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Performance analysis of multirate systems: A direct frequency-domain identification approach

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ABSTRACT

Frequency-domain performance analysis of intersample behavior in sampled-data and multirate systems is challenging due to the lack of a frequency-separation principle, and systematic identification techniques are lacking. The aim of this paper is to develop an efficient technique for identifying the full intersample performance in the frequency-domain for closed-loop multirate systems, in particular the Performance Frequency Gain (PFG). Through local modeling techniques, aliased frequency components are effectively disentangled when identifying the PFG, which is directly facilitated by frequency-lifting the multirate system to a multivariable time-invariant representation. The developed method accurately and directly identifies the PFG in a single identification experiment. Finally, the developed method is experimentally validated on a prototype motion system, showing accurate identification of frequency-domain representations for the multirate system, including the PFG.

1. Introduction

The performance of sampled-data systems is naturally defined in the continuous-time, i.e., the intersample performance. Unlike on-sample performance, which only considers performance at specific sampling instances, intersample performance additionally evaluates the system's behavior in between the sampling instances. Sampled-data systems include essentially all physical systems which are controlled by digital controllers [1], for example networked control systems [2] and precision mechatronics [3]. The on-sample performance may vary significantly from the intersample performance, depending on the sampling time of the digital controller, the dynamics of the system, and the disturbances present. With this in mind, high-performance digital control designs should consider the intersample behavior of the system.

The intersample behavior of digital control systems can be considered through the use of sampled-data control techniques. First, the controller can be designed in continuous-time and subsequently discretized [4, Chapter 8], where the digital control implementation is not considered during the design. Second, the system itself can be discretized in combination with a discrete-time control design, which ignores the intersample response of the system. Third, direct sampled-data control design [1,5] overcomes these disadvantages by simultaneously considering the intersample response and the digital control implementation. Finally, multirate control design samples the system at an increased sampling rate relative to the controller's, which in combination with a discrete-time control design addresses the intersample response to a certain degree [1,6,7]. However, both the first and second

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option do not consider the full intersample behavior, and the direct control approaches require an accurate continuous-time or fast-rate model, which is generally unknown or non-trivial to determine.

Effective frequency-domain modeling of sampled-data and multirate systems to capture the full intersample behavior is challenging, as their linear periodically time-varying nature [1] prevents direct application of Linear Time-Invariant (LTI) methods due to signal aliasing. Therefore, alternative frequency-domain representations for sampled-data systems are developed in [8,9], and include the Performance Frequency Gain (PFG) [10]. In addition, the PFG allows for an equivalent multirate definition [3]. The PFG captures the full intersample behavior of sampled-data and multirate systems, and can be effectively used for control design [3]. However, there is currently no efficient method to determine the PFG, as it either requires a continuous-time or fast-rate model of the system, which is generally unknown or difficult to identify, or an identification experiment for each input frequency, since aliasing is not accounted for.

Alternatively, the PFG can be identified indirectly through first identifying the underlying fast-rate system, followed by evaluating the PFG as described in [3]. The identification of the underlying fast-rate system can be done using either fast-rate outputs [11] or downsampled outputs [12]. On the other hand, indirectly identifying the PFG using this two-step approach can lead to inaccurate results, and requires internal feedback signals that might be unavailable.

Although model-based sampled-data and multirate control design methods are broadly present, no effective and systematic frequency-domain identification techniques for these models are currently present. The aim of this paper is to develop a fast, accurate, and inexpensive frequency-domain identification technique for closed-loop multirate systems that model the full intersample performance. The key idea in this paper is to disentangle aliased frequency components for closed-loop multirate systems through local modeling techniques [13,14]. Frequency lifting the multirate system to a multivariable time-invariant representation [15,16] directly facilitates the application of local modeling techniques, which are originally developed for LTI systems. Furthermore, the multivariable time-invariant representations are used to directly compute the PFG, which can readily be used for intersample performance evaluation in multirate control design, such as those in [6,7]. The contributions include the following.

- (C1) The representation of the closed-loop PFG through the use of frequency-lifted time-invariant representations of multirate systems.
- (C2) Effective single-experiment frequency-domain identification of these time-invariant representations through local modeling techniques, enabling direct evaluation of the PFG through C1.
- (C3) Validation of the developed framework on an experimental setup.

This work extends [11] by being directly capable of computing the PFG for the multirate system, and in addition is suitable for systems with lightly-damped resonant dynamics.

Notation: Signals sampled at a fast sampling rate are denoted by subscript h and signals sampled at a slow sampling rate by subscript l . The N -points and M -points Discrete Fourier Transform (DFT) for finite-time fast-rate and slow-rate signals are respectively given by

$$X_h(e^{j\omega_k T_h}) = \sum_{n=0}^{N-1} x_h(n) e^{-j \frac{2\pi n k}{N}}, \quad X_l(e^{j\omega_k T_l}) = \sum_{m=0}^{M-1} x_l(m) e^{-j \frac{2\pi m k}{M}} \quad (1)$$

with sampling times T_h and T_l , discrete-time indices for fast-rate signals $n \in \mathbb{Z}_{[0, N-1]}$ and slow-rate signals $m \in \mathbb{Z}_{[0, M-1]}$ with integers \mathbb{Z} and N, M the amount of data points of the fast-rate and slow-rate signals and frequency bin $k \in \mathbb{Z}_{[0, N-1]}$, which relates to the frequency grid

$$\omega_k = \frac{2\pi k}{NT_h} = \frac{2\pi k}{MT_l} \in [0, \omega_{s,h}), \quad (2)$$

with fast-rate sampling frequency $\omega_{s,h} = \frac{2\pi}{T_h}$ in rad/s. The sampling times of the slow-rate and fast-rate signals relate as $T_l = FT_h$, with downsampling factor $F \in \mathbb{Z}_{>0}$ resulting in the slow-rate sampling frequency $\omega_{s,l} = \frac{2\pi}{T_l} = \omega_{s,h}/F$. Hence, the signal lengths relate as $N = FM$.

2. Problem definition

In this section, the problem is defined. The control setting is introduced, with its corresponding frequency-domain input–output analysis. Finally, the problem considered in this paper is defined.

2.1. Control setting

The control setting in Fig. 1 is considered, where a fast-rate LTI system G is under control with a slow-rate controller K_d . The system G is sampled at the fast sampling rate $\omega_{s,h}$, whereas the controller K_d is sampled at a reduced sampling rate $\omega_{s,l} = \omega_{s,h}/F$. The exogenous signals w_h contain any external signals, e.g., references or noise sources. The performance variable z_h is for example the tracking error of the system. Note that while the excitation signal w_h and the performance variable z_h are assumed to be known, they may contain noise. The LTI system G is described by

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \quad (3)$$

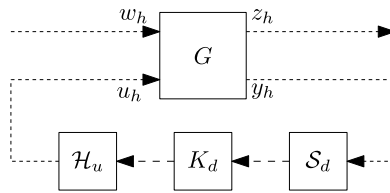


Fig. 1. Multirate system, with fast-rate system G and slow-rate controller K_d , which utilizes upsampler \mathcal{H}_u and downsampler \mathcal{S}_d .

For ease of notation, the elements G_{11} , G_{12} , G_{21} , and G_{22} are assumed single-input single-output throughout the paper. The notation can be straightforwardly extended to multivariable systems. The interpolator \mathcal{H}_u consists of a zero-order hold filter and an upsampler $\mathcal{H}_u = \mathcal{I}_{ZOH}(q)S_u$, with upsampler [17]

$$v_h(n) = S_u v_l(m) = \begin{cases} v_l\left(\frac{n}{F}\right) & \text{for } \frac{n}{F} \in \mathbb{Z}, \\ 0 & \text{for } \frac{n}{F} \notin \mathbb{Z}. \end{cases} \quad (4)$$

The zero-order hold filter is defined by

$$\mathcal{I}_{ZOH}(q) = \sum_{f=0}^{F-1} q^{-f}, \quad (5)$$

with fast-rate shift operator $qv_h(n) = v_h(n+1)$. The downsampler \mathcal{S}_d is described by [17]

$$v_l(m) = \mathcal{S}_d v_h(n) = v_h(Fm). \quad (6)$$

2.2. Frequency-domain analysis of multirate system

In this section, the frequency-domain behavior of multirate systems is described, and it is shown that the frequency-separation principle does not hold. First, after absorbing the feedback controller $\mathcal{H}_u K_d \mathcal{S}_d$ into the system G , the fast-rate input–output behavior of the closed-loop multirate system is described as

$$z_h = \left(G_{11} + G_{12} \mathcal{H}_u (I - K_d \mathcal{S}_d G_{22} \mathcal{H}_u)^{-1} K_d \mathcal{S}_d G_{21} \right) w_h. \quad (7)$$

Taking the DFT (1) on both sides of (7), the output is described in the frequency-domain by Oomen et al. [3]

$$Z_h(e^{j\omega_k T_h}) = G_{11}(e^{j\omega_k T_h}) W_h(e^{j\omega_k T_h}) + G_{12}(e^{j\omega_k T_h}) \mathcal{I}_{ZOH}(e^{j\omega_k T_h}) Q_d(e^{j\omega_k T_h}) \frac{1}{F} \sum_{f=0}^{F-1} G_{21}(e^{j(\omega_k + (f/F)\omega_{s,h}) T_h}) W_h(e^{j(\omega_k + (f/F)\omega_{s,h}) T_h}), \quad (8)$$

where

$$Q_d(e^{j\omega_k T_h}) = (I - K_d(e^{j\omega_k T_h}) G_{22,l}(e^{j\omega_k T_h}))^{-1} K_d(e^{j\omega_k T_h}). \quad (9)$$

The slow-rate system $G_{22,l} = \mathcal{S}_d G_{22} \mathcal{H}_u$ in (9) is described in the frequency domain as Vaidyanathan [17]

$$G_{22,l}(e^{j\omega_k T_h}) = \frac{1}{F} \sum_{f=0}^{F-1} \left(G_{22}(e^{j(\omega_k + (f/F)\omega_{s,h}) T_h}) \cdot \mathcal{I}_{ZOH}(e^{j(\omega_k + (f/F)\omega_{s,h}) T_h}) \right). \quad (10)$$

A key observation is that the intersample behavior $Z_h(e^{j\omega_k T_h})$ is influenced by F frequencies of the input $W_h(e^{j(\omega_k + (f/F)\omega_{s,h}) T_h})$ due to aliasing. Conversely, each frequency of the input $W_h(e^{j\omega_k T_h})$ influences F frequencies of the output [18, Theorem 3]. This dependency makes it unclear how to analyze the frequency-domain intersample behavior for multirate systems. As a result, there is a need for a systematic approach for analyzing this behavior, particularly in a way that is useful for control design.

2.3. Problem definition

The problem considered in this paper is as follows. Given a fast-rate excitation signal w_h and performance variable z_h of the multirate system shown in Fig. 1, directly identify the relevant frequency-domain representations that include the full intersample performance, i.e., the PFG.

3. Method

In this section, a frequency-domain representation of the multirate system is introduced through time-invariant representations, constituting contribution C1. Furthermore, the time-invariant representation is identified in a single identification experiment through local modeling, leading to contribution C2. The developed approach is then summarized in a procedure.

3.1. Intersample performance analysis through the PFG

An effective frequency-domain approach for analyzing the intersample performance for the multirate system in Fig. 1 is the PFG, which analyzes the total power output for a single-frequency input. The PFG is given by

$$\mathcal{P}(e^{j\omega_d T_h}) = \frac{\|z_h\|_p}{\|w_h\|_p}, \quad (11)$$

where the signal space \mathcal{W} consists of single complex sinusoidal disturbances having frequency ω_d and amplitude c , i.e.,

$$\mathcal{W} = \left\{ w_h(n) | w_h(n) = c e^{j\omega_d n T_h}, 0 < \|c\|_2 < \infty \right\}, \quad (12)$$

The power $\|x_h\|_p$ in (11) is given by

$$\|x_h\|_p = \sqrt{\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \|x_h(n)\|_2^2}. \quad (13)$$

Alternatively, the PFG is calculated in the frequency-domain as presented in Lemma 1.

Lemma 1. The PFG (11) is equivalent to

$$\mathcal{P}(e^{j\omega_d T_h}) = \frac{\|Z_h\|_p}{\|W_h\|_p}, \quad (14)$$

with

$$\|X_h\|_p = \sqrt{\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \|X_h(e^{j\omega_k T_h})\|_2^2}. \quad (15)$$

Proof. From Parseval's theorem it is known that

$$\sum_{n=0}^{N-1} \|x(n)\|_2^2 = \frac{1}{N} \sum_{k=0}^{N-1} \|X(e^{j\omega_k T_h})\|_2^2, \quad (16)$$

which, when substituted into (13), and subsequently in (11) for w_h and z_h directly leads to (14). \square

Note that an input signal $w_h \in \mathcal{W}$ (12) consists of a single sinusoid, and hence results in a DFT (1) magnitude of

$$\|W_h(e^{j\omega_k T_h})\|_2 = \begin{cases} cN & \text{for } \omega_k = \omega_d, \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

Therefore, the power of the input signal is determined by substituting (17) in (15), resulting in

$$\|W_h\|_p = c\sqrt{N}. \quad (18)$$

The PFG (11) represents the full intersample behavior of the multirate system in Fig. 1, since it takes into account all output frequencies, including aliased ones, for a single input frequency. Due to the aliasing of signals in (8), identifying the PFG is time-consuming since the excitation signal w_h is limited to a single frequency.

3.2. Direct PFG identification through frequency-lifting

The multirate PFG is directly identified through frequency-lifting the multirate system to a multivariable time-invariant representation. The frequency-lifted signal $\tilde{X}(e^{j\omega_k T_h}) = \mathcal{L}_f X(e^{j\omega_k T_h})$ is defined as

$$\tilde{X}(e^{j\omega_k T_h}) = \begin{bmatrix} X(e^{j\omega_k T_h}) \\ X(e^{j\omega_k T_h} \phi) \\ \vdots \\ X(e^{j\omega_k T_h} \phi^{F-1}) \end{bmatrix} \in \mathbb{C}^F, \quad (19)$$

where $\phi = e^{j2\pi/F}$ corresponds to a frequency shift of $\omega_{s,h}/F$ rad/s. Note that therefore the i th entry of \tilde{X} is essentially the original signal shifted in frequency by $i\omega_{s,h}/F$. By frequency-lifting the exogenous inputs $\tilde{w} = \mathcal{L}_f w_h$ and performance outputs $\tilde{z} = \mathcal{L}_f z_h$, the system becomes

$$\tilde{G} = \left[\begin{array}{c|c} \mathcal{L}_f G_{11} \mathcal{L}_f^{-1} & \mathcal{L}_f G_{12} \\ \hline G_{21} \mathcal{L}_f^{-1} & G_{22} \end{array} \right]. \quad (20)$$

The frequency-lifted system is shown in Fig. 2. By absorbing the feedback controller $\mathcal{H}_u K_d S_d$ into the interconnection, the closed-loop transfer describing the frequency-lifted input–output behavior is

$$\tilde{M} = \mathcal{L}_f G_{11} \mathcal{L}_f^{-1} + \mathcal{L}_f G_{12} \mathcal{H}_u K_d S_d (I - G_{22} \mathcal{H}_u K_d S_d)^{-1} G_{21} \mathcal{L}_f^{-1}, \quad (21)$$

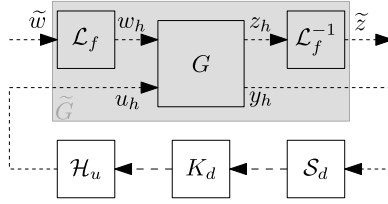


Fig. 2. Frequency-lifted multirate system, where frequency-lifting operators \mathcal{L}_f and \mathcal{L}_f^{-1} transform the multirate system to a time-invariant multivariable system.

which is LTI and multivariable $\widetilde{M}(e^{j\omega_k T_h}) \in \mathbb{C}^{F \times F}$ [16]. Therefore, the DFT of the input–output behavior is described by

$$\widetilde{Z}(e^{j\omega_k T_h}) = \widetilde{M}(e^{j\omega_k T_h}) \widetilde{W}(e^{j\omega_k T_h}), \quad (22)$$

where $\widetilde{M}(e^{j\omega_k T_h}) \in \mathbb{C}^{F \times F}$.

The frequency-lifted closed-loop \widetilde{M} is directly related to the PFG of the multirate system, which leads to the main result in this section in Theorem 1.

Theorem 1. The PFG (11) of closed-loop multirate system in Fig. 1 for frequencies $\omega_d \in [0, \omega_{s,h})$ is equivalent to

$$P(e^{j\omega_d T_h}) = \sqrt{\sum_{f=0}^{F-1} \left\| \widetilde{M}_{[f+1,1]}(e^{j\omega_d T_h}) \right\|_2^2}, \quad (23)$$

where $X_{[i,j]}(e^{j\omega_k T_h})$ denotes the (i, j) th element of matrix X .

Proof. First, $\|Z_h\|_P$ in the PFG (14) is computed by splitting the sum in (15) into F frequency bands, and represented through the use of the frequency-lifted outputs as

$$\sum_{k=0}^{N-1} \|Z_h(e^{j\omega_k T_h})\|_2^2 = \sum_{k=0}^{M-1} \sum_{f=0}^{F-1} \|Z_h(e^{j\omega_k T_h} \phi^f)\|_2^2 = \sum_{k=0}^{M-1} \sum_{f=0}^{F-1} \|\widetilde{Z}_{[f+1]}(e^{j\omega_k T_h})\|_2^2. \quad (24)$$

Second, under excitation signal $w_h \in \mathcal{W}$ with magnitude (17) and utilizing (22) the magnitude of these frequency-lifted outputs are

$$\|\widetilde{Z}_{[f+1]}(e^{j\omega_k T_h})\|_2 \Big|_{w_h \in \mathcal{W}} = \begin{cases} cN \left\| \widetilde{M}_{[f+1,1]}(e^{j\omega_k T_h}) \right\|_2, & \text{for } \omega_k = \omega_d. \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

By substitution of (24) and (25) in (15), $\|Z_h\|_P$ is formulated as

$$\|Z_h\|_P \Big|_{w \in \mathcal{W}} = c\sqrt{N} \sqrt{\sum_{f=0}^{F-1} \left\| \widetilde{M}_{[f+1,1]}(e^{j\omega_d T_h}) \right\|_2^2}. \quad (26)$$

Finally, substitution of (18) and (26) in (14) leads to the main result (23). \square

Remark 1. Note that while Theorem 1 utilizes the first column of $\widetilde{M}(e^{j\omega_k T_h})$, any f th column can be used as well by frequency shifting its result with ϕ^{-f} .

By direct identification of the PFG as shown in Theorem 1, the frequency-lifted representation directly allows for intersample performance evaluation.

3.3. Multivariable identification through local modeling

In this section, the frequency-lifted system \widetilde{M} is effectively identified in a single identification experiment by disentangling aliased frequency components through multivariable local modeling techniques.

In a local frequency window $r \in \mathbb{Z}_{[-n_w, n_w]}$, the frequency-lifted output \widetilde{Z} in (22) is approximated as

$$\widehat{\widetilde{Z}}(e^{j\omega_{k+r} T_h}) = \widehat{\widetilde{M}}(e^{j\omega_{k+r} T_h}) \widehat{\widetilde{W}}(e^{j\omega_{k+r} T_h}) + \widehat{\widetilde{T}}(e^{j\omega_{k+r} T_h}) \quad (27)$$

where $\widehat{\widetilde{M}}(e^{j\omega_{k+r} T_h}) \in \mathbb{C}^{F \times F}$ approximates the closed-loop \widetilde{M} in (21), and transient term $\widehat{\widetilde{T}}(e^{j\omega_{k+r} T_h}) \in \mathbb{C}^F$, which is introduced due to finite-length signals, and includes leakage effects. The multivariable system $\widehat{\widetilde{M}}$ and transient $\widehat{\widetilde{T}}$ are modeled using the local models

$$\widehat{\widetilde{M}}(e^{j\omega_{k+r} T_h}) = D^{-1}(e^{j\omega_{k+r} T_h}) N(e^{j\omega_{k+r} T_h}), \quad \widehat{\widetilde{T}}(e^{j\omega_{k+r} T_h}) = D^{-1}(e^{j\omega_{k+r} T_h}) L(e^{j\omega_{k+r} T_h}), \quad (28)$$

where local lifted system numerator $N(e^{j\omega_{k+r}T_h}) \in \mathbb{C}^{F \times F}$, denominator $D(e^{j\omega_{k+r}T_h}) \in \mathbb{C}^{F \times F}$, and transient numerator $L(e^{j\omega_{k+r}T_h}) \in \mathbb{C}^F$ are given by

$$\begin{aligned} N(e^{j\omega_{k+r}T_h}) &= \widehat{M}(e^{j\omega_k T_h}) + \sum_{s=1}^{R_n} N_s(k)r^s, \\ L(e^{j\omega_{k+r}T_h}) &= \widehat{T}(e^{j\omega_k T_h}) + \sum_{s=1}^{R_l} L_s(k)r^s, \\ D(e^{j\omega_{k+r}T_h}) &= I + \sum_{s=1}^{R_d} D_s(k)r^s, \end{aligned} \quad (29)$$

with complex coefficients $N_s(k) \in \mathbb{C}^{F \times F}$, $L_s(k) \in \mathbb{C}^F$, and $D_s(k) \in \mathbb{C}^{F \times F}$. The decision parameters

$$\Theta(k) : \begin{cases} \widehat{M}(e^{j\omega_k T_h}) & \in \mathbb{C}^{F \times F}, \\ \widehat{T}(e^{j\omega_k T_h}) & \in \mathbb{C}^F, \\ N_s(k) & \in \mathbb{C}^{n_u F + n_y \times n_u F}, \\ L_s(k) & \in \mathbb{C}^{n_u F + n_y}, \\ D_s(k) & \in \mathbb{C}^{n_u F + n_y \times n_u F + n_y}, \end{cases} \quad (30)$$

are determined by minimizing the weighted difference between approximated outputs (27) and measured outputs $Z(e^{j\omega_{k+r}T_h})$, resulting in the linear least squares problem

$$\begin{aligned} \hat{\Theta}(k) &= \arg \min_{\Theta(k)} \sum_{r=-n_w}^{n_w} \left\| D(e^{j\omega_{k+r}T_h}) \left(\tilde{Z}(e^{j\omega_{k+r}T_h}) - \widehat{Z}(e^{j\omega_{k+r}T_h}) \right) \right\|_2^2 \\ &= \arg \min_{\Theta(k)} \sum_{r=-n_w}^{n_w} \left\| D(e^{j\omega_{k+r}T_h}) \tilde{Z}(e^{j\omega_{k+r}T_h}) - N(e^{j\omega_{k+r}T_h}) \widetilde{W}(e^{j\omega_{k+r}T_h}) - L(e^{j\omega_{k+r}T_h}) \right\|_2^2, \end{aligned} \quad (31)$$

which has a unique closed-form solution [19].

Remark 2. An unweighted version of (31) can also be minimized, either through direct non-linear optimization or by utilizing iterative reweighted methods like the Sanathanan–Koerner algorithm [20]. Such optimization techniques generally do not ensure convergence to a global minimizer. In addition, the weighted least-squares criterion (31) is particularly effective for practical applications [19,21].

Remark 3. Typically, the cost function (31) has a unique closed-form solution only if the excitation signal $\widetilde{W}(e^{j\omega_{k+r}T_h})$ is sufficiently ‘rough’ within the window $r \in \mathbb{Z}_{[-n_w, n_w]}$ [22]. For instance, orthogonal random-phase multisines [23] for \tilde{w} or random-phase multisines for w_h meet this criterion.

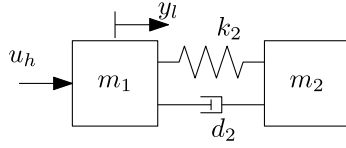
The frequency-lifted transfer function $\widehat{M}(e^{j\omega_k T_h})$ is now identified by evaluating the unique closed-form solution (31) for all frequency bins $k \in \mathbb{Z}_{[0, N-1]}$. The developed method, where the PFG is directly evaluated through frequency-lifted time-invariant representations of multirate systems (C1), which are identified with local modeling techniques (C2), is summarized in [Procedure 1](#).

Procedure 1 (Frequency-Domain Identification of PFG through Lifting and Local Modeling).

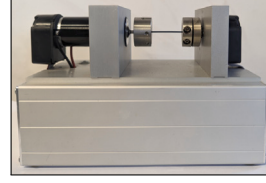
1. Construct excitation signal w_h , see [Remark 3](#).
 2. Excite multirate system in [Fig. 1](#) with w_h and record fast-rate performance variable z_h .
 3. Take DFT (1) of the exogenous signal w_h and performance variable z_h , resulting in W_h and Z_h .
 4. Lift fast-rate signals W_h and Z_h into $\widetilde{W} = \mathcal{L}_f W_h$ and $\tilde{Z} = \mathcal{L}_f Z_h$ using (19).
 5. For frequency bins $k \in \mathbb{Z}_{[0, N-1]}$ identify the PFG $\mathcal{P}(e^{j\omega_k T_h})$ as follows.
 - (a) Identify frequency-lifted closed-loop $\widehat{M}(e^{j\omega_k T_h})$ by minimizing the local modeling cost function (31), which has a unique global minimizer.
 - (b) Compute the PFG through [Theorem 1](#), specifically by using (23).
-

4. Experimental validation

In this section, the developed approach for direct frequency-domain identification of the intersample performance for multirate systems is experimentally validated. It is shown that the developed approach can directly identify the intersample performance in the frequency-domain through PFGs, while traditional representations or approaches cannot.



(a) Sketch of experimental setup.



(b) Photograph of experimental setup.

Fig. 3. Experimental setup.

4.1. Experimental setup

The developed approach is validated on the prototype motion system shown in Fig. 3, which consists of two rotating masses connected with a flexible shaft. The first mass is actuated by a DC motor, and its position is measured using an incremental encoder.

The multirate system is operating in closed-loop control as shown in Fig. 4, which shows the system is performing a constant velocity reference tracking task $r_h(n) = 20 \cdot 2\pi \cdot n$. In addition, the output of the system P is disturbed by the exogenous noise signal w_h , resulting in the system

$$G = \begin{bmatrix} -1 & -P \\ -1 & -P \end{bmatrix} : \begin{bmatrix} w_h \\ u_h \end{bmatrix} \mapsto \begin{bmatrix} z_h \\ y_h \end{bmatrix} \quad (32)$$

For validation purposes, an FRF of the system is made using single-rate feedback control with a sampling rate of $\frac{\omega_{s,h}}{2\pi} = f_h = 240$ Hz, where 54 000 samples of the excitation signal w_h , input u_h , and output y_h are used. The FRF is made in closed-loop with the indirect approach, in combination with the local rational modeling approach from [14], with rational degrees $R_d = R_n = R_m = 3$ and window size $n_w = 150$. The FRF is seen in Fig. 5(a). The feedback controller K_d sampled at $\frac{\omega_{s,L}}{2\pi} = f_l = 80$ Hz stabilizes the downsampled system $S_d P H_u$ with a bandwidth of 2 Hz. Additionally, it tries to suppress any disturbance effects due to the rotational movement introduced by the reference $r_h(n) = 20 \cdot 2\pi \cdot n$, for example a mass imbalance. For this purpose, the loop gain is increased at 20 Hz through an inverse Notch filter. An FRF of the controller is shown in Fig. 5(b).

For the experimental validation, the intersample performance of the tracking error is studied through identifying the PFG

$$\mathcal{P}(e^{j\omega_k T_h})_{w_h \in \mathcal{W}} = \frac{\|z_h\|_P}{\|w_h\|_P} = \frac{\|e_h\|_P}{\|w_h\|_P}. \quad (33)$$

In addition, the PFG is compared to the slow-rate sensitivity $S(e^{j\omega_k T_l}) : W_l(e^{j\omega_k T_l}) \mapsto Z_l(e^{j\omega_k T_l})$, which is given by

$$S(e^{j\omega_k T_l}) = (1 + K_d(e^{j\omega_k T_l}) P_l(e^{j\omega_k T_l}))^{-1}, \quad (34)$$

where P_l is calculated similarly to (10) using the true FRF shown in Fig. 5(a). Further experimental settings are seen in Table 1.

4.2. Intersample analysis of the experimental setup

In this section, it is shown that traditional slow-rate FRFs or the on-sample behavior cannot accurately represent the intersample behavior of an experimental multirate system, in contrast to the PFG (33).

First, it is shown that the slow-rate sensitivity does not represent the frequency-domain intersample performance by comparing it to the PFG. The PFG is computed analytically using Oomen et al. [3, Lemma 4] and a fast-rate validation FRF $P(e^{j\omega_k T_h})$ measured at $f_h = 240$ Hz. Specifically, the PFG is computed as

$$\mathcal{P}(e^{j\omega_k T_h}) = \sqrt{\sum_{f=0}^{F-1} |c_f(e^{j\omega_k T_h})|^2}, \quad (35)$$

where $c_f(e^{j\omega_k T_h}) \in \mathbb{C}$ is given by

$$c_f(e^{j\omega_k T_h}) = \begin{cases} G_{11}(e^{j\omega_k T_h}) + \frac{1}{F} G_{12}(e^{j\omega_k T_h}) \mathcal{I}_{ZOH}(e^{j\omega_k T_h}) \cdot Q_d(e^{j\omega_k T_l}) G_{21}(e^{j\omega_k T_h}), & f = 0, \\ \frac{1}{F} G_{12}(e^{j\omega_k T_h} \phi^f) \mathcal{I}_{ZOH}(e^{j\omega_k T_h} \phi^f) Q_d(e^{j\omega_k T_l}) G_{21}(e^{j\omega_k T_h}), & f \neq 0, \end{cases} \quad (36)$$

with frequency shift $\phi = e^{j2\pi/F} = e^{j\omega_{s,h} T_h / F}$, and G_{ij} is defined in (32) for the experimental setup. Both the slow-rate sensitivity (34) and the PFG, which is calculated using (35) and the validation FRF in Fig. 5(a), are shown in Fig. 6.

Second, the difference in the slow-rate sensitivity and the PFG is illustrated through performing three time-domain validation experiments by applying single-sinusoidal disturbances having frequencies 20, 60, and 100 Hz to w_h , which show a different between on-sample and intersample performance. Note that for slow-rate sampling $w_l = S_d w_h$ these signals have the same frequency of 20 Hz. The Cumulative Power Spectrum (CPS) of the fast-rate and slow-rate performance variables are respectively defined as

$$\text{CPS}_h(e^{j\omega_k T_h}) = \sum_{i=0}^k Z'_h(e^{j\omega_i T_h}) f_{r,h}, \quad \text{CPS}_l(e^{j\omega_k T_l}) = \sum_{i=0}^k Z'_l(e^{j\omega_i T_l}) f_{r,l}, \quad (37)$$

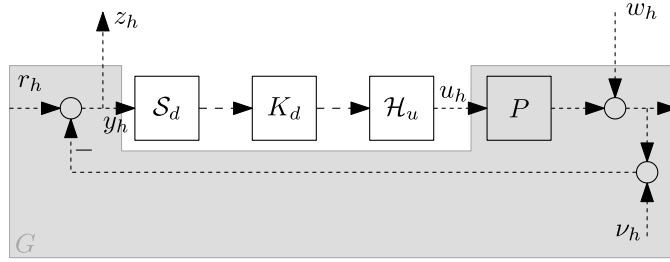
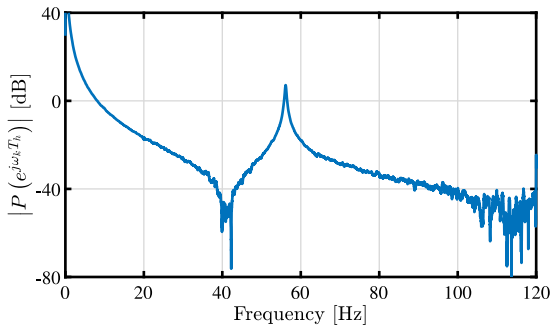
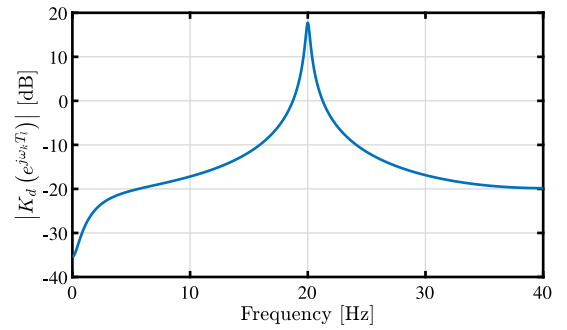


Fig. 4. Closed-loop control setup during experimental validation.



(a) Validation single-rate FRF measurement of the experimental setup $P(e^{j\omega_k T_h})$ measured at $f_h = 240$ Hz (\rightarrow).



(b) FRF of the feedback controller for the experimental setup $K_d(e^{j\omega_k T_l})$ with $f_l = 80$ Hz (\rightarrow).

Fig. 5. Frequency Response Functions (FRFs) of the system P and controller K_d .

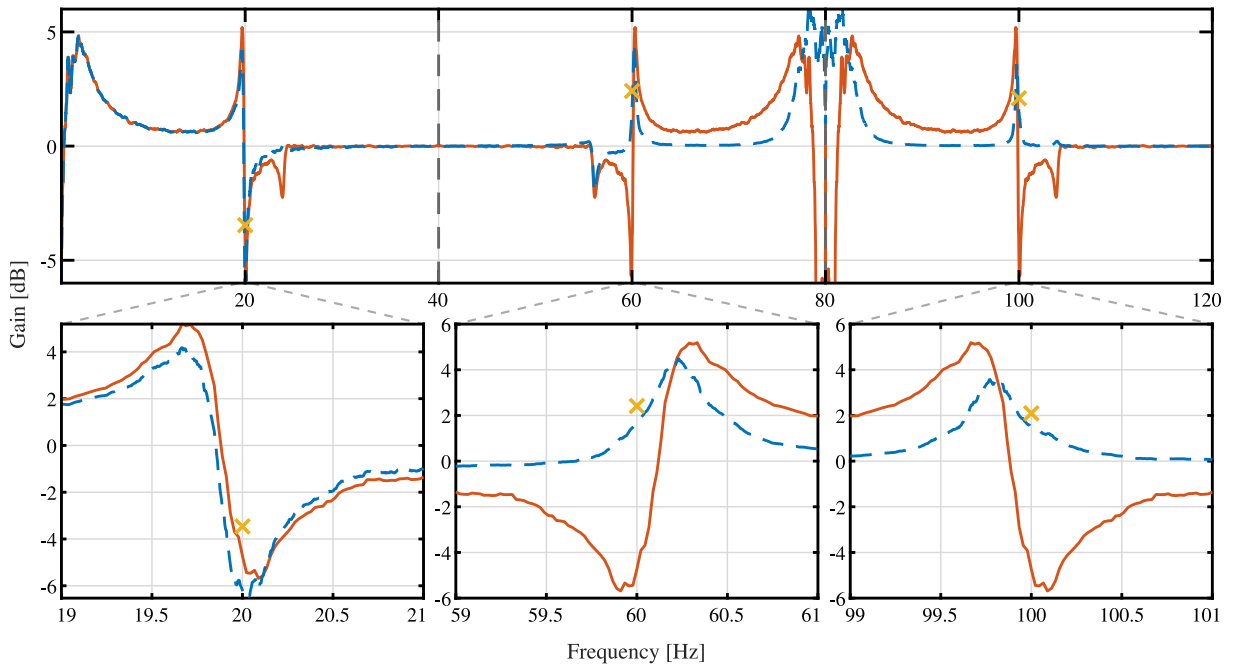


Fig. 6. The slow-rate sensitivity $S(e^{j\omega_k T_l})$ (34) (\rightarrow) cannot assess intersample performance, unlike the PFG $P(e^{j\omega_k T_h})$ derived with the validation FRF and (35) (\dashrightarrow). For instance, it deviates over a factor 2 (6 dB) at 60 and 100 Hz from the PFG. The PFG is validated for several single-sinusoidal measurements (\times).

Table 1

Experimental settings.

Variable	Abbreviation	Value	Unit
Fast sampling frequency	$f_h = \omega_{s,h} / (2\pi)$	240	Hz
Slow sampling frequency	$f_l = \omega_{s,l} / (2\pi)$	80	Hz
Downsampling factor	F	3	–
Number of input samples	N	10 800	–
Number of output samples	M	3600	–

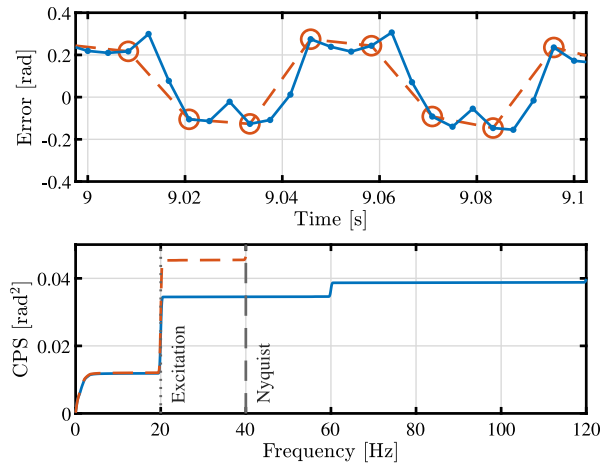
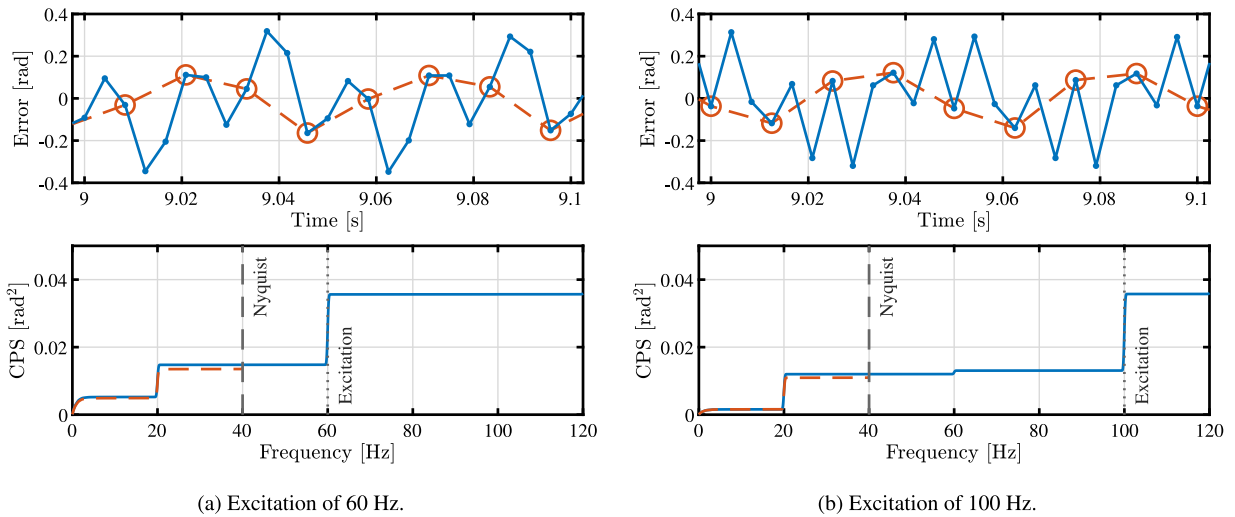


Fig. 7. For excitation of 20 Hz, the on-sample performance z_l (○) is relatively similar to the intersample performance z_h (●) (top). The on-sample CPS_{*l*} (○) and intersample CPS_{*h*} (●) show that they both have a similar dominant component at 20 Hz (bottom).



(a) Excitation of 60 Hz.

(b) Excitation of 100 Hz.

Fig. 8. When exciting the system beyond the Nyquist frequency, the intersample performance z_h (●) is significantly worse than the on-sample performance z_l (○) (top). The intersample CPS_{*h*} (●) is significantly higher than the on-sample CPS_{*l*} (○), since the dominant components beyond the Nyquist frequency are not observed by the on-sample CPS_{*l*} (bottom).

where the power spectral densities $Z'_h(e^{j\omega_i T_h})$ and $Z'_l(e^{j\omega_i T_l})$ can for example be determined using Welch's method [24] with frequency resolutions $f_{r,h}$ and $f_{r,l}$. The performance variables and their CPS determined with Welch's method for applying the three single-sinusoidal disturbances are shown in Figs. 7 and 8.

From the slow-rate sensitivity and the PFG in Fig. 6, in addition to the performance variables for single-sinusoidal disturbances in Figs. 7, and 8, it becomes clear that the slow-rate sensitivity and the on-sample behavior cannot accurately represent the intersample behavior of the multirate system, which is concluded from the following observations.

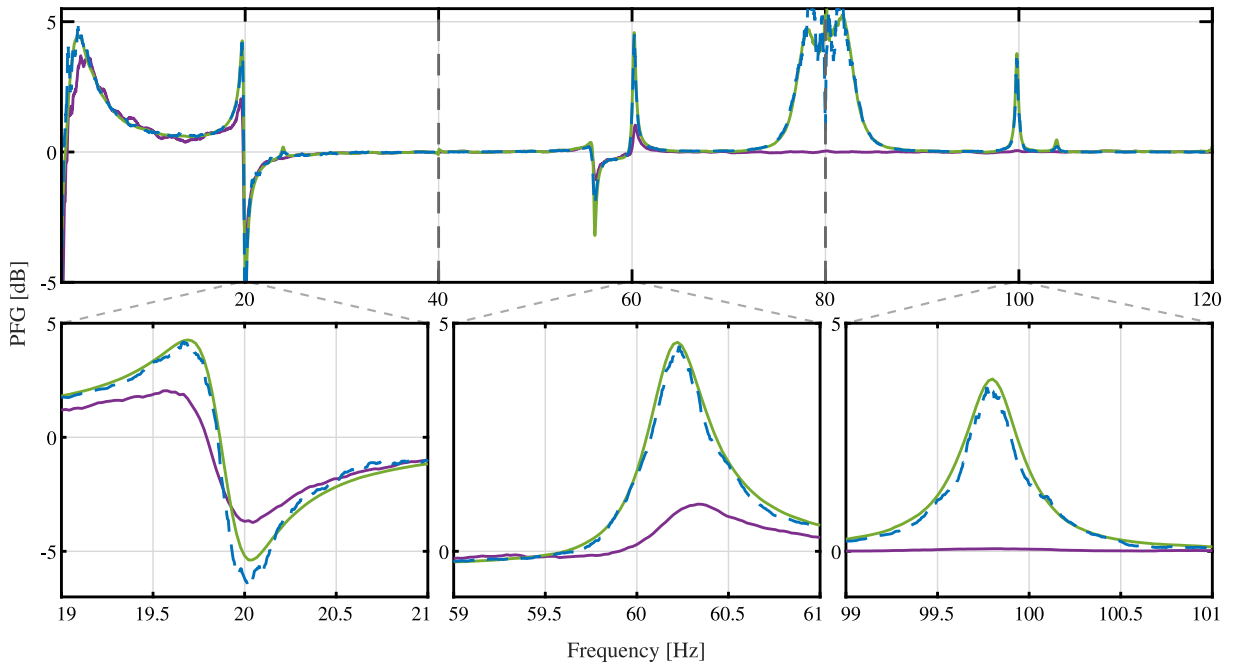


Fig. 9. The true PFG determined with the validation FRF and (35) (---) is accurately identified by the developed approach (—) using Theorem 1, in contrast to a direct approach that neglects the multirate behavior (—).

- While the PFG and the slow-rate sensitivity in Fig. 6 are relatively similar below 20 Hz, for the intersample performance beyond 20 Hz they are significantly different.
- While the on-sample performance in Figs. 7 and 8 is good, supported by the suppression of -3.17 dB of the slow-rate sensitivity in Fig. 6, the intersample performance in Fig. 8 deteriorates significantly for 60 and 100 Hz. Specifically, the root mean squared intersample performance deteriorates respectively by factors 1.63 and 1.79 compared to the on-sample performance.

4.3. Experimental identification of PFG

In this section, the PFG is directly and accurately identified in a single identification experiment for the experimental setup using the developed method. In addition, the developed method is compared to an approach that neglects the multirate behavior by assuming single-rate sampling. Therefore, the compared methods are the following.

- An approach that neglects the multirate behavior by identifying the closed-loop $w_h \mapsto z_h$ directly through local rational modeling.
- The developed approach that identifies the frequency-lifted system \tilde{M} and calculates the PFG using (23).

Both methods utilize rational degrees $R_d = R_n = R_m = 3$ and window size $n_w = 60$. Additional settings during experimentation are shown in Table 1. The methods are compared to the validation PFG computed in Section 4.2. The identified PFG by both methods is shown in Fig. 9. The PFG is accurately identified in a single identification experiment using the developed approach, while the approach that neglects the multirate behavior is not able to accurately identify the PFG as shown in Fig. 9.

In conclusion, the observations show that the PFG is essential for quantifying the intersample performance of a multirate or sampled-data system, which is accurately and directly identified by the developed approach. The approach that neglects the multirate behavior does not represent the full intersample behavior of the experimental system, because it does not accurately identify its PFG.

5. Conclusions

The results in this paper enable direct single-experiment identification of frequency-domain intersample performance in closed-loop multirate systems. The PFG, which is a frequency-domain representation for sampled-data or multirate systems, is directly evaluated using the frequency-lifted system, which is a time-invariant representation of the multirate system. The multiple aliased frequencies in the multivariable time-invariant representation are effectively disentangled through multivariable local modeling techniques. The time-invariant representation is directly identified in a single identification experiment. For an experimental prototype motion system it is shown that slow-rate or on-sample FRFs cannot be used to analyze the intersample performance of a

multirate system. In sharp contrast, the PFG represents the intersample performance of the experimental system in the frequency domain, which is accurately and directly identified in a single identification experiment by the developed approach. Therefore, the developed method is a key enabler for sampled-data and multirate control by directly identifying the performance of digital control systems in the frequency-domain, including the intersample performance.

CRedit authorship contribution statement

Max van Haren: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Lennart Blanken:** Writing – review & editing, Investigation. **Tom Oomen:** Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

References

- [1] T. Chen, B.A. Francis, *Optimal Sampled-Data Control Systems*, Springer, 1995, <http://dx.doi.org/10.1007/978-1-4471-3037-6>.
- [2] J.P. Hespanha, P. Naghshtabrizi, Y. Xu, A survey of recent results in networked control systems, *Proc. IEEE* 95 (2007) 138–172, <http://dx.doi.org/10.1109/JPROC.2006.887288>.
- [3] T. Oomen, M. van de Wal, O. Bosgra, Design framework for high-performance optimal sampled-data control with application to a wafer stage, *Internat. J. Control* 80 (6) (2007) 919–934, <http://dx.doi.org/10.1080/00207170701216329>.
- [4] K.J. Åström, B. Wittenmark, *Computer-Controlled Systems: Theory and Design*, Dover Publications, 2011.
- [5] B. Bamieh, J. Pearson, B.A. Francis, A. Tannenbaum, A lifting technique for linear periodic systems with applications to sampled-data control, *Systems Control Lett.* 17 (2) (1991) 79–88, [http://dx.doi.org/10.1016/0167-6911\(91\)90033-B](http://dx.doi.org/10.1016/0167-6911(91)90033-B).
- [6] J. Salt, P. Albertos, Model-based multirate controllers design, *IEEE Trans. Control Syst. Technol.* 13 (6) (2005) 988–997, <http://dx.doi.org/10.1109/TCST.2005.857410>.
- [7] M. Cimino, P.R. Pagilla, Design of linear time-invariant controllers for multirate systems, *Automatica* 46 (8) (2010) 1315–1319, <http://dx.doi.org/10.1016/j.automatica.2010.05.003>.
- [8] M. Araki, Y. Ito, T. Hagiwara, Frequency response of sampled-data systems, *Automatica* 32 (4) (1996) 483–497, [http://dx.doi.org/10.1016/0005-1098\(95\)00162-X](http://dx.doi.org/10.1016/0005-1098(95)00162-X).
- [9] Y. Yamamoto, P. Khargonekar, Frequency response of sampled-data systems, *IEEE Trans. Autom. Control* 41 (20) (1996) 166–176, <http://dx.doi.org/10.1109/9.481516>.
- [10] O. Lindgarde, B. Lennartson, Performance and robust frequency response for multirate sampled-data systems, in: *Proceedings of the 1997 American Control Conference*, 1997, pp. 3877–3881, <http://dx.doi.org/10.1109/ACC.1997.609611>.
- [11] M. van Haren, L. Blanken, T. Oomen, Frequency domain identification of multirate systems: A lifted local polynomial modeling approach, in: *IEEE 61st Conference on Decision and Control, CDC*, 2022, pp. 2795–2800, <http://dx.doi.org/10.1109/CDC51059.2022.9992919>.
- [12] M. van Haren, M. Mae, L. Blanken, T. Oomen, Lifted frequency-domain identification of closed-loop multirate systems: Applied to dual-stage actuator hard disk drives, *Mechatronics* 108 (2025) 103311, <http://dx.doi.org/10.1016/j.mechatronics.2025.103311>.
- [13] R. Pintelon, J. Schoukens, *System Identification: A Frequency Domain Approach*, John Wiley & Sons, 2012.
- [14] T. McKelvey, G. Guérin, Non-parametric frequency response estimation using a local rational model, in: *Proceedings of the 16th IFAC Symposium on System Identification*, 2012, pp. 49–54, <http://dx.doi.org/10.3182/20120711-3-BE-2027.00299>.
- [15] C. Zhang, J. Zhang, K. Furuta, Analysis of H_2 and H_∞ performance of discrete periodically time-varying controllers, *Automatica* 33 (4) (1997) 619–634, [http://dx.doi.org/10.1016/S0005-1098\(96\)00224-5](http://dx.doi.org/10.1016/S0005-1098(96)00224-5).
- [16] S. Bittanti, P. Colaneri, *Periodic Systems*, in: *Communications and Control Engineering*, Springer, London, 2009, <http://dx.doi.org/10.1007/978-1-84800-911-0>.
- [17] P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, 1993.
- [18] J. Salt, A. Sala, A new algorithm for dual-rate systems frequency response computation in discrete control systems, *Appl. Math. Model.* 38 (23) (2014) 5692–5704, <http://dx.doi.org/10.1016/j.apm.2014.04.054>.
- [19] R. Voorhoeve, A. van der Maas, T. Oomen, Non-parametric identification of multivariable systems: A local rational modeling approach with application to a vibration isolation benchmark, *Mech. Syst. Signal Process.* 105 (2018) 129–152, <http://dx.doi.org/10.1016/j.ymssp.2017.11.044>.
- [20] C. Sanathanan, J. Koerner, Transfer function synthesis as a ratio of two complex polynomials, *IEEE Trans. Autom. Control* 8 (1) (1963) 56–58, <http://dx.doi.org/10.1109/TAC.1963.1105517>.
- [21] D. Verbeke, J. Schoukens, Frequency response measurements with local parametric modeling, *IEEE Trans. Instrum. Meas.* 69 (6) (2020) 3249–3261, <http://dx.doi.org/10.1109/TIM.2019.2924570>.
- [22] J. Schoukens, G. Vandersteen, K. Barbé, R. Pintelon, Nonparametric preprocessing in system identification: a powerful tool, *Eur. J. Control* 15 (3) (2009) <http://dx.doi.org/10.3166/ejc.15.260-274>.
- [23] T.P. Dobrowiecki, J. Schoukens, P. Guillaume, Optimized excitation signals for MIMO frequency response function measurements, *IEEE Trans. Instrum. Meas.* 55 (6) (2006) 2072–2079, <http://dx.doi.org/10.1109/TIM.2006.887036>.
- [24] P. Welch, The use of fast Fourier transform for the estimation of power spectra: A method based on time averaging over short, modified periodograms, *IEEE Trans. Audio Electroacoust.* 15 (2) (1967) 70–73, <http://dx.doi.org/10.1109/TAU.1967.1161901>.