A Game-Theoretical Approach to Boycott Modelling

Quantitative impacts of boycotts by cooperative game theory

MSc Thesis

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Quantitative impacts of boycotts by cooperative game theory

by

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Cover: Li et al., 2017 (Modified).

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Abstract

Almost everyone is familiar with games such as poker and checkers, but games can also be found in non-entertaining settings, such as competing companies in a market or conflict resolutions between countries. However, what happens when players want to avoid working together? This thesis provides two game-theoretical models of boycotts based on existing literature and is complemented with new insights. We discuss the dynamic process of a consumer boycott, where the effectiveness can be determined by the predefined consumers' and firm's thresholds or by the maximum boycott duration. Besides, we consider a static model using the Shapley value to analyze the impacts of boycotts, which are balanced for 2*-player boycotts but not when three players are involved. We extended the existing models by finding supermodularity as a sufficient requirement for the boycott to never be profitable for the involved players, by introducing a direct formula for the coalition value in a boycott game (the boycott-contraction), and by analyzing real-world data, where we observe the effect of boycotts on trading activities with uninvolved players.*

Samenvatting

Bijna iedereen is wel bekend met spellen als poker en dammen, maar er bestaan ook spellen die niet zijn ontworpen voor puur vermaak en gewoonweg voorkomen in het dagelijkse leven, zoals concurrerende bedrijven in een markt of conflictonderhandelingen tussen landen. Maar wat gebeurt er als spelers hierin niet willen samenwerken? Deze scriptie bespreekt twee speltheoretische modellen van boycots, gebaseerd op bestaande literatuur en aangevuld met nieuwe inzichten. We bekijken het dynamische proces van een consumentenboycot, waarbij de effectiviteit kan worden bepaald door vooraf gedefinieerde grenswaarden of door de maximale boycotduur van beide partijen te berekenen. Daarnaast bevat dit verslag een statisch model, waarin we de impact van boycots analyseren aan de hand van de Shapleywaarde, die perfect in balans is voor boycots met twee spelers, maar dat niet is wanneer er drie spelers bij betrokken zijn. We hebben de bestaande modellen uitgebreid door de voorwaarde van supermodulariteit te koppelen aan de winstgevendheid van een boycot, door een directe formule op te stellen voor de coalitiewaarde in een boycotspel (de boycotcontractie), en door het analyseren van werkelijke data uit de praktijk, waar we het effect van boycots op handelsverkeer met niet-betrokken spelers observeren.

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Nomenclature

Abbreviations

Abbreviation	Definition
BC	Balanced contributions axiom
BC^-	Weak balanced contributions axiom
BIB	Balanced impacts of boycotts axiom
BIB^-	Weak balanced impacts of boycotts axiom
BIB^w	w-balanced impacts of boycotts axiom
DP	Disjointly productive players axiom
E	Efficiency axiom
Lin	Linearity axiom
Mar	Marginality axiom
Null	Null player axiom
PBC	Proportional balanced contributions axiom
PBIB	Proportional balanced impacts of boycotts axiom
Sym	Symmetry axiom
TU-game	Cooperative game with transferable utility
WBC^w	w-weighted balanced contributions axiom

Symbols

Symbol	Definition
2^N	Set of all possible coalitions in the game with player set N
B_i	Decision variable for player i to participate in the boycott
C_i	Boycotting costs for player i
Δ_v	Harsanyi dividend corresponding to a game with utility function v
G_i	Potential boycotting gain for player i
λ	Boycott efficiency parameter: ratio of $\bar{n}$ over n
MC_i^v	Marginal contribution of player i in a game with utility function v
n	Number of players in the game
$\bar{n}$	Equilibrium boycott population
n^s	Firm threshold
n^*	Number of players participating in the boycott
N	Set of players in the game
$\mathscr{N}$	The set of all finite and non-empty subsets of $\mathscr{U}$
NCP_c	Consumers' net cumulative profit of winning the boycott
NCP_f	Firm's net cumulative profit of winning the boycott
$\bar{p}_i$	Participation threshold
π_1	Firm's profit from strategy 1
π_2	Firm's profit from strategy 2
φ	TU-value (payoff-vector)
S_1	Firm's strategy 1
S_2	Firm's strategy 2
Sh	Shapley value
Sh^p	Proportional Shapley value

Symbol	Definition
Sh^w	Weighted Shapley value
T^c	Consumers' maximum boycott duration
T^f	Firm's maximum boycott duration
U_1	Consumers' utility for product 1
U_2	Consumers' utility for product 2
U_b	Utility for boycotting consumers
$\mathscr{U}$	The collection of all players
v	Utility (value) function $v : 2^N \to \mathbb{R}$
$\mathbb{V}(N)$	The set of all TU-games (N, v)
$\mathbb{V}_+(N)$	The set of all totally positive TU-games (N, v)
$\mathbb{V}_0(N)$	The set of all TU-games (N, v) with strictly positive/negative values
W	All possible (positive) weight distributions on the collection of all players
$\emptyset$	Empty coalition

1
Introduction

Poker, rock-paper-scissors, checkers, and computer games. Simply a list of some entertaining games most of us are familiar with. However, games can also be found in non-entertaining settings, such as psychological games on a personal level, in which the game is played with words, with good or bad feelings as payoffs (Berne, 1964). Similarly, in biological games, different species compete in natural selection, which can be seen as genes playing a game, with survival as the ultimate payoff (Smith, 1982). Economic, political, and military games are probably more relatable to you - competing companies in a market, negotiations for peace, and conflict resolution between countries are all examples of games in action. Studying game theory can give us valuable insights into the optimal choice for the players in the game and provide us with better predictions.

Games can be defined by a number of interacting players (strategic decision-makers) who might form coalitions, undertake actions in situations of uncertainty, and finally get some payoff, which can be beneficial (e.g., some benefit or prize money) or detrimental (e.g., some punishment or lost money). Game theory can be presented in many classes of games, of which Ferguson (2020) and Peters (2008) chose the following categorization of the main types of games:

- Non-cooperative games are games in which the players can not negotiate, which makes finding an optimal strategy of utmost importance. This class can be subdivided into the following categories:
  - Zero-sum games are games in which the sum of payoffs equals zero. In a game with two players, this implies that one player's gain corresponds to the other player's loss.
    - Perfect information means that each player knows the results of all previously occurred events in the game. Often, this happens because the players play sequentially, such as in the game of chess. We call these combinatorial games.
    - Imperfect information indicates that players do not know all previous moves. Often, this happens because the players play simultaneously, such as in rock-paper-scissors, or when players do not know the other players' private information (e.g. cards), such as in poker.
  - Non-zero-sum games have a net payoff which does not equal zero. The Prisoner's Dilemma is one of the most famous examples in this category.
- Cooperative games (coalitional-form games) involve a large number of players working together in order to obtain a larger payoff. In this type of game, the concept of strategy becomes irrelevant - the coalition and its value are now the main features to work with.

This thesis will focus on the last type of game: the coalitional form. Nowadays, small companies, multinationals, and even countries and governments work together as coalitions to benefit from each other or, in terms of game theory, obtain a larger payoff. We all know about the United Nations - countries working together on developing the world economy and global security. You may have also heard of the recent collaboration between Albert Heijn and HEMA. These are examples of coalitions in which players work together, but we can also think of examples in which not every player in a coalition is cooperating as well as another. Anyhow, the most frequently asked question and the central issue in game theory will always be how to divide the profits among all coalition members.

Let $\mathscr{U}$ be defined as the collection of all players and $\mathscr{N}$ as the set of all non-empty and finite subsets of $\mathscr{U}$. We will label all players in a game with the integers 1 to n, where the total number of players in the game is indicated by $n \geq 2$. The player set will be denoted by $N = \{1, ..., n\} \subseteq \mathscr{N}$. A coalition S is then defined as a subset of the player set N, while 2^N represents the set of all possible coalitions (Ferguson, 2020). By convention, we also use the symbol $\emptyset$ to refer to the empty coalition, which consists of zero players. The grand coalition is represented by the player set N.

> *Example* 1. In a game with three players, where $N = \{1, 2, 3\}$, we have eight possible coalitions S, including the empty and grand coalitions: $2^N = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, N\}$.

In cooperative games, we usually deal with the payoff of a game, which can be positive (profit) or negative (loss). In practice, some measure of usefulness or utility can be derived from this payoff, which differs for each player. Therefore, utility is not always equivalent to cash payoffs since there can be a difference in how poor and wealthy players value the same amount of money, resulting in a different utility. This can be described by a utility function $v : 2^N \to \mathbb{R}$. For each coalition $S \subseteq N$, the quantity $v(S)$ may be seen as the power, the value, or the worth of this coalition (Ferguson, 2020). Another frequently used term for $v(S)$ is the safety level of coalition S, which refers to the amount S can guarantee for itself, even in the worst possible circumstances.

The pair (N, v), consisting of the set of players N and the utility function v, represents the coalitional form of an n-person game (Ferguson, 2020). Peters (2008) states that when the utility can be (partially) transferred from each coalition (consisting of at least one player) to another coalition without any loss using a common currency; this is called transferable utility. The mathematical equivalent of this definition can be read in Definition 1.

Definition 1 (Peters, 2008)**.** A cooperative game with transferable utility (TU-game) is a pair (N, v), where $N \in \mathscr{N}$ is the set of players, and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, is a function assigning utilities to each coalition S.

$\mathbb{V}(N)$ denotes the set of all possible TU-games, where $\mathbb{V}_0(N) \subset \mathbb{V}(N)$ contains all TU-games with strictly positive (only profits) or strictly negative (only losses) values v.

We assume the utility function v to be superadditive: $v(S)+v(T) \leq v(S \cup T)$ for disjoint coalitions $S, T \subseteq N$. This assumption is not needed for most of the theory, but it seems to be a natural condition.

After finding the value of each coalition, a logical follow-up question is how this amount should be split among all coalition members. In 1953, Shapley proposed a value concept that assigns a unique payoff vector to each coalition-form game. This function, φ, is a function that allocates an n-tuple of real numbers to any possible utility function corresponding to an n-person game: $\varphi(v) = (\varphi_1(v), \varphi_2(v), ..., \varphi_n(v))$, where $\varphi_i(v)$ denotes the payoff of player i in the game with utility function v. To guarantee a fair distribution of payoffs among all players, we require the function φ to satisfy the following four axioms.

Axioms (Shapley, 1953).

- Efficiency - E. $\sum_{i \in N} \varphi_i(v) = v(N)$.

  Explanation. The total profit is fully distributed among all players.

- Symmetry - Sym. If $\forall S \subseteq N \backslash \{i, j\}$ we have $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\varphi_i(v) = \varphi_j(v)$.

  Explanation. If two players make an equal individual contribution to every coalition, they receive an equal payoff.

- Linearity - Lin. If u, v are utility functions, then $\varphi(u + v) = \varphi(u) + \varphi(v)$.

  Explanation. When two games are combined into a new game, the payoff of the new game equals the sum of payoffs in the two original games.

- Null player - Null. If $\forall S \subseteq N \backslash \{i\}$ we have $v(S \cup \{i\})$, then $\varphi_i(v) = 0$.

  Explanation. If a player neither helps nor harms any coalition he joins, his payoff is zero.

These axioms of fairness led to formulating of the Shapley value in the following theorem, which contributed to his Nobel Memorial Prize in Economic Sciences in 2012.

Theorem 2 (Shapley, 1953). The Shapley value, $Sh = (Sh_1, ..., Sh_n)$ for $i = 1, ..., n$, is the unique function satisfying E, Sym, Lin, and Null and is given by

$$Sh_i(N, v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S| - 1)!(n - |S|)!}{n!} \left[v(S) - v(S \backslash \{i\}) \right]. \tag{1.1}$$

Explanation. (1.1) can be interpreted as follows: in a game with n players, there exist $n!$ permutations. Draw one arbitrary permutation p, with probability $\frac{1}{n!}$ and consider player i. Player i stands in place $p(i)$ in line, where he joins a coalition with $p(i) - 1$ players - together, they form coalition S. Player i's marginal contribution to the coalition is given by $v(S) - v(S \backslash \{i\})$, which is the second term in (1.1). To calculate player i's contribution, it does not matter in which order the first $p(i) - 1$ players are standing and in which order the last $n - p(i)$ will join.

Example 2. Based on the United Nations Security Council, consider a smaller council consisting of five voting nations (A,B,C,D,E), of which two have veto power (nations A and B). To make a decision, at least three votes are needed (winning coalition). We will use the Shapley value to calculate each nation's power (payoff).

Let $v(S) = 0$ for a losing coalition S, and $v(S) = 1$ for a winning coalition. Since the second term of the Shapley value (1.1) represents the increased value of a coalition, only the coalitions that become winning after the entry of a nation play a role in the summation. For one of the small nations, e.g. nation C, his entry only makes coalition {A,B} winning. One of the veto nations, e.g. nation A, makes all of {B,C},{B,D},{B,E},{B,D,E},{B,C,E},{B,C,D},{B,C,D,E} winning coalitions. Therefore, we obtain the following Shapley values:

$$
\begin{aligned}
Sh_{\mathsf{A}}(N,v) &= 3\cdot\frac{(3-1)!(5-3)!}{5!}+3\cdot\frac{(4-1)!(5-4)!}{5!}+\frac{(5-1)!(5-5)!}{5!}=0.450,\\
Sh_{\mathsf{C}}(N,v) &= \frac{(3-1)!(5-3)!}{5!}=0.033.
\end{aligned}
$$

Hence, we observe that the three smaller nations combined have only 10% of the power, whereas both nations with veto power each have 45%.

The example above fits well with today's world, where many economic, political, and military cooperative games are being 'played' between nations. After the invasion of Russia in Ukraine in February 2022, which originated from a long-standing political and territorial disagreement between the two countries, many nations condemned the invasion and imposed economic sanctions on Russia. This resulted in boycotting Russian export products such as coal, oil, and other goods. According to Eurostat, the trade between Russia and Europe dramatically decreased to half of the pre-war levels within one year. Meanwhile, Russia decided to boycott countries helping Ukraine during the war, which made this a two-sided boycotting situation. These boycotts affected more than just the trading relationships: many countries also banned Russian aeroplanes from their airspace, and many big companies decided to close their stores in Russia.

Boycotts impact not only the boycotted party but it also affects the boycotting party, which is why a player should consider the situation thoroughly before initiating a boycott. In this research, we will study models based on cooperative game theory. In Chapter 2, we study a dynamic model in which the success of a consumer boycott can be determined by a participation threshold or the boycott duration. Next, we will look at a static model in Chapter 3. This framework focuses on the impact of a boycott by using the Shapley value to calculate each player's payoff. In the last section of this chapter, we will develop new results and extensions of this model. Finally, in Chapter 4, we make the connection to real-world data about boycotts.

This thesis is written as part of the master's degree in Applied Mathematics at the Delft University of Technology.

2

Effectiveness of consumer boycotts

In this chapter, we study "On the Sources of Consumer Boycotts Ineffectiveness" by Delacote (2009) in which we analyze the effectiveness of a consumer boycott - a situation in which players, individually or collectively, decide to stop buying a specific product from another player or stop buying from a specific company to express their discontent. After having introduced some preliminary remarks and definitions in Section 2.1, in Section 2.2, we consider the (in)effectiveness of boycotts by describing the dynamical coordination between players as a process with threshold effects (Diermeier and Mieghem, 2005), where a boycott is effective if the targeted player modifies its behaviour consistent with the objective of the boycotting players. Section 2.3 introduces the term "boycott duration", which brings valuable insights to anticipate on the conduct of the boycott. Besides, we have corrected some minor errors and clarified some terse arguments by providing several examples of different levels.

2.1. Preliminaries

First, we need to make some assumptions and define some variables. We assume that there are n players (consumers) in the game, who each have their own amount of boycotting costs C_i and potential gain G_i ($i = 1, ..., n$). Each consumer i's decision is represented by the variable $B_i = 0$ (not cooperating in the boycott) or $B_i = 1$ (cooperating in the boycott). Therefore, the total number of boycotting consumers on time t equals

$$n^*(t) = \sum_{i=1}^{n} B_i(t).$$

In this model, we will assume that a firm threshold n^s exists at which the boycott can be seen as successful. This firm threshold is unknown to the consumers in the game, but they have an (optimistic or pessimistic) estimate for this firm threshold, which is the same for all consumers. The firm does know the value of n^s. The probability of success for the boycott at time t, given by $\mathbb{P}[n^*(t) \geq n^s]$, is zero if none of the consumers decides to boycott ($\mathbb{P}[0 \geq n^s] = 0$) - if all consumers participate in the boycott, the success probability equals one ($\mathbb{P}[n \geq n^s] = 1$). In contrast to the firm threshold, the latest number of boycotting consumers n^* is public knowledge. We assume that this belief structure ($\mathbb{P}[n \geq n^s]$) can be modelled with a cumulative distribution function - the choice for this function depends on the consumers' belief about n^s.

One of the major problems of non-organized boycotts is free riding: a consumer has to choose between participating in the boycott, which can be quite costly due to the switch to an imperfect substitute product, and ignoring the boycott, after which the boycott can succeed or fail. The individual decision of one consumer only has a minimal impact on the success probability of the boycott, making the boycott's success most uncertain. Therefore, it is very tempting for a consumer to ignore the boycott but still hope for it to succeed - this is called free riding.

In order to avoid the problem of free riding in the model, we now disregard the consumers' gain from a boycott success without participating and therefore assume that a consumer only profits from a boycott when he participates. This means that for a non-free riding consumer, the choice to boycott or ignore the boycott depends on the expected values of cooperating in the boycott and not cooperating in the boycott: consumer i only participates in the boycott if

$$\begin{aligned} \mathbb{E}[\text{cooperating in the boycott}] &\geq \mathbb{E}[\text{not cooperating in the boycott}], \\ \iff \mathbb{P}\left[\sum_{j\neq i} B_j(t) + 1 \geq n^s\right] G_i - C_i &\geq 0. \end{aligned} \tag{2.1}$$

> **Explanation.** The first term on the left side of the equation represents the probability of a successful boycott after deciding to participate (which explains the "$+1$"). Multiplying this probability with the potential gain after succeeding and subtracting the certain costs results in an expected value. Note that the "$+1$"-term will be negligible in situations with a large enough n, so from now on, we will leave this term out.

Let $\bar{p}_i = \frac{C_i}{G_i}$ be the participation threshold of consumer i, which represents the ratio of boycotting costs over potential gain - we assume this value to lie between 0 and 1. Then the following equation follows from (2.1):

$$\mathbb{P}\left[\sum_{j\neq i} B_j(t) \geq n^s\right] \quad \geq \quad \bar{p}_i. \tag{2.2}$$

> **Explanation.** (2.2) requires that the boycott's success probability is higher than the participation threshold $\bar{p}_i$ of consumer i.

As we can see, the probability of a successful boycott should be greater than the participation threshold of consumer i for him to participate in the boycott. Since all consumers have the same estimated value for the firm threshold n^s, and the number of boycotting players n^* is public knowledge, the only difference between all consumers is their ratio between costs and gains (p_i). This line of reasoning leads to the formulation of Lemma 3.

Lemma 3 (Delacote, 2009)**.** A consumer will participate in the boycott if his potential gain exceeds his boycotting costs, which is denoted by the following decision variable:

$$\begin{cases} B_i(t) = 0 & \text{if } \mathbb{P}[n^*(t) \geq n^s] < \bar{p}_i, \\ B_i(t) = 1 & \text{if } \mathbb{P}[n^*(t) \geq n^s] \geq \bar{p}_i. \end{cases} \tag{2.3}$$

As Lemma 3 indicates, we further assume that all consumers enter sequentially in the boycott: consumers with low costs of boycotting and, therefore, low participation thresholds first, and later, the consumers with a higher participation threshold may enter. This triggers a chain reaction. The following section will analyze whether this chain reaction goes through the entire population or will be extinguished.

2.2. Firm threshold

Now that all necessary assumptions have been made, we can start analyzing the model in which the firm threshold n^s decides the success of a boycott. This model can be seen as a game of imperfect information since the consumers do not know from all other consumers whether they will boycott the firm or not. As we can derive from (2.3), consumers will keep joining the boycott until no consumer is left for whom the probability of boycott success exceeds the participation threshold. When $n^*(t+1) = n^*(t)$, we have reached the equilibrium boycott population, which is defined as follows:

Definition 4 (Delacote, 2009)**.** The last consumer who joins the boycott, that is the consumer whose participation threshold $\bar{p}_i$ is closest underneath the probability of boycott success, is called $\bar{n}$. This value is called the equilibrium boycott population.

> **Explanation.** In case of a boycott involving a large enough number of players with diverse participation thresholds between 0 and 1, the function of $\bar{p}_i$ approaches a continuous function, making it safe to claim that $\bar{n}$ then is defined as $\bar{n} \ : \ \mathbb{P}[\bar{n} \geq n^s] = \bar{p}_{\bar{n}}$.

When the equilibrium boycott population exceeds the firm threshold ($\bar{n} \geq n^s$), the boycott has succeeded.

	A	B	C	D	E	F	G	H
C_i	0.0	0.5	1.5	1.0	2.0	4.0	8.0	5.5
G_i	2.0	10.0	12.0	6.0	10.0	10.0	10.0	6.0
$\bar{p}_i$	0.000	0.050	0.125	0.167	0.200	0.400	0.800	0.917

Table 2.1: Boycotting costs, potential gains, and participation thresholds of eight players, belonging to Example 3.

Example 3. Suppose we have eight consumers who are all considering a boycott and will decide to participate or ignore the boycott according to the model's rules. All consumers in this model believe that the probability of a successful boycott is given by the uniform distribution - their individual participation thresholds are given in Table 2.1.

- $t = 1:\quad n^* = 0$ boycotting consumers, so $\mathbb{P}[n^*(1) \geq n^s] = 0.000$. A joins the boycott since the probability of success is higher than or equal to this consumer's participation threshold.
- $t = 2:\quad n^* = 1$ boycotting consumer, so $\mathbb{P}[n^*(2) \geq n^s] = 0.125$. B and C join the boycott.
- $t = 3:\quad n^* = 3$ boycotting consumers, so $\mathbb{P}[n^*(3) \geq n^s] = 0.375$. D and E join the boycott.
- $t = 4:\quad n^* = 5$ boycotting consumers, so $\mathbb{P}[n^*(4) \geq n^s] = 0.625$. F joins the boycott.
- $t = 5:\quad n^* = 6$ boycotting consumers, so $\mathbb{P}[n^*(5) \geq n^s] = 0.750$. No consumers join anymore.

Thus, in this example, the equilibrium boycott population equals $\bar{n} = 6$ consumers. Indeed, player F is the last player who joined the boycott and his participation threshold $\bar{p}_{\mathsf{F}} = 0.400$ is closest underneath the probability of success (0.750).

Next, we will look into a consumer boycott with more players, after which we will discuss and analyze the observations.

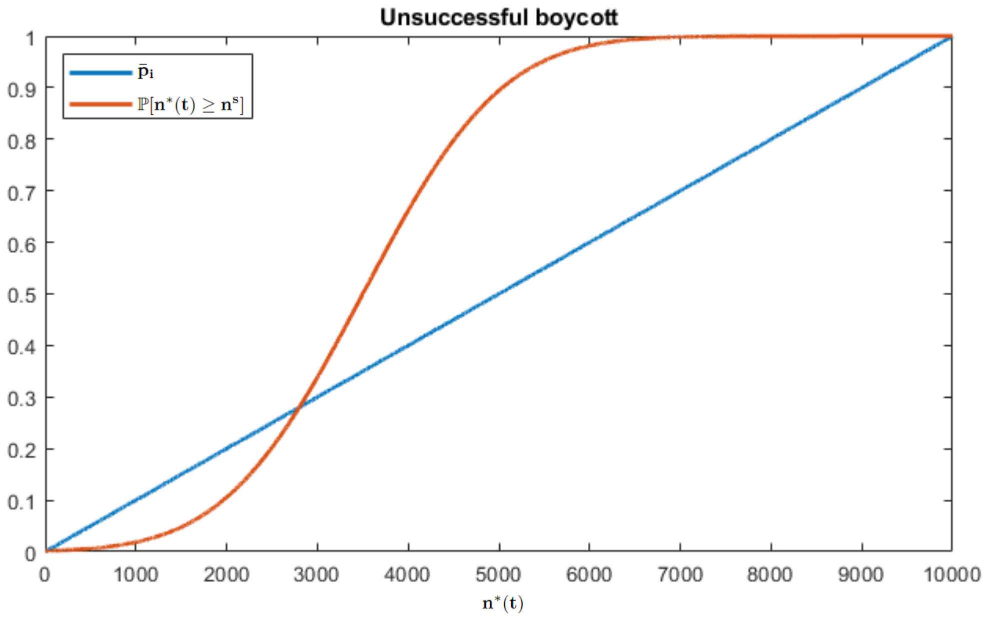

Figure 2.1: Example 4: an unsuccessful boycott, $N = 10.000$, $\bar{n} = 0$.
Parameters: $\bar{p}_i \sim U(0,1)$ and $\mathbb{P}[n^* \geq n^s] \sim N(3.500, 1.200)$.

Example 4. Suppose we have 10.000 consumers who are all considering a boycott and will decide to participate or ignore the boycott according to the model's rules. The participation thresholds $\bar{p}_i$ of all consumers are uniformly distributed, $U(0, 1)$. All consumers in this model believe that the probability of a successful boycott is given by the normal distribution, $N(3.500, 1.200)$.

When we look at the results of these distributions in Figure 2.1, we can see that already from the start of the boycott, we have no consumers for whom the probability of success (marked by the orange function) is higher than their participation thresholds (marked by the blue function). According to the model's decision variable, as defined in (2.3), not a single consumer will decide to boycott.

Hence, we have obtained $\bar{n} = 0$. The source code of Figure 2.1 can be found in Appendix A.

The boycott in Example 4 failed because the boycott success function $\mathbb{P}[n^*(t) \geq n^s]$ (in red) increased faster than the participation threshold function $\bar{p}_i$ (in blue) in Figure 2.1, which always leaves (2.3) at zero. According to this model, the only way to make this boycott succeed is when there would be a sufficient degree of communication, such that at $t = 0$, at least 2.802 consumers decide to join the boycott. After this jump start, the tables have turned: the participation threshold function is now lower than the boycott success function, which would lead to a boycott in which every consumer eventually joins ($\bar{n} = 10.000$). However, without communication, no consumer will decide to take the first step because $\forall i \in N \ \mathbb{P}[n^*(t) \geq n^s] < \bar{p}_i$.

Coordination failures like this, which refer to the incapability of direct coordination, are a major problem of boycotts. Even a potentially successful boycott, as in Example 4, may fail because of the lack of coordination. Large-scale communication is usually impossible, but it would definitely help the cause. When communicating extensively about the boycotting plans, all the players will be much more optimistic about the probability of the boycott's success. This could eventually lead to a potential jump start for a boycott.

Example 5. Suppose we have 10.000 consumers who are all considering a boycott and will decide to participate or ignore the boycott according to the model's rules. The participation thresholds $\bar{p}_i$ of all consumers are normally distributed, $N(4.000, 1.000)$, so quite some players have low participation thresholds, which can easily be the case when a boycott is (almost) costless for a consumer. All consumers in this model are also more optimistic and believe that the probability of a successful boycott is given by the exponential distribution, $Exp(1/2.000)$. The boycott is successful if at least half of the consumers join the boycott: $n^s = 5.000$.

When we look at the results of these distributions in Figure 2.2, we can see that immediately, a large number of consumers decides to participate in the boycott. The equilibrium boycott population equals $\bar{n} = 5.531$, so this boycott has succeeded. The source code of Figure 2.2 can be found in Appendix A.

The originator of this idea, Granovetter (1978), described the equilibrium boycott population as the point where the $\mathbb{P}[n^* \geq n^s]$-function (in red) first crosses the $\bar{p}_i$-function (in blue) from above. Indeed, for a given $\bar{p}_i$, the blue function indicates how many consumers are in the boycott, and the red function shows the believed success probability of the boycott - as long as the red function is above the blue function, the boycott population keeps increasing. With this in mind, it is easy to see that Figure 2.1 represents an unsuccessful boycott ($\mathbb{P}[n^* \geq n^s]$ first crosses $\bar{p}_i$ from below) and that Figure 2.2 represents a successful boycott ($\mathbb{P}[n^* \geq n^s]$ first crosses $\bar{p}_i$ from above at $\bar{n} = 5.531$).

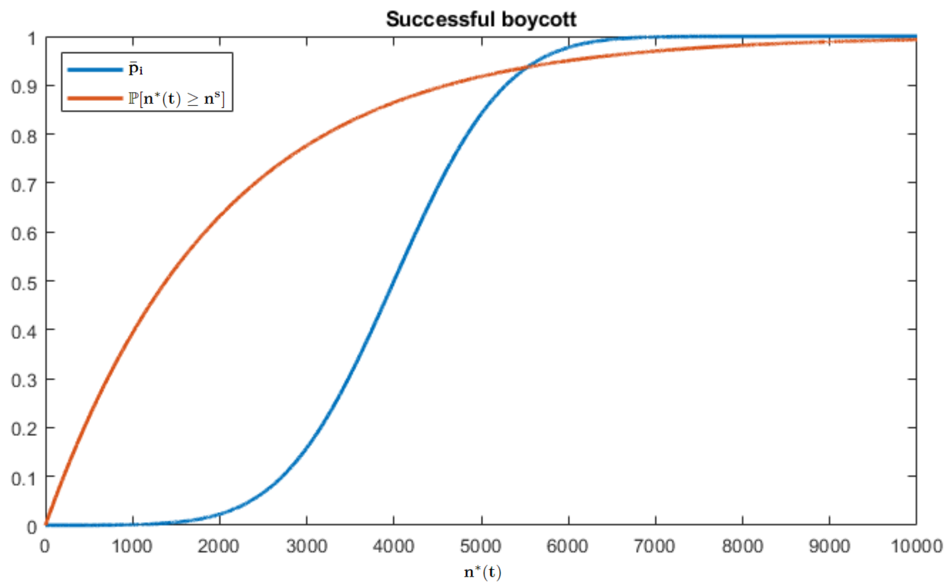

Figure 2.2: Example 5: a successful boycott, $N = 10.000$, $\bar{n} = 5.531$.
Parameters: $\bar{p}_i \sim N(4.000, 1.000)$ and $\mathbb{P}[n^* \geq n^s] \sim Exp(1/2.000)$.

2.3. Maximum boycott duration

Now that we have discussed boycott problems such as free riding and coordination failure and studied how all consumers behave and react to developments in the boycott, it is also interesting to analyze the firm's behaviour. In Section 2.1, we assumed that a boycott is successful when the firm threshold n^s is achieved. In this section, we release this assumption and search for a natural condition for ending the boycott.

First, let us define strategy S_1 as the firm's current strategy (manufacturing product 1), which is most profitable for the company but faces resistance from many consumers. Strategy S_2 is the alternative strategy (manufacturing product 2) that is less profitable for the firm and for which all boycotting consumers are hoping. The firm's profits from strategies S_1 and S_2 are denoted by $\pi_1 > \pi_2$, respectively. Next, we will introduce three utilities: U_1 marks the consumers' utility for product 1, U_2 indicates how much consumers prefer product 2, and U_b denotes the utility for a boycotting consumer who has to buy an imperfect substitute product from another company. Since the boycott aims to force the firm to switch to strategy $S2$, it is only natural to assume that $U_2 > U_1$ and $U_2 > U_b$. In contrary to Section 2.2, we now assume the best-case scenario with perfect information, which means that all profits and costs are common knowledge and that some organization somehow managed to coordinate between all consumers.

When a boycott occurs, and the equilibrium boycott population $\bar{n}$ (Definition 4) has been reached, there can still be a share of consumers who did not join the boycott. We call this share the boycott efficiency parameter $\lambda = \frac{\bar{n}}{n}$. In other words, the residual profit of the firm during a boycott is given by $\lambda\pi_1$. In this situation, such a boycott can be seen as an adapted war of attrition model: a game in which two players simultaneously have to decide between staying in the game and withdrawing in each time step, without changing conditions during play - until one player withdraws. This brings us to the definition of the main subject of Section 2.3.

Definition 5 (Delacote, 2009)**.** The maximum boycott duration is the moment in time after which a player would never stay in the game.

From the firm's perspective, we can describe the net cumulative payoff NCP_f of winning the boycott after T periods by taking the difference between strategy $S1$'s profit and $S2$'s profit. The profit of strategy $S1$ can be calculated by summing the firm's residual profit $\lambda\pi_1$ during the boycott and its entire profit π_1 after winning the boycott - strategy $S2$'s profit is the sum of its full profit π_2 over time since the consumers would not boycott this strategy. Since €1 today is worth more than €1 in six months, we will use $r < 1$ as a discounting factor to calculate the net present value:

$$\begin{aligned} NCP_f(T) &= \text{[Profit of strategy } S1] - \text{[Profit of strategy } S2], \\ &= \left[\sum_{t=0}^{T-1} r^t \lambda\pi_1 + \sum_{t=T}^{\infty} r^t \pi_1\right] - \sum_{t=0}^{\infty} r^t \pi_2. \end{aligned} \tag{2.4}$$

When the firm's net cumulative payoff NCP_f drops below zero, it would be wise to change strategy since, from then on, maintaining strategy $S1$ is not the optimal choice anymore for the firm. The moment in time at which $NCP_f = 0$ denotes the firm's maximum boycott duration T^f. It is fair to claim that in real-world situations, the firm does not have to decide at specific moments in time, but it can decide to maintain or switch the strategy anytime. Therefore, we will transform (2.4) into continuous time by replacing the summations with integrals. Below, we will evaluate the expression $NCP_f(T^f) = 0$ to find an explicit expression for the firm's maximum boycott duration T^f.

$$\begin{aligned} & NCP_f(T^f) = 0, \\ \implies & \int_0^{T^f} r^t \lambda\pi_1 \text{ dt} + \int_{T^f}^{\infty} r^t \pi_1 \text{ dt} - \int_0^{\infty} r^t \pi_2 \text{ dt} = 0, && \text{((2.4) in continuous time)} \\ \implies & \lambda\pi_1 \left[\frac{r^t}{\ln r}\right]_0^{T^f} + \pi_1 \left[\frac{r^t}{\ln r}\right]_{T^f}^{\infty} - \pi_2 \left[\frac{r^t}{\ln r}\right]_0^{\infty} = 0, && \text{(computing the integrals)} \\ \implies & \frac{1}{\ln r}\left(\lambda\pi_1 r^{T^f} - \lambda\pi_1 - \pi_1 \rho^{T^f} + \pi_2\right) = 0, && \text{(evaluating the antiderivatives)} \\ \implies & (\lambda\pi_1 - \pi_1)\rho^{T^f} = \lambda\pi_1 - \pi_2, \\ \implies & r^{T^f} = \frac{\pi_2 - \lambda\pi_1}{(1-\lambda)\pi_1}, \\ \implies & T^f = \frac{1}{\ln r} \ln\left(\frac{\pi_2 - \lambda\pi_1}{(1-\lambda)\pi_1}\right), && \text{for } \pi_2 > \lambda\pi_1. \end{aligned}$$

The previous equation yields no solution in the case of $\pi_2 \leq \lambda\pi_1$, so to be still able to define a value of T^f in this situation, we will look at the interpretation of this expression. When the profit of strategy $S2$, π_2, is less than or equal to the residual profit of strategy $S1$ during a boycott, $\lambda\pi_1$, the firm will always keep using strategy $S1$. Even during a boycott, this strategy is still more profitable than an alternative strategy, so the firm would never withdraw, and its maximum boycott duration would go to infinity.

Lemma 6. The maximum boycott duration of the firm is given by:

$$T^f = \begin{cases} \frac{1}{\ln r} \ln \left(\frac{\pi_2 - \lambda\pi_1}{(1-\lambda)\pi_1} \right), & \text{if } \pi_2 > \lambda\pi_1, \\ \infty, & \text{if } \pi_2 \leq \lambda\pi_1. \end{cases} \tag{2.5}$$

In the same way as above, we can define the net cumulative payoff NCP_c of winning the boycott from the consumers' perspective. The only difference in the equation is that instead of using the firm's profits, we are now using the players' utilities, as defined at the start of this section.

$$NPC_c(T) = \left[\sum_{t=0}^{T-1} r^t U_b + \sum_{t=T}^{\infty} r^t U_2 \right] - \sum_{t=0}^{\infty} r^t U_1. \tag{2.6}$$

Completely analogue to the derivation of equation (2.5), we can derive the consumers' maximum boycott duration T^c as follows:

$$T^c = \frac{1}{\ln r} \ln \left(\frac{U_1 - U_b}{U_2 - U_b} \right), \qquad \text{for } U_2 > U_1 > U_b.$$

Under the previously made assumptions, the only situation in which the equation above yields no solution, is when $U_2 > U_b > U_1$. We can interpret this case as follows: When the consumers' utility for the substitute product (U_b) is higher than their utility for the original product (U_1), the players themselves are not facing any boycotting costs, so it is likely that they can hold on to this boycott forever, which lets their maximum boycotting duration go to infinity.

Lemma 7. The maximum boycott duration of the consumers is given by:

$$T_c = \begin{cases} \frac{1}{\ln r} \ln \left(\frac{U_1 - U_b}{U_2 - U_b} \right), & \text{if } U_2 > U_1 > U_b, \\ \infty, & \text{if } U_2 > U_b > U_1. \end{cases} \tag{2.7}$$

In a game of perfect information, it can easily be decided which side has the longest maximum boycott duration. This gives valuable insight into the decision-making process in the boycott since, for any side, it is better to withdraw immediately at $t = 0$ than to withdraw later.

Example 6. Suppose an energy firm makes $\pi_1 = 2.000$ profit using a polluting strategy S_1 and $\pi_2 = 1.500$ profit using a cleaner strategy S_2. Many environmentalists are trying to force the firm to use $S2$, which is why the utility for strategy $S2$ is higher than strategy $S1$: $U_1 = 2$, $U_2 = 11$. During the boycott, the environmentalists need to consume their energy somewhere else, which has utility $U_b = 1$. Due to the boycott, the firm only keeps a share of $\lambda = 0.72$ of its original profit when all antagonists participate in the boycott. The discount factor is given by $r = 0.98$.

This is a game of perfect information since all information is common knowledge. Using these data, we can calculate both the firm's and environmentalists' maximum boycott duration, which turns out to be $T^f = 111$ days and $T^c = 114$ days. Since the duration of the environmentalists exceeds the firm's duration, the boycott will be successful. Therefore, with this information in hand, it would be a wise decision for the firm to withdraw immediately when the boycott starts to avoid unnecessary losses.

In comparison with the example above, naturally, all variables will differ from one case to another. Therefore, we present a table in which it can be quickly seen whether a boycott will be successful or not, provided a game of perfect information. After considering the assumptions we made in this chapter, four possible scenarios remain, which can be seen in Table 2.2.

		$\pi_2 \leq \lambda\gamma\pi_1$	$\pi_2 > \lambda\gamma\pi_1$
		$\implies T^f \to \infty$	$\implies T^f > 0$
$U_2 > U_1 > U_b$	$\implies T_c > 0$	There is a boycott, but it does not hurt the firm enough to consider a strategy change. Strategy 1 will always be kept.	There is a boycott that hurts the firm. In this case, who lasts the longest? Boycott successful if $T_c \geq T^f$. Boycott unsuccessful if $T_c < T^f$.
$U_2 > U_b > U_1$	$\implies T_c \to \infty$	All players prefer U_b over U_1, while the boycott does not really hurt the firm. This case ends in an eternal boycott.	All players prefer U_b over U_1. The firm is hurt by the boycott, so it withdraws. The boycott is successful.

Table 2.2: Consumers' utilities and the firm's profits determine the outcome of the boycott, adapted from Delacote (2009).

3

Boycott impacts concerning the Shapley value

In this chapter, we study "Impacts of boycotts concerning the Shapley value and extensions" by Besner (2022) and underlying articles where Besner describes the impact of boycotts using a static cooperative game theory model. Compared to the previous chapter about consumer boycotts, the boycotts in cooperative game theory are usually less massive in terms of the number of players, eliminating major problems such as free riding and coordination failure. Besner's article focuses on the impact a boycott can have on both the boycotting player and the player being boycotted since it is important to realize that the boycotting player also punishes himself.

In Section 3.1, we introduce the Harsanyi dividend as a new term, which will eventually be one of the building blocks of this chapter. Section 3.2 contains all preliminary assumptions and theories needed to prove Besner's main result about balanced impacts concerning the Shapley value in Section 3.3.1. Subsections 3.3.2 and 3.3.3 prove the same result for other members of the class of weighted Shapley values but are entirely analogue to the previous subsection, which is why we will not go into too much detail here. Finally, we conclude in Section 3.4 with our own extension of the model, in which we will examine the model's relationship with supermodularity, its extensibility, and a direct formula for the coalition value in a boycott game.

3.1. Harsanyi dividends

After the contributions of Neumann and Morgenstern (1944) and Shapley (1953) to the field of cooperative game theory, subsequent Nobel Memorial Prize laureate Harsanyi (1959) wanted to set up an analytical model for 2-person non-constant and n-person games. Not much later, in 1963, Harsanyi used his social model to generalize the Shapley value (Theorem 2). A fundamental assumption that Harsanyi used in his model is that in every possible subset of players (coalition), all players will agree to cooperate to protect their common interests. In other words, each player is not a member of only one coalition but simultaneously a member of all possible coalitions.

Harsanyi (1959) introduced a concept to quantify each coalition's contribution to the grand coalition N's worth, which he did inductively: the dividend of a 1-person coalition simply equals its value, while all other coalitions' dividends equal their value minus the Harsanyi dividends of all proper subcoalitions.

Definition 8 (Harsanyi, 1959)**.** Let $(N, v) \in \mathbb{V}(N)$. For all coalitions $S \subseteq N$, the Harsanyi dividends $\Delta_v(S)$ are defined inductively by

$$\Delta_v(S) = \begin{cases} v(S) - \sum_{R \subset S} \Delta_v(R), \text{ if } |S| \geq 1, \text{ and} \\ 0, \text{ if } S = \emptyset. \end{cases} \tag{3.1}$$

Explanation. The Harsanyi dividends can be interpreted as the pure surplus of cooperation: What is the added value of this specific coalition, compared to all smaller coalitions consisting of these players (proper subcoalitions)? Therefore, the value $v(S)$ of coalition S is equal to the sum of all added values (Harsanyi dividends) $\Delta_v(S)$.

Example 7. Consider a 3-player game, $N = \{\text{A}, \text{B}, \text{C}\}$, where player B can not win anything on his own but makes a significant difference when cooperating. The utility function v is given by $v(\emptyset) = 0$, $v(\text{A}) = 1$, $v(\text{B}) = 0$, $v(\text{C}) = 4$, $v(\text{A,B}) = 7$, $v(\text{A,C}) = 5$, $v(\text{B,C}) = 4$, $v(N) = 12$.

Then we obtain $\Delta_v(\text{A}) = 1$, $\Delta_v(\text{B}) = 0$ and $\Delta_v(\text{C}) = 4$. Besides, only one of the 2-player coalitions results in an extra value: $\Delta_v(\text{A,B}) = 6$ and $\Delta_v(\text{A,C}) = \Delta_v(\text{B,C}) = 0$. In the end, even the grand coalition N gives an extra surplus: $\Delta_v(N) = 1$.

However, Harsanyi dividends can also be negative and contribute negatively to the worth of the grand coalition. This is caused by how Harsanyi dividends are defined, in which the effect of overlapping coalitions (with at least one common player) is not considered. Therefore, the Harsanyi dividend of a bigger coalition might have to compensate with a negative dividend such that Definition 8 still holds. This implies that the grand coalition N has no cooperation benefit if its worth is equal to or smaller than the sum of all Harsanyi dividends of proper subcoalitions. Many researchers find the possibility of negative Harsanyi dividends unsatisfying, which is why they have come up with several other definitions of the surplus value (Besner, 2022). In this thesis, we will keep working with the original Harsanyi dividend, of which an example of its negative values can be seen below.

Example 8. Consider a 3-player game ($N = \{\text{A}, \text{B}, \text{C}\}$), in which player B is selling 10 products that have no intrinsic value to player B itself. The product has intrinsic value to players A and C, who are both willing to buy 8 products. The utility function v, representing the intrinsic value, is given by $v(\emptyset) = v(\text{A}) = v(\text{B}) = v(\text{C}) = 0$ since no single player can increase his value on his own. Players A and B, and players B and C can trade together - players A and C cannot do anything in a coalition: $v(\text{A,B}) = v(\text{B,C}) = 8$, and $v(\text{A,C}) = 0$.

Then we obtain $\Delta_v(\text{A}) = \Delta_v(\text{B}) = \Delta_v(\text{C}) = 0$, $\Delta_v(\text{A,C}) = 0$, $\Delta_v(\text{A,B}) = \Delta_v(\text{B,C}) = 8$, and $\Delta_v(N) = -6$. This makes sense since all coalitions that cannot increase their value on their own have zero added value. Creating a coalition with players A and B, or players B and C, yields a surplus of 8, since they can increase their value. However, these two coalitions are overlapping, so in the grand coalition $N = \{\text{A,B,C}\}$, it is not possible that both players A and C still buy 8 products each since player B only had 10 products available. Therefore, the Harsanyi dividend of the grand coalition has to compensate for this, resulting in a negative surplus for N.

By calculating the values of Harsanyi dividends in easy examples, we recognized a pattern, which led to an equivalent formula in direct form. In Lemma 9, we will prove the equivalence of this formula to Definition 8, which will be very useful for future calculations and proofs.

Lemma 9. The Harsanyi dividends, as defined in Definition 8, are equivalent to

$$\Delta_v(S) = \sum_{R \subseteq S} (-1)^{|S|-|R|} \, v(R). \tag{3.2}$$

Proof. Let $(N, v) \in \mathbb{V}(N)$. We will prove this lemma using the principle of mathematical induction on the set size $|S|$. Obviously, the lemma is true for $|S| = 0$, which implies that S is the empty set. Both (3.1) and (3.2) result in $\Delta_v(S) = 0$, so the base case holds.

For the induction step, assume that the lemma holds for all $|S| < k \in \mathbb{N}$ (induction hypothesis). Below, we will prove that the lemma then also holds for $|S| = k$.

$$\begin{aligned}
\Delta_v(S) &= v(S) - \sum_{R \subset S} \Delta_v(R), && \text{(Definition 8)} \\
&= v(S) - \sum_{R \subset S} \sum_{Q \subseteq R} (-1)^{|R|-|Q|} \, v(Q), && \text{(induction hypothesis)} \\
&= v(S) - \sum_{Q \subset S} \left(v(Q) \sum_{\substack{R \subseteq S \\ R \supseteq Q}} (-1)^{|R|-|Q|} \right), && \text{(commutative property of addition)} \\
&= v(S) - \sum_{Q \subset S} \left(v(Q) \sum_{k=0}^{|S|-|Q|-1} \binom{|S|-|Q|}{k} (-1)^k \right), && \text{(counting sets)} \\
&= v(S) - \sum_{Q \subset S} (-1)^{|S|-|Q|-1} \, v(Q), && \text{(Newton's Binomial Theorem)} \\
&= \sum_{Q \subseteq S} (-1)^{|S|-|Q|} \, v(Q).
\end{aligned}$$

Hence, by the principle of mathematical induction, Lemma 9 holds $\forall |S| \in \mathbb{N}$. □

3.2. Preliminaries

In this section, we will introduce some necessary assumptions and theories. In Chapter 1, we learnt about the Shapley value Sh (Theorem 2), which gives a fair distribution of the total amount among all players. In 1959, Harsanyi proposed an equivalent definition of the Shapley value, expressed in terms of all the coalitions' dividends. He showed that the Shapley value coincides with an equal division of Harsanyi dividends within a coalition, which is the definition we will use from now on. In order to prove this equivalence, we will make use of the following lemma.

Lemma 10. For $k, n \in \mathbb{N}$ such that $0 < k \leq n$, the following identity is true:

$$\sum_{x=0}^{n-k} \frac{(k+x-1)!}{x!} = \frac{n!}{k \cdot (n-k)!}. \tag{3.3}$$

Proof. We will prove this lemma using the principle of mathematical induction on the set size $|N| = n$, starting from $n = k$. The identity is obviously true for $n = k$ since both sides of the identity yield the same result:

$$\sum_{x=0}^{0} \frac{(k+x-1)!}{x!} = (k-1)! \qquad \text{and} \qquad \frac{k!}{k \cdot 0!} = (k-1)!$$

For the induction step, we assume that the identity holds for all $n < m \in \mathbb{N}$ (induction hypothesis). Now, we will prove that the identity then also holds for $n = m$.

$$\begin{aligned}
\sum_{x=0}^{m-k} \frac{(k+x-1)!}{x!} &= \sum_{x=0}^{m-k-1} \frac{(k+x-1)!}{x!} + \frac{(k+(m-k)-1)!}{(m-k)!}, \\
&= \frac{(m-1)!}{k\cdot(m-1-k)!} + \frac{(m-1)!}{(m-k)!}, && \text{(induction hypothesis)} \\
&= \frac{(m-k)\cdot(m-1)!}{k\cdot(m-k)!} + \frac{k\cdot(m-1)!}{k\cdot(m-k)!}, && \text{(equalizing denominators)} \\
&= \frac{m\cdot(m-1)!}{k\cdot(m-k)!} + \frac{k\cdot(m-1)!}{k\cdot(m-k)!}, \\
&= \frac{m!}{k\cdot(m-k)!}.
\end{aligned}$$

Hence, by the principle of mathematical induction, Lemma 10 holds $\forall n \in \mathbb{N},\ n \geq k$. □

An equivalent expression for Lemma 10, obtained by dividing both sides by $(k-1)!$, is $\sum_{x=0}^{n-k} \binom{k+x-1}{x} = \binom{n}{k}$. A graphical interpretation of the latter can be seen in Pascal's triangle, in which the lemma then states that the answer to any diagonal summation can be found in the row below, one spot to the right (Table 3.1).

$n=0$							1						
$n=1$						1		1					
$n=2$					1		2		1				
$n=3$				1		3		3		1			
$n=4$			1		4		6		4		1		
$n=5$		1		5		10		10		5		1	
$n=6$	1		6		15		20		15		6		1

Table 3.1: Pascal's Triangle, filled with binomial coefficients $\binom{n}{k}$, where k denotes the element-number in the n^{th} row. The red and blue numbers show two examples of a result of Lemma 10: the answer to any diagonal summation can be found in the row below, one spot to the right.

Next, we will state our proof for Harsanyi's expression of the Shapley value. An alternative proof via the Möbius inversion formula can be found in Caulier, Mauleon, and Vannetelbosch (2015).

Theorem 11 (Harsanyi, 1959)**.** The Shapley value in Definition 2 (Shapley, 1953) can be written as

$$Sh_i(N,v) = \sum_{\substack{S\subseteq N \\ S\ni i}} \frac{\Delta_v(S)}{|S|}, \quad \forall i \in N. \tag{3.4}$$

Proof. Let $(N,v) \in \mathbb{V}(N)$. In the comments next to the computation below, we will use 'nCr' and 'CPA' as abbreviations for 'binomial coefficient' and 'commutative property of addition', respectively.

$$\begin{aligned}
Sh_i(N,v) &= \sum_{\substack{S\subseteq N \\ S\ni i}} \frac{(|S|-1)!(n-|S|)!}{n!} \left[v(S) - v(S\backslash\{i\})\right], && \text{(Theorem 2)} \\
&= \sum_{\substack{S\subseteq N \\ S\ni i}} \left(\frac{(|S|-1)!(n-|S|)!}{n!} \left(\sum_{R\subseteq S} \Delta_v(R) - \sum_{R\subseteq S\backslash\{i\}} \Delta_v(R) \right) \right), && \text{(Definition 8)} \\
&= \sum_{\substack{S\subseteq N \\ S\ni i}} \left(\frac{(|S|-1)!(n-|S|)!}{n!} \sum_{\substack{R\subseteq S \\ R\ni i}} \Delta_v(R) \right),
\end{aligned}$$

$$= \sum_{\substack{R \subseteq N \\ R \ni i}} \left(\Delta_v(R) \sum_{\substack{S \subseteq N \\ S \supseteq R}} \frac{(|S|-1)!(n-|S|)!}{n!} \right), \qquad \text{(CPA)}$$

$$= \sum_{\substack{R \subseteq N \\ R \ni i}} \left(\Delta_v(R) \sum_{x=0}^{n-|R|} \frac{(|R|+x-1)!(n-|R|-x)!}{n!} \cdot \binom{n-|R|}{x} \right), \qquad \text{(counting sets)}$$

$$= \sum_{\substack{R \subseteq N \\ R \ni i}} \left(\Delta_v(R) \sum_{x=0}^{n-|R|} \frac{(|R|+x-1)!(n-|R|-x)!}{n!} \cdot \frac{(n-|R|)!}{(n-|R|-x)! \cdot x!} \right), \qquad \text{(nCr)}$$

$$= \sum_{\substack{R \subseteq N \\ R \ni i}} \left(\Delta_v(R) \cdot \frac{(n-|R|)!}{n!} \sum_{x=0}^{n-|R|} \frac{(|R|+x-1)!}{x!} \right),$$

$$= \sum_{\substack{R \subseteq N \\ R \ni i}} \Delta_v(R) \cdot \frac{(n-|R|)!}{n!} \cdot \frac{n!}{|R| \cdot (n-|R|)!)}, \qquad \text{(Lemma 10)}$$

$$= \sum_{\substack{R \subseteq N \\ R \ni i}} \frac{\Delta_v(R)}{|R|}.$$

Hence, we can conclude that Harsanyi's expression for the Shapley value in Theorem 11 coincides with Shapley's definition in Theorem 2. □

The Shapley value, as given in Theorem 11, is based on an equal division of Harsanyi dividends amongst the members of each coalition. Another way to split these dividends is proportionally: more valuable players receive a higher payoff than less valuable players in the game. Therefore, we can define a Shapley value that uses endogenously given weights that are given by the individual values of the players, which is given by the following definition.

Definition 12 (Besner, 2016 & Béal et al., 2018)**.** The proportional Shapley value Sh^p is given by

$$Sh_i^p(N,v) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \, \Delta_v(S), \quad \forall i \in N. \tag{3.5}$$

Both the Shapley value, as given in Theorem 11, and the proportional Shapley value, as given in Definition 12, are special cases of the same family of weighted Shapley values. Shapley introduced this family in 1953, in which each player is assigned an exogenously given positive weight.

Definition 13 (Shapley, 1953)**.** Let $W = \{f : \mathscr{U} \to \mathbb{R}_+\}$ be a set that contains all possible (positive) weight distributions on the collection of all players. The (positively) weighted Shapley values Sh^w are given by

$$Sh_i^w(N,v) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{w_i}{\sum_{j \in S} w_j} \Delta_v(S), \quad \forall i \in N, w \in W. \tag{3.6}$$

In the remainder of this section, we will use the following axioms for TU-values.

Axioms (Besner, 2022, Peyton Young, 1985 & Myerson, 1980)**.**

- Efficiency - E. $\sum_{i\in N} \varphi_i(v) = v(N)$.

  > **Explanation.** The total profit is fully distributed among all players (similar to Chapter 1).

- Marginality - Mar. Let the marginal contribution of a player $i \in N$ to a coalition $S \subset N \backslash \{i\}$ be given by $MC_i^v(S) = v(S \cup \{i\}) - v(S)$. Then for all $(N, v),\ (N, w) \in \mathbb{V}(N)$, with $MC_i^v(S) = MC_i^w(S)\ \forall S \subseteq N \backslash \{i\}$, we have

$$\varphi_i(N, v) = \varphi_i(N, w).$$

  > **Explanation.** A player with identical marginal contributions in different games receives the same payoff in both games.

- Balanced contributions - BC. For all $(N, v) \in \mathbb{V}(N)$ and distinct players $i, j \in N$:

$$\varphi_i(N, v) - \varphi_i(N \backslash \{j\}, v) = \varphi_j(N, v) - \varphi_j(N \backslash \{i\}, v).$$

  > **Explanation.** When player i leaves the coalition, the loss that player j incurs is identical to the loss of player i, when player j decides to leave the coalition.

The balanced contributions axiom BC can be rewritten into three variants. The proportional balanced contributions - PBC (Besner, 2016 & Béal et al., 2018), the w-weighted balanced contributions - WBC^w (Myerson, 1980), and the weak balanced contributions - BC$^-$ (Casajus, 2017) axioms can be obtained as follows:

- PBC, by dividing the left-hand side by $v(\{i\})$ and the right-hand side by $v(\{j\})$.
- WBC^w, by dividing the left-hand side by w_i and the right-hand side by w_j, with $w \in W$.
- BC$^-$, by applying $\text{sign}(x) = \begin{cases} -1, & \text{if x<0,} \\ 0, & \text{if x=0,} \\ 1, & \text{if x>0,} \end{cases}$ to both sides of the equation.

In Section 3.3, we will refer to the following theorems, of which we designed our own proof.

Theorem 14. (Myerson, 1980). Sh is the unique TU-value that satisfies E and BC.

Proof. Let $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$.

I. Existence: Sh satisfies both E and BC:

$$\begin{aligned} \sum_{i\in N} Sh_i(N, v) &= \sum_{i\in N} \sum_{\substack{S\subseteq N \\ S\ni i}} \frac{\Delta_v(S)}{|S|}, && \text{(Theorem 11)} \\ &= \sum_{S\subseteq N} \Delta_v(S), \\ &= v(N). && \text{(Definition 8)} \end{aligned}$$

$$\begin{aligned}
Sh_i(N,v) - Sh_i(N\backslash\{j\},v) &= \sum_{\substack{S\subseteq N\\ S\ni i}} \frac{\Delta_v(S)}{|S|} - \sum_{\substack{S\subseteq N\backslash\{j\}\\ S\ni i}} \frac{\Delta_v(S)}{|S|}, && \text{(Theorem 11)}\\
&= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,j\}}} \frac{\Delta_v(S)}{|S|},\\
&= \sum_{\substack{S\subseteq N\\ S\ni j}} \frac{\Delta_v(S)}{|S|} - \sum_{\substack{S\subseteq N\backslash\{i\}\\ S\ni j}} \frac{\Delta_v(S)}{|S|},\\
&= Sh_j(N,v) - Sh_j(N\backslash\{i\},v). && \text{(Theorem 11)}
\end{aligned}$$

II. Uniqueness: Let φ be a TU-value on $\mathbb{V}(N)$ that satisfies E and BC. We will prove that $\varphi = Sh$ using the principle of mathematical induction on the set size $|N|$. Obviously, the statement holds for $|N| = 1$, where we use E to obtain the result:

$$\varphi_k(N,v) = \sum_{i\in N} \varphi_i(N,v) = v(N) = \sum_{i\in N} Sh_i(N,v) = Sh_k(N,v), \qquad k \in N.$$

For the induction step, assume that the statement holds for all $|N| < n \in \mathbb{N}$ (induction hypothesis). Below, we will prove that the statement then also holds for $|N| = n$. Let $i,j \in N$. Then:

$$\begin{aligned}
Sh_i(N,v) - Sh_j(N,v) &= Sh_i(N\backslash\{j\},v) - Sh_j(N\backslash\{i\},v), && \text{(BC)}\\
&= \varphi_i(N\backslash\{j\},v) - \varphi_j(N\backslash\{i\},v), && \text{(induction hypothesis)}\\
&= \varphi_i(N,v) - \varphi_j(N,v). && \text{(BC)}
\end{aligned}$$

Since this equality holds $\forall i,j \in N$, there exists a constant $c \in \mathbb{R}$ such that $Sh_i(N,v) - \varphi_i(N,v) = c$, $\forall i \in N$. By applying E, we obtain:

$$\begin{aligned}
c &= \frac{1}{n}\sum_{i\in N} c,\\
&= \frac{1}{n}\sum_{i\in N} \left(Sh_i(N,v) - \varphi_i(N,v)\right),\\
&= \frac{1}{n}\left(\sum_{i\in N} Sh_i(N,v) - \sum_{i\in N}\varphi_i(N,v)\right),\\
&= \frac{1}{n}(v(N) - v(N)),\\
&= 0.
\end{aligned}$$

Hence, if a TU-value exists that satisfies E and BC, this value equals the Shapley value. Therefore, Sh is the unique TU-value that satisfies E and BC. □

Both Theorem 15 and 16 are variations on Theorem 14, with another member from the family of weighted Shapley values as the only difference. Their respective proofs are left out since they are analogous to the proof of Theorem 14. The proof of Theorem 17 is provided in Casajus (2021).

Theorem 15. (Myerson, 1980). Let $w \in W$, $(N,v) \in \mathbb{V}(N)$.
Sh^w is the unique TU-value that satisfies E and WBC^w.

Theorem 16. (Besner, 2016). Let $(N, v) \in \mathbb{V}_0(N)$.
Sh^p is the unique TU-value that satisfies E and PBC.

Theorem 17. (Casajus, 2021). Let $(N, v) \in \mathbb{V}(N)$.
Sh is the unique TU-value that satisfies E, Mar, and BC$^-$.

3.3. Balanced impacts of boycotts concerning the Shapley family

One of the main ideas Besner (2022) presents in his study is that of disjointly productive players. We call two players disjointly productive when the marginal contribution of one player to the coalition does not depend on the presence or absence of the other player.

Definition 18 (Besner, 2022)**.** For all $(N, v) \in \mathbb{V}(N)$, two distinct players $i, j \in N$ are called disjointly productive in (N, v) if, for all $S \subseteq N \backslash \{i, j\}$, we have

$$v(S \cup \{i, j\}) - v(S \cup \{j\}) = v(S \cup \{i\}) - v(S). \tag{3.7}$$

When the marginal contribution of a player is not affected by the presence or absence of another player, so when they are disjointly productive, a coalition containing both players will have no advantage of cooperation: their surplus value equals zero.

Lemma 19 (Besner, 2022)**.** Let $(N, v) \in \mathbb{V}(N)$. Two distinct players $i, j \in N$ are disjointly productive in (N, v) if and only if for all $S \subseteq N$, we have $\Delta_v(S) = 0$ if $\{i, j\} \subseteq S$.

Proof. $\Rightarrow$ Let $(N, v) \in \mathbb{V}(N)$ and $S \subseteq N$. Let $i, j \in N$ be disjointly productive. Observe that

$$\begin{aligned}
\Delta_v(S) &= \sum_{R \subseteq S} (-1)^{|S|-|R|} v(R), \\
&= \sum_{\substack{R \subseteq S \\ i \in R \\ j \in R}} (-1)^{|S|-|R|} v(R) + \sum_{\substack{R \subseteq S \\ i \in R \\ j \notin R}} (-1)^{|S|-|R|} v(R) + \sum_{\substack{R \subseteq S \\ i \notin R \\ j \in R}} (-1)^{|S|-|R|} v(R) + \sum_{\substack{R \subseteq S \\ i \notin R \\ j \notin R}} (-1)^{|S|-|R|} v(R), \\
&= \sum_{R \subseteq S \backslash \{i,j\}} (-1)^{|S|-|R|} \, [v(R \cup \{i, j\}) - v(R \cup \{i\}) - v(R \cup \{j\}) + v(R)], \\
&= \sum_{R \subseteq S \backslash \{i,j\}} (-1)^{|S|-|R|} \cdot 0, \\
&= 0,
\end{aligned}$$

where the final steps are obtained by making use of the initial assumption and Definition 18.

$\Leftarrow$ Now assume that $\Delta_v(S \cup \{i, j\}) = 0$ for all coalitions $S \subset N \backslash \{i, j\}$. Then we obtain:

$$\begin{aligned}
v(S \cup \{i, j\}) - v(S \cup \{j\}) - v(S \cup \{i\}) + v(S) &= \\
\sum_{R \subseteq S \cup \{i,j\}} \Delta_v(S) - \sum_{R \subseteq S \cup \{j\}} \Delta_v(S) - \sum_{R \subseteq S \cup \{i\}} \Delta_v(S) + \sum_{R \subseteq S} \Delta_v(S) &= \\
\sum_{R \subseteq S} \Delta_v(S \cup \{i, j\}) &= 0,
\end{aligned}$$

where we use Definition 8 to obtain the final result. Hence, the lemma holds. $\square$

When a boycott occurs, one player withdraws from any relations with another player, often as a form of financial punishment or political protest. In other words, the boycotting and boycotted players change their behaviour towards each other compared with the original game. Therefore, it does not matter which player is the boycotting player since the impact on the payoff changes equally for both players. Note that a boycott between two players can be seen as those two players becoming disjointly productive in (N, v).

Definition 20 (Besner, 2022)**.** Let $(N, v) \in \mathbb{V}(N)$, and $i, j \in N$ be distinct players. A TU-game (N, v^{ij}) such that i is disjointly productive in relation to j is called the (i, j)-boycott game corresponding to (N, v) if

$$v^{ij}(S) = v(S) \qquad \forall S \subseteq N, \quad \{i, j\} \nsubseteq S. \tag{3.8}$$

In Subsection 3.4.3, we will see that such a function v^{ij} always exists.

3.3.1. Shapley value

In this subsection, we will focus on using the Shapley value, as defined in Theorem 11, to discuss the balanced impacts of a boycott. In order to do so, we will introduce two new axioms in addition to the previously mentioned axioms in Section 3.2.

Axioms (Besner, 2022)**.**

- Disjointly productive players - DP. For all $(N, v) \in \mathbb{V}(N)$, and $i, j \in N$ such that i and j are disjointly productive players in (N, v), we have

$$\varphi_i(N, v) = \varphi_i(N \backslash \{j\}, v).$$

> **Explanation.** One player's payoff is not affected by the other player leaving the game.

- Balanced impacts of boycotts - BIB. For all $(N, v) \in \mathbb{V}(N)$, and the (i, j)-boycott games (N, v^{ij}) corresponding to (N, v), we have

$$\varphi_i(N, v) - \varphi_i(N, v^{ij}) = \varphi_j(N, v) - \varphi_j(N, v^{ij}).$$

> **Explanation.** The impact of a boycott is equal for both the boycotting and the boycotted players.

As we can see, the BC axiom (Section 3.2) requires games on different player sets, whilst the BIB axiom is defined on the same player set N. Nonetheless, we can formulate a relationship between the BC and BIB axioms by using the DP axiom.

Lemma 21 (Besner, 2022)**.** DP and BIB imply BC.

Proof (Besner, 2022). Let $(N, v) \in \mathbb{V}(N)$, $i, j \in N$ be distinct players, (N, v^{ij}) be the (i, j)-boycott game corresponding to (N, v), and φ be a TU-value satisfying DP and BIB. Then we have

$$\varphi_i(N, v^{ij}) \overset{\text{(DP)}}{=} \varphi_i(N \backslash \{j\}, v^{ij}) \overset{\text{(Def. 20)}}{=} \varphi_i(N \backslash \{j\}, v),$$
$$\varphi_j(N, v^{ij}) \overset{\text{(DP)}}{=} \varphi_j(N \backslash \{i\}, v^{ij}) \overset{\text{(Def. 20)}}{=} \varphi_j(N \backslash \{i\}, v).$$

Substituting these results into BIB yields the expression for BC.

$$\begin{aligned} \varphi_i(N,v) - \varphi_i(N,v^{ij}) &= \varphi_j(N,v) - \varphi_j(N,v^{ij}), \\ \implies \varphi_i(N,v) - \varphi_i(N\backslash\{j\},v) &= \varphi_j(N,v) - \varphi_j(N\backslash\{i\},v). \end{aligned}$$

Hence, DP and BIB imply BC. □

Now, we can state the main result of Besner (2022) below.

Theorem 22 (Besner, 2022)**.** Let $(N,v) \in \mathbb{V}(N)$.
Sh is the unique TU-value that satisfies E, DP, and BIB.

Proof (Besner, 2022). Let $(N,v) \in \mathbb{V}(N)$.

I. Existence: By Theorem 14, Sh satisfies E. Sh also satisfies DP:

$$Sh_i(N,v) = \sum_{\substack{S\subseteq N \\ S\ni i}} \frac{\Delta_v(S)}{|S|} \overset{\text{(Lemma 19)}}{=} \sum_{\substack{S\subseteq N \\ S\ni i \\ S\not\ni j}} \frac{\Delta_v(S)}{|S|} = Sh_i(N\backslash\{j\},v).$$

By Lemma 19, we obtain, for $i,j \in N$, the following equation, which satisfies BIB:

$$Sh_i(N,v) - Sh_i(N,v^{ij}) = \sum_{\substack{S\subseteq N \\ S\supseteq\{i,j\}}} \frac{\Delta_v(S)}{|S|} = Sh_j(N,v) - Sh_j(N,v^{ij}).$$

II. Uniqueness: Let φ be a TU-value on $\mathbb{V}(N)$ that satisfies E, DP, and BIB. By Lemma 21, DP and BIB imply that φ also satisfies BC. Since φ satisfies E and BC, Theorem 14 implies that φ is then uniquely determined. □

To create a greater understanding of a situation in which Theorem 22 applies, we provide Example 9.

Example 9 (Nouweland et al., 1996 & Besner, 2022). The Terrestrial Flight Telephone System (TFTS) is a public service for airline passengers. In short, the telephone connection of a passenger is established by connecting the aircraft's communication equipment to a near ground station, which in turn is connected to the existing network. The main question for the cooperating countries remains: how to divide the revenues that are generated through TFTS?

For a simple case with five countries, let us model this situation as a coalitional game (N,v) with player set $N = \{\text{A,B,C,D,E}\}$. Installing the necessary equipment costs 15 per ground station and 3 per aeroplane. The number of ground stations and aeroplanes per country is provided in Table 3.2 - Table 3.3 shows the revenues a_{ij} per country $i \in N$, which are generated when an aeroplane from country $j \in N$ flies over country i. If the countries within a coalition $S \subseteq N$ decide to work together, the worth $v(S)$ of such a coalition is given by the jointly earned revenues:

$$v(S) = \sum_{i\in S} a_{ii} + \sum_{i\in S} \sum_{\substack{j\in S \\ j\neq i}} a_{ij}. \tag{3.9}$$

A complete overview of all coalition values, calculated via (3.9), is provided in the upper part of Table 3.4. By using this table and Definition 8, we can calculate this game's Harsanyi dividend for each coalition:

Country	Costs of ground stations and aeroplanes	
A	4 stations $\cdot$ $15 + 150$ aeroplanes $\cdot$ $3 =$	510
B	4 stations $\cdot$ $15 + 210$ aeroplanes $\cdot$ $3 =$	690
C	3 stations $\cdot$ $15 + 250$ aeroplanes $\cdot$ $3 =$	795
D	5 stations $\cdot$ $15 + 400$ aeroplanes $\cdot$ $3 =$	1,275
E	5 stations $\cdot$ $15 + 420$ aeroplanes $\cdot$ $3 =$	1,335

Table 3.2: Operating costs of the TFTS, belonging to Example 9. Adapted from Nouweland et al. (1996).

Ground	aeroplanes					
station	**A**	**B**	**C**	**D**	**E**	**Sum**
A	430	2	6	46	12	496
B	1	603	27	27	68	726
C	8	24	834	129	138	1,133
D	286	12	91	1,072	264	1,725
E	28	164	174	259	1,688	2,313

Table 3.3: Revenues generated by a country's base stations, belonging to Example 9. Adapted from Nouweland et al. (1996).

$$\Delta_v(S) = \begin{cases} a_{ii}, & \text{if } S = \{i\}, \quad i \in N, \\ a_{ij} + a_{ji}, & \text{if } S = \{i,j\},\ i,j \in N,\ i \neq j, \\ 0, & \text{if } S \subseteq N, \quad |S| > 2. \end{cases} \tag{3.10}$$

From this point on, we leave the example of Nouweland et al. (1996) and follow Besner's approach to creating a boycott. We assume that country A boycotts country D: its ground stations will no longer support telephone traffic from aeroplanes from country D, and the aeroplanes of country A will no longer make telephone calls over country D. Hence, we obtain $\Delta_v(\text{A,D}) = 0$ - all other dividends remain the same. By Lemma 19, Table 3.3, and (3.9), the (A,D)-boycott game (N, v^{AD}), as defined in Definition 20, is given in the lower part of Table 3.4.

Besner's main result in Theorem 22 states the balanced impacts of boycotts property of the Shapley value. Below, we compare three ways to divide the grand coalition's value (sum of revenues) amongst the five countries. $P(N,v)$ splits this value proportional to the countries' costs in Table 3.2, R just keeps the profits at the corresponding ground stations, as in Table 3.3. On the next page, we will analyze the boycott's impact on countries A and D by using payoff functions P, R, and Sh.

$v(\{A\}) = 430$	$v(\{A,B\}) = 1,036$	$v(\{A,B,C\}) = 1,935$	$v(\{A,B,C,D\}) = 3,598$
$v(\{B\}) = 603$	$v(\{A,C\}) = 1,278$	$v(\{A,B,D\}) = 2,479$	$v(\{A,B,C,E\}) = 4,207$
$v(\{C\}) = 834$	$v(\{A,D\}) = 1,834$	$v(\{A,B,E\}) = 2,996$	$v(\{A,B,D,E\}) = 4,962$
$v(\{D\}) = 1,072$	$v(\{A,E\}) = 2,158$	$v(\{A,C,D\}) = 2,902$	$v(\{A,C,D,E\}) = 5,465$
$v(\{E\}) = 1,688$	$v(\{B,C\}) = 1,488$	$v(\{A,C,E\}) = 3,318$	$v(\{B,C,D,E\}) = 5,574$
	$v(\{B,D\}) = 1,714$	$v(\{A,D,E\}) = 4,085$	
	$v(\{B,E\}) = 2,523$	$v(\{B,C,D\}) = 2,819$	$v(\{A,B,C,D,E\}) = 6,393$
	$v(\{C,D\}) = 2,126$	$v(\{B,C,E\}) = 3,720$	
	$v(\{C,E\}) = 2,834$	$v(\{B,D,E\}) = 4,157$	
	$v(\{D,E\}) = 3,283$	$v(\{C,D,E\}) = 4,649$	
$v^{AD}(\{A\}) = 430$	$v^{AD}(\{A,B\}) = 1,036$	$v^{AD}(\{A,B,C\}) = 1,935$	$v^{AD}(\{A,B,C,D\}) = 3,266$
$v^{AD}(\{B\}) = 603$	$v^{AD}(\{A,C\}) = 1,278$	$v^{AD}(\{A,B,D\}) = 2,147$	$v^{AD}(\{A,B,C,E\}) = 4,207$
$v^{AD}(\{C\}) = 834$	$v^{AD}(\{A,D\}) = 1,502$	$v^{AD}(\{A,B,E\}) = 2,996$	$v^{AD}(\{A,B,D,E\}) = 4,630$
$v^{AD}(\{D\}) = 1,072$	$v^{AD}(\{A,E\}) = 2,158$	$v^{AD}(\{A,C,D\}) = 2,570$	$v^{AD}(\{A,C,D,E\}) = 5,133$
$v^{AD}(\{E\}) = 1,688$	$v^{AD}(\{B,C\}) = 1,488$	$v^{AD}(\{A,C,E\}) = 3,318$	$v^{AD}(\{B,C,D,E\}) = 5,574$
	$v^{AD}(\{B,D\}) = 1,714$	$v^{AD}(\{A,D,E\}) = 3,753$	
	$v^{AD}(\{B,E\}) = 2,523$	$v^{AD}(\{B,C,D\}) = 2,819$	$v^{AD}(\{A,B,C,D,E\}) = 6,061$
	$v^{AD}(\{C,D\}) = 2,126$	$v^{AD}(\{B,C,E\}) = 3,720$	
	$v^{AD}(\{C,E\}) = 2,834$	$v^{AD}(\{B,D,E\}) = 4,157$	
	$v^{AD}(\{D,E\}) = 3,283$	$v^{AD}(\{C,D,E\}) = 4,649$	

Table 3.4: The value of every possible coalition $S \subseteq N$ before (upper part) and after (upper part) country A boycotting country D, belonging to Example 9. Adapted from Nouweland et al. (1996) & Besner (2022).

$$\begin{aligned} P_A(N,v) - P_A(N,v^{AD}) = 37 &\neq 92 = P_D(N,v) - P_D(N,v^{AD}), \\ R_A(N,v) - R_A(N,v^{AD}) = 46 &\neq 286 = R_D(N,v) - R_D(N,v^{AD}), \\ Sh_A(N,v) - Sh_A(N,v^{AD}) = 166 &= 166 = Sh_D(N,v) - Sh_D(N,v^{AD}). \end{aligned}$$

Hence, we see here two examples (P and R) in which the impacts of a boycott are way bigger for country D. However, the impact of the boycott is equal for both countries when using the Shapley value, which confirms its balanced impacts of boycotts property.

3.3.2. Proportional Shapley value

In many real-life situations, the influence of multiple players on the same coalition differs from player to player - for example, when the players' values are determined by their capital or military strength. Therefore, the results from Subsection 3.3.1 can be reproduced entirely analogously to describe the proportional (to the players' individual values) balanced impacts of a boycott. First, we will introduce a variation on the BIB axiom:

Axiom (Besner, 2022)**.**

- Proportional balanced impacts of boycotts - PBIB. For all $(N,v) \in \mathbb{V}_0(N)$ and the (i,j)-boycott games (N,v^{ij}) corresponding to (N,v), we have

$$\frac{\varphi_i(N,v) - \varphi_i(N,v^{ij})}{v(\{i\})} = \frac{\varphi_j(N,v) - \varphi_j(N,v^{ij})}{v(\{j\})}.$$

Analogous to Subsection 3.3.1, the DP and PBIB axioms imply the PBC axiom. The proof is omitted because it is entirely analogous to the proof of Lemma 27.

Lemma 23. (Besner, 2022). DP and PBIB imply PBC.

This leads to a variant of Besner's main result which we have just proven in Theorem 22.

Theorem 24. (Besner, 2022). Let $(N,v) \in \mathbb{V}_0(N)$.
Sh^p is the unique TU-value that satisfies E, DP, and PBIB.

Proof (Besner, 2022). Let $(N,v) \in \mathbb{V}_0(N)$.

I. Existence: By Theorem 16, Sh^p satisfies E. Sh also satisfies DP, which has been proven in Theorem 22. By Lemma 19, we obtain the following equation, for $i,j \in N$, satisfying PBIB.

$$\frac{Sh_i^p(N,v) - Sh_i^p(N,v^{ij})}{v(\{i\})} = \sum_{\substack{S\subseteq N \\ S\supseteq\{i,j\}}} \frac{\Delta_v(S)}{\sum_{k\in S} v(\{k\})} = \frac{Sh_j^p(N,v) - Sh_j^p(N,v^{ij})}{v(\{j\})}.$$

II. Uniqueness: This part of the proof is entirely analogous to the second part of the proof of Theorem 22, where BC is replaced by PBC, Lemma 21 is replaced by Lemma 23, and Theorem 14 is replaced by Theorem 16. □

3.3.3. Weighted Shapley value

It can also happen that the players in a coalition do not have a balanced impact on the boycott (Subsection 3.3.1), but also do not have proportional balanced impacts, which are based on the players' individual values (Subsection 3.3.2). This can happen, for instance, in situations in which players show different levels of effort in cooperating. To this end, we can use a weight vector $w \in W$, after which the results from Subsection 3.3.1 can be reproduced again to describe the weighted balanced impacts of a boycott. Therefore, we will introduce another variation on the BIB axiom:

Axiom (Besner, 2022)**.**

- w-balanced impacts of boycotts - BIB^w. For all $(N, v) \in \mathbb{V}(N)$, distinct players $i, j \in N$, the (i,j)-boycott games (N, v^{ij}) corresponding to (N, v), and $w \in W$, we have

$$\frac{\varphi_i(N,v) - \varphi_i(N,v^{ij})}{w_i} = \frac{\varphi_j(N,v) - \varphi_j(N,v^{ij})}{w_j}.$$

Analogous to previous subsections, the DP and BIB^w axioms imply the BC^w axiom. The proof is again omitted because it is similar to the proof of Lemma 21.

Lemma 25. (Besner, 2022). DP and BIB^w imply WBC^w.

This leads to another variant of Besner's main result in Theorem 22.

Proposition 26. (Besner, 2022). Let $(N, v) \in \mathbb{V}(N)$, $w \in W$.
Sh^w is the unique TU-value that satisfies E, DP and BIB^w.

Proof (Besner, 2022). Let $(N, v) \in \mathbb{V}(N)$ and $w \in W$.

I. Existence: By Theorem 15, Sh^w satisfies E. Sh^w also satisfies DP, which has been proven in Theorem 22. By Lemma 19, we obtain, for $i, j \in N$, the following equation, which satisfies BIB^w:

$$\frac{Sh_i^w(N,v) - Sh_i^w(N,v^{ij})}{w_i} = \sum_{\substack{S \subseteq N \\ S \supseteq \{i,j\}}} \frac{\Delta_v(S)}{\sum_{k \in S} w_k} = \frac{Sh_j^w(N,v) - Sh_j^w(N,v^{ij})}{w_j}.$$

II. Uniqueness: This part of the proof is entirely analogous to the second part of the proof of Theorem 22, where BC is replaced by WBC^w, Lemma 21 by Lemma 25, and Theorem 14 by Theorem 15. □

Now, we have replaced the individual values of the players that we used in Subsection 3.3.2 by exogenously given weights $w \in W$. However, the game is now dependent on this weight vector. To eliminate this dependence, we do have to weaken the initial axiom and, thus, the final theorem. Instead of requiring w-balanced impacts, we only need the boycotting and boycotted players to have equal signs.

Axiom (Besner, 2022)**.**

- Weak balanced impacts of boycotts - BIB$^-$. For all $(N, v) \in \mathbb{V}(N)$ and the (i,j)-boycott games (N, v^{ij}) corresponding to (N, v), we have

$$\text{sign}\left(\varphi_i(N,v) - \varphi_i(N,v^{ij})\right) = \text{sign}\left(\varphi_j(N,v) - \varphi_j(N,v^{ij})\right).$$

Analogous to Lemma 21, the following lemma states that the DP and BIB$^-$ axioms imply the BC$^-$ axiom.

Lemma 27. (Besner, 2022). DP and BIB$^-$ imply BC$^-$.

Hence, we obtain the final result in terms of the weighted Shapley value, where the dependence on $w \in W$ has been eliminated.

Theorem 28 (Besner, 2022)**.** Let $(N, v) \in \mathbb{V}(N)$. A TU-value φ satisfies E, DP, Mar, and BIB$^-$ if and only if there exists a $w \in W$ such that $\varphi = Sh^w$.

Proof (Besner, 2022). Let $(N, v) \in \mathbb{V}(N)$, and $w \in W$.

I. Existence: Observing the original definition of the Shapley value in Theorem 2 term-wise shows us that also Sh^w satisfies Mar. Besides, by Proposition 26, we have that Sh^w satisfies E, DP, and BIB^w. Therefore, also BIB$^-$ is satisfied.

II. Uniqueness: Let φ be a TU-value satisfying E, DP, Mar, and BIB$^-$. By Lemma 27, φ then also satisfies BC$^-$. Then, by Theorem 17, uniqueness is accomplished. □

3.4. Extensions of the model

The article of Besner (2022) contained an original view on boycotts from a game theoretical point of view. However, the article also left some open ends unresolved, which is what we will dive into in this Section. We will look at the relationship between Harsanyi dividends and supermodularity in Subsection 3.4.1, a multiplayer boycott in Subsection 3.4.2, overflowing into Subsection 3.4.3, where we introduce some new findings.

3.4.1. Supermodularity

As we can read in Section 3.3, Besner (2022) transforms a TU-game into a boycott game by setting the Harsanyi dividend of all coalitions involving both the boycotting and boycotted players to zero. However, we know that the Shapley value is defined as a weighted sum of Harsanyi dividends and that Harsanyi dividends might be negative. With this in mind, the suspicion arose that without any assumptions, there could exist a theoretical example of a game in which the boycotting and boycotted players are both profiting from the boycott.

This suspicion seemed correct, as seen in the following example of a 3-player game.

Example 10. Consider a 3-player game (N, v) with $N = \{A,B,C\}$. From the given value function in Table 3.5, we can see that player B brings extra value when cooperating but that players A and C do not create extra value at all when joining a coalition.

When players A and C boycott each other, we set the Harsanyi value of the corresponding coalitions to zero, and so we also update the value function. Then we see that both the boycotting and the boycotted players A and C profit from this boycott, while player B is the major victim of this situation.

(N, v)			
Coalition	**Value**	**Dividend**	**Payoff**
{A}	3	3	$Sh_{A}(N, v) = 3.17$
{B}	1	1	$Sh_{B}(N, v) = 2.67$
{C}	2	2	$Sh_{C}(N, v) = 2.17$
{A,B}	5	1	
{A,C}	3	-2	
{B,C}	4	1	
{A,B,C}	8	2	

(N, v^{AC})			
Coalition	**Value**	**Dividend**	**Payoff**
{A}	3	3	$Sh_{A}(N, v) = 3.50$
{B}	1	1	$Sh_{B}(N, v) = 2.00$
{C}	2	2	$Sh_{C}(N, v) = 2.50$
{A,B}	5	1	
{A,C}	5	0	
{B,C}	4	1	
{A,B,C}	8	0	

Table 3.5: The impact of a boycott in which both the boycotting and boycotted players make a profit due to the boycott, belonging to Example 10.

In the table above, we can see that the balanced impact of boycotts property still applies, but the case in which both the boycotting and the boycotted players profit from a boycott seems unrealistic. Therefore, it makes sense to find the necessary assumption in order to have that a boycott is never profitable for both parties.

The critical element appears to be the negative Harsanyi value since changing a negative value to zero indeed leads to an improvement in the Shapley value of the involved player(s). As a matter of fact, when we look at the definition of the Shapley value in Theorem 11, we see that a boycott would never be profitable for both players when all Harsanyi dividends are positive: a summation of non-negative terms cannot get a higher outcome by changing terms to zero - the outcome will then be less than or equal to the original summation.

A TU-game (N, v) in which all Harsanyi dividends are positive, that is $\Delta_v(S) \geq 0 \ \forall S \subseteq N$, is called a totally positive game (Vasil'ev, 1975). The set of all totally positive TU-games is denoted by $\mathbb{V}_+(N)$. However, should we then constrain ourselves to totally positive games to obtain that a boycott is detrimental for both the boycotting and the boycotted players, or does a weaker requirement exist that also suffices? Supermodularity could be such a requirement that also suffices. Supermodular coalition games were first introduced by Shapley (1971) under the name of a 'convex game' and are used to evaluate the agents' motivation change after other agents' decisions.

Definition 29 (Peters, 2008)**.** Let $\mathscr{N}$ be finite and (N, v) be a TU-game. Then (N, v) is a supermodular TU-game if and only if $\forall i \in N$ and $\forall S \subseteq T \subseteq N \backslash \{i\}$:

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T). \tag{3.11}$$

> **Explanation.** A supermodular game shows an increasing marginal contribution for coalition membership: as the coalition grows, a player becomes more motivated to join since the value he will be adding grows with the size of the coalition.

A logical follow-up question would be how totally positive and supermodular TU-games relate to each other. It turns out to be that a game with positive Harsanyi dividends implies the game to be supermodular.

Theorem 30. (N, v) is a totally positive game $\implies$ (N, v) is a supermodular game.

Proof. Let (N, v) be a totally positive game, which means that $\Delta_v(T) \geq 0 \ \forall T \subseteq N$. Next, let $i \in N$ be an arbitrary player and let $S \subseteq T \subseteq N \setminus \{i\}$. Then the following is true:

$$\begin{aligned} v(S \cup \{i\}) - v(s) &= \sum_{R \subseteq S \cup \{i\}} \Delta_v(R) - \sum_{R \subseteq S} \Delta_v(R), \\ &= \sum_{R \subseteq S} \Delta_v(R \cup \{i\}), \\ &\leq \sum_{R \subseteq T} \Delta_v(R \cup \{i\}), \\ &= \sum_{R \subseteq T \cup \{i\}} \Delta_v(R) - \sum_{R \subseteq T} \Delta_v(R), \\ &= v(T \cup \{i\}) - v(T). \end{aligned}$$

Hence, a game (N, v) with positive Harsanyi dividends implies supermodularity. Note that the converse is not true in general, which we can see in the 3-player game below (Table 3.6). That game is supermodular, since joining a bigger coalition is at least as profitable as joining a smaller coalition. However, the Harsanyi dividend of the grand coalition is negative: $\Delta_v(N) < 0$. □

The implication of Theorem 30 could also work in the other direction when we impose an extra requirement on the game (N, v). This requirement imposes conditions on the marginal contributions MC_i^v of coalitions

Coalition	Value function	Dividends
{A}	1	1
{B}	1	1
{C}	1	1
{A,B}	3	1
{A,C}	3	1
{B,C}	3	1
{A,B,C}	5	−1

Table 3.6: Example in which a supermodular TU-game can have Harsanyi dividends with a negative value, belonging to the proof of Theorem 30.

with an even number of players compared to coalitions with an odd number of players. We will shortly state this condition but not evaluate it deeply since this is a purely theoretical condition without a possible interpretation that makes any sense:

$$\begin{cases} \forall S \subseteq N : |S| \text{ odd} & : \exists i \in S : \sum_{\substack{R\subseteq S \\ |R| \text{ even}}} MC_i^v(R) \geq \sum_{\substack{R\subseteq S \\ |R| \text{ odd}}} MC_i^v(R), \\ \forall S \subseteq N : |S| \text{ even} & : \exists i \in S : \sum_{\substack{R\subseteq S \\ |R| \text{ odd}}} MC_i^v(R) \geq \sum_{\substack{R\subseteq S \\ |R| \text{ even}}} MC_i^v(R). \end{cases}$$

Although the implication of Theorem 30 does not work in both ways, it could still be that the weaker requirement of supermodularity, compared to being totally positive, is enough to make a boycott detrimental for both the boycotting and boycotted players. This turns out to be the case.

Theorem 31. Let (N, v) be a supermodular TU-game, in which we use the Shapley value to determine the payoffs. Then, a boycott is never profitable for both the boycotting and boycotted players.

Proof. Let (N, v) be a supermodular TU-game and let us define the following function $c(R)$ for $R \subseteq N$ and distinct players $i, j \in N$:

$$c(R) = \begin{cases} \sum_{k=0}^{|N|-|R|} \binom{|N|-|R|}{k} \frac{(-1)^k}{|R|+k}, & \text{if } R \text{ contains 2 elements from } \{i,j\}, \\ \sum_{k=0}^{|N|-|R|-1} \binom{|N|-|R|-1}{k} \frac{(-1)^{k+1}}{|R|+k+1}, & \text{if } R \text{ contains 1 element from } \{i,j\}, \\ \sum_{k=0}^{|N|-|R|-2} \binom{|N|-|R|-2}{k} \frac{(-1)^{k+2}}{|R|+k+2}, & \text{if } R \text{ contains 0 elements from } \{i,j\}. \end{cases}$$

Now, from the definition of $c(R)$ above, we can observe that we can make groups of four related sets that have the same outcome, apart from their sign. In other words, we can see that the following relationship ($\star$) holds: $c(R) = -c(R\backslash\{i\}) = -c(R\backslash\{j\}) = c(R\backslash\{i,j\})$, since:

$$\begin{aligned} c(R) &= \sum_{k=0}^{|N|-|R|} \binom{|N|-|R|}{k} \frac{(-1)^k}{|R|+k} \\ c(R\backslash\{j\}) = c(R\backslash\{i\}) &= \sum_{k=0}^{|N|-|R\backslash\{i\}|-1} \binom{|N|-|R\backslash\{i\}|-1}{k} \frac{(-1)^{k+1}}{|R\backslash\{i\}|+k+1} = -c(R) \\ c(R\backslash\{i,j\}) &= \sum_{k=0}^{|N|-|R\backslash\{i,j\}|-2} \binom{|N|-|R\backslash\{i,j\}|-2}{k} \frac{(-1)^{k+2}}{|R\backslash\{i,j\}|+k+2} = c(R) \end{aligned}$$

Next, assume that players i and j decide to boycott each other, which lets their payoffs change. The definition of an (i, j)-boycott game (N, v^{ij}) (Definition 20) states that, after a boycott, the Harsanyi dividends of all coalitions which contain both i and j are set to zero. Hence, the change in payoff can be determined by calculating the difference between the payoffs before and after the boycott. In the comments next to the computation below, we will use 'CPA' as an abbreviation for 'commutative property of addition'.

$$\begin{aligned}
Sh_i(v^{ij}) - Sh_i(v) &= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,j\}}} \frac{\Delta_v(S)}{|S|}, && \text{(Theorem 11)}\\
&= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,j\}}} \sum_{R\subseteq S} \frac{(-1)^{|S|-|R|}v(R)}{|S|}, && \text{(Lemma 9)}\\
&= \sum_{R\subseteq N} \left(v(R) \sum_{\substack{R\subseteq S\subseteq N\\ S\supseteq\{i,j\}}} \frac{(-1)^{|S|-|R|}}{|S|} \right), && \text{(CPA)}\\
&= \sum_{R\subseteq N} c(R)v(R), && \text{(substituting } c(R))\\
&= \sum_{\substack{R\subseteq N\\ R\supseteq\{i,j\}}} [c(R)v(R) + c(R\backslash\{i\})v(R\backslash\{i\})\\
&\qquad + c(R\backslash\{j\})v(R\backslash\{j\}) + c(R\backslash\{i,j\})v(R\backslash\{i,j\})],\\
&= \sum_{\substack{R\subseteq N\\ R\supseteq\{i,j\}}} c(R)\left[v(R) - v(R\backslash\{i\}) - v(R\backslash\{j\}) + v(R\backslash\{i,j\})\right], && (\star)\\
&\geq \sum_{\substack{R\subseteq N\\ R\supseteq\{i,j\}}} c(R)\cdot 0 \;=\; 0. && \text{(supermodularity)}
\end{aligned}$$

Hence, we conclude that supermodularity is a sufficient requirement for a boycott to never be profitable for both the boycotting and boycotted players. □

3.4.2. Asymmetric boycotts

One of the things that Besner (2022) did not take into account in his study was the impact of multiple players boycotting one or more other players at the same time. To examine the extensibility of the Shapley value together with the balanced impacts of boycotts property, as we did for boycotts involving only two players in Section 3.3, in this subsection, we will consider a game in which two players are boycotting one player: an asymmetric boycott. Therefore, let us first define such a game.

Definition 32. Let $(N, v) \in \mathbb{V}(N)$ and $i, j, k \in N$ be distinct players. A TU-game $(N, v^{ij,k})$ such that i, j are disjointly productive in relation to k, is called the (ij, k)-boycott game corresponding to (N, v) if

$$v^{ij,k}(S) = v(S) \qquad \forall S \subseteq N \text{ with } \{i,k\} \not\subseteq S \text{ and } \{j,k\} \not\subseteq S.$$

Analogue to Section 3.3, we will now define adapted balanced contributions, disjointly productive players and balanced impacts of boycotts axiom, in which we will require that the boycott's impact is equal on both sides of the boycott.

Axioms.

- Balanced contributions* - BC*. For all $(N, v) \in \mathbb{V}(N)$, and distinct players $i, j, k \in N$ in (N, v), we have

$$(\varphi_i(N, v) - \varphi_i(N\backslash\{k\}, v)) + (\varphi_j(N, v) - \varphi_j(N\backslash\{k\}, v)) = \varphi_k(N, v) - \varphi_k(N\backslash\{i, j\}, v).$$

- Disjointly productive players* - DP*. For all $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$ disjointly productive in relation to $k \in N$ in (N, v), we have

$$\begin{aligned}\varphi_i(N, v) &= \varphi_i(N\backslash\{k\}, v),\\ \varphi_j(N, v) &= \varphi_j(N\backslash\{k\}, v),\\ \varphi_k(N, v) &= \varphi_k(N\backslash\{i, j\}, v).\end{aligned}$$

- Balanced impacts of boycotts* - BIB*. For all $(N, v) \in \mathbb{V}(N)$ and the (ij, k)-boycott game $(N, v^{ij,k})$ corresponding to (N, v), we have

$$\left(\varphi_i(N, v) - \varphi_i(N, v^{ij,k})\right) + \left(\varphi_j(N, v) - \varphi_j(N, v^{ij,k})\right) = \varphi_k(N, v) - \varphi_k(N, v^{ij,k}).$$

Then, the DP* and BIB* axioms above imply the BC* axiom.

Lemma 33. DP* and BIB* imply BC*.

Proof. Let $(N, v) \in \mathbb{V}(N)$, $i, j, k \in N$ be distinct players, and $(N, v^{ij,k}$ be the (ij, k)-boycott game corresponding to (N, v). Let φ be a TU-value that satisfies DP* and BIB*. Then we have

$$\begin{aligned}\varphi_i(N, v^{ij,k}) &\overset{(\text{DP}^*)}{=} \varphi_i(N\backslash\{k\}, v^{ij,k}) \overset{(\text{Def. }32)}{=} \varphi_i(N\backslash\{k\}, v),\\ \varphi_j(N, v^{ij,k}) &\overset{(\text{DP}^*)}{=} \varphi_j(N\backslash\{k\}, v^{ij,k}) \overset{(\text{Def. }32)}{=} \varphi_j(N\backslash\{k\}, v),\\ \varphi_k(N, v^{ij,k}) &\overset{(\text{DP}^*)}{=} \varphi_k(N\backslash\{i, j\}, v^{ij,k}) \overset{(\text{Def. }32)}{=} \varphi_k(N\backslash\{i, j\}, v).\end{aligned}$$

Substituting these results into BIB* yields the expression for BC*.

$$\begin{aligned}&\left(\varphi_i(N, v) - \varphi_i(N, v^{ij,k})\right) + \left(\varphi_j(N, v) - \varphi_j(N, v^{ij,k})\right) = \varphi_k(N, v) - \varphi_k(N, v^{ij,k}),\\ \Longrightarrow\ &(\varphi_i(N, v) - \varphi_i(N\backslash\{k\}, v)) + (\varphi_j(N, v) - \varphi_j(N\backslash\{k\}, v)) = \varphi_k(N, v) - \varphi_k(N\backslash\{i, j\}, v).\end{aligned}$$

Hence, DP* and BIB* imply BC*. □

The final step in examining the extensibility of Besner's approach would be to analyze for the asymmetric boycott whether Sh is again the unique TU-value that satisfies E, DP* and BIB*. However, this does not seem to be true. Sh does satisfy both the E and DP* axioms but fails the BIB* axiom:

$$\begin{aligned}\begin{matrix}\left(Sh_i(N, v) - Sh_i(N, v^{ij,k})\right)\\ +\left(Sh_j(N, v) - Sh_j(N, v^{ij,k})\right)\end{matrix} &= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,k\}}} \frac{\Delta_v(S)}{|S|} + \sum_{\substack{S\subseteq N\\ S\supseteq\{j,k\}}} \frac{\Delta_v(S)}{|S|},\\ &= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,k\}\\ \text{or } S\supseteq\{j,k\}}} \frac{\Delta_v(S)}{|S|} + \sum_{\substack{S\subseteq N\\ S\supseteq\{i,j,k\}}} \frac{\Delta_v(S)}{|S|},\\ &= \left(Sh_k(N, v) - Sh_k(N, v^{ij,k})\right) + \sum_{\substack{S\subseteq N\\ S\supseteq\{i,j,k\}}} \frac{\Delta_v(S)}{|S|}.\end{aligned}$$

Hence, looking at the extra summation in the final line (which can be beneficial or detrimental), we see that the boycott's impact on players i and j is not the same as the boycott's impact on player k. This is caused by the coalitions that contain all of the players i, j and k: in the Shapley value, all dividends are divided equally amongst all coalitions members, so when a coalition S loses its added value ($\Delta_v(S)$), it can never happen that this hurts player k exactly twice as much as players i and j. Accordingly, no weighted Shapley value will satisfy any adjusted axioms here either.

Another way to obtain the same result is to consider the asymmetric boycott as a sequential boycott: first player i decides to boycott player k, and directly afterwards, player j decides to do the same. In this case, the game (N, v) first changes to the (i, k)-boycott game, which has the following impact on all players:

$$\begin{aligned} Sh_i(N,v) - Sh_i(N,v^{ik}) &= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,k\}}} \frac{\Delta_v(S)}{|S|} \; = \; Sh_k(N,v) - Sh_k(N,v^{ik}), \\ Sh_j(N,v) - Sh_j(N,v^{ik}) &= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,j,k\}}} \frac{\Delta_v(S)}{|S|}. \end{aligned}$$

In the first line, we see that the BIB property holds for the boycott involving two players, which we could expect after establishing Theorem 22. In the second line, we now also consider the boycott's impact on a player who is not directly involved in the boycott. Because the Harsanyi dividend of all coalitions containing players i and k are set to zero after the boycott, the impact of this boycott on player j equals the change in dividend of coalitions containing players i, j and k. Next, also player j decides to boycott player k, which brings us to the $(N, v^{ij,k})$-boycott game. Note that the impact on the players directly involved will be less than in the previous boycott since some coalitions involving those players already became non-value-adding in the first boycott:

$$\begin{aligned} Sh_j(N,v^{ik}) - Sh_j(N,v^{ij,k}) &= \sum_{\substack{S\subseteq N\\ S\supseteq\{j,k\}}} \frac{\Delta_{v^{ik}}(S)}{|S|} \; = \; \sum_{\substack{S\subseteq N\\ S\supseteq\{j,k\}\\ S\not\ni i}} \frac{\Delta_v(S)}{|S|} \; = \; Sh_k(N,v^{ik}) - Sh_k(N,v^{ij,k}), \\ Sh_i(N,v^{ik}) - Sh_i(N,v^{ij,k}) &= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,j,k\}}} \frac{\Delta_{v^{ik}}(S)}{|S|} \; = \; 0. \end{aligned}$$

Finally, we can calculate the impact of both boycotts on each player by summing up the previous equations:

$$\begin{aligned} Sh_i(N,v) - Sh_i(N,v^{ij,k}) &= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,k\}}} \frac{\Delta_v(S)}{|S|} + 0 &&= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,k\}}} \frac{\Delta_v(S)}{|S|}, \\ Sh_j(N,v) - Sh_j(N,v^{ij,k}) &= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,j,k\}}} \frac{\Delta_v(S)}{|S|} + \sum_{\substack{S\subseteq N\\ S\supseteq\{j,k\}\\ S\not\ni i}} \frac{\Delta_v(S)}{|S|} &&= \sum_{\substack{S\subseteq N\\ S\supseteq\{j,k\}}} \frac{\Delta_v(S)}{|S|}, \\ Sh_k(N,v) - Sh_k(N,v^{ij,k}) &= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,k\}}} \frac{\Delta_v(S)}{|S|} + \sum_{\substack{S\subseteq N\\ S\supseteq\{j,k\}\\ S\not\ni i}} \frac{\Delta_v(S)}{|S|} &&= \sum_{\substack{S\subseteq N\\ S\supseteq\{i,k\}\\ \text{or } S\supseteq\{j,k\}}} \frac{\Delta_v(S)}{|S|}. \end{aligned}$$

Hence, we observe the same impact on all players as in the other approach. We can conclude again that the impact of the asymmetric boycott is not balanced under the Shapley value.

3.4.3. Boycott-contractions

Even though we have seen the non-balanced impacts of multiple players-boycotts, multiple players-boycotts are still a fascinating topic to study. However, the way in which the value functions of those boycotts are defined, via setting certain Harsanyi dividends to zero and summing them up per coalition S, is pretty time-consuming. This raises the question of whether this can be done in a better way. To find the answer to this question, we first take one step back to (ij)-boycott games.

The definition of the (ij)-boycott game (Definition 20) tells us that the value function does not change for coalitions that do not contain both the boycotting and boycotted players. The way in which the value function changes for coalitions containing both i and j is described in the following lemma.

Lemma 34. The coalition value in an (ij)-boycott game can be calculated by the following boycott-contraction:

$$v^{ij}(S) = \begin{cases} v(S), & \text{if } \{i,j\} \not\subseteq S, \\ v(S\backslash\{i\}) + v(S\backslash\{j\}) - v(S\backslash\{i,j\}), & \text{if } \{i,j\} \subseteq S. \end{cases} \tag{3.12}$$

Proof. Let $(N, v) \in \mathbb{V}(N)$. Definition 20 gives the definition of $v^{ij}(S)$ for $S \subseteq N$ with $\{i,j\} \not\subseteq S$. Now, let us assume that $\{i,j\} \subseteq S \subseteq N$. Then, we obtain the following results:

$$\begin{aligned} v^{ij}(S) &= \sum_{R\subseteq S} \Delta_{v^{ij}}(R), && \text{(Definition 8)} \\ &= v(S) - \sum_{\substack{R\subseteq S \\ R\supseteq\{i,j\}}} \Delta_v(S), && \text{(Definition 20)} \\ &= v(S) - \sum_{\substack{R\subseteq S \\ R\supseteq\{i,j\}}} \left(\sum_{Q\subseteq R} (-1)^{|R|-|Q|} v(Q) \right), && \text{(Lemma 9)} \\ &= v(S) - \sum_{Q\subseteq S} \left(v(Q) \sum_{\substack{Q\subseteq \mathbf{R}\subseteq S \\ R\supseteq\{i,j\}}} (-1)^{|R|-|Q|} \right). && \text{(commutative property of addition)} \end{aligned}$$

Inspecting the summation over all possible sets R above yields the following result, where we use $k = |R| - |Q|$, $n = |S| - |Q \cup \{i,j\}|$ and δ_n as the Kronecker delta function, which is 1 for $n = 0$ and 0 otherwise:

$$\begin{aligned} \sum_{\substack{Q\subseteq \mathbf{R}\subseteq S \\ R\supseteq\{i,j\}}} (-1)^{|R|-|Q|} &= \begin{cases} \sum_{k=0}^{n} \binom{n}{k}(-1)^{m} \;\;= \;\;\delta_n, & \text{if } Q \text{ contains } 2 \text{ elements from } \{i,j,k\}, \\ \sum_{k=0}^{n} \binom{n}{k}(-1)^{m+1} = -\delta_n, & \text{if } Q \text{ contains } 1 \text{ element from } \{i,j,k\}, \\ \sum_{k=0}^{n} \binom{n}{k}(-1)^{m+2} = \;\;\delta_n, & \text{if } Q \text{ contains } 0 \text{ elements from } \{i,j,k\}, \end{cases} \\ &= \begin{cases} -1, & \text{if } Q = S\backslash\{i\}, Q = S\backslash\{j\}, \\ \;\;1, & \text{if } Q = S,\; Q = S\backslash\{i,j\}, \\ \;\;0, & \text{else.} \end{cases} \end{aligned}$$

When we go back to the initial equation in this proof, we see that almost all terms disappear when we substitute the outcome of the summation above back into the equation, except the coalition values for which $\delta_n = 1$, namely $v(S)$, $v(S\backslash\{i\})$, $v(S\backslash\{j\})$, and $v(S\backslash\{i,j\})$. We obtain the following result:

$$\begin{aligned} v^{ij}(S) &= v(S) - [\; v(S) - v(S\backslash\{i\}) - v(S\backslash\{j\}) + v(S\backslash\{i,j\}) \;], \\ &= v(S\backslash\{i\}) + v(S\backslash\{j\}) - v(S\backslash\{i,j\}). \end{aligned}$$

Hence, Lemma 34's expression for the coalition value $v^{ij}(S)$ is true $\forall S \subseteq N$. □

By establishing Lemma 34, we can now calculate the value function of all coalitions $S \subseteq N$ after a boycott between two players. Now, let us return to the situation we studied in the previous subsection: the asymmetric boycott involving three players. Would such a lemma also hold for this kind of boycott?

Lemma 35. The coalition value in an (ij,k)-boycott game can be calculated by the following boycott-contraction:

$$v^{ij,k}(S) = \begin{cases} v(S), & \text{if } \{i,k\} \not\subseteq S \text{ and } \{j,k\} \not\subseteq S, \\ v(S\backslash\{i,j\}) + v(S\backslash\{k\}) - v(S\backslash\{i,j,k\}), & \text{if } \{i,k\} \subseteq S \text{ or } \{j,k\} \subseteq S. \end{cases} \quad (3.13)$$

Proof. Let $(N,v) \in \mathbb{V}(N)$. Definition 32 gives the definition of $v^{ij,k}(S)$ for $S \subseteq N$ when $\{i,k\} \not\subseteq S$ and $\{j,k\} \not\subseteq S$. Now, let us assume that $\{i,k\} \subseteq S$ or $\{j,k\} \subseteq S$, for $S \subseteq N$. Then we obtain the following results:

$$\begin{aligned} v^{ij,k}(S) &= \sum_{R \subseteq S} \Delta_{v^{ij,k}}(R), && \text{(Definition 8)} \\ &= v(S) - \sum_{\substack{R \subseteq S \\ R \supseteq \{i,k\} \\ \text{or} \{j,k\}}} \Delta_v(S), && \text{(Definition 20)} \\ &= v(S) - \sum_{\substack{R \subseteq S \\ R \supseteq \{i,k\} \\ \text{or} \{j,k\}}} \left(\sum_{Q \subseteq R} (-1)^{|R|-|Q|} v(Q) \right), && \text{(Lemma 9)} \\ &= v(S) - \sum_{Q \subseteq S} \left(v(Q) \sum_{\substack{Q \subseteq \mathbf{R} \subseteq S \\ R \supseteq \{i,k\} \\ \text{or} \{j,k\}}} (-1)^{|R|-|Q|} \right). && \text{(commutative property of addition)} \end{aligned}$$

Inspecting the summation over all possible sets R above yields the following result, where we use $k = |R| - |Q|$ and $n = |S| - |Q \cup \{i,j,k\}|$ and δ_n as the Kronecker delta function, which is 1 for $n = 0$ and 0 otherwise:

$$\sum_{\substack{Q \subseteq \mathbf{R} \subseteq S \\ R \supseteq \{i,k\} \\ \text{or} \{j,k\}}} (-1)^{|R|-|Q|} = \sum_{\substack{Q \subseteq \mathbf{R} \subseteq S \\ R \supseteq \{i,k\}}} (-1)^{|R|-|Q|} + \sum_{\substack{Q \subseteq \mathbf{R} \subseteq S \\ R \supseteq \{j,k\}}} (-1)^{|R|-|Q|} - \sum_{\substack{Q \subseteq \mathbf{R} \subseteq S \\ R \supseteq \{i,j,k\}}} (-1)^{|R|-|Q|}.$$

The first two summations on the right-hand side are already known since we calculated similar summations in the proof of Lemma 34. The last summation above is slightly different but can be calculated in an analogue way:

$$\begin{aligned} \sum_{\substack{Q \subseteq \mathbf{R} \subseteq S \\ R \supseteq \{i,j,k\}}} (-1)^{|R|-|Q|} &= \begin{cases} \sum_{k=0}^{n} \binom{n}{k} (-1)^{m} = -\delta_n, & \text{if } Q \text{ contains } 3 \text{ elements from } \{i,j,k\}, \\ \sum_{k=0}^{n} \binom{n}{k} (-1)^{m+1} = \delta_n, & \text{if } Q \text{ contains } 2 \text{ elements from } \{i,j,k\}, \\ \sum_{k=0}^{n} \binom{n}{k} (-1)^{m+2} = -\delta_n, & \text{if } Q \text{ contains } 1 \text{ element from } \{i,j,k\}, \\ \sum_{k=0}^{n} \binom{n}{k} (-1)^{m+3} = \delta_n, & \text{if } Q \text{ contains } 0 \text{ elements from } \{i,j,k\}, \end{cases} \\ &= \begin{cases} -1, & \text{if } Q = S\backslash\{i\},\ Q = S\backslash\{j\},\ Q = S\backslash\{k\}, \\ 1, & \text{if } Q = S,\ Q = S\backslash\{i,j\},\ Q = S\backslash\{i,k\},\ Q = S\backslash\{j,k\}, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

When summing up all results (the summation's outcome above and the summations which we can deduct from Lemma 34's proof), we obtain:

$$\sum_{\substack{Q\subseteq \mathbf{R}\subseteq S\\ R\supseteq\{i,k\}\\ \text{or}\{j,k\}}} = \begin{cases} -1, & \text{if } Q = S\backslash\{i,j\},\ Q = S\backslash\{k\}, \\ 1, & \text{if } Q = S,\ Q = S\backslash\{i,j,k\}, \\ 0. & \text{else.} \end{cases}$$

Substituting this result back into the original equation yields:

$$\begin{aligned} v^{ij,k}(S) &= v(S) - [\ v(S) - v(S\backslash\{i,j\}) - v(S\backslash\{k\}) + v(S\backslash\{i,j,k\})\ ], \\ &= v(S\backslash\{i,j\}) + v(S\backslash\{k\}) - v(S\backslash\{i,j,k\}). \end{aligned}$$

Hence, Lemma 35's expression for the coalition value $v^{ij,k}(S)$ is true $\forall S \subseteq N$. □

Since we now have developed two direct formulas for the value function of coalitions in boycotts that does not contain Harsanyi dividends, we recognize a pattern. In both the boycott involving just two players and the asymmetric boycott with three players, we see that the value function of a coalition S in a boycott equals the sum of three terms: the original value of S without one side of the boycott, the original value of S without the other side of the boycott, minus the original value of S without both sides of the boycott. This can lead to a generalized contraction for arbitrary, distinct sets $A, B \subseteq N$ in an (A, B)-boycott.

Definition 36. Let $(N, v) \in \mathbb{V}(N)$, and distinct player sets $A, B \subseteq N$. A TU-game $(N, v^{A,B})$ such that every player $i \in A$ is disjointly productive in relation to any player $j \in B$ is called the (A, B)-boycott game corresponding to (N, v) if

$$v^{A,B}(S) = v(S), \qquad \text{if } \forall i \in A : \forall j \in B : \{i,j\} \nsubseteq S. \tag{3.14}$$

Definition 37. Let $(N, v^{A,B})$ be an (A, B)-boycott game corresponding to (N, v). Then $\{i, j\}$ is a boycotting pair if $i \in A$ and $j \in B$.

Let us now state the following conjecture, introducing the generalized boycott-contraction.

Conjecture 38. The coalition value in an (A, B)-boycott game can be calculated by the following generalized boycott-contraction:

$$v^{A,B}(S) = \begin{cases} v(S), & \text{if } \forall i \in A : \forall j \in B : \{i,j\} \nsubseteq S, \\ v(S\backslash A) + v(S\backslash B) - v(S\backslash\{A \cup B\}), & \text{if } \exists i \in A : \exists j \in B : \{i,j\} \subseteq S. \end{cases} \tag{3.15}$$

This conjecture is believed to be true but has yet to be proven. In the last part of this subsection, we will provide the building blocks of Conjecture 38's proof and make clear why it is believed to be correct.

To start with, Definition 36 gives the definition of $v^{A,B}(S)$ for $S \subseteq N$ for which $\forall i \in A : \forall j \in B : \{i,j\} \nsubseteq S$ holds. Now, let us assume that $\exists i \in A : \exists j \in B : \{i,j\} \subseteq S$. Then we obtain the following results, in a completely analogue way as earlier this subsection:

$$\begin{aligned}
v^{A,B}(S) &= \sum_{R\subseteq S} \Delta_{v^{A,B}}(R), && \text{(Definition 8)}\\
&= v(S) - \sum_{\substack{R\subseteq S\\ \exists i\in A:\exists j\in B:\\ R\supseteq\{i,j\}}} \Delta_v(R), && \text{(Definition 20)}\\
&= v(S) - \sum_{\substack{R\subseteq S\\ \exists i\in A:\exists j\in B:\\ R\supseteq\{i,j\}}} \left(\sum_{Q\subseteq R}(-1)^{|R|-|Q|}v(Q)\right), && \text{(Lemma 9)}\\
&= v(S) - \sum_{Q\subseteq S}\left(v(Q)\sum_{\substack{Q\subseteq\mathbf{R}\subseteq S\\ \exists i\in A:\exists j\in B:\\ R\supseteq\{i,j\}}}(-1)^{|R|-|Q|}\right). && \text{(commutative property of addition)}
\end{aligned}$$

Analogue to the proofs of Lemma 34 and Lemma 35, we will now evaluate the summation over all possible sets R. Here we use $n = |S| - |Q|$ and $k = |R| - |Q|$. The goal is to establish that

$$\sum_{\substack{Q\subseteq\mathbf{R}\subseteq S\\ \exists i\in A:\exists j\in B:\\ R\supseteq\{i,j\}}}(-1)^{|R|-|Q|} \stackrel{?}{=} \begin{cases} -1, & \text{if } Q = S\backslash A,\ Q = S\backslash B,\\ 1, & \text{if } Q = S,\ Q = S\backslash\{A\cup B\},\\ 0, & \text{else.}\end{cases} \tag{3.16}$$

If we can prove that this statement holds, then the final result of Conjecture 38 can easily be obtained in a similar way as previous results. Let us now evaluate the summation above. Identical to previous lemmas, the trick here seems to lie in the process of counting all the possible sets R. We can distinguish three situations, in which we use the convention that $\binom{n}{k} = 0$ when $k > n$:

1. **Q contains no players from A and B.** In that case, at least 1 boycotting pair needs to be added to Q to create the smallest set R possible. The number of possible sets R can be calculated via the product of 3 binomial coefficients:

$$\sum_{\substack{Q\subseteq\mathbf{R}\subseteq S\\ \exists i\in A:\exists j\in B:\\ R\supseteq\{i,j\}}}(-1)^{|R|-|Q|} = \sum_{i=2}^{n}\left((-1)^i\cdot\sum_{k=1}^{\min\{|A|,i-1\}}\sum_{l=1}^{\min\{|B|,i-k\}}\binom{|A|}{k}\binom{|B|}{l}\binom{n-|A|-|B|}{i-k-l}\right).$$

2. **Q contains only players from A - not from B.** In that case, at least 1 player from B needs to be added to Q to create the smallest set R possible. The number of possible sets R can be calculated via the product of 2 binomial coefficients:

$$\sum_{\substack{Q\subseteq\mathbf{R}\subseteq S\\ \exists i\in A:\exists j\in B:\\ R\supseteq\{i,j\}}}(-1)^{|R|-|Q|} = \sum_{i=1}^{n}\left((-1)^i\cdot\sum_{l=1}^{\min\{|B|,i\}}\binom{|B|}{l}\binom{n-|B|}{i-l}\right).$$

Of course, this situation is symmetrical, so when Q contains only players from B, the result can be obtained in an analogue way.

3. Q **contains at least one boycotting pair.** In that case, the smallest set R possible is Q. The number of possible sets R can be calculated via a binomial coefficient:

$$\sum_{\substack{Q\subseteq \mathbf{R}\subseteq S\\ \exists i\in A:\exists j\in B:\\ R\supseteq\{i,j\}}} (-1)^{|R|-|Q|} = \sum_{i=0}^{n}(-1)^i \cdot \binom{n}{i}.$$

As we can see from the three distinguished situations above, most of their expressions are pretty complex. Therefore, we did not find an algebraic way to prove that they are equal to plus/minus the Kronecker delta function, as we did in the proofs of Lemma 34 and 35. Therefore, we wrote a MATLAB code to calculate the outcome of the right-hand side summations for all possible set sizes of $2 \leq |S| \leq 50$. For situations 1 and 2, MATLAB confirmed that the outcomes are equal to Kronecker delta functions $\delta_{n-|A|-|B|}$ and $-\delta_{n-|B|}$, respectively. In the first situation, this function will result in a 1 for $Q = S\backslash(A \cup B)$ and 0 for all other $Q \subseteq S$. The function in the second situation yields a 1 for $Q = S\backslash A$ and $Q = S\backslash A$, and 0 for all other $Q \subseteq S$. The source code of these computations can be found in Appendix A. Finally, the third situation did not require a computation simulation since we can recognize this expression immediately as the Kronecker delta function δ_n, yielding only a 1 for $Q = S$ and 0 otherwise.

Hence, for $2 \leq |S| \leq 50$, we have numerically obtained the required result (3.16) - which strengthens our belief that Conjecture 38 holds as the generalized boycott-contraction for all boycott sizes.

4

Macroeconomic considerations

In this chapter, we will make the connection between the game-theoretical models and some real-world data - an insightful analysis missing in all studied literature. To this end, we will discuss several examples deducted from real-world data, considering the boycott of the Russian Federation in 2022.

(N, v)				(N, v^{AB})			
Coalition	**Value**	**Dividend**	**Payoff**	**Coalition**	**Value**	**Dividend**	**Payoff**
{A}	0	0	$Sh_{\text{A}} = 46.67$	{A}	0	0	$Sh_{\text{A}} = 3.33$
{B}	0	0	$Sh_{\text{B}} = 51.67$	{B}	0	0	$Sh_{\text{B}} = 48.33$
{C}	0	0	$Sh_{\text{C}} = 1.67$	{C}	0	0	$Sh_{\text{C}} = 48.33$
{A,B}	100	100		{A,B}	0	0	
{A,C}	0	0		{A,C}	0	0	
{B,C}	10	10		{B,C}	90	90	
{A,B,C}	100	-10		{A,B,C}	100	10	

Table 4.1: A boycott in which the boycott hits the initiating player the most, belonging to Example 11.

Example 11. Let us consider a situation with three countries, $N = \{\text{A,B,C}\}$, where country A decides to boycott country B (Table 4.1). Before the boycott, countries A and B were big trading partners, and C was just a small player trading small amounts with country B. Because of the boycott, country A and B do not trade anymore, so country B decides to improve its trading relationship with country C: country C now gets to buy all the goods for a slightly lower price. Since country A is not boycotting country C, country C is being smart by making also trades with country A now. This means that the coalition of $\{\text{A,C}\}$ still has value zero since the goods have to be bought from country B first, but in the coalition of $\{\text{A,B,C}\}$, the countries still trade via different routes.

We use again the Shapley value to calculate the payoffs, and we see immediately that the impact of this (A,B)-boycott is -43.33 for country A, -3.33 for country B, and 46.67 for country C. In other words, it can happen that the boycotting country feels the impact of its own boycott the most! Next, we also see that country C, which is not directly involved in the boycott, is profiting the most from this boycott.

This example may sound familiar. Since February 2022, the Russian Federation has been boycotted by many other countries in the world in response to the illegal invasion of Ukraine. We will now consider a 3-player game involving the European Union (E), the Russian Federation (R), and the Republic of Türkiye (T). The value function is defined in proportion to the trading results (sum of the total import and export data) before the boycott (February 2022) and after the boycott (July 2023), where we use data provided by Eurostat, OECD, Tradingeconomics, and Bruegel. Since we did not observe a significant difference in domestic trading between the two points in time in the data, we set all individual values to zero. Also, countries do not provide sufficient data to analyze the complete trading flow, which is why coalitions of more than two countries are considered non-value-adding. Therefore, we only present the numbers of 2-player and n-player coalitions in the following tables.

Coalition	Value	Payoff	Coalition	Value	Payoff	Boycott impact
{E,R}	29	$Sh_E = 22.00$	{E,R}	6	$Sh_\text{E} = 11.83$	-10.17
{E,T}	15	$Sh_\text{R} = 17.00$	{E,T}	17	$Sh_\text{R} = 7.33$	-9.67
{R,T}	5	$Sh_\text{T} = 10.00$	{R,T}	8	$Sh_\text{T} = 12.83$	2.83
{E,R,T}	49		{E,R,T}	32		

Table 4.2: Boycott game with a utility function v that is proportional to real-world data between February 2022 and July 2023.

Because of the neutral status of the Republic of Türkiye, from which the European Union and the Russian Federation benefit to different degrees, we see that this boycott is not entirely balanced, as we saw in the theoretical results in Chapter 3. However, both the European Union and the Russian Federation still feel the negative impact of this boycott. We also see that the Republic of Türkiye is a country that is not directly involved but still profits slightly from this boycott, as both countries increased their trading amount with this country - just as we have seen in Example 11.

According to an article published by European Union (2023), *Financial Times - EU urged to crack down on imports of Indian fuels made with Russian oil*, the theoretical situation in which a country not directly involved is profiting from the boycott is actually happening: the European Union has bought large volumes of fuel from India, which was made of Russian crude oil. Let us now add India to this game to see if we can see even a bigger profiteer and what happens to the boycott's impact on the European Union and the Russian Federation.

Coalition	Value	Payoff	Coalition	Value	Payoff	Boycott impact
$\{E, R\}$	29	$Sh_E = 26.00$	$\{E, R\}$	6	$Sh_E = 18.00$	-8.00
$\{E, I\}$	8	$Sh_R = 10.00$	$\{E, I\}$	13	$Sh_R = 20.50$	0.00
$\{E, T\}$	15	$Sh_T = 10.00$	$\{E, T\}$	17	$Sh_T = 13.00$	2.50
$\{R, I\}$	7	$Sh_I = 10.00$	$\{R, I\}$	27	$Sh_I = 20.50$	12.50
$\{R, T\}$	5		$\{R, T\}$	8		
$\{I, T\}$	1		$\{I, T\}$	1		
$\{E, R, I, T\}$	65		$\{E, R, I, T\}$	72		

Table 4.3: Boycott game with a utility function v that is proportional to real-world data between February 2022 and July 2023.

Naturally, we observe a higher game value $v(N)$ since more countries participate in the trading process. To analyze the changes in the game, we have to look at the last column: boycott impact. The boycott's impact on the European Union became lower, which makes sense since we just named an article that confirmed that a significant part of Europe's fuel consumption shifted from the Russian Federation to India. However, the Russian Federation is profiting big time from its trading relationship with India, which keeps them from feeling the boycott's impact in this situation. Finally, as expected, we can observe that India profits the most from this boycott. A graphical representation of the seaborne oil exports, Russia's main export product, confirms this thought (Figure 4.1).

One of the reasons we did not find any complete real-world data about boycotts in known studies is probably the difficulty of measuring economies. Import and export data are not openly available for every country - and when they are, countries use different methods to express these results. Countries can decide to adjust their results for inflation, currency exchange rates, and seasonality, while others do not. Next, even if all countries would have the exact same measurement method, we can not draw conclusions based only on these numbers since many other factors influence these numbers, such as developments in the global economy. Finally, in these cases, we only considered the 3 or 4 biggest players in terms of countries and unions, which means that the game does not provide all information on the real-world situation.

However, it is essential to realize that a theoretical boycott with balanced impacts does not always happen in the real world due to intelligent countries finding other ways to import and export goods. Also Besner (2022) mentioned that his study did not consider the impacts of a boycott on those not directly targeted. They not only lose their share of the cooperation profits of the coalitions containing both the boycotting and

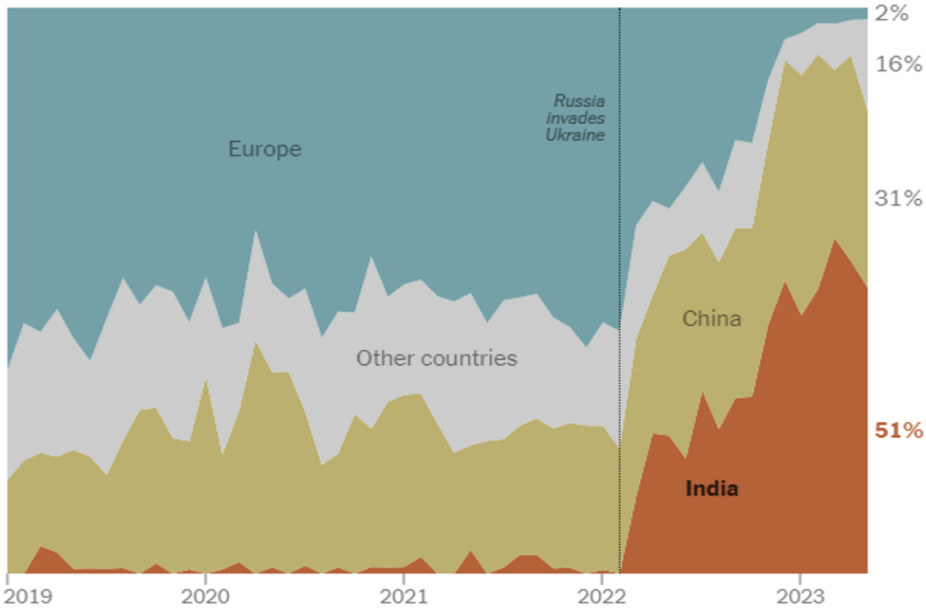

Figure 4.1: Where Russia's seaborne oil exports go .Source: Gamio et al., 2023.

the boycotted player, but as we have seen, also the trading relationships between countries/unions might be shifting due to the new situation.

5

Conclusion

The main goal of this thesis was to study and improve boycott models from a game-theoretical point of view. In this conclusion, we will state the results shown in this report, starting with some notes about consumer boycotts, followed by observations of boycott impacts in a static model, after which we will summarize our observations in the real-world boycott data. Finally, we make some suggestions for follow-up research.

In the first part of this thesis, we pointed out that consumer boycotts involve a dynamic process between all involved consumers, where free riding and coordination failure can play a significant role in the boycott's effectiveness. It turns out to be plausible that a consumer will join the boycott if he believes that, at a particular moment in time, the probability of boycott success is bigger than his ratio of costs over potential gain (his participation threshold). This leads to an equilibrium boycott population boycotting the firm, where the firm surrenders if this population is bigger than the firm threshold. Instead of awaiting the boycott, one can also anticipate what the final loss or profit will be from a boycott by evaluating the maximum boycott duration to be able to make an appropriate decision earlier in the process.

In the second part of this thesis, we considered the impacts of a boycott in a static model concerning the family of weighted Shapley values. By using Harsanyi dividends in modelling the boycott, we have seen that the impacts of a 2-player boycott are balanced between the boycotting and the boycotted player. This property does not hold for asymmetric boycotts involving 3 players. Sometimes, a boycotted player can profit from a boycott, which is unlikely in the theoretical model, so we found supermodularity as a sufficient requirement for the utility function to ensure that a boycott is never profitable for the involved players. Finally, we introduced the boycott-contraction, a direct formula to calculate the coalition value in a boycott game, which turned out to be true for boycotts involving 2 or 3 players. We also suspect validity for all boycott sizes, supported by numerical simulations, which all confirmed the conjecture.

In real-world data, we see different things than we did in the theoretical models. Despite the difficulties faced in the measurement of economies, we can see that sometimes the negative impact of a boycott is bigger for the boycotting player than for the boycotted player. This can happen when the boycotted player finds other ways to continue its trading activities - usually with an uninvolved player who becomes the boycott's big (financial) winner.

Even though we briefly mentioned a boycott's impact on uninvolved players, this study did not focus on this topic. These players also lose their share of the cooperation gains of the coalitions in which they cooperate with the involved players, although some players can also profit from a boycott, as we have seen in real-world data. Besides, there is a broad area of research in looking for surplus values similar to the Harsanyi dividend, with different properties, which raises the question of how other surplus values would behave in this kind of boycott model. Finally, the newly introduced generalized boycott-contraction has not been proven algebraically yet in this research, but we already provided the building blocks of such proof. In other words, after studying this thesis, there are plenty of opportunities for follow-up research.

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A

Implementations

```
%% Source code to generate Figure 2.1
x = 0:1:10000;
y1 = x/10000;                                          %uniform distribution

mu = 3500;
sigma = 1200;
y2 = 0.5*(1+erf((x-mu)/(sigma*sqrt(2))));              %normal distribution

figure(1)
plot(x,y1,'LineWidth',2)
hold on
plot(x,y2,'LineWidth',2)
title('Unsuccessful boycott due to coordination failure')
xlabel('Players')
legend({'Pbar_i','P[n*>=ns]'},'Location','northwest')
hold off

%% Source code to generate Figure 2.2
x = 0:1:10000;
y1 = 0.5*(1+erf((x-mu)/(sigma*sqrt(2))));              %normal distribution

mu = 4000;
sigma = 1000;
y2 = expcdf(x,2000);                                   %exponential distribution

figure(1)
plot(x,y1,'LineWidth',2)
hold on
plot(x,y2,'LineWidth',2)
title('Successful boycott')
xlabel('Players')
legend({'Pbar_i','P[n*>=ns]'},'Location','northwest')
hold off
```

```
%% Subsection 3.4.3 - Generalized boycott-contraction - Situation 1
clear all; clc;
check = zeros(1,1);
for i=2:50
    S = i;                                             %Define |S|
    for j=0:i-2
        Q = j;                                         %Define |Q|
        for k=1:S-Q-1
            A = k;                                     %Define |A|
            for l=1:S-Q-A
                B = l;                                 %Define |B|
```

```
                x3 = zeros(1,1);
                for a=2:S-Q                                         %Summation over i
                    x2 = zeros(1,1);
                    x1 = zeros(1,1);
                    for b=1:min(A,a-1)                              %Summation over k
                        for c=1:min(B,a-b)                          %Summation over l
                            if (S-Q-A-B>=0) && (a-b-c>=0) && (b<=A) && ... %Checking for convention
                                    (c<=B) && ((a-b-c)<=(S-Q-A-B))
                                x1 = x1 + (factorial(A)/factorial(b)/... %Calculation
                                    factorial(A-b))*(factorial(B)/...
                                    factorial(c)/factorial(B-c))*...
                                    (factorial(S-Q-A-B)/factorial(a-b-c)/...
                                    factorial(S-Q-A-B-a+b+c));
                            end
                        end
                    end
                    x2 = (-1)^a * x1;
                    x3 = x3 + x2;
                end
                x3 = round(x3);                                     %Correcting for round-off errors
                check(end+1) = ((x3==1) && (Q==S-A-B)) + ...        %Checking whether the result
                    ((x3==0) && (Q~=S-A-B));                        %coincides with the Kr.Delta f.
            end
        end
    end
end
check = check(2:end);
nnz(~check)                                                         %Indicates #wrong outcomes

%% Subsection 3.4.3 - Generalized boycott-contraction - Situation 2
clear all;
x0 = zeros(1,1);
check = zeros(1,1);
for i=1:50
    S = i;                                                          %Define |S|
    for j=1:i-1
        Q = j;                                                      %Define |Q|
        for l=1:S-Q
            B = l;                                                  %Define |B|
            [i j l];

            x3 = zeros(1,1);
            for a=1:S-Q                                             %Summation over i
                x2 = zeros(1,1);
                x1 = zeros(1,1);
                for c=1:min(B,a)                                    %Summation over l
                    if (S-Q-B>=0) && (a-c>=0) && (c<=B) && ...      %Checking for convention
                            ((a-c)<=(S-Q-B))
                        x1 = x1 + (factorial(B)/factorial(c)/...    %Calculation
                            factorial(B-c))*...
                            (factorial(S-Q-B)/factorial(a-c)/...
                            factorial(S-Q-B-a+c));
                    end
                end
                x2 = (-1)^a * x1;
                x3 = x3 + x2;
            end
            x3 = round(x3);                                         %Correcting for round-off errors
            check(end+1) = ((x3==-1) && (Q==S-B)) + ...             %Checking whether the result
                ((x3==0) && (Q~=S-B));                              %coincides with the Kr.Delta f.
        end
    end
end
check = check(2:end);
nnz(~check)                                                         %Indicates #wrong outcomes
```