suffices to prove the claim for $\omega = f\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}$ where $1\le i_1<\cdots<i_k\le n$ and $\epsilon = g\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_{k-1}}$ where $1\le j_1<\cdots<j_{k-1}\le n$.

In that case we find that

$$
\begin{aligned}
&\omega\cdot(\mathrm{d}\rho\wedge g\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_{k-1}}) \\
&= fg(\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k})\cdot(\mathrm{d}\rho\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_{k-1}}) \\
&= \sum_{r=1}^{k}(-1)^{r+1}fg(\mathrm{d}x^{i_r}\cdot\mathrm{d}\rho)(\mathrm{d}x^{i_1}\wedge\cdots\wedge\widehat{\mathrm{d}x^{i_r}}\wedge\cdots\wedge\mathrm{d}x^{i_k})\cdot(\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_{k-1}}) \\
&= \iota(\mathrm{d}\rho^*)\omega\cdot\epsilon.
\end{aligned}
$$

Here the third line follows by remembering that the inner product can be seen as the determinant of a matrix, and that we can develop this determinant to the row of $\mathrm{d}\rho$. The last equality follows by simply expanding $\iota(\mathrm{d}\rho^*)\omega$. □

A.2 Some analytic and algebraic results

Lemma A.2.1. *Let $f\in C^2([0,\infty))$ be such that $|f(x)|,|f'(x)|,|f''(x)|\le Ce^{-ax}$ for some constants $C,a>0$. Then*

$$f(0) = \int_0^\infty xf''(x)\mathrm{d}x.$$

Proof. We simply compute $\lim_{R\to\infty}\int_0^R xf''(x)\mathrm{d}x$. Using integration by parts, we obtain

$$\int_0^R xf''(x)\mathrm{d}x = [xf'(x)]_0^R - \int_0^R f'(x)\mathrm{d}x = Rf'(R) - (f(R)-f(0)).$$

By the assumption $f(R)$ and $f'(R)$ go to 0 exponentially as $R\to\infty$. But then also $Rf'(R)\to 0$ as $R\to\infty$. We conclude that

$$\int_0^\infty xf''(x)\mathrm{d}x = \lim_{R\to\infty} Rf'(R) - f(R) + f(0) = f(0).$$

□

Lemma A.2.2. *Let $a,b,c,d\ge 0$. Then*

$$ab+cd\le(a^2+c^2)^{1/2}(b^2+d^2)^{1/2}.$$

Proof. Note that

$$ab + cd = ((ab + cd)^2)^{1/2} = (a^2b^2 + c^2d^2 + 2abcd)^{1/2}$$

and

$$0 \leq (ad - bc)^2 = a^2d^2 + b^2c^2 - 2abcd$$

from which it follows that $2abcd \leq a^2d^2 + b^2c^2$. Combining the two, we find that

$$ab + cd \leq (a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2)^{1/2} = (a^2 + c^2)^{1/2}(b^2 + d^2)^{1/2}$$

as desired. $\square$

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