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FOURIER MULTIPLIERS AND WEAK DIFFERENTIAL SUBORDINATION OF MARTINGALES IN UMD BANACH SPACES

IVAN YAROSLAVTSEV

ABSTRACT. In this paper we introduce the notion of weak differential subordination for martingales and show that a Banach space X is a UMD Banach space if and only if for all $p \in (1, \infty)$ and all purely discontinuous X -valued martingales M and N such that N is weakly differentially subordinated to M , one has the estimate $\mathbb{E}\|N_\infty\|^p \leq C_p \mathbb{E}\|M_\infty\|^p$. As a corollary we derive the sharp estimate for the norms of a broad class of even Fourier multipliers, which includes e.g. the second order Riesz transforms.

1. INTRODUCTION

Applying stochastic techniques to Fourier multiplier theory has a long history (see e.g. [2, 3, 6, 9, 11, 20, 21, 30]). It turns out that the boundedness of certain Fourier multipliers with values in a Banach space X is equivalent to this Banach space being in a special class, namely in the class of UMD Banach spaces. Burkholder in [11] and Bourgain in [9] showed that the Hilbert transform is bounded on $L^p(\mathbb{R}; X)$ for $p \in (1, \infty)$ if and only if X is UMD. The same type of assertion can be proven for the Beurling-Ahlfors transform, see the paper [21] by Geiss, Montgomery-Smith and Saksman. Examples of UMD spaces include the reflexive range of L^q -, Sobolev and Besov spaces.

A more general class of Fourier multiplier has been considered in recent works of Bañuelos and Bogdan [3] and Bañuelos, Bielaszewski and Bogdan [2]. They derive sharp estimates for the norm of a Fourier multiplier with symbol

$$(1.1) \quad m(\xi) = \frac{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z) \phi(z) V(dz) + \frac{1}{2} \int_{S^{d-1}} (\xi \cdot \theta)^2 \psi(\theta) \mu(d\theta)}{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z) V(dz) + \frac{1}{2} \int_{S^{d-1}} (\xi \cdot \theta)^2 \mu(d\theta)}, \quad \xi \in \mathbb{R}^d,$$

on $L^p(\mathbb{R}^d)$. Here we will extend their result to $L^p(\mathbb{R}^d; X)$ for UMD spaces X . More precisely, we will show that a Fourier multiplier T_m with a symbol of the form (1.1) is bounded on $L^p(\mathbb{R}^d; X)$ if V is a Lévy measure, μ is a Borel positive measure, $|\phi|, |\psi| \leq 1$, and that then the norm of T_m does not exceed the UMD_p constant of X . In Subsection 4.2, several examples of symbols m of the form (1.1) are given, and we will see that for some particular symbols m the norm of T_m equals the UMD constant.

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To prove the generalization of the results in [2, 3] we will need additional geometric properties of a UMD Banach space. In the fundamental paper [14], Burkholder showed that a Banach space X is UMD if and only if for some $\beta > 0$ there exists a zigzag-concave function $U : X \times X \rightarrow \mathbb{R}$ (i.e., a function U such that $U(x+z, y+\varepsilon z)$ is concave in z for any sign ε and for any $x, y \in X$) such that $U(x, y) \geq \|y\|^p - \beta^p \|x\|^p$ for all $x, y \in X$. Such a function U is called a *Burkholder function*. In this situation, we can in fact take β equal to the UMD_p constant of X (see Subsection 2.1 and Theorem 3.8). By exploiting appropriate Burkholder functions U one can prove a wide variety of interesting results (see [4–7, 12, 13, 52] and the works [34–43] by Osękowski). For our purposes the following result due to Burkholder [12] (for the scalar case) and Wang [52] (for the Hilbert space case) is of special importance:

Theorem 1.1. *Let H be a Hilbert space, $(d_n)_{n \geq 0}, (e_n)_{n \geq 0}$ be two H -valued martingale difference sequences such that $\|e_n\| \leq \|d_n\|$ a.s. for all $n \geq 0$. Then for each $p \in (1, \infty)$,*

$$\mathbb{E} \left\| \sum_{n \geq 0} e_n \right\|^p \leq (p^* - 1)^p \mathbb{E} \left\| \sum_{n \geq 0} d_n \right\|^p.$$

Here and in the sequel $p^* = \max(p, p')$, where $\frac{1}{p} + \frac{1}{p'} = 1$. This result cannot be generalized beyond the Hilbertian setting; see [39, Theorem 3.24(i)] and [23, Example 4.5.17]. In the present paper we will show the following UMD variant of Theorem 1.1:

Theorem 1.2. *Let X be a UMD space, $(d_n)_{n \geq 0}, (e_n)_{n \geq 0}$ be two X -valued martingale difference sequences, $(a_n)_{n \geq 0}$ be a scalar-valued adapted sequence such that $|a_n| \leq 1$ and $e_n = a_n d_n$ for all $n \geq 0$. Then for each $p \in (1, \infty)$*

$$\mathbb{E} \left\| \sum_{n \geq 0} e_n \right\|^p \leq \beta_{p,X}^p \mathbb{E} \left\| \sum_{n \geq 0} d_n \right\|^p,$$

where $\beta_{p,X}$ is the UMD_p -constant of X (notice that Burkholder proved the identity $\beta_{p,H} = p^* - 1$ for a Hilbert space H , see [12]). Theorem 1.2 generalizes a famous Burkholder's result [10, Theorem 2.2] on martingale transforms, where $(a_n)_{n \geq 0}$ was supposed to be predictable. The main tool for proving Theorem 1.2 is a Burkholder function with a stricter zigzag-concavity: now we also require $U(x+z, y+\varepsilon z)$ to be concave in z for any ε such that $|\varepsilon| \leq 1$. In the finite dimensional case one gets it for free thanks to the existence of an explicit formula of U (see Remark 5.6 and [52]). Here we show the existence of such a Burkholder function in infinite dimension.

For the applications of our abstract results to the theory of Fourier multipliers we extend Theorem 1.2 to the continuous time setting. Namely, we show an analogue of Theorem 1.2 for purely discontinuous martingales (i.e. martingales which quadratic variations are pure jump processes, see Subsection 3.2).

An extension of Theorem 1.2 to general continuous-time martingales is shown in the paper [54]. Nevertheless, the sharp estimate in this extension for the case of continuous martingales remains an open problem. This problem is in fact of interest in Harmonic Analysis. If true, this sharp estimate can be used to study a larger class of multipliers, including the Hilbert transform \mathcal{H}_X . Garling in [20] proved that

$$\|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \leq \beta_{p,X}^2,$$

and it is a long-standing open problem (see [23, pp.496–497]) to prove a linear estimate of the form

$$\|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \leq C\beta_{p,X}$$

for some constant C . Here we will show that the latter estimate would indeed follow if one can show the existence of a Burkholder function with certain additional properties. At present, the existence of such Burkholder functions is known only in the Hilbert space case (see Remark 5.6).

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2. PRELIMINARIES

2.1. UMD Banach spaces. A Banach space X is called a *UMD space* if for some (or equivalently, for all) $p \in (1, \infty)$ there exists a constant $\beta > 0$ such that for every $n \geq 1$, every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega; X)$, and every scalar-valued sequence $(\varepsilon_j)_{j=1}^n$ such that $|\varepsilon_j| = 1$ for each $j = 1, \dots, n$ we have

$$\left(\mathbb{E}\left\|\sum_{j=1}^n \varepsilon_j d_j\right\|^p\right)^{\frac{1}{p}} \leq \beta \left(\mathbb{E}\left\|\sum_{j=1}^n d_j\right\|^p\right)^{\frac{1}{p}}.$$

The least admissible constant β is denoted by $\beta_{p,X}$ and is called the *UMD_p constant* or, if the value of p is understood, the *UMD constant*, of X . It is well-known that UMD spaces obtain a large number of good properties, such as being reflexive. Examples of UMD spaces include all finite dimensional spaces and the reflexive range of L^q -spaces, Besov spaces, Sobolev spaces and Schatten class spaces. Example of spaces without the UMD property include all nonreflexive Banach spaces, e.g. $L^1(0, 1)$, $L^\infty(0, 1)$ and $C([0, 1])$. We refer the reader to [15, 23, 44, 48] for details.

2.2. Martingales. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ which satisfies the usual conditions (see [28, Definition 1.2.25] and [46]). Then \mathbb{F} is right-continuous and the following proposition holds:

Proposition 2.1. *Let X be a Banach space. Then any martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ admits a càdlàg version, namely there exists a version of M which is right-continuous and has left limits.*

Let $t > 0$. For a Banach space X we define the *Skorohod space* $D([0, t]; X)$ of all right-continuous functions $f : \mathbb{R}_+ \rightarrow X$ with left limits. The following lemma follows from [45, Problem V.6.1] (see also [50]).

Lemma 2.2. *Let X be a Banach space, $t > 0$. Then $(D([0, t]; X), \|\cdot\|_\infty)$ is a Banach space.*

Proof of Proposition 2.1. One can find the proof in [51, Proposition 2.2.2], but we will repeat it here for the convenience of the reader. Without loss of generality suppose that $M_\infty := \lim_{t \rightarrow \infty} M_t$ exists a.s. and in $L^1(\Omega; X)$. Also we can assume that there exists $t > 0$ such that $M_t = M_\infty$. Let $(\xi^n)_{n \geq 1}$ be a sequence of simple functions in $L^1(\Omega; X)$ such that $\xi^n \rightarrow M_t$ in $L^1(\Omega; X)$ as $n \rightarrow \infty$. For each $n \geq 1$ define a martingale $M^n : \mathbb{R}_+ \times \Omega \rightarrow X$ such that $M_s^n = \mathbb{E}(\xi^n | \mathcal{F}_s)$ for each $s \geq 0$. Fix $n \geq 1$. Since ξ^n takes its values in a finite dimensional subspace of X , M^n takes

its values in the same finite dimensional subspace as well, and therefore by [17] (or [46, p.8]) it has a càdlàg version. But $M_t^n = \xi^n \rightarrow M_t$ in $L^1(\Omega; X)$ as $n \rightarrow \infty$, so by the Doob maximal inequality [28, Theorem 1.3.8(i)], $M^n \rightarrow M$ in the ucp topology (the topology of the uniform convergence on compacts in probability). By taking an appropriate subsequence we can assume that $M^n \rightarrow M$ a.s. uniformly on $[0, t]$, and consequently, uniformly on \mathbb{R}_+ . Therefore, by Lemma 2.2 M has a càdlàg version. \square

Thanks to Proposition 2.1 we can define ΔM_t and M_{t-} for each $t \geq 0$,

$$\begin{aligned}\Delta M_t &:= M_t - \lim_{\varepsilon \rightarrow 0} M_{(t-\varepsilon) \vee 0}, \\ M_{t-} &:= \lim_{\varepsilon \rightarrow 0} M_{t-\varepsilon}, \quad M_{0-} := M_0.\end{aligned}$$

2.3. Quadratic variation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ that satisfies the usual conditions, H be a Hilbert space. Let $M : \mathbb{R}_+ \times \Omega \rightarrow H$ be a local martingale. We define a *quadratic variation* of M in the following way:

$$(2.1) \quad [M]_t := \mathbb{P} - \lim_{\text{mesh} \rightarrow 0} \sum_{n=1}^N \|M(t_n) - M(t_{n-1})\|^2,$$

where the limit in probability is taken over partitions $0 = t_0 < \dots < t_N = t$. The reader can find more about a quadratic variation in [27, 32, 46].

2.4. Stochastic integration. Let X be a Banach space, H be a Hilbert space. For each $h \in H$, $x \in X$ we denote a linear operator $g \mapsto \langle g, h \rangle x$, $g \in H$, by $h \otimes x$. The process $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ is called *elementary progressive* with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ if it is of the form

$$(2.2) \quad \Phi(t, \omega) = \sum_{k=1}^K \sum_{m=1}^M \mathbf{1}_{(t_{k-1}, t_k] \times B_{m,k}}(t, \omega) \sum_{n=1}^N h_n \otimes x_{kmn}, \quad t \geq 0, \omega \in \Omega,$$

where $0 \leq t_0 < \dots < t_K < \infty$, for each $k = 1, \dots, K$ the sets B_{1k}, \dots, B_{Mk} are in $\mathcal{F}_{t_{k-1}}$ and vectors h_1, \dots, h_N are orthogonal.

Let $M : \mathbb{R}_+ \times \Omega \rightarrow H$ be a martingale. Then we define a *stochastic integral* $\Phi \cdot M : \mathbb{R}_+ \times \Omega \rightarrow X$ of Φ with respect to M in the following way:

$$(2.3) \quad (\Phi \cdot M)_t = \sum_{k=1}^K \sum_{m=1}^M \mathbf{1}_{B_{m,k}} \sum_{n=1}^N \langle (M(t_k \wedge t) - M(t_{k-1} \wedge t)), h_n \rangle x_{kmn}, \quad t \geq 0.$$

The reader can find more on stochastic integration in a finite dimensional case in [27]. The following lemma is a multidimensional version of [27, Theorem 26.6(v)].

Lemma 2.3. *Let d be a natural number, H be a d -dimensional Hilbert space, $M : \mathbb{R}_+ \times \Omega \rightarrow H$ be a martingale, $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, \mathbb{R})$ be elementary progressive. Then $[\Phi \cdot M] \lesssim_d \|\Phi\|^2 \cdot [M]$ a.s.*

Proof. Let $(h_n)_{n=1}^d$ be an orthogonal basis of H , $\Phi_1, \dots, \Phi_d : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be such that $\Phi = \sum_{n=1}^d \Phi_n h_n$, and $M_1, \dots, M_d : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be martingales such that $M = \sum_{n=1}^d M_n h_n$. Notice that thanks to the definition of a quadratic variation (2.1) one

has that $[M] = [M_1] + \cdots + [M_d]$. Then since a quadratic variation is a positive-definite quadratic form (see [27, Theorem 26.6]), thanks to [27, Theorem 26.6(v)] one has for each $t \geq 0$ a.s.,

$$\begin{aligned} [\Phi \cdot M]_t &= [\Phi_1 \cdot M_1 + \cdots + \Phi_d \cdot M_d]_t \lesssim_d [\Phi_1 \cdot M_1]_t + \cdots + [\Phi_d \cdot M_d]_t \\ &= (\|\Phi_1\|^2 \cdot [M_1])_t + \cdots + (\|\Phi_d\|^2 \cdot [M_d])_t \\ &\lesssim_d (\|\Phi\|^2 \cdot [M])_t. \end{aligned}$$

□

Using Lemma 2.3 one can extend stochastic integral to the case of general Φ . In particular, the following lemma on stochastic integration can be shown.

Lemma 2.4. *Let d be a natural number, H be a d -dimensional Hilbert space, $p \in (1, \infty)$, $M, N : \mathbb{R}_+ \times \Omega \rightarrow H$ be L^p -martingales, $F : H \rightarrow H$ be a measurable function such that $\|F(h)\| \leq C\|h\|^{p-1}$ for each $h \in H$ and some $C > 0$. Let $N_- : \mathbb{R}_+ \times \Omega \rightarrow H$ be such that $(N_-)_t = N_{t-}$ for each $t \geq 0$. Then $F(N_-) \cdot M$ is a martingale and for each $t \geq 0$,*

$$(2.4) \quad \mathbb{E}|(F(N_-) \cdot M)_t| \lesssim_{p,d} C(\mathbb{E}\|N_t\|^p)^{\frac{p-1}{p}} (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}.$$

Proof. First notice that $F(N_-)$ is predictable. Therefore, thanks to Lemma 2.3 and [27, Theorem 26.12], in order to prove that $F(N_-)$ is stochastically integrable with respect to M and that $F(N_-) \cdot M$ is a martingale it is sufficient to show that $\mathbb{E}(\|F(N_-)\|^2 \cdot [M])_t^{\frac{1}{2}} < \infty$. Without loss of generality suppose that $M_0 = N_0 = 0$ a.s. and $C = 1$. Then

$$\begin{aligned} \mathbb{E}(\|F(N_-)\|^2 \cdot [M])_t^{\frac{1}{2}} &\leq \mathbb{E}(\|N_{t-}\|^{2(p-1)} \cdot [M]_t)^{\frac{1}{2}} \leq \mathbb{E}\left(\sup_{0 \leq s \leq t} \|N_s\|^{p-1} [M]_t^{\frac{1}{2}}\right) \\ (2.5) \quad &\stackrel{(i)}{\leq} (\mathbb{E} \sup_{0 \leq s \leq t} \|N_s\|^p)^{\frac{p-1}{p}} (\mathbb{E}[M]_t^{\frac{p}{2}})^{\frac{1}{p}} \\ &\stackrel{(ii)}{\lesssim_p} (\mathbb{E}\|N_t\|^p)^{\frac{p-1}{p}} (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}} < \infty, \end{aligned}$$

where (i) follows from the Hölder inequality, and (ii) holds thanks to [27, Theorem 26.12] and [28, Theorem 1.3.8(iv)].

Now let us show (2.4):

$$\begin{aligned} \mathbb{E}|(F(N_-) \cdot M)_t| &\stackrel{(i)}{\lesssim_p} \mathbb{E}|F(N_-) \cdot M|_t^{\frac{1}{2}} \stackrel{(ii)}{\lesssim_d} \mathbb{E}(\|F(N_-)\|^2 \cdot [M])_t^{\frac{1}{2}} \\ &\stackrel{(iii)}{\lesssim_p} (\mathbb{E}\|N_t\|^p)^{\frac{p-1}{p}} (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \end{aligned}$$

Here (i) follows from [27, Theorem 26.12], (ii) holds thanks to Lemma 2.3, and (iii) follows from (2.5). □

3. UMD BANACH SPACES AND WEAK DIFFERENTIAL SUBORDINATION

From now on the scalar field \mathbb{K} can be either \mathbb{R} or \mathbb{C} .

3.1. Discrete case. In this section we assume that X is a Banach space over the scalar field \mathbb{K} and with a separable dual X^* . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\mathbb{F} := (\mathcal{F}_n)_{n \geq 0}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Definition 3.1. Let $(f_n)_{n \geq 0}$, $(g_n)_{n \geq 0}$ be X -valued local martingales. For each $n \geq 1$ we define $df_n := f_n - f_{n-1}$, $dg_n := g_n - g_{n-1}$.

- (i) g is differentially subordinated to f if one has that $\|dg_n\| \leq \|df_n\|$ a.s. for all $n \geq 1$ and $\|g_0\| \leq \|f_0\|$ a.s.
- (ii) g is weakly differentially subordinated to f if for each $x^* \in X^*$ one has that $|\langle dg_n, x^* \rangle| \leq |\langle df_n, x^* \rangle|$ a.s. for all $n \geq 1$ and $|\langle g_0, x^* \rangle| \leq |\langle f_0, x^* \rangle|$ a.s.

The following characterization of Hilbert spaces can be found in [39, Theorem 3.24(i)]:

Theorem 3.2. A Banach space X is isomorphic to a Hilbert space if and only if for some (equivalently, for all) $1 < p < \infty$ there exists a constant $\alpha_{p,X} > 0$ such that for any pair of X -valued local martingales $(f_n)_{n \geq 0}$, $(g_n)_{n \geq 0}$ such that g is differentially subordinated to f one has that

$$(3.1) \quad \mathbb{E}\|g_n\|^p \leq \alpha_{p,X}^p \mathbb{E}\|f_n\|^p$$

for each $n \geq 1$.

By the Pettis measurability theorem [23, Theorem 1.1.20], we may assume that X is separable. Then weak differential subordination implies differential subordination. Indeed, let $(x_k)_{k \geq 1}$ be a dense subset of X , $(x_k^*)_{k \geq 1}$ be a sequence of linear functionals on X such that $\langle x_k, x_k^* \rangle = \|x_k\|$ and $\|x_k^*\| = 1$ for each $k \geq 1$ (such a sequence exists by the Hahn-Banach theorem). Let $(g_n)_{n \geq 0}$ be weakly differentially subordinated to $(f_n)_{n \geq 0}$. Then for each $n \geq 1$ a.s.

$$\|dg_n\| = \sup_{k \geq 1} |\langle dg_n, x_k^* \rangle| \leq \sup_{k \geq 1} |\langle df_n, x_k^* \rangle| = \|df_n\|.$$

By the same reasoning $\|g_0\| \leq \|f_0\|$ a.s. This means that the weak differential subordination property is more restrictive than the differential subordination property. Therefore, under the weak differential subordination, one could expect that the assertions of the type (3.1) characterize a broader class of Banach spaces X . Actually we will prove the following theorem, which extends [13, Theorem 2] to the UMD case.

Theorem 3.3. A Banach space X is a UMD space if and only if for some (equivalently, for all) $1 < p < \infty$ there exists a constant $\beta > 0$ such that for all X -valued local martingales $(f_n)_{n \geq 0}$ and $(g_n)_{n \geq 0}$ such that g is weakly differentially subordinated to f one has

$$(3.2) \quad \mathbb{E}\|g_n\|^p \leq \beta^p \mathbb{E}\|f_n\|^p, \quad n \geq 1.$$

If this is the case then the smallest admissible β is the UMD constant $\beta_{p,X}$.

Theorem 1.2 is contained in this result as a special case.

The proof of Theorem 3.3 consists of several steps.

Proposition 3.4. Let X be a Banach space. Let $(f_n)_{n \geq 0}$, $(g_n)_{n \geq 0}$ be two X -valued local martingales. Then g is weakly differentially subordinated to f if and only if there exists an adapted scalar-valued process $(a_n)_{n \geq 0}$ such that $|a_n| \leq 1$ a.s. for all $n \geq 1$, $dg_n = a_n df_n$ a.s. and $g_0 = a_0 f_0$ a.s.

For the proof we will need two lemmas.

Lemma 3.5. *Let X be a Banach space, $\ell_1, \ell_2 \in X^*$ be such that $\ker(\ell_1) \subset \ker(\ell_2)$. Then there exists $a \in \mathbb{K}$ such that $\ell_2 = a\ell_1$.*

Proof. If $\ell_2 = 0$, then the assertion is obvious and one can take $a = 0$. Suppose that $\ell_2 \neq 0$. Then $\text{codim}(\ker(\ell_2)) = 1$ (see [29, p.80]), and there exists $x_0 \in X \setminus \ker(\ell_2)$ such that $x_0 \oplus \ker(\ell_2) = X$. Notice that since $\text{codim}(\ker(\ell_1)) \leq 1$ and $\ker(\ell_1) \subset \ker(\ell_2)$, one can easily conclude that $\ker(\ell_1) = \ker(\ell_2)$. Let $a = \ell_2(x_0)/\ell_1(x_0)$. Fix $y \in X$. Then there exists $\lambda \in \mathbb{K}$ such that $y - \lambda x_0 \in \ker(\ell_1) = \ker(\ell_2)$. Therefore

$$\ell_2(y) = \ell_2(\lambda x_0) + \ell_2(y - \lambda x_0) = a\ell_1(\lambda x_0) + a\ell_1(y - \lambda x_0) = a\ell_1(y),$$

hence $\ell_2 = a\ell_1$. \square

Lemma 3.6. *Let X be a Banach space, (S, Σ, μ) be a measure space. Let $f, g : S \rightarrow X$ be strongly measurable such that $|\langle g, x^* \rangle| \leq |\langle f, x^* \rangle|$ μ -a.s. for each $x^* \in X^*$. Then there exists a measurable function $a : S \rightarrow \mathbb{K}$ such that $\|a\|_\infty \leq 1$ and $g = af$.*

Proof. By the Pettis measurability theorem [23, Theorem 1.1.20] we can assume X to be separable. Let $(x_m)_{m \geq 1}$ be a dense subset of X . By the Hahn-Banach theorem we can find a sequence $(x_m^*)_{m \geq 1}$ of linear functionals on X such that $\langle x_m, x_m^* \rangle = \|x_m\|$ and $\|x_m^*\| = 1$ for each $m \geq 1$. Let $Y_0 = \mathbb{Q} - \text{span}(x_1^*, x_2^*, \dots)$, and let $Y = \text{span}(x_1^*, x_2^*, \dots)$ be a separable closed subspace of X^* . Then $X \hookrightarrow Y^*$ isometrically. Fix a set of full measure S_0 such that for all $x^* \in Y_0$, $|\langle g, x^* \rangle| \leq |\langle f, x^* \rangle|$ on S_0 . Fix $x^* \in Y$. Let $(y_k)_{k \geq 1}$ be a sequence in Y_0 such that $y_k \rightarrow x^*$ in Y as $k \rightarrow \infty$. Then on S_0 we have that $|\langle g, y_k^* \rangle| \rightarrow |\langle g, x^* \rangle|$ and $|\langle f, y_k \rangle| \rightarrow |\langle f, x^* \rangle|$. Consequently for each $s \in S_0$,

$$(3.3) \quad |\langle g(s), x^* \rangle| \leq |\langle f(s), x^* \rangle|, \quad x^* \in Y.$$

Therefore the linear functionals $f(s), g(s) \in X \hookrightarrow Y^*$ are such that $\ker g(s) \subset \ker f(s)$, and hence by Lemma 3.5 there exist $a(s)$ defined for each fixed $s \in S_0$ such that $g(s) = a(s)f(s)$. By (3.3) one has that $|a(s)| \leq 1$.

Let us construct a measurable version of a . Y_0 is countable since it is a \mathbb{Q} -span of a countable set. Let $Y_0 = (y_m)_{m \geq 1}$. For each $m > 1$ construct $A_m \in \Sigma$ as follows:

$$A_m = \{s \in S : \langle g(s), y_m \rangle \neq 0, \langle g(s), y_{m-1} \rangle = 0, \dots, \langle g(s), y_1 \rangle = 0\}$$

and put $A_1 = \{s \in S : \langle g(s), y_1 \rangle \neq 0\}$. Obviously on the set $S \setminus \cup_{m=1}^\infty A_m$ one has that $g = 0$, so one can redefine $a := 0$ on $S \setminus \cup_{m=1}^\infty A_m$. For each $m \geq 1$ we redefine $a := \frac{\langle g, y_m \rangle}{\langle f, y_m \rangle}$ on A_m . Then a constructed in such a way is Σ -measurable. \square

Proof of Proposition 3.4. The proposition follows from Lemma 3.6: the assumption of this lemma holds for df_n and dg_n for any $n \geq 1$, and for f_0 and g_0 . So according to Lemma 3.6 there exists a sequence $(a_n)_{n \geq 0}$ which is a.s. bounded by 1, such that $dg_n = a_n df_n$ for each $n \geq 1$ and $g_0 = a_0 f_0$ a.s. Moreover, again thanks to Lemma 3.6, a_n is \mathcal{F}_n -measurable, so $(a_n)_{n \geq 0}$ is adapted. \square

Definition 3.7. *Let E be a linear space over the scalar field \mathbb{K} .*

- (i) *A function $f : E \rightarrow \mathbb{R}$ is called convex if for each $x, y \in E$, $\lambda \in [0, 1]$ one has that $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.*
- (ii) *A function $f : E \rightarrow \mathbb{R}$ is called concave if for each $x, y \in E$, $\lambda \in [0, 1]$ one has that $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$.*

- (iii) A function $f : E \times E \rightarrow \mathbb{R}$ is called *biconcave* if for each $x, y \in E$ one has that the mappings $e \mapsto f(x, e)$ and $e \mapsto f(e, y)$ are concave.
- (iv) A function $f : E \times E \rightarrow \mathbb{R}$ is called *zigzag-concave* if for each $x, y \in E$ and $\varepsilon \in \mathbb{K}$, $|\varepsilon| \leq 1$ the function $z \mapsto f(x + z, y + \varepsilon z)$ is concave.

Note that our definition of zigzag-concavity is a bit different from the classical one (e.g. as in [23]): usually one sets in the definition $|\varepsilon| = 1$. The reader should pay attention to this extension: thanks to this additional property Theorem 3.8 below is more general than [23, Theorem 4.5.6]. This improvement will later allow us to prove the main theorem of this section.

In [14] Burkholder showed that the UMD property is equivalent to the existence of a certain biconcave function $V : X \times X \rightarrow \mathbb{R}$. With a slight variation of his argument (see Remark 3.11) one can also show the equivalence with the existence of a certain zigzag-concave function with a better structure.

Theorem 3.8 (Burkholder). *For a Banach space X the following are equivalent*

- (1) X is a UMD Banach space;
- (2) for each $p \in (1, \infty)$ there exists a constant $\beta > 0$ and a zigzag-concave function $U : X \times X \rightarrow \mathbb{R}$ such that

$$(3.4) \quad U(x, y) \geq \|y\|^p - \beta^p \|x\|^p, \quad x, y \in X.$$

The smallest admissible β for which such U exists is $\beta_{p, X}$.

Proof. The proof is essentially the same as the one given in [23, Theorem 4.5.6], but the construction of U is a bit different. The only difference is allowing $|\varepsilon| \leq 1$ instead of $|\varepsilon| = 1$ for the appropriate scalars ε .

For each $x, y \in X$ we define $\mathbb{S}(x, y)$ as a set of all pairs (f, g) of discrete martingales such that

- (1) $f_0 \equiv x, g_0 \equiv y$;
- (2) there exists $N \geq 0$ such that $df_n \equiv 0, dg_n \equiv 0$ for $n \geq N$;
- (3) $(dg_n)_{n \geq 1} = (\varepsilon_n df_n)_{n \geq 1}$ for some sequence of scalars $(\varepsilon_n)_{n \geq 1}$ such that $|\varepsilon_n| \leq 1$ for each $n \geq 1$.

Then we define $U : X \times X \rightarrow \mathbb{R} \cup \{\infty\}$ as follows:

$$(3.5) \quad U(x, y) := \sup \{ \mathbb{E}(\|g_\infty\|^p - \beta^p \|f_\infty\|^p) : (f, g) \in \mathbb{S}(x, y) \}.$$

The rest of the proof repeats the one given in [23, Theorem 4.5.6]. \square

Remark 3.9. Notice that function U constructed above coincides with the one in the proof of [23, Theorem 4.5.6]. This is due to the fact that the function

$$(\varepsilon_n)_{n=1}^N \mapsto \left(\mathbb{E} \left\| g_0 + \sum_{n=1}^N \varepsilon_n df_n \right\|^p \right)^{\frac{1}{p}}$$

is convex on the \mathbb{K} -cube $\{(\varepsilon_n)_{n=1}^N : |\varepsilon_1|, \dots, |\varepsilon_N| \leq 1\}$ because of the triangle inequality, therefore it takes its supremum on the set of the domain endpoints, namely on the set $\{(\varepsilon_n)_{n=1}^N : |\varepsilon_1|, \dots, |\varepsilon_N| = 1\}$.

Remark 3.10. Analogously to [23, (4.31)] by (3.5) we have that $U(\alpha x, \alpha y) = |\alpha|^p U(x, y)$ for each $x, y \in X$, $\alpha \in \mathbb{K}$. Therefore $U(0, 0) = 0$, and hence for each

$x \in X$ and each scalar ε such that $|\varepsilon| \leq 1$, by the zigzag-concavity of U in the point $(0, 0)$

$$(3.6) \quad U(x, \varepsilon x) = \frac{1}{2}U(0 + x, 0 + \varepsilon x) + \frac{1}{2}U(0 - x, 0 - \varepsilon x) \leq U(0, 0) = 0.$$

Let $\xi, \eta \in L^0(\Omega; X)$ be such that $|\langle \eta, x^* \rangle| \leq |\langle \xi, x^* \rangle|$ for each $x^* \in X^*$ a.s. Then thanks to Lemma 3.6 and (3.6), $U(\xi, \eta) \leq 0$ a.s.

Remark 3.11. For each zigzag-concave function $U : X \times X \rightarrow \mathbb{R}$ one can construct a biconcave function $V : X \times X \rightarrow \mathbb{R}$ as follows:

$$(3.7) \quad V(x, y) = U\left(\frac{x-y}{2}, \frac{x+y}{2}\right), \quad x, y \in X.$$

Indeed, by the definition of U , for each $x, y \in X$ the functions

$$\begin{aligned} z \mapsto V(x+z, y) &= U\left(\frac{x-y}{2} + \frac{z}{2}, \frac{x+y}{2} + \frac{z}{2}\right), \\ z \mapsto V(x, y+z) &= U\left(\frac{x-y}{2} - \frac{z}{2}, \frac{x+y}{2} + \frac{z}{2}\right) \end{aligned}$$

are concave. Moreover, for each $x, y \in X$ and $a, b \in \mathbb{K}$ such that $|a+b| \leq |a-b|$ one has that the function

$$z \mapsto V(x+az, y+bz) = U\left(\frac{x-y}{2} + \frac{(a-b)z}{2}, \frac{x+y}{2} + \frac{(a+b)z}{2}\right)$$

is concave since $\left|\frac{a+b}{a-b}\right| \leq 1$.

Remark 3.12. Due to the explicit representation (3.5) of U we can show that for each $x_1, x_2, y_1, y_2 \in X$,

$$|U(x_1, y_1) - U(x_2, y_2)| \leq \|x_1 - x_2\|^p + \beta_{p,X}^p \|y_1 - y_2\|^p.$$

Therefore U is continuous, and consequently V is continuous as well.

Remark 3.13. Notice that if X is finite dimensional then by Theorem 2.20 and Proposition 2.21 in [19] there exists a unique translation-invariant measure λ_X on X such that $\lambda_X(\mathbb{B}_X) = 1$ for the unit ball \mathbb{B}_X of X . We will call λ_X a *Lebesgue measure*. Thanks to the Alexandrov theorem [18, Theorem 6.4.1] $x \mapsto V(x, y)$ and $y \mapsto V(x, y)$ are a.s. Fréchet differentiable with respect to λ_X , and by [26, Proposition 3.1] and Remark 3.12 for a.a. $(x, y) \in X \times X$ for each $u, v \in X$ there exists the directional derivative $\frac{\partial V(x+tu, y+tv)}{\partial t}$. Moreover,

$$(3.8) \quad \frac{\partial V(x+tu, y+tv)}{\partial t} = \langle \partial_x V(x, y), u \rangle + \langle \partial_y V(x, y), v \rangle,$$

where $\partial_x V$ and $\partial_y V$ are the corresponding Fréchet derivatives with respect to the first and the second variable. Thanks to (3.8) and Remark 3.11 one obtains that for a.e. $(x, y) \in X \times X$, for all $z \in X$ and $a, b \in \mathbb{K}$ such that $|a+b| \leq |a-b|$,

$$(3.9) \quad \begin{aligned} V(x+az, y+bz) &\leq V(x, y) + \frac{\partial V(x+atz, y+btz)}{\partial t} \\ &= V(x, y) + a\langle \partial_x V(x, y), z \rangle + b\langle \partial_y V(x, y), z \rangle. \end{aligned}$$

Lemma 3.14. *Let X be a finite dimensional Banach space, $V : X \times X \rightarrow \mathbb{R}$ be as defined in (3.7). Then there exists $C > 0$ which depends only on V such that for a.e. pair $x, y \in X$,*

$$\|\partial_x V(x, y)\|, \|\partial_y V(x, y)\| \leq C(\|x\|^{p-1} + \|y\|^{p-1}).$$

Proof. We show the inequality only for $\partial_x V$, the proof for $\partial_y V$ being analogous. First we prove that there exists $C > 0$ such that $\|\partial_x V(x, y)\| \leq C$ for a.e. $x, y \in X$ such that $\|x\|, \|y\| \leq 1$. Let us show this by contradiction. Suppose that such C does not exist. Since V is continuous by Remark 3.12, and since a unit ball in X is a compact set, there exists $K > 0$ such that $|V(x, y)| < K$ for all $x, y \in X$ such that $\|x\|, \|y\| \leq 2$. Let $x_0, y_0 \in X$ be such that $\|x_0\|, \|y_0\| \leq 1$ and $\|\partial_x V(x_0, y_0)\| > 3K$. Therefore there exists $z \in X$ such that $\|z\| = 1$ and $\langle \partial_x V(x_0, y_0), z \rangle < -3K$. Hence we have that $\|x_0 + z\| \leq 2$ and because of the concavity of V in the first variable

$$V(x_0 + z, y_0) \leq V(x_0, y_0) + \langle \partial_x V(x_0, y_0), z \rangle \leq K - 3K \leq -2K.$$

Consequently, $|V(x_0 + z, y_0)| > K$, which contradicts with our suggestion.

Now fix $C > 0$ such that $|\partial_x V(x, y)| \leq C$ for all $x, y \in X$ such that $\|x\|, \|y\| \leq 1$. Fix $x, y \in X$. Without loss of generality assume that $\|x\| \geq \|y\|$. Let $L = \|x\|$. Then $\|\partial_x V(\frac{x}{L}, \frac{y}{L})\| \leq C$. Let $z \in X$ be such that $\|z\| = 1$. Then by Remark 3.10,

$$\begin{aligned} |\langle \partial_x V(x, y), z \rangle| &= \left| \lim_{t \rightarrow 0} \frac{V(x+tz, y) - V(x, y)}{t} \right| = \left| \lim_{t \rightarrow 0} \frac{L^p V(\frac{x}{L} + \frac{t}{L}z, \frac{y}{L}) - L^p V(\frac{x}{L}, \frac{y}{L})}{L \frac{t}{L}} \right| \\ &= L^{p-1} \left| \lim_{t \rightarrow 0} \frac{V(\frac{x}{L} + tz, \frac{y}{L}) - V(\frac{x}{L}, \frac{y}{L})}{t} \right| = L^{p-1} \left| \langle \partial_x V\left(\frac{x}{L}, \frac{y}{L}\right), z \rangle \right| \\ &\leq L^{p-1} C \leq C(\|x\|^{p-1} + \|y\|^{p-1}). \end{aligned}$$

Therefore since z was arbitrary, $\|\partial_x V(x, y)\| \leq C(\|x\|^{p-1} + \|y\|^{p-1})$. The case $\|x\| < \|y\|$ can be done in the same way. \square

Lemma 3.15. *Let X be a finite dimensional Banach space, $1 < p < \infty$, $(f_n)_{n \geq 0}$, $(g_n)_{n \geq 0}$ be X -valued martingales on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_n)_{n \geq 0}$ and assume that $(g_n)_{n \geq 0}$ is weakly differentially subordinated to $(f_n)_{n \geq 0}$. Let $Y = X \oplus \mathbb{R}$ be the Banach space with the norm as follows:*

$$\|(x, r)\|_Y := (\|x\|_X^p + |r|^p)^{\frac{1}{p}}, \quad x \in X, r \in \mathbb{R}.$$

Then there exist two sequences $(f^m)_{m \geq 1}$ and $(g^m)_{m \geq 1}$ of Y -valued martingales on an enlarged probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with an enlarged filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_n)_{n \geq 0}$ such that

- (1) f_n^m, g_n^m have absolutely continuous distributions with respect to the Lebesgue measure on Y for each $m \geq 1$ and $n \geq 0$;
- (2) $f_n^m \rightarrow (f_n, 0)$, $g_n^m \rightarrow (g_n, 0)$ pointwise as $m \rightarrow \infty$ for each $n \geq 0$;
- (3) if for some $n \geq 0$ $\mathbb{E}\|f_n\|^p < \infty$, then for each $m \geq 1$ one has that $\mathbb{E}\|f_n^m\|^p < \infty$ and $\mathbb{E}\|f_n^m - (f_n, 0)\|^p \rightarrow 0$ as $m \rightarrow \infty$;
- (4) if for some $n \geq 0$ $\mathbb{E}\|g_n\|^p < \infty$, then for each $m \geq 1$ one has that $\mathbb{E}\|g_n^m\|^p < \infty$ and $\mathbb{E}\|g_n^m - (g_n, 0)\|^p \rightarrow 0$ as $m \rightarrow \infty$;
- (5) for each $m \geq 1$ we have that $(g_n^m)_{n \geq 0}$ is weakly differentially subordinated to $(f_n^m)_{n \geq 0}$.

Proof. First of all let us show that we may assume that f_0 and g_0 are nonzero a.s. For this purpose we can modify f_0 and g_0 as follows:

$$\begin{aligned} f_0^\varepsilon &= f_0 + \varepsilon x \mathbf{1}_{f_0=0}, \\ g_0^\varepsilon &= g_0 + \varepsilon x \mathbf{1}_{f_0=0} + \varepsilon f_0 \mathbf{1}_{g_0=0, f_0 \neq 0}, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary and $x \in X$ is fixed. This small perturbation does not destroy the weak differential subordination property. Moreover, if we let $f_n^\varepsilon := f_0^\varepsilon + \sum_{k=1}^n df_k$, $g_n^\varepsilon := g_0^\varepsilon + \sum_{k=1}^n dg_k$ for any $n \geq 1$, then $f_n^\varepsilon \rightarrow f_n$ and $g_n^\varepsilon \rightarrow g_n$ a.s., and $f_n^\varepsilon - f_n \rightarrow 0$ and $g_n^\varepsilon - g_n \rightarrow 0$ in $L^p(\Omega; X)$ as $\varepsilon \rightarrow 0$.

From now we assume that f_0 and g_0 are nonzero a.s. This in fact means that random variable a_0 from Proposition 3.4 is nonzero a.s. as well. Let \mathbb{B}_Y be the unit ball of Y , $(\mathbb{B}_Y, \mathcal{B}(\mathbb{B}_Y), \hat{\mathbb{P}})$ be a probability space such that $\hat{\mathbb{P}} := \lambda_Y|_{\mathbb{B}_Y}$ has the uniform Lebesgue distribution on \mathbb{B}_Y (see Remark 3.13). Fix some scalar product $\langle \cdot, \cdot \rangle : Y \times Y \rightarrow \mathbb{R}$ in Y . We will construct a random operator $T : \mathbb{B}_Y \rightarrow \mathcal{L}(Y)$ as follows:

$$T(b, y) := \langle b, y \rangle b \quad b \in \mathbb{B}_Y, y \in Y.$$

Note that for each fixed $b \in \mathbb{B}_Y$ the mapping $y \mapsto \langle b, y \rangle b$ is a linear operator on Y . Moreover,

$$(3.10) \quad \sup_{b \in \mathbb{B}_Y} \|T(b, \cdot)\|_{\mathcal{L}(Y)} < \infty.$$

Now let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) := (\Omega \times \mathbb{B}_Y, \mathcal{F} \otimes \mathcal{B}(\mathbb{B}_Y), \mathbb{P} \otimes \hat{\mathbb{P}})$. For each $m \geq 1$ define an operator-valued function $Q_m : \bar{\Omega} \rightarrow \mathcal{L}(Y)$ as follows: $Q_m := I + \frac{1}{m}T$.

Fix $\varepsilon > 0$. For each $n \geq 0$ define $\tilde{f}_n^\varepsilon := (f_n, \varepsilon)$, $\tilde{g}_n^\varepsilon := (g_n, \varepsilon a_0)$. Then $(\tilde{f}_n^\varepsilon)_{n \geq 0}$ and $(\tilde{g}_n^\varepsilon)_{n \geq 0}$ are Y -valued martingales which are nonzero a.s. for each $n \geq 0$ and are such that $(\tilde{g}_n^\varepsilon)_{n \geq 0}$ is weakly differentially subordinated to $(\tilde{f}_n^\varepsilon)_{n \geq 0}$. For each $m \geq 1$ define Y -valued martingales f^m and g^m in the following way:

$$\begin{aligned} f_n^m &:= Q_m \tilde{f}_n^\varepsilon, \quad m \geq 1, n \geq 0, \\ g_n^m &:= Q_m \tilde{g}_n^\varepsilon, \quad m \geq 1, n \geq 0. \end{aligned}$$

Let us illustrate that for each $m \geq 1$, f^m and g^m are martingales with respect to the filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_n)_{n \geq 0} := (\mathcal{F}_n \otimes \mathcal{B}(\mathbb{B}_Y))_{t \geq 0}$: for each $n \geq 1$ we have

$$\begin{aligned} \mathbb{E}(f_n^m | \bar{\mathcal{F}}_{n-1}) &= \mathbb{E}(Q_m \tilde{f}_n^\varepsilon | \mathcal{F}_{n-1} \otimes \mathcal{B}(\mathbb{B}_Y)) \stackrel{(i)}{=} Q_m \mathbb{E}(\tilde{f}_n^\varepsilon | \mathcal{F}_{n-1} \otimes \mathcal{B}(\mathbb{B}_Y)) \\ &\stackrel{(ii)}{=} Q_m \tilde{f}_{n-1}^\varepsilon = f_{n-1}^m, \end{aligned}$$

where (i) holds since Q_m is $\mathcal{B}(\mathbb{B}_Y)$ -measurable, and (ii) holds since \tilde{f}_n^ε is independent of $\mathcal{B}(\mathbb{B}_Y)$. The same can be proven for g^m . Thanks to (3.10) one has that $\lim_{m \rightarrow \infty} \sup_{b \in \mathbb{B}_Y} \|Q_m - I\|_{\mathcal{L}(Y)} = 0$ and hence (2), (3) and (4) hold for \tilde{f}^ε and \tilde{g}^ε .

Let us prove (5). For each $m \geq 1$ and $n \geq 1$ one has:

$$dg_n^m = dQ_m \tilde{g}_n^\varepsilon = dQ_m a_n \tilde{f}_n^\varepsilon = a_n dQ_m \tilde{f}_n^\varepsilon = a_n df_n^m.$$

The same holds for g_0^m and f_0^m .

Now we will show (1). Let us fix a set $A \subset Y$ of Lebesgue measure zero. Then for each fixed $n \geq 0$ and $m \geq 1$,

$$(3.11) \quad \begin{aligned} \mathbb{E} \mathbf{1}_{f_n^m \in A} &= \int_{\Omega} \int_{\mathbb{B}_Y} \mathbf{1}_{\tilde{f}_n^\varepsilon + \frac{1}{m} \langle b, \tilde{f}_n^\varepsilon \rangle b \in A} d\hat{\mathbb{P}}(b) d\mathbb{P} \\ &= \int_{\Omega} \int_{\mathbb{B}_Y} \mathbf{1}_{\frac{1}{m} \langle b, \tilde{f}_n^\varepsilon \rangle b \in A - \tilde{f}_n^\varepsilon} d\hat{\mathbb{P}}(b) d\mathbb{P}, \end{aligned}$$

where $F - y$ is a translation of a set $F \subset Y$ by a vector $y \in Y$. For each fixed $y \in Y \setminus \{0\}$ the distribution of a Y -valued random variable $b \mapsto \langle b, y \rangle b$ is absolutely

continuous with respect to λ_Y . Since $\hat{\mathbb{P}}(A - y) = 0$ for each $y \in Y \setminus \{0\}$, one has

$$(3.12) \quad \int_{\mathbb{B}_Y} \mathbf{1}_{\frac{1}{m}\langle b, y \rangle b \in A - y} d\hat{\mathbb{P}}(b) = 0.$$

Recall that $\mathbb{P}\{\tilde{f}_n^\varepsilon = 0\} = 0$, therefore due to (3.12) a.s.

$$\int_{\mathbb{B}_Y} \mathbf{1}_{\frac{1}{m}\langle b, \tilde{f}_n^\varepsilon \rangle b \in A - \tilde{f}_n^\varepsilon} d\hat{\mathbb{P}}(b) = 0.$$

Consequently the last double integral in (3.11) vanishes. The same works for g^m .

Now to construct such a sequence for $((f_n, 0))_{n \geq 0}$ and $((g_n, 0))_{n \geq 0}$ one needs to construct it for different ε and take an appropriate subsequence. \square

Proof of Theorem 3.3. The ‘‘if’’ part is obvious thanks to the definition of a UMD Banach space. Let us prove the ‘‘only if’’ part. As in the proof of the lemma above, without loss of generality suppose that X is separable and that the set $\bigcup_n (\{f_n = 0\} \cup \{g_n = 0\})$ is of \mathbb{P} -measure 0. If it does not hold, we consider $Y := X \oplus \mathbb{R}$ instead of X with the norm of $(x, r) \in Y$ given by $\|(x, r)\|_Y = (\|x\|_X^p + |r|^p)^{1/p}$. Notice that then $\beta_{p,Y} = \beta_{p,X}$. We can suppose that a_0 is nonzero a.s., so we consider $(f_n^\varepsilon)_{n \geq 0} := (f_n \oplus \varepsilon)_{n \geq 0}$ and $(g_n^\varepsilon)_{n \geq 0} := (g_n \oplus \varepsilon a_0)_{n \geq 0}$ with $\varepsilon > 0$, and let ε go to zero.

One can also restrict to a finite dimensional case. Indeed, since X is a separable reflexive space, X^* is separable as well. Let $(Y_m)_{m \geq 1}$ be an increasing sequence of finite-dimensional subspaces of X^* such that $\bigcup_m Y_m = X^*$ and $\|\cdot\|_{Y_m} = \|\cdot\|_{X^*}$ for each $m \geq 1$. Then for each fixed $m \geq 1$ there exists a linear operator $P_m : X \rightarrow Y_m^*$ of norm 1 defined as follows: $\langle P_m x, y \rangle = \langle x, y \rangle$ for each $x \in X, y \in Y_m$. Then since Y_m is a closed subspace of X^* , [23, Proposition 4.33] yields $\beta_{p', Y_m^*} \leq \beta_{p', X^*}$, consequently again by [23, Proposition 4.33] $\beta_{p, Y_m^*} \leq \beta_{p, X^{**}} = \beta_{p, X}$. So if we prove the finite dimensional version, then

$$\mathbb{E}\|P_m g_n\|^p \leq \beta_{p,X}^p \mathbb{E}\|P_m f_n\|^p, \quad n \geq 0,$$

for each $m \geq 1$, and due to the fact that $\|P_m x\|_{Y_m^*} \nearrow \|x\|_X$ for each $x \in X$ as $m \rightarrow \infty$, we would obtain (3.2) in the general case.

Let β be the UMD constant of X , and let $U, V : X \times X \rightarrow \mathbb{R}$ be as defined in Theorem 3.8 and in (3.7) respectively, $(a_n)_{n \geq 0}$ be as defined in Proposition 3.4. By Lemma 3.15 we can suppose that f_n and g_n have distributions which are absolutely continuous with respect to the Lebesgue measure. Then

$$(3.13) \quad \begin{aligned} \mathbb{E}(\|g_n\|^p - \beta\|f_n\|^p) &\stackrel{(i)}{\leq} \mathbb{E}U(f_n, g_n) = \mathbb{E}U(f_{n-1} + df_n, g_{n-1} + a_n df_n) \\ &\stackrel{(ii)}{=} \mathbb{E}V(g_{n-1} + f_{n-1} + (a_n + 1)df_n, g_{n-1} - f_{n-1} + (a_n - 1)df_n) \\ &\stackrel{(iii)}{\leq} \mathbb{E}V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}) \\ &\quad + \mathbb{E}\langle \partial_x V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}), (a_n + 1)df_n \rangle \\ &\quad + \mathbb{E}\langle \partial_y V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}), (a_n - 1)df_n \rangle \\ &\stackrel{(iv)}{=} \mathbb{E}V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}) \\ &\stackrel{(v)}{=} \mathbb{E}U(f_{n-1}, g_{n-1}). \end{aligned}$$

Here (i) and (iii) hold by Theorem 3.8 and (3.9) respectively, (ii) and (v) follow from the definition of V . Let us prove (iv). We will show that

$$(3.14) \quad \mathbb{E}\langle \partial_x V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}), (a_n + 1)df_n \rangle = 0.$$

Since both f_n and $a_n f_n$ are martingale differences, $(a_n + 1)df_n$ is a martingale difference as well. Therefore $\mathbb{E}((a_n - 1)df_n | \mathcal{F}_{n-1}) = 0$. Note that according to Lemma 3.14 a.s.

$$\|\partial_x V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1})\| \lesssim_V \|f_n\|^{p-1} + \|g_n\|^{p-1}.$$

Therefore by the Hölder inequality $\langle \partial_x V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}), (a_n + 1)df_n \rangle$ is integrable. Since $\partial_x V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1})$ is \mathcal{F}_{n-1} -measurable,

$$\begin{aligned} & \mathbb{E}\left(\langle \partial_x V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}), (a_n + 1)df_n \rangle | \mathcal{F}_{n-1}\right) \\ &= \left\langle \partial_x V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}), \mathbb{E}((a_n + 1)df_n | \mathcal{F}_{n-1}) \right\rangle \\ &= \langle \partial_x V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}), 0 \rangle = 0, \end{aligned}$$

so (3.14) holds. By the same reason

$$\mathbb{E}\langle \partial_y V(g_{n-1} + f_{n-1}, g_{n-1} - f_{n-1}), (a_n - 1)df_n \rangle = 0,$$

and (iv) follows.

Notice that thanks to Remark 3.10 $\mathbb{E}(f_0, g_0) \leq 0$. Therefore from the inequality (3.13) by an induction argument we get

$$\mathbb{E}(\|g_n\|^p - \beta^p \|f_n\|^p) \leq \mathbb{E}U(f_n, g_n) \leq \mathbb{E}U(f_{n-1}, g_{n-1}) \leq \dots \leq \mathbb{E}U(f_0, g_0) \leq 0.$$

This terminates the proof. \square

3.2. Continuous time case. Now we turn to continuous time martingales. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ that satisfies the usual conditions.

Definition 3.16. Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a local martingale. Then M is called purely discontinuous if $[M]$ is a pure jump processes (i.e. $[M]$ has a version that is a constant a.s. in time). Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a local martingale. Then M is called purely discontinuous if for each $x^* \in X^*$ a local martingale $\langle M, x^* \rangle$ is purely discontinuous.

The reader can find more on purely discontinuous martingales in [25, 27].

Definition 3.17. Let $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ be local martingales. Then we say that N is weakly differentially subordinated to M if for each $x^* \in X^*$ one has that $[\langle M, x^* \rangle] - [\langle N, x^* \rangle]$ is an a.s. nondecreasing function and $|\langle N_0, x^* \rangle| \leq |\langle M_0, x^* \rangle|$ a.s.

The following theorem is a natural extension of Proposition 3.4.

Theorem 3.18. Let X be a Banach space. Then X is a UMD space if and only if for some (equivalently, for all) $1 < p < \infty$ there exists $\beta > 0$ such that for each purely discontinuous X -valued local martingales $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ such that N is weakly differentially subordinated to M one has

$$(3.15) \quad \mathbb{E}\|N_t\|^p \leq \beta^p \mathbb{E}\|M_t\|^p.$$

If this is the case then the smallest admissible β equals the UMD constant $\beta_{p,X}$.

Lemma 3.19. *Let X be a finite dimensional Banach space, $1 < p < \infty$, $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ be local martingales on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ such that N is weakly differentially subordinated to M . Let $Y = X \oplus \mathbb{R}$ be a Banach space such that $\|(x, r)\|_Y = (\|x\|_X^p + |r|^p)^{\frac{1}{p}}$ for each $x \in X$, $r \in \mathbb{R}$. Then there exist two sequences $(M^m)_{m \geq 1}$ and $(N^m)_{m \geq 1}$ of Y -valued martingales on an enlarged probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with an enlarged filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t \geq 0}$ such that*

- (1) M_t^m, N_t^m have absolutely continuous distributions with respect to the Lebesgue measure on Y for each $m \geq 1$ and $t \geq 0$;
- (2) $M_t^m \rightarrow (M_t, 0), N_t^m \rightarrow (N_t, 0)$ pointwise as $m \rightarrow \infty$ for each $t \geq 0$;
- (3) if for some $t \geq 0$ $\mathbb{E}\|M_t\|^p < \infty$, then for each $m \geq 1$ one has that $\mathbb{E}\|M_t^m\|^p < \infty$ and $\mathbb{E}\|M_t^m - (M_t, 0)\|^p \rightarrow 0$ as $m \rightarrow \infty$;
- (4) if for some $t \geq 0$ $\mathbb{E}\|N_t\|^p < \infty$, then for each $m \geq 1$ one has that $\mathbb{E}\|N_t^m\|^p < \infty$ and $\mathbb{E}\|N_t^m - (N_t, 0)\|^p \rightarrow 0$ as $m \rightarrow \infty$;
- (5) for each $m \geq 1$ we have that N^m is weakly differentially subordinated to M^m .

Proof. The proof is essentially the same as one of Lemma 3.15. \square

Proof of Theorem 3.18. We use a modification of the argument in [52, Theorem 1], where the Hilbert space case was considered. Thanks to the same methods as were applied in the beginning of the proof of Theorem 3.3 and using Lemma 3.19 instead of Lemma 3.15, one can suppose that X is finite-dimensional and M_t and N_t are nonzero a.s. for each $t \geq 0$. We know that $\mathbb{E}U(M_t, N_t) \geq \mathbb{E}(\|N_t\|^p - \beta^p \|M_t\|^p)$ for each $t \geq 0$. On the other hand, thanks to the fact that $[\langle M, x^* \rangle]$ and $[\langle N, x^* \rangle]$ are pure jump for each $x^* \in X^*$ and the finite-dimensional version of Itô formula [27, Theorem 26.7], one has

$$(3.16) \quad \begin{aligned} \mathbb{E}U(M_t, N_t) &= \mathbb{E}U(M_0, N_0) + \mathbb{E} \int_0^t \langle \partial_x U(M_{s-}, N_{s-}), dM_s \rangle \\ &\quad + \mathbb{E} \int_0^t \langle \partial_y U(M_{s-}, N_{s-}), dN_s \rangle + \mathbb{E}I, \end{aligned}$$

where

$$I = \sum_{0 < s \leq t} [\Delta U(M_s, N_s) - \langle \partial_x U(M_{s-}, N_{s-}), \Delta M_s \rangle - \langle \partial_y U(M_{s-}, N_{s-}), \Delta N_s \rangle].$$

Note that since a.s.

$$\Delta |\langle N, x^* \rangle|^2 = \Delta [\langle N, x^* \rangle] \leq \Delta [\langle M, x^* \rangle] = \Delta |\langle M, x^* \rangle|^2$$

for each $x^* \in X^*$, one has that thanks to Lemma 3.6 for each $s \geq 0$, for a.e. $\omega \in \Omega$ there exists $a_s(\omega)$ such that $|a_s(\omega)| \leq 1$ and $\Delta N_s(\omega) = a_s(\omega) \Delta M_s(\omega)$. Therefore for each $s \geq 0$ by (3.9) \mathbb{P} -a.s.

$$\begin{aligned} &\Delta U(M_s, N_s) - \langle \partial_x U(M_{s-}, N_{s-}), \Delta M_s \rangle - \langle \partial_y U(M_{s-}, N_{s-}), \Delta N_s \rangle \\ &= V(M_{s-} + N_{s-} + (a_s + 1)\Delta M_s, N_{s-} - M_{s-} + (a_s - 1)\Delta M_s) \\ &\quad - V(M_{s-} + N_{s-}, N_{s-} - M_{s-}) \\ &\quad - \langle \partial_x V(M_{s-} + N_{s-}, N_{s-} - M_{s-}), (a_s + 1)\Delta M_s \rangle \\ &\quad - \langle \partial_y V(M_{s-} + N_{s-}, N_{s-} - M_{s-}), (a_s - 1)\Delta M_s \rangle \leq 0, \end{aligned}$$

so $I \leq 0$ a.s., and $\mathbb{E}I \leq 0$. Also

$$\begin{aligned} \int_0^t \langle \partial_x U(M_{s-}, N_{s-}), dM_s \rangle + \int_0^t \langle \partial_y U(M_{s-}, N_{s-}), dN_s \rangle \\ = \int_0^t \langle \partial_x V(M_{s-} + N_{s-}, N_{s-} - M_{s-}), d(M_s + N_s) \rangle \\ + \int_0^t \langle \partial_y V(M_{s-} + N_{s-}, N_{s-} - M_{s-}), d(N_s - M_s) \rangle, \end{aligned}$$

so by Lemma 2.4 and Lemma 3.14 it is a martingale that starts at zero, and therefore its expectation is zero as well. Consequently, thanks to (3.4), (3.16) and Remark 3.10,

$$\mathbb{E}\|N_t\|^p - \beta_{p,X}^p \mathbb{E}\|M_t\|^p \leq \mathbb{E}U(M_t, N_t) \leq \mathbb{E}U(M_0, N_0) \leq 0,$$

and therefore (3.15) holds. \square

As one can see, in our proof we did not need the second order terms of the Itô formula thanks to the nature of the quadratic variation of a purely discontinuous process. Nevertheless, Theorem 3.18 holds for arbitrary martingales M and N , but with worse estimates (see [54]). The connection of Theorem 3.18 for continuous martingales with the Hilbert transform will be discussed in Section 5.

4. FOURIER MULTIPLIERS

In [3] and [2] the authors exploited the differential subordination property to show boundedness of certain Fourier multipliers in $\mathcal{L}(L^p(\mathbb{R}^d))$. It turned out that it is sufficient to use the weak differential subordination property to obtain the same assertions, but in the vector-valued situation.

4.1. Basic definitions and the main theorem. Let $d \geq 1$ be a natural number. Recall that $\mathcal{S}(\mathbb{R}^d)$ is a space of Schwartz functions on \mathbb{R}^d . For a Banach space X with a scalar field \mathbb{C} we define $\mathcal{S}(\mathbb{R}^d) \otimes X$ as the space of all functions $f : \mathbb{R}^d \rightarrow X$ of the form $f = \sum_{k=1}^K f_k \otimes x_k$, where $K \geq 1$, $f_1, \dots, f_K \in \mathcal{S}(\mathbb{R}^d)$, and $x_1, \dots, x_K \in X$. Notice that for each $1 \leq p < \infty$ the space $\mathcal{S}(\mathbb{R}^d) \otimes X$ is dense in $L^p(\mathbb{R}^d; X)$.

We define the *Fourier transform* \mathcal{F} and the *inverse Fourier transform* \mathcal{F}^{-1} on $\mathcal{S}(\mathbb{R}^d)$ as follows:

$$\begin{aligned} \mathcal{F}(f)(t) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle t, u \rangle} f(u) du, \quad f \in \mathcal{S}(\mathbb{R}^d), t \in \mathbb{R}^d, \\ \mathcal{F}^{-1}(f)(t) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\langle t, u \rangle} f(u) du, \quad f \in \mathcal{S}(\mathbb{R}^d), t \in \mathbb{R}^d. \end{aligned}$$

It is well-known that for any $f \in \mathcal{S}(\mathbb{R}^d)$ we have $\mathcal{F}(f), \mathcal{F}^{-1}(f) \in \mathcal{S}(\mathbb{R}^d)$, and $\mathcal{F}^{-1}(\mathcal{F}(f)) = f$. The reader can find more details on the Fourier transform in [22].

Let $m : \mathbb{R}^d \rightarrow \mathbb{C}$ be measurable and bounded. Then we can define a linear operator T_m on $\mathcal{S}(\mathbb{R}^d) \otimes X$ as follows:

$$(4.1) \quad T_m(f \otimes x) = \mathcal{F}^{-1}(m\mathcal{F}(f)) \cdot x, \quad f \in \mathcal{S}(\mathbb{R}^d), x \in X.$$

The operator T_m is called a *Fourier multiplier*, while the function m is called the *symbol* of T_m . If X is finite-dimensional then T_m can be extended to a bounded linear operator on $L^2(\mathbb{R}^d; X)$. The question is usually whether one can extend T_m to a bounded operator on $L^p(\mathbb{R}^d; X)$ for a general $1 < p < \infty$ and a given X .

Here the answer will be given for m of quite a special form and X with the UMD property.

Let V be a Lévy measure on \mathbb{R}^d , that is $V(\{0\}) = 0$, $V \neq 0$ and

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) V(dx) < \infty.$$

Let $\phi \in L^\infty(\mathbb{R}^d; \mathbb{C})$ be such that $\|\phi\|_{L^\infty(\mathbb{R}^d; \mathbb{C})} \leq 1$. Also let $\mu \geq 0$ be a finite Borel measure on the unit sphere $S^{d-1} \subset \mathbb{R}^d$, and $\psi \in L^\infty(S^{d-1}; \mathbb{C})$ satisfies $\|\psi\|_{L^\infty(S^{d-1}; \mathbb{C})} \leq 1$.

In the sequel we set $\frac{a}{0} = 0$ for each $a \in \mathbb{C}$. The following result extends [2, Theorem 1.1] to the UMD Banach space setting.

Theorem 4.1. *Let X be a UMD Banach space. Then the Fourier multiplier T_m with a symbol*

$$(4.2) \quad m(\xi) = \frac{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z) \phi(z) V(dz) + \frac{1}{2} \int_{S^{d-1}} (\xi \cdot \theta)^2 \psi(\theta) \mu(d\theta)}{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z) V(dz) + \frac{1}{2} \int_{S^{d-1}} (\xi \cdot \theta)^2 \mu(d\theta)}, \quad \xi \in \mathbb{R}^d,$$

has a bounded extension on $L^p(\mathbb{R}^d; X)$ for $1 < p < \infty$. Moreover, then for each $f \in L^p(\mathbb{R}^d; X)$

$$(4.3) \quad \|T_m f\|_p \leq \beta_{p,X} \|f\|_p.$$

Remark 4.2. The coefficient $\frac{1}{2}$ in both numerator and denominator of (4.1), even though it looks wired and useless (because one can always transform μ to 2μ), exists because of the strong connection with the Lévy–Khintchin representation of Lévy processes (see e.g. [1, Part 4.1]).

The proof is a modification of the arguments given in [2] and [3], but instead of real-valued process we will work with processes that take their values in a finite dimensional space. For the convenience of the reader the proof will be given in the same form and with the same notations as the original one. However, we will need to justify here some steps, so we cannot just skip the proof. First of all as that was done in [2], we reduce to the case of symmetric V and $\mu = 0$, and proceed as in the proof of [3, Theorem 1].

In the rest of the section we may assume that X is finite dimensional, since it is sufficient to show (4.3) for all f with values in X_0 for each finite dimensional subspace X_0 of X .

Let ν be a positive finite symmetric measure on \mathbb{R}^d , $\tilde{\nu} = \nu/|\nu|$. Let T_i and Z_i , $i = \pm 1, \pm 2, \pm 3, \dots$, be a family of independent random variables, such that each T_i is exponentially distributed with parameter $|\nu|$ (i.e. $\mathbb{E}T_i = 1/|\nu|$), and each Z_i has $\tilde{\nu}$ as a distribution. Let $S_i = T_1 + \dots + T_i$ for a positive i and $S_i = -(T_{-1} + \dots + T_i)$ for a negative i . For each $-\infty < s < t < \infty$ we define $X_{s,t} := \sum_{s < S_i \leq t} Z_i$ and $X_{s,t-} := \sum_{s < S_i < t} Z_i$. Note that $\mathcal{N}(B) = \#\{i : (S_i, Z_i) \in B\}$ defines a Poisson measure on $\mathbb{R} \times \mathbb{R}^d$ with the intensity measure $\lambda \otimes \nu$, and $X_{s,t} = \int_{s < v \leq t} x \mathcal{N}(dv, d\nu)$ (see e.g. [49]). Let $N(s, t) = \mathcal{N}((s, t] \times \mathbb{R}^d)$ be the number of *signals* S_i such that $s < S_i \leq t$. The following Lemmas 4.3–4.6 are multidimensional versions of [3, Lemma 1–5], which can be proven in the same way as in the scalar case.

Lemma 4.3. *Let $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow X$ be Borel measurable and be either nonnegative or bounded, and let $s \leq t$. Then*

$$\mathbb{E} \sum_{s < S_i \leq t} F(S_i, X_{s, S_i-}, X_{s, S_i}) = \mathbb{E} \int_s^t \int_{\mathbb{R}^d} F(v, X_{s, v-}, X_{s, v-+z}) \nu(dz) dv.$$

We will consider the following filtration:

$$\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathbb{R}} = \{\sigma\{X_{s,t} : s \leq t\}\}_{t \in \mathbb{R}}.$$

Recall that for measures ν_1 and ν_2 on \mathbb{R}^d the expression $\nu_1 * \nu_2$ means the *convolution* of these measures (we refer the reader [8, Chapter 3.9] for the details). Also for each $n \geq 1$ we define $\nu_1^{*n} := \underbrace{\nu_1 * \dots * \nu_1}_{n \text{ times}}$. For each $t \in \mathbb{R}$ define

$$p_t = e^{*t(\nu - |\nu|\delta_0)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\nu - |\nu|\delta_0)^{*n} = e^{-t|\nu|} \sum_{n=0}^{\infty} \frac{t^n}{n!} \nu^{*n}.$$

The series converges in the norm of absolute variation of measures. As in [3, (18)] and [2, (3.9)] p_t is symmetric, and

$$\frac{\partial}{\partial t} p_t = (\nu - |\nu|\delta_0) * p_t, \quad t \in \mathbb{R}.$$

Also $p_{t_1+t_2} = p_{t_1} * p_{t_2}$ for each $t_1, t_2 \in \mathbb{R}$. In fact for all $t \leq u$ the measure p_{u-t} is the distribution of $X_{t,u}$ and $X_{t,u-}$. Put

$$\Psi(\xi) = \int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1) \nu(dz), \quad \xi \in \mathbb{R}^d.$$

Thanks to the symmetry of ν one has as well that

$$\Psi(\xi) = \int_{\mathbb{R}^d} (\cos \xi \cdot z - 1) \nu(dz) = \Psi(-\xi) \leq 0.$$

Therefore Ψ is bounded on \mathbb{R}^d , and due to [2, (3.12)] we have that the characteristic function of p_t is of the following form:

$$\hat{p}_t(\xi) = e^{t\Psi(\xi)}, \quad \xi \in \mathbb{R}^d.$$

(The reader can find more on characteristic functions in [8, Chapter 3.8].)

Let $g \in L^\infty(\mathbb{R}^d; X)$. Then for $x \in \mathbb{R}^d$, $t \leq u$, we define the *parabolic extension* of g by

$$P_{t,u}g(x) := \int_{\mathbb{R}^d} g(x+y) p_{u-t}(dy) = g * p_{u-t}(x) = \mathbb{E}g(x + X_{t,u}).$$

For $s \leq t \leq u$ we define the *parabolic martingale* by

$$G_t = G_t(x; s, u; g) := P_{t,u}g(x + X_{s,t}).$$

Lemma 4.4. *We have that G_t is a bounded \mathbb{F} -martingale.*

Let $\phi \in L^\infty(\mathbb{R}^d; \mathbb{C})$ be symmetric. For each $x \in \mathbb{R}^d$, $s \leq t \leq u$, and $f \in C_c(\mathbb{R}^d; X)$ we define F_t as follows:

$$F_t = F_t(x; s, u; f, \phi) := \sum_{s < S_i \leq t} [P_{S_i, u}f(x + X_{s, S_i}) - P_{S_i, u}f(x + X_{s, S_i-})] \phi(X_{s, S_i} - X_{s, S_i-})$$

$$- \int_s^t \int_{\mathbb{R}^d} [P_{v,u}f(x + X_{s,v-} + z) - P_{u,v}f(x + X_{s,v-})] \phi(z) \nu(dz) dv.$$

Lemma 4.5. *We have that $F_t = F_t(x; s, u; f, \phi)$ is an \mathbb{F} -martingale for $t \in [s, u]$. Moreover, $\mathbb{E}\|F_t\|^p < \infty$ for each $p > 0$.*

Lemma 4.6. $G_t(x; s, u; g) = F_t(x; s, u; g, 1) + P_{s,u}g(x)$.

Analogously to [3, (21)-(22)] one has that for each $x^* \in X^*$ the quadratic variations of $\langle F_t(x; s, u; f, \phi), x^* \rangle$ and $\langle G_t(x; s, u; g), x^* \rangle$ satisfy the following a.s. identities,

$$[\langle F, x^* \rangle]_t = \sum_{s < S_i \leq t} \left(\langle P_{S_i, u}f(x + X_{s, S_i}) - P_{S_i, u}f(x + X_{s, S_i-}), x^* \rangle \right)^2 \phi^2(\Delta X_{s, S_i}),$$

$$[\langle G, x^* \rangle]_t = |\langle P_{s, u}g(x), x^* \rangle|^2 + \sum_{s < S_i \leq t} \left(\langle P_{S_i, u}g(x + X_{s, S_i}) - P_{S_i, u}g(x + X_{s, S_i-}), x^* \rangle \right)^2.$$

It follows that for each $f \in C_c(\mathbb{R}^d; X)$, $(F_t(x; s, u; f, \phi))_{t \in [s, u]}$ is weakly differentially subordinated to $(G_t(x; s, u; f))_{t \in [s, u]}$ and by Theorem 3.18 one has for each $t \in [s, u]$

$$\mathbb{E}\|F_t(x; s, u; f, \phi)\|^p \leq \beta_{p, X}^p \mathbb{E}\|G_t(x; s, u; f)\|^p.$$

Note that $G_u(x; s, u; f) = f(x + X_{s, u})$, so

$$(4.4) \quad \begin{aligned} \int_{\mathbb{R}^d} \mathbb{E}\|F_u(x; s, u; f, \phi)\|^p dx &\leq \beta_{p, X}^p \int_{\mathbb{R}^d} \mathbb{E}\|f(x + X_{s, u})\|^p dx \\ &= \beta_{p, X}^p \|f\|_{L^p(\mathbb{R}^d; X)}^p. \end{aligned}$$

Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Consider the linear functional on $L^{p'}(\mathbb{R}^d; X^*)$:

$$L^{p'}(\mathbb{R}^d; X^*) \ni g \mapsto \int_{\mathbb{R}^d} \mathbb{E}\langle F_u(x; s, u; f, \phi), g(x + X_{s, u}) \rangle dx.$$

Then by Hölder's inequality and (4.4) one has

$$\int_{\mathbb{R}^d} \mathbb{E}|\langle F_u(x; s, u; f, \phi), g(x + X_{s, u}) \rangle| dx \leq \beta_{p, X} \|f\|_{L^p(\mathbb{R}^d; X)} \|g\|_{L^{p'}(\mathbb{R}^d; X^*)}.$$

By Theorem 1.3.10 and Theorem 1.3.21 in [23], $(L^{p'}(\mathbb{R}^d; X^*))^* = L^p(\mathbb{R}^d; X)$, so there exists $h \in L^p(\mathbb{R}^d; X)$ such that for each $g \in L^{p'}(\mathbb{R}^d; X^*)$

$$\int_{\mathbb{R}^d} \mathbb{E}\langle F_u(x; s, u; f, \phi), g(x + X_{s, u}) \rangle dx = \int_{\mathbb{R}^d} \langle h(x), g(x) \rangle dx,$$

and

$$(4.5) \quad \|h\|_{L^p(\mathbb{R}^d; X)} \leq \beta_{p, X} \|f\|_{L^p(\mathbb{R}^d; X)}.$$

In particular, since X is finite dimensional

$$(4.6) \quad \int_{\mathbb{R}^d} \mathbb{E}F_u(x; s, u; f, \phi)g(x + X_{s, u}) dx = \int_{\mathbb{R}^d} h(x)g(x) dx, \quad g \in L^{p'}(\mathbb{R}^d).$$

For each $s < 0$ define $m_s : \mathbb{R}^d \rightarrow \mathbb{C}$ as follows

$$m_s(\xi) = \begin{cases} \left(1 - e^{2|s|\Psi(\xi)}\right) \frac{1}{\Psi(\xi)} \int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1) \phi(z) \nu(dz), & \Psi(\xi) \neq 0, \\ 0, & \Psi(\xi) = 0. \end{cases}$$

Let $u = 0$. Then analogously to [3, (32)], by (4.6) one obtains

$$\mathcal{F}(h)(\xi) = m_s(\xi)\mathcal{F}(f)(\xi), \quad \xi \in \mathbb{R}^d.$$

Let T_{m_s} be the Fourier multiplier on $L^2(\mathbb{R}^d; X)$ with symbol m_s (that is bounded by 1). By (4.5) one obtains that T_{m_s} extends uniquely to a bounded operator on $L^p(\mathbb{R}^d; X)$ with $\|T_{m_s}\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))} \leq \beta_{p,X}$. Let T_m be the multiplier on $L^2(\mathbb{R}^d; X)$ with the symbol m given by

$$m(\xi) = \begin{cases} \frac{1}{\Psi(\xi)} \int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1)\phi(z)\nu(dz), & \Psi(\xi) \neq 0, \\ 0, & \Psi(\xi) = 0. \end{cases}$$

Note that m is a pointwise limit of m_s as $s \rightarrow -\infty$. Also note that $T_{m_s}f \rightarrow T_m f$ in $L^2(\mathbb{R}^d; X)$ as $s \rightarrow -\infty$ for each $f \in C_c(\mathbb{R}^d; X)$ by Plancherel's theorem. Therefore by Fatou's lemma one has that for each $f \in C_c(\mathbb{R}^d; X)$ the following holds:

$$\|T_m f\|_{L^p(\mathbb{R}^d; X)} \leq \varliminf_{s \rightarrow -\infty} \|T_{m_s} f\|_{L^p(\mathbb{R}^d; X)} \leq \beta_{p,X} \|f\|_{L^p(\mathbb{R}^d; X)},$$

hence T_m uniquely extends to a bounded operator on $L^p(\mathbb{R}^d; X)$ with

$$\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))} \leq \beta_{p,X}.$$

4.2. Examples of Theorem 4.1. In this subsection X is a UMD Banach space, $p \in (1, \infty)$. The examples will be mainly the same as were given in [2, Chapter 4] with some author's remarks. Recall that we set $\frac{a}{0} = 0$ for any $a \in \mathbb{C}$.

Example 4.7. Let V_1, V_2 be two nonnegative Lévy measures on \mathbb{R}^d such that $V_1 \leq V_2$. Let

$$m(\xi) = \frac{\int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z))V_1(dz)}{\int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z))V_2(dz)}, \quad \xi \in \mathbb{R}^d.$$

Then $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))} \leq \beta_{p,X}$.

Example 4.8. Let μ_1, μ_2 be two nonnegative measures on S^{d-1} such that $\mu_1 \leq \mu_2$. Let

$$m(\xi) = \frac{\int_{S^{d-1}} (\xi \cdot \theta)^2 \mu_1(d\theta)}{\int_{S^{d-1}} (\xi \cdot \theta)^2 \mu_2(d\theta)}, \quad \xi \in \mathbb{R}^d.$$

Then $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))} \leq \beta_{p,X}$.

Example 4.9 (Beurling-Ahlfors transform). Let $d = 2$. Put $\mathbb{R}^2 = \mathbb{C}$. Then the Fourier multiplier T_m with a symbol $m(z) = \frac{\bar{z}^2}{|z|^2}$, $z \in \mathbb{C}$, has the norm at most $2\beta_{p,X}$ on $L^p(\mathbb{R}^d; X)$. This multiplier is also known as the *Beurling-Ahlfors transform*. It is well-known that $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^2; X))} \geq \beta_{p,X}$. There is quite an old problem whether $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^2; X))} = \beta_{p,X}$. This question was firstly posed by Iwaniec in [24] in \mathbb{C} . Nevertheless it was neither proved nor disproved even in the scalar-valued case. We refer the reader to [1] and [23] for further details.

Example 4.10. Let $\alpha \in (0, 2)$, μ be a finite positive measure on S^{d-1} , ψ be a measurable function on S^{d-1} such that $|\psi| \leq 1$. Let

$$m(\xi) = \frac{\int_{S^{d-1}} |(\xi \cdot \theta)|^\alpha \psi(\theta) \mu(d\theta)}{\int_{S^{d-1}} |(\xi \cdot \theta)|^\alpha \mu(d\theta)}, \quad \xi \in \mathbb{R}^d.$$

Then analogously to [2, (4.7)], $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))} \leq \beta_{p,X}$.

Example 4.11 (Double Riesz transform). Let $\alpha \in (0, 2]$. Let

$$m(\xi) = \frac{|\xi_1|^\alpha}{|\xi_1|^\alpha + \dots + |\xi_d|^\alpha}, \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,$$

Then according to Example 4.10, $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))} \leq \beta_{p, X}$. Note that if $\alpha = 2$, then T_m is a double Riesz transform. (In the forthcoming paper [53] it is shown that the norm $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))}$ does not depend on α and equals the $\text{UMD}_p^{\{0,1\}}$ constant of X).

Example 4.12. Let $\alpha \in [0, 2]$, $d \geq 2$. Let

$$m(\xi) = \frac{|\xi_1|^\alpha - |\xi_2|^\alpha}{|\xi_1|^\alpha + \dots + |\xi_d|^\alpha}, \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,$$

Then by Example 4.10, $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))} \leq \beta_{p, X}$. Moreover, if $d = 2$, $\alpha \in [1, 2]$, then $\max_{\xi \in \mathbb{R}^2} m(\xi) = 1$, $\min_{\xi \in \mathbb{R}^2} m(\xi) = -1$ and $m|_{S^1} \in W^{2,1}(S^1)$. Therefore due to Proposition 3.4, Proposition 3.8 and Remark 3.9 in [21] one has $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^2; X))} \geq \beta_{p, X}$. This together with Theorem 4.1 implies $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^2; X))} = \beta_{p, X}$, which extends [21, Theorem 1.1], where the same assertion was proven for $\alpha = 2$.

Example 4.13. Let μ be a nonnegative Borel measure on S^{d-1} , $\psi \in L^\infty(S^{d-1}, \mu)$, $\|\psi\|_\infty \leq 1$. Let

$$m(\xi) = \frac{\int_{S^{d-1}} \ln(1 + (\xi \cdot \theta)^{-2}) \psi(\theta) \mu(d\theta)}{\int_{S^{d-1}} \ln(1 + (\xi \cdot \theta)^{-2}) \mu(d\theta)}, \quad \xi \in \mathbb{R}^d.$$

Then $\|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))} \leq \beta_{p, X}$.

5. HILBERT TRANSFORM AND GENERAL CONJECTURE

In this section we assume that X is a finite dimensional Banach space to avoid difficulties with stochastic integration. Many of the assertions below can be extended to the general UMD Banach space case by using the same techniques as in the proof of Theorem 3.3.

5.1. Hilbert transform and Burkholder functions. It turns out that the generalization of Theorem 3.18 to the case of continuous martingales is connected with the boundedness of the Hilbert transform. The Fourier multiplier $\mathcal{H} \in \mathcal{L}(L^2(\mathbb{R}))$ with the symbol $m \in L^\infty(\mathbb{R})$ such that $m(t) = -i \operatorname{sign}(t)$, $t \in \mathbb{R}$, is called the *Hilbert transform*. This operator can be extended to a bounded operator on $L^p(\mathbb{R})$, $1 < p < \infty$ (see [47] and [23, Chapter 5.1] for the details).

Let X be a Banach space. Then one can extend the Hilbert transform \mathcal{H} to $\mathcal{S}(\mathbb{R}) \otimes X$ in the same way as it was done in (4.1). Denote this extension by \mathcal{H}_X . By [9, Lemma 2] and [20, Theorem 3] the following holds true:

Theorem 5.1 (Bourgain, Burkholder). *Let X be a Banach space. Then X is a UMD Banach space if and only if \mathcal{H}_X can be extended to a bounded operator on $L^p(\mathbb{R}; X)$ for each $1 < p < \infty$. Moreover, then*

$$(5.1) \quad \sqrt{\beta_{p, X}} \leq \|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \leq \beta_{p, X}^2.$$

The proof of the right-hand side of (5.1) is based on the following result.

Proposition 5.2. *Let X be a finite dimensional Banach space, B_1, B_2 be two real-valued Wiener processes, $f_1, f_2 : \mathbb{R}_+ \times \Omega \rightarrow X$ be two stochastically integrable functions. Let us define $M := f_1 \cdot B_1 + f_2 \cdot B_2$, $N := f_2 \cdot B_1 - f_1 \cdot B_2$. Then for each $T \geq 0$*

$$(\mathbb{E}\|N_T\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E}\|M_T\|^p)^{\frac{1}{p}}.$$

Proof. The theorem follows from [54]. Nevertheless we wish to illustrate an easier and more specific proof. Let $\tilde{B}_1, \tilde{B}_2 : \tilde{\Omega} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be two Wiener process defined on an enlarged probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with an enlarged filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$ such that \tilde{B}_1 and \tilde{B}_2 are independent of \mathcal{F} . Then by applying the decoupling theorem [23, Theorem 4.4.1] twice (see also [31]) and the fact that $-\tilde{B}_1$ is a Wiener process

$$\begin{aligned} \mathbb{E}\|N_T\|^p &= \mathbb{E}\|(f_2 \cdot B_1)_T - (f_1 \cdot B_2)_T\|^p \leq \beta_{p,X}^p \mathbb{E}\|(f_2 \cdot \tilde{B}_1)_T - (f_1 \cdot \tilde{B}_2)_T\|^p \\ &= \beta_{p,X}^p \mathbb{E}\|(f_1 \cdot (-\tilde{B}_2))_T + (f_2 \cdot \tilde{B}_1)_T\|^p \\ &\leq \beta_{p,X}^{2p} \mathbb{E}\|(f_1 \cdot B_1)_T + (f_2 \cdot B_2)_T\|^p \\ &= \beta_{p,X}^{2p} \mathbb{E}\|M_T\|^p. \end{aligned}$$

□

Let $p \in (1, \infty)$. A natural question is whether there exists a constant $C_p > 0$ such that

$$(5.2) \quad \|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \leq C_p \beta_{p,X}.$$

Then the following theorem is applicable.

Theorem 5.3. *Let X be a Banach space, $p \in (1, \infty)$. Then there exists $C_p \geq 1$ such that (5.2) holds if there exists **some** Burkholder function $U : X \times X \rightarrow \mathbb{R}$ such that U is continuous and a.s. twice Fréchet differentiable, $U(x, y) \geq \|y\|^p - (C_p \beta_{p,X})^p \|x\|^p$ for any $x, y \in X$, $U(\alpha x, \alpha y) = |\alpha|^p U(x, y)$ for any $\alpha \in \mathbb{R}$ and $x, y \in X$, and the function*

$$t \mapsto U(x + tz_1, y + tz_2) + U(x + tz_2, y - tz_1), \quad t \in \mathbb{R},$$

or, equivalently,

$$t \mapsto U(x + tz_1, y + tz_2) + U(x - tz_2, y + tz_1), \quad t \in \mathbb{R},$$

is concave for each $x, y, z_1, z_2 \in X$ at $t = 0$.

For the proof of Theorem 5.3 we will need a variant of the Itô formula for a general basis of a finite dimensional linear space.

Definition 5.4. *Let d be a natural number, E be a d -dimensional linear space, $(e_n)_{n=1}^d$ be a basis of E . Then $(e_n^*)_{n=1}^d \subset E^*$ is called the corresponding dual basis of $(e_n)_{n=1}^d$ if $(e_n, e_m^*) = \delta_{nm}$ for each $m, n = 1, \dots, d$.*

Note that the corresponding dual basis is uniquely determined. Moreover, if $(e_n^*)_{n=1}^d$ is the corresponding dual basis of $(e_n)_{n=1}^d$, then, the other way around, $(e_n)_{n=1}^d$ is the corresponding dual basis of $(e_n^*)_{n=1}^d$ (here we identify E^{**} with E in the natural way).

The following theorem is a variation of [27, Theorem 26.7] which does not use the Hilbert space structure of a finite dimensional space.

Theorem 5.5 (Itô formula). *Let d be a natural number, X be a d -dimensional Banach space, $f \in C^2(X)$, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a martingale. Let $(x_n)_{n=1}^d$ be a basis of X , $(x_n^*)_{n=1}^d$ be the corresponding dual basis. Then for each $t \geq 0$*

$$(5.3) \quad \begin{aligned} f(M_t) &= f(M_0) + \int_0^t \langle \partial_x f(M_{s-}), dM_s \rangle \\ &+ \frac{1}{2} \int_0^t \sum_{n,m=1}^d f_{x_n, x_m}(M_{s-}) d[\langle M, x_n^* \rangle, \langle M, x_m^* \rangle]_s^c \\ &+ \sum_{s \leq t} (\Delta f(M_s) - \langle \partial_x f(M_{s-}), \Delta M_s \rangle). \end{aligned}$$

Proof. To apply [27, Theorem 26.7] one needs only to endow X with a proper Euclidean norm $\|\cdot\|$. Define $\|x\| = \sum_{n=1}^d \langle x, x_n^* \rangle^2$ for each $x \in X$. Then $(x_n)_{n=1}^d$ is an orthonormal basis of $(X, \|\cdot\|)$, $M = \sum_{n=1}^d \langle M, x_n^* \rangle x_n$ is a decomposition of M in this orthonormal basis, and therefore (5.3) is equivalent to the formula in [27, Theorem 26.7]. \square

Proof of Theorem 5.3. Let M and N be as in Proposition 5.2. By the approximation argument we can suppose that M and N have absolutely continuous distributions. Let d be the dimension of X . Then by the Itô formula in Theorem 5.5

$$(5.4) \quad \begin{aligned} \mathbb{E}\|N_t\|_X^p - (C_p \beta_{p,X})^p \mathbb{E}\|M_t\|_X^p &\leq \mathbb{E}U(M_t, N_t) = \mathbb{E}U(M_0, N_0) \\ &+ \mathbb{E} \int_0^t \langle \partial_x U(M_s, N_s), dM_s \rangle \\ &+ \mathbb{E} \int_0^t \langle \partial_y U(M_s, N_s), dN_s \rangle + \frac{1}{2} \mathbb{E}I, \end{aligned}$$

where

$$(5.5) \quad \begin{aligned} I &= \int_0^t \sum_{i,j=1}^d (U_{x_i, x_j}(M_s, N_s) d[\langle x_i^*, M_s \rangle, \langle x_j^*, M_s \rangle] \\ &+ 2U_{x_i, y_j}(M_s, N_s) d[\langle x_i^*, M_s \rangle, \langle y_j^*, N_s \rangle] \\ &+ U_{y_i, y_j}(M_s, N_s) d[\langle y_i^*, N_s \rangle, \langle y_j^*, N_s \rangle]), \end{aligned}$$

where $(x_i)_{i=1}^d = (y_i)_{i=1}^d \subset X$ is the same basis of X , and $(x_i^*)_{i=1}^d = (y_i^*)_{i=1}^d \subset X^*$ are the same corresponding dual bases of X^* .

Notice that by Remark 3.10 $\mathbb{E}U(M_0, N_0) \leq 0$ since $\|N_0\| \leq \|M_0\|$ a.s. and $C_p, \beta_{p,X} \geq 1$, and that

$$\mathbb{E} \left(\int_0^t \langle \partial_x U(M_s, N_s), dM_s \rangle + \int_0^t \langle \partial_y U(M_s, N_s), dN_s \rangle \right) = 0,$$

since due to the same type of discussion as was done in the proof of Theorem 3.18, $\int_0^\cdot \langle \partial_x U(M_s, N_s), dM_s \rangle + \int_0^\cdot \langle \partial_y U(M_s, N_s), dN_s \rangle$ is a martingale which starts at zero.

Let us now prove that $I \leq 0$. For each $i = 1, 2, \dots, d$ we define $f_i^1 := \langle x_i^*, f_1 \rangle$ and $f_i^2 := \langle x_i^*, f_2 \rangle$. Then for each $i, j = 1, 2, \dots, d$ one has that

$$(5.6) \quad d[\langle x_i^*, M_s \rangle, \langle x_j^*, M_s \rangle] = d[\langle y_i^*, N_s \rangle, \langle y_j^*, N_s \rangle] = (f_i^1 f_j^1 + f_i^2 f_j^2) dt,$$

and

$$(5.7) \quad d[\langle x_i^*, M_s \rangle, \langle y_j^*, N_s \rangle] = (f_i^1 f_j^2 - f_i^2 f_j^1) dt.$$

Notice also that for each $x, y \in X$

$$\begin{aligned}
 \frac{\partial^2}{\partial u^2} U(x + uf_1, y + uf_2)|_{u=0} &= \sum_{i,j=1}^d ((U_{x_i^*, x_j^*}(x, y) f_i^1 f_j^1 + 2U_{x_i^*, y_j^*}(x, y) f_i^1 f_j^2 \\
 &\quad + U_{y_i^*, y_j^*}(x, y) f_i^2 f_j^2), \\
 (5.8) \quad \frac{\partial^2}{\partial u^2} U(x + uf_2, y - uf_1)|_{u=0} &= \frac{\partial^2}{\partial u^2} U(x - uf_2, y + uf_1)|_{u=0} \\
 &= \sum_{i,j=1}^d ((U_{x_i^*, x_j^*}(x, y) f_i^2 f_j^2 - 2U_{x_i^*, y_j^*}(x, y) f_i^2 f_j^1 + U_{y_i^*, y_j^*}(x, y) f_i^1 f_j^1).
 \end{aligned}$$

Therefore by (5.5), (5.6), (5.7), and (5.8) we have that

$$\begin{aligned}
 I &= \int_0^t \sum_{i,j=1}^d ((U_{x_i^*, x_j^*}(M_{s-}, N_{s-})(f_i^1 f_j^1 + f_i^2 f_j^2) + 2U_{x_i^*, y_j^*}(M_{s-}, N_{s-})(f_i^1 f_j^2 - f_i^2 f_j^1) \\
 &\quad + U_{y_i^*, y_j^*}(M_{s-}, N_{s-})(f_i^1 f_j^1 + f_i^2 f_j^2)) dt = \int_0^t \frac{\partial^2}{\partial u^2} U(M_{s-} + uf_1, N_{s-} + uf_2)|_{u=0} \\
 &\quad + \frac{\partial^2}{\partial u^2} U(M_{s-} + uf_2, N_{s-} - uf_1)|_{u=0} ds \\
 &= \int_0^t \frac{\partial^2}{\partial u^2} \left(U(M_{s-} + uf_1, N_{s-} + uf_2) + U(M_{s-} + uf_2, N_{s-} - uf_1) \right) \Big|_{u=0} ds,
 \end{aligned}$$

and thanks to the concavity of $U(x + uf_1, y + uf_2) + U(x + uf_2, y - uf_1)$ in point $u = 0$ for each $x, y \in X$ one deduces that a.s. $I \leq 0$. Then thanks to (5.4) one has that

$$(5.9) \quad \mathbb{E} \|N_t\|_X^p - (C_p \beta_{p,X})^p \mathbb{E} \|M_t\|_X^p \leq \mathbb{E} U(M_t, N_t) \leq 0.$$

Now one can prove that (5.9) implies (5.2) in the same way as it was done for instance in [20, Theorem 3], [6, p.592] or [15, Chapter 3]. \square

Remark 5.6. Note that if X is a finite dimensional Hilbert space, then one gets condition (iii) in Theorem 5.3 for free from [52]. Indeed, let $U : X \times X \rightarrow \mathbb{R}$ be as in [52, p. 527], namely

$$U(x, y) = p(1 - 1/p^*)^{p-1} (\|y\| - (p^* - 1)\|x\|)(\|x\| + \|y\|)^{p-1}, \quad x, y \in X.$$

Then U is a.s. twice Fréchet differentiable, and thanks to the property (c) of U , which is given on [52, p. 527], for all nonzero $x, y \in X$ there exists a constant $c(x, y) \geq 0$ such that

$$\begin{aligned}
 \langle \partial_{xx} U(x, y), (h, h) \rangle + 2\langle \partial_{xy} U(x, y), (h, k) \rangle + \langle \partial_{yy} U(x, y), (k, k) \rangle \\
 \leq -c(x, y)(\|h\|^2 - \|k\|^2), \quad h, k \in X.
 \end{aligned}$$

Therefore for any $z_1, z_2 \in X$

$$\begin{aligned}
 \frac{\partial^2}{\partial t^2} [U(x + tz_1, y + tz_2) + U(x + tz_2, y - tz_1)] \Big|_{t=0} \\
 = \langle \partial_{xx} U(x, y), (z_1, z_1) \rangle + 2\langle \partial_{xy} U(x, y), (z_1, z_2) \rangle + \langle \partial_{yy} U(x, y), (z_2, z_2) \rangle \\
 \quad + \langle \partial_{xx} U(x, y), (z_2, z_2) \rangle - 2\langle \partial_{xy} U(x, y), (z_2, z_1) \rangle + \langle \partial_{yy} U(x, y), (z_1, z_1) \rangle \\
 \leq -c(x, y)(\|z_1\|^2 - \|z_2\|^2) - c(x, y)(\|z_2\|^2 - \|z_1\|^2) = 0.
 \end{aligned}$$

5.2. General conjecture. By Theorem 5.3 the estimate (5.2) is a direct corollary of the following conjecture.

Conjecture 5.7. *Let X be a finite dimensional Banach space, $p \in (1, \infty)$. Then there exists $C_p \geq 1$ such that for each pair of continuous martingales $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ such that N is weakly differentially subordinated to M one has that for each $t \geq 0$*

$$(5.10) \quad (\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq C_p \beta_{p,X} (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}.$$

Remark 5.8. Notice that in [54] the estimate (5.10) is proven with the constant $\beta_{p,X}^2$ instead of $C_p \beta_{p,X}$. Moreover, it is shown in [54] that C_p can not be less than 1.

We wish to finish by pointing out some particular cases in which Conjecture 5.7 holds. These results are about stochastic integration with respect to a Wiener process. Recall that we assume that X is a finite dimensional space. Later we will need a couple of definitions.

Let $W^H : \mathbb{R}_+ \times H \rightarrow L^2(\Omega)$ be an H -cylindrical Brownian motion, i.e.

- $(W^H h_1, \dots, W^H h_d) : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ is a d -dimensional Wiener process for all $d \geq 1$ and $h_1, \dots, h_d \in H$,
- $\mathbb{E}W^H(t)h W^H(s)g = \langle h, g \rangle \min\{t, s\} \forall h, g \in H, t, s \geq 0$.

(We refer the reader to [16, Chapter 4.1] for further details). Let X be a Banach space, $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be elementary progressive of the form (2.2). Then we define a *stochastic integral* $\Phi \cdot W^H : \mathbb{R}_+ \times \Omega \rightarrow X$ of Φ with respect to W^H in the following way:

$$(\Phi \cdot W^H)_t = \sum_{k=1}^K \sum_{m=1}^M \mathbf{1}_{B_{mk}} \sum_{n=1}^N (W^H(t_k \wedge t)h_n - W^H(t_{k-1} \wedge t)h_n)x_{kmn}, \quad t \geq 0.$$

The following lemma is a multidimensional variant of [28, (3.2.19)] and it is closely connected with Lemma 2.3.

Lemma 5.9. *Let $X = \mathbb{R}$, $\Phi, \Psi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, \mathbb{R})$ be elementary progressive. Then for all $t \geq 0$ a.s.*

$$[\Phi \cdot W^H, \Psi \cdot W^H]_t = \int_0^t \langle \Phi^*(s), \Psi^*(s) \rangle ds.$$

The reader can find more on stochastic integration with respect to an H -cylindrical Brownian motion in the UMD case in [33].

Theorem 5.10. *Let X be a finite dimensional Banach space, W^H be an H -cylindrical Brownian motion, $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be stochastically integrable with respect to W^H function. Let $A \in \mathcal{L}(H)$ be self-adjoint. Then*

$$(5.11) \quad (\mathbb{E}\|((\Phi A) \cdot W^H)_\infty\|_X^p)^{\frac{1}{p}} \leq \beta_{p,X} \|A\| (\mathbb{E}\|(\Phi \cdot W^H)_\infty\|_X^p)^{\frac{1}{p}}.$$

Notice that by Lemma 5.9 for each $x^* \in X^*$ and $0 \leq s < t < \infty$ a.s.

$$\begin{aligned} [((\Phi A) \cdot W^H, x^*)]_t - [((\Phi A) \cdot W^H, x^*)]_s &= \int_s^t \|A\Phi^*(r)x^*\|^2 dr \\ &\leq \|A\|^2 \int_s^t \|\Phi^*(r)x^*\|^2 dr \end{aligned}$$

$$= \|A\|^2([\langle \Phi \cdot W^H, x^* \rangle]_t - [\langle \Phi \cdot W^H, x^* \rangle]_s).$$

Hence if $\|A\| \leq 1$, then $(\Phi A) \cdot W^H$ is weakly differentially subordinated to $\Phi \cdot W^H$, and therefore Theorem 5.10 provides us with a special case of Conjecture 5.7.

Proof of Theorem 5.10. Due to [33, Theorem 3.6] it is enough to show (5.11) for elementary progressive process Φ . Let $(h_n)_{n \geq 1}$ be an orthogonal basis of H , and let Φ be of the form (2.2). For each $n \geq 1$ we define $P_n \in \mathcal{L}(H)$ as an orthonormal projection onto $\text{span}(h_1, \dots, h_n)$, and set $A_n := P_n A P_n$. Notice that $\|A_n\| \leq \|A\|$. Then

$$\|((\Phi A - \Phi A_n) \cdot W^H)_\infty\|_{L^p(\Omega; X)} \rightarrow 0, \quad n \rightarrow \infty,$$

so it is sufficient to prove (5.11) for A with a finite dimensional range, and we can suppose that there exists $d \geq 1$ such that $\text{ran } A \subset \text{span}(h_1, \dots, h_d)$. This implies that A is compact self-adjoint, so we can change the first d vectors h_1, \dots, h_d of the orthonormal basis in such a way that $A = \sum_{n=1}^d \lambda_n h_n \otimes h_n$, where $(\lambda_n)_{n \geq 1}$ is a real-valued sequence. Without loss of generality we can assume that $|\lambda_1| \geq \dots \geq |\lambda_d|$ and $|\lambda_1| = \|A\|$. Notice that under this change of coordinates Φ remains elementary progressive (perhaps of a different form). Therefore by the martingale transform theorem [23, Theorem 4.2.25]:

$$\begin{aligned} \mathbb{E}\|((\Phi A) \cdot W^H)_\infty\|_X^p &= \mathbb{E}\left\| \sum_{n=1}^d ((\Phi A h_n) \cdot W^H(h_n))_\infty \right\|_X^p \\ &= \|A\|^p \mathbb{E}\left\| \sum_{n=1}^d \frac{\lambda_n}{\|A\|} ((\Phi h_n) \cdot W^H(h_n))_\infty \right\|_X^p \\ &\leq \beta_{p,X}^p \|A\|^p \mathbb{E}\left\| \sum_{n=1}^d ((\Phi h_n) \cdot W^H(h_n))_\infty \right\|_X^p. \end{aligned}$$

The last inequality holds because of structure of Φ so that one can rewrite $(\Phi h_n) \cdot W^H(h_n)$ as a summation in time, and because $(W^H(h_n))_{n \geq 1}$ is a sequence of independent Wiener processes. \square

Remark 5.11. Theorem 5.10 in fact can be shown using [21, Proposition 3.7.(i)].

Remark 5.12. An analogue of Theorem 5.10 for antisymmetric A (i.e. A such that $A^* = -A$) remains open. It is important for instance for the possible estimate (5.2). Indeed, in Proposition 5.2 the Hilbert space H can be taken 2-dimensional, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ is such that $\Phi \begin{pmatrix} a \\ b \end{pmatrix} = a f_1 + b f_2$ for each $a, b \in \mathbb{R}$. Then $M = \Phi \cdot W^H$, $N = (\Phi A) \cdot W^H$, and if one shows (5.11) for an antisymmetric operator A , then one automatically gains (5.2).

The next theorem shows that Conjecture 5.7 holds for stochastic integrals with respect to a one-dimensional Wiener process.

Theorem 5.13. *Let X be a finite dimensional Banach space, $W : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a one-dimensional Wiener process, $\Phi, \Psi : \mathbb{R}_+ \times \Omega \rightarrow X$ be stochastically integrable with respect to W , $M = \Phi \cdot W$, $N = \Psi \cdot W$. Let N be weakly differentially subordinated to M . Then for each $p \in (1, \infty)$,*

$$(5.12) \quad \mathbb{E}\|N_\infty\|^p \leq \beta_{p,X}^p \mathbb{E}\|M_\infty\|^p.$$

Proof. Without loss of generality suppose that there exists $T \geq 0$ such that $\Phi \mathbf{1}_{[T, \infty]} = \Psi \mathbf{1}_{[T, \infty]} = 0$. Since N is weakly differentially subordinated to M , by the Itô isomorphism for each $x^* \in X^*$, $0 \leq s < t < \infty$ we have a.s.

$$\begin{aligned} [\langle x^*, N \rangle]_t - [\langle x^*, N \rangle]_s &= \int_s^t |\langle x^*, \Psi(r) \rangle|^2 dr \\ &\leq \int_s^t |\langle x^*, \Phi(r) \rangle|^2 dr = [\langle x^*, M \rangle]_t - [\langle x^*, M \rangle]_s. \end{aligned}$$

Therefore we can deduce that $|\langle x^*, \Psi \rangle| \leq |\langle x^*, \Phi \rangle|$ a.s. on $\mathbb{R}_+ \times \Omega$. By Lemma 3.6 there exists progressively measurable $a : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that $|a| \leq 1$ on $\mathbb{R}_+ \times \Omega$ and $\Psi = a\Phi$ a.s. on $\mathbb{R}_+ \times \Omega$. Now for each $n \geq 1$ set $a_n : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$, $\Phi_n : \mathbb{R}_+ \times \Omega \rightarrow X$ be elementary progressively measurable such that $|a_n| \leq 1$, $a_n \rightarrow a$ a.s. on $\mathbb{R}_+ \times \Omega$ and $\mathbb{E} \int_0^T \|\Phi(t) - \Phi_n(t)\|^2 dt \rightarrow 0$ as $n \rightarrow \infty$. Then by the triangle inequality

$$(5.13) \quad \left(\mathbb{E} \int_0^T \|\Psi(t) - a_n(t)\Phi_n(t)\|^2 dt \right)^{\frac{1}{2}} \leq \left(\mathbb{E} \int_0^T \|\Phi(t)\|^2 (a(t) - a_n(t))^2 dt \right)^{\frac{1}{2}} + \left(\mathbb{E} \int_0^T \|\Phi(t) - \Phi_n(t)\|^2 a_n^2 dt \right)^{\frac{1}{2}},$$

which vanishes as $n \rightarrow \infty$ by the dominated convergence theorem. For each $n \geq 1$ the inequality

$$\left(\mathbb{E} \|((a_n \Phi_n) \cdot W)_\infty\|^p \right)^{\frac{1}{p}} \leq \beta_{p,X} \left(\mathbb{E} \|(\Phi_n \cdot W)_\infty\|^p \right)^{\frac{1}{p}}$$

holds thanks to the martingale transform theorem [23, Theorem 4.2.25]. Then (5.12) follows from the previous estimate and (5.13) when one lets n go to infinity. \square

Remark 5.14. Let W be a one-dimensional Wiener process, \mathbb{F} be a filtration which is generated by W . Let $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ be \mathbb{F} -martingales such that $M_0 = N_0 = 0$ and N is weakly differentially subordinated to M . Then thanks to the Itô isomorphism [33, Theorem 3.5] there exist progressively measurable $\Phi, \Psi : \mathbb{R}_+ \times \Omega \rightarrow X$ such that $M = \Phi \cdot W$ and $N = \Psi \cdot W$, and thanks to Theorem 5.13

$$\mathbb{E} \|N_\infty\|^p \leq \beta_{p,X}^p \mathbb{E} \|M_\infty\|^p, \quad p \in (1, \infty).$$

This shows that on certain probability spaces the estimate (5.10) automatically holds with a constant $C_p = 1$.

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