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Dekking, Michel

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Morphic words, Beatty sequences and integer images of the Fibonacci language

Michel Dekking

Delft University of Technology, Faculty EEMCS, P.O. Box 5031, 2600 GA Delft, the Netherlands

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ABSTRACT

Morphic words are letter-to-letter images of fixed points x of morphisms on finite alphabets. There are situations where these letter-to-letter maps do not occur naturally, but have to be replaced by a morphism. We call this a decoration of x. Theoretically, decorations of morphic words are again morphic words, but in several problems the idea of decorating the fixed point of a morphism is useful. We present two of such problems. The first considers the so called AA sequences, where α is a quadratic irrational, A is the Beatty sequence defined by $A(n) = \lfloor \alpha n \rfloor$, and AA is the sequence $(A(A(n)))$. The second example considers homomorphic embeddings of the Fibonacci language into the integers, which turns out to lead to generalized Beatty sequences with terms of the form $V(n) = p\lfloor \alpha n \rfloor + qn + r$, where p, q and r are integers.

1. Introduction

The goal of this paper is to show that if one suspects an infinite word on a finite alphabet to be a morphic word, i.e., the letter-to-letter image of a fixed point of a morphism, then the way to achieve this is not to try to do this directly, but indirectly. By the latter we mean that one replaces the search for a fixed point and a letter-to-letter map by a search for a fixed point and a more general object: a morphism. To emphasize this principle, we call this morphism a *decoration*, and the infinite word will then be a *decoration* of a fixed point. It is well-known that the class of decorations of fixed points of morphisms is equal to the class of morphic words, see, e.g., Corollary 7.7.5 in the monograph by Allouche and Shallit [2]. Their proof, although algorithmic, is somewhat indirect. We will be using a 'natural' algorithm to go from the decorated fixed point to a morphic word, given, e.g., in [16]. We describe this algorithm in the proof of Corollary 9.

We illustrate the usefulness of this 'decoration principle' by giving two examples: iterated Beatty sequences in Section 3 and integer images of the Fibonacci language in Section 4. In that section we solve the Frobenius problem for homomorphic embeddings of the Fibonacci language in the set of integers, which means that we give a precise description of the complement of this embedding.

Although the two examples are seemingly unrelated, they are connected by the appearance of generalized Beatty sequences, which we define in Section 2.

In the appendix we give a different proof that the difference sequence of the iterated Beatty sequence AA defined by $AA(n) = \lfloor n \lfloor n\sqrt{2} \rfloor \rfloor$ is a morphic word. This leads to a morphic word on an alphabet of size 4. We conjecture that this is the smallest size possible, which is equivalent to the conjecture that the difference sequence of AA is not a fixed point of a morphism.

E-mail address: F.M.Dekking@math.tudelft.nl.

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For some general results for a special class of decorations of fixed points of morphisms see [15]. In [15] the decorations are so called marked morphisms, which in some sense are the opposite of the decorations that one will encounter in the present paper. We mention also that decorations of morphisms are closely connected to HD0L-systems. See [19] for some recent results on these in the context of Beatty sequences, which in some sense are also opposite to our results.

2. Generalized Beatty sequences

Let α be an irrational number larger than 1, then A defined by $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$ is known as the *Beatty sequence of* α. Here, $\lfloor\cdot\rfloor$ denotes the floor function. Following [3] we call any sequence V of the form

$$V(n) = pA(n) + qn + r \quad \text{for } n \geq 1$$

where p, q, r are integers, a *generalized Beatty sequence*, for short a GBS.

If S is a sequence, we denote its sequence of first order differences as ΔS, i.e., ΔS is defined by

$$\Delta S(n) = S(n+1) - S(n), \quad \text{for } n = 1, 2 \ldots$$

How does one recognize GBS's? In general this is not easy, but there is a useful characterization for quadratic irrational numbers α, which have the property that $\alpha \in (0, 1)$ and their algebraic conjugate $\overline{\alpha} \notin (0, 1)$. These are known as the Sturm numbers. In general, the sequences of first differences

$$c_\alpha := \big(\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor\big) = \big(\Delta A(n)\big)$$

are called *Sturmian sequences*. The characterization is derived from the following key result, which is also proved in the monographs [2] and [17].

Proposition 1. ([10], [1]) *Let α be a Sturm number. Then there exists a morphism σ_α on the alphabet $\{0, 1\}$, such that $\sigma_\alpha(c_\alpha) = c_\alpha$.*

In the following we will consider the variants of σ_α on various other alphabets than $\{0, 1\}$, but will not indicate this in the notation. As noted in [3], the following lemma follows directly from Proposition 1 by realising that

$$V = pA + q\,\mathrm{Id} + r \;\Rightarrow\; V(n+1) - V(n) = p(A(n+1) - A(n)) + q = p\,c_\alpha(n) + q.$$

Lemma 2. (Allouche and Dekking [3]) *Let α be a Sturm number. Let $V = (V(n))_{n\geq 1}$ be the generalized Beatty sequence defined by $V(n) = p(\lfloor n\alpha \rfloor) + qn + r$, and let ΔV be the sequence of its first differences. Then ΔV is the fixed point of σ_α on the alphabet $\{q, p+q\}$.*

3. Iterated Beatty sequences

Recall that a Beatty sequence is a sequence $A = (A(n))_{n\geq 1}$, with $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$, where α is a positive real number. What Beatty observed is that when $B = (B(n))_{n\geq 1}$ is the sequence defined by $B(n) = \lfloor n\beta \rfloor$, with α and β satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \tag{1}$$

then A and B are *complementary* sequences, that is, the sets $\{A(n) : n \geq 1\}$ and $\{B(n) : n \geq 1\}$ are disjoint and their union is the set of positive integers. In particular if $\alpha = \varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean, this gives that the sequences $(\lfloor n\varphi \rfloor)_{n\geq 1}$ and $(\lfloor n\varphi^2 \rfloor)_{n\geq 1}$ are complementary.

In this paper we look at sequences as functions from $\mathbb{N}$ to $\mathbb{N}$. In this way compositions $Z = XY$ of two sequences X and Y are defined as the sequence given by $Z(n) = X(Y(n))$ for $n \in \mathbb{N}$.

A well known result on the composition of Beatty sequences in the golden mean case is the following.

Theorem 3. (Carlitz-Scoville-Hoggatt [7]) *Let $U = (U(n))_{n\geq 1}$ be a composition of the sequences $A = (\lfloor n\varphi \rfloor)_{n\geq 1}$ and $B = (\lfloor n\varphi^2 \rfloor)_{n\geq 1}$, containing i occurrences of A and j occurrences of B, then for all $n \geq 1$*

$$U(n) = F_{i+2j}A(n) + F_{i+2j-1}n - \lambda_U,$$

where F_k are the Fibonacci numbers ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$) and λ_U is a constant.

This means that any composition of A and B can be written as an integer linear combination $pA + q\mathrm{Id} + r$, where Id is defined by $\mathrm{Id}(n) = n$. The length 2 compositions in Theorem 3 give

$$AA = B - 1 = A + \mathrm{Id} - 1, \quad AB = A + B = 2A + \mathrm{Id}, \quad BA = A + B - 1 = 2A + \mathrm{Id} - 1, \quad BB = A + 2B = 3A + 2\mathrm{Id}.$$

A result as Theorem 3 does not hold for all quadratic irrationals. If we take, for example, $\alpha = \sqrt{2}$, i.e., we consider the Beatty sequence given by $A(n) = \lfloor n\sqrt{2}\rfloor$, then the complementary Beatty sequence B is given by $B(n) = \lfloor n(2+\sqrt{2})\rfloor$. It is proved in [8] (see also [14]) that for $n \geq 1$

$$AB(n) = \lfloor \sqrt{2}\lfloor n(2+\sqrt{2})\rfloor\rfloor = A(n) + B(n) = 2A(n) + 2n.$$

However, no expression for AA is given.[1] In fact, one can easily prove that there do not exist integers p, q and r such that $AA = pA + q\mathrm{Id} + r$. This follows from Lemma 2 in Section 2, since the first order difference sequence of AA takes more than 2 values. Still, expressions for AA are known involving the sequence $\lfloor \sqrt{2}\{n\sqrt{2}\}\rfloor$, see Theorem 1 in [13], and see [5].

Why does the golden mean always yield GBS's for the difference sequences of the compositions of A and B, but the silver mean does not? Our Theorem 5 clarifies the situation.

From now on we focus on the iterated Beatty sequence AA given by $AA(n) = \lfloor \lfloor n\alpha\rfloor \alpha\rfloor$. It has been studied by many authors. See, among others, [7], [8], [13], [4], [5]. The main effort in these papers has been to express AA as a linear combination of A, Id and the constant function.

Here is an important basic result on the iterates of the Beatty sequence $A(n) = \lfloor n\alpha\rfloor$ for algebraic α of degree d. In the following, $\{n\alpha\} = n\alpha - \lfloor n\alpha\rfloor$ is the fractional part of $n\alpha$.

Theorem 4. (Fraenkel [13]) *Let $d \geq 1$ and $a_0, ..., a_d, n, K, L, M \in \mathbb{Z}$. Suppose that $a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 = 0$ has a real nonzero root α. Let $A(n) = \lfloor n\alpha\rfloor$. Then*
$A\big(M + Ln + \sum_{i=0}^{d-2} A^i(Ka_{i+2}A(n))\big) = (L - Ka_1)A(n) - Ka_0 n + D$, where D is bounded in n, namely,
$D = \lfloor M\alpha + (L + Ka_0\alpha^{-1})\{n\alpha\} - \theta\alpha\rfloor$, where $\theta = \sum_{i=1}^{d-2}(Ka_{i+2}A(n)\alpha^i - A^i(Ka_{i+2}A(n)))$.

We are interested only in the case $d = 2$. Let $(x-\alpha)(x-\overline{\alpha})$ be the minimal polynomial of a quadratic irrational α.

Theorem 5. *Let $\alpha > 1$ be a quadratic irrational with minimal polynomial in $\mathbb{Z}[x]$. Let $A(n) = \lfloor n\alpha\rfloor$. The sequence AA is a generalized Beatty sequence if and only if $|\overline{\alpha}| < 1$.*

Proof. If one substitutes $K = 1$, $L = M = 0$, $d = 2$ and $a_2 = 1$ in Theorem 4, one obtains

$$AA(n) = -a_1 A(n) - a_0 n + D(n),$$

where $(x-\alpha)(x-\overline{\alpha}) = x^2 + a_1 x + a_0$, and

$$D(n) = \left\lfloor \frac{a_0}{\alpha}\{n\alpha\}\right\rfloor.$$

The theorem now follows, since $\alpha\overline{\alpha} = a_0$, and since the sequence $(\{n\alpha\})$ is equidistributed over $[0, 1]$.

Here we used that $\theta = 0$ for $d \leq 2$, as indicated by Fraenkel in the Notes on page 642 of [13]. □

Example 6. Let $\alpha = 1 + \sqrt{2}$, with corresponding $A(n) = \lfloor n(1+\sqrt{2})\rfloor$. As in the proof of Theorem 5 one computes that $A(A(n)) = 2A(n) + n - 1$. The number $\alpha - 2 = \sqrt{2} - 1$ is a Sturm number. The corresponding Sturmian sequence is fixed point of the morphism $\sigma_{\alpha-2}$ given by $0 \to 01, 1 \to 010$, as follows from a computation by continued fractions ([10],[2]). Since $A(n) = \lfloor n(\alpha-2)\rfloor + 2n$, ΔA is fixed point of the morphism given by $2 \to 23, 1 \to 232$. Since $AA(n+1) - AA(n) = 2(A(n+1) - A(n)) + 1$, ΔAA is fixed point of the morphism given by $5 \to 57, 7 \to 575$. So in this particular case ΔAA is pure morphic. □

What is the structure of AA if $\alpha > 1$ and $|\overline{\alpha}| > 1$?

We determine this for the Fraenkel family, also known as the *metallic means*, which are the positive solutions to $x^2 + (t-2)x = t$, where the natural number t is the parameter. For $t = 1$ one obtains the golden mean, for $t = 2$ the silver mean $\sqrt{2}$.

Theorem 7. *Let $\alpha = \big(2 - t + \sqrt{t^2+4}\big)/2$, for $t = 2, 3, \ldots$, and let $A(n) = \lfloor n\alpha\rfloor$ for $n \geq 1$. Then ΔAA is a morphic word. In fact, ΔAA is a decoration δ of a fixed point of a morphism τ, both defined on the alphabet $\{1, 2 \ldots, t+1\}$. For $t = 2$ and $t = 3$ the morphisms τ and δ are given respectively by*

$$\tau(1) = 12,\ \tau(2) = 131,\ \tau(3) = 121,\ \ \delta(1) = 13,\ \delta(2) = 222,\ \delta(3) = 132,$$

$$\tau(1) = 123,\ \tau(2) = 124,\ \tau(3) = 1141,\ \tau(4) = 1241,\ \ \delta(1) = 113,\ \delta(2) = 122,\ \delta(3) = 2122,\ \delta(4) = 1222.$$

[1] Neither for BA. The sequence BA has about the same complexity as AA, since $BA = AA + 2A$, as implied by $B(n) = A(n) + 2n$ for all $n \geq 1$.

For $t \geq 4$ the morphism τ is given[2] *by* $\tau(1) = 1...[t-1]\,t,\ \tau(2) = 1...[t-1]\,[t+1]$*, and for* $j = 3, ..., t-1$

$$\tau(j) = 1...[t-j]\,[t-j+1]\,[t-j+1]\,[t-j+2]\ldots[t-2]\,[t+1],$$
$$\tau(t) = 112\ldots[t-2]\,[t+1]\,1, \quad \tau(t+1) = 1223\ldots[t-2]\,[t+1]\,1.$$

For $t \geq 4$ the morphism δ is given by $\delta(1) = 1^{t-1}\,3,\ \delta(2) = 1^{t-2}\,22,\ \delta(j) = 1^{t-j}\,2\,1^{j-2}\,2$ *for* $j = 3, ..., t-1$*, and* $\delta(t) = 2\,1^{t-2}\,22,\ \delta(t+1) = 12\,1^{t-3}\,22$.

In the proof of this theorem we need the combinatorial Lemma 8. We know that ΔA is fixed point of the morphism σ on the alphabet $\{1, 2\}$ given by

$$\sigma(1) = 1^{t-1}2, \quad \sigma(2) = 1^{t-1}21, \tag{2}$$

as can be found in Crisp et al. [10], or Allouche and Shallit [2]. Here one uses that α has a very simple continued fraction expansion: $\alpha = [1; t, t, t, \ldots]$.

Lemma 8. *Let $t \geq 2$ be an integer. For $t = 2$, define the three words* $u_1 = 121,\ v = 2112$*, and* $w = 1212$.
For $t \geq 3$, define the $t-1$ words $u_j = 1^{t-j}21^j$ *for* $j = 1, ..., t-1$*, and the two words* $v = 21^t2,\ w = 121^{t-1}2$.
Let σ be the morphism in (2), then for $t = 2$, one has $\sigma(u_1) = u_1v,\ \sigma(v) = u_1wu_1,\ \sigma(w) = u_1vu_1$.
For $t = 3$ one has $\sigma(u_1) = u_1u_2v,\ \sigma(u_2) = u_1u_2w,\ \sigma(v) = u_1u_1wu_1,\ \sigma(w) = u_1u_2wu_1$.
For $t \geq 4$ one has $\sigma(u_1) = u_1...u_{t-1}v,\ \sigma(u_2) = u_1...u_{t-1}w$*, and for* $j = 3, ..., t-1$ *one has*

$$\sigma(u_j) = u_1...u_{t-j}u_{t-j+1}u_{t-j+1}u_{t-j+2}\ldots u_{t-2}w,$$
$$\sigma(v) = u_1u_1u_2\ldots u_{t-2}wu_1, \quad \sigma(w) = u_1u_2u_2u_3\ldots u_{t-2}wu_1.$$

Proof. First we take $t = 2$. Then σ is given by $\sigma(1) = 12,\ \sigma(2) = 121$. One easily verifies the statement of the lemma: $\sigma(u_1) = 1212112 = u_1v,\ \sigma(v) = 1211212121 = u_1wu_1,\ \sigma(w) = 1212112121 = u_1vu_1$.

The case $t = 3$ follows from an analogous computation.

Next, the case $t \geq 4$. We first mention four relations, directly implied by the definitions, which will be used in the proof:

$$v = 21\sigma(1), \quad w = 12\sigma(1), \quad w = u_{t-1}2, \quad u_1 = \sigma(2).$$

We also use repeatedly

$$\sigma(1^j) = u_1\ldots u_{j-1}1^{t-j}2 \quad \text{for } j = 2, \ldots t-1,$$

which can be proved by induction: $\sigma(1^{j+1}) = u_1\ldots u_{j-1}1^{t-j}2\sigma(1) = u_1\ldots u_{j-1}1^{t-j}21^{t-1}2 = u_1\ldots u_j1^{t-j-1}2$.

We then have

$$\sigma(u_1) = \sigma(1^{t-1}21) = u_1\ldots u_{t-2}12\sigma(2)\sigma(1) = u_1\ldots u_{t-2}121^{t-1}21\sigma(1) = u_1\ldots u_{t-1}21\sigma(1) = u_1...u_{t-1}v,$$
$$\sigma(u_2) = \sigma(1^{t-2}211) = u_1\ldots u_{t-3}1121^{t-1}211^{t-2}12\sigma(1) = u_1\ldots u_{t-1}12\sigma(1) = u_1...u_{t-1}w.$$

Now for u_j, with $3 \leq j \leq t-1$: (interpreting $u_1\ldots u_0$ as an empty prefix in the case $j = t-1$; so in that case the outcome is $\sigma(u_{t-1}) = u_1u_2u_2u_3\ldots u_{t-2}w$ (if $t \geq 4$).)

$$\begin{aligned}
\sigma(u_j) &= \sigma(1^{t-j})\,\sigma(21^j) \\
&= u_1\ldots u_{t-j-1}1^j2\ 1^{t-1}21\ \sigma(1^j) \\
&= u_1\ldots u_{t-j-1}\,u_{t-j}\ 1^{j-1}21\ 1^{t-1}2\ \sigma(1^{j-1}) \\
&= u_1\ldots u_{t-j-1}u_{t-j}\,u_{t-j+1}\ 1^{j-1}2\sigma(1^{j-1}) \\
&= u_1\ldots u_{t-j-1}u_{t-j}u_{t-j+1}\ 1^{j-1}2\ 1^{t-1}2\ \sigma(1^{j-2}) \\
&= u_1\ldots u_{t-j-1}u_{t-j}u_{t-j+1}\ u_{t-j+1}\ 1^{j-2}2\ \sigma(1^{j-2}) \\
&= u_1\ldots u_{t-j-1}u_{t-j}u_{t-j+1}u_{t-j+1}\ 1^{j-2}2\ 1^{t-1}2\ \sigma(1^{j-3}) \\
&= u_1\ldots u_{t-j-1}u_{t-j}u_{t-j+1}u_{t-j+1}\ u_{t-j+2}\ 1^{j-3}2\ \sigma(1^{j-3}) \\
&= \cdots
\end{aligned}$$

[2] For readability, we denote the letters $t-j$ as $[t-j]$.

$$= u_1 \dots u_{t-j-1} u_{t-j} u_{t-j+1} u_{t-j+1} \dots u_{t-2}\ 12\ \sigma(1)$$
$$= u_1 \dots u_{t-j} u_{t-j+1} u_{t-j+1} u_{t-j+2} \dots u_{t-2}\ w.$$

For v and w one derives:

$$\sigma(v) = \sigma(2)\,\sigma(1^{t-1})\sigma(12) = u_1\ u_1 \dots u_{t-2} 12\,\sigma(1)\,\sigma(2) = u_1 u_1 u_2 \dots u_{t-2}\ w\, u_1$$
$$\sigma(w) = \sigma(u_{t-1}2) = u_1 u_2 u_2 u_3 \dots u_{t-2} w\, \sigma(2) = u_1 u_2 u_2 u_3 \dots u_{t-2} w\, u_1. \quad \square$$

Proof of Theorem 7. In view of the complexity of the proof we first give the proof for the case $t = 3$, i.e., the case $\alpha = (\sqrt{13} - 1)/2$, the bronze mean.

We then have to show that ΔAA is a decoration δ of a fixed point of a morphism τ, both defined on the alphabet $\{1, 2, 3, 4\}$, where τ is given by

$$\tau(1) = 123, \quad \tau(2) = 124, \quad \tau(3) = 1141, \quad \tau(4) = 1241,$$

and the decoration δ is given by

$$\delta(1) = 113, \quad \delta(2) = 122, \quad \delta(3) = 2122, \quad \delta(4) = 1222.$$

The words from Lemma 8 are in this case

$$u_1 = 1121, \quad u_2 = 1211, \quad v = 21112, \quad w = 12112,$$

and their images under σ are

$$\sigma(u_1) = u_1 u_2 v, \quad \sigma(u_2) = u_1 u_2 w, \quad \sigma(v) = u_1 u_1 w u_1 \quad \sigma(w) = u_1 u_2 w u_1.$$

The coding $u_1 \mapsto 1, u_2 \mapsto 2, v \mapsto 3, w \mapsto 4$ transforms σ working on $\{u_1, u_2, v, w\}$ into τ.

Let L be the map that assigns to any word its length, so, e.g., $L(u_1) = 4$, $L(v) = 5$.

CLAIM: 1) The word ΔA can be written as $\Delta A = x_1 x_2 \dots$ where each x_i is an element from $\{u_1, u_2, v, w\}$.
2) The word $r := L(x_1)L(x_2)\dots$ is fixed point of the morphism $\sigma_{4,5}$ given by $4 \to 445$, $5 \to 4454$.

Proof of part 1) of the claim: we know that ΔA is the unique fixed point of the morphism $\sigma = \sigma_{1,2}$ given by $1 \to 112$, $2 \to 1121$. Since $1121 = u_1$ is a prefix of ΔA, the word $\sigma^n(u_1)$ is also a prefix of ΔA for all $n \geq 1$. So with Lemma 8 this proves the CLAIM, part 1). Part 2) of the claim then follows from $L(u_1) = L(u_2) = 4$, $L(v) = L(w) = 5$, which induces the morphism $\sigma_{4,5}$ for the infinite word r of lengths.

How do we obtain ΔAA from ΔA? Since $A(\mathbb{N}) = AA(\mathbb{N}) \cup AB(\mathbb{N})$, a disjoint union, one obtains AA from A by removing the integers $AB(n)$, which, of course, have index $B(n)$ in the sequence A. The difference sequence ΔB of this sequence is the unique fixed point of the morphism $\sigma_{4,5}$, since $\beta = \alpha + 3$. It follows then from the CLAIM that the integers $AB(n)$ occur at positions which correspond to the third letter in the word x_i. Here it is the *third* letter, because the first term of the sequence $(A(B(n)) = 5, 10, 15, 22, \dots$ occurs at position 4 in the sequence $(A(n)) = 1, 2, 3, 5, \dots$. Removal of the $AB(n)$ is then performed by adding the third and the fourth letter in the x_i. This operation turns $u_1 = 1121$ into $\delta(1) = 113$, $u_2 = 1211$ into $\delta(2) = 122$, $v = 21112$ into $\delta(3) = 2122$, and $w = 12112$ into $\delta(4) = 1222$. The conclusion is that this decoration δ turns the unique fixed point of τ into ΔAA. This ends the proof for the case $t = 3$.

For general t, the coding $u_1 \mapsto 1, \dots, u_{t-1} \mapsto t-1, v \mapsto t, w \mapsto t+1$ transforms σ working on $\{u_1, \dots, u_{t-1}, v, w\}$ into τ. An analogous claim as for the $t = 3$ case holds, and now the map L satisfies

$$L(u_1) = L(u_2) = \dots = L(u_{t-1}) = t + 1, \quad L(v) = L(w) = t + 2,$$

which induces the morphism $\sigma_{t+1,t+2}$ for the infinite word r of lengths. One continues in the same way, using now that $\beta = \alpha + t$. This time, the integers $AB(n)$ occur at positions in A with correspond to the t^{th} letter in the words x_i from $\{u_1, \dots, u_{t-1}, v, w\}$. Here it is the t^{th} letter, because the first term of the sequence $(A(B(n))$ occurs at position $B(1) = t + 1$ in the sequence $(A(n))$. Here $B(1) = \lfloor \beta \rfloor = \lfloor \alpha + t \rfloor = t + 1$, since a simple computation shows that $1 < \alpha < 2$ for all t.

Removal of the $AB(n)$ is then performed by adding the t^{th} and the $(t+1)^{\text{th}}$ letter in the x_i. This operation turns $u_1 = 1^{t-1}21$ into $\delta(1) = 1^{t-1}3$, $u_2 = 1^{t-2}211$ into $\delta(2) = 1^{t-2}22$ and $u_j = 1^{t-j}21^j$ into $\delta(j) = 1^{t-j}21^{j-2}2$, for $j = 3, \dots, t-1$. Moreover, the two words $v = 21^t2$, $w = 121^{t-1}2$ are turned into $\delta(t) = 2\,1^{t-2}\,22$, respectively $\delta(t+1) = 12\,1^{t-3}\,22$.

The conclusion is that this decoration δ maps the fixed point of τ to the first differences ΔAA. $\square$

Corollary 9. *Here is a way to write $\Delta AA = 11312221222\dots$ as a morphic word for the case $t = 3$, i.e., $\alpha = (\sqrt{13} - 1)/2$, the bronze mean. Let θ on $\{1, \dots, 6\}$ be the morphism given by*

$$\theta: \quad 1 \to 123,\ 2 \to 164,\ 3 \to 5145,\ 4 \to 1645,\ 5 \to 123,\ 6 \to 164.$$

Let the letter-to-letter morphism λ be given by

$$\lambda: \quad 1 \to 1,\ 2 \to 1,\ 4 \to 2,\ 5 \to 2,\ 6 \to 2,\ 3 \to 3.$$

Then $\Delta AA = \lambda(\theta^\infty(1))$.

Proof. This corollary is derived from Theorem 7 by using the 'natural' algorithm given, for example, by Honkala in [16], Lemma 4. Honkala's requirement of 'cyclicity' in that lemma is not necessary.

To make this paper more self-contained we give a description of this 'natural' algorithm.

Let x be a fixed point of a morphism τ on an alphabet A, and $\delta: A \to B$ a decoration. Let $d(a) := |\delta(a)|$, so that $\delta(a) = \delta_1(a)\dots\delta_{d(a)}(a)$ for each $a \in A$. The 'natural' algorithm consists of replacing each letter a in x by $d(a)$ copies of the letter a, denoted as $C_1(a), \dots, C_{d(a)}(a)$. The letter to letter map λ on the alphabet $A_C := \{C_j(a) : a \in A,\ j=1,\dots,d(a)\}$ is then defined as $\lambda(C_j(a)) = \delta_j(a)$. The morphism τ induces a large number of morphisms θ on A_C, by first mapping for each a the word $C(a) := C_1(a), \dots C_{d(a)}(a)$ to the concatenation of words $C_\tau(a) := C(\tau_1(a))\dots C(\tau_{t(a)}(a))$, when $\tau(a) = \tau_1(a)\dots\tau_{t(a)}(a)$, and then splitting $C_\tau(a)$ into $d(a)$ words, defining $\theta(C_i(a))$ as the i^{th} word in this splitting. The splitting should be done in such a way that a primitive morphism θ results.

In the proof of Corollary 9 the alphabet A_C has a priori $d(1)+d(2)+d(3)+d(4) = 14$ letters. The situation is special here, since $d(a) = t(a)$ for $a = 1,2,3,4$. This suggests to define θ by splitting the $C_\tau(a)$ into the words $C(\tau_1(a))$ till $C(\tau_{t(a)}(a))$. After projecting letters with the same θ-image and the same λ-image on a single letter, the number of letters reduces to 6, and one obtains the morphism θ in the corollary. □

Fraenkel's Theorem 4 with the 'defect' function $D = D(n)$ suggests that the ΔAA sequences can take many values. This is not the case.

Proposition 10. *For any irrational α larger than 1 the sequence $\Delta AA = \big(\lfloor\lfloor(n+1)\alpha\rfloor\alpha\rfloor - \lfloor\lfloor n\alpha\rfloor\alpha\rfloor\big)$ takes values in an alphabet of size two, three or four.*

Proof. We illustrate the proof with the case $1 < \alpha < 2$. Then $s := \Delta A$ is a Sturmian word taking values $d = 1$ or $d = 2$. So

$$\Delta AA(n) = A(A(n+1)) - A(A(n)) = A(A(n)+d) - A(A(n)), \text{ where } d = 1 \text{ or } 2.$$

We put $i := A(n)$. In case $d = 1$, $A(A(n)+d) - A(A(n)) = A(i+1) - A(i) = 1$ or 2. In case $d = 2$, $A(A(n)+d) - A(A(n)) = A(i+2) - A(i) = A(i+2) - A(i+1) + A(i+1) - A(i)$. So either $A(A(n)+d) - A(A(n)) = 2$ or 3, or $A(A(n)+d) - A(A(n)) = 3$ or 4, respectively if 11, 12 and 21 are the subwords of length 2 of s, or if 12, 21 and 22 are the subwords of length 2 of s. What we found is that ΔAA takes values in $\{1,2,3\}$ if $1 < \alpha < 3/2$, and ΔAA takes values in $\{1,2,3,4\}$ if $3/2 < \alpha < 2$. In some cases ΔAA may take only 2 values, for example, if α is the golden mean.

The proof for other values of α is similar, exploiting balancedness of the Sturmian word $s = \Delta A$. □

Example. Take $\alpha = \sqrt{11}/2 = 1.658\dots$. Then $AA(n) = 1,4,6,9,13,14,18,21,23\dots$, so ΔAA takes the four values 1, 2, 3 and 4.

Remark. Once more, let $\alpha = \sqrt{2}$. The differences $x_{2,k} := (AB)^k - (BA)^k$, where $k \ge 1$, are the 'commutator' functions. They are extensively studied in [9]. They are all similar to $x_{2,1}$, which is equal to $x_{2,1} = AB - BA = 2\mathrm{Id} - AA$. One can derive from this that all commutator functions are morphic words.

4. Embeddings of the Fibonacci language into the integers

Let $\mathcal{L}$ be a language, i.e., a sub-semigroup of the free semigroup generated by a finite alphabet under the concatenation operation. A homomorphism of $\mathcal{L}$ into the natural numbers is a map $\mathrm{S}: \mathcal{L} \to \mathbb{N}$ satisfying

$$\mathrm{S}(vw) = \mathrm{S}(v) + \mathrm{S}(w), \quad \text{for all } v, w \in \mathcal{L}.$$

The classical Frobenius problem asks whether the complement of $\mathrm{S}(\mathcal{L})$ in the natural numbers will be infinite or finite, and in the latter case the value of the largest element in this complement. In the classical Frobenius problem $\mathcal{L}$ is the full language consisting of all words over a finite alphabet. We will solve this problem when $\mathcal{L} = \mathcal{L}_{\mathrm{F}}$ i.e., the set of all words occurring in x_{F}, where x_{F} is the Fibonacci word, the infinite word fixed by the morphism $0 \to 01$, $1 \to 0$.

Recall that $\varphi = (1+\sqrt{5})/2$. The key ingredient in this section is the lower Wythoff sequence $(\lfloor n\varphi\rfloor)_{n\ge1} = 1,3,4,6,8,9,$ $11,12,14,16,17,19,\dots$. The following result is proved in [12].

Theorem 11. **([12])** *Let* $\mathrm{S} : \mathcal{L}_{\mathrm{F}} \to \mathbb{N}$ *be a homomorphism. Define* $a = \mathrm{S}(0), b = \mathrm{S}(1)$. *Then* $\mathrm{S}(\mathcal{L}_{\mathrm{F}})$ *is the union of the two generalized Beatty sequences* $\big((a-b)\lfloor n\varphi \rfloor + (2b-a)n\big)$ *and* $\big((a-b)\lfloor n\varphi \rfloor + (2b-a)n + a - b\big)$.

What remains to be done is to determine the complement of the set $\mathrm{S}(\mathcal{L}_{\mathrm{F}})$ in $\mathbb{N}$. We shall show that the corresponding infinite word is always a morphic word, by representing it as a decoration of a fixed point of a morphism. It appears that this is a matter of complicated bookkeeping, especially when the two values S(0) and S(1) are small.

There are three morphisms f, g and h that play an important role in this section, where it is convenient to look at a and b both as integers and as abstract letters. The morphisms are given by

$$f : \begin{cases} a \to ab \\ b \to a \end{cases}, \qquad g : \begin{cases} a \to baa \\ b \to ba \end{cases}, \qquad h : \begin{cases} a \to aab \\ b \to ab \end{cases}.$$

Lemma 12. *Let* x_{F} *be the Fibonacci sequence on the alphabet* $\{a, b\}$, *fixed point of* f. *Then the fixed point* x_{G} *of* g *is the sequence* $b\,x_{\mathrm{F}}$, *and the fixed point* x_{H} *of* h *is* $a\,x_{\mathrm{F}}$.

Proof. Berstel and Séébold prove in [6] (Theorem 3.1) that for any morphic Sturmian word c_α on the alphabet $\{a, b\}$ both ac_α and bc_α are again morphic words. Their proof is constructive, and they give the morphisms g and h for $c_\alpha = x_{\mathrm{F}}$ in their Example 1. □

Here is a result that gives an idea of the proof in general for the case $\mathrm{S}(0) > \mathrm{S}(1)$.

Theorem 13. *Let* $\mathrm{S} : \mathcal{L}_{\mathrm{F}} \to \mathbb{N}$ *be a homomorphism determined by* $a = \mathrm{S}(0), b = \mathrm{S}(1)$. *Suppose that*

$$a + 2 < 2b + 1 < 2a - 1.$$

Then the first differences of the complement $\mathbb{N} \setminus \mathrm{S}(\mathcal{L}_{\mathrm{F}})$ *of* $\mathrm{S}(\mathcal{L}_{\mathrm{F}})$ *is the word obtained by decorating the fixed point* x_{H} *of the morphism* h *by the morphism* δ *given by*

$$\delta(a) = 1^{b-2}\,2\,1^{a-b-2}\,2, \quad \delta(b) = 1^{2b-a-2}\,2\,1^{a-b-2}\,2.$$

Proof. The sequence of first differences of a generalized Beatty sequence $\big(p\lfloor n\varphi \rfloor + qn + r\big)$ is the fixed point of the Fibonacci morphism f on the alphabet $\{2p+q, p+q\}$. This follows directly form Lemma 2, see also Lemma 8 in [3]. So the two generalized Beatty sequences $G_1 := \big((a-b)\lfloor n\varphi \rfloor + (2b-a)n\big)$ and G_2, given by $G_2(n) = G_1(n) + a - b$ in Theorem 11 have the property that $\Delta G_1 = \Delta G_2$ is the fixed point x_{F} of the Fibonacci morphism on the alphabet with symbols $2(a-b) + 2b - a = a$ and $a - b + 2b - a = b$.

We illustrate the proof by first considering the case $a = 8$, $b = 5$. In this case we have

$$G_1 = 5, 13, 18, 26, 34, 39, 47, 52, 60, \ldots, \quad G_2 = G_1 + 3 = 8, 16, 21, 29, 37, 42, 50, 55, 63, \ldots\ldots$$

Partition the positive integers $\mathbb{N}$ into adjacent sets V_i, $i = 1, 2, \ldots$ defined by

$$V_i = \{G_2(i-1) + 1, \ldots, G_2(i)\}.$$

Here we put $G_2(0) = 0$. As a consequence, $\mathrm{Card}(V_i) = 8$ if $x_{\mathrm{H}}(i) = a$ and $\mathrm{Card}(V_i) = 5$ if $x_{\mathrm{H}}(i) = b$, where $x_{\mathrm{H}} = x_{\mathrm{H}}(1)x_{\mathrm{H}}(2)\cdots = a\,a\,b\,a\,a\,b\ldots$ is the fixed point of h. The reason that the directive sequence is x_{H} instead of x_{F} is that the *last* element of each V_i is equal to $G_2(i)$ for $i = 1, 2, \ldots$.

			V_1								V_2							V_3					
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
▽	▽	▽	▽	⊡	▽	▽	⊞	▽	▽	▽	▽	⊡	▽	▽	⊞	▽	⊡	▽	▽	⊞	▽	▽	▽

In the table above, the integers in $G_1(\mathbb{N})$ are marked with ⊡, those in $G_2(\mathbb{N})$ with ⊞, and those in the complement with a ▽. By construction, *all* the V_i with cardinality 8 have the same pattern ▽▽▽▽⊡▽▽⊞ for their members. Also *all* V_i with cardinality 5 have the same pattern ▽⊡▽▽⊞. Note that the last two symbols are ▽⊞, for both size 5 and size 8 V_i's, *and* their first symbols are ▽ for both. This implies that if we glue the patterns together, then the infinite sequence of differences of the positions of ▽ in the infinite pattern yields first differences of the sequence of elements in $\mathbb{N} \setminus (G_1(\mathbb{N}) \cup G_2(\mathbb{N}))$. For V_i of size 8 these differences (including the 'jump over' last value 2) are given by 1,1,1,2,1,2, and for V_i of size 5 by 2,1,2. It follows that the first differences are obtained by decorating the fixed point x_{H} by the morphism δ given by

$$\delta: \quad a \to 111212, \quad b \to 212.$$

For the general case one considers sets V_i of consecutive integers of size a or size b, where the order is again dictated by the fixed point x_{H} of h. The corresponding patterns have exactly one symbol ⊞ at the end, and exactly one symbol ⊡

positioned $a-b$ places before the end. It follows again that over the V_i's the first differences of the complement set end in 2 (the 'jump over' value), are preceded by $a-b-2$ 1's, which is preceded by a 2. The first differences start with a number of 1's, which is $(a-2)-1-(a-b-2)-1=b-2$ for the V_i's of length a, and $(b-2)-1-(a-b-2)-1=2b-a-2$ for the V_i's of length b. This yields the decoration δ stated in the theorem. □

We now give an example of the difficulties one encounters when S(0) or S(1) are (relatively) small.

Theorem 14. *Let* $\mathrm{S}:\mathcal{L}_\mathrm{F}\to\mathbb{N}$ *be the homomorphism determined by* $a=\mathrm{S}(0)=3,\ b=\mathrm{S}(1)=1$. *Then the sequence of first differences of the complement* $\mathbb{N}\setminus\mathrm{S}(\mathcal{L}_\mathrm{F})$ *of* $\mathrm{S}(\mathcal{L}_\mathrm{F})$ *is the word obtained by decorating the fixed point* x_H *of* h *by* $\delta:\{a,b\}\to\{7,11\}$ *given by* $\delta(a)=7,11$, *and* $\delta(b)=11$.

Proof. According to Theorem 11, $\mathrm{S}(\mathcal{L}_\mathrm{F})$ is the union of the two sets $G_1(\mathbb{N})$ and $G_2(\mathbb{N})$ given by

$$G_1(\mathbb{N})=\{2\lfloor n\varphi\rfloor-n,\ n\ge 1\}=1,4,5,8,11,12\ldots,\quad G_2(\mathbb{N})=\{2\lfloor n\varphi\rfloor-n+2,\ n\ge 1\}=3,6,7,10,13,14\ldots.$$

The first differences $\Delta G_1=\Delta G_2$ are the Fibonacci word on the alphabet $\{3,1\}$. Imitating the proof of the previous theorem, we obtain the following table, induced by the morphism h given by $1\to 331, 3\to 31$. One has $\mathrm{Card}(V_i)=a=3$ if $x_\mathrm{H}(i)=a$ and $\mathrm{Card}(V_i)=b=1$ if $x_\mathrm{H}(i)=b$, where $x_\mathrm{H}=x_\mathrm{H}(1)x_\mathrm{H}(2)\cdots=a\,a\,b\,a\,a\,b\,a\,b\,a\,a\ldots$ is the fixed point of h.

	V_1			V_2		V_3		V_4			V_5		V_6		V_7		V_8		V_9			V_{10}	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$\boxdot$	$\triangledown$	$\boxplus$	$\boxdot$	$\boxdot$	$\boxplus$	$\boxplus$	$\boxdot$	$\triangledown$	$\boxplus$	$\boxdot$	$\boxdot$	$\boxplus$	$\boxplus$	$\boxdot$	$\boxdot$	$\boxplus$	$\boxplus$	$\boxdot$	$\triangledown$	$\boxplus$	$\boxdot$	$\boxdot$	$\boxplus$

There are at least two things wrong with this:

[E1] The V_i's of length 3 do not all have the same pattern,
[E2] There are patterns that do not contain a $\triangledown$.

To counter these problems, we go from the letters $a=3, b=1$ to the words $h(3), h(1)$, yielding a partition with W_i's of length 7 and 4. The table we obtain is

			W_1							W_2					W_3						W_4		
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$\boxdot$	$\triangledown$	$\boxplus$	$\boxdot$	$\boxdot$	$\boxplus$	$\boxplus$	$\boxdot$	$\triangledown$	$\boxplus$	$\boxdot$	$\boxdot$	$\boxplus$	$\boxplus$	$\boxdot$	$\boxdot$	$\boxplus$	$\boxplus$	$\boxdot$	$\triangledown$	$\boxplus$	$\boxdot$	$\boxdot$	$\boxplus$

Problem [E1] is caused by the fact that V_i's of length 3 have different patterns depending on whether they are followed by a V_i of length 1 or of length 3. Problem [E1] is now solved with the W_i's, since 33 can only occur as a prefix of $h(1)=331$, and 31 can only occur as a suffix of either $h(1)$ or $h(3)$.

However, [E2] is not yet solved, since W_3 does not contain a $\triangledown$. The way to tackle this is to pass to the square of h, i.e., take the W_i''s of length 18 and 11 corresponding to $h^2(1)=33133131$ and $h^2(3)=33131$.

It is obvious from the corresponding patterns, that the differences of the complement $\mathbb{N}\setminus\mathrm{S}(\mathcal{L}_\mathrm{F})$ are given by the decoration $W_1'\to 7,11, W_3'\to 11$ of the W_i''s. But since $h^2(x_\mathrm{H})=x_\mathrm{H}$, this is the same as decorating the letters $a\to 7,11$, and $b\to 11$ in x_H. □

Remark 15. Theorem 25 in [3] states that the three sequences $(2\lfloor n\varphi\rfloor-n,\ n\ge 1)=(1,4,5,8,11,12\ldots)$, $(2\lfloor n\varphi\rfloor-n+2n\ge 1)$, and $z:=(4\lfloor n\varphi\rfloor+3n+2,\ n\ge 0)=(2,9,20,27,\ldots)$ form a complementary triple. From Lemma 2 applied with the Sturm number $\alpha=\varphi-1$ one deduces that $\Delta z=7x_{11,7}$ the Fibonacci sequence on the alphabet $\{11,7\}$, preceded by the letter 7 (see also Lemma 8 in [3], which states that if $V(n)=p(\lfloor n\varphi\rfloor)+qn+r$ then $\Delta V=x_{2p+q,p+q}$).

On the other hand, we have Theorem 14, telling us that $\Delta z=\delta(x_\mathrm{H})$, where δ is the decoration $a\to 7,11$, and $b\to 11$. Applying the 'natural' algorithm to $\delta(x_\mathrm{H})$, we obtain that $\delta(x_\mathrm{H})$ is the morphic word obtained by applying the letter-to-letter map $\lambda(a)=11,\ \lambda(b)=7$ to the fixed point x_G of the morphism g. Thus

$$\delta(x_\mathrm{H})=\lambda(x_\mathrm{G})=\lambda(bx_\mathrm{F})=\lambda(b)\lambda(x_\mathrm{F})=7\lambda(x_\mathrm{F})=7x_{11,7}.$$

Conclusion: Theorem 14 is essentially equal to Theorem 25 in [3], but has a completely different proof. □

We let C be the increasing sequence of integers in the complement of $\mathrm{S}(\mathcal{L}_\mathrm{F})$, so $C(\mathbb{N})=\mathbb{N}\setminus\mathrm{S}(\mathcal{L}_\mathrm{F})$.

Theorem 16. *Let* $\mathrm{S}:\mathcal{L}_\mathrm{F}\to\mathbb{N}$ *be a homomorphism. Then the sequence* ΔC *of first differences of the complement* $\mathbb{N}\setminus\mathrm{S}(\mathcal{L}_\mathrm{F})$ *of* $\mathrm{S}(\mathcal{L}_\mathrm{F})$ *is a fixed point of a morphism on an alphabet of two letters decorated by a morphism* δ.

Proof. The homomorphism S is determined by $a := S(0), b := S(1)$.

Case 1: $a \geq 4, b = 1$. Here we follow the proof of Theorem 14. The V_i are given by $V_i = \{G_2(i-1)+1, \ldots, G_2(i)\}$. Problem [E1], mentioned in the proof of Theorem 14, is more severe in this case, as the pattern of the V_i's of length a depends both on V_{i-1} and V_{i+1}. If these have both length 1, then the distance to the next element in V_{i+1} with symbol $\triangledown$ is 5, otherwise it is 4. To make the process context free, we choose the W_i corresponding to the two words

$$v := h^2(a) = aa1aa1a1, \quad w := h^2(1) = aa1a1.$$

Context-freeness now occurs because $1a1$ occurs uniquely inside v and w. One checks that the decoration δ is then given by

$$\delta(v) = 1^{a-3}\,4\,1^{a-4}\,4\,1^{a-3}\,4\,1^{a-4}\,5\,1^{a-4}\,4, \quad \delta(w) = 1^{a-3}\,4\,1^{a-4}\,5\,1^{a-4}\,4.$$

Since v and w start and end with the same words, this decoration yields ΔC, when applied to x_{H} on the alphabet $\{v, w\}$.

Case 2: $a = 1, b \geq 5$. This[3] is a variant of Case 1. The sequence ΔG_1 is fixed point of the Fibonacci morphism on the alphabet $\{1, b\}$, and so $b\,\Delta G_1$ is fixed point of g on $\{1, b\}$. Problem [E1] is now that the 'jump over' from $b11$ to $b1b$ is 6, but the 'jump over' from $b11$ to $b11$ equals 7. The adequate partition elements W_i correspond to the words v or w:

$$v := g^2(1) = b\,1\,b\,1\,1\,b\,1\,1, \quad w := g^2(b) = b\,1\,b\,1\,1.$$

The decoration δ is given by

$$\delta(v) = 1^{b-4}\,6\,1^{b-5}\,7\,1^{b-5}\,6, \quad \delta(w) = 1^{b-4}\,6\,1^{b-5}6.$$

Case 3: $a > b \geq 2$. The partition elements are defined as $V_i = \{G_2(i-1)+1, \ldots, G_2(i)\}$, where we put $G_2(0) = 0$. This gives $\mathrm{Card}(V_i) = a$ if $x_{\mathrm{H}}(i) = a$ and $\mathrm{Card}(V_i) = b$ if $x_{\mathrm{H}}(i) = b$, where $x_{\mathrm{H}} = x_{\mathrm{H}}(1)x_{\mathrm{H}}(2)\cdots = a\,a\,b\,a\,a\,b\ldots$ is the fixed point of h. To get rid of problem [E1], we coarsen the partition to blocks W_i corresponding to the words $h(a) = aab$ and $h(b) = ab$. The problem disappears because aa uniquely occurs as a prefix of aab, and ab uniquely as a suffix of aab or ab. Problem [E2] will not occur, since any 5 consecutive integers will contain an element of C (as $b \geq 2$, and no bb occurs in x_{H}), and the smallest cardinality of a W_i is $a + b \geq 5$. Also, since both aab and ab start with a, and both end in b, the patterns of the W_i will concatenate consistently, so that the decoration δ obtained from the patterns of the W_i acting as a morphism on x_{H}, will yield the difference sequence of C.

Case 4: $b > a \geq 2$. The partition elements are defined as $V_i = \{G_1(i-1)+1, \ldots, G_1(i)\}$, where we put $G_1(0) = 0$. This gives $\mathrm{Card}(V_i) = a$ if $x_{\mathrm{G}}(i) = a$ and $\mathrm{Card}(V_i) = b$ if $x_{\mathrm{G}}(i) = b$, where $x_{\mathrm{G}} = x_{\mathrm{G}}(1)x_{\mathrm{G}}(2)\cdots = b\,a\,b\,a\,a\,b\ldots$ is the fixed point of g. The rest of the proof follows Case 3, replacing h by g (noting that this time aa uniquely occurs as a *suffix* of $g(a)$, and ab only occurs split over a suffix of $g(\ell)$ and a prefix of $g(\ell')$, for $\ell, \ell' = a, b$). □

We illustrate Case 4 with the following example.

Example 17. Let $a = 5$, $b = 9$. Then $G_1 = 9, 14, 23, \ldots$ and $G_2 = 5, 10, 19, \ldots$. The partition elements are W_1 of cardinality 14 corresponding to $g(b) = ba = 95$, and W_2 of cardinality 19, corresponding to $g(a) = baa = 955$.

The patterns of these sets are ▽▽▽▽⊞▽▽▽⊡⊞▽▽▽⊡ and ▽▽▽▽⊞▽▽▽⊡⊞▽▽▽⊡⊞▽▽▽⊡.

It follows that the decoration δ is given by $\delta(9) = \delta(b) = 1112113112$, $\delta(5) = \delta(a) = 1112113113112$. □

The representation in Theorem 16 is by no means unique. As an example, let the morphism $\hat{g}_2$ on $\{1, 2, 3\}$ be given by

$$\hat{g}_2(1) = 12,\ \hat{g}_2(2) = \hat{g}_2(3) = 132.$$

The morphism $\hat{g}_2$ is the 2-block morphism of g under the coding $ba \to 1, ab \to 2, aa \to 3$ (cf. [18] and [11]). The use of $\hat{g}_2$ gives an alternative way to solve problem [E2], leading, for example, in Example 17 to the fact that ΔC is the decoration of the fixed point of $\hat{g}_2$ by the morphism δ given by

$$\delta(1) = 1112113,\ \delta(2) = 112,\ \delta(3) = 113.$$

Finally we mention another way in which the representation in Theorem 16 is not unique. In fact, one can show that every ΔC is a decoration of the single word x_{G}. Let $\bar{f}$ be the time reversal of the Fibonacci morphism f, i.e., $\bar{f}$ is defined by $\bar{f}(0) = 10,\ \bar{f}(1) = 0$. One verifies that

$$g = \bar{f}\,f, \quad h = f\,\bar{f}.$$

This leads to

[3] We leave the case $a = 1, b = 4$ as an exercise to the reader. In this case the decoration δ turns out to be $v \to 11$, $w \to 17$.

$$\bar{f}(x_{\rm H}) = \bar{f}h(x_{\rm H}) = \bar{f}f\bar{f}(x_{\rm H}) = g\bar{f}(x_{\rm H}) \quad \Rightarrow \quad \bar{f}(x_{\rm H}) = x_{\rm G},$$

since $x_{\rm G}$ is the unique fixed point of g. As a corollary one obtains that if z is a decoration of $x_{\rm H}$ by δ, then z is *also* a decoration of $x_{\rm G}$: replace δ by $\delta' = \bar{f}\delta$.

5. Appendix

In this section we give an alternative proof of Theorem 7, when $t = 2$, i.e., the case $\alpha = \sqrt{2}$.

Theorem 18. *Let $\alpha = \sqrt{2}$, $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$. Then $\Delta AA = 1, 3, 2, 2, 2, 1, 3, 1, 3, 2, \ldots$ is a decoration δ of a fixed point of a morphism σ, both defined on the alphabet $\{1, 2, 3\}$. Here σ is given by*

$$\sigma(1) = 123, \quad \sigma(2) = 1, \quad \sigma(3) = 121,$$

and the decoration δ is given by

$$\delta(1) = 13, \quad \delta(2) = 2, \quad \delta(3) = 22.$$

Proof. Step 1. In this step we 'refine' the sequence $x := \Delta A$ to a sequence y on 4 symbols, which codes the occurrence of the terms of AA in A.

From [10] (or see [17]) one deduces that $x = \Delta A$ is the fixed point of the morphism γ given by

$$\gamma(1) = 12, \quad \gamma(2) = 121.$$

We define the extended morphism γ_E on the alphabet $\{1, 2, 3, 4\}$ by

$$\gamma_E(1) = 13, \quad \gamma_E(2) = 24, \quad \gamma_E(3) = 241 \quad \gamma_E(4) = 132.$$

Note that $\gamma = \pi\gamma_E$, where $\pi(1) = \pi(2) = 1$, and $\pi(3) = \pi(4) = 2$. We define

$$y = 1, 3, 2, 4, 1, 2, 4, 1, 3, 2, 1, 3, \ldots,$$

the fixed point of γ_E with $y_1 = 1$. We claim that y has the property that the letters 1 and 2 alternate in y. Indeed, the words 132 and 12 are the only words in y with prefix 1 and suffix 2 containing no other 1's or 2's, and these are mapped to

$$\gamma_E(12) = 1324, \quad \gamma_E(132) = 1324124,$$

in which 1's and 2's alternate, and similarly the words 241 and 21 are mapped to 2413213 and 2413 in which 2's and 1's alternate. Since in the first case the first occurring letter is 1 and the last is 2, and in the second case the first occurring letter is 2 and the last is 1, it follows by induction that the letters 1 and 2 in $\gamma_E^n(1)$ alternate for all n.

We are interested in the positions 3,6,10,13,... of the letter 2 in y. Let x' be defined by $x'_n = x_n - 1$. Then $x' = 0, 1, 0, 1, 0, 0, 1 \ldots$ is a Sturmian word with slope $\sqrt{2} - 1$. Its binary complement $\tilde{x}' = 1, 0, 1, 0, 1, 1, 0$ is a Sturmian word with slope $\tilde{\alpha}' = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2}$. By Lemma 9.1.3 in [2], the positions of 1's in $\tilde{x}'$ are given by the Beatty sequence $b = (\lfloor n\beta \rfloor)$, where

$$\beta = 1/\tilde{\alpha}' = 1/(2 - \sqrt{2}) = 1 + \frac{1}{2}\sqrt{2}.$$

But the 1's in $\tilde{x}'$ correspond to the 1's *and* 2's in y, and since these alternate, the positions of the 2's in y are given by the sequence

$$(b_{2n}) = (\lfloor 2n\beta \rfloor) = (\lfloor n(2 + \sqrt{2}) \rfloor).$$

Thus we found that the 2's in y exactly occur at the Beatty complement B of A.

Step 2. In this step we partition the 'refinement' y of the word $x = \Delta A$ in three words w_1, w_2, w_3, which will tell us how ΔAA behaves. We claim that the three words

$$w_1 = 132, \quad w_2 = 4, \quad w_3 = 124$$

partition y. This follows directly from $\gamma_E(y) = y$ by noting that

$$\gamma_E(w_1) = 1324124 = w_1w_2w_3, \quad \gamma_E(w_2) = 132 = w_1, \quad \gamma_E(w_3) = 1324132 = w_1w_2w_1.$$

This equation induces a morphism σ on the alphabet {1,2,3}, by replacing w_j with j:

$$\sigma(1) = 123, \quad \sigma(2) = 1, \quad \sigma(3) = 121.$$

How do we obtain ΔAA from ΔA? Since $A(\mathbb{N}) = AA(\mathbb{N}) \cup AB(\mathbb{N})$, a disjoint union, one obtains AA from A by removing the integers $AB(n)$, which, of course, have index $B(n)$ in the sequence A. In Step 1 we showed that this sequence of indices corresponds to the positions of 2's in y. Now if such a 2 occurs in $w_1 = 132$, then the differences $x_k, x_{k+1}, x_{k+2} = 1, 2, 1$ in x turn into differences 1,3 in ΔAA, since the second 1 disappears because of the removal of the A-number corresponding to x_{k+2}, and this 1 must be added to $x_{k+1} = 2$. The other possibility is that such a 2 occurs in $w_3 = 124$, and now the removal of the A-number corresponding to x_{k+1} leads to differences 2,2 in ΔAA. The conclusion is that the decoration δ given by $\delta(1) = 13, \delta(2) = 2$ and $\delta(3) = 22$ turns the fixed point of σ into ΔAA. □

Corollary 19. *Here is a way to write $\Delta AA = 1, 3, 2, 2, 2, 1, 3, 1, 3, 2, \ldots$ as a morphic word (derived from the previous theorem). Let θ on $\{a, b, c, d\}$ be the morphism given by $\theta: a \to adc,\ b \to adc,\ c \to ad,\ d \to bc$.*

Let the letter-to-letter morphism λ be given by $\lambda: a \to 1,\ b \to 2,\ c \to 2,\ d \to 3$. Then $\Delta AA = \lambda(\theta^\infty(a))$.

Declaration of competing interest

There are no conflicts of interest, that I know of.

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References

[1] C. Allauzen, Une caractérisation simple des nombres de Sturm, J. Théor. Nr. Bordx. 10 (1998) 237–241.
[2] J.-P. Allouche, J. Shallit, Automatic Sequences, Cambridge Univ. Press, 2003.
[3] J.-P. Allouche, F.M. Dekking, Generalized Beatty sequences and complementary triples, Mosc. J. Comb. Number Theory 8 (2019) 325–341, https://doi.org/10.2140/moscow.2019.8.325.
[4] S. Artstein-Avidan, A.S. Fraenkel, V. Sos, A two-parameter family of an extension of Beatty sequences, Discrete Math. 308 (2008) 4578–4588, https://doi.org/10.1016/j.disc.2007.08.070.
[5] C. Ballot, On functions expressible as words on a pair of Beatty sequences, J. Integer Seq. 20 (2017) 17.4.2.
[6] J. Berstel, P. Séébold, A remark on morphic Sturmian words, RAIRO Theor. Inform. Appl. 28 (1994) 255–263.
[7] L. Carlitz, R. Scoville, V.E. Hoggatt Jr., Fibonacci representations, Fibonacci Q. 10 (1972) 1–28, Also see L. Carlitz, R. Scoville, V.E. Hoggatt Jr., Addendum to the paper: "Fibonacci representations" Fibonacci Q. 10 (1972) 527–530.
[8] L. Carlitz, R. Scoville, V.E. Hoggatt Jr., Pellian representations, Fibonacci Q. 10 (5) (1972) 449–488.
[9] J.J. Chew III, S.M. Tanny, Further results on iterated Beatty functions, J. Differ. Equ. Appl. 7 (3) (2001) 413–434, https://doi.org/10.1080/10236190108808279.
[10] D. Crisp, W. Moran, A. Pollington, P. Shiue, Substitution invariant cutting sequences, J. Théor. Nr. Bordx. 5 (1993) 123–137.
[11] F.M. Dekking, Morphisms, symbolic sequences, and their standard forms, J. Integer Seq. 19 (2016) 16.1.1.
[12] M. Dekking, The Frobenius problem for homomorphic embeddings of languages into the integers, Theor. Comput. Sci. 732 (2018) 73–79, https://doi.org/10.1016/j.tcs.2018.04.023.
[13] A.S. Fraenkel, Iterated floor function, algebraic numbers, discrete chaos, Beatty subsequences, semigroups, Trans. Am. Math. Soc. 341 (1994) 639–664, https://doi.org/10.1090/S0002-9947-1994-1138949-9.
[14] A.S. Fraenkel, C. Kimberling, Generalized Wythoff arrays, shuffles and interspersions, Discrete Math. 126 (1994) 137–149, https://doi.org/10.1016/0012-365X(94)90259-3.
[15] A. Frid, Applying a uniform marked morphism to a word, Discret. Math. Theor. Comput. Sci. 3 (1999) 125–140.
[16] J. Honkala, The equality problem for infinite words generated by primitive morphisms, Inf. Comput. 207 (2009) 900–907, https://doi.org/10.1016/j.ic.2009.01.002.
[17] M. Lothaire, Algebraic Combinatorics on Words, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, 2002.
[18] M. Queffélec, Substitution Dynamical Systems – Spectral Analysis, 2nd ed., Lecture Notes in Mathematics, vol. 1294, Springer, Berlin, 2010.
[19] P. Séébold, Sturmian images of non Sturmian words and standard morphisms, Theor. Comput. Sci. 711 (2018) 92–104, https://doi.org/10.1016/j.tcs.2017.11.011.